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Thermodynamics of spacetime and unimodular gravity

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In this review we discuss emergence of unimodular gravity (or, more precisely, Weyl transverse gravity) from thermodynamics of spacetime. By analyzing three different ways to obtain gravitational equations of motion by thermodynamic arguments, we show that the results point to unimodular rather than fully diffeomorphism invariant theories and that this is true even for modified gravity. The unimodular character of dynamics is especially evident from the status of cosmological constant and energy-momentum conservation.

Keywords: thermodynamics of spacetime; alternative theories of gravity; unimodular gravity; Weyl transverse gravity.

1. Introduction

The relation between thermodynamics and gravitational physics was first described for the case of black holes. It was found that the Einstein equations imply a set of laws of black hole dynamics analogous to those of thermodynamics [1] and that black holes posses entropy proportional to the horizon area [2]. Furthermore, due to quantum effects they6 emit black body radiation and, therefore, posses a finite temperature [3]. Analogous thermodynamic properties have been later described for other geometries with causal horizons, e.g. de Sitter spacetimes [4], Rindler wedges [5] and causal diamonds [6].

A novel perspective on the connection between thermodynamics and gravity was offered by the idea that equations governing gravitational dynamics can be recovered from thermodynamic arguments [7]. To do so, one needs to construct a local (approximate) causal horizon in every spacetime point and assign to them temperature and entropy. This approach has been later refined [8,9] and even shown to work for some alternative theories of gravity [10,11,12,14,13,15] and for the cases when gravity is sourced by quantum fields [14,15,16].

The aim of this review is to argue that thermodynamics of spacetime leads naturally to unimodular theories of gravity rather than fully diffeomorphism invariant ones. Unimodular gravity (UG) is classically equivalent to general relativity (GR), but its symmetry group is reduced to transverse (metric determinant preserving) diffeomorphisms. Similarly, there exists a unimodular version of any local, diffeomorphism invariant theory of gravity. One of the appeals of UG lies in the fact, that vacuum energy does not couple to gravity, solving some (but not all) of the problems related to the value of the cosmological constant [17,18,19,20]. Instead of a fully fledged theory of gravity, UG can also be understood as gauge fixed Weyl transverse gravity (WTG), a theory invariant under transverse diffeomorphisms and Weyl transformations. Just like UG, WTG is classically equivalent to general relativity (GR). Moreover, the Weyl symmetry ensures radiative stability of the value of the cosmological constant [20,21], curing a problem common to both GR and UG.

Various aspects of the relation of thermodynamics of spacetime and UG have been noted in several works [11,22,23,24]. Here, we try to provide a broad perspective on this connection by discussing the emergence of UG in three distinct thermodynamic derivations. The most straightforward one is based on the seminal Jacobson's paper concerning thermodynamics of Rindler wedges [7] (with some later improvements taken into account [8,9]). The second one obtains gravitational dynamics from thermodynamics of local causal diamonds. As it turns out, the properties of these objects allow us to derive equations of motion for a wide class of modified theories of gravity [13,15]. The last approach also considers causal diamonds, but describes the sources of gravity as quantum fields and, thus, leads to semiclassical gravitational dynamic [16]. In all the cases, we find distinctly unimodular behavior of the resulting gravitational dynamics.

The paper is organized as follows. In section 2 we briefly introduce UG and WTG. Section 3 concerns emergence of UG (or WTG) from thermodynamics of spacetime. Finally, in section 4 we sum up the results and discuss possible future developments.

For simplicity of notation we consider four spacetime dimensions although a generalization to an arbitrary dimension is trivial. We work with metric signature $(-,+,+,+)$. Definitions of the curvature-related quantities follow [25]. Lower case Greek letters denote abstract spacetime indices. Unless otherwise explicitly stated, we use the SI units.

2. Unimodular and Weyl transverse gravity

The origin of UG can be traced all the way back to Einstein [26]. The basic idea is to restrict the full diffeomorphism invariance of GR only to transverse diffeomorphisms, i.e., those generated by an arbitrary divergence-free vector field ξ^{μ},

$$\begin{aligned} g'_{\mu\nu} &= g_{\mu\nu} + 2\xi_{(\mu;\nu)}, \\ \xi^{\mu}{}_{;\mu} &= 0. \end{aligned} \tag{1}$$

Such diffeomorphisms do not change the metric determinant, which one keeps fixed to $g = -\omega_0^2$, where ω_0 is an arbitrary real number (Einstein's original choice was $g = -1$). The simplest action for UG reads [18]

$$S = \int_{\Omega} \left[\frac{c^4}{16\pi G} \left(R - 2\bar{\Lambda} \right) + \mathcal{L}_{matter} \right] \omega_0 \mathrm{d}^4 x, \tag{2}$$

where Ω is a spacetime manifold, $\bar{\Lambda}$ is a constant and L_{matter} denotes the matter Lagrangian. When varying the action with respect to the metric, the allowed variations are only those with $\delta g = 0$. Otherwise, they would break the fixed determinant condition, $g = -\omega_0^2$. Varying the UG action with respect to the metric under this condition yields traceless equations of motion (EoMs)

$$R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu} \right). \tag{3}$$

These equations are equivalent to nine of the ten EoMs of GR. However, the tenth equation corresponding to the trace part of Einstein equations is missing. To obtain it, one needs to perform a covariant divergence of the EoMs. Together with the contracted Bianchi identities, it implies

$$\frac{8\pi G}{c^4} T_{\mu}{}^{\nu}{}_{;\nu} = \frac{1}{4} \left(\frac{8\pi G}{c^4} T_{;\mu} - R_{;\mu} \right). \tag{4}$$

From this equation, we see that, unlike in GR, EoMs of UG do not imply a divergence-free energy-momentum tensor. Instead, a milder condition holds

$$T_{\mu}{}^{\nu}{}_{;\nu} = \mathcal{J}_{;\mu}, \tag{5}$$

for some scalar function $\mathcal{J}$. This function then represents a measure of local non-conservation of energy-momentum. Integrating equation (4) and substituting the result into the traceless EoMs yields

$$G_{\mu\nu} + \left(\Lambda + \frac{8\pi G}{c^4} \mathcal{J} \right) g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{6}$$

where $G_{\mu\nu} = R_{\mu\nu} - R g_{\mu\nu}/2$ is the Einstein tensor and Λ denotes an arbitrary integration constant. While several models with $\mathcal{J} \neq 0$ have been studied in the context of UG [27,28], for most commonly considered matter fields (perfect fluid, scalar field, electromagnetic field,...), the local energy-momentum conservation is guaranteed by matter EoMs. Then, one has $\mathcal{J} = 0$ regardless of gravitational EoMs. In such cases, the standard Einstein equations are recovered,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{7}$$

although the restriction on metric determinant, $g = -\omega_0^2$, remains in place. We can see that integration constant Λ plays the role of the cosmological constant. However, in contrast with GR, it is not related to the constant parameter in the Lagrangian, $\bar{\Lambda}$, which does not affect EoMs in any way. Instead, the value of Λ is arbitrary and free to vary between solutions. Furthermore, the traceless EoMs are clearly invariant under the shift of energy-momentum tensor by $Cg_{\mu\nu}$, where C can be any constant. Since $Cg_{\mu\nu}$ is precisely the form of vacuum energy, it does not couple to gravity in unimodular theories. In this way, UG avoids large contributions of vacuum energy to the cosmological constant [20]. Nevertheless, it has been argued that the value of cosmological constant in UG is radiatively unstable in the same way as in GR [19]. In other words, higher order loop contributions to Λ tend to be much larger than its observed value and the theory requires an infinite amount of fine-tuning. Besides the status of the cosmological constant, UG is classically fully equivalent to GR. The equivalence of both theories on the quantum level has been studied in several works, with differing conclusions [17,18,19,29].

In the previous discussion of UG we have fixed the metric determinant to a constant value “by hand”. However, more refined treatments has been put forward, e.g. including the fixed determinant condition in the action via a Lagrange multiplier [18] or even restoring the full diffeomorphism invariance by adding dynamical degrees of freedom besides the metric [30]. Unfortunately, none of these proposals resolves the problem of radiative stability of Λ. However, instead of treating UG as a fundamental theory, one can understand it as a gauge fixed Weyl invariant theory of gravity [20,31]. To see this, consider an auxiliary metric

$$\tilde{g}_{\mu\nu} = \left(\frac{-g}{\omega_0^2}\right)^{\frac{1}{4}} g_{\mu\nu}, \tag{8}$$

so that $\tilde{g} = -\omega_0^2$. Note that the transformation from $g_{\mu\nu}$ to $\tilde{g}_{\mu\nu}$ is clearly non-invertible (we lose one degree of freedom corresponding to the determinant). In terms of the auxiliary metric, the UG action reads

$$S = \int_\Omega \left[\frac{c^4}{16\pi G}\left(\tilde{R} - 2\bar{\Lambda}\right) + \mathcal{L}_{matter}\right] \omega_0 \mathrm{d}^4 x, \tag{9}$$

where $\tilde{R}$ is an auxiliary Ricci scalar constructed from $\tilde{g}_{\mu\nu}$ and the corresponding Levi-Civita connection (i.e., a torsion-free connection obeying $\tilde{\nabla}_\rho \tilde{g}_{\mu\nu} = 0$). To obtain the corresponding EoMs, one can now perform an arbitrary metric variation $\delta g^{\mu\nu}$. The result is

$$\tilde{R}_{\mu\nu} - \frac{1}{4}\tilde{R} g_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu}\right). \tag{10}$$

Let us remark that instead of $\omega_0 \mathrm{d}^4 x$ with a constant ω_0 we might consider a more general position-dependent background spacetime volume element, $|\omega(x)|\,\mathrm{d}^4 x$.

The volume element is non-dynamical,[a] indicating an effective, non-fundamental description of physics [20] (much like the presence of non-dynamical Minkowski metric in special relativistic theories).

The action and EoMs written in terms of $\tilde{g}_{\mu\nu}$ are clearly invariant under Weyl transformations

$$g'_{\mu\nu} = e^{2\sigma} g_{\mu\nu}, \tag{11}$$

where σ is some scalar function, as well as under transverse diffeomorphisms (defined with respect to the auxiliary Levi-Civita connection)

$$\begin{aligned} g'_{\mu\nu} &= g_{\mu\nu} + 2\xi_{(\mu;\nu)}, \\ \tilde{\nabla}_\mu \xi^\mu &= 0, \end{aligned} \tag{12}$$

but they are not fully diffeomorphism invariant. By introducing the Weyl invariance, we expanded UG into a theory known as WTG. This theory reduces to UG in the unimodular gauge, $g = -\omega^2$. Therefore, it has the same classical solutions as GR. The behavior of local energy conservation and cosmological constant is the same as in UG [33]. However, the additional Weyl symmetry renders Λ radiatively stable [21], eliminating one of the main drawbacks of UG.

3. Gravitational dynamics from thermodynamics

In the following, we discuss three distinct ways to obtain gravitational dynamics from thermodynamic arguments. The first one is similar to the original derivation presented by Jacobson for local Rindler wedges. The second works with causal diamonds and has the advantage of being applicable even to modified theories of gravity. The last one also uses causal diamonds, but treats the sources of gravity as quantum fields, leading to semiclassical gravitational dynamics. As we will see, in every case we recover a unimodular theory of gravity rather than a fully diffeomorphism invariant one.

3.1. *Thermodynamics of local Rindler wedges*

First, we review a version of the original Jacobson's derivation [7]. The crucial ingredient of any thermodynamic derivation is a local approximate causal horizon that can be constructed in every regular spacetime point. Here, we consider the simplest option, a local approximate Rindler wedge. In an arbitrary spacetime point P introduce a locally flat coordinate system. Choose a small (with respect to the local curvature length scale) patch $\mathcal{B}$ of a spacelike 2-surface passing through P. Then select a small piece of the one branch of the null boundary of the causal past of $\mathcal{B}$ and denote it by Σ. The construction is sketched in Figure 1.

[a] In principle, dynamics for the volume form, independent of the metric, can be introduced, without spoiling any of the salient features of the theory [32].

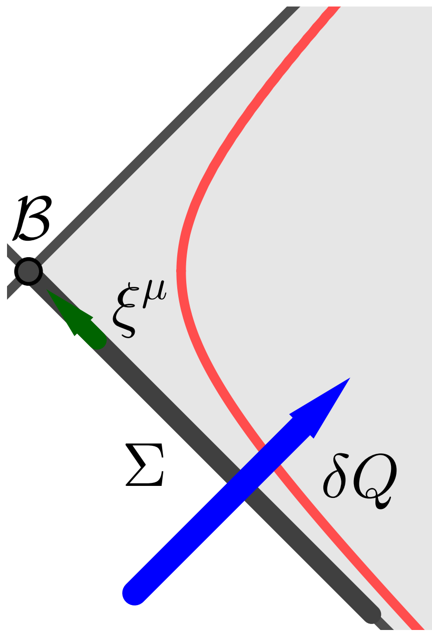

Fig. 1. A sketch of the local approximate Rindler wedge. The bifurcate spacelike 2-surface $\mathcal{B}$ is denoted by a small black circle, red line is an example trajectory of a uniformly accelerated observer perceiving the horizon. The selected part of one branch of the past causal horizon Σ is represented by a thick black line. The blue arrow shows the direction of the heat flux across the horizon and the green one represents null vector ξ^μ normal to Σ.

Now, consider a timelike hypersurface Σ' swept out by wordlines of a class of observers with constant acceleration a, so that Σ' approaches Σ in the limit of $a \to \infty$. If the length of wordlines forming Σ' is much larger than c^2/a, this hypersurface has a well-defined Unruh temperature, $T_{Unruh} = \hbar a/2\pi k_B c$ [34]. The validity of the Unruh temperature formula further requires that the local inertial vacuum state of quantum fields is well approximated by the standard Minkowski vacuum. Assuming this is equivalent to the validity of Einstein equivalence principle[b].

In the presence of matter, one can state the heat flux δQ across Σ in terms of the energy-momentum tensor. Then, invoking the equilibrium Clausius relation, $\mathrm{d}S_{Clausius} = \delta Q/T_{Unruh}$, one obtains an expression for the Clausius entropy flux across Σ'. Taking the limit of $a \to \infty$ then yields the Clausius entropy flux across the null hypersurface Σ

$$\Delta S_{Clausius} = \frac{2\pi k_B}{\hbar c} \int_\Sigma \lambda T_{\mu\nu} \xi^\mu \xi^\nu \mathrm{d}^3\Sigma, \tag{13}$$

where λ denotes a parameter along the (approximate) geodesics forming Σ (different parametrization of each of the geodesics is allowed) and ξ^μ is a future-oriented null

[b]For the statements and hierarchy of various formulations of the equivalence principle, see [35].

vector field orthogonal to Σ. This formula is not restricted to Rindler wedges but holds for essentially any bifurcate null surfaces for which the spacetime curvature effects can be neglected [9]. Notably, since its definition relies on the Unruh effect, Clausius entropy is semiclassical rather than classical, which is evident from the presence of $\hbar$ in the equation.

In addition to the Clausius entropy associated with the matter flux, we assume that the local Rindler horizon possesses entropy proportional to the area of its cross-section, $S_{horizon} = \eta \mathcal{A}\left(\mathcal{B}\right)$, where η is constant throughout the spacetime (this assumption implicitly demands that the strong equivalence principle holds [8,23]). A natural interpretation of this entropy in terms of quantum entanglement has been presented in the context of the search for microscopic origin of black hole entropy [36,37,38]. The idea is that an observer on one side of any causal horizon measures entanglement entropy proportional to its area. This entanglement is often assumed to be the only source of horizon's entropy in the context of thermodynamics of spacetime [7,8]. Nevertheless, we stress that our reasoning does not depend on the way we interpret the horizon's entropy.

Therefore, finding an expression for $\Delta S_{horizon}$ is simply a matter of describing evolution of the horizon's cross-section area. Its change can be described in terms of expansion θ of the geodesic congruence forming Σ, yielding for the change of entropy

$$\Delta S_{horizon} = \eta \int_{\Sigma} \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \mathrm{d}^3\Sigma. \tag{14}$$

To evaluate $\mathrm{d}\theta/\mathrm{d}\lambda$ we use the Raychaudhuri equation (to simplify the discussion, we assume vanishing expansion and shear at $\mathcal{B}$, but this assumption can be relaxed [8]). Let us stress that this equation is a geometric identity completely independent of gravitational dynamics and we are thus free to invoke it without making a circular argument. The resulting expression for the entropy change reads

$$\Delta S_{horizon} = -\eta \int_{\Sigma} \lambda R_{\mu\nu} \xi^{\mu} \xi^{\nu} \mathrm{d}^3\Sigma. \tag{15}$$

If we assume that the local Rindler wedge is in thermal equilibrium, the Clausius entropy crossing the horizon must be compensated by the change of its (entanglement) entropy

$$0 = \Delta S_{Clausius} + \Delta S_{horizon} = \int_{\Sigma} \lambda \left(\frac{2\pi k_B}{\hbar c} T_{\mu\nu} - \eta R_{\mu\nu} \right) \xi^{\mu} \xi^{\nu} \mathrm{d}^3\Sigma. \tag{16}$$

Since this identity in spacetime point P holds for any null vector ξ^{μ} (one just needs to consider all the possible Rindler wedges crossing P), the bracket inside the integral needs to be equal to a term of the form $\eta \Phi g_{\mu\nu}$, where Φ is some scalar function (such a term can appear because $g_{\mu\nu}\xi^{\mu}\xi^{\nu} = 0$). The strong equivalence principle guarantees that we obtain the same condition for every spacetime point P. Hence, throughout the spacetime it holds

$$R_{\mu\nu} + \Phi g_{\mu\nu} = \frac{2\pi k_B}{\eta \hbar c} T_{\mu\nu}. \tag{17}$$

At this point, most of the treatments in the literature impose the local energy-momentum conservation, i.e., that $T_{\mu}{}^{\nu}{}_{;\nu} = 0$, and calculate Φ from the contracted Bianchi identities, arriving at the Einstein equations. However, one cannot derive condition $T_{\mu}{}^{\nu}{}_{;\nu} = 0$ from thermodynamics of spacetime. Thus, it represents an additional, nontrivial assumption. A simpler way to specify Φ, without introducing any new conditions, is by taking a trace of the equations. Then, we obtain

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = \frac{2\pi k_B}{\eta\hbar c}\left(T_{\mu\nu} - \frac{1}{4}Tg_{\mu\nu}\right). \tag{18}$$

Lastly, we define the Newton's gravitational constant in terms of η as $G = k_B c^3/4\hbar\eta$ (by requiring the correct Newtonian limit of the equations). This implies that the entropy of Rindler horizon corresponds to Bekenstein entropy

$$S_{horizon} = \eta\mathcal{A} = \frac{k_B c^3 \mathcal{A}}{4G\hbar} = S_{Bekenstein}. \tag{19}$$

Hence, the recovery of gravitational dynamics from thermodynamics requires that black hole horizons and Rindler horizons (and, indeed, local causal horizons of any construction) have the same entropy per unit area, $\eta = k_B c^3/4G\hbar$. Conversely, one might take the view that traceless Einstein equations together with the first law of thermodynamics imply universal entropy density for any causal horizon (for a detailed discussion, see [5,6]).

In total, we have obtained the following equations for gravitational dynamics

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{4}Tg_{\mu\nu}\right), \tag{20}$$

which are identical to EoMs of UG. Furthermore, the behavior of the Rindler wedge set-up under Weyl transformations seems to suggest that these are in fact EoMs of WTG in the unimodular gauge [39] (we will address this observation in a future work). In any case, the cosmological constant appears as an arbitrary integration constant (whether we assume local energy-momentum conservation or not) and its value can vary between solutions, exactly as in unimodular theories.

3.2. *Thermodynamics of local causal diamonds*

While seminal papers concerning thermodynamics of spacetime considered causal horizons realized as local Rindler wedges [7,11,8], there are some drawbacks to this approach. First, the boundary of the 2-dimensional spacelike patch $\mathcal{B}$ used to define the local Rindler wedge is chosen arbitrarily. Second, the finite strip of a Rindler horizon we considered does not form a boundary of some interior region causally disconnected from the exterior. Therefore, it is not quite clear whether it should posses entanglement entropy [13] (assuming one chooses to interpret Bekenstein entropy in terms of entanglement). Moreover, it has been argued that Clausius entropy flux associated with a local Rindler wedge cannot be correctly interpreted in terms of quantum von Neumann entropy [40]. Lastly, the entropy of Rindler horizons, in contrast with black holes, does not acquire quantum corrections logarithmic in area [38]. Consequently, thermodynamics of local Rindler horizons cannot provide any insights into low energy quantum gravity effects [24,38]. Fortunately, all the above mentioned problems disappear when one replaces local Rindler horizons with spherical local causal horizons (in principle, closed horizons of any shape would be suitable, but calculations quickly become unmanageable beyond spherical symmetry). Then the boundary of the horizon's spatial cross-section is an approximate 2-sphere determined by a single length scale. The 2-sphere encloses an interior region that can plausibly posses entropy due to its entanglement with the exterior and von Neumann and Clausius entropy of the matter inside turn out to be in agreement [23].

3.2.1. *Geodesic local causal diamonds*

In the following, we will consider a local spherical horizon constructed as small geodesic local causal diamonds (GLCD)[c] To construct a GLCD centered at point P, consider an arbitrary unit timelike vector n^μ and send out geodesics of parameter length l in all directions orthogonal to n^μ. These geodesics form a spacelike geodesic ball Σ_0. The GLCD is then defined as the spacetime region causally determined by Σ_0 (see figure 2). There exists an approximate (up to $O\left(l^3\right)$ terms) spherically symmetric conformal isometry preserving the GLCD generated by a conformal Killing vector [16]

$$\zeta = \frac{1}{2l}\left(\left(l^2 - t^2 - r^2\right)\frac{\partial}{\partial t} - 2rt\frac{\partial}{\partial r}\right). \tag{21}$$

It is easy to see that the null boundary of the GLCD is a conformal Killing horizon.

3.2.2. *Physical process derivation*

We discuss two different derivations of gravitational dynamics from thermodynamics of GLCD's, one developed by Svesko [15] and the other by Jacobson [16]. We present both approaches in a slightly modified form to emphasize their unimodular features. The first method is based on studying a physical process, namely the flux of Clausius entropy across the GLCD's horizon [15]. We can calculate it in the same way as for a local Rindler wedge [9,23]. The total entropy flux between times 0 (corresponding to the bifurcation surface) and $t>0$ equals [9]

$$\Delta S_{Clausius} = \frac{2\pi k_B}{\hbar c}\int_0^t \mathrm{d}t' t' \int_{\mathcal{S}\mathrm{d}^2\mathcal{A}(t)} T_{\mu\nu}\xi^\mu\xi^\nu, \tag{22}$$

where $\mathcal{B}(t)$ is a spatial cross-section of the GLCD's null boundary at time t (an approximate 2-sphere of radius $l-ct$) and ξ^μ denotes a future-oriented null vector

[c]Note that one can choose to work with null cones instead. However, since the results are equivalent in both cases [13,15] (at least in the semiclassical setting), we concentrate on thermodynamics of GLCD's in this work.

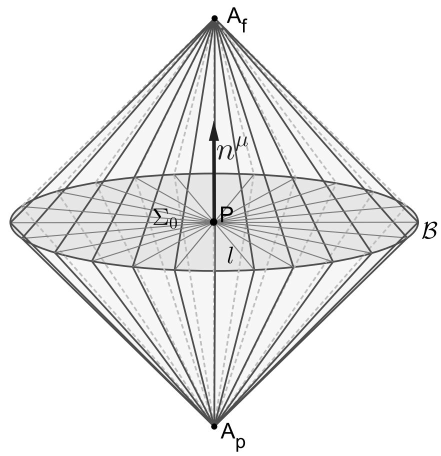

Fig. 2. A schematic picture of a GLCD with angular coordinate θ suppressed. Diamond's base Σ_0 is a spacelike geodesic ball, formed by geodesics of parameter length l sent out from point P (represented by the grey lines inside the base). Boundary $\mathcal{B}$ of Σ_0 is approximately a two-sphere. The ball is orthogonal to a timelike vector n^μ. The tilted lines represent geodesic generators of the diamond's null boundary. The generators all start from past apex A_p (corresponding to coordinate time $-l/c$) and again converge together in future apex A_f (coordinate time l/c). Thus, the diamond's base is the spatial cross section of the future domain of dependence of A_p at coordinate time $t=0$ and, likewise, the cross section of the past domain of dependence of A_f.

field orthogonal to Σ. We could now again assume that the horizon possesses entropy proportional to its area. However, the GLCD settings allow us to treat a more general entropy formula [13,15][d]

$$S_{horizon}\left(t\right)=\int_{\mathcal{B}(t)}S_{\mu\nu}\epsilon^{\mu\nu}\mathrm{d}^2\mathcal{A}, \tag{23}$$

where $\epsilon^{\mu\nu}=m^\mu n^\nu-n^\mu m^\nu$ denotes a bi-normal to $\mathcal{B}\left(t\right)$ (with $m=\partial/\partial r$ being a unit radial vector) and $S_{\mu\nu}$ is some entropy density tensor. This represents a generalization of Bekenstein entropy which we recover by choosing [41]

$$S_{\mu\nu}=\frac{k_Bc^3}{16\hbar G}\left(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}\right)\zeta^{\rho;\sigma}. \tag{24}$$

More generally, one might consider entropy density defined by the Wald's Noether charge prescription [41]. For any theory whose Lagrangian density $\mathcal{L}$ does not depend on derivatives of the Riemann tensor, it holds [13,41]

$$S_{\mu\nu}=\frac{k_Bc^3}{8\hbar G}\left(-2P_{\mu\nu\rho\sigma}\zeta^{\rho;\sigma}+4P_{\mu\nu}{}^{\rho}{}_{\sigma;\rho}\zeta^{\sigma}\right), \tag{25}$$

[d]Thermodynamics of local Rindler wedges has also been extended in this way [11,12]. However, all the proposed extensions suffer drawbacks which are not encountered for GLCD's and null cones [13]. These drawbacks can be traced back to the lack of spherical symmetry for Rindler wedges and absence of a well defined "interior region".

where $P^{\mu\nu\rho\sigma} = \left(16\pi G/c^4\right)\partial\mathcal{L}/\partial R_{\mu\nu\rho\sigma}$ depends on the metric and the Riemann tensor. For the Wald entropy difference between times $t = 0$ and $t = \varepsilon \ll l/c$ we obtain [13]

$$\Delta S_{horizon}\left(t\right) = \int_{\mathcal{B}(\varepsilon)} S_{\mu\nu}\zeta^{\nu;\mu}\mathrm{d}^2\mathcal{A} - \int_{\mathcal{B}(0)} S_{\mu\nu}\zeta^{\nu;\mu}\mathrm{d}^2\mathcal{A} \approx -\frac{k_B c^3}{4\hbar G}\int_0^{\varepsilon}\mathrm{d}t\int_{\mathcal{B}(t)}\mathrm{d}^2\mathcal{A}\xi^{\mu} \left[-2P^{\mu\nu\rho\sigma}{}_{;\nu\rho}\zeta_{\sigma} - \left(P_{\mu\nu\rho\sigma}{}^{;\rho} + P_{\mu\sigma\rho\rho}{}^{;\rho}\right)\zeta^{\nu;\sigma} + P_{\mu\nu\rho\sigma}\left(R^{\sigma\rho\nu\iota}\xi_{\iota} + f^{\nu\rho\sigma}\right)\right], \tag{26}$$

where term $R_{\rho\mu\sigma\nu}\zeta^{\sigma}$ comes from the Killing identity and $f_{\rho\mu\nu}$ accounts for the fact that this identity is not satisfied by ζ^{μ}. This happens, on one side, because ζ^{μ} is only a conformal Killing vector even in flat spacetime. On the other side, $f_{\rho\mu\nu}$ includes terms appearing due to the isometry generated by ζ^{μ} being only approximate in curved spacetime (up to $O\left(l^3\right)$ curvature dependent terms). For the same reasons, the term $-\left(P_{\mu\nu\rho\sigma}{}^{;\rho} + P_{\mu\sigma\rho\rho}{}^{;\rho}\right)\zeta^{\nu;\sigma}$ does not vanish, although it would be the case for a true Killing vector. However, it has been shown that the spherical symmetry combined with correcting the definition of ζ^{μ} by higher order terms in l leads to [15]

$$-\frac{k_B c^3}{4\hbar G}\int_0^{\varepsilon}\mathrm{d}t\int_{\mathcal{B}(t)}\mathrm{d}^2\mathcal{A}k_+^{\mu}\left[-\left(P_{\mu\nu\rho\sigma}{}^{;\rho} + P_{\mu\sigma\rho\rho}{}^{;\rho}\right)\xi^{\nu;\sigma} + P_{\mu\nu\rho\sigma}f^{\nu\rho\sigma}\right] = \Delta S_{flat}, \tag{27}$$

corresponding to the change in Wald entropy of the GLCD in a flat spacetime. This contribution can be understood as irreversible change of entropy [13,15]. Since the heat flux compensates only the reversible part of the entropy flux [8,13], the entropy balance

$$\Delta S_{Wald} - \Delta S_{flat} + \Delta S_{Clausius} = 0, \tag{28}$$

requires

$$\int_0^{\varepsilon}\mathrm{d}t\int_{\mathcal{B}(t)}\mathrm{d}^2\mathcal{A}\xi^{\mu}\left(\frac{k_B c^3}{2\hbar G}P_{\mu\nu\rho\sigma}{}^{;\nu\rho}\zeta^{\sigma} - \frac{k_B c^3}{4\hbar G}P_{\mu\nu\rho\sigma}R^{\sigma\rho\nu\iota}\zeta_{\iota} + t\frac{2\pi k_B}{\hbar c}T_{\mu\nu}\xi^{\mu}\xi^{\nu}\right) = 0. \tag{29}$$

Approximating $P^{\mu\nu\rho\sigma}$, $R_{\sigma\rho\nu\iota}$ and $T_{\mu\nu}$ by their values in the GLCD's origin P allows us to carry out the integration[e]

$$2\pi l^2\varepsilon^2\frac{k_B c^3}{4\hbar G}\left(2P_{\mu\sigma\rho\nu}{}^{;\sigma\rho} - P_{\mu\sigma\rho\iota}R^{\iota\rho\sigma}{}_{\nu} + \frac{8\pi G}{c^4}T_{\mu\nu}\right)\left(n^{\mu}n^{\nu} + \frac{1}{3}\left(g^{\mu\nu} + n^{\mu}n^{\nu}\right)\right) + O\left(\varepsilon^3\right) = 0. \tag{30}$$

Since a GLCD can be constructed with respect to any timelike direction, the previous equation must hold for every unit timelike vector n^{μ} defined in point P. The entropy balance for every such GLCD centered in P thus demands

$$P_{(\mu|\sigma\rho\iota}R^{\iota\rho\sigma}{}_{|\nu)} - 2P_{(\mu|\sigma\rho|\nu)}{}^{;\sigma\rho} - \frac{1}{4}g^{\kappa\lambda}\left(P_{\kappa\sigma\rho\iota}R^{\iota\rho\sigma}{}_{\lambda} - 2P_{\kappa\sigma\rho\lambda}{}^{;\sigma\rho}\right)g_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{4}Tg_{\mu\nu}\right). \tag{31}$$

[e]In the original paper [15], the integration is not considered. However, we choose to perform it as it directly leads to traceless equations of motion without the presence of any undetermined terms.

The equivalence principle ensures that this equation can be derived in every spacetime point. In the end, we have recovered traceless equations of motion for any local theory of gravity whose Lagrangian is independent of the derivatives of the Riemann tensor and satisfies $P^{\mu\nu\rho\sigma} = \left(16\pi G/c^4\right) \partial\mathcal{L}/\partial R_{\mu\nu\rho\sigma}$. However, although our starting point was the entropy density derived from a fully diffeomorphism invariant Lagrangian with a constant term corresponding to the cosmological constant, only the traceless part of equations of motion has been recovered. Even if we impose local energy-momentum conservation to recover the full equations of motion, the cosmological term only appears as an integration constant unrelated to the parameter in the original Lagrangian. This behavior points towards a unimodular theory of gravity rather than a fully diffeomorphism invariant one. Thus, the unimodular character of thermodynamically derived equations of gravitational dynamics persists even when one starts from a more general Wald expression for entropy density (this remains true even when quantum gravity corrections to entropy are taken into account [24]). The role of Weyl symmetry in this case will be discussed in a future work.

3.2.3. *Maximal vacuum entanglement hypothesis derivation*

The last derivation we consider is based on studying a small variation of the GLCD set-up rather than on the entropy flux. Such a variation is expected to behave in accord with the maximal vacuum entanglement hypothesis: "When the geometry and quantum fields are simultaneously varied from maximal symmetry, the entanglement entropy in a small geodesic ball is maximal at fixed volume [16]."

Consider a small GLCD in a maximally symmetric spacetime in vacuum. The maximal vacuum entanglement hypothesis states that the entanglement entropy of the GLCD is then maximal [16]. Consequently, a variation of entanglement entropy vanishes to first order. Under a simultaneous variation of spacetime geometry and matter fields, the change in entanglement entropy has two components. The change due to variation of the geometry is, as discussed in subsection 3.1 proportional to the area, $\delta S_{horizon} = \eta\delta\mathcal{A}$. The variation of the area of 2-sphere $\mathcal{B}$ at fixed volume of geodesic ball Σ_0[f] reads

$$\delta\mathcal{A} = -\frac{4\pi}{15} l^4 \left(G_{\mu\nu} n^\mu n^\nu - \Lambda + \delta\Lambda\right), \tag{32}$$

where Λ is the cosmological constant of the original maximally symmetric spacetime and $\delta\Lambda$ its variation inside the GLCD (in general dependent on the position in spacetime and the length scale l). Varying cosmological constant has already been introduced in the context of thermodynamics of GLCD's in GR [6,42], but it becomes especially natural in the light of its relation to UG[g]. In UG (and WTG) two

[f] For a physical justification of this condition, see [14,42].
[g] Of course, one could carry out the derivation without assuming varying cosmological constant and return to the issue once the unimodular nature of the dynamics is shown (we have followed

solutions of EoMs can be characterised by different Λ, as it is merely an arbitrary integration constant.

The second component appears due to variation of matter fields away from vacuum. We can obtain it by evaluating the variation of vacuum density matrix and then calculating the corresponding change in von Neumann entropy. For non-conformal matter fields with a UV fixed point this procedure yields [6,16,43,44]

$$\delta S_{matter} = \frac{2\pi k_B}{\hbar c}\frac{4\pi l^4}{15}\left(\delta\langle T_{\mu\nu}\rangle n^\mu n^\nu + \delta X\right), \tag{33}$$

where X is a spacetime scalar that in general depends on l. For conformal fields it holds $X = 0$.

In total, the entanglement equilibrium condition $\delta S_{horizon} + \delta S_{matter} = 0$ states

$$-\frac{4\pi}{15}l^4\eta\left(G_{\mu\nu}n^\mu n^\nu - \Lambda + \delta\Lambda\right) + \frac{2\pi k_B}{\hbar c}\frac{4\pi l^4}{15}\left(\delta\langle T_{\mu\nu}\rangle n^\mu n^\nu + \delta X\right) = 0. \tag{34}$$

Since the unit timelike vector n^μ is arbitrary, it must hold

$$G_{\mu\nu} + \Lambda g_{\mu\nu} - \delta\Lambda g_{\mu\nu} - \frac{2\pi k_B}{\hbar c\eta}\left(\delta\langle T_{\mu\nu}\rangle - \delta X g_{\mu\nu}\right) = 0. \tag{35}$$

Taking a trace of the equations, we find a condition on the variation of the cosmological constant

$$\delta\Lambda = \frac{1}{4}R - \Lambda + \frac{2\pi k_B}{\hbar c\eta}\left(\frac{1}{4}\delta\langle T\rangle - \delta X\right), \tag{36}$$

and, setting $G = c^3/4k_B\hbar$ as before, we obtain the traceless equations equivalent to EoMs of UG

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = \frac{8\pi G}{c^4}\left(\delta\langle T_{\mu\nu}\rangle - \frac{1}{4}\delta\langle T\rangle g_{\mu\nu}\right). \tag{37}$$

Once again, the strong equivalence principle ensures that the equations are valid throughout the spacetime. The main new feature of this approach is that on the right hand side appears an expectation value of energy-momentum tensor of quantum fields. Hence, the equations we obtained should be interpreted as semiclassical EoMs of UG.

To conclude, we remark that if all the fields present in the spacetime are conformal, then $\delta\langle T\rangle = \delta X = 0$, $R = 4\Lambda$ and, consequently, $\delta\Lambda = 0$. In this way, we see that a variation of the cosmological constant occurs if and only if one deals with a non-conformal field theory. Interestingly, one of the explanations of the origin of Λ proposed in the context of UG relies on the effects of non-conformal fields in the early universe [28]. However, we presently do not know whether this or some similar proposal can be connected with the maximal vacuum entanglement hypothesis.[h]

this approach in an earlier paper [23]). However, for the sake of clarity, we take advantage of the insights we gained by the previously discussed methods.

[h]Let us note that an interpretation of δX within the unimodular framework unrelated to the cosmological constant has been recently put forward [45].

4. Discussion

All three ways to derive gravitational EoMs from thermodynamics we reviewed in this work lead to unimodular theories of gravity (this is in fact true for any thermodynamic approach known to the authors). In each case we found traceless EoMs that only constrain divergence of the energy-momentum tensor to be equal to a gradient of some scalar. If we impose divergence-free energy-momentum tensor as an additional condition, we recover equations of the same form as EoMs of the corresponding fully diffeomorphism invariant theory. However, the cosmological constant appears as an integration constant of arbitrary value, in correspondence with its behavior in unimodular theories of gravity. Notably, even when we use thermodynamics of spacetime to study semiclassical gravity or modified gravitational dynamics, the result possesses an unimodular character. As we explored in a previous work [24], this remains true even when we consider low energy quantum gravity effects. In this case it even appears that the equivalence between unimodular and fully diffeomorphism invariant gravitational dynamics breaks down.

Moreover, based on the results available in the literature, thermodynamically derived gravitational dynamics seems to be invariant under Weyl transformations. This suggests that thermodynamic derivations lead directly to WTG rather than UG (or to appropriate generalizations of these theories). However, we leave a systematic study of this possibility for a future work.

To conclude, thermodynamics of spacetime, when considered as a way to gain insight into the nature of gravitational dynamics, clearly point to its unimodular (or rather Weyl transverse) character. Interestingly, UG, and especially WTG, offer some advantages over GR in the behaviour of the cosmological constant, while reproducing all of its classical predictions. It would be interesting to check whether thermodynamic methods have something to say about the origin of the cosmological constant. Likewise, the suggested breakdown of the equivalence between unimodular and fully diffeomorphism invariant gravitational dynamics due to quantum gravity effects deserves further attention.

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