THE RELATIVE WHITNEY TRICK AND ITS APPLICATIONS

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Abstract. We introduce a geometric operation, which we call the relative Whitney trick, that removes a single double point between properly immersed surfaces in a 4-manifold with boundary. Using the relative Whitney trick we prove that every link in a homology sphere is homotopic to a link that is topologically slice in a contractible topological 4-manifold. We further prove that any link in a homology sphere is order k Whitney tower concordant to a link in S^3 for all k. Finally, we explore the minimum Gordian distance from a link in S^3 to a homotopically trivial link. Extending this notion to links in homology spheres, we use the relative Whitney trick to make explicit computations for 3-component links and establish bounds in general.

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1. Introduction

The Whitney trick is a fundamental technique of geometric topology and its general failure in 4-manifolds is widely cited as the reason that topology in this dimension is so interesting and unusual. In ambient dimension four, the (topological) Whitney trick seeks to remove a pair of oppositely signed transverse intersection points between two locally flat immersed surfaces. In this article, we will introduce a geometric technique called the *relative Whitney trick* that removes a single point of intersection between properly immersed locally flat surfaces in a 4-manifold with boundary. The details of the relative Whitney trick are given in Section 2, but we sketch the procedure now. Suppose S_1 and S_2 are surfaces in a 4-manifold W and $p \in S_1 \cap S_2$. We find an immersed disk with an embedded arc of its boundary lying on the boundary of the 4-manifold and use it to guide a regular homotopy of S_1 that removes the point of intersection p, at the cost of changing ∂S_1 by a homotopy along that arc in the boundary; see Figure 1.

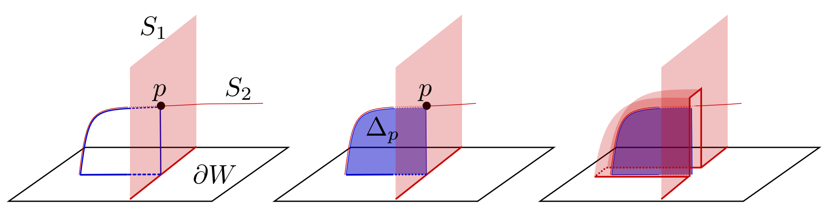

Figure 1. A schematic for the relative Whitney trick. Possible singularities on the interior of the relative Whitney disk not depicted.

In comparison, the ordinary Whitney trick begins with two intersection points $p, q \in S_1 \cap S_2$ with opposite sign that are paired by a Whitney disk. This immersed disk guides the (ordinary) Whitney trick, which is a regular homotopy of S_1 with the effect of removing both intersection points p and q; see Figure 2. Any singularities present in the guiding Whitney disk will yield new singularities in the result of the Whitney move, and similarly for the relative Whitney trick.

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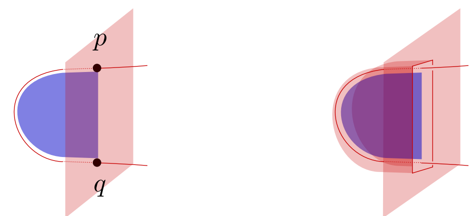

FIGURE 2. A schematic for the ordinary Whitney trick. Possible singularities on the interior of the Whitney disk not depicted.

Our applications of the relative Whitney trick all concern the study of links in homology 3-spheres. In this article, links $L = L_1 \cup \cdots \cup L_n$ will be ordered and oriented. Given a homology 3-sphere Y, recall it bounds a contractible topological 4-manifold [11, Theorem 1.4'], which we usually denote by X. The uniqueness of this 4-manifold, up to homeomorphisms fixing the boundary, follows from a standard argument using topological surgery theory and the 5-dimensional h-cobordism theorem.

1.1. **Slicing links in homology spheres up to homotopy.** We say a link L is *slice* if it bounds a collection of disjoint locally flat embedded disks D in X, and is moreover *freely slice* if $\pi_1(X \smallsetminus D)$ is a free group generated by the meridians of L. Two links L and L' are *freely homotopic* if there exists a continuous function $F\colon S^1 \times \{1, \dots, n\} \times [0,1] \to Y$ with $F(S^1, i, 0) = L_i$ and $F(S^1, i, 1) = L'_i$ for all $i = 1, \dots, n$.

The relative Whitney trick will be used in the proof of the following theorem, the first main result of the article.

Theorem 1.1. *Every link in a homology sphere is freely homotopic to a freely slice link.*

This result should be compared to the work of Austin-Rolfsen [2], who proved that any knot in a homology sphere is freely homotopic to a knot with trivial Alexander polynomial. Combined with a result of Freedman-Quinn [12, Theorem 11.7B], the Austin-Rolfsen result shows that any knot in a homology sphere can be reduced by a free homotopy to a freely slice knot. (For a historical discussion of Alexander polynomial 1 knots see [3, §21.6.3].) To extend this to links we will use the relative Whitney trick, together with the methods of topological surgery theory and a generalization of results of Cha-Kim-Powell [5] which give a sufficient condition for a link in a homology sphere to be freely slice. This condition arises from a surgery-theoretic link slicing approach as we now outline.

A link $L = L_1 \cup \cdots \cup L_n$ in a 3-manifold Y is a *boundary link* if there exists a collection of pairwise disjoint surfaces $F = F_1 \cup \cdots \cup F_n$ in Y, where F_i is a Seifert surface for L_i. Such F is called a *boundary link Seifert surface* for the boundary link L. A surgery-theoretic strategy to slice L is to construct a 4-manifold W with boundary M_L, the 0-surgery on L, such that when we glue 2-handles to the boundary, reversing the 0-surgery, we obtain a contractible 4-manifold. The desired slice disks are then the cocores of the 2-handles.

In an attempt to construct such a W, begin by pushing a boundary link Seifert surface $F = F_1 \cup \cdots \cup F_n$ for L into the contractible 4-manifold bounded by Y. Excise a tubular neighbourhood of the pushed in surface to obtain a 4-manifold X_F with free fundamental group generated by the meridians of L, and whose boundary decomposes as $\partial X_F = (S^1 \times \bigcup_i (F_i \setminus \mathring{D}^2)) \cup X_L$, where X_L denotes the link exterior. Let $H = H_1 \cup \cdots \cup H_n$ be a collection of 3-dimensional handlebodies where H_i has the same genus as F_i, and let $\varphi\colon F \cong \partial H$ denote a collection of homeomorphisms $\varphi_i\colon F_i \cong \partial H_i$. Form a new manifold $V_F := X_F \cup (S^1 \times H)$ by using the homeomorphism

$$\mathrm{id} \times \varphi\colon S^1 \times \bigcup_i (F_i \setminus \mathring{D}^2) \cong S^1 \times \bigcup_i (\partial H_i \setminus \phi(\mathring{D}^2))$$

to glue only along this part of the boundary. The resulting 4-manifold V_F has boundary the 0-surgery M_L, as desired, and it is moreover possible to choose φ so that $\pi_1(V_F)$ is still free and generated by the meridians of L. We would now like to know that there exist framed embedded 2-spheres in V_F that can be removed by surgery to kill the second homology.

As in Cha-Kim-Powell, we will obtain these embedded 2-spheres via a theorem of Freedman-Quinn [12, Theorem 6.1]. This theorem states that the presence of certain configurations of framed immersed spheres (specifically π_1-*null immersions of a union of transverse pairs with algebraically trivial intersections*; see Appendix A) imply the existence of the embedded spheres required for surgery. To build these configurations, Cha-Kim-Powell find properly immersed disks in the exterior of the pushed in boundary link Seifert surface, bounded by a basis for that surface, and cap them off with properly embedded discs in $H \subseteq V_F$ that are dual to the cores of the handlebodies H_i in V_F. They derive conditions on $L \subseteq S^3$ sufficient to ensure the described immersed sphere collection satisfies the hypotheses of [12, Theorem 6.1]. In Section 4, we produce a straightforward generalization of their conditions for links in a homology sphere and in Appendix A we confirm that links in homology spheres satisfying the generalized conditions are freely slice.

Thus our real challenge in the proof of Theorem 1.1 becomes finding a way to freely homotope an arbitrary link to one satisfying the generalized Cha-Kim-Powell conditions. For this we will need a mechanism for separating properly immersed disk collections in 4-manifold, at the expense of changing the link on the boundary by a free homotopy. In Section 3, we use the relative Whitney trick to achieve this goal, proving the following (in fact, we prove a more general statement in Proposition 3.2).

Proposition 1.2. *If L is a link in a homology sphere Y, and X is a contractible 4-manifold bounded by Y, then L is freely homotopic to a link whose components bound disjoint locally flat immersed disks in X. Moreover, if X is smooth, then these disks may be smoothly immersed.*

We end this subsection by pointing out that it is not known whether Theorem 1.1 can be extended to the smooth category or not. Concretely, we ask the following question.

Question 1.3. Let L be a link in a homology sphere Y and X be a *smooth* contractible 4-manifold bounded by Y. Is L freely homotopic to a link J in Y so that the link J bounds a collection of disjoint *smooth* disks in X? Can this question be answered if L is a knot?

In the case that L is a knot, the question was answered affirmatively by the first named author, under the assumption that X admits a handle structure with no 3-handles [9, Theorem 1.5]. Compare this with [8, Remark 1.6], where Daemi shows the answer to Question 1.3 is negative if one requires X to be only a homology ball. Indeed, he shows there is a knot in $Y\# -Y$, where Y is the Poincaré homology sphere, such that the knot, even up to free homotopy, does not bound a smooth immersed disk in any smooth homology ball bounded by $Y\# -Y$.

We finally remark that if a link $L \subset Y$ bounds a collection of disjoint piecewise linear disks in a 4-manifold then that link may be changed by a homotopy in Y to a link bounding a collection of disjoint smooth disks in that 4-manifold; see e.g. [16, Proof of Proposition 1.3] for a technique to achieve this by "tubing into the singularities". Thus we compare Question 1.3 with results proving the non-existence of piecewise linear disks for certain knots in the boundaries of contractible 4-manifolds [24, 1, 16, 15, 25], and suggest our question is a natural refinement of the general problem of finding piecewise linear slice disks in contractible 4-manifolds.

1.2. **Whitney tower concordance.** Our second application of the relative Whitney trick concerns *homology concordance* of links. Links L and J in homology spheres Y and Y' are *homology concordant* if there is a disjointly embedded union of locally flat annuli each one bounded by a component of L and a component of J in a homology cobordism from Y to Y'. It is conjectured by the first named author [10] that every link in a homology sphere is homology concordant to a link in S^3. This conjecture is particularly intriguing because the corresponding statement is known to be false in the smooth category [16, 15, 25, 8]. Evidence for this conjecture was provided in [9, 10] and we provide a similar type of evidence in this article. Our evidence will come in the language of Whitney tower concordance. A formal definition of a Whitney tower appears in Section 5; see also [6]. Informally, a *Whitney tower* is a 2-complex given by starting with an immersed surface in a 4-manifold (a union of annuli in a homology cobordism in our case) and iteratively pairing up intersection points with Whitney disks, while accepting that each added Whitney disk will introduce more intersection points which must be paired with new Whitney disks. The *order* of a Whitney tower records roughly how far into this tower one must go before seeing intersection points which are not paired with Whitney disks. Two links L and J in homology spheres Y and Y' are *order k Whitney tower concordant* and we write $L \simeq_k J$ if they bound a collection of immersed annuli which extend to an order k Whitney tower in a simply connected homology $S^3 \times [0,1]$; cf. [6, Definition 3.2]. Hence if two links are homology concordant then they are order k Whitney tower concordant for all k. We will use the relative Whitney trick to prove the following.

Theorem 1.4. *If L is link in a homology sphere and k is a nonnegative integer, then there is a link J in S^3 such that $L \simeq_k J$.*

We point out an interesting consequence. Consider now an n-component link L in a homology sphere. By Theorem 1.4 there is a link J in S^3 so that L and J cobound an order n Whitney tower. According to [22, Theorem 4], if n-component links cobound an order n Whitney tower, then this Whitney tower can be used to guide a sequence of Whitney tricks to produce a disjoint union of immersed annuli. As observed in [22, Remark 3], these annuli can be made smooth. We arrive at the following corollary.

Corollary 1.5. *If L is link in a homology sphere Y, then there is a link J in S^3 and a simply connected homology cobordism from Y to S^3 such that the components of L and J cobound disjoint immersed annuli in the cobordism. Moreover, if the cobordism is smooth, then these annuli may be smoothly immersed.*

1.3. **Gordian distance and link homotopy.** Freedman-Teichner [13] say $L \subset Y$ is *4D-homotopically trivial* if it bounds disjoint immersed disks in a contractible 4-manifold. Suppose L intersects a 3-ball $B \subset Y$ so that $(B, B \cap L)$ is orientation preserving homeomorphic to one of the tangles in Figure 3. A *crossing change* is the local tangle replacement operation of replacing a positive crossing with a negative crossing, or vice-versa. The reader may notice that this construction depends on the choice of identification of B to the 3-ball, but this subtlety will not be relevant in our analysis.

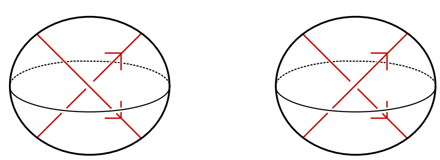

Figure 3. Left: A positive crossing. Right: A negative crossing.

We define the *homotopy trivializing number* $n_h(L)$ to be the minimum number of crossing changes required to transform L into a 4D-homotopically trivial link. If L is a link in S^3, then the homotopy trivializing number of L is the minimum Gordian distance from L to a homotopically trivial link (in the sense of Milnor [18]).

In Subsection 6.1, we use the relative Whitney trick to prove that this number can be computed by counting the number of intersections between distinct generically immersed disks bounded by the link in a contractible 4-manifold. More concretely, we prove the following.

Proposition 1.6. *If L is a link in a homology sphere Y, then*

$$n_h(L) = \min\left\{\sum_{i<j} \#(D_i \cap D_j)\right\}$$

where the minimum is taken over all collections of immersed disks $D_1 \cup \cdots \cup D_n$ in the contractible 4-manifold bounded by Y, with boundary the link, and meeting one-another transversely.

In Subsection 6.2, we combine Proposition 1.6 with results of Habegger-Lin [14] to obtain upper and lower bounds for $n_h(L)$. Moreover, for links with 3 or fewer components we completely determine $n_h(L)$. We remark that the upper bound we establish depends only on the linking numbers and the number of components, and in particular, it is independent of any of the higher order link homotopy invariants of Milnor [18].

Theorem 1.7. *Let L be a link in a homology sphere and $\Lambda(L) := \sum_{i<j} |\operatorname{lk}(L_i, L_j)|$. If L is a 2-component link, then $n_h(L) = \Lambda(L)$. If L is a 3-component link, then*

$$n_h(L) = \begin{cases} \Lambda(L) & \text{if } \Lambda(L) \neq 0 \\ 2 & \text{if } \Lambda(L) = 0 \text{ and } \mu_{123}(L) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In general, there is some $C_n \in \mathbb{N}$ so that for every n-component link L,

$$\Lambda(L) \leq n_h(L) \leq \Lambda(L) + C_n.$$

1.4. **Comparison with the ordinary Whitney trick.** In high-dimensions, ordinary Whitney disks are usually assumed to be (and can always be arranged to be) embedded with interiors disjoint from the submanifolds containing their boundary arcs (with the framing condition guaranteed by the opposite signs of the paired intersections). In dimension four, ordinary Whitney disks are generally assumed to contain self-intersections and intersections with other surfaces (as well as framing obstructions) which are then studied or controlled.

In dimension four, it is not always possible to find a Whitney disk. Their existence is obstructed by the self-intersection invariant in a quotient of the fundamental group ring of the ambient 4-manifold. In comparison, under appropriate conditions, it is always possible to find relative Whitney disks. For instance if the map on fundamental groups induced by the inclusion of the boundary is surjective then every point of intersection will admit a relative Whitney disk. Notice that our setting of homology spheres bounding contractible 4-manifolds certainly satisfies this condition.

Organization of the paper. The reader will have noticed that Theorems 1.1, 1.4, and 1.7 seem disparate. They are related by their reliance on the relative Whitney trick as a means to separate immersed disks. In Section 2, we give a precise description of the relative Whitney trick, and in Section 3, we use it to separate immersed disks, proving Proposition 1.2. These two sections are prerequisite to the remaining sections of the paper, which are then more or less independent of each other. In Section 4, we state a sufficient condition, Theorem 4.4, for freely slicing a boundary link in a homology sphere generalizing [5, Theorem A], and use it to prove Theorem 1.1. The proof of Theorem 4.4 uses the same ideas as appear in [5] and so is delayed until Appendix A. In Section 5, we apply the relative Whitney trick to the construction of Whitney towers and prove Theorem 1.4. In Section 6, we relate $n_h(L)$ to the minimum number of intersection points amongst immersed disks bounded by a link L and prove Theorem 1.7.

Notation and conventions. In this article, links are ordered and oriented. All manifolds are assumed oriented and compact. We will denote by $-Y$ the manifold Y with reversed orientation.

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2. The relative Whitney trick

In this section, we review some of the terminology and conventions we will use throughout the paper, working in the category of topological manifolds. We then give a detailed description of the relative Whitney trick.

2.1. **Conventions for topological manifolds.** We recall some definitions used in [19, Section 3]. A continuous map between topological manifolds is a *generic immersion* if it is locally a smooth immersion (in particular a generic immersion is locally flat away from double points). For $(\Sigma, \partial\Sigma)$ a compact surface with boundary and $(W, \partial W)$ a compact 4-manifold with boundary, any continuous map $(\Sigma, \partial\Sigma) \to (W, \partial W)$ is homotopic to a generic immersion; this follows from [12, Theorem 8.2], see [19, Proposition 3.1] for an argument. We will henceforth assume generic immersions of surfaces in 4-manifolds have at worst double points. We will call the image of a generically immersed surface simply an *immersed surface* from now on and call it an *embedded surface* if it has no double points. If $p \in S$ is a double point of an immersed surface $S \subseteq W$ then by definition there is a neighbourhood U of p in W such that $S \cap U$ is homeomorphic to $(\mathbb{R}^2 \times \{0\}) \cup (\{0\} \times \mathbb{R}^2) \subseteq \mathbb{R}^4$. The submanifolds $S_1 \subseteq S$ and $S_2 \subseteq S$ identified with $\mathbb{R}^2 \times \{0\}$ and $\{0\} \times \mathbb{R}^2$ (respectively) under this homeomorphism are called *local sheets* of S near p. If $S \subseteq W$ is an immersed surface, it follows that S has a neighbourhood homeomorphic to a self plumbing of a vector bundle over S; see [19, Remark 3.2]. Such a neighbourhood is called a *tubular neighbourhood* and the bundle is called the *normal vector bundle*. All the surfaces in this section are compact and each component has boundary. As a consequence, if the normal vector bundle is orientable then it is also trivializable. If a choice of trivialization has been made, we will say S is *framed*.

We say two (locally flat) immersed submanifolds are *transverse* if they are locally transverse in the sense of smooth manifolds. Note that if connected embedded surfaces N_1 and N_2 meet transversely, this is equivalent to saying $N_1 \cup N_2$ is immersed, so in particular it makes sense to talk about local sheets near transverse intersections of embedded surfaces. We say a proper submanifold $(N, \partial N) \subseteq (W, \partial W)$ is *transverse* to the boundary if it is locally transverse to the boundary in the sense of smooth manifolds. Transversality is generic in the topological category as follows. Given locally flat proper submanifolds N_1, N_2 of a 4-manifold W that are transverse to the boundary ∂W, there is an isotopy of W, supported in any given neighbourhood of $N_1 \cap N_2$ taking N_1 to a submanifold N_1' that is transverse to N_2; see [20, 21] and [12, Section 9.5].

2.2. **The relative Whitney trick.** Let S be a generically immersed oriented surface in an oriented 4-manifold, (possibly disconnected and possibly with corners). Assume S is transverse to ∂W. Then $\partial S \subseteq W$ is embedded, so that $S \cap \partial W \subseteq \partial S$ is a union of embedded circles and arcs with endpoints at corners of S.

Let p be a double point of S and write S_1, S_2 for two local sheets of the immersion near p. Assume both sheets belong to components of S that have nonempty intersection with ∂W. For $i = 1, 2$, choose embedded arcs $\alpha_i \subseteq S$ running from some $q_i \in \partial S \cap \partial W$ to p; see Figure 4 for a labelled schematic.

We assume that $\alpha_i \cap (S_1 \cup S_2) \subseteq S_i$, and that α_1 and α_2 are disjoint from each other and from all double points of S other than their common endpoint at p. Suppose that there exists an embedded arc $\alpha_3 \subseteq \partial W$ running from q_2 to q_1 with interior disjoint from ∂S such that the concatenation $\alpha_1 * \alpha_2 * \alpha_3$ is nullhomotopic in W. Let Δ_p be an immersed disk in W transverse to S bounded by $\alpha_1 * \alpha_2 * \alpha_3$.

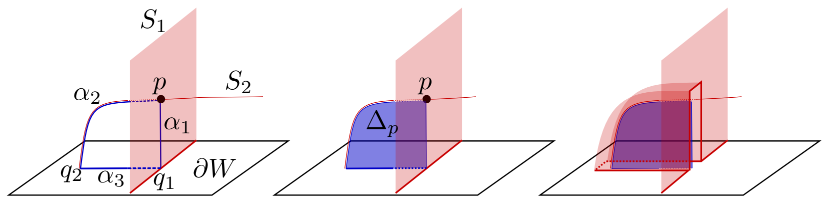


FIGURE 4. Left to right: A transverse point of intersection p in $S_1 \cap S_2$. A relative Whitney disk, Δ_p for p. The result of modifying S_1 by the relative Whitney trick.

Definition 2.1. Let S be an immersed oriented surface in an oriented 4-manifold W and p be a double point of S. Any disk Δ_p as above is called a *relative Whitney disk* for p, and we call the corresponding α_3 the *relative Whitney arc* of Δ_p.

As we are working in the topological category, we take a moment to justify our later use of smooth concepts, such as tangent and normal vectors, by arranging the Whitney disk into a generic position near its boundary. Let $U_1, U_2 \subseteq S$ be closed regular neighbourhoods of $\alpha_1, \alpha_2 \subseteq S$ respectively; each U_i is thus homeomorphic to a closed disk, so that α_i is interior to U_i except for its endpoint at q_i. We may assume, for some local sheets S_1 and S_2 of p, that $U_i \cap (S_1 \cup S_2) = S_i$. By taking these neighbourhoods small enough we arrange that $U_1 \cap U_2 = \{p\}$ and U_1 and U_2 are disjoint from every double point of S other than p. As a consequence U_1 and U_2 are each embedded in W. Since S is immersed, so is $U := U_1 \cup U_2$. Thus U has a tubular neighbourhood $N_U \subseteq W$ homeomorphic to the result of plumbing together $U_1 \times D^2$ and $U_2 \times D^2$ and we may regard U_1 and U_2 as smoothly embedded disks in N_U with the smooth structure pulled back from this plumbing. After changing Δ_p by a homotopy fixing its boundary we may assume that $\Delta_p \cap N_U \subseteq N_U$ is a smooth submanifold with corners. In the complement of the interior of N_U we now see a properly immersed disk, $\Delta_p \smallsetminus \operatorname{int}(N_U)$. Let $N \subseteq W \smallsetminus \operatorname{int}(N_U)$ be its tubular neighbourhood. We now have that $N_\Delta := N_U \cup N$ is the result of gluing together N_U and a tubular neighbourhood of $\Delta_p \smallsetminus \operatorname{int}(N_U)$ (a self plumbing of a disk) along a 3-ball in their shared boundary. Being the result of gluing together two smooth 4-manifolds along a shared submanifold in their boundary, N_Δ is a smooth 4-manifold and Δ_p, U_1 and U_2 are smoothly immersed surfaces in N_Δ.

Introducing some notation, for points $a, b, c \in \mathbb{R}^2$ denote by $\overline{ab}$ the straight line segment between a and b, and by Δ_{abc} the triangle with vertices at a, b, and c. We will parametrize Δ_p by the triangle $\Delta_{xyz} \subseteq \mathbb{R}^2$ of Figure 5, sending the line segments $\overline{xy}$, $\overline{yz}$, and $\overline{zx}$ to α_1, α_2, and α_3 respectively. By taking the neighbourhood N_U small enough we may assume that $\Delta_p \cap N_U$ is parametrized by the region R depicted in Figure 5. We also construct a slightly larger immersed disk $\Delta_p^+ \subset N_\Delta$ as follows. Let w be the normal vector to $\alpha_2 \subseteq N_\Delta$ given by the outward tangent to $\Delta_p \subseteq N_\Delta$. Define Δ_p^+ by extending Δ_p in the w-direction along the length of the arc α_2. We parametrize Δ_p^+ by the larger triangle $\Delta_{xyz}^+ := \Delta_{xy^+z^+} \subseteq \mathbb{R}^2$ of Figure 5. Note that at p, the vector w is tangent to α_1 and to S_1.

By restricting the normal bundle E of Δ_p^+ to $\alpha_1 * \alpha_2$ we obtain a 2-plane bundle $E|_{\alpha_1 * \alpha_2}$ over $\overline{xy} \cup \overline{yz}$. At each point in α_i let v_i be the tangent direction in S_i normal to α_i, and let u_i be a common normal to both Δ_p and S_i, chosen to vary continuously (i.e. to form a section of $E|_{\alpha_i}$). Moreover, choose u_i so that $u_1 = v_2$ and $u_2 = v_1$ at p. Together these give a framing of $E|_{\alpha_1 * \alpha_2}$. Since Δ_{xyz}^+ deformation retracts to $\overline{xy} \cup \overline{yz}$, this framing extends to a framing of E. By the tubular neighbourhood theorem we find an identification of a (possibly smaller) tubular neighbourhood of Δ_p^+ with a plumbed disk, which we parametrize by an immersion $\Phi : \Delta_{xyz}^+ \times \mathbb{R}^2 \looparrowright N_\Delta$ such that:

- $\Phi\left(\overline{xy} \times \{(0,0)\}\right) = \alpha_1$, $\Phi|_{\overline{xy^+} \times \mathbb{R} \times \{0\}}$ is an embedding with image in U_1,

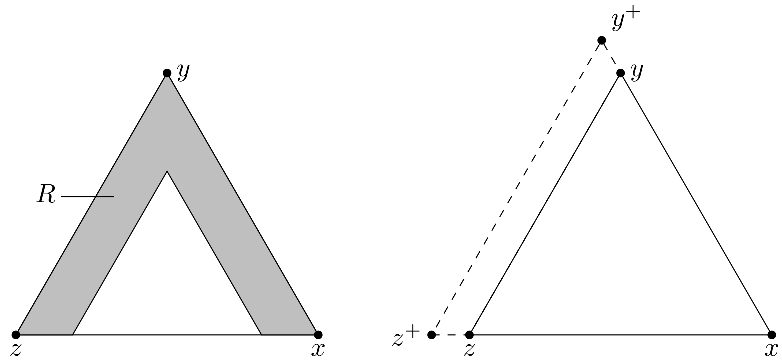


FIGURE 5. Left: The triangle Δ_{xyz} used to parametrize Δ_p, together with R, the preimage of $\Delta_p \cap N_U$. Right: A larger triangle $\Delta^+_{xyz} := \Delta_{xy^+z^+}$.

- $\Phi\left(\overline{xy^+} \times \{(0,0)\}\right) \subseteq U_1$ is α_1 extended slightly on S_1 along the direction tangent to α_1 at p,
- $\Phi(\overline{yz} \times \{(0,0)\}) = \alpha_2$, $\Phi|_{\overline{yz}\times\{0\}\times\mathbb{R}}$ is an embedding with image in U_2,
- otherwise $\Phi\left(\left(\overline{xy^+} \cup \overline{yz}\right) \times \mathbb{R}^2\right)$ is disjoint from S, and
- $\Phi(\overline{zx} \times \{(0,0)\}) = \alpha_3$ and $\Phi\left(\overline{zx} \times \mathbb{R}^2\right) \subseteq \partial W$ is a tubular neighbourhood of α_3.

We now take U_1, remove a neighbourhood of α_1, and replace it with pushed-off copies of Δ^+_p together with a thickened α_2, pushed up in the outwards facing tangent direction to Δ_p. Precisely, we form the following:

$$U'_1 := \left(U_1 \smallsetminus \Phi\left(\overline{xy^+} \times [-1,1] \times \{0\}\right)\right) \cup \Phi\left(\Delta^+_{xyz} \times \{-1,1\} \times \{0\}\right) \cup \Phi\left(\overline{y^+z^+} \times [-1,1] \times \{0\}\right).$$

We will refer to the act of modifying S by replacing U_1 by U'_1 the *relative Whitney trick* along Δ_p.

Remark 2.2. Here are some observations.

(1) U'_1 and U_2 are disjoint.

(2) The image $\Phi(\Delta^+_{xyz} \times [-1,1] \times \{0\})$ parametrizes a homotopy from U_1 to U'_1. If Δ_p is not embedded then U'_1 is not embedded.

(3) The homotopy describing the relative Whitney trick changes ∂S by a *finger move* as depicted in Figure 6. In general, a finger move between embedded arcs A_1 and A_2 in a 3-manifold is band-sum operation from A_1 to a meridional circle of A_2. This operation is specified by a choice of embedded arc from A_1 to A_2, together with a choice of framing for that arc relative to a fixed framing on the boundary of the arc. In our case α_3 is this arc from A_1 to A_2, and the framing is determined by Φ. For the purposes of this paper, we will not need to keep track of this specific framing, but we note that for future applications it might be desirable to do so.

(4) If F is an immersed surface in W (for example a subsurface of S), meeting the interior of Δ_p transversely n times, then the relative Whitney trick adds $2n$ points of intersection to $U'_1 \cap F$. Moreover, for every point of self-intersection of Δ_p, there are four new points of self intersection are added to U'_1 by performing the relative Whitney trick.

(5) If W is a smooth 4-manifold in which S and Δ_p are smoothly immersed, then the result of modifying S by the Whitney trick using Δ_p is still smoothly immersed (after smoothing corners).

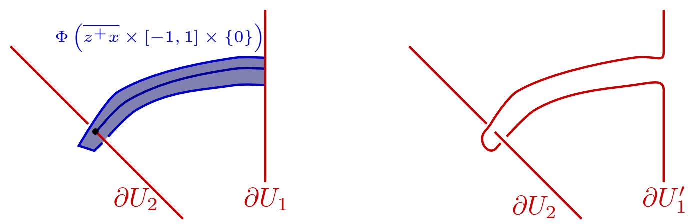


FIGURE 6. The relative Whitney trick affects $\partial S \cap \partial W$ by a finger move.

3. SEPARATING AN IMMERSED DISK COLLECTION

Let L be a link in a 3-manifold, whose components bound immersed disks in a bounded 4-manifold. In this section, we explain how to use the relative Whitney trick to homotope away all intersections between these disks.

Lemma 3.1. *Let W be a 4-manifold, Y be a connected 3-manifold in ∂W, and $L = L_1 \cup L_2$ be a link in Y whose components are nullhomotopic in W. If the inclusion induced map $\pi_1(Y) \to \pi_1(W)$ is surjective, then there exists a link $J = J_1 \cup J_2$ in Y which is freely homotopic to L and bounds disjoint immersed disks in W. If W is smooth, then these disks may be smoothly immersed.*

Moreover, if D_1 and D_2 are transverse immersed disks in W bounded by L, then we may choose a homotopy from L to J which restricts to an isotopy on each component of L and changes a crossing between L_1 and L_2 exactly once for each point in $D_1 \cap D_2$.

Proof. Let D_1 and D_2 be transverse immersed disks bounded by L. We will show how to reduce the number $|D_1 \cap D_2|$ by one, via the relative Whitney trick. Notice that each point in $D_1 \cap D_2$ can be thought of as a double point in $D_1 \cup D_2$, and so it makes sense to apply the relative Whitney trick as outlined in Section 2. As in Figure 6 this relative Whitney trick has the effect of changing a single crossing between $L_1 = \partial D_1$ and $L_2 = \partial D_2$.

First, we alter D_1 and D_2 by a homotopy which is constant on the boundary so that D_1 and D_2 intersect transversely. Let $p \in D_1 \cap D_2$, $q_1 \in L_1$, $q_2 \in L_2$, α_1 be an embedded arc in D_1 running from q_1 to p, and α_2 be an embedded arc in D_2 running from p to q_2. We may assume that α_1 and α_2 are disjoint from all double points of D_1 and D_2 and miss all points in $D_1 \cap D_2$ other than their common endpoint at p. Since $\pi_1(Y) \to \pi_1(W)$ is surjective, there is an embedded arc α_3 in Y running from q_2 to q_1 so that the concatenation $\alpha_1 * \alpha_2 * \alpha_3$ is nullhomotopic in W. Thus, there is a relative Whitney disk Δ_p for p.

Performing the relative Whitney trick using Δ_p to modify D_1 will add two points to $D_1 \cap D_2$ for each point in $\Delta_p \cap D_2$. As the objective is to reduce the number $|D_1 \cap D_2|$, our next goal must be the removal of all points in $\Delta_p \cap D_2$. Let $r \in \Delta_p \cap D_2$. Pick an embedded arc β in Δ_p from r to a point t interior to $\alpha_2 \subseteq D_2$ and disjoint from all double points and points of intersection. Perform a 4-dimensional *finger move* on D_2 along β (cf. [12, §1.5]); a visualization of how this move changes D_2 appears in Figure 7. We will refer to the modified D_2 by the same name. The cost of the finger move is to add two new points of self-intersection to D_2, but this will not concern us. Repeat this process at each point in $\Delta_p \cap D_2$. A similar procedure may be used to remove points in $D_1 \cap \Delta_p$, although this is not required in the proof of the lemma.

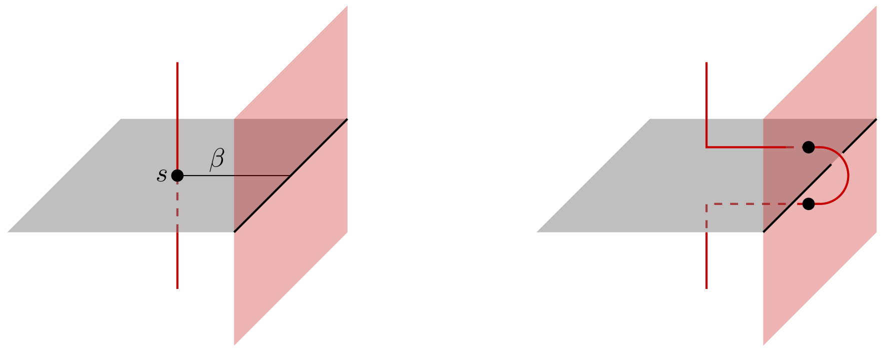


FIGURE 7. Left: A point s in the intersection of Δ_p (gray) with D_2 (red) together with an embedded arc β in Δ_p from s to $\alpha_2 \subseteq \Delta_2$. Right: A homotopy of D_2 removes the point of intersection at $\phi(s)$ and introduces two new points of self intersection for D_2.

Now perform the relative Whitney trick using Δ_p to modify D_1. As we have $D_2 \cap \Delta_p = \emptyset$, the relative Whitney trick reduces $|D_1 \cap D_2|$ by one. By iterating the procedure above, we achieve that $|D_1 \cap D_2| = 0$. Since each application of the relative Whitney trick reduces $|D_1 \cap D_2|$ by one and changes one crossing between L_1 and L_2, the last claim of the lemma follows.

For the statement regarding smoothly immersed disks, if W is smooth, then we can arrange that D_1, D_2 as well as each Δ_p are smoothly immersed. As a consequence of Remark 2.2 (5), this would result in the disjoint disks produced being smoothly immersed. □

When one tries to extend Lemma 3.1 to links with more components, a new complication arises. Assume the hypotheses of the lemma, but now assume that $D_1, \dots, D_n$ is a collection of n transverse immersed disks. Consider two of these immersed disks D_i and D_j. We wish to separate them by removing intersection points with the relative Whitney trick. Let $p \in D_i \cap D_j$ and Δ_p be a relative Whitney disk for p. If Δ_p intersects a disk D_k for some $k \notin \{i, j\}$, then performing the relative Whitney trick using Δ_p to modify D_i will produce new points in $D_i \cap D_k$. Hence, we would need a way to arrange that the relative Whitney disk Δ_p is disjoint from D_k for all $k \notin \{i, j\}$. Also, note that if $k \in \{i, j\}$, as in the proof of Lemma 3.1, we may perform a finger move on D_k to remove the intersection points in $\Delta_p \cap D_k$ for the price of introducing more self intersections.

Suppose $k \notin \{i, j\}$ and let $r \in \Delta_p \cap D_k$. Our strategy to remove r is to modify the disk Δ_p itself by a relative Whitney trick using some relative Whitney disk Δ_r for r. Of course, the use of Δ_r may add new intersection points to Δ_p, according to what intersects Δ_r, so we pause to consider this. Intersection points in $\Delta_r \cap D_i$ and $\Delta_r \cap D_j$ are not a problem because performing the relative Whitney trick using Δ_r to modify Δ_p only adds points to $\Delta_p \cap D_i$ and $\Delta_p \cap D_j$, and these points can be removed by performing finger moves as in the proof of Lemma 3.1. In fact, an intersection point in $\Delta_r \cap D_k$ is also not a problem, as we can now perform a finger move on D_k to remove it, at the cost of adding two self-intersections to D_k. So intersections between Δ_r and each of D_i, D_j and D_k are all unproblematic for us, whereas on Δ_p we could only deal with the first two types. The above argument allows us to extend Lemma 3.1 to 3-component links. For links with more than 3-components there may be further intersection types in Δ_r that we cannot yet deal with. This suggests an induction: adding relative Whitney disks at each stage, and increasing the number of intersection types we know how to remove with finger moves. Eventually we know how to deal with all intersection types. Then a series of finger moves and relative Whitney tricks will remove the intersection point r, with the only cost a possible increase in self-intersection for the disks. We now give more detailed proof. Note that the following statement is more general than Proposition 1.2.

Proposition 3.2. *Let W be a 4-manifold, Y be a connected 3-manifold in ∂W, and $L = L_1 \cup \cdots \cup L_n$ be a link in Y whose components are nullhomotopic in W. If the inclusion induced map $\pi_1(Y) \to \pi_1(W)$ is surjective, then there exists a link $J = J_1 \cup \cdots \cup J_n$ in Y which is freely homotopic to L and bounds disjoint immersed disks in W. If W is smooth, then these disks may be smoothly immersed.*

Moreover, if $D_1, \ldots, D_n$ are transverse immersed disks in W bounded by L, then we may choose a homotopy from L to J which restricts to an isotopy on each component of L and for any $i \neq j$ changes a crossing between the i^{th} and j^{th} components exactly once for each intersection point in $D_i \cap D_j$.

Proof. Let $D_1, \ldots, D_n$ be transverse immersed disks bounded by L. Consider two of these immersed disks D_i and D_j and let $p \in D_i \cap D_j$. We will modify $D_1 \cup \cdots \cup D_n$ by a homotopy which is constant on the boundary and which does not change $|D_k \cap D_\ell|$ for any $k \neq \ell$. Afterwards we will produce a relative Whitney disk Δ_p associated with p so that $\Delta_p \cap D_k = \emptyset$ for all $k \neq i$. Once we have accomplished this, the relative Whitney trick using Δ_p to modify D_i preserves $|D_k \cap D_\ell|$ for all $\{k, \ell\} \neq \{i, j\}$ and will reduce this number by one if $\{k, \ell\} = \{i, j\}$. As in Lemma 3.1, it also affects L by a homotopy obtained by changing a crossing between the i^{th} and the j^{th} component.

We will call the point $p \in D_i \cap D_j$ an *order zero* intersection point. Choose a relative Whitney disk Δ_p associated with p and call it an *order one* relative Whitney disk. As in the proof of Lemma 3.1, a relative Whitney disk exists since the induced map $\pi_1(Y) \to \pi_1(W)$ is surjective. In fact, for any double point, we can find an associated relative Whitney disk. We define the set of *acceptable numbers* for Δ_p to be $A_p = \{i, j\}$. An intersection point $r \in \Delta_p \cap D_k$ is called *unacceptable* for Δ_p if $k \notin A_p$. We now make an inductive definition. Let $m \in \mathbb{N}$, suppose that Δ_q is an order m relative Whitney disk with set of acceptable numbers $A_q \subseteq \{1, \ldots, n\}$ and that $r \in \Delta_q \cap D_k$ is some unacceptable intersection point for Δ_q. We will call the point r an *order m* intersection point. An associated relative Whitney disk Δ_r is called an *order $m+1$* relative Whitney disk. The set of *acceptable numbers* for Δ_r is defined to be $A_r := A_q \cup \{k\}$. It follows from induction that if Δ_q is an order m relative Whitney disk, then $|A_q| = m + 1$. In particular, if Δ_q is an order $n-1$ relative Whitney disk, then $A_q = \{1, \ldots, n\}$ and every intersection point $r \in \Delta_q \cap D_k$ is acceptable.

Let $\mathcal{D}_1 = \{\Delta_p\}$. We make an inductive construction. Let $m \in \mathbb{N}$ and suppose $\mathcal{D}_m$ is a set of order m relative Whitney disks. For each $\Delta_q \in \mathcal{D}_m$ and for each unacceptable order m intersection $r \in \Delta_q$, choose an order $m+1$ relative Whitney disk Δ_r. Write $\mathcal{D}_{m+1}$ for the set consisting of a single choice of Δ_r for each such q and r. Write $\mathcal{D} := \mathcal{D}_0 \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_{n-1}$. We now organize $\mathcal{D}$ into a tree with root Δ_p by declaring that for any $\Delta_q \in \mathcal{D}_m$ and any unacceptable intersection point $r \in \Delta_q \cap D_k$ the relative Whitney disk Δ_r is a descendent of Δ_q. Notice that a vertex Δ_q on this graph is a leaf if and only if it has no unacceptable intersections. Consequentially, any order $n-1$ relative Whitney disk in $\mathcal{D}$ is a leaf.

Suppose that $\Delta_r \in \mathcal{D}_m$ is a leaf of order $m > 1$. We will homotope $D_1 \cup \cdots \cup D_n$ while preserving $D_k \cap D_\ell$ for all $k \neq \ell$, preserving $\mathcal{D}$, and without introducing any new unacceptable intersection points. The end result will have that the interior of Δ_r is disjoint from $D_1 \cup \cdots \cup D_n$.

Proceeding, since Δ_r is order $m > 1$, it is a descendent of some Δ_q. Thus for some k, we have that $r \in \Delta_q \cap D_k$ where r is an unacceptable intersection point of order $m-1$, and $A_r = A_q \cup \{k\}$. Suppose $\Delta_r \cap D_\ell$ is nonempty for some ℓ. If $\ell = k$, then for each point perform a finger move on D_k, along an embedded arc in Δ_r, to remove this intersection point, with the cost of producing two new self-intersections in D_k. If $\ell \neq k$, then we perform a finger move on D_ℓ, along an embedded arc in Δ_r, to remove this point of intersection, with the cost of producing two new points in $\Delta_q \cap D_\ell$. Since Δ_r is a leaf, ℓ must be in $A_r = A_q \cup \{k\}$. As $\ell \neq k$, it must be that $\ell \in A_q$, and so these two new points of intersection are acceptible. We have now arranged that the interior of Δ_r is disjoint from $D_1 \cup \cdots \cup D_n$. Importantly, we have created no new unacceptable intersections.

We now modify Δ_q by the relative Whitney trick using Δ_r. This reduces the number of unacceptable intersections in Δ_q by one and eliminates the leaf at Δ_r. The possible cost of this procedure is to produce new self-intersections in D_k and new acceptable intersections in Δ_q. The affect on the boundary of this move is to change the relative Whitney arc associated with Δ_q by a homotopy passing it through a component of L. In particular, the original link L is preserved.

Iterate the modification of the previous three paragraphs until the tree $\mathcal{D}$ has no leaves of order $m > 1$. This means that $\mathcal{D} = \{\Delta_p\}$ and so Δ_p is a leaf. Thus $\Delta_p \cap D_k = \emptyset$ for all $k \notin \{i, j\}$. By performing finger moves as in Lemma 3.1, we arrange that Δ_p is disjoint from D_j at a cost of adding self intersections to D_j. Modifying D_i by the relative Whitney trick using Δ_p reduces $|D_i \cap D_j|$ by exactly one at the possible expense of increasing the self-intersections of D_i. On the boundary this relative Whitney move affects a single crossing change between L_i and L_j as claimed.

Again, the same argument from the proof of Lemma 3.1 gives the statement regarding smoothly immersed disks. □

Remark 3.3. The idea of finding increasingly high order relative Whitney disks used in the proof of Proposition 3.2 is reminiscent of the concept of a Whitney tower; see for example [6, 4]. It motivates and is an example of what we call a relative Whitney tower, a concept whose formal definition appears in Section 5.

4. Every link in a homology sphere is freely homotopic to a slice link

Cha-Kim-Powell [5] have obtained conditions that ensure a link in S^3 is freely slice. We now describe a generalization of these conditions for links in a general homology sphere. Links satisfying the generalized conditions are also freely slice, as we confirm in Appendix A. We then describe how the disk separation results from Section 3 are used to modify any link in a homology sphere by a homotopy to satisfy the generalized Cha-Kim-Powell conditions. This will prove Theorem 1.1, the main theorem of the paper. We first give some definitions. Recall that a link L in a homology sphere Y is a *boundary link* if it bounds pairwise disjoint Seifert surfaces in Y and a collection of such surfaces is called a *boundary link Seifert surface* for L. Lastly, a link L is *freely slice* if it bounds a collection of disjoint locally flat disks D in the contractible topological 4-manifold X bounded by Y such that $\pi_1(X \smallsetminus D)$ is a free group generated by the meridians of L.

Definition 4.1. Given a boundary link L together with a boundary link Seifert surface F, a *good basis* is a collection $\{a_i, b_i\}_{1\leq i\leq g}$ of simple closed curves on F, whose geometric intersections are symplectic, that represent a basis for $H_1(F;\mathbb{Z})$, and such that the Seifert matrix of F with respect to this basis is reducible by a sequence of elementary S-reductions to the *null matrix*

$$\left[\begin{array}{cc:cc:cc:c:cc}
0 & \epsilon_1 & 0 & * & 0 & * & \cdots & 0 & * \\
1-\epsilon_1 & 0 & 0 & * & 0 & * & & 0 & * \\
\hdashline
0 & 0 & 0 & \epsilon_2 & 0 & * & \cdots & 0 & * \\
* & * & 1-\epsilon_2 & 0 & 0 & * & & 0 & * \\
\hdashline
0 & 0 & 0 & 0 & 0 & \epsilon_3 & \cdots & 0 & * \\
* & * & * & * & 1-\epsilon_3 & 0 & & 0 & * \\
\hdashline
\vdots & & \vdots & & \vdots & & \ddots & \vdots & \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \epsilon_g \\
* & * & * & * & * & * & & 1-\epsilon_g & 0
\end{array}\right]$$

where $\epsilon_i \in \{0, 1\}$ for each i and each $*$ represents some integer.

Definition 4.2. Let L be a boundary link in a homology sphere Y, F be a boundary link Seifert surface for L, and $\{a_i, b_i\}_{1\leq i\leq g}$ be a good basis for L on F. For each i, let b_i' be the result of pushing b_i off of F such that it has zero linking with a_i, and $(b_i')^+$ be a zero linking parallel copy of b_i'.

We say that $\{a_i, b_i\}_{1\leq i\leq g}$ is a *good disky basis* for L if there exist immersed disks

$$\{\Delta_j^+, \Delta_i \mid 1 \leq j \leq 2g, 1 \leq i \leq g\}$$

in the contractible 4-manifold bounded by Y, such that $\partial\Delta_i^+ = a_i$, $\partial\Delta_{g+i}^+ = (b_i')^+$, and $\partial\Delta_i = b_i'$ for each i and all disks are pairwise disjoint except possibly for intersections among $\{\Delta_j^+\}_{1\leq j\leq 2g}$.

Remark 4.3. Cha-Kim-Powell use the condition of 'homotopy trivial$^+$' in addition to being a good basis. As in the proof of [5, Proposition 4.3] it follows from [5, Remark 3.2 (1) and Lemma 3.3] that a homotopy trivial$^+$ good basis for a boundary link in S^3 satisfies the conditions of Definition 4.2.

We can now state a version of the Cha-Kim-Powell theorem for links in general homology spheres.

Theorem 4.4. *A boundary link in a homology sphere with a good disky basis is freely slice.*

The proof of Theorem 4.4 is essentially the same as Cha-Kim-Powell's proof of [5, Theorem A]. A sketch of this argument, together with the minor adjustments required to confirm their argument transfers over to general homology spheres, are found in Appendix A. The current section will proceed assuming that Theorem 4.4 is proved.

Next we show how a link in a 3-manifold can be modified by a homotopy to a boundary link with a good disky basis.

Proposition 4.5. *Let W be a simply connected 4-manifold, Y be a connected 3-manifold in ∂W, and $L = L_1 \cup \cdots \cup L_n$ be a link in Y whose components are nullhomologous in Y. Then L is freely homotopic to a link $J = J_1 \cup \cdots \cup J_n$ so that there exists a collection of disjoint immersed disks $\{D_i, D_i^+\}_{1\leq i\leq n}$ in W such that $\partial D_i = J_i$ and $\partial D_i^+ = (J_i)^+$, where $(J_i)^+$ is a zero linking parallel copy of J_i.*

Proof. Pick n distinct points $p_1, \ldots, p_n$ in Y. As each L_i is nullhomologous in Y, we see that L_i is freely homotopic to a product of commutators $\prod_{j=1}^{g_i}[\alpha_{i,j}, \beta_{i,j}]$ for some $\alpha_{i,j}, \beta_{i,j} \in \pi_1(Y, p_i)$. By Proposition 3.2, there is a link

$$\mathcal{L} := \bigcup_{i=1}^{n}\bigcup_{j=1}^{g_i} a_{i,j} \cup b_{i,j}$$

such that for each i, j, the components $a_{i,j}$ and $b_{i,j}$ are freely homotopic to $\alpha_{i,j}$ and $\beta_{i,j}$ respectively, and there are disjoint immersed disks

$$\{\Delta_{a_{i,j}}, \Delta_{b_{i,j}} \mid 1 \leq i \leq n, 1 \leq j \leq g_i\}$$

in W bounded by $\mathcal{L}$. Thus, for each i, j, there exist embedded arcs $c_{i,j}$ and $d_{i,j}$ from p_i to a point on $a_{i,j}$ and $b_{i,j}$ respectively, so that, as elements in $\pi_1(Y, p_i)$

$$\alpha_{i,j} = c_{i,j} * a_{i,j} * c_{i,j}^{-1} \quad \text{and} \quad \beta_{i,j} = d_{i,j} * b_{i,j} * d_{i,j}^{-1}.$$

As in Figure 8, we construct a collection of disjoint surfaces $F_1, \ldots, F_n$ in Y so that for each i,

- F_i is a Seifert surface for some knot, denoted by J_i, which is freely homotopic to L_i,
- F_i has a symplectic basis $\{A_{i,j}, B_{i,j}\}_{1\leq j\leq g_i}$ so that as curves in Y, $A_{i,j} = a_{i,j}$ and $B_{i,j}' = b_{i,j}$ for each j. Here, the curve $B_{i,j}'$ is the positive pushoff of $B_{i,j}$ with respect to F_i.
- The framing of $a_{i,j} = A_{i,j}$ induced by F_i extends over $\Delta_{a_{i,j}}$ and the framing of $b_{i,j} = B_{i,j}'$ induced by F_i^+, the result of pushing F_i off itself, extends over $\Delta_{b_{i,j}}$. (This can be arranged by adding twists to the bands of F_i.)

Finally, for each i, the disk D_i required by the theorem is produced by starting with F_i and performing ambient surgery using the disks $\{\Delta_{a_{i,j}}\}_{1\leq j\leq g_i}$. The disk D_i^+ is produced similarly by starting with F_i^+ and performing ambient surgery using $\{\Delta_{b_{i,j}}\}_{1\leq j\leq g_i}$. This completes the proof. □

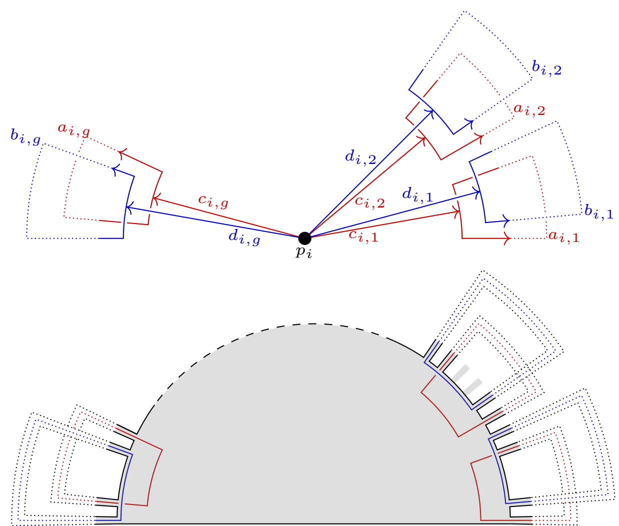


FIGURE 8. Above: The elements $\alpha_{i,1}, \beta_{i,1}, \ldots, \alpha_{i,g}, \beta_{i,g} \in \pi_1(Y, p_i)$ are given by conjugating the components of a link $a_{i,1}, b_{i,1}, \ldots, a_{i,g}, b_{i,g}$ whose components bound disjoint immersed disks in a contractible 4-manifold by embedded arcs $c_{i,1}, d_{i,1}, \ldots c_{i,g}, d_{i,g}$. Below: A knot L_i which is freely homotopic to $\prod_{j=1}^{g_i}[\alpha_{i,j}, \beta_{i,j}]$ which bounds a Seifert surface S_i so that $a_{i,1}, \ldots, a_{i,g}$ sit on S_i and $b_{i,1}, \ldots, b_{i,g}$ sit on the normal pushoff S_i'.

We are ready to prove Theorem 1.1, which we restate here.

Theorem 1.1. *Every link in a homology sphere is freely homotopic to a freely slice link.*

Proof. Let L be a link in a homology sphere Y, and X be the unique contractible 4-manifold bounded by Y. As in the proof of Proposition 4.5, pick n distinct points $p_1, \ldots, p_n$ in Y. Since Y is a homology sphere, each L_i is freely homotopic to a product of commutators $\prod_{j=1}^{g_i}[\alpha_{i,j}, \beta_{i,j}]$ for some $\alpha_{i,j}, \beta_{i,j} \in \pi_1(Y, p_i)$. So far the proof is similar to the proof of Proposition 4.5, but now we replace the reference to Proposition 3.2 with the reference to Proposition 4.5, so that we obtain a link

$$\mathcal{L} := \bigcup_{i=1}^{n} \bigcup_{j=1}^{g_i} a_{i,j} \cup b_{i,j}$$

such that for each i, j, the components $a_{i,j}$ are $b_{i,j}$ are freely homotopic to $\alpha_{i,j}$ and $\beta_{i,j}$ respectively. Moreover, there are disjoint immersed disks

$$\left\{ \Delta_{a_{i,j}}, \Delta_{b_{i,j}}, \Delta^+_{a_{i,j}}, \Delta^+_{b_{i,j}} \mid 1 \le i \le n, 1 \le j \le g_i \right\}$$

in X bounded by $\mathcal{L} \cup \mathcal{L}^+$ where $\mathcal{L}^+$ is a zero linking parallel copy of $\mathcal{L}$.

As in the proof of Proposition 4.5, we obtain a boundary link $J = J_1 \cup \cdots \cup J_n$ which is freely homotopic to L, a boundary link Seifert surface $F = F_1 \cup \cdots \cup F_n$ for J, and a symplectic basis $\{A_{i,j}, B_{i,j}\}_{1 \le i \le n, 1 \le j \le g_i}$ on F such that $A_{i,j} = a_{i,j}$ and $B'_{i,j} = b_{i,j}$ for each i, j. Again, here $B'_{i,j}$ is the positive pushoff of $B_{i,j}$ with respect to F_i, and we may assume that the Seifert framings on $A_{i,j}$ and $B_{i,j}$ are the zero framings.

We claim that J satisfies all of the assumptions of Theorem 4.4. Since linking numbers can be computed in terms of intersections of bounded disks, we see that the Seifert matrix for F with respect to the basis $\{A_{i,j}, B_{i,j}\}_{1\leq i\leq n, 1\leq j\leq g_i}$ has the form

$$\left[\begin{array}{cc|cc|cc|c|cc}
0 & \epsilon_1 & 0 & * & 0 & * & \cdots & 0 & * \\
1-\epsilon_1 & 0 & 0 & * & 0 & * & & 0 & * \\
\hline
0 & 0 & 0 & \epsilon_2 & 0 & * & \cdots & 0 & * \\
* & * & 1-\epsilon_2 & 0 & 0 & * & & 0 & * \\
\hline
0 & 0 & 0 & 0 & 0 & \epsilon_3 & \cdots & 0 & * \\
* & * & * & * & 1-\epsilon_3 & 0 & & 0 & * \\
\hline
\vdots & & \vdots & & \vdots & & \ddots & \vdots & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \epsilon_g \\
* & * & * & * & * & * & & 1-\epsilon_g & 0
\end{array}\right]$$

with each $*$ and each ϵ_i equal to zero. We have now produced a good basis. The existence of the disjoint disks $\{\Delta_{a_{i,j}}, \Delta_{b_{i,j}}, \Delta^+_{b_{i,j}}\}_{1\leq i\leq n, 1\leq j\leq g_i}$ implies that $\{a_{i,j}, b_{i,j}\}_{1\leq i\leq n, 1\leq j\leq g_i}$ is a good disky basis. Thus, by Theorem 4.4 we conclude that J is freely slice. □

5. Whitney tower concordance and links in homology spheres

In this section, we will explain how to use the relative Whitney trick to construct Whitney towers. We will begin by recalling the definition of a (non-relative) Whitney disk, and Whitney tower; see also [4, Section 2.1], for example.

Let S be an immersed oriented surface in a 4-manifold W with double points p and q of opposite signs. Let α_1 and α_2 be embedded arcs in S, running from p to q and q to p respectively. Assume that α_1 and α_2 meet the double point set of S only at $\{p,q\}$, that $\alpha_1 \cap \alpha_2 = \{p,q\}$, and that near both p and q the arcs are in different local sheets. Let Δ be an immersed disk in W bounded by the circle $\gamma := \alpha_1 * \alpha_2$, and with interior transverse to S. The normal bundle of Δ has a unique trivialisation; its restriction to γ determines a choice of framing for the trivial 2-plane bundle over γ. At each point on α_i let v_i be the tangent direction in S normal to α_i, and let u_i be a common normal to both Δ and S, chosen to be a section of the normal bundle of Δ restricted to α_i. These can moreover be chosen so that at p and q, we have $u_1 = v_2$ and $u_2 = v_1$. These combine to determine a second framing for the trivial 2-plane bundle over γ. If the two framings described above agree, then Δ is a called a *Whitney disk* pairing the double points at p and q.

A *Whitney tower* is a special type of union of immersed surfaces. The precise definition is recursive. A union of properly immersed oriented surfaces in a 4-manifold W which are transverse to each other is a Whitney tower. Let T be a Whitney tower and Δ be a Whitney disk pairing two intersections of opposite signs between surfaces in T. Suppose also that Δ is disjoint from the boundary of every surface in T. Then $T \cup \Delta$ is a Whitney tower.

The various immersed surfaces which make up a Whitney tower have an associated *order*. The initial surfaces in a Whitney tower T are called *order 0* surfaces in T. A point in the intersection of an order k and an order ℓ surface in T is called an *order* $k+\ell$ intersection. A Whitney disk pairing two order k intersections is called an *order* $k+1$ Whitney disk. If all intersection points of order less than k are paired by Whitney disks, then T is called an *order* k Whitney tower.

Given an intersection point p in a Whitney tower T, it may be that in the 4-manifold W there is no Whitney disk pairing p with another intersection point in T; as a consequence, there is a filtration of link concordance, as we now describe. Suppose that W is a 4-manifold with $\partial W = \partial_+ W \cup -\partial_- W$. Two n-component links $L \subseteq \partial_+ W$ and $J \subseteq \partial_- W$ are *order* k *Whitney tower concordant in* W if there is an order k Whitney tower T in W so that the order 0 surfaces of T are n immersed annuli $A_1, \ldots, A_n$ with $\partial A_i = L_i \cup -J_i$.

Definition 5.1. If L and J are links in homology spheres that are order k Whitney tower concordant in a simply connected homology cobordism between the homology spheres, then we say that L and J are *order k Whitney tower concordant* and write $L \simeq_k J$.

Remark 5.2. Particularly for links in S^3, the equivalence relation from Definition 5.1 has been the subject of deep study and is known to be highly nontrivial. The reader is directed to [4, 6, 7], for example, for further background and results.

The main goal of this section is to prove Theorem 1.4. For convenience, we recall the statement.

Theorem 1.4. *If L is link in a homology sphere and k is a nonnegative integer, then there is a link J in S^3 such that $L \simeq_k J$.*

In order to construct the link J in Theorem 1.4, as well as the needed Whitney tower concordance, we extend the idea of a relative Whitney disk to an object analogous to a Whitney tower with relative Whitney disks in place of Whitney disks.

A *relative Whitney tower* is recursively defined as follows. A union of properly immersed oriented surfaces in a 4-manifold W which are transverse to each other is a relative Whitney tower. Let T be a relative Whitney tower and Δ be a relative Whitney disk associated with a double point in T. Suppose that Δ is disjoint from the boundary of every surface in T other than the endpoints of its relative Whitney arc. Then $T \cup \Delta$ is a relative Whitney tower.

Similarly to Whitney towers, relative Whitney towers have an associated order. The initial surfaces in a Whitney tower T are called *order 0 surfaces* of T. A point in the intersection of an order k and an order ℓ surface in T is called an *order $k+\ell$ intersection*. A relative Whitney disk associated to an order k intersection is called an *order $k+1$ relative Whitney disk*. If all intersection points of order less than k have relative Whitney disks in T, then T is called an *order k relative Whitney tower.*

Remark 5.3. The proof of Proposition 3.2 involved the construction of an object which is a relative Whitney tower.

The proof of Theorem 1.4 will require two lemmas which we hope also provide evidence that relative Whitney towers are interesting and useful. First, in Lemma 5.4 we show that they exist much more readily than Whitney towers. Secondly, in Lemma 5.5 we explain how a relative Whitney tower can be modified to produce a Whitney tower.

Lemma 5.4. *Let W be a 4-manifold and $S = S_1 \cup \cdots \cup S_n$ be a union of properly immersed connected oriented surfaces in W. If Y is a connected submanifold of ∂W such that $\pi_1(Y) \to \pi_1(W)$ is surjective, and $\partial S_i \cap Y \neq \emptyset$ for each i, then for any nonnegative integer k, there is an order k relative Whitney tower T whose order 0 surfaces are precisely S and for which all relative Whitney arcs are contained in Y.*

Lemma 5.5. *Let W be a 4-manifold and T be an order k relative Whitney tower. If Y is a connected submanifold of ∂W and contains all relative Whitney arcs of T, then there exists an order k Whitney tower T' such that the order 0 surfaces of T and T' differ by a homotopy which is constant outside of a small neighbourhood of Y.*

Armed with these lemmas we can prove Theorem 1.4.

Proof of Theorem 1.4 (assuming Lemmas 5.4 and 5.5). Let L be a link in a homology sphere Y, let k be a nonnegative integer, and let X be the contractible 4-manifold bounded by Y. Let W be a 4-manifold obtained by removing an open 4-ball from X. Note that W is a simply connected homology cobordism from Y to S^3. Since W is simply connected, the components of L are freely homotopic in W to the components of the unlink in S^3. Thus, there exists a collection of immersed annuli $A_1, \dots, A_n$ so that A_i is bounded by L_i and the i^{th} component of the unlink in S^3.

By Lemma 5.4, there is an order k relative Whitney tower T whose order 0 surfaces are precisely $A_1, \dots, A_n$. Moreover, we can ensure that all relative Whitney arcs of T are contained in S^3. By Lemma 5.5, there is an order k Whitney tower T' whose order 0 surfaces are homotopic to $A_1, \dots, A_n$ by a homotopy which is constant away from a neighbourhood of S^3. In particular, the i^{th} component of the order 0 surface of T' is bounded by L_i in Y and by some knot J_i in S^3. Setting $J = J_1 \cup \dots \cup J_n$, we conclude that $L \simeq_k J$, completing the proof. □

Next, we prove Lemmas 5.4 and 5.5. The proof of Lemma 5.4 follows a relatively straightforward induction, which we now present.

Proof of Lemma 5.4. Let W be a 4-manifold and suppose Y is a submanifold of ∂W such that $\pi_1(Y) \to \pi_1(W)$ is surjective. Consider also a collection of connected oriented properly immersed surfaces $S_1, \dots, S_n$ in W such that $\partial S_i \cap Y \neq \emptyset$ for each i.

We will inductively prove that for every nonnegative integer k, there is an order k relative Whitney tower with order 0 surfaces $S_1, \dots, S_n$ and whose relative Whitney arcs are contained in Y. When $k = 0$ there is nothing to prove.

Let T be an order k relative Whitney tower satisfying the above properties. Let p be an order k intersection point in T. Then p is contained in the intersection of two surfaces A and B in T of order a and b respectively where $a + b = k$. If $a = 0$, then A is an order 0 surface S_i and by assumption, there is an embedded arc α in A running from p to a point q in Y. If $a > 0$, then A is a relative Whitney disk of T and there is an embedded arc α in A running from p to a point q on the associated relative Whitney arc. A schematic including α appears to the left of Figure 9. By assumption this relative Whitney arc is contained in Y and hence so is q. For any other relative Whitney disk Δ in T, we have that $\partial\Delta \cap A$ is either empty or is an embedded arc in A with one endpoint in ∂A and the other interior to A. Thus, we can arrange that except for its endpoint on ∂A, α is disjoint from the boundary of every surface in T. Similarly, there is an embedded arc β contained in B running from p to a point r in Y.

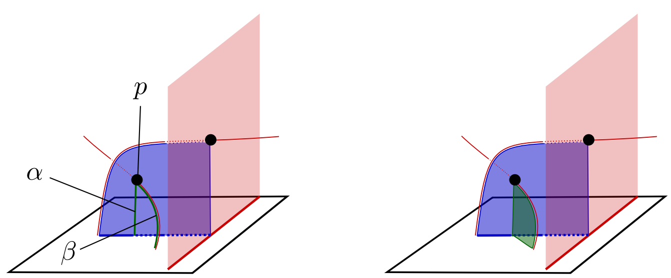


FIGURE 9. Left: A relative Whitney disk in T containing an intersection point p along with arcs α and β in different sheets of T from p to the boundary. Right: A relative Whitney disk associated with p.

Since $\pi_1(Y) \to \pi_1(W)$ is onto, there is an embedded arc γ in Y from r to q so that $\alpha * \beta * \gamma$ bounds an immersed disk Δ_p in W. Thus every order k intersection point in Y admits a relative Whitney disk with relative Whitney arc in Y. After isotoping the interior of Δ_p we have that it is disjoint from the boundary of each surface in T. By replacing T by $T \cup \Delta_p$ we arrange that the order k intersection now has a relative Whitney disk in T. Observe that Δ_p has order $k+1$, so any intersections in its interior are of order at least $k+1$; adding Δ_p to T adds no new intersections of order up to k. By adding a relative Whitney disk to T for every such intersection point, we produce an order $k+1$ relative Whitney tower and complete the proof. □

Next we explain how to use a relative Whitney tower to find an honest Whitney tower, and prove Lemma 5.5. Our argument will employ objects interpolating between Whitney towers and relative Whitney towers. They are akin to both Whitney towers and relative Whitney towers, allowing for both Whitney disks and relative Whitney disks.

A *mixed Whitney tower* is defined recursively as follows. A union of properly immersed surfaces in a 4-manifold W which are transverse to each other is a mixed Whitney tower. Let T be a mixed Whitney tower and Δ be either a Whitney disk pairing two intersection points between surfaces in T or a relative Whitney disk associated with a single intersection between surfaces in T. If Δ is a Whitney disk then we require Δ be disjoint from the boundary of any surface in T. If Δ is a relative Whitney disk then Δ is disjoint from the boundary of any surface in T away from the endpoints of its relative Whitney arc. Then $T \cup \Delta$ is a mixed Whitney tower.

Mixed Whitney towers have an associated order. The initial surfaces in a mixed Whitney tower T are called *order 0 surfaces* in T. A point in the intersection of an order k and an order ℓ surface in T is called an *order* $k+\ell$ *intersection.* A Whitney disk pairing two order k intersections is called an *order* $k+1$ *Whitney disk.* A relative Whitney disk associated to an order k intersection is called an *order* $k+1$ *relative Whitney disk.* If all intersection points of order less than k have either associated Whitney disks or relative Whitney disks in T, then T is called an *order* k *mixed Whitney tower.*

Proof of Lemma 5.5. For an order k mixed Whitney tower T and a natural number $\ell \leq k$, if T has no relative Whitney disks of order greater than ℓ then we say T *transitions at* ℓ. Clearly a relative Whitney tower of order k is a mixed Whitney tower which transitions at k. Additionally, a mixed Whitney tower is a Whitney tower if and only if it transitions at 0. Thus if we can explain how to lower the parameter ℓ at which a given mixed Whitney tower transitions, then induction will complete the proof. Precisely, let W be a 4-manifold containing a mixed Whitney tower T of order k which transitions at ℓ. Moreover, suppose Y is a submanifold of ∂W and contains all relative Whitney arcs of T. We claim that there exists a Whitney tower T' of order k which transitions at $\ell-1$, so that all relative Whitney arcs are contained in Y, and so that the order 0 surfaces of T and T' differ by a homotopy which is constant outside of a small neighbourhood of Y.

The proof of this claim will proceed by changing T by homotopies introducing new intersection points so that order ℓ relative Whitney disks become Whitney disks. Let p be an order $\ell-1$ intersection point sitting in the intersection of surfaces A and B in T and let Δ_p be an associated order ℓ relative Whitney disk. The move drawn schematically in Figure 10 changes A by a homotopy, and adds a new intersection point in $A\cap B$. A subdisk Δ' of Δ_p forms a Whitney disk pairing p and this new intersection point. We proceed to explain this move.

As in Section 2, and using the same notation, we find an immersion $\Phi : \Delta_{xyz}\times\mathbb{R}^2 \to W$ parametrizing a tubular neighbourhood of Δ_p. Since T contains no relative Whitney disks of order greater than ℓ, we see that T contains no relative Whitney disks associated with intersection points on Δ_p. As a consequence, if we take the regions Q and R in Figure 11 close enough to $\overline{xz}$ then Φ will be an embedding when restricted to $Q\cup R\times\mathbb{R}^2$ and will have image disjoint from T except that $\Phi(Q\cup R\times\{(0,0)\})\subseteq\Delta_p$, $\Phi(\overline{xx'}\times\mathbb{R}\times\{0\})\subseteq A$ and $\Phi(\overline{zz''}\times\{0\}\times\mathbb{R})\subseteq B$.

Modify A and B using $\Phi(Q\times[-1,1]\times\{0\})$ and $\Phi(R\times\{0\}\times[-1,1])$ as guides. That is, set

$$A' := \left(A \smallsetminus \Phi\left(\overline{xx'}\times[-1,1]\times\{0\}\right)\right)\cup\Phi\left(Q\times\{-1,1\}\times\{0\}\right)$$
$$\cup\,\Phi\left(\left(\overline{x'z''}\cup\overline{z''z}\right)\times[-1,1]\times\{0\}\right),$$

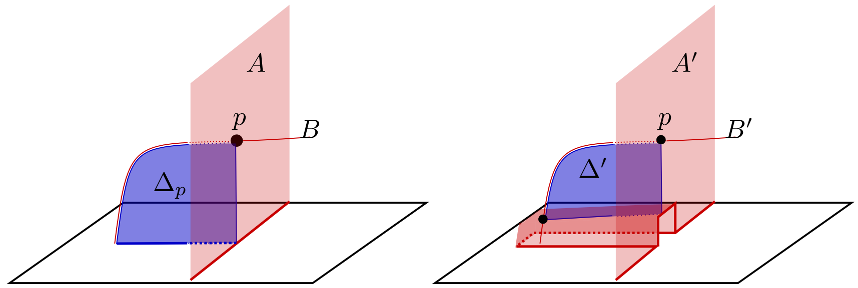


FIGURE 10. Left: A relative Whitney disk associated to the point $p \in A \cap B$. Right: After changing A by a homotopy, we see a new point of intersection and a Whitney disk.

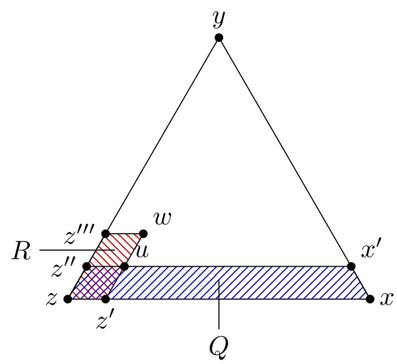


FIGURE 11. The triangle Δ_{xyz}, with a highlighted quadrilaterals Q and R having vertex sets $\{x, x', z'', z\}$ and $\{z, z', w, z'''\}$ respectively.

and

$$B' := \left(B \smallsetminus \Phi\left(\overline{zz''} \times \{0\} \times [-1,1]\right)\right) \cup \Phi\left(R \times \{0\} \times \{-1,1\}\right) \\ \cup \Phi\left(\left(\overline{z'''w} \cup \overline{wz'}\right) \times \{0\} \times [-1,1]\right).$$

The embedded (but non-disjoint) cubes $\Phi(Q \times [-1,1] \times \{0\})$ and $\Phi(R \times \{0\} \times [-1,1])$ parametrize a homotopy from $A \cup B$ to $A' \cup B'$ which is constant outside of $\Phi(Q \cup R \times [-1,1] \times [-1,1])$. The relative Whitney arc associated to Δ_p is parametrized by $\Phi(\overline{xy})$, and by assumption this is contained in Y. Thus, by taking the tubular neighbourhood of Δ_p small enough, and the regions Q and R close enough to $\overline{xz}$, we can arrange that $\Phi(Q \cup R \times [-1,1] \times [-1,1])$ is contained in a small neighbourhood of Y. The homotopy constructed is constant outside of this small neighbourhood of Y.

By direct inspection, $A' \cap B' = (A \cap B) \cup \{q\}$ where $q = \Phi(u, 0, 0)$. Moreover, Δ', the closure of $\Phi(\Delta_{xyz} \smallsetminus (Q \cup R))$, gives a Whitney disk pairing p and q. Thus if T' is given by replacing A, B, and Δ_p by A', B', and Δ' then T' is still an order k mixed Whitney tower and it has one fewer relative Whitney disks of order ℓ than T.

By iterating the procedure above, we replace all order ℓ relative Whitney disks in T with Whitney disks. The result is an order k mixed Whitney tower which transitions at $\ell - 1$. Induction on ℓ now completes the proof. □

6. Homotopy trivializing numbers for links in homology spheres.

In the method we described for separating an immersed disk collection in Section 3, we obtained a precise relationship between the number of crossing changes between link components in the boundary, and the number of intersection points removed in the cobounding disk collection. In this section, we study this relationship in more detail. We introduce two link homotopy invariants (which we then prove to be the same). We provide precise calculations of these invariants for links of up to 3 components and more generally we provide bounds for these invariants.

6.1. **The homotopy trivializing number coincides with the disk intersection number.**

Definition 6.1. Let $L = L_1 \cup \cdots \cup L_n$ be a link in a homology sphere Y. The *disk intersection number* of L is

$$n_d(L) := \min \left\{ \sum_{i<j} \#(D_i \cap D_j) \right\}$$

where the minimum is taken over all collections of immersed disks $D_1 \cup \cdots \cup D_n$ in the contractible 4-manifold bounded by Y, with boundary the link, and meeting one-another transversely.

Recall that a link in a homology sphere is called *4D-homotopically trivial* if it bounds disjoint immersed disks in the unique contractible 4-manifold bounded by the homology sphere.

Definition 6.2. The *homotopy trivializing number* of a link L is given by minimizing the Gordian distance d_G from L to a 4D-homotopically trivial link. That is,

$$n_h(L) := \min\{d_G(L, J) \mid J \text{ is 4D-homotopically trivial}\}.$$

In more detail, we say that $d_G(L, J) \leq m$ if there is a collection of disjoint 3-balls $B_1, \ldots, B_m$ in Y so that $(B_i, B_i \cap L)$ is orientation preserving homeomorphic to one of the tangles in Figure 3 and that J is isotopic to the result of changing L by a homotopy supported on $B_1 \cup \cdots \cup B_m$ replacing each positive crossing by a negative crossing and conversely.

We now recall Proposition 1.6 with this added language and give a proof.

Proposition 1.6. *For any link L in any homology sphere Y there is an equality $n_d(L) = n_h(L)$.*

Proof. Let L be an n-component link in Y with $n_d(L) = m$. Let X be the contractible 4-manifold bounded by Y and $D = D_1 \cup \cdots \cup D_n \subseteq W$ be a collection of immersed disks bounded by the components of L with $\sum_{i<j} \#(D_i \cap D_j) = m$. Since X is contractible, the inclusion induced map $\pi_1(Y) \to \pi_1(X)$ is trivially surjective and we may apply Proposition 3.2 to see that there is a homotopy consisting of m crossing changes from L to a new link J which is 4D-homotopically trivial. Thus, $n_d(L) \geq n_h(L)$, proving one of the needed inequalities.

Now assume that $n_h(L) = m$ and let J be a 4D-homotopically-trivial link obtained from L by making m crossing changes. If we cap off the trace of the homotopy from L to J in $Y \times [0, 1]$ with a collection of disjoint immersed disks in X bounded by the components of J then we see a collection of immersed disks $D_1 \cup \cdots \cup D_n$ bounded by the components of L with $\sum_{i<j} \#(D_i \cap D_j) \leq m$. Thus, $n_d(L) \leq n_h(L)$. □

Remark 6.3. The disk intersection number $n_d(L)$ is invariant under link homotopy. As a consequence of Proposition 1.6, so is $n_h(L)$. We take a moment to explain. Let L and J be links in Y. If they are link homotopic, then the trace of a link homotopy from L to J is an union of disjoint immersed annuli in $Y \times [0, 1]$ cobounded by $L \times \{0\}$ and $J \times \{1\}$. Let X be the contractible 4-manifold bounded by Y. By capping $J \times \{1\}$ with a collection of immersed disks bounded by J in X we see that $n_d(L) \leq n_d(J)$. By symmetry $n_d(L) = n_d(J)$.

6.2. **Computations of the homotopy trivializing number.** Any 4D-homotopically trivial link has vanishing linking numbers and each time when we perform a crossing change the linking number changes at most by one. Hence we have the following obvious lower bound on the homotopy trivializing number:

$$\Lambda(L) := \sum_{i<j} |\operatorname{lk}(L_i, L_j)| \leq n_h(L). \tag{1}$$

Moreover, since the linking number $\operatorname{lk}(L_i, L_j)$ can be computed by taking any immersed disks bounded by L_i and L_j in a homology ball, and counting points of intersection with sign, we see that $n_h(L) = n_d(L) \equiv \Lambda(L) \pmod 2$.

For a link with 2 or 3 components we determine $n_h(L)$ completely. For links of more than 3 components we find a bound on the difference between $n_h(L)$ and $\Lambda(L)$. Remarkably, this upper bound depends only on the number of components of L, and in particular is independent of the higher order link homotopy invariants of Milnor [18]. We restate Theorem 1.7.

Theorem 1.7. *Let L be a link in a homology sphere. The the following holds.*

- *If L is a 2-component link, then*
$$n_h(L) = \Lambda(L).$$
- *If L is a 3-component link, then*
$$n_h(L) = \begin{cases} \Lambda(L) & \text{if } \Lambda(L) \neq 0 \\ 2 & \text{if } \Lambda(L) = 0 \text{ and } \mu_{123}(L) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$
- *In general, there is some C_n so that for every n-component link L,*
$$\Lambda(L) \leq n_h(L) \leq \Lambda(L) + C_n.$$

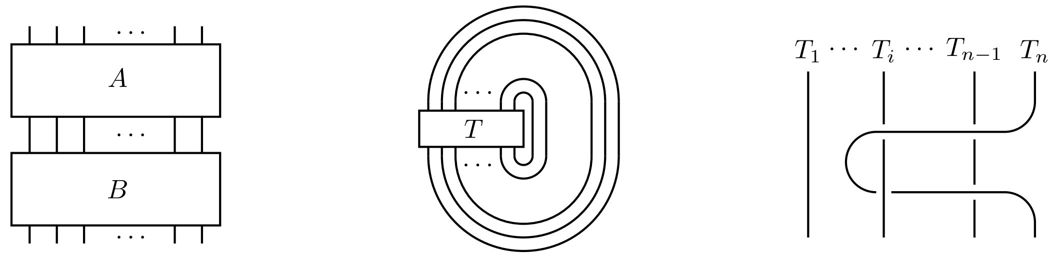
Remark 6.4. By Corollary 1.5, for any link L in a homology sphere, there is a link J in S^3 such that in a simply connected homology cobordism from Y to S^3 the components of L and J cobound disjoint immersed annuli. It follows that $n_d(L) = n_d(J)$. By Proposition 1.6, we have that $n_h(L) = n_h(J)$. Lastly, recall that if two links in homology spheres cobound disjoint immersed annuli, then they have the same pairwise linking number and Milnor's triple linking number. Therefore it suffices to prove Theorem 1.7 for links in S^3, and for which the notion of 4D-homotopically trivial and Milnor's more classical notion of link homotopically trivial are the same.

The proof of Theorem 1.7 passes though the string link classification of Habegger-Lin [14]. We take a moment and recall some of their definitions and tools.

Pick n points $p_1, \dots, p_n$ in the disk D^2. An n-component *string link* $T = T_1 \cup \cdots \cup T_n$ is a disjoint union of embedded arcs in $D^2 \times [0,1]$ with T_i running from $p_i \times \{0\}$ to $p_i \times \{1\}$. Let $\mathcal{LH}_n$ denote the set of n-component string links in S^3.

The notion of link homotopy extends in an obvious way to string links; let $\mathcal{SLH}_n$ denote the set of string links up to link homotopy. Importantly, $\mathcal{SLH}_n$ is a group under the stacking operation of Figure 12a. The definition of homotopy trivializing number n_h extends to string links and, just as for links in S^3, depends only on the link homotopy class. Notice that n_h is subadditive under the stacking operation. The operation of Figure 12b given by sending a string link T to its closure $\widehat{T}$ gives a surjection $\mathcal{SLH}_n \twoheadrightarrow \mathcal{LH}_n$. For any string link T, it is now immediate that

$$n_h(\widehat{T}) \leq n_h(T). \tag{2}$$



(a) $A * B$: The result of stacking string links A and B.

(b) The closure $\widehat{T}$ of a string link T.

(c) $\phi\colon RF(n-1) \to \mathcal{SLH}_n$ sends x_i to the string link T above.

FIGURE 12. The stacking and closure operations, together with the map ϕ.

The key tool we will use in our proof of Theorem 1.7 is the split exact sequence of [14, Lemma 1.8]:

$$1 \longrightarrow RF(n-1) \xrightarrow{\ \phi\ } \mathcal{SLH}_n \underset{s}{\overset{\psi}{\rightleftarrows}} \mathcal{SLH}_{n-1} \longrightarrow 1. \tag{3}$$

Here, $RF(n-1)$ indicates the *reduced free group*. That is, $RF(n-1)$ is the quotient of the free group $F(n-1)$ with generators $x_1, \dots, x_{n-1}$ so that for each i any conjugate of x_i commutes with any other conjugate of x_i. The map $\phi\colon RF(n-1) \to \mathcal{SLH}_n$ is the homomorphism sending a generator x_i to the string link of Figure 12c. The fact that this is well defined follows from work in [14]. The map $\psi\colon \mathcal{SLH}_n \to \mathcal{SLH}_{n-1}$ is given by deleting the n^{th} component of a string link. The splitting $s\colon \mathcal{SLH}_{n-1} \to \mathcal{SLH}_n$ of ψ is given by adding to an $(n-1)$-component string link T an unknotted component which does not interact with the components of T. In summary, any $T \in \mathcal{SLH}_n$ decomposes as $T = s(\psi(T)) * \phi(\gamma)$ for some $\gamma \in RF(n-1)$.

Definition 6.5. Let $w = x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \dots x_{i_\ell}^{\epsilon_\ell}$ be a word in the letters $x_1^{\pm 1}, \dots, x_n^{\pm 1}$. The *trivializing number* of w, denoted by $Z(w)$, is the minimum number of deletions needed to reduce w to a word representing the trivial element of the free group $F(n)$.

Let $\gamma \in RF(n)$ be an element of the reduced free group. The *reduced trivializing number* of γ, denoted by $RZ(\gamma)$, is the minimum of $Z(w)$ among all words $w \in F(n)$ which represent γ.

The proof of Theorem 1.7 will be inductive with the inductive step requiring bounds on $n_h(\phi(\gamma))$ where γ is an element of $RF(n)$. Since $n_h(\phi(x_i^{\pm 1})) = 1$ for each generator of $RF(n)$, it follows that

$$n_h(\phi(\gamma)) \leq RZ(\gamma). \tag{4}$$

During the proof of Theorem 1.7, we will furthermore derive an upper bound on $RZ(\gamma)$ in terms of the classical concepts of *basic commutators* and their *weights*, so we recall the definition of these now.

Definition 6.6. Writing $\{x_1, \dots, x_n\}$ for a generating set of $F(n)$, the ordered set $\{c_1, c_2, \dots\}$ of *basic commutators*, along with associated integer valued *weights* $w(c_1) \leq w(c_2) \leq \dots$, are defined by the following:

- If $i = 1, \dots, n$, then $c_i = x_i$ and $w(c_i) = 1$.
- If $i < j$, then $w(c_i) \leq w(c_j)$.
- If $i > n$, then $c_i = [c_\ell, c_j]$ for some $\ell < j < i$. Additionally $w(c_i) = w(c_\ell) + w(c_j)$.
- If $c_i = [c_\ell, c_j]$ and $c_j = [c_r, c_s]$, then $r \leq \ell$.
- Every $[c_i, c_j]$ satisfying the conditions above is a basic commutator.

The next lemma studies the reduced trivializing number for basic commutators.

Lemma 6.7. *Let $c_i \in F(n)$ be a basic commutator. Then, considering c_i as an element of the reduced free group $RF(n)$, we have*

$$\begin{cases} RZ(c_i^a) = |a| & \text{if } 1 \leq i \leq n, \\ RZ(c_i^a) \leq w(c_i) & \text{otherwise.} \end{cases}$$

Proof. First suppose $1 \leq i \leq n$. Then $c_i^a = x_i^a$ is a length $|a|$ word, and so $RZ(c_i^a) \leq |a|$. In order to see the reverse inequality, note that $\phi(c_i^a)$ is an $(n+1)$-component string link with $\Lambda(\phi(c_i^a)) = |a|$. Thus by inequalities (1) and (4) we have $RZ(c_i^a) \geq n_h(\psi(c_i^a)) \geq \Lambda(\psi(c_i^a)) = |a|$.

The proof of the result when $n < i$ begins with an inductive argument showing that each basic commutator c_i is a product of conjugates of a single generator x_t for some t and that $RZ(c_i) \leq w(c_i)$. When $w(c_i) = 1$, then $c_i = x_i$ and so we are done. When $w(c_i) > 1$, then $c_i = [c_\ell, c_j]$ where $w(c_i) = w(c_\ell) + w(c_j)$. In particular, $w(c_j) < w(c_i)$. We can therefore inductively assume that c_j is a product of conjugates of some x_t. Thus, $c_i = [c_\ell, c_j] = (c_\ell c_j c_\ell^{-1})c_j^{-1}$, and so since both $(c_\ell c_j c_\ell^{-1})$ and c_j^{-1} are products of conjugates of x_t, we have now expressed c_i as a product of conjugates of x_t.

Notice next that in the expression $(c_\ell c_j c_\ell^{-1})c_j^{-1}$ if we make $2 \cdot RZ(c_\ell)$ letter deletions then we may replace the c_ℓ and c_ℓ^{-1} subwords with words representing the trivial element. As a consequence $RZ(c_i) \leq 2 \cdot RZ(c_\ell)$. By the definition of basic commutators we have $w(c_\ell) \leq w(c_j)$ and $w(c_i) = w(c_\ell) + w(c_j)$. Therefore $w(c_\ell) \leq \frac{1}{2}w(c_i)$. By our inductive assumption, $RZ(c_\ell) \leq w(c_\ell)$. Thus

$$RZ(c_i) \leq 2 \cdot RZ(c_\ell) \leq 2w(c_\ell) \leq w(c_i).$$

This completes the inductive argument.

Midway through that induction, we saw that both $c_\ell c_j c_\ell^{-1}$ and c_j^{-1} are products of conjugates of a single x_t for some t. By the definition of $RF(n)$, they commute. Thus,

$$c_i^a = \left((c_\ell c_j c_\ell^{-1})c_j^{-1}\right)^a = (c_\ell c_j^a c_\ell^{-1})c_j^{-a}$$

Similarly to our inductive argument, by making $2 \cdot RZ(c_\ell)$ letter deletions, the c_ℓ and c_ℓ^{-1} subwords appearing in the expression above can be reduced to words representing the trivial element of $RF(n)$. Thus, just as in the inductive argument, $RZ(c_i^a) \leq 2 \cdot RZ(c_\ell) \leq 2w(c_\ell) \leq w(c_i)$. □

In [14, Lemma 1.3], Habegger-Lin prove that the reduced free group $RF(n)$ is nilpotent of class n. In the proof of Theorem 1.7, we will use a simple expression for elements of $RF(n)$ that is obtained by combining their result with the following classical theorem; see e.g. [17, Theorem 5.13A].

Theorem 6.8 (P. Hall's Basis Theorem)**.** *Any $\gamma \in F(n)/F(n)_{n+1}$ can be expressed as*

$$\gamma = c_1^{a_1} c_2^{a_2} \dots c_N^{a_N} \in F(n)/F(n)_{n+1}$$

where N is the number of basic commutators of weight at most n and $a_1, \dots, a_N \in \mathbb{Z}$.

We now have everything we need to complete the proof of Theorem 1.7.

Proof of Theorem 1.7. Let L be a 2-component link in S^3. Since the linking number is a complete homotopy invariant for 2-component links [18], we have that $n_h(L) = \Lambda(L)$.

Now let L be an 3-component link in S^3. Assume first that $\Lambda(L) \neq 0$ and after reordering the components of L and changing the orientations of a component, if needed, we may assume that $\operatorname{lk}(L_2, L_3) > 0$. Let $T \subseteq D^2 \times [0,1]$ be a string link with $\widehat{T} = L$. Using short exact sequence (3), we have that $T = s(\psi(T)) * \phi(\gamma) \in \mathcal{SLH}_3$ for some $\gamma \in RF(2)$. Note that the linking number of $\psi(T) \in \mathcal{SLH}_2$ is equal to $\operatorname{lk}(L_1, L_2)$. Therefore $n_h(s(\psi(T))) = |\operatorname{lk}(L_1, L_2)|$.

As mentioned above, the reduced free group $RF(2)$ is nilpotent of class 2 [14, Lemma 1.3]. Hence by Theorem 6.8, we may express γ in terms of basic commutators: $\gamma = x_1^a * x_2^b * [x_1, x_2]^c$ for some $a, b, c \in \mathbb{Z}$. It follows by inspection that $a = \operatorname{lk}(L_1, L_3)$ and $b = \operatorname{lk}(L_2, L_3) > 0$. In $RF(2)$, x_2 and $x_1x_2x_1^{-1}$ commute as do x_1 and $x_2x_1^{-1}x_2^{-1}$. Thus,

$$\begin{aligned}\gamma &= x_1^a x_2^b [x_1, x_2]^c \\ &= x_1^a x_2^b ((x_1x_2x_1^{-1})x_2^{-1})^c \\ &= x_1^a (x_1(x_2x_1^{-1}x_2^{-1}))^c x_2^b \\ &= x_1^a x_1^c (x_2x_1^{-1}x_2^{-1})^c x_2^b \\ &= x_1^a x_1^c x_2 x_1^{-c} x_2^{b-1}\end{aligned}$$

Deleting $|a| = |\operatorname{lk}(L_1, L_3)|$ instances of x_1 and $1+|b-1| = b = \operatorname{lk}(L_2, L_3)$ instances of x_2 reduces this word to $x_1^c x_1^{-c}$ which is the trivial word. Thus, by using inequality (4), we obtain $n_h(\phi(\gamma)) \leq RZ(\gamma) \leq |\operatorname{lk}(L_1, L_3)| + |\operatorname{lk}(L_2, L_3)|$. Finally, by using the above inequalities combined with inequality (2), we conclude that

$$\begin{aligned} n_h(L) \leq n_h(T) \leq n_h(s(\psi(T))) + n_h(\phi(\gamma)) \\ \leq |\operatorname{lk}(L_1, L_2)| + |\operatorname{lk}(L_1, L_3)| + |\operatorname{lk}(L_2, L_3)| = \Lambda(L).\end{aligned}$$

This gives the claimed result when L has 3 components and $\Lambda(L) \neq 0$.

Now suppose that the 3-component link L has $\Lambda(L) = 0$ and $\mu_{123}(L) \neq 0$. As above, if $T \subseteq D^2 \times [0, 1]$ is a string link with $\widehat{T} = L$, then by short exact sequence (3), we have that $T = s(\psi(T)) * \phi(\gamma) \in \mathcal{SLH}_3$ for some $\gamma \in RF(2)$. In this case, $\psi(T)$ is a 2-component string link with vanishing linking numbers so that $\psi(T)$ is homotopically trivial. Furthermore, for some $c \in \mathbb{Z}$,

$$\gamma = [x_1, x_2]^c = (x_1x_2x_1^{-1}x_2^{-1})^c = x_1^c (x_2x_1^{-1}x_2^{-1})^c = x_1^c x_2 x_1^{-c} x_2^{-1}$$

where the third equality follows since x_1 commutes with $(x_2x_1^{-1}x_2^{-1})$. The resulting word reduces to the trivial element of the free group after two letter deletions (specifically, deleting x_2 and x_2^{-1}). As we have explained above, this affects $\phi(\gamma)$ by two crossing changes. Thus, $n_h(L) \leq n_h(T) \leq 2$. Since $\mu_{123}(L) \neq 0$, L is not homotopically-trivial, and so $n_h(L) > 0$. As $n_h(L) \equiv \Lambda(L) \pmod 2$, we conclude that $n_h(L) = 2$, as claimed.

Next we address the case that L has 3 components and $\Lambda(L) = \mu_{123}(L) = 0$. In this case, Milnor [18] concludes that L is link homotopically trivial and so $n_h(L) = 0$, as claimed.

We now move on to the proof of the statement concerning links with 4 or more components. We begin with the inductive assumption that there is a constant C_n so that for every string link $Q \in \mathcal{SLH}_n$ there is an inequality $n_h(Q) \leq \Lambda(Q) + C_n$. Let $T \in \mathcal{SLH}_{n+1}$, then by short exact sequence (3), we have that $T = s(\psi(T)) * \phi(\gamma)$ for some $\gamma \in RF(n)$.

As before, combining the fact that $RF(n)$ is nilpotent of class n with Theorem 6.8, we may express $\gamma \in RF(n)$ in terms of basic commutators:

$$\gamma = c_1^{a_1} c_2^{a_2} \dots c_N^{a_N}$$

where N is the number of basic commutators of weight at most n and $a_1, \dots, a_N \in \mathbb{Z}$. By inspection $a_i = \operatorname{lk}(L_{n+1}, L_i)$ for $i = 1, \dots n$. Appealing to Lemma 6.7, we have that

$$n_h(\phi(\gamma)) \leq RZ(\gamma) \leq \sum_{i=1}^{N} RZ(c_i^{a_i}) \leq \sum_{i=1}^{n} |a_i| + \sum_{i=n+1}^{N} w(c_i).$$

Thus, by the above inequality combined with the inductive hypothesis we get

$$\begin{aligned} n_h(T) \leq n_h(s(\psi(T))) + n_h(\phi(\gamma)) &\leq \Lambda(s(\psi(T))) + C_n + \sum_{i=1}^{n} |a_i| + \sum_{i=n+1}^{N} w(c_i) \\ &= \Lambda(T) + C_n + \sum_{i=n+1}^{N} w(c_i). \end{aligned}$$

Setting $C_{n+1} = C_n + \sum_{i=n+1}^{N} w(c_i)$ completes the induction. □

Remark 6.9. The statement of Theorem 1.7 does not give a precise value for the sequence of numbers C_n when $n > 3$. We note that by combining the recurrence relation $C_{n+1} = C_n + \sum_{i=n+1}^{N} w(c_i)$ (from the end of the proof) with Witt's formula for the number of basic commutators of a fixed weight [23] one can find an upper bound for the C_n constructed in the proof, and these upper bounds could themselves function as the C_n in the statement of the theorem. However, a formula produced this way would be far from sharp. This is because any basic commutator with repeated indices ($[x_2, [x_1, x_2]]$ for instance) is zero in $RF(n)$ and so should not be counted in a formula for C_n. Thus to obtain a less crude formula for C_n we would desire a Witt-type formula counting only the number of basic commutators without repeated indices. While such a result might be within reach of current technology, it is definitely beyond the scope of this paper.

Appendix A. Freely slicing boundary links

Cha-Kim-Powell [5] describe a set of conditions on a link in S^3 that ensure the link is freely slice. In Section 4, we generalized these conditions to links in a general homology 3-sphere Y and claimed in Theorem 4.4 that our conditions guaranteed the link was freely slice in the contractible 4-manifold X bounded by Y. The proof of this is a close imitation of the argument from Cha-Kim-Powell [5, §4 & §5] and, as such, we only sketch the argument below. An attempt has been made to include enough detail to follow the argument, but without repeating too much of what already appears in [5].

We begin by recalling some terminology and a theorem from Freedman-Quinn [12]. A *transverse pair* is two copies of $S^2 \times D^2$ plumbed together at one point. This model is a neighbourhood of the pair of spheres

$$(S^2 \times \{pt\}) \cup (\{pt\} \times S^2) \subseteq S^2 \times S^2.$$

Take the disjoint union $N_1, \ldots, N_\ell$ of copies of the transverse pair and perform further plumbings between the copies, possibly including self-plumbings, then map the result into a topological 4-manifold W via a continuous map that is a homeomorphism to its image. The result of this process is a map $f\colon \coprod_i N_i \to W$ which is called an *immersion of a union of transverse pairs.*

An immersion of a union of transverse pairs is said to have *algebraically trivial intersections* if the images of the further plumbings we performed can be arranged in pairs by Whitney disks in W (that may *a priori* meet $\coprod_i N_i$). Such a map f is called π_1*-null* if the inclusion induced map $\pi_1(f(\coprod_i N_i)) \to \pi_1(W)$ is trivial.

If middle-dimensional homology classes can be represented in this arrangement, then one is able to use a result of Freedman-Quinn [12, Theorem 6.1] to conclude that f is s-cobordant rel. boundary to an embedding. In the particular case of interest to us, this gives the following.

Theorem A.1. *Suppose W is a compact topological 4-manifold, bounded by M_L, with $\pi_1(W)$ free and generated by the meridians of L. Let $f\colon \coprod_i N_i \to W$ be a π_1-null immersion of a union of transverse pairs with algebraically trivial intersections, and inducing an isomorphism $f_*\colon H_2(\coprod_i N_i) \to H_2(W)$.*

Then there exists a compact topological 4-manifold W', bounded by M_L, with $\pi_1(W')$ free and generated by the meridians of L, and a locally flat embedding $f' \colon \coprod_i N_i \hookrightarrow W'$ inducing an isomorphism $f'_* \colon H_2(\coprod_i N_i) \to H_2(W')$.

We now follow the standard surgery-theoretic approach to slice L, sketched in the introduction. Recall that the 0-surgery on L is denoted by M_L.

Proposition A.2. *Let L be a boundary link with a good disky basis in a homology sphere Y, then M_L bounds a compact oriented 4-manifold W such that*

(1) *$\pi_1(W)$ is free and generated by the meridians of L, and*
(2) *$H_2(W;\mathbb{Z})$ is free and represented by a π_1-null immersion of a union of transverse pairs with algebraically trivial intersections.*

The argument we now use is almost identical to that appearing in [5, Section 5].

Summary of the proof of Proposition A.2. Let $F = F_1 \cup \cdots \cup F_n$ be a boundary link Seifert surface for L and let $\{a_i, b_i\}_{1 \le i \le g}$ be a good disky basis, with

$$\left\{\Delta_j^+, \Delta_i \mid 1 \le j \le 2g, 1 \le i \le g\right\}$$

the immersed disks as in Definition 4.2. Recalling these conditions, for each i, $\partial\Delta_i^+ = a_i$, $\partial\Delta_{g+i}^+ = (b_i')^+$, and $\partial\Delta_i = b_i'$, where b_i' is the result of pushing b_i off F such that it has zero linking with a_i, and $(b_i')^+$ is a zero linking parallel copy of b_i'. These disks are all disjoint except that the disks $\{\Delta_j^+\}_{1 \le j \le 2g}$ might intersect each other. Write X for the contractible 4-manifold bounded by Y.

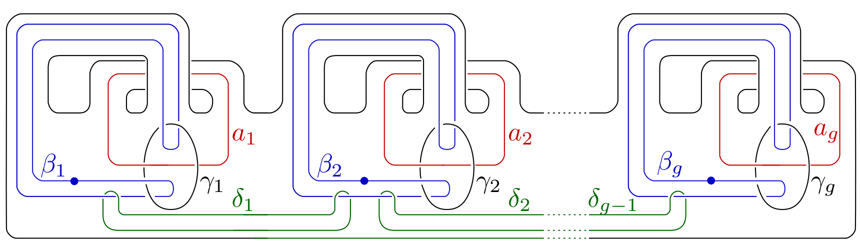


FIGURE 13. Curves a_i, β_i, γ_i and δ_i sitting in a produce neighbourhood of a Seifert surface for L. Attaching a 1-handle using the dotted β_i curves and attaching 2-handles along the 0-framings of a_i, γ_i, and δ_i.

For each i, let $\beta_i \cup \gamma_i$ be the Bing double of b_i appearing in Figure 13. Attach 1-handles to X along β_i and 2-handles to X along the 0-framings of a_i, γ_i, and δ_i to get a 4-manifold W. A straightforward argument shows that W has boundary M_L and has fundamental group freely generated by the meridians of L; see [5, Claim A] for details. Clearly $H_2(W;\mathbb{Z}) \cong \mathbb{Z}^{2g}$. This basis is generated by framed immersed spheres $\Sigma_1, \ldots, \Sigma_{2g}$ described as follows. For each i, take Σ_{2i-1} to be the union of Δ_i^+ and the core of the 2-handle attached to a_i. For each i, we can and will assume b_i' and $(b_i')^+$ lie on the gray surface bounded by γ_i depicted in Figure 14 (left). We use this to define a planar surface P_i bounded by γ_i, b_i' and $(b_i')^+$ as in Figure 14 (right). Take Σ_{2i} to be the union of Δ_{g+i}^+, Δ_i, the core of the handle attached to γ_i, and P_i; cf. [5, Claim B].

For each i, a regular neighbourhood of $\Sigma_{2i-1} \cup \Sigma_{2i}$ can now be viewed as an immersed transverse pair. The same arguments from [5, Claim C] and [5, Claim D] now reveal that $\cup_{i=1}^{2g} \Sigma_i$ has algebraically trivial intersections and is π_1-null. □

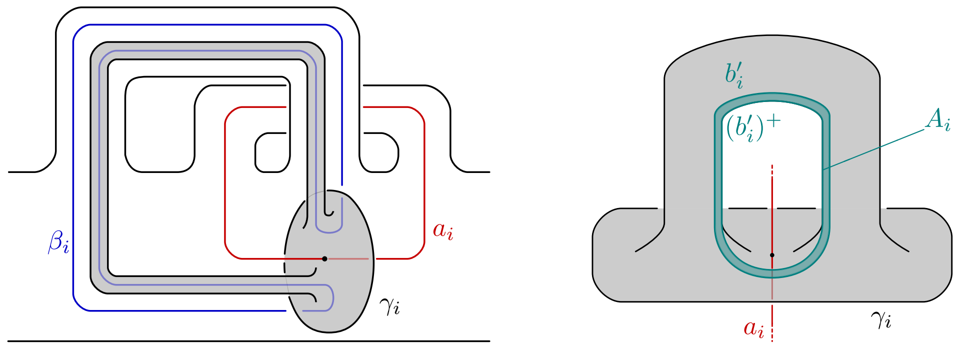


FIGURE 14. Left: A section of the surface F containing $\{a_i, b_i\}$ (b_i not depicted). The curves β_i and γ_i form a bing double of the curve b_i in a neighbourhood of F. A gray genus one surface disjoint from β_i with boundary γ_i is also depicted. Right: A close-up of the gray surface. The annulus A_i on the gray surface with boundary b_i and b_i' is depicted. The complement of the interior of the annulus in the gray surface is the planar surface P_i.

Finally, we can confirm that Cha-Kim-Powell [5, Theorem A] generalizes as claimed.

Proof of Theorem 4.4. Let W be the 4-manifold and $f\colon \coprod_i N_i \to W$ the immersion of a union of transverse pairs representing $H_2(W;\mathbb{Z})$ described in Proposition A.2. Applying Theorem A.1, we obtain W' and f'. Note that the image of f' consists of a tubular neighbourhood of locally flat embedded 2-spheres representing generators for $H_2(W';\mathbb{Z}) \cong H_2(W;\mathbb{Z})$. These embedded 2-spheres come in transverse pairs and we now perform surgery on one sphere from each transverse pair. Since the second sphere from each transverse pair intersected the surgered sphere geometrically once, these surgeries preserve $\pi_1(W')$. Thus, we obtain W'' with boundary M_L, with $H_2(W'';\mathbb{Z}) = 0$, and with $\pi_1(W'')$ freely generated by the meridians of L. Now attach 2-handles to M_L along the meridians of the link components, with framing so that the 0-surgery is reversed. This has Y as the effect of surgery, and by glueing across meridians we ensure that $\pi_1(W'') = 0$. The resultant 4-manifold is contractible and has boundary Y. The link L has slice disks given by the cocores of the 2-handles we have just attached so it is slice and moreover freely slice as $\pi_1(W'')$ is free. $\square$

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