Primal and Dual Combinatorial Dimensions

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August 24, 2021

Abstract

We give tight bounds on the relation between the primal and dual of various combinatorial dimensions, such as the pseudo-dimension and fat-shattering dimension, for multi-valued function classes. These dimensional notions play an important role in the area of learning theory. We first review some (folklore) results that bound the dual dimension of a function class in terms of its primal, and after that give (almost) matching lower bounds. In particular, we give an appropriate generalization to multi-valued function classes of a well-known bound due to Assouad (1983), that relates the primal and dual VC-dimension of a binary function class.

1 Introduction

The Vapnik-Chervonenkis (VC) dimension [11] is a fundamental combinatorial dimension in learning theory used to characterize the complexity of *learning* a class X consisting of functions $f : Y \to \{0, 1\}$ where X and Y are given (possibly infinite) sets. Informally, the VC-dimension captures how rich or complex a class of functions is. Many extensions of the VC-dimension to multi-valued functions $f : X \to Z$, for some given $Z \subseteq \mathbb{R}$, have been proposed in the literature, such as the Vapnik-dimension (also known as the uniform pseudo-dimension) [10], the Pollard-dimension (also known as pseudo-dimension) [9, 5], and the fat-shattering dimension [6]. All these combinatorial dimensions are formally defined in Section 2.

Every (primal) class of functions can be identified with a *dual class* whose functions are of the form $g_y: X \to Z$ for $y \in Y$ defined by $g_y(f) = f(y)$ for $f \in X$. When interpreting a function class as a matrix A whose rows and columns are indexed by X and Y, respectively, the dual class is simply given by the transpose matrix A^{\top} . The (VC, pseudo-, etc..) dimension of the dual class is defined as the dimension of the matrix A^T .

Assouad [2] showed the following relation between the primal VC-dimension VC(A) and the dual VC-dimension $VC^*(A)$:

$$\operatorname{VC}^*(A) \le 2^{\operatorname{VC}(A)+1} - 1.$$
 (1.1)

This has turned out to be a very useful inequality, e.g., in the context of so-called sample compression schemes [8]. In case $VC^*(A)$ is a power of two, this immediately yields $VC^*(A) \leq 2^{VC(A)}$. It is known that this bound is tight for all values of $VC^*(A)$, see, e.g., [7].

The purpose of this work is to understand the relation between the primal and dual of combinatorial dimensions for multi-valued function classes, in particular, for multi-valued functions where $Z = \{0, 1, ..., k\}$ for $k \in \mathbb{N}$. For the pseudo-dimension, as explained in Section 3, it can be shown that

$$\operatorname{Pdim}^{*}(A) \le k \cdot \left(2^{\operatorname{Pdim}(A)+1} - 1\right),$$
 (1.2)

which naturally generalizes Assouad's bound in (1.1).¹ Again, when $Pdim^*(A)$ is a power of two, this yields

$$\operatorname{Pdim}^*(A) \le k \cdot 2^{\operatorname{Pdim}(A)}.$$
(1.3)

Our first contribution is that the bound in (1.3) is in fact tight for every value of k and Pdim(A) (Theorem 4.2). In case Pdim(A) = 1, we give an improved bound of k + 2 (Theorem 4.1), and also show that this is tight (Theorem 4.2). We obtain similar bounds for the fat-shattering dimension (Theorem 4.5).

Remark 1.1. It is sometimes believed that Assouad's bound also holds for combinatorial dimensions beyond the VC-dimension, see, e.g., [4]. Our results show that this is, unfortunately, not correct.

Outline. We continue in Section 2 with all the necessary definitions and notations, in particular the formal definitions of all combinatorial dimensions considered in this work. Then, in Section 3, we outline known results regarding the relations between various combinatorial dimensions and their duals. After that, in Section 4, we summarize our results, followed by their proofs in Section 5.

2 Preliminaries

For $k \geq 1$, we set $[k] := \{1, \ldots, k\}$ and $[k]_0 := [k] \cup \{0\}$. Let X and Y be disjoint sets and let $Z \subseteq \mathbb{R}$ be a subset of the reals. Consider a function $A : X \times Y \to Z$. For $x \in X$, we define $A_x : Y \to Z$ by $A_x(y) = A(x, y)$ and refer to A_x as a row of A. For $y \in Y$, we define $A_y : X \to Z$ by $A_y(x) = A(x, y)$ and refer to A_y as a column of A. The transpose of A is defined as the function $A^\top : Y \times X \to Z$ given by $A^\top(y, x) = A(x, y)$. As suggested by this terminoly, we view A as a (possibly infinite) matrix with rows indexed by X, columns indexed by Y and with A^\top as its transpose.

A matrix $A: X \times Y \to Z$ with $Z = \{0, 1\}$ is said to be *Boolean*. Let $d \ge 1$ be a positive integer. We denote by $B_d: X \times Y \to \{0, 1\}$ the Boolean matrix which is defined as follows:

- 1. $X = [2^d]$ and Y = [d].
- 2. For every function $b : [d] \to \{0,1\}$, there exists an $x \in [2^d]$ such that, for every $y \in [d]$, we have $B_d(x,y) = b(y)$.

Note that B_d is unique modulo renaming rows and columns.

Definition 2.1 (Shattered sets). Let $A : X \times Y \to Z$, with $Z \subseteq \mathbb{R}$, be a matrix and let $J \subseteq Y$ be a subset of its columns.

- 1. Suppose that $Z = \{0, 1\}$. We say that J is VC-shattered by A if, for every function $b : J \to \{0, 1\}$, there exists an $x \in X$ such that, for every $y \in J$, we have B(x, y) = b(y).
- 2. We say that J is P-shattered by A if there exists a function $\mathbf{t} : J \to \mathbb{R}$ such that the following holds: for every function $b : J \to \{0, 1\}$, there exists an $x \in X$ such that, for every $y \in J$, we have $A(x, y) \ge \mathbf{t}(y)$ iff b(y) = 1.
- 3. Let $\gamma > 0$. We say that J is P_{γ} -shattered by A if there exists a function $\mathbf{t} : J \to \mathbb{R}$ such that the following holds: for every function $b : J \to \{0, 1\}$, there exists an $x \in X$ such that, for every $y \in J$, we have

$$A(x,y) \left\{ \begin{array}{ll} \geq \mathbf{t}(y) + \gamma & \text{if } b(y) = 1 \\ < \mathbf{t}(y) - \gamma & \text{if } b(y) = 0 \end{array} \right.$$

- 4. We say that J is V-shattered by A if there exists a number $t \in \mathbb{R}$ such that the following holds: for every function $b: J \to \{0, 1\}$, there exists an $x \in X$ such that, for every $y \in J$, we have $A(x, y) \ge t$ iff b(y) = 1.
- 5. Let $\gamma > 0$. We say that J is V_{γ} -shattered by A if there exists a number $t \in \mathbb{R}$ such that the following holds: for every function $b: J \to \{0, 1\}$, there exists an $x \in X$ such that, for every $y \in J$, we have

$$A(x,y) \begin{cases} \geq t + \gamma & \text{if } b(y) = 1\\ < t - \gamma & \text{if } b(y) = 0 \end{cases}$$

 $^{^1\}mathrm{We}$ refer to this as a folklore result, rather than a contribution of this work.

We will refer to $\mathbf{t}: J \to \mathbb{R}$ occuring in the definition of P- and the P_{γ} -shattered sets as the thresholds used for shattering J. Similarly, we will refer to $t \in \mathbb{R}$ occuring in the definition of V- and the V_{γ} -shattered sets as the uniform threshold used for shattering J.

Definition 2.2 (Combinatorial dimensions). Let $A: X \times Y \to Z$ be a matrix. Let $\tau \in \{VC, P, P_{\gamma}, V, V_{\gamma}\}$ be one of the shattering types mentioned in Definition 2.1. The (primal) τ -dimension of A is the size of a largest set $J \subseteq Y$ that is τ -shattered by A (resp. ∞ if there exist τ -shatterable sets of unbounded size). The dual τ -dimension of A is defined as the τ -dimension of A^{\perp} .

We use the notations VC(A), Pdim(A), $P_{\gamma}(A)$, Vdim(A) and $V_{\gamma}(A)$ for the (primal) dimensions of type $\tau = VC, P, P_{\gamma}, V, V_{\gamma}$, respectively. Here, VC(A) is the VC-dimension [11], Pdim(A) the pseudo-dimension $[9, 5], P_{\gamma}(A)$ the fat-shattering dimension [6], Vdim(A) the Vapnik-dimension [10] and $V_{\gamma}(A)$ the fatshattered version of the Vapnik-dimension, see, e.g., [1]. The corresponding dual dimensions are denoted by $VC^*(A)$, $Pdim^*(A)$, $P^*_{\gamma}(A)$, $Vdim^*(A)$ and $V^*_{\gamma}(A)$, respectively.

The matrix obtained by thresholding the columns of $A: X \times Y \to Z$ at $\mathbf{t}: Y \to \mathbb{R}$ is defined as the Boolean matrix $B: X \times Y \to \{0,1\}$ such that, for all $x \in X$ and $y \in Y$, we have B(x,y) = 1 iff $A(x,y) \ge \mathbf{t}(y)$. For $I \subseteq X$ and $J \subseteq Y$, we denote the restriction of A to $I \times J$ by $A_{I,J}$. In other words: $A_{I,J}$ is the submatrix of A whose rows are indexed by I and whose columns are indexed by J. A witness for the inequality $Pdim(A) \ge d$ is defined as a tripel (I, J, \mathbf{t}) such that the following holds:

- 1. I is a subset of X of size 2^d , J is a subset of Y of size d and $\mathbf{t}: J \to \mathbb{R}$.
- 2. Every pattern $b: J \to \{0, 1\}$ occurs in exactly one row of the Boolean matrix obtained by thresholding the columns of $A_{I,J}$ at t, i.e., $A_{I,J}$ equals B_d up to permutation of its rows.

Remark 2.3. Let $k \ge 1$ be a positive integer. Consider a matrix $A: X \times Y \to [k]_0$. It is easy to see that, if a set $J \subseteq Y$ can be P-shattered by A with thresholds $\mathbf{t}: J \to \mathbb{R}$, then it can also be P-shattered with (suitably chosen) thresholds $\mathbf{t}: J \to [k]$. An analogous remark applies to V-shattering with a uniform threshold t.

When analyzing the P- or the V-dimension of a matrix with entries in $[k]_0$, we will assume that thresholds are taken from [k] whenever we find that convenient.

3 Known relations

In this section we review some known relations between the combinatorial dimensions defined in Section 2.

Bounding P- in terms of V-dimension 3.1

It follows directly from the definitions that

$$\operatorname{Vdim}(A) \le \operatorname{Pdim}(A) \text{ and } \operatorname{V}_{\gamma}(A) \le \operatorname{P}_{\gamma}(A).$$
 (3.1)

This raises the question whether we can bound the P- in terms of the V-dimension (resp. the P_{γ} in terms of the V_{γ} -dimension). The gap between $\operatorname{Pdim}(A)$ and $\operatorname{Vdim}(A)$ cannot be bounded in general, as the following well-known example shows.

Example 3.1. Let X be the set of all monotone² functions from [0, 1] to [0, 1], Y = [0, 1] and A(x, y) = x(y)for $x \in X$. Then $\operatorname{Vdim}(A) = 1$ and $\operatorname{Pdim}(A) = \infty$.

In order to bound the P- in terms of the V-dimension, the focus will therefore be on matrices of the form $A: X \times Y \to [k]_0$. According to the following results of Ben-David et al. [3] (here expressed in our notation), the P- can exceed the V-dimension by factor k, but not by a larger factor³:

Theorem 3.2 ([3]). For every matrix $A: X \times Y \to [k]_0$, we have

$$\operatorname{Pdim}(A) \le k \cdot \operatorname{Vdim}(A). \tag{3.2}$$

²A function $f:[0,1] \to [0,1]$ is monotone if $f(x) \le f(y)$ for all $x \le y$. ³See [3, Theorem 7-8] and the proof of [3, Theorem 7].

Theorem 3.3 ([3]). For every $d \ge 1$ and every $k \ge 1$, there exists a matrix $A: X \times Y \to [k]_0$ such that

$$\operatorname{Vdim}(A) = d$$
 and $\operatorname{Pdim}(A) = k \cdot d$.

Alon et al. [1] have bounded P_{γ} - in terms of the $V_{\gamma/2}$ -dimension.

Theorem 3.4 ([1]). For every matrix $A: X \times Y \to [0,1]$ and every $0 < \gamma \leq 1/2$, we have⁴

$$P_{\gamma}(A) \leq \left(\left\lceil \frac{1}{\gamma} \right\rceil - 1 \right) \cdot V_{\gamma/2}(A) \leq \left(\left\lceil \frac{1}{\gamma} \right\rceil - 1 \right) \cdot Pdim(A) \quad .$$

$$(3.3)$$

Proof. The thresholds t_1, \ldots, t_d used for P_{γ} -shattering $d := P_{\gamma}(A)$ many columns of A must belong to the interval $[\gamma, 1 - \gamma]$. Any threshold t_i can be rounded to the closest multiple of γ . Denote the latter by \hat{t}_i . The inequality (3.3) becomes now evident from the following observations. First, by using the thresholds \hat{t}_i instead of t_i , the width of shattering may drop from γ to $\gamma/2$ (but not beyond). Second, $\hat{t}_1, \ldots, \hat{t}_d$ can take on at most

$$r := \left\lceil \frac{1 - 2\gamma}{\gamma} \right\rceil + 1 = \left\lceil \frac{1}{\gamma} \right\rceil - 1$$

different values. By the pidgeon-hole principle, there is some $t \in {\hat{t}_1, \ldots, \hat{t}_d}$ that can be used for $V_{\gamma/2}$ -shattering d/r many points.

3.2 Bounding dual dimension in terms of its primal

A well-known result due to Assouad [2] already mentioned in Section 1, which we will refer to as Assouad's bound, states that one can upper bound the dual VC-dimension in terms of the (primal) VC-dimension.

Theorem 3.5 ([2]). For every matrix $A: X \times Y \to \{0, 1\}$, we have

$$\operatorname{VC}^{*}(A) \le 2^{\operatorname{VC}(A)+1} - 1.$$
 (3.4)

Note that, under the assumption that $VC^*(A)$ is a power of two, this means

$$\mathrm{VC}^*(A) \le 2^{\mathrm{VC}(A)}.\tag{3.5}$$

The bound in (3.5) is known to be tight for every value of VC(A), see, e.g., [7].

In Appendix A we show that the Assouad's bound also holds for Vdim(A) and $V_{\gamma}(A)$, based on the notion of *uniform* Ψ -dimension as defined in [1]. These observations are summarized in the following result.

Corollary 3.6 (Folklore). For every matrix $A: X \times Y \rightarrow [0,1]$, we have

$$Vdim^{*}(A) \le 2^{Vdim(A)+1} - 1 \quad and \quad V_{\gamma}^{*}(A) \le 2^{V_{\gamma}(A)+1} - 1.$$
(3.6)

If $Vdim^*(A)$, respectively $V^*_{\gamma}(A)$, is a power of two, this means

$$\operatorname{Vdim}^{*}(A) \leq 2^{\operatorname{Vdim}(A)} \quad and \quad V_{\gamma}^{*}(A) \leq 2^{\operatorname{V}_{\gamma}(A)}.$$

$$(3.7)$$

Combining Theorem 3.2 (applied to A^{\top}) with Corollary 3.6, we directly obtain the following result:

Theorem 3.7 (Folklore). For every matrix $A: X \times Y \to [k]_0$, the following holds:

- 1. $\operatorname{Pdim}^*(A) \le k \cdot (2^{\operatorname{Vdim}(A)+1} 1) \le k \cdot (2^{\operatorname{Pdim}(A)+1} 1).$
- 2. If $\operatorname{Vdim}^*(A)$ is a power of two, then $\operatorname{Pdim}^*(A) \leq k \cdot 2^{\operatorname{Vdim}(A)} \leq k \cdot 2^{\operatorname{Pdim}(A)}$.

Similarly, combining Theorem 3.4 with Corollary 3.6, we directly obtain the following result.

⁴In [3], one finds a factor $2\lceil 1/(2\gamma)\rceil$ at the place of factor $\lceil 1/\gamma\rceil$. We find the latter (and slightly smaller) factor preferable because of its simpler form.

Corollary 3.8 (Folklore). For every matrix $A: X \times Y \rightarrow [0,1]$, the following holds:

$$P_{\gamma}^{*}(A) \leq \left(\left\lceil \frac{1}{\gamma} \right\rceil - 1 \right) \cdot \left(2^{V_{\gamma/2}(A) + 1} - 1 \right) \leq \left(\left\lceil \frac{1}{\gamma} \right\rceil - 1 \right) \cdot \left(2^{\operatorname{Pdim}(A) + 1} - 1 \right).$$

4 Our results

In this section we describe our new contributions, that complement those mentioned in Section 3. We first discuss results related to the pseudo-dimension. We start with a result showing that the upper bound on $Pdim^*(A)$ in Theorem 3.7 can be improved by a factor 2 (roughly) for matrices A with Vdim(A) = 1.

Theorem 4.1. Let $A: X \times Y \to [k]_0$ with $k \ge 1$ be a matrix with $\operatorname{Vdim}(A) = 1$. Then $\operatorname{Pdim}^*(A) \le k + 2$.

The next result implies that the upper bound on $\operatorname{Pdim}^*(A)$ in the second statement of Theorem 3.7 is tight for matrices with $\operatorname{Vdim}(A) \geq 2$, as well as the upper bound on $\operatorname{Pdim}^*(A)$ in Theorem 4.1 whenever $\operatorname{Vdim}(A) \geq 1$.

Theorem 4.2. The following two lower bounds hold:

1. For every $d \ge 2$ and every $k \ge 1$, there exists a matrix $A: X \times Y \to [k]_0$ such that

 $\operatorname{Pdim}(A) = d$, $\operatorname{Vdim}^*(A) = 2^d$ and $\operatorname{Pdim}^*(A) = k \cdot 2^d$.

2. For every $k \ge 1$, there exists a matrix $A : X \times Y \to [k]_0$ such that $\operatorname{Vdim}(A) = \operatorname{Pdim}(A) = 1$ and $\operatorname{Pdim}^*(A) = k + 2$.

In combination with the a technical tool defined in Section 5.2, we also obtain the following corollary. It stands in stark contrast to Assouad's bound for the VC-dimension.

Corollary 4.3. There exist a matrix $A: X \times Y \to [0,1]$, such that $\operatorname{Pdim}(A) = 1$ and $\operatorname{Pdim}^*(A) = \infty$.

We next move to our results for the fat-shattering dimensions. The first result here implies that upper bound on $P_{\gamma}(A)$ from Theorem 3.4 is tight up to a small constant factor:

Theorem 4.4. For every $d \ge 1$, there exists a matrix $A : X \times Y \rightarrow [0,1]$ such that Pdim(A) = d and, for all $k \ge 1$,

$$\mathcal{P}_{1/(2\mathbf{k})}(A) \ge k \cdot d \; \; .$$

Finally, our last results state that the bound on $P^*_{\gamma}(A)$ from Corollary 3.8 is tight up to a small constant factor.

Theorem 4.5. The following two lower bounds hold:

1. For every $d \ge 2$, there exists a matrix $A: X \times Y \to [0,1]$ such that $\operatorname{Pdim}(A) = d$ and, for all $k \ge 1$,

$$P_{1/(2k)}^*(A) = k \cdot 2^d$$
.

2. There exists a matrix $A: X \times Y \to [0,1]$ such that Pdim(A) = 1 and, for all $k \ge 2$,

$$P_{1/(2k)}^*(A) = k+2$$

5 Proofs

Section 5.1 is devoted to the proof of Theorem 4.1. In Section 5.2, we make some considerations which will allow for an easier presentation of our lower bound constructions, that are given in Sections 5.3 and 5.4.

5.1 Proof of Theorem 4.1

For k = 1, the assertion of the theorem collapses to the claim that $VC^*(A) \leq 3$ for every Boolean matrix A with $V\dim(A) = 1$. This is an immediate consequence of (3.4). Suppose now that $k \geq 2$. It suffices to show that $P\dim^*(A) \geq k + 3$ implies that $V\dim(A) \geq 2$ (i.e., we give a proof by contradiction). Pick a witness (I, J, \mathbf{t}) for $P\dim^*(A) \geq k + 3$. More concretely:

- $I = \{x_1, ..., x_{k+3}\}, J \subseteq Y$ with $|J| = 2^{k+3}$ and $\mathbf{t} : I \to [k]$, say $\mathbf{t}(x_i) = t_i$.
- The matrix obtained by thresholding the rows of $A_{I,J}$ at **t** equals B_{k+3}^{\top} .

We may assume that, after renumbering the rows appropriately, one has $t_1 \leq \ldots \leq t_{k+3}$. We decompose the rows of $A_{I,J}$ into maximal blocks such that the same threshold is assigned to every row from the same block. Since any threshold t_i is taken from [k], the total number k' of blocks is bounded by k. A block that is different from the first and from the last block is said to be an *inner block*. We proceed by case analysis:

Case 1: One of the blocks contains 4 rows.

Then $\operatorname{Vdim}^*(A) \ge 4$ is immediate. Thus $\operatorname{Vdim}(A) \ge \lfloor \log \operatorname{Vdim}^*(A) \rfloor \ge 2$.

Case 2: The first or the last block contains 3 rows.

For reasons of symmetry, we may assume that the first block contains 3 rows. Consider the following (4×2) -submatrix of B_{k+3}^{\top} :

$$\begin{array}{cccc}
0 & 0 \\
0 & 1 \\
1 & 0 \\
\hline
1 & 1
\end{array}$$

The first three rows are taken from the first block and the last row is taken from the last block. The separation line between the third and the last row is only intended to illustrate the transition from one block to another. Remember that the rows of the first block of $A_{I,J}$ are thresholded at t_1 while the rows of the last block are thresholded at $t_{k'} > t_1$. Hence, if we threshold all rows (or all columns) of $A_{I,J}$ at t_1 , then the above submatrix of B_{k+3}^{\top} will remain unchanged. We may conclude from this discussion that $V\dim(A) \geq 2$.

Case 3: One of the inner blocks contains 2 rows, say block *b*.

The argument is similar to that given in Case 2. The relevant submatrix of B_{k+3}^{\top} (with one row of the first block, two rows of block b, one row of the last block and two separation lines inbetween) now looks as follows:

0	0
0	1
1	0
1	1

Since $t_1 < t_b < t_{k'}$, thresholding all rows (or all columns) of $A_{I,J}$ at t_b will leave the above submatrix of B_{k+3}^{\top} unchanged. We may conclude that $\operatorname{Vdim}(A) \geq 2$.

Since $A_{I,J}$ has k + 3 rows (with $k \ge 2$), it is easy to argue that one of the three above cases must occur. Suppose first that k = 2. Then there at most 2 blocks and 5 rows. It follows that the first or the last block contains at least 3 rows. Suppose now that $k \ge 3$. If the first and the last block contain at most two rows, respectively, then at least k-1 rows are left for the $k'-2 \le k-2$ inner blocks. By the pidgeon-hole principle, there must be an inner block with two rows. This completes the proof of Theorem 4.1.

5.2 Preliminaries for lower bound constructions

Consider again the Boolean matrix B_d with d columns and 2^d rows that had been defined in Section 2. It is evident that B_d satisfies the following conditions:

i) **Distinctness Condition:** The rows of B_d are pairwise distinct.

ii) General Balance Condition: For any $k \in [d]$, any choice of k distinct columns of B_d and any pattern $\mathbf{b} \in \{0, 1\}^k$, there are exactly 2^{d-k} rows of B_d which realize the pattern \mathbf{b} within the chosen columns.

The general balance condition implies the following:

- iii) 1st Balance Condition: Each column of B_d has as many zeros as ones.
- iv) **2nd Balance Condition:** For any two distinct columns of B_d , any pattern from $\{0,1\}^2$ is realized within these columns by the same number of rows.

Remark 5.1 (Proof templates). Consider a matrix $A : X \times Y \to [k]_0$. The following template for proving assertions like $Pdim(A) \leq d$ will prove itself quite useful.

- Assume for contradiction that $Pdim(A) \ge d+1$.
- Pick a witness (I, J, t) for this inequality.
- Exploit the fact that the matrix B obtained by thresholding the columns of $A_{I,J}$ at t must be equal to B_{d+1} .
- Prove that B violates one of the conditions that B_{d+1} must satisfy.

Sometimes the following (slightly simpler) template can be used instead:

- Take a fixed but arbitrary function $\mathbf{t}: Y \to [k]$.
- Let B be the matrix obtained by thresholding the columns of A at t.
- Show that no more than d columns of B have at least 2^d zeros and at least 2^d ones.

This also shows that $\operatorname{Pdim}(A) \leq d$ because no submatrix of B with d+1 columns and 2^{d+1} rows has a chance to satisfy the first balance condition.

We next introduce matrices that, though not being Boolean, are close relatives of the matrix B_d .

Definition 5.2. Let k and d_1, \ldots, d_k be positive integers and let $D = d_1 + \ldots + d_k$ denote their sum. The B_D -based matrix with k column blocks of sizes d_1, \ldots, d_k is the matrix $A : X \times Y \to [k]_0$, where $X = [2^D]$ and Y = [D], that results from the following procedure:

- 1. Decompose the D columns of B_D into k blocks of sizes d_1, \ldots, d_k . The blocks are consecutively numbered from 1 to k.
- 2. Obtain A from B_D by replacing any 1-entry (resp. 0-entry) in a column belonging to block $b \in [k]$ by b (resp. by b 1).

The B_D^{\top} -based matrix with k row blocks of sizes d_1, \ldots, d_k is defined analogously.

Note that the matrix A resulting from the above procedure has the property that, for any two columns y_1 in block b_1 and y_2 in block $b_2 > b_1$ and any row x, we have $A(x, y_1) \leq A(x, y_2)$. We will refer to this property as *block monotonicity*.

At this point we also bring into play the matrix A, which is defined as the matrix A augmented with a row of zeros. Formally, we assume that $0 \notin X$ and define $\dot{A} : (X \cup \{0\}) \times Y \to Z$ as the extension of A which satisfies $\dot{A}(0, y) = 0$ for all $y \in Y$. The (technical) use of \dot{A} will become clear in Section 5.4 (in particular, this is explained after Definition 5.7), but it is already included in the statements that follow.

Lemma 5.3. Let $D = d_1 + \ldots + d_k$ and let A be the B_D -based matrix with k column blocks of sizes d_1, \ldots, d_k . Then $\operatorname{Vdim}(A) = \operatorname{Vdim}(\dot{A}) = \max_{j \in [k]} d_j$ and $\operatorname{Pdim}(A) = \sum_{j=1}^k d_j$. *Proof.* We first show that the pseudo-dimension of A equals D. Let $\mathbf{t} : [D] \to [k]$ be the mapping that assigns to every column in block $j \in [k]$ the threshold j. Then the matrix obtained by thresholding the columns of A at \mathbf{t} equals B_D . It follows that $\operatorname{Pdim}(A) \geq \operatorname{VC}(B_D) = D$. Of course $\operatorname{Pdim}(A)$ cannot exceed D so that $\operatorname{Pdim}(A) = D$.

Next, set $d_{max} = \max_{j \in [k]} d_j$. Pick some index $j_{max} \in [k]$ such that $d_{j_{max}} = d_{max}$. We still have to show that $\operatorname{Vdim}(A) = \operatorname{Vdim}(\dot{A}) = d_{max}$. Thresholding the columns of A at the uniform threshold j_{max} , we obtain a matrix B that equals B_D within block j_{max} . This shows that $\operatorname{Vdim}(A) \ge d_{max}$. The inequality $\operatorname{Vdim}(\dot{A}) \le d_{max}$ can be seen as follows. Pick a fixed but arbitrary $J \subseteq [D]$ of size $1 + d_{max}$ and a fixed but arbitrary uniform threshold $t \in [k]$. Let B be the matrix obtained by thresholding the columns of \dot{A} at t. The set J must contain two columns belonging to two different blocks, say column y_1 in block b_1 and column y_2 in block $b_2 > b_1$. By the block-monotonicity of A (which implies block-monotonicity for \dot{A} as well), no row of B can assign label 1 to y_1 and label 0 to y_2 . Since J and t were arbitrary choices, it follows that no set of size $1 + d_{max}$ can be V-shattered by \dot{A} .

Setting $d_1 = \ldots = d_k = d$ in Lemma 5.3, we obtain the following result (which is almost the same as Theorem 3.2):

Corollary 5.4. For every $d \ge 1$ and every $k \ge 1$, there exists a matrix $A : X \times Y \to [k]_0$ such that $\operatorname{Vdim}(A) = \operatorname{Vdim}(\dot{A}) = d$ and $\operatorname{Pdim}(A) = k \cdot d$.

5.3 Proof of Theorem 4.2

Theorem 4.2 is a direct consequence of the following two results:

Lemma 5.5. Let $d \ge 2$ and $k \ge 1$ be given. For $D = k \cdot 2^d$, let A be the B_D^{\top} -based matrix with k row blocks of size 2^d , respectively. Then

$$\operatorname{Pdim}(A) = \operatorname{Pdim}(A) = d$$
, $\operatorname{Vdim}^*(A) = 2^d$ and $\operatorname{Pdim}^*(A) = k \cdot 2^d$.

Proof. The identities $\operatorname{Vdim}^*(A) = 2^d$ and $\operatorname{Pdim}^*(A) = k \cdot 2^d$ are immediate from Lemma 5.3. Hence it suffices to verify the identity $\operatorname{Pdim}(A) = \operatorname{Pdim}(\dot{A}) = d$. We can infer from $\operatorname{Vdim}^*(A) = 2^d$ that $\operatorname{Pdim}(\dot{A}) \geq \operatorname{Pdim}(A) \geq \operatorname{Vdim}(A) \geq d$. Hence the proof can be accomplished by showing that $\operatorname{Pdim}(\dot{A}) \leq d$. For sake of brevity, set

$$s = 2^d$$
 and $\bar{d} = 1 + d$

Assume for contradiction that $Pdim(\dot{A}) \geq \bar{d}$ and fix some witness (I, J, t) for this inequality, i.e.,

- 1. $I \subseteq [D]_0, |I| = 2^{\overline{d}}, J \subseteq [2^D], |J| = \overline{d} \text{ and } \mathbf{t} : J \to [k] \text{ assigns a threshold to each column of } A_{I,J}$.
- 2. The matrix B obtained by thresholding the columns of $A_{I,J}$ at t equals $B_{\bar{d}}$.

Before we proceed with the proof, we fix some notation. For b = 1, ..., k, let I_b denote the set of row indices in I that belong to block b of A. Set $I_0 = I \cap \{0\}$ and note that

$$\sum_{b=0}^{k} |I_b| = |I| = 2^{\overline{d}} = 2s \quad \text{(twice the block size)}$$

while, for every $b \in [k]$, we clearly have $0 \leq |I_b| \leq s$. Let b_0, b'_0 (resp. b_1, b'_1) denote the smallest and second-smallest (resp. largest and second-largest) $b \in [k]$ such that $|I_b| \neq 0$.

An obvious question is whether $0 \in I$, that is, whether the extra all-zeros row is among the rows of B. We claim that this is not the case. Assume for contradiction that $0 \in I$. We proceed by case analysis:

Case 1: $1 \le |I_{b_0}| \le s - 1$.

Pick an arbitrary but fixed column $j \in J$ of B. In order to satisfy the first balance condition, the threshold t_j must be large enough so that in block b_0 of column j only 0-entries are found.⁶ It follows that any row of B belonging to block b_0 has 0-entries only and therefore coincides with the extra all-zeros row. This is in contradiction with the distinctness condition.

⁵Recall from the definition of \dot{A} that this matrix is obtained from A by adding an all-zeros row which is indexed by 0.

⁶A single 1-entry in this block would imply that we have only 1-entries in all subsequent blocks.

Case 2: $|I_{b_0}| = s$.

Pick an arbitrary but fixed column $j \in J$ of B. In order to satisfy the first balance condition, the threshold t_j must be large enough so that in block b_0 of column j at least s - 1 0-entries are found. Pick another column $j' \neq j$ in B (also with at least s - 1 0-entries in block b_0). Then the pattern 00 occurs in columns j and j' of B at least s - 1 times (one time in row 0 and at least s - 2 times in block b_0). But then B must have at least 4(s-1) rows in order to satisfy the second balance condition. Hence $4(s-1) \leq 2s$ (because B has 2s rows). It follows that $s \leq 2$, which is in contradiction with our assumptions that $d \geq 2$ and $s = 2^d \geq 4$.

In any case, we arrived at a contradiction, which proves the above claim that $0 \notin I$. In order to accomplish the proof, we still have to derive a final contradiction. We proceed by case distinction again.

Case A: $|I_{b_0}| \ge 2$.

In order to satisfy the first balance condition, the threshold t_j of any column $j \in J$ must be large enough so that in block b_0 of this column only 0-entries are found. Thus all rows of B belonging to block b_0 realize the all-zeros pattern, which is in contradiction with the distinctness condition.

Case B: $|I_{b_0}| = 1$ and $|I_{b'_0}| \le s - 1$.

The argument is similar. Now the single row of B belonging to block b_0 and all rows of B belonging to block b'_0 realize the all-zeros pattern.

- **Case C:** $|I_{b_1}| \ge 2$ or $(|I_{b_1}| = 1 \text{ and } |I_{b'_1}| \le s 1)$. Then, for reasons of symmetry, the last two rows of B both realize the all-ones pattern.
- **Case D:** $|I_{b_0}| = |I_{b_1}| = 1$ and $|I_{b'_0}| = |I_{b'_1}| = s$.

Since |I| = 2s, this case can occur only if $b'_0 = b'_1$. But then $2s = |I| = |I_{b_0}| + |I_{b'_0}| + |I_{b_1}| = s + 2$ so that s = 2. This contradicts to our assumption $d \ge 2$ and $s = 2^d \ge 4$.

In any case, we arrived at a contradiction.

Lemma 5.6. Let $k \geq 2$ and let A be the B_{k+2}^{\top} -based matrix with k row blocks of sizes

$$d_{j} = \begin{cases} 2 & \text{for } j = 1, k \\ 1 & \text{for } j = 2, \dots, k - 1 \end{cases}$$

Then $\operatorname{Pdim}(A) = \operatorname{Pdim}(\dot{A}) = 1$ and $\operatorname{Pdim}^*(A) = k + 2$.

Proof. The identity $\operatorname{Pdim}^*(A) = k + 2$ is immediate from Lemma 5.3. Clearly $\operatorname{Pdim}(\dot{A}) \ge \operatorname{Pdim}(A) \ge 1$. Hence it suffices to show that $\operatorname{Pdim}(\dot{A}) \le 1$. The rows of \dot{A} have indices $0, 1, \ldots, k + 2$ and index 0 is reserved for the all-zeros row. Assume for contradiction that $\operatorname{Pdim}(\dot{A}) \ge 2$ and pick a witness (I, J, t) for this inequality so that the following holds:

- $J = \{j_1, j_2\} \subset [2^{k+2}], I \subseteq [k+2]_0$ with |I| = 4 and $\mathbf{t} : J \to [k]$, say $\mathbf{t}(j_1) = t_1$ and $\mathbf{t}(j_2) = t_2$.
- The matrix B obtained by thresholding the columns of $A_{I,J}$ at t equals B_2 (with rows indexed by I and columns indexed by J).

Consequently *B* satisfies the distinctness condition and the balance conditions. Consider the smallest index i_1 and the second-smallest index i_2 in *I*. Note that, since |I| = 4 and the last block of *B* is of size 2, neither i_1 nor i_2 belongs to the last block, i.e., either $i_1, i_2 \in \{0, 1, 2\}$ or i_2 belongs to one of the inner blocks consisting of a single row only. In order to establish the first balance condition for the matrix *B*, the thresholds t_1 and t_2 must be large enough so that only zeros are found in the first two components (indexed by i_1 and i_2) of the columns j_1 and j_2 . Thus the first two rows of *B* both realize the all-zeros pattern, which is in contradiction with the distinctness condition.

Lemma 5.6 does not cover the case k = 1 in the second assertion of Theorem 4.2. But this case is easy to handle: setting $A = B_3^{\top}$, we obtain $\operatorname{Pdim}^*(A) = \operatorname{VC}^*(A) = 3$ and $\operatorname{Pdim}(A) = \operatorname{VC}(A) = 1$.

5.4 Proofs of Theorems 4.4 and 4.5

Matrices A with the properties as prescribed by Theorems 4.4 and 4.5 are easy to construct by means of a suitable operation that merges matrices of a given matrix family into a single matrix.

Definition 5.7 (Merge-operation). Let $(A_k)_{k\geq 1}$ with $A_k : X_k \times Y_k \to [k]_0$ be a given family of matrices. Let X (resp. Y) denote the disjoint union of the sets X_k (resp. Y_k) with $k \geq 1$. Assume that $X \cap Y = \emptyset$. For every $x \in X$, let k(x) denote the unique k such that $x \in X_k$. The notation k(y) is understood analogously. The matrix $A : X \times Y \to [0, 1]$ given by

$$A(x,y) = \begin{cases} \frac{A_{k(x)}(x,y)}{k(x)} & \text{if } k(y) = k(x) \\ 0 & \text{otherwise} \end{cases}$$

is called the merge of the family $(A_k)_{k\geq 1}$.

The merge-operation reveals why we introduce the matrix \dot{A} : The pseudo-dimension (or any other combinatorial dimension for that matter) of the matrix A restricted to the columns Y_k is nothing more than the pseudo-dimension of the functions in A_k augmented with an infinite number of functions that are zero everywhere. The pseudo-dimension of this function class clearly equals the pseudo-dimension of the matrix \dot{A}_k . The merge-operation has the following properties:

Lemma 5.8. Let A be the merge of the family $(A_k)_{k\geq 1}$. Then the following holds:

- 1. $P_{1/(2k)}(A) \ge Pdim(A_k).$
- 2. Let $d_0 \in \mathbb{N}$. If $\sup_k \operatorname{Pdim}(\dot{A}_k) \leq d_0$, then $\operatorname{Pdim}(A) \leq d_0$.
- 3. $V_{1/(2k)}(A) \ge Vdim(A_k)$.
- 4. Let $d_0 \in \mathbb{N}$. If $\sup_k \operatorname{Vdim}(\dot{A}_k) \leq d_0$, then $\operatorname{Vdim}(A) \leq d_0$.

Proof. We only prove the first two assertions of the lemma; the other two assertions are quite similar.

Note that, for k = k(x) = k(y), A coincides with A_k except for scaling down the values $0, 1, \ldots, k$ by factor k. Since A_k takes integer values, each set that can be P-shattered by A_k can actually be $P_{1/2}$ -shattered. After down-scaling, the width of shattering becomes 1/(2k). From these observations, the first assertion of the lemma easily follows.

We proceed with the proof of the second assertion. Set d := Pdim(A). Fix some witness (I, J, \mathbf{t}) so that the following holds:

- 1. $I \subset X$, $|I| = 2^d$, $J \subset Y$, |J| = d and $\mathbf{t} : J \to \mathbb{N}$ assigns a threshold $t_y := \mathbf{t}(y)$ to every $y \in J$.
- 2. The matrix B obtained by thresholding the columns of $A_{I,J}$ at t equals B_d (with rows indexed by I and columns indexed by J).

It follows that B satisfies the distinctness condition and the balance conditions.

Claim 1: For every $y \in J$, we have $t_y > 0$.

Proof. $t_y \leq 0$ would imply that column y of B has no 0-entry, which is in contradiction with the first balance condition.

Claim 2: The mapping $y \mapsto k(y)$ assigns the same value to all $y \in J$.

Proof. Assume to the contrary that there exist $y_1, y_2 \in J$ such that $k(y_1) \neq k(y_2)$. Then, for every row x of B, at least one of the entries $B[x, y_1]$ and $B[x, y_2]$ equals 0 (because k(x) cannot be equal to both, $k(y_1)$ and $k(y_2)$). By the first balance condition, any column in B has as many 0- as 1-entries. Since this is particularly true for the columns y_1 and y_2 , it follows that, for every row x of B, exactly one of the entries $B[x, y_1]$ and $B[x, y_2]$ equals 0. Thus column y_2 of B is the entry-wise logical negation of the column y_1 . This, however, is in contradiction with the second balance condition.

Claim 3: Let k_1 denote the common k-value of y_1, \ldots, y_d . Then any row x in B with $k(x) \neq k_1$ has 0-entries only.

Proof. This is straighforward.

We conclude from Claims 2 and 3 that $Pdim(A) = d \leq Pdim(\dot{A}_{k_1})$ and, by assumption, the latter quantity is at most d_0 , which concludes the proof.

Theorem 4.4 is now a direct consequence of Lemma 5.8 in combination with Corollary 5.4, while Theorem 4.5 is a direct consequence of Lemma 5.8 in combination with Lemmas 5.5 and 5.6. In order to prove Corollary 4.3, note that Lemma 5.6 tells us that for every k there exists a matrix A_k such that $\operatorname{Pdim}(\dot{A}_k) = 1$ and $\operatorname{Pdim}^*(\dot{A}_k) \geq \operatorname{Pdim}^*(A_k) = k + 2$. We may then apply Lemma 5.8.

Acknowledgements. The first author thanks Tim Roughgarden for discussions that (indirectly) lead to the questions studied in this work.

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A On the derivation of Assouad's bound for uniform dimensions

We say that $J \subseteq Y$ is *VC-shattered* by $A: X \times Y \to \{0, 1, *\}$ if, for every function $b: J \to \{0, 1\}$, there exists an $x \in X$ such that, for every $y \in J$, we have B(x, y) = b(y). We first note that (3.4) is also valid for every matrix of the form $A: X \times Y \to \{0, 1, *\}$: the central observation in the proof is that B_d contains $B_{\lfloor \log d \rfloor}^{\top}$ as a submatrix. This implies that $VC(A) \geq \lfloor \log VC^*(A) \rfloor$, which is equivalent to (3.4).

Consider now a matrix of the general form $A: X \times Y \to Z$ with $Z \subseteq \mathbb{R}$. Making use of the concept of uniform Ψ -dimensions from [3], the result of Assouad can be extended to several other combinatorial dimensions. Let Ψ denote a family of substitutions of the form $\psi : \mathbb{R} \to \{0, 1, *\}$. Denote by $\psi(A)$ the matrix obtained from A by performing the substitution ψ entry-wise. The uniform Ψ -dimension of A is then defined as

$$\Phi_U(A) = \sup_{\psi \in \Psi} \operatorname{VC}(\psi(A)) \; .$$

Let Ψ_Y denote the set of all collections $\bar{\psi} = (\psi_y)_{y \in Y}$ with $\psi_y \in \Psi$. Denote by $\bar{\psi}(A)$ the matrix obtained from A by replacing each entry A(x, y) with $\psi_y(A(x, y))$. The *(non-uniform)* Ψ -dimension of A is defined as

$$\Phi(A) = \sup_{\bar{\psi} \in \Psi_Y} \operatorname{VC}(\bar{\psi}(A)) \; .$$

As usual, we get the corresponding dual dimensions by setting $\Phi^*(A) = \Phi(A^{\top})$ and $\Phi^*_U(A) = \Phi_U(A^{\top})$. Note that $\psi(A^{\top}) = \psi(A)^{\top}$ while $\bar{\psi}(A^{\top})$ is not generally equal to $\bar{\psi}(A)^{\top}$.

As noted in [3], several popular combinatorial dimensions can be viewed as (uniform or non-uniform) ψ -dimension. Here we are particularly interested in the *P*-, *P*_{γ}, *V*-and *V*_{γ}-dimension:

Remark A.1. We next explain how to interpret known dimensions as special cases of the Ψ -dimension.

- 1. If Ψ is the set of mappings ψ_t of the form $\psi_t(a) = sgn(a-t)$ for some $t \in \mathbb{R}$, then $\Phi(A) = Pdim(A)$ and $\Phi_U(A) = Vdim(A)$ (see [3]).
- 2. If Ψ is the set of mappings ψ_t of the form

$$\psi_t(a) = \begin{cases} 1 & \text{if } a \ge t + \gamma \\ 0 & \text{if } a < t - \gamma \\ * & \text{otherwise} \end{cases}$$

for some $t \in \mathbb{R}$ and $\gamma > 0$, then $\Phi(A) = P_{\gamma}(A)$ and $\Phi_U(A) = V_{\gamma}(A)$.

The following calculation, with ψ ranging over all functions in Ψ , shows that Theorem 3.5 can be extended to any uniform Ψ -dimension at the place of the VC-dimension:

$$\begin{split} \Phi_U^*(A) &= \Phi_U(A^{\top}) = \sup_{\psi} \operatorname{VC}(\psi(A^{\top})) = \sup_{\psi} \operatorname{VC}(\psi(A)^{\top}) \\ &= \sup_{\psi} \operatorname{VC}^*(\psi(A)) \le \sup_{\psi} \left(2^{\operatorname{VC}(\psi(A))+1} - 1 \right) \\ &= 2^{\sup_{\psi} \operatorname{VC}(\psi(A))+1} - 1 = 2^{\Phi_U(A)+1} - 1 \end{split}$$

We remark that a similar argument for the non-uniform Ψ -dimension fails as it then no longer holds that $\bar{\psi}(A^{\top}) = \bar{\psi}(A)^{\top}$ (which is the argument we use in the third equality above).