# Sparse Fourier Transform by traversing Cooley-Tukey FFT computation graphs 

Karl Bringmann<br>Saarland Uni. \& MPI<br>Vasileios Nakos<br>Saarland Uni. \& MPI

Michael Kapralov<br>EPFL<br>Amir Yagudin*<br>MIPT

Mikhail Makarov<br>EPFL<br>Amir Zandieh<br>MPI-Informatics

December 5, 2021


#### Abstract

Computing the dominant Fourier coefficients of a function/vector is a common task in many fields, such as signal processing, learning theory and computational complexity. In the Sparse Fast Fourier Transform (Sparse FFT) problem, one is given oracle access to a $d$-dimensional vector $x$ of size $N$, and is asked to compute the best $k$-term approximation of $\widehat{x}$, the Discrete Fourier Transform of $x$, quickly and using few samples of the input vector $x$. Sparse FFT has received a significant amount of attention over the past years. However, while the sample complexity of the problem is quite well understood, all previous approaches either suffer from an exponential dependence of runtime on the dimension $d$ or can only tolerate a trivial amount of noise. This is in sharp contrast with the classical FFT algorithm of Cooley and Tukey, which is stable and completely insensitive to the dimension of the input vector: its runtime is $O(N \log N)$ in any dimension $d$.

In this work, we make progress in high-dimensional FFTs by introducing a new Sparse FT toolkit and using it to obtain new algorithms, both on the exact, as well as in the case of bounded $\ell_{2}$ noise. This toolkit includes i) a new strategy for exploring a pruned FFT computation tree that reduces the cost of filtering, ii) new structural properties of adaptive aliasing filters recently introduced by Kapralov, Velingker and Zandieh'SODA'19, and iii) a novel lazy estimation argument, suited to reducing the cost of estimation in FFT tree-traversal approaches. Our robust algorithm can be viewed as a highly optimized sparse, stable extension of the Cooley-Tukey FFT algorithm.

Finally, we explain the barriers we have faced by proving a conditional quadratic lower bound on the running time of the well-studied non-equispaced Fourier transform problem. Among other consequences, this resolves a natural and frequently asked question in computational Fourier transforms, see for example Problem 21 from IITK Workshop on Algorithms for Data Streams, Kanpur 2006. Lastly, we provide a preliminary experimental evaluation comparing the runtime of our algorithm to FFTW and SFFT 2.0.


[^0]
## Contents

1 Introduction ..... 1
2 Computational Tasks and Formal Results Statement ..... 4
3 Preliminaries and notation ..... 6
3.1 Fourier Transform basics ..... 6
3.2 Notation for manipulating FFT computation trees ..... 6
4 Techniques and Comparison with the Previous Technology ..... 7
4.1 Previous Techniques ..... 7
4.2 Our Techniques ..... 9
4.3 Explanation of the barriers faced ..... 13
5 Roadmap ..... 15
6 Machinery from Previous work: Adaptive Aliasing Filters ..... 15
6.1 One-dimensional Fourier transform ..... 16
$6.2 d$-dimensional Fourier transform ..... 17
7 Kraft-McMillan inequality and averaging claims ..... 17
8 Exactly $k$-sparse Case ..... 18
8.1 Warm Up ..... 18
8.2 The Almost Quadratic-Time Algorithm ..... 25
9 Lower Bound on Non-Equispaced Fourier Transform ..... 31
10 Robust analysis of adaptive aliasing filters ..... 35
10.1 One-dimensional case ..... 35
10.2 Extension to $d$ dimensions ..... 39
11 Robust Sparse Fourier Transform I ..... 41
11.1 Computational Primitives for the Robust Setting ..... 42
11.2 Main Algorithm ..... 43
11.3 Proving the Correctness of our Computational Primitives ..... 62
12 Robust Sparse Fourier Transform II ..... 67
13 Experiments ..... 82
13.1 FFT Backtracking vs Vanilla FFT Tree Pruning ..... 83
13.2 Sparse FFT Backtracking vs FFTW ..... 84
13.3 Comparison to SFFT 2.0 in Dimension One ..... 85
14 Acknowledgements ..... 86

## 1 Introduction

Computing the largest in magnitude Fourier coefficients of a function without computing all of its Fourier transform, or reconstructing a sparse vector/signal $x$ from partial Fourier measurements are common and well-studied tasks across science and engineering, as they appear in a variety of disciplines. Possibly the earliest work on the topic was by Gaspard de Prony in 1795, who showed that any $k$-sparse vector can be efficiently reconstructed from its first $2 k$ Discrete Fourier transform (DFT) coefficients. These ideas have been re-discovered/used both in the context of decoding BCH codes Wol67, as well as in the context of computer algebra by Ben-Or and Tiwari [BOT88]. In the context of learning theory, and in particular learning decision trees, Kushilevitz and Mansour [KM93] devised an algorithm that detects the largest Fourier coefficients of a function defined over the Boolean hypercube, building upon GL89. The work of AGS03] uses sparse Fourier transform techniques in cryptography, namely for proving hard-core predicates for one-way functions. In 2002, a sublinear-time efficient algorithm for learning the $k$ largest DFT coefficients was proposed in [GGI ${ }^{+} 02$ ]; this line of work has resulted in (near-)optimal algorithms [GMS05, HIKP12a, Kap16, Kap17] for the DFT case. In terms of its applications to signal processing and reconstruction, arguably the most prominent is the work of Candes, Donoho, Romberg and Tao [Don06, CT06, CRT06, which has far-reaching applications in fields such as medical imaging and spectroscopy LDSP08, KY11], and created the area of compressed sensing; the reader may consult the text [FR13] for a thorough view on the topic.

Formally, the Sparse Fourier Transform problem is the following. Given oracle access to a size $N d$-dimensional vector $x$, find a vector $\hat{\chi}$ such that

$$
\|\widehat{x}-\widehat{\chi}\|_{p} \leq C \cdot \min _{k \text {-sparse vectors }}\| \| \widehat{x}-\widehat{z} \|_{q},
$$

where $C$ is the approximation factor, and $\|\cdot\|_{p},\|\cdot\|_{q}$ are norms. The number of oracle accesses to $x$ shall be referred to as sample complexity. The most well studied case in the literature is the case where $C=1+\epsilon$ (or constant) and $p=q=2$, referred to as the $\ell_{2} / \ell_{2}$ guarantee. Other wellstudied cases are the so-called $\ell_{\infty} / \ell_{2}$ guarantee, where $C=\frac{1}{\sqrt{k}}, p=\infty, q=2$, as well as the $\ell_{2} / \ell_{1}$ guarantee, see [CT06, IK14, NSW19. Our focus in this paper is the $\ell_{2} / \ell_{2}$ guarantee. Frequently, the $k$ largest in magnitude coordinates of $\widehat{x}$ are referred to as the head of the signal, while all the other coordinates are referred to as the tail of the signal, or as noise. With this vocabulary, the $\ell_{2} / \ell_{2}$ guarantee asks to recover the head of $\widehat{x}$ up to $\epsilon$ times the noise level.

The research on the topic, especially over the last fifteen years, has been extensive [KM93, LMN93, BFJ ${ }^{+} 94$, Man94, Man95, GGI 02 , GMS05, CT06, IGS07, Iwe10, Aka10, CGV13, HIKP12a, HIKP12b, BCG ${ }^{+}$12, PR13, IKP14, PR14, Bou14, IK14, OPR15, PS15, JENR15, CKPS16, HR16, Kap16, CKSZ17, Kap17, CI17, MZIC17, KVZ19, AZKK19, NSW19, OHR19, JLS20. Our understanding on the sample complexity of the problem is quite good: we know that $O(k \operatorname{poly}(\log N))$ samples are sufficient for finding in time near linear in $N$ a vector $\widehat{\chi}$ satisfying any of the aforementioned guarantees [T06, HR16, NSW19]. Regarding the particularly interesting case of $d=1$, the research effort of the community has produced time efficient algorithms as well. The fastest algorithm, due to the celebrated work of Hassanieh, Indyk, Katabi, and Price HIKP12a, runs in time $O(k \log (N / k) \log N)$ and achieves the same sample complexity as well. We know also how to achieve $O(k \log N)$ sample complexity and $O(k \operatorname{poly}(\log N))$ running time Kap17. On the other extreme, when $d=\log N$, i.e. in the case of the Walsh-Hadamard transform, almost optimal running time is known to be achievable, even deterministically CI17.

Along with the running time, the sample complexity, and the error guarantee, of particular interest is also the sensitivity of the algorithm to the underlying field. When we are concerned
with Fourier transforms over $\mathbb{Z}_{n}^{d} \mathbb{1}^{1}$, this corresponds to the sensitivity to the dimension $d$. Indeed, virtually all Sparse Fourier transform algorithms have a running time which suffers from an exponential dependence on $d$ (in particular $\log ^{\Omega(d)} N$ ), and the techniques either in dimension $d=1$ or $d=\log N$ heavily rely on the structure of the corresponding group. At the same time, given that the Cooley-Tukey FFT algorithm itself is completely dimension-independent, a natural question is whether this independence transfers also to the Sparse Fourier transform setting. Concretely, is the curse of dimensionality an inherent problem, or an artifact of previous techniques? A major practical motivation is that a quest for removing the curse of dimensionality can ultimately lead to new insights for designing empirical, efficient algorithms in dimensions $d=3,4$, which are mostly relevant in applications in NMR-spectroscopy and MRI imaging. It is known in theory that fast algorithms exist in low dimensions Kap16, but they incur a multiplicative $2^{\Omega(d \log d)}$ factor in the running time, which is prohibitive even for $d=3$; a algorithm with better dependence on the $d$ and $k$ could thus be of practical importance as well.

Which sampling patterns enable sublinear-time Sparse Fourier recovery? The question on efficient dimension-independent Sparse Fourier transform can also be viewed as a question on the sampling patterns, i.e. collections of samples in time domain, which permit sublinear-time, robust recovery. The classical Prony's argument, see for example [Sau18], postulates that any $2 k$ points from $x$ in an arithmetic progression suffice to recover $\widehat{x}$ if it is $k$-sparse. However, due to the fact that the argument proceeds by solving a polynomial equation, the overall algorithm is highly unstable, and, additionally suffers from the curse of dimensionality. In $d=1$, a small random collection of $O(k \log N)$-length arithmetic progressions suffices to design rather time-efficient Sparse Fourier transform algorithms [GMS05, HIKP12b, HIKP12a, IKP14, Kap17]; in the case of general $d$, however, this approach, even if one counts only the sample complexity, meets a barrier of $2^{\Omega(d \log d)} k \log N$ IK14. On the other hand, a random, unstructured collection of samples CT06, NSW19] can lead to sample-efficient algorithms for the problem, but getting sublinear time in the absence of any (additive) structure in the sampling pattern seems out of reach, if not impossible. In this paper we investigate the power of the only known sampling patterns, i.e. those introduced in [KVZ19], that allow sublinear-time and dimension-independent Fourier sparse recovery and show how to obtain a robust algorithm along with a component of unstructured samples. For fully unstructured patterns, i.e. random points, even estimating the values of the coefficients given the location of the indices is likely to require quadratic time as we show later in this work.

Adaptive aliasing filters. A step towards dimension-independence was made in [KVZ19. The authors give a $O\left(k^{3} \cdot \operatorname{poly}(\log N)\right)$-time algorithm which recovers exactly $k$-sparse signals, and thus does not scale exponentially with the dimension. Their approach is based on pruning an FFT computation graph, using a new tool called adaptive aliasing filters. However, the aforementioned algorithm had two disadvantages: i) the time was cubic and there was no evidence whether this was optimal under some reasonable assumption, and ii) was not able to go beyond the barrier of exactly $k$-sparse signals (or, noise level poly $(N)$ times smaller than the energy of the head). Somehow relevant is an algorithm due to Mansour [Man95], which performs breadth-first search in the Cooley-Tukey FFT computation tree with random sampling, and can get poly $(k)$ running time for exactly $k$-sparse signals, but pays an additional multiplicative SNR factor for general signals [Man95]. We also mention a beautiful $O(k \cdot \operatorname{poly}(\log N)$ )-time algorithm for exactly $k$ sparse signals from [GHI ${ }^{+} 13$ ], which requires a distributional assumption on the support of the

[^1]input signal in Fourier domain and unfortunately suffers from the restriction $k=O\left(N^{\frac{1}{d}}\right)$; already in dimension $d=O(\log N / \log \log N)$, this guarantees correctness only for $k \leq \operatorname{poly}(\log N)$. We bring the readers' attention to a very recent paper JLS20 which studies multidimensional Sparse Fourier transforms in the continuous setting, but the algorithm presented still has an exponential dependence on the dimension $d$, in both running time and sample complexity.

Our techniques: new methods for traversing pruned Cooley-Tukey FFT computation graphs. We introduce a variety of new techniques for the high dimensional Sparse Fourier Transform problem that enable us to go beyond some of the barriers faced in previous works. In particular, we augment the Sparse FT toolkit with the following set of techniques.

1. FFT backtracking: a novel technique for traversing Cooley-Tukey FFT computation graphs of $k$-sparse vectors. This allows us to spend fewer resources on "cheap" subproblems, and return to previously processed ones in order to correct potential errors.
2. New structural properties of adaptive aliasing filters. Roughly speaking, these new properties indicate that the collection of aforementioned filters acts in a specific sense as a near-isometry on an arbitrary vector.
3. A novel lazy estimation argument, which allows us to postpone estimation of identificated frequencies in our explorative algorithm, estimating them only when it is cheap on average to do so.
4. A connection of the closely related non-equispaced Fourier transform task with the Orthogonal Vectors problems, a central problem in fine-grained complexity.

Our results. The new algorithmic techniques that we introduce lead to fast, sample efficient and robust algorithms for the high dimensional Sparse FFT problem:

- An almost quadratic-time (in the sparsity $k$ ) algorithm for exactly $k$-sparse signals. This shaves off almost a factor of $k$ from the previous best sublinear-time, dimension-independent algorithm of [KVZ19]. As we argue below (see our lower bound results), overcoming this quadratic time barrier will likely require a major paradigm shift in Sparse FFT technology.
- A quadratic sample complexity, sublinear-time, dimension-independent Sparse Fourier transforms that recovers the head of the signal under bounded $\ell_{2}$ noise, i.e. when every frequency in the head is larger than the energy of the tail. Even under this seemingly restricted noise model, designing an efficient algorithm turns out to be non-trivial, requiring a constellation of new techniques. Previous algorithms were either i) robust and dimension-independent but not sublinear-time CT06, IK14, NSW19, ii) sublinear-time and robust but not dimensionindependent GMS05, HIKP12a, Kap16, or iii) sublinear-time and dimension-independent but not robust to any form of noise [KVZ19. We also discuss all the barriers we have faced, including the barrier on handling noise of larger magnitude, in Section 4.3 .

It seems likely that our quadratic time Sparse FFT algorithm for exact signals will be hard to improve upon barring a major paradigm shift in Sparse FFT technology. Indeed, most such algorithms implement an iterative refinement scheme that at every point subtracts the signal recovered so far from the input. Fast schemes for such subtraction (semi-equispaced Fourier transform algorithms) are only known for a very restricted class of sampling patterns. In this work we show that structural assumptions on the sampling patterns are indeed needed, assuming the Orthogonal Vectors hypothesis. Specifically, on the lower bound side we give

- A new hardness result on computational Fourier transforms, postulating that the well-studied non-equispaced FT problem cannot be solved in strongly subquadratic time even in dimension 1, unless the Orthogonal Vectors hypothesis fails. This lower bound i) separates it from the analogous (and well-studied as well) problem of semi-equispaced Fourier transform, which features a near-linear time algorithm in constant dimension, and ii) as we stress in Section 2 , provides evidence that the quadratic time barrier for the $k$-sparse case that our algorithm meets is most likely impenetrable by any explorative approach which successively peels off elements.
Additionally, our lower bound applies to sparse multipoint evaluation: it shows that a $k$-sparse polynomial of maximum degree $n$ cannot be evaluated on a set of $k$ complex numbers faster in strongly subquadratic time in $k$ whenever $k$ is smaller than a fractional power of $n$, unless SETH fails.


## 2 Computational Tasks and Formal Results Statement

This section contains the computational tasks studied in this paper, our results, and a preparations section for the lower bound, namely Theorem 22. We will be concerned with $N$-length $d$-dimensional vectors $x:[n]^{d} \rightarrow \mathbb{C}$, where $N=n^{d}$ and $n$ is a power of 2 . Thus, $N, n, d$ will remain unaltered throughout the paper. We will use the notation $[n]$ to denote the set of integer numbers $\{0,1, \ldots, n-$ $1\}$. We will use a non-standard notation $\widetilde{O}(f)=O(f$ poly $(\log N))$, where $f$ is some parameter and $N$ will be the size of our underlying vector $x$, which will be the largest parameter of interest. For a vector $x$, we denote $\|\widehat{x}\|_{0}=\left|\left\{\boldsymbol{f} \in[n]^{d}: \widehat{x}_{\boldsymbol{f}} \neq 0\right\}\right|$, and $\widehat{x}_{T}$, for a set $T \subseteq[n]^{d}$, to be the vector that results from zeroing out every coordinate of $x$ outside of $T$. We let $\widehat{x}_{-k}$ be the vector that occurs after zeroing out the top $k$ coordinates in magnitude, breaking ties arbitrarily. All logarithms are base 2. For the algorithm we present, we shall assume exact arithmetic operations over $\mathbb{C}$ in unit time throughout the paper, although the analysis goes through with $\frac{1}{\text { poly }(N)}$ precision as well.

Theorem 1 (Near-Quadratic Time Fourier Transform for Sparse Signals). Given oracle access to $x:[n]^{d} \rightarrow \mathbb{C}$ with $\|\widehat{x}\|_{0} \leq k$, we can find $\widehat{x}$ in deterministic time

$$
\widetilde{O}\left(k^{2} \cdot 2^{\Theta(\sqrt{\log k \cdot \log \log N})}\right) .
$$

Conjecture 1. (Orthogonal Vectors Hypothesis(OVH) Wil05, AWW14]) For every $\epsilon>0$, there exists a constant $c \geq 1$ such that $\mathrm{OV}_{k, d}$ requires $\Omega\left(k^{2-\epsilon}\right)$ time whenever $d \geq c \log k$.

It is known that a collapse of the Orthogonal Vectors Hypothesis would have groundbreaking implications in algorithm design, see GIKW19 and ABDN18.

Theorem 2 (Lower Bound for Non-Equispaced Fourier Transform). Assume that for all $k<$ $n$ and $\epsilon$ there exists an algorithm that solves the Non-Equispaced Fourier Transform in time $O\left(k^{2-\delta} \operatorname{poly}(\log (n / \epsilon))\right)$ for some constant $\delta>0$. Then the Orthogonal Vectors hypothesis fails.

A lower bound for sparse polynomial multipoint evaluation also follows immediately.
Theorem 3 (Lower bound for Sparse Polynomial Multipoint Evaluation over $\mathbb{C}$ ). Assume that for all $k<n$ and $\epsilon$ there exists an algorithm for sparse polynomial multipoint evaluation which runs in time $k^{2-\delta} \operatorname{poly}(\log (n / \epsilon))$. Then the Orthogonal Vector Hypothesis fails.
$\star$ Sparse Fourier Transform in the exact case
Input: $\quad$ Integers $n, d, k$ and $N=n^{d}$, and oracle access to a vector $x \in \mathbb{C}^{n^{d}}$ satisfying $\|\widehat{x}\|_{0} \leq k$.
Question: Compute $\widehat{x}$.
$\star \ell_{2} / \ell_{2}$ Sparse Fourier Transform
Input: Integers $n, d, k$ and $N=n^{d}$, parameter $\epsilon<1$, and oracle access to a vector $x \in \mathbb{C}^{n^{d}}$.
Question: Compute a vector $\widehat{\chi} \in \mathbb{C}^{n^{d}}$ such that $\|\widehat{x}-\widehat{\chi}\|_{2} \leq(1+\epsilon)\|\widehat{x}-k\|_{2}$.
$\star$ Non-Equispaced Fourier Transform
Input: $\quad$ Integers $n, d$, parameter $\epsilon<1$, two sets $F, T \subseteq[n]^{d}$ with $|F|=|T|=k$, and a vector $x \in \mathbb{C}^{n^{d}}$ supported on $T$.
Question: Compute additive $\pm \epsilon\|\widehat{x}\|_{2}$ approximations to each of $\widehat{x}_{\boldsymbol{f}}$, for $\boldsymbol{f} \in F$.
$\star$ Sparse Polynomial Multipoint Evaluation
Input: Integers $n, k$, parameter $\epsilon<1$, a polynomial $p$ of degree $n$ and sparsity $k$, each coefficient of which is of magnitude 1 , as well as points $a_{1}, a_{2}, \ldots, a_{k} \in \mathbb{C}^{n}$ of magnitude 1 .
Question: Compute additive $\pm \epsilon$ approximations to each of $p\left(a_{i}\right)$, for all $i=1,2, \ldots, k$.
$\star$ Orthogonal vectors, $\mathrm{OV}_{k, d}$
Input: $\quad A, B \subseteq\{0,1\}^{d}$, with $|A|=|B|=k$
Question: Determine whether there exists $a \in A, b \in B$ such that $\langle a, b\rangle=0$.
Figure 1: Computational tasks considered in this paper.

Significance of our lower bound for computational Fourier Transforms. Non-equispaced Fourier transform falls into a class of Fourier transforms referred to as non-uniform. These transforms are an extensively studied topic in signal processing and numerical analysis GR87, FS03, GL04, with numerous applications in imaging, signal interpolation and solutions of differential equations; the reader may consult the texts [BM96, PST01, BM12]. An implementation of nonuniform Fourier transforms is also available in Matlab [Mat]; alternative implementations are also available [KKP09]. Usually, researchers recognize three different types of non-uniform Fourier transforms, see lecture notes Can for a categorization. While the first two types admit near-linear time solutions in $k$ [DR93], the type-III transform, which is also the transform relevant to this paper, more likely does not have a strongly subquadratic algorithm. This shows a separation between the semi-equispaced case (type-I and type-II) and the non-equispaced case.

In what follows, we quantify the notion of "bounded $\ell_{2}$ noise" used in our robust SparseFFT results.

High SNR model. A vector $x:[n]^{d} \rightarrow \mathbb{C}$ satisfies the $k$-high SNR assumption, if there exist vectors $w, \eta:[n]^{d} \rightarrow \mathbb{C}$ such that i) $\widehat{x}=\widehat{w}+\widehat{\eta}$, ii) $\operatorname{supp}(\widehat{w}) \cap \operatorname{supp}(\widehat{\eta})=\varnothing$, iii) $|\operatorname{supp}(\widehat{w})| \leq k$ and iv) $\left|\widehat{w}_{f}\right| \geq 3 \cdot\|\widehat{\eta}\|_{2}^{2}$, for every $f \in \operatorname{supp}(\widehat{w})$.

Theorem 4 (Robust Sparse Fourier Transform with Near-quadratic Sample Complexity). Given oracle access to $x:[n]^{d} \rightarrow \mathbb{C}$ in the $k$-high SNR model and parameter $\epsilon>0$, we can solve the $\ell_{2} / \ell_{2}$ Sparse Fourier Transform problem with high probability in $N$ using

$$
m=\widetilde{O}\left(\frac{k^{2}}{\epsilon}+k^{2} \cdot 2^{\Theta(\sqrt{\log k \cdot \log \log N})}\right)
$$

[^2]samples from $x$ and $\widetilde{O}\left(\frac{k^{3}}{\epsilon}\right)$ running time.
We re-iterate that even though the noise model we consider might seem restrictive, it turns out to be quite challenging requiring whole new constellation of ideas. Additionally, in subsection 4.3 we explain how we are led to consider this particular notion, and why handling lower SNR seems like a hard barrier for algorithms which explore a pruned Cooley-Tukey FFT computation tree (which is also the only known class of algorithms that enables sublinear and dimension-independent recovery). Finally, we explain the discrepancy between running time and sample complexity via Theorem 2 .

Experimental Evaluation. Lastly, we present the results of our preliminary experimental evaluation in Section 13 ,

## 3 Preliminaries and notation

### 3.1 Fourier Transform basics

We will often identify $[n]^{d} \rightarrow \mathbb{C}$ with $\mathbb{C}^{n^{d}}$ for convenience (and use the two interchangeably depending on the context).
Definition 1 (Fourier transform). For any positive integers $d$ and $n$, the Fourier transform of a signal $x \in \mathbb{C}^{n^{d}}$ is denoted by $\widehat{x}$, where $\widehat{x}_{\boldsymbol{f}}=\sum_{\boldsymbol{t} \in[n] d} x_{\boldsymbol{t}} e^{-2 \pi i \frac{f^{\top} \boldsymbol{t}}{n}}$ for any $\boldsymbol{f} \in[n]^{d}$. Here $\boldsymbol{f}^{\top} \boldsymbol{t}=$ $\sum_{q=0}^{d-1} f(q) t(q)$.

Recall that by Parseval's theorem we have $\|\widehat{x}\|_{2}^{2}=n^{d} \cdot\|x\|_{2}^{2}$. Furthermore, recall convolutionmultiplication duality $\widehat{(x \star y)}=\widehat{x} \cdot \widehat{y}$, where $x \star y \in C^{n^{d}}$ is the convolution of $x$ and $y$ and defined by the formula $\left.(x \star y)_{\boldsymbol{t}}=\sum_{\boldsymbol{\tau} \in[n]^{d}} x_{\boldsymbol{\tau}} \cdot y_{(\boldsymbol{t}-\boldsymbol{\tau}} \bmod n\right)$ for all $\boldsymbol{t} \in[n]^{d}$, where the modulus is taken coordinate-wise. We will also need the following well-known theorem on Fourier subsampled matrices.

Theorem 5. (Restricted Isometry Property of subsampled Fourier matrices, [HR17, Theorem 3.7]) Let $q=\Theta\left(s \log ^{3} N\right)$. Then with high probability in $N$, the time domain points $\left\{x_{\boldsymbol{t}}\right\}_{\boldsymbol{t} \in Q}$ for a random multiset $Q \subseteq[n]^{d}$ with $q$ uniform samples are sufficient to $(1 \pm \epsilon)$-approximate the energy of any $s$-sparse vector $\widehat{x}$, where $\epsilon$ is some sufficiently small absolute constant. Formally, simultaneously for all s-sparse vectors: $\frac{N^{2}}{q} \sum_{t \in Q}\left|x_{t}\right|^{2} \in\left[(1-\epsilon)\|\widehat{x}\|_{2}^{2},(1+\epsilon)\|\widehat{x}\|_{2}^{2}\right]$.

### 3.2 Notation for manipulating FFT computation trees

Recall that given a signal $x:[n]^{d} \rightarrow \mathbb{C}$, the execution of the FFT algorithm produces a binary tree, henceforth referred to as $T_{N}^{\text {full }}$. The root of $T_{N}^{\text {full }}$ corresponds to the universe $[n]^{d}$, while the children of the root correspond to $[n / 2] \times[n]^{d-1}$; note that FFT recurses by peeling off the least significant bit. Every node $v$ has a label $\boldsymbol{f}_{v} \in \mathbb{Z}_{n}^{d}$ associated to it, with the following rules.

1. The root has label $\boldsymbol{f}_{\text {root }}=(\underbrace{0,0, \ldots, 0}_{d \text { entries }})$, and corresponds to the universe $[n]^{d}$.
2. The children of a node $v$ corresponding to the universe $\left[n / 2^{\ell}\right] \times[n]^{d^{\prime}}$, with $0 \leq d^{\prime} \leq d-$ $1,0 \leq \ell \leq \log n-1$, let them be $v_{\text {left }}, v_{\text {right }}$ have the following properties. Both correspond to universe $\left[n / 2^{\ell+1}\right] \times[n]^{d^{\prime}}$, and $v_{\text {right }}$ has label $\boldsymbol{f}_{v_{\text {right }}}=\boldsymbol{f}_{v}$, while $v_{\text {left }}$ has label $\boldsymbol{f}_{v_{\text {left }}}=$ $\boldsymbol{f}_{v}+(\underbrace{0,0, \ldots, 0}_{d^{\prime}}, 2^{\ell}, \underbrace{0,0, \ldots, 0}_{d-d^{\prime}-1})$.
3. The children of a node $v$ corresponding to universe $[1] \times[n]^{d^{\prime}}$ with $d^{\prime}>0$, are $v_{\text {left }}, v_{\text {right }}$, corresponding to universe $[n / 2] \times[n]^{d^{\prime}-1}$ and have labels $\boldsymbol{f}_{v_{\text {right }}}=\boldsymbol{f}_{v}$ and $\boldsymbol{f}_{v_{\text {left }}}=\boldsymbol{f}_{v}+$ $(\underbrace{0,0, \ldots, 0}_{d^{\prime}-1}, 1, \underbrace{0,0, \ldots, 0}_{d-d^{\prime}})$ respectively.
4. A node $v$ corresponding to universe [1] is called a leaf in $T_{N}^{\text {full }}$.

The above rules create a binary tree of depth $\log N$, which corresponds to the FFT computation tree. We demonstrate $T_{N}^{\text {full }}$ that corresponds to the 2-dimensional FFT computation on universe [4] $\times[4]$ in Figure 2 . Subtrees $T$ of $T_{N}^{\text {full }}$ can be defined as usual. For every node $v \in T$, the level of $v$, denoted by $l_{T}(v)$, is the distance from the root to $v$. We denote by $\operatorname{leaves}(T)$ the set of all leaves of tree $T$, and for every $v \in \operatorname{LEAVES}(T)$, we define its weight $w_{T}(v)$ with respect to $T$ to be the number of ancestors of $v$ in tree $T$ with two children. The levels (distances from the root) on which the aforementioned ancestors lie will be called $\operatorname{Anc}(v, T)$. Furthermore, the sub-path of $v$ with respect to $T$ will be the children of the aforementioned ancestors which are not ancestors of $v$. Additionally, for a node $v \in T$ we denote the subtree of $T$ rooted at $v$ by $T_{v}$.

The following definition will be particularly important for our algorithms.
Definition 2 (Frequency cone of a leaf of $T$ ). For every subtree $T$ of $T_{N}^{\text {full }}$ and every node $v \in T$, we define the frequency cone of $v$ with respect to $T$ as,

$$
\text { FreqCone }_{T}(v):=\left\{\boldsymbol{f}_{u}: \text { for every leaf } u \text { in subtree of } T_{N}^{\text {full }} \text { rooted at } v\right\} .
$$

Furthermore, we define $\operatorname{supp}(T):=\bigcup_{u \in \operatorname{leaves}(T)} \operatorname{Freq}^{\operatorname{Cone}}{ }_{T}(u)$.
The splitting tree of a set $S \subseteq[n]^{d}$ is the subtree of $T_{N}^{\text {full }}$ that contains all nodes $v \in T_{N}^{\text {full }}$ such that $S \cap$ FreqCone $_{T_{N}^{\text {full }}}(v) \neq \varnothing$.

## 4 Techniques and Comparison with the Previous Technology

This section is devoted to highlighting the differences between previous work and our technical contributions.

### 4.1 Previous Techniques

Most previous sublinear-time Sparse Fourier transform algorithms GMS05, HIKP12a, Kap16, Kap17 rely on emulating the hashing of signal $\widehat{x}$ by picking a structured set of samples (in low dimensions, the samples correspond to arithmetic progressions) and processing them with the help of bandpass filters, i.e. functions which approximate the $\ell_{\infty}$ box in frequency domain and are simultaneously sparse in time domain. However, while those filters are particularly efficient in low dimensions, their performance deteriorates when the number of dimensions increases: indeed, a $d$-dimensional $\ell_{\infty}$ box has $2^{d}$ faces, and hence this approach suffers inevitably from the curse of dimensionality. On the other hand, an unstructured collection of $O(k \cdot \operatorname{poly}(\log N))$ samples [CT06, NSW19] suffice, showing that the sample complexity is dimension-independent; the cost that one needs to pay, however, is $\Omega(N)$ running time.

To (partially) remedy the aforementioned state of affairs, the approach of KVZ19] departs from both the aforementioned approaches, and performs pruning in the Cooley-Tukey FFT computation graph, in a way that suffices for recovery of exactly $k$-sparse vectors. The main technical innovation of that work is the introduction of adaptive aliasing filters, a new class of filters that allow isolating
a given frequency from a given set of $k$ other frequencies using $O(k)$ samples in time domain and in $O(k \log N)$ time. Those filters are revised in Section 6 .

Definition $3\left((v, T)\right.$-isolating filter, see Definition 6). Consider a subtree $T$ of $T_{N}^{\text {full }}$, and a leaf $v$ of $T$. A filter $G:[n]^{d} \rightarrow \mathbb{C}$ is called $(v, T)$-isolating if the following conditions hold:

- For all $\boldsymbol{f} \in \operatorname{FreqCone}_{T}(v)$, we have $\widehat{G}(\boldsymbol{f})=1$.
- For every $\boldsymbol{f}^{\prime} \in \bigcup_{\substack{u \in \operatorname{LEaves}(T) \\ u \neq v}} \operatorname{FreqCone}_{T}(u)$, we have $\widehat{G}_{v}\left(\boldsymbol{f}^{\prime}\right)=0$.

As shown in KVZ19], for a given tree $T$ and a node $v$ one can construct isolating filters $G$ such that $\|G\|_{0}=O\left(2^{w_{T}(v)}\right)$, and $\widehat{G}(\boldsymbol{f})$ is computable in $\widetilde{O}(1)$ time (see also Lemma 5 ). The sparsity of $G$ in time domain, i.e. $\|G\|_{0}$, corresponds to the number of accesses to $x$ needed in order to get our hands on $(\widehat{G} \cdot \widehat{x})_{\boldsymbol{f}}$ for a fixed $\boldsymbol{f}$.

Given the above, the algorithm maintains at all times a tree $T \subseteq T_{N}^{\text {full }}$ and a vector $\hat{\chi}$ such that $\operatorname{supp}(\widehat{x}-\widehat{\chi}) \subseteq \cup_{u \in T} \operatorname{Freq} \operatorname{Cone}(u)$, and $\operatorname{supp}(\widehat{\chi}) \subseteq \operatorname{supp}(\widehat{x})$. The aim is to gradually discover $\operatorname{supp}(\widehat{x})$, by peeling off one element $\boldsymbol{f} \in \operatorname{supp}(\widehat{x})$ at a time whenever it reaches at a leaf of $T_{N}^{\text {full }}$. Upon termination, it will be the case that $\widehat{\chi}=\widehat{x}$. While $T$ is not empty, the algorithm picks the lowest weight node $v^{*} \in T$, and construct a $\left(v^{*}, T\right)$-isolating filter. Subsequently, it needs to check whether $(\widehat{x}-\widehat{\chi})_{\text {FreqCone }\left(v^{*}\right)}$, i.e. the residual vector projected on FreqCone $\left(v^{*}\right)$, is the all zeros vector or not. This can be phrased as performing a zero test on $(\widehat{x}-\widehat{\chi})_{\text {FreqCone }\left(v^{*}\right)}$. This check can be performed efficiently using a (deterministic) collection of $O\left(k \log ^{3} N\right)$ samples which satisfy the Restricted Isometry Property (RIP) of order $k$; its pseudocode, named ZeroTest, is depicted in Algorithm 1. If $(\widehat{x}-\widehat{\chi})_{\text {FreqCone }\left(v^{*}\right)}$ is indeed the all zeros vector, then $v^{*}$ is removed from $T$. Otherwise, exploration proceeds by adding the two children of $v^{*}$ to $T$. The sample complexity of ZeroTest is then

$$
O\left(2^{w_{T}\left(v^{*}\right)} \cdot k \cdot \operatorname{poly}(\log N)\right),
$$

namely, one needs to multiply the time domain support size of the isolating filter $G$ with the number of samples needed to satisfy RIP of order $k$. If $v^{*}$ is a leaf in $T^{\text {full }}$, i.e. a node at $\operatorname{depth} \log N$, instead of performing a call to ZeroTest, the algorithm instead estimates $\widehat{x}_{\boldsymbol{f}_{v^{*}}}$ immediately using the $\left(v^{*}, T\right)$-isolating filter, see Algorithm 2 for a pseudocode. This requires only $O\left(2^{w_{T}\left(v^{*}\right)}\right)$ samples.

So far, we have a primitive for estimation, and a primitive for testing whether a frequency cone contains a part of the support of $\widehat{x}$. But how does exploration proceed? Using Kraft's equality on $T$ (Theorem 6), and in particular Kraft averaging (Lemma 6) it is straightforward to see that always $2^{w_{T}\left(v^{*}\right)}=\widehat{O}(k)$, and hence the sample complexity of ZEROTEST (which is the most expensive out of the two primitives) is bounded by $O\left(k^{2} \cdot \operatorname{poly}(\log N)\right)$. Since $\widehat{x}$ is $k$-sparse, this process will eventually terminate after exploring $O(k \log N)$ nodes, resulting in $O\left(k^{3}\right.$ poly $\left.(\log N)\right)$ sample complexity. To obtain a similar bound for the running time requires some more care, since one needs to subtract $\widehat{\chi}$ from the measurements; owing to the fact that adaptive aliasing filters are "sharp" filters and efficiently computable, with some additional work one can still obtain $O\left(k^{3}\right.$ poly $\left.(\log N)\right)$ running time, see paragraph "Accessing the residual signal" and Subsection 2.1 from [KVZ19.

Unfortunately, as we have already pointed out, the algorithm in [KVZ19] works only for exactly $k$-sparse signals, and also demands cubic time and sample complexity. Our new toolkit shows that all three can be remedied (though not completely simultaneously).

We also mention that a modified version of Man95 can be employed to recover exactly $k$ sparse signals in $\widetilde{O}\left(k^{3}\right)$ time. The algorithm presented in Man95 performs breadth-first search in the Cooley-Tukey FFT computation graph, rather than exploring by picking the lowest weight leaf.

Opposed to [KVZ19], the algorithm in [Man95] uses Dirac comb filters to learn all the non-empty frequency cones in the same level at once. However, the techniques in that paper cannot go beyond cubic time for $k$-sparse signals, and as can be seen in [Man95, Section 6], extending the result to robust signals pays a multiplicative signal-to-noise ratio factor on top of $k^{3}$.

### 4.2 Our Techniques

Our first technique shows how to traverse the Cooley-Tukey FFT computation graph in a way that achieves almost quadratic time complexity.
FFT backtracking. The first crucial observation is that the vanilla FFT traversal algorithm in orded to decide whether a subtree contains a non-zero frequency performs a zero test with RIP of order $k$, and this might be unnecessary. Indeed, if we are at a node $v$ for which $\left\|\widehat{x}_{\text {FreqCone }(v)}\right\|_{0}=$ $O(1)$, i.e. there are at most $O(1)$ elements in FreqCone $(v)$, we only need to perform RIP of order $O(1)$ (since at all times we maintain the invariant that we can isolate $u$ from all the other frequency cones on which the signal is non-zero). Thus, maybe there is a way to approximately learn $\left\|\widehat{x}_{\text {FreqCone }(v)}\right\|_{0}$, for nodes $v$ explored during the execution of the algorithm, and perform a low-budget zero test accordingly, for example whenever $\left\|\widehat{x}_{\text {FreqCone }(v)}\right\|_{0}$ is smaller than a threshold?

Indeed, we demonstrate that this intuition is correct, and give a preliminary, warm up algorithm with $\widetilde{O}\left(k^{2.5}\right)$ runtime, see Subsection 8.1. The idea is the following. The algorithm maintains at all times a subtree $T$, as well as a vector $\hat{\chi}$, such that $\operatorname{supp}(\widehat{x}-\widehat{\chi}) \subseteq \cup_{u \in T} \operatorname{FreqCone}(u)$, and $\operatorname{supp}(\widehat{\chi}) \subseteq \operatorname{supp}(\widehat{x})$. The crucial difference is that, in order to explore $T$, after the minimum-weight node $v \in \operatorname{LEAVES}(T)$ is picked, the algorithm now runs a vanilla FFT traversal (as described in the previous subsection) in the subtree $T_{v}$, under the assumption that $\left\|\widehat{x}_{\text {FreqCone }(v)}\right\|_{0} \leq b$. The latter assumption can be right or wrong. Once the execution on $T_{v}$ is finished and the algorithm backtracks to $v$, it performs a zero test with budget $k$ to test whether $(\widehat{x}-\widehat{\chi})_{\text {FreqCone }(v)}=\varnothing$, i.e. we correctly recovered everything on the subtree $T_{v}$. If this $k$-budget zero test returns True, we remove $v$ from $T$, otherwise its two children are added to $T$ (since such a turn of events would mean that we have underestimated $\left\|\widehat{x}_{\text {FreqCone }(v)}\right\|_{0}$, i.e. our assumption on its sparsity being smaller than $b$ was incorrect). The benefit of this approach is that if our assumption was correct, all calls to ZeroTest in exploring $T_{v}$ will be correct and cheap, since we make use of RIP of order $b$, instead of order $k$, which results in shaving off a multiplicative factor of $k / b$. Putting everything carefully in place, this type of argumentation leads to running time $\widetilde{O}\left(k^{2.5}\right)$, as already mentioned.

Performing multi-layer backtracking. The aforementioned approach essentially uses one layer of backtracking, or, if one prefers, just one threshold. We optimize this approach, so that the algorithm backtracks more aggresively, by considering multiple values $b_{1}, b_{2} \ldots$, corresponding to the possible assumptions on the sparsity of $\widehat{x}_{\text {FreqCone }(v)}$, for some node $v$ picked during the execution of the algorithm. For a parameter $\alpha<1$ we will use thresholds $b_{0}:=k, b_{1}:=\alpha k, b_{2}:=\alpha^{2} k, \ldots, b \frac{\log k}{\log (1 / \alpha)}=$ $O(1)$. Our new algorithm will be recursive, and at all times a call to the algorithm corresponds to $\operatorname{exploring}$ a subtree $T_{v}$ with some budget $b:=b_{j}$, i.e. under the assumption $\left\|\widehat{x}_{\text {FreqCone }(v)}\right\|_{0} \leq b$. The algorithm maintains a subtree $T_{v}$, initialized at $\{v\}$. At all times, it picks the minimum weight node $z \in T_{v}$ and considers the two children of $z$, let them be $z_{\text {left }}, z_{\text {right }}$. Then, it runs itself recursively on $T_{z_{\text {left }}}, T_{z_{\text {right }}}$ with budget $b_{j+1}=\alpha b$. When the recursive calls return, yielding candidate vectors $\widehat{\chi}_{\text {left }}, \widehat{\chi}_{\text {right }}$, it performs a zero test on each of $z_{\text {left }}, z_{\text {right }}$ with RIP of order $b$, in order to check whether $\widehat{x}_{\text {FreqCone }\left(z_{\text {left }}\right)}-\widehat{\chi}_{\text {left }}$ is the all zeros vector (similarly for the right child). If the zero test on $z_{\text {left }}$ is False, we add $z_{\text {left }}$ to $T_{v}$; similarly for $z_{\text {right }}$. If both zero tests are True, then we remove $z$. This continues either until $T_{v}=\varnothing$ or until the number of nodes that have ever been inserted in $T_{v}$
becomes too large (in particular if there is $\Omega(b / \alpha)$ leaves). In the first case, the algorithm returns the found vector, otherwise it returns the all zeros vector, since insertion of too many nodes into $T_{v}$ means that we have underestimated the sparsity of $\widehat{x}_{\text {FreqCone }(v)}$, as we argue in Subsection 8.2 . The check on the number of nodes that have been ever inserted in $T_{v}$ is crucial for detecting early whether we have understimated $\left\|\widehat{x}_{\text {FreqCone }(v)}\right\|_{0}$, and thus crucial for keeping the running time low.

Upon performing a call with arguments a node $v$ and a budget $b$, it could be the case that $\left\|\widehat{x}_{\text {FreqCone }(v)}\right\|_{0} \leq b$ does not hold; however, this misassumption is not detected by that call, and a vector which is not equal to $\widehat{x}_{\text {FreqCone }(v)}$ is returned to the above recursion level. Nevertheless, although undetectable at the time, this discrepancy will be detected in some recursion level above, where we make use of higher budget; definitely at the very first level where we perform RIP of order $k$. Proving correctness of the above process can be done by using induction on $\left\|\widehat{x}_{\text {FreqCone }(v)}\right\|_{0}$ and the level of $v$ inside $T_{N}^{\text {full }}$, along with the fact that the very first call is invoked on root of $T_{N}^{\text {full }}$ with budget $k$, and always $\|\widehat{x}\|_{0} \leq k$. This means that even if the algorithm makes a lot of mistakes in a lower recursion level, this will be detected when backtracking at the topmost recursion level (on which zero tests with RIP of order $k$ are performed), and the budgets of descedants of $v$ will be increased; this constitutes "progress". Roughly speaking, the algorithm tries to gradually learn up to a multiplicative $\alpha$ factor all $\left\|\widehat{x}_{\text {FreqCone }(v)}\right\|_{0}$ by backtracking. Analyzing the dynamics of this process and optimizing over $\alpha$, we obtain $\widetilde{O}\left(k^{2} \cdot 2^{O(\sqrt{\log k \cdot \log \log N})}\right)$ running time. This proves our first result, namely Theorem 1. The algorithm and its analysis appear in Subsection 8.2.

Let us now proceed with the techniques needed for the robust algorithm. First of all, in the robust case we should substitute ZeroTest with an analogous HeavyTest routine. The role of this routine is to determine whether $\left\|(\widehat{x}-\widehat{\chi})_{\text {FreqCone(v) }}\right\|_{2} \geq\|\widehat{\eta}\|_{2}$, where $v$ is any node that appears during the execution of the algorithm. If the latter inequality holds, this means that there are elements of the head of $\widehat{x}$ inside FreqCone $(v)$ that are yet to be recovered. Pseudocode for this routine is presented in Algorithm 6, and the guarantees of this routine are spelled out in Lemma 18 , The algorithm is very similar to ZeroTest, with the difference that we now need to take a collection of random samples, since a deterministic collection of samples sastisfying RIP does not suffice to control the non-sparse component, i.e. the contribution of the tail under filtering. Furthermore, what is demanded is a control on how a $(v, T)$-isolating filter $\widehat{G}$ acts on $\widehat{x}_{\cup_{u \in T \backslash\{v\}}}$ FreqCone(u), i.e. on parts of the signal living inside frequency cones which $u$ is not isolated from. In words, one would like to appropriately control the energy of $\left(\widehat{G} \cdot \widehat{x}_{\cup_{u \notin T}} \operatorname{FreqCone}(u)\right)$, where $\cdot$ corresponds to element-wise vector multiplication.

Collectively, adaptive aliasing filters act as near-isometries. Adaptive aliasing filters are particularly effective for non-obliviously isolating elements of the head with respect to each other. However, in standard sparse recovery tasks, one desires control of the tail energy that participates in the measurement. This is a relatively easy (or at least well-understood) task in Sparse Fourier schemes which operate via $\ell_{\infty}$-box filters HIKP12a, HIKP12b, IKP14, IK14, Kap17, but a nontrivial task using adaptive aliasing filters. The reason is that the tail via the latter filtering is hashed in a non-uniform way. The hashing depends on the arithmetic structure of the elements used to construct the filters, as well as their arithmetic relationship with the elements in the tail. This non-uniformity is essentially the main driving reason for the "exactly $k$-sparse" assumption in [KVZ19]. Our starting point is the observation that for every tree $T \subseteq T_{N}^{\text {full }}$, the $(v, T)$-isolating filters for $v \in \operatorname{LEAVES}(T)$, satisfy the following orthonormality condition in dimension one, see subsection 10.1.

Lemma 1. (Gram Matrix of adaptive alliasing filters in $d=1$ ) Let $T \subseteq T_{n}^{\text {full }}$, let $G_{v}$ be the $(v, T)$-isolating filter of leaf $v \in \operatorname{LEAVES}(T)$, as per (1). Let $v$ and $v^{\prime}$ be two distinct leaves of $T$. Then,
1.

$$
\left\|\widehat{G}_{v}\right\|_{2}^{2}:=\sum_{\xi \in[n]}\left|\widehat{G}_{v}(\xi)\right|^{2}=\frac{n}{2^{w_{T}(v)}} .
$$

2. (cross terms) the adaptive aliasing filters corresponding to $v$ and $v^{\prime}$ are orthogonal, i.e.

$$
\left\langle\widehat{G}_{v}, \widehat{G}_{v^{\prime}}\right\rangle:=\sum_{\xi \in[n]} \widehat{G}_{v}(\xi) \cdot \overline{\widehat{G}_{v^{\prime}}(\xi)}=0 .
$$

This already postulates that adaptive aliasing filters are relatively well-behaved: for a tree $T$ all leaves of which have roughly the same weight, it must be the case that $x \mapsto \quad\left\{\left\langle\widehat{G}_{v}, \widehat{x}\right\rangle\right\}_{v \in \operatorname{LEAVES}(T)}$ is a near-orthonormal transformation. Of course, this is too much to ask in general. The crucial property that we will make use of is captured in the following Lemma, see Subsection 10.2 .
Lemma 2. (see Lemma 17) Consider a tree $T \subseteq T_{N}^{f u l l}$. For every leafv of $T$ we let $\widehat{G}_{v}$ be a Fourier domain $(v, T)$-isolating filter. Then for every $\boldsymbol{\xi} \in[n]^{d}$,

$$
\sum_{v \in \operatorname{LEAVES}(T)}\left|\widehat{G}_{v}(\boldsymbol{\xi})\right|^{2}=1
$$

Using standard arguments, the above gives the following Lemma.
Lemma 3. For $z:[n]^{d} \rightarrow \mathbb{C}$, let $z^{\rightarrow \boldsymbol{a}}$ be the cyclic shift of $z$ by a, i.e. $z^{\rightarrow \boldsymbol{a}}(\boldsymbol{f}):=z(\boldsymbol{f}-\boldsymbol{a})$, where the subtraction happens modulo $n$ in every coordinate. For a tree $T \subseteq T_{N}^{\text {full }}$,

$$
\mathbb{E}_{\boldsymbol{a} \sim U_{[n]]^{d}}}\left[\sum_{v \in \operatorname{LEAVES}(T)}\left|\left\langle\widehat{G_{v}}, \widehat{z \rightarrow a}\right\rangle\right|^{2}\right]=\|z\|_{2}^{2},
$$

i.e. on expectation over a random shift the total collection of filters is an isometry.

Thus, although the tail is hashed in a way that is dependent on the head of the signal, what we can prove is that on expectation over a random shift the total amount of noise is controllable. Using the last property we can ensure that HeavyTest in the high-SNR regime we consider i) does not introduce false positives, i.e. does not engage in exploration in subtrees that contain no sufficient amount of energy, and ii) prevents false negatives. Guarantee i) translates to a bound on the running time of the algorithm, while ii) ensures correct execution of the algorithm. Note that due to the explorative nature of algorithm and the fact that missing a heavy element increases the total noise in the system (since we stop isolating with respect to it afterwards, it contributes as noise in subsequent measurements), accumulation of false negatives can totally destroy the guarantees of our approach. We note that this phenomenon of the tail not hashed independently of the signal occurs also in one-dimensional continuous Sparse Fourier Transform PS15, although for a very different reason; in their setting handling such an irregularity is significantly easier, mostly due to the fact that errors do not accumulate as in our explorative algorithm.

Identification and estimation are interleaved. In contrast to more standard sparse recovery tasks where usually identification and estimation can be decoupled, our algorithm needs to have a precise way to perform estimation upon identification of a coordinate. That happens due to the explorative nature of our algorithm, which does not allow us to perform estimation at the very end. This is relatively easy in the exactly $k$-sparse case, but in the robust case, due to the presence of noise it is much more challenging. Whenever we identify a frequency and isolate it from the other head elements, we can pick $\tilde{O}(k)$ random samples and estimate it up to $1 / \sqrt{k}$ fraction of the tail energy. Although this precision is sufficient for our algorithm to go through, it would lead us to an undesirable cubic sample complexity in total. The next two techniques are introduced in order to handle this situation.

Lazy Estimation. One additional crucial difference between the exactly $k$-sparse case and the robust case is estimation. In the former, when we had a tree $T$ and the minimim-weight leaf $v \in T$ was also a leaf in $T_{N}^{\text {full }}$, we needed $\widetilde{O}\left(2^{w_{T}(v)}\right)$ samples in order to perfectly estimate $\widehat{x}_{\boldsymbol{f}_{v}}$. However, in the robust case, perfect estimation is impossible, and as is usual in sparse recovery tasks, we should estimate it up to additive error $O\left(\frac{1}{\sqrt{k}}\|\widehat{\eta}\|_{2}\right)$ (recall that we write $\widehat{x}=\widehat{w}+\widehat{\eta}$, where $\eta$ is the tail of the signal). In order to achieve this type of guarantee, one way is to take $\widetilde{O}(k)$ random samples from $G_{v} \star x$, where $G_{v}$ is the $(v, T)$-isolating filter. This would yield $\widetilde{O}\left(k \cdot 2^{w_{T}(v)}\right)$ samples for estimation, a $k$ factor worse than what is needed in the exactly $k$-sparse case. In total, the sample complexity (and running time) would be $k$ times more expensive, getting us back to $\widetilde{O}\left(k^{3}\right)$.

Let's see how it is possible to shave the aforementioned multiplicative $k$ factor in the sample complexity. Imagine that upon finding such a leaf $v$, our algorithm does not estimate it immediately, but rather decides to postpone estimation for later. Instead, it marks it as a fully identified frequency, without removing it from $T$ and proceeds in exploring $T$ further. From now on, instead of picking the lowest weight leaf in $T$ at any time, it picks the lowest weight unmarked leaf in $T$. Of course, it could be the case that this rule causes the leaf picked to have weight much more than $\log k$, significantly increasing the cost of filtering. Consider however the following strategy. While the minimum weight unmarked leaf in $T$ has weight at most $\log k+2$, we pick and it and continue exploring. Whenever the aforementioned condition does not hold, the total Kraft mass ${ }^{3}$ occupied by the marked leaves in $T$ is at least $1-k \cdot \frac{1}{2 k}=\frac{1}{2}$. When this happens, we show that we can extract a large subset of the marked nodes, see Lemma 7, which can be well-estimated on average using only a polylogarithmic number of samples. This suffices for the $\ell_{2} / \ell_{2}$ guarantee, and furthermore reduces the number of marked nodes (and hence the Kraft mass occupied by marked nodes) causing our algorithm to proceed without increasing the cost of filtering. A more involved demonstration of this idea appears in section 11 .

Multi-scale Estimation. The lazy estimation technique presented above can estimate $k$ heavy frequencies of $\widehat{x}$ up to average additive error of $O\left(\frac{\|\widehat{\eta}\|_{2}}{\sqrt{k}}\right)$ using quadratic samples only if we use the vanilla tree exploration strategy which always picks the lowest weight unmarked leaf of tree $T$ and explores its children. This exploration strategy ensures that leaves get identified and consequentky marked in ascending weight order. Thus, there will be a point where the Kraft mass occupied by marked leaves is sufficiently large (recall that marked leaves have weight bounded by $\log k+2$ ). However, as we already mentioned, the tree exploration employed in KVZ19 results in cubic sample complexity even in the exactly $k$-sparse case. On the other hand, our new exploration

[^3]strategy (FFT backtracking) does not necessarily guarantee that the identified leaves will have large Kraft mass and bounded weight at the same time.

To make both lazy estimation and backtracking tree exploration techniques work together and achieve near quadratic total sample complexity, we devise a multi-scale estimation scheme. Our estimation strategy is to estimate every heavy frequency not once, but multiple times, each time to a different accuracy. More precisely, let's assume we are exploring a node $v \in T$ under the assumption that $\left\|\widehat{x}_{\text {FreqCone }(v)}\right\|_{0} \leq b$, and this assumption is correct. For every found frequency $\boldsymbol{f}$, we estimate $\widehat{x}(\boldsymbol{f})$, to precision $\frac{\|\widehat{\eta}\|_{2}}{\sqrt{b}}$ instead of $\frac{\|\hat{\|}\|_{2}}{\sqrt{k}}$, which would be the standard thing to do. However, sticking to this error precision will not give the desired $\ell_{2} / \ell_{2}$ guarantee: for small $b$, it blows up the error by a factor of $\sqrt{\frac{k}{b}}$, and it could be that all $\boldsymbol{f} \in \operatorname{supp}(\widehat{x})$ are estimated in a low-budget subproblem, due to recursion. Nevertheless, we can use these coarse-grained estimates to only locate the support of $\widehat{x}$ inside a subtree, and return it to the parent subproblem, i.e. to the above recursion level. The parent subproblem will mark those recovered frequencies, ignore their values, and continue its execution normally (pick the lowest leaf, perform lazy estimation etc). At some point, when the Kraft mass occupied by the parent subproblem is large enough, those frequencies will be estimated up to higher precision, i.e. $\frac{\|\eta\|_{2}}{\sqrt{b / \alpha}}$. When it finishes execution, it will return those elements to the above recursion level, so on so forth. This type of argumentation can be used to glue together lazy estimation and FFT backtracking. An illustration of this idea takes place in Section 12.

Lower Bound: Encoding OV via non-equispaced Fourier Transform. Given sets of vectors

$$
A=\left\{a_{0}, a_{1}, \ldots, a_{k-1}\right\}, B=\left\{b_{0}, b_{1}, \ldots, b_{k-1}\right\} \subseteq\{0,1\}^{d},
$$

we build $|A|$ points in time domain and $|B|$ points in frequency domain as follows. We pick sufficiently large $M, q, N$ (for details see Section 9) and define for $j \in[k]$ :

$$
t_{j}:=\sum_{r \in[d]} a_{j}(r) \cdot M^{r q}, \quad f_{j}:=\sum_{r \in[d]} b_{j}(r) \cdot \frac{N}{M^{r q+1}},
$$

Subsequently, we look at the indicator vector of the set $\left\{t_{0}, t_{1}, \ldots, t_{k-1}\right\}$, let it be $x$. Asking for the values $\widehat{x}_{f_{0}}, \widehat{x}_{f_{1}}, \ldots, \widehat{x}_{f_{k-1}}$ corresponds exactly to the non-equispaced Fourier transform problem.

Using the aforementioned evaluations we show that it is possible to extract the values

$$
V_{j, h}:=\sum_{\ell \in[k]}\left\langle a_{\ell}, b_{j}\right\rangle^{h}, \text { for } j \in[n], h \in[d] .
$$

For a fixed $j$, the values of $V_{j, h}$ can be expresed in terms of $Z_{r}:=\left|\left\{\ell \in[k] \mid\left\langle a_{\ell}, b_{j}\right\rangle=r\right\}\right|$, via multiplication by a $d \times d$ Vandermonde matrix. Since the entries involved in this matrix and $V_{j, h}$ have $\operatorname{poly}(d, \log k)$ bits, we can then solve for $Z_{0}$ in $\operatorname{poly}(d, \log k)$ time, where $Z_{0}$ corresponds to the number of vectors $a \in A$ which are orthogonal to $B$. Repeating this over all $j \in[k]$ yields whether there exists a pair of orthogonal vectors.

### 4.3 Explanation of the barriers faced

Discussion on the limits of the explorative approach, or why the quadratic barrier is impenetrable. On a high level, the explorative approach we take maintains a vector $\widehat{\chi}$ such that


Figure 2: An example of the FFT binary tree $T_{N}^{\text {full }}$ with $n=4$ and dimension $d=2$, (thus $N=16$ ). The universe corresponding to the nodes at each level of the tree is shown on the right side and the labeles of each node appears next to it.
$\operatorname{supp}(\widehat{\chi}) \subseteq \operatorname{supp}(\widehat{x})$ at all times ${ }^{4}$. Whenever the algorithm reaches a leaf $v \in T_{N}^{\text {full }}$ (see definitions in the Preliminaries Section), it estimates it and adds it to $\widehat{\chi}$. Subsequently, it proceeds by trying to recover the residual vector $\widehat{x}-\widehat{\chi}$. Now, imagine that we have recovered a constant fraction, say $1 / 10$, of $\widehat{x}$, and want to proceed further in order to recover the remaining part of $x$, i.e. $\widehat{x}-\widehat{\chi}$, which is an $\Omega(k)$-sparse vector. In order even to test whether $\widehat{x}-\hat{\chi}$ is the zero vector, we need to pick a set of $\Omega(k)$ random samples, satisfying for example the Restricted Isometry Property of order $k$, from $x-\chi$. In turn, this means that we need to compute the values $\chi_{\boldsymbol{t}}$ for all $\boldsymbol{t}$ in the aforementioned collection of random samples, and subtract them from the corresponding values of $x$. Since both $\operatorname{supp}(\widehat{\chi})$ and the samples needed for RIP are in principle unstructured sets of size $\Omega(k)$, the computation of the relevant $\chi_{t}$ is exactly the classical non-equispaced Fourier transform, for which no strongly subquadratic algorithm in available. We explain this unavailability by providing a quadratic lower bound on this task based on the well-established Orthogonal Vectors hypothesis, see Theorem 2. This also provides evidence that the quadratic time barrier is the limit of our explorative approach. Indeed, at all times we need to decide whether to explore a subtree or not by testing whether $\widehat{x}-\widehat{\chi}$ is the zero vector projected on that subtree. Since subtracting the effect of $\widehat{\chi}$ from the measurements, i.e. evaluating $\chi$ on an unstructured set of samples, cannot be done in strictly subquadratic time unless OVH fails, a subquadratic algorithm for exactly $k$-sparse FFT by traversing a pruned Cooley-Tukey FFT computation tree would most likely yield a subquadratic algorithm for the Orthogonal Vectors problem.

Discussion on the high-SNR regime. We shall illustrate a potential scenario where we might miss most frequencies in the head of the signal if we run our algorithm on an input signal that is not in the high-SNR regime. Note that throughout the exploration algorithm, we always maintain a set of nodes, such that the union of the frequency cones of those nodes covers the head of the signal. The frequencies which are not covered are essentially treated as noise, and we do not isolate with respect to them. Due to the fact that the adaptive aliasing filters hash the noise in a non-uniform way, it could be that our HEAVYTest primitive misclassifies a subtree as "frequency-inactive", i.e. no head

[^4]element inside it, although it contains one. In such a scenario, it is natural to abandon exploration inside the subtree. This would cause the noise in the system to increate by the magnitude of the missed head element (since we shall not isolate with respect to it anymore). Subsequently, this can potentially lead to a chain reaction, leading to successively missing head elements, and successively increasing the noise in the system, ending up to not recovering anything. However, our HeavyTest primitive is strong and ensures that we never miss a heavy frequency of signals that are not in the high-SNR regime as long as we perform oversampling by a factor $k$.

On the other hand, note that in order to achieve the $\ell_{2} / \ell_{2}$ guarantee on signals that are not in high-SNR regime, we need to set the threshold of HEAVYTEST to $1 / k$ fraction of the tail norm as opposed to the tail norm. Hence, another conceivable bad scenario is that, with such low threshold, the tail of the signal can make some frequency-inactive cones to appear heavy, introducing false positives. This can blow up the running time of the algorithm to super-polynomial in $k$.

Discrepancy between the runtime of our robust algorithm and its sample complexity. The only way we know how to perform dimension-independent estimation is via random sampling, as implemented in the HEAVYTEST routine. If we perform standard (non-lazy estimation) this would yield an additional multiplicative $k$ factor, as claimed in the first paragraph of Techniques III. Remedying this via lazy estimation shaves the multiplicative $k$ factor from the sample complexity, but does not do so in the running time. In particular, we run again into the same issue of subtracting $\widehat{\chi}$ from the buckets (which corresponds to an unstructured set of samples), i.e. the solution of a non-equispaced Fourier transform instance. As we've proven a quadratic time lower bound for the latter problem, this indicates that this discrepancy is most likely unavoidable with this approach.

## 5 Roadmap

The roadmap of this paper is the following. We follow an incremental approach, trying to introduce the techniques one by one, to the extent that is possible. In Section 6 we revise adaptive aliasing filters from [KVZ19]. In Section 7 we give the facts related to Kraft's inequality which we are going to use throughout our algorithms. Section 8 is devoted to the exact case, and in particular, illustrating our FFT backtracking technique. For convenience of the reader, as also discussed in the Techniques Section, we first design a warm-up algorithm, and then utilize the full power of our technique. In Section 9 we give the conditional lower bound on non-equispaced Fourier transform. In Section 10, the new structural properties of adaptive aliasing filters are inferred. In section 11 we introduce our first robust Sparse Fourier transform algorithm, illustrating techniques II-III and partly technique I. Lastly, in Section 12 we obtain our final robust Sparse FT algorithm, which uses techniques I-IV. For that reason, the algorithm is presented last.

## 6 Machinery from Previous work: Adaptive Aliasing Filters

In this section, we recall the class of adaptive aliasing filters that were introduced in [KVZ19]. These filters form the basis of our sparse recovery algorithm. For simplicity, we begin by introducing the filters in one-dimensional setting and then show how they naturally extend to the multidimensional setting (via tensoring).

### 6.1 One-dimensional Fourier transform

Our algorithm extensively relies on binary partitioning the frequency domain. In $d=1$, the following definitions are the one-dimensional analogues (special cases) of the ones in subsection 3.2 . We re-iterate them here, for completeness. The following is a re-interpretation of the splitting tree of a set in dimension 1 .

Definition 4 (Splitting tree). For every $S \subseteq[n]$, the splitting tree $T=\operatorname{Tree}(S, n)$ of a set $S$ is a binary tree that is the subtree of $T_{n}^{\text {full }}$ that contains, for every $j \in[\log n]$, all nodes $v \in T_{n}^{\text {full }}$ at level $j$ such that $\left\{f \in S: f \equiv f_{v}\left(\bmod 2^{j}\right)\right\} \neq \varnothing$.

Our Sparse FFT algorithm requires a filter $G$ that satisfies a refined isolating property due to the fact that throughout the execution of the algorithm, the identity of $\operatorname{supp}(\widehat{x})$ is only partially known. The following is a re-interpretation of the frequency cone of a node in dimension 1.
Definition 5 (Frequency cone of a leaf of $T$ ). Consider a subtree $T$ of $T_{n}^{\text {full }}$, and vertex $v \in T$ which is at level $l_{T}(v)$ from the root, the frequency cone of $v$ with respect to $T$ is defined as,

$$
\text { FreqCone }_{T}(v):=\left\{f_{u}: \text { for every leaf } u \text { in subtree of } T_{n}^{\text {full }} \text { rooted at } v\right\} .
$$

Note that under this definition, the frequency cone of a vertex $v$ of $T$ corresponds to the subtree rooted at $v$ when $T$ is embedded inside $T_{n}^{\text {full }}$. Next we present the definition of an isolating filter, introduced in KVZ19.

Definition $6\left((v, T)\right.$-isolating filter). Consider a subtree $T$ of $T_{n}^{\text {full }}$, and leaf $v$ of $T$, a filter $G$ : $[n] \rightarrow \mathbb{C}^{n}$ is called $(v, T)$-isolating if the following conditions hold:

- For all $f \in \operatorname{FreqCone}_{T}(v)$, we have $\widehat{G}(f)=1$.
- For every $f^{\prime} \in \bigcup_{\substack{u \in \operatorname{Leaves}(T) \\ u \neq v}} \operatorname{FreqCone}_{T}(u)$, we have $\widehat{G}_{v^{\prime}}\left(f^{\prime}\right)=0$.

Note that in particular, for all signals $x \in \mathbb{C}^{n}$ with $\operatorname{supp}(\widehat{x}) \subseteq \bigcup_{u \in \operatorname{LEAVES}(T)} \operatorname{FreqCone}_{T}(u)$ and $t \in[n]$,

$$
\sum_{j \in[n]} x(j) G_{v}(t-j)=\frac{1}{n} \sum_{f \in \operatorname{FreqCone}_{T}(v)} \widehat{x}_{f} e^{2 \pi i \frac{f t}{n}}
$$

The main technical construction of [KVZ19] is captured by the following Lemma.
Lemma 4 (Filter properties, [KVZ19]). Let $n$ be an integer power of two, $T$ a subtree of $T_{n}^{\text {full }}, v a$ leaf in $T$. Let $f:=f_{v}$ be the label of node $v$. Then the filter $G_{v}:[n] \rightarrow \mathbb{C}$ with Fourier Transform

$$
\begin{equation*}
\widehat{G}_{v}(\xi)=\frac{1}{2^{w_{T}(v)}} \prod_{\ell \in \operatorname{Anc}(v, T)}\left(1+e^{2 \pi i \frac{(\xi-f)}{2^{\ell+1}}}\right) \tag{1}
\end{equation*}
$$

is a $(v, T)$-isolating filter. Furthermore,

- $\left|\operatorname{supp}\left(G_{v}\right)\right|=2^{w_{T}(v)}$, and the filter $G$ can be constructed in $O\left(2^{w_{T}(v)}+\log n\right.$ ) time (in the time domain).
- Computing $\widehat{G}_{v}(\xi)$ for $\xi \in[n]$ can be done in $O(\log n)$ time.


## $6.2 d$-dimensional Fourier transform

In this subsection, we present the extension of adaptive aliasing filters to higher dimensions (by tensoring). It was shown in KVZ19 that multidimensional construction of these filters is extremely efficient and incurs no loss in the dimensionality.

Definition 7 (Multidimensional $(v, T)$-isolating filter). For every subtree $T$ of $T_{N}^{\text {full }}$ and vertex $v \in T$, a filter $G_{v} \in \mathbb{C}^{n^{d}}$ is called $(v, T)$-isolating if $\widehat{G}_{v}(\boldsymbol{f})=1$ for every $\boldsymbol{f} \in \mathrm{FreqCone}_{T}(v)$ and $\widehat{G}_{v}\left(\boldsymbol{f}^{\prime}\right)=0$ for every $\boldsymbol{f}^{\prime} \in \operatorname{supp}(T) \backslash \operatorname{FreqCone}_{T}(v)$.

In particular, for every signal $x \in \mathbb{C}^{n^{d}}$ with $\operatorname{supp}(\widehat{x}) \subseteq \operatorname{supp}(T)$ and for all $\boldsymbol{t} \in[n]^{d}$,

$$
\sum_{\boldsymbol{j} \in[n]^{d}} x(\boldsymbol{j}) G_{v}(\boldsymbol{t}-\boldsymbol{j})=\frac{1}{N} \sum_{\boldsymbol{f} \in \mathrm{FreqCone}_{T}(v)} \widehat{x}_{\boldsymbol{f}} e^{2 \pi i \frac{\boldsymbol{f}^{T} \boldsymbol{t}}{n}}
$$

We need the following lemma which is the main result of this section and shows that isolating filters can be constructed efficiently.

Lemma 5 (Construction of a multidimensional isolating filter - Lemma 4.2 of [KVZ19]). Let $T$ of $T_{N}^{\text {full }}$, and consider $v \in \operatorname{LEAVES}(T)$. There exists a deterministic construction of a $(v, T)$-isolating filter $G_{v}$ such that

1. $\left|\operatorname{supp}\left(G_{v}\right)\right|=2^{w_{T}(v)}$.
2. $G_{v}$ can be constructed in time $O\left(2^{w_{T}(v)}+\log N\right)$.
3. For any frequency $\boldsymbol{\xi} \in[n]^{d}, \widehat{G}_{v}(\boldsymbol{\xi})$, i.e. the Fourier transform of $G_{v}$ at frequency $\boldsymbol{\xi}$, can be computed in time $O(\log N)$.

## 7 Kraft-McMillan inequality and averaging claims

For our needs, we are going to make use of the following standard claim from coding theory, referred to as Kraft's or Kraft-McMillan inequality. The most general version is an inequality, but in the case of binary trees (complete codes in coding theory vocabulary), it becomes an equality.
Theorem 6 (Kraft's equality). Let $T \subseteq T_{N}^{\text {full }}$, it holds that

$$
\sum_{u \in \operatorname{LEAVES}(T)} 2^{-w_{T}(u)}=1
$$

For a tree $T \subseteq T_{N}^{\text {full }}$ and a set $S \subseteq \operatorname{LEAVES}(T)$, we shall refer to the Kraft mass of $S$ with respect to $T$ as the quantity $\sum_{u \in S} 2^{-w_{T}(u)}$.

We shall frequently use the following straightforward Lemma, which we shall refer to as Kraft averaging. This ideas has appeared in KVZ19.
Lemma 6 (Kraft averaging). Let $T \subseteq T_{N}^{\text {full, with } L \text { leaves. Then there exists a } u^{*} \in \operatorname{LEAVES}(T) ~}$ such that $w_{T}\left(u^{*}\right) \leq \log _{2} L$.

The following fine-grained version of Kraft averaging is an indispensable building block of our lazy estimation technique, and constitutes one of the important departures from the approach in [KVZ19]. The reader may postpone reading it at the moment, since its first usage will be in section 11. Neverthless, we decided to keep all the claims regarding Kraft's inequality in a separate section, for compactness reasons.

Lemma 7 (Fine-grained Kraft Averaging). Consider a subtree $T$ of $T_{N}^{\text {full }}$ and a positive integer $b$ such that $|\operatorname{LEAVes}(T)| \leq b$. Let $S:=\left\{v \in \operatorname{Leaves}(T): 2^{w_{T}(v)} \leq 2 b\right\}$, i.e. the leaves of $T$ with weight at most $\log _{2}(2 b)$. Then there exists a subset $L \subseteq S$ such that

$$
\frac{\max _{v \in L} 2^{w_{T}(v)}}{|L|} \leq \frac{1}{\theta},
$$

where $\theta \leq \frac{1}{4+2 \log _{2} b}$.
Informally (but somewhat imprecisely), the claim postulates that for any subtree $T$ of $T_{N}^{\text {full }}$ with $|\operatorname{leaves}(T)|=k$, there exist either 1 node of weight 1 , or 2 nodes of weight of 2 , or $\ldots$ at least $2^{j} / \log k$ nodes of weight $j$, or $\ldots k / \log k$ nodes of weight $\log k$. We now proceed with its proof.

Proof. First note that one can show the preconditions of claim imply that $\sum_{u \in S} 2^{-w_{T}(u)} \geq \frac{1}{2}$. For every $j=0,1, \ldots\left\lceil\log _{2}(2 b)\right\rceil$, let $L_{j}$ denote the subset of $S$ defined as $L_{j}:=\left\{u: u \in S, w_{T}(u)=j\right\}$. We can write,

$$
\sum_{u \in S} 2^{-w_{T}(u)}=\sum_{j=0}^{\left\lceil\log _{2}(2 b)\right\rceil} \frac{\left|L_{j}\right|}{2^{j}}
$$

Therefore by the assumption of the claim, we have that there must exist a $j \in\left\{0,1, \ldots\left\lceil\log _{2}(2 b)\right\rceil\right\}$ such that $\frac{\left|L_{j}\right|}{2^{j}} \geq \frac{1}{2\left\lceil\log _{2}(2 b)\right\rceil}$. Because $\theta \leq \frac{1}{4+2 \log _{2}|S|}$, there must exist a set $L \subseteq S$ such that $|L| \geq \theta \cdot \max _{v \in L} 2^{w_{T}(v)}$.

## 8 Exactly $k$-sparse Case

This section is devoted to proving Theorem 1. In subsection 8.1] we describe a preliminary algorithm which uses the idea of trying to learn the sparsities of $\widehat{x}$ when projected on the different subtrees in $T_{N}^{\text {full }}$, and perform recovery/exploration with respect to those. In essence, it performs only mild backtracking. In subsection 8.2 we give our result on exactly $k$-sparse signals, which utilizes the full power of FFT backtracking.

### 8.1 Warm Up

The goal of this subsection is to prove the following result.
Theorem 7. The sparse Fourier transform problem with an exactly $k$-(Fourier sparse) signal $x$ : $[n]^{d} \rightarrow \mathbb{C}$, i.e., $\|\widehat{x}\|_{0} \leq k$ can be solved in $m=O\left(k^{2.5} \operatorname{poly}(\log N)\right)$ time, deterministically.

We are going to analyze algorithm $\operatorname{SparseFT}-\operatorname{Warm} \operatorname{Up}(x, k)$, see pseudocode 4 .
We let $\widehat{x}_{v}$ be the vector $\widehat{x}_{\operatorname{Freq} \operatorname{Cone}(v)}$, i.e. the vector supported on frequencies in the frequency cone corresponding to $v$. The algorithm keeps a vector $\widehat{\chi}$, which at the end of the execution will equal $\widehat{x}$. For this overview, we can imagine that for every $v \in T_{N}^{\text {full }}$ the vector $\widehat{\chi}_{v}$, i.e. $\widehat{\chi}_{\text {FreqCone }(v)}$ serves as our estimate for $\widehat{x}_{v}$. Initially, all these vectors are set to $\{0,1\}^{n^{d}}$. The execution of our algorithm ensures that we can always keep sparse representations of them. A node $v$ is called frequency-active if $\widehat{\chi}_{v} \neq \widehat{x}_{v}$, i.e there is is still frequency content to recover inside FreqCone $(v)$. Note also that $\operatorname{supp}\left(\widehat{x}_{v}\right)=\operatorname{FreqCone}(v) \cap \operatorname{supp}(\widehat{x})$. We will say that $v$ is "heavy" if $\left\|\widehat{x}_{v}\right\|_{0}>b$.

Parameters/ variables $n, d, \chi$ are treated as global.

The algorithm $\operatorname{SparseFT}-\operatorname{WarmUp}(x, k)$ maintains at all times a subtree of $T_{N}^{\text {full }}$ which is referred to as Frontier, with the invariant that at all times

$$
\operatorname{supp}(\widehat{x}-\widehat{\chi}) \subseteq \bigcup_{u \in \text { Frontier }} \operatorname{FreqCone}(u)
$$

At all times, it picks the lowest weight leaf in Frontier, let it be $v$, and executes the PromiseSparseFT routine, which is a variant of the algorithm in KVZ19, on its children, under the assumption that $\left\|(\widehat{x}-\widehat{\chi})_{v}\right\|_{0} \leq b$. The latter assumption can be either true or false. When it returns, it performs a ZeroTest on $v$ : if the result of the ZeroTest is true, then it removes $v$ from Frontier, otherwise adds the children of $v$ to Frontier. Intuitively, the algorithm tries to learn the heavy nodes in the tree, i.e. those $v$ which satisfy $\left\|\widehat{x}_{v}\right\|_{0}>b$; the set Frontier corresponds to those nodes, or in particular a small superset of those.

```
Algorithm 1 ZeroTest \((x, \widehat{\chi}, v, T, s)\)
    \(f:=f_{v}\)
    \(G \leftarrow G_{1} \times G_{2} \times \ldots \times G_{d}\) the \((v, T)\) isolating filter in Lemma 5
    RIP \(_{s}:=\) a set of \(O\left(s \log ^{3} N\right)\) samples, which suffice for \(s\)-RIP, see Theorem 5
    \(h_{\boldsymbol{f}}^{\Delta} \leftarrow \sum_{\xi \in[n]^{d}}\left(e^{2 \pi \frac{\xi^{T} \Delta}{n}} \widehat{\chi}(\boldsymbol{\xi}) \cdot \prod_{q=1}^{d} \widehat{G}_{q}(\xi(q))\right)\), for all \(\Delta \in \operatorname{RIP}_{s}\)
    \(H_{\boldsymbol{f}}^{\Delta} \leftarrow \sum_{\boldsymbol{j} \in[n] d} x(\boldsymbol{j}) G(\Delta-\boldsymbol{j})-h^{\Delta}(\boldsymbol{j})\), for all \(\Delta \in \operatorname{RIP}_{s}\)
    if \(\sum_{\Delta \in \mathrm{RIP}_{s}}\left|H_{\boldsymbol{f}}^{\Delta}\right|^{2}=0\) then
        Return True
    else
        Return False
```

```
\(\overline{\operatorname{Algorithm} 2} \operatorname{EstimateFreq}(x, \widehat{\chi}, v, S)\)
    \(f:=f_{v}\)
    \(G \leftarrow G_{1} \times G_{2} \times \ldots \times G_{d}\) the \((v, S)\) isolating filter in Lemma 5
    \(h_{\boldsymbol{f}} \leftarrow \sum_{\xi \in[n]^{d}}\left(\widehat{\chi}(\boldsymbol{\xi}) \cdot \prod_{q=1}^{d} \widehat{G}_{q}(\xi(q))\right)\)
    Return \(N \cdot \sum_{\boldsymbol{j} \in[n] d} x(\boldsymbol{j}) G(-\boldsymbol{j})-h(\boldsymbol{j})\)
```

The following simple observation is the building block for estimation in KVZ19, and follows by the filter isolation properties presented in Lemma 5 .
Lemma 8. (Estimation) Let a signal $x, \widehat{\chi}$, a tree $S \subseteq T_{N}^{\text {full }}$ such that $\operatorname{supp}(\widehat{x}-\widehat{\chi}) \subseteq \operatorname{supp}(S)$, and a leaf $v \in S$ which is also a leaf of $T_{N}^{\text {full }}$. Then the procedure EstimateFreq $(x, \widehat{\chi}, v, S)$ returns $(\widehat{x}-\widehat{\chi})\left(\boldsymbol{f}_{v}\right)$. Furthermore, the routine requires

- $O\left(2^{w_{S}(v)}\right)$ sample complexity, and
- $\widetilde{O}\left(\|\widehat{\chi}\|_{0}+2^{w_{S}(v)}\right)$ running time.

The following Lemma is also one of the primitives in KVZ19.
Lemma 9. (Testing recovery on a subtree, see also [KVZ19, Lemma 7]) Let signals $x, \widehat{\chi}$, a tree $T \subseteq T_{N}^{\text {full }}$ such that $\operatorname{supp}(\widehat{x}-\widehat{\chi}) \subseteq \operatorname{supp}(T)$, and a leaf $v \in T$. Then, if $\left\|(\widehat{x}-\widehat{\chi})_{v}\right\|_{0} \leq s$, the call $\operatorname{ZeroTest}(x, \widehat{\chi}, v, T, s)$ determines correctly whether $\widehat{x}_{v}=\widehat{\chi}$ or not. If $\widehat{x}_{v}=\widehat{\chi}_{v}$, then the call ZeroTest ( $x, \widehat{\chi}, v, T, s$ ) always returns True.

Furthermore, the routine requires


Figure 3: Illustration of an instance of PromiseSparseFT( $x, \widehat{\chi}, v, \operatorname{SideTree}, b)$. The goal is to recover the residual vector $\widehat{x}-\widehat{\chi}$ on the subtree rooted at $v$. Here, SideTree $=\left\{v_{1}, v_{2}, v_{3}\right\}$, and the isolating filters constructed in PromiseSparseFT( $x, \widehat{\chi}, v$, SideTree, $b$ ) will isolate the corresponding nodes also from the nodes in SideTree. Furthermore, we are exploring the tree rooted at $v$ under the assumption that $\left\|(\widehat{x}-\chi)_{v}\right\|_{0} \leq b$, which means that inside this PromiseSparseFT call we shall invoke ZeroTest with budget $b$. If our assumption is correct and, furthermore, $\operatorname{supp}(\widehat{x}-\widehat{\chi}) \subseteq \operatorname{FreqCone}(v) \cup_{u \in \operatorname{SideTree}} \operatorname{FreqCone}(u)$, then our algorithm will correctly find the residual signal on the corresponding subtree, i.e. will recover $(\widehat{x}-\widehat{\chi})_{v}$.


Figure 4: An illustration of $\partial C$, appearing in proof of Lemma 13). The collection of ancestors of blue nodes (including the blue nodes) constitute the heavy nodes in $T_{N}^{\text {full }}$, i.e. $\mathcal{C}:=\left\{v \in T_{N}^{\text {full }}\right.$ : $\left.\left\|\widehat{x}_{v}\right\|_{0}>b\right\}$. The blue nodes correspond to $\partial \mathcal{C}:=\left\{v \in T_{N}^{\text {full }}: \nexists v^{\prime} \in C, v\right.$ is an ancestor of $\left.v^{\prime}\right\}$. The nodes that could be inserted in Frontier are only nodes in $\mathcal{C}$ and the children of nodes in $\partial \mathcal{C}$, see Lemma 13. The size of $\partial \mathcal{C}$ is always $O(k / b)$, and thus $|\mathcal{C}|=O((k / b) \log N)$, which in turn says that at all times $\mid$ Frontier $\mid=O((k / b) \log N)$.

```
Algorithm 3 PromiseSparseFT( \(x, \widehat{\chi}, v\), SideTree, \(b\) )
    //SideTree is a subset of the set of nodes which are siblings of some ancestor of \(v\)
    \(/ / b\) is the estimate (budget) of \((\widehat{x}-\hat{\chi})_{v}\)
    //Oracle Access to \(x\).
    \(S \leftarrow\{v\}\)
    NodesExplored \(\leftarrow 1\)
    \(\widehat{\chi_{\text {out }}} \leftarrow\{0\}^{n^{d}}\)
    repeat
        if NodesExplored \(>6 \cdot b \log N\) then
            //Have explored more than the estimated sparsity
            return \(\{0\}^{n^{d}}\)
        \(z:=\) leaf in \(S\) with the smallest weight.
        NodesExplored \(\leftarrow\) NodesExplored +1
        if \(z\) is a leaf in \(T_{N}^{\text {full }}\) then
            \(\widehat{\chi_{o u t}}\left(\boldsymbol{f}_{z}\right) \leftarrow \operatorname{EstimATEFREQ}\left(x, \widehat{\chi}+\widehat{\chi_{o u t}}, z, S \cup\{\operatorname{SideTreE}\}\right)\)
            \(S \leftarrow S \backslash\{z\}\)
        else if \(\operatorname{ZeroTest}(x, \widehat{\chi}+\widehat{\chi o u t}, z, S \cup\{\operatorname{SideTree}\}, b)=\) True then
            \(S \leftarrow S \backslash\{z\}\)
        else
            \(z_{\text {left }}:=\) left child of \(z\)
            \(z_{\text {right }}:=\) right of \(z\)
            \(S \leftarrow S \cup\left\{z_{\text {left }}, z_{\text {right }}\right\}\)
    until \(S=\varnothing\)
    Return \(\widehat{\chi_{o u t}}\)
    return
```

- $O\left(2^{w_{T}(v)} \cdot\left|\operatorname{RIP}_{s}\right|\right)$ sample complexity, and
- $\widetilde{O}\left(\|\widehat{\chi}\|_{0} \cdot\left|\operatorname{RIP}_{s}\right|+2^{w_{T}(v)} \cdot\left|\operatorname{RIP}_{s}\right|\right)$ running time ${ }^{5}$.

Recall that $\mathrm{RIP}_{s}$ is a set of samples satisfying s-RIP, see Theorem 5
We proceed by analyzing algorithm PromiseSparseFT.
Lemma 10. (Correctness of the Promise problem) Consider an invocation of the algorithm PromiseSparseFT $(x, \widehat{\chi}, v, \operatorname{SideTree}, b)$. If i) $\left\|(\widehat{x}-\widehat{\chi})_{v}\right\|_{0} \leq b$, and ii) SideTree isolates $v$ from every other frequency-active node in $T_{N}^{\text {full }}$, i.e. $\operatorname{supp}(\widehat{x}-\widehat{\chi}) \subseteq \operatorname{FreqCone}(v) \subseteq \cup_{u \in \operatorname{SideTreE}} \operatorname{FreqCone}(u)$, then at the end of the call it holds that $\widehat{\chi}_{v}=\widehat{x}_{v}$, i.e. we have recovered $\widehat{x}$ perfectly on the subtree rooted at $v$.

Proof. First of all, note that all calls to ZeroTest will be executed correctly since $\left\|(\widehat{x}-\widehat{\chi})_{v}\right\|_{0} \leq b$ and $u$ is isolated from every other frequency-active node in $T_{N}^{\text {full }}$. This means that for any non-leaf $z$ we have that i) if $(\widehat{x}-\widehat{\chi})_{z}=\{0\}^{n^{d}}$, this will be detected and the algorithm will remove $z$ from $S_{v}$, and ii) if $(\widehat{x}-\widehat{\chi})_{z} \neq\{0\}^{n^{d}}$, this will be detected and the algorithm will proceed by adding the

[^5]```
Algorithm 4 SparseFT-WarmUp \((x, k)\)
    Frontier := \{root \(\}\)
    \(b:=\lceil\sqrt{k}\rceil\)
    \(/ /\) Frontier is at all times a tree containing the set of nodes \(v\) with budget \(k\) and \(\widehat{\chi}_{v} \neq \widehat{x}_{v}\).
    \(\widehat{\chi} \leftarrow\{0\}^{n^{d}}\)
    repeat
        Pick \(v \in\) Frontier with the smallest weight.
        \(\operatorname{SideTree}_{v}:=\) the sub-path of node \(v\) with respect to Frontier
        if \(\operatorname{ZeroTest}\left(x, \widehat{\chi}, v, \operatorname{SideTreE}_{v}, k\right)\) then
            //Have recovered \(\widehat{x}_{v}\) perfectly
            Frontier \(\leftarrow\) Frontier \(\backslash\{v\}\).
            continue
        \(v_{\text {left }}:=\) left child of \(v\)
        \(v_{\text {right }}:=\) right child of \(v\)
        \(P_{\text {left }}:=\operatorname{SideTREE}_{v} \cup\left\{v_{\text {right }}\right\}\)
        \(P_{\text {right }}:=\operatorname{SideTreE}_{v} \cup\left\{v_{\text {left }}\right\}\)
        \(/ / P_{\text {left }}\left(\right.\) resp. \(\left.P_{\text {right }}\right)\) is guaranteed to isolate \(v_{\text {left }}\) (resp. \(v_{\text {right }}\) ) from every other frequency-
    active node in \(T_{N}^{\text {full }}\).
        \(\widehat{\chi_{\text {left }}} \leftarrow \operatorname{PromiseSparseFT}\left(x, \widehat{\chi}, v_{\text {left }}, P_{\text {left }}, b\right)\)
        \(\widehat{\chi_{\text {right }}} \leftarrow \operatorname{PromiseSparseFT}\left(x, \widehat{\chi}, v_{\text {right }}, P_{\text {right }}, b\right)\)
        if \(\operatorname{ZeroTest}\left(x, \widehat{\chi}+\widehat{\chi_{\text {left }}}+\widehat{\chi_{\text {right }}}, v, \operatorname{SideTreE}_{v}, k\right)\) then
            //Have recovered \(\widehat{x}_{v}\) perfectly
            Frontier \(\leftarrow\) Frontier \(\backslash\{v\}\)
            \(\widehat{\chi} \leftarrow \widehat{\chi}+\widehat{\chi_{\text {left }}}+\widehat{\chi_{\text {right }}}\)
        else
            Frontier \(=\) Frontier \(\cup\left\{v_{\text {left }}, v_{\text {right }}\right\}\)
    until Frontier \(=\varnothing\)
    Return \(\widehat{\chi}_{v}\).
```

children of $u$ in $S_{v}$. Furthermore, if $z$ is also a leaf in $T_{N}^{\text {full }}$, EstimateFreq in Line 14 , will set $\widehat{\chi}\left(\boldsymbol{f}_{z}\right)=\widehat{x}\left(\boldsymbol{f}_{z}\right)$; subsequently $z$ will be removed from the tree. The above properties ensure that estimation and identification is always correct, and thus, when $S_{v}=\varnothing$, we have perfectly recovered the signal in $\operatorname{FreqCone}(v)$, i.e. $\widehat{\chi}_{v}=\widehat{x}_{v}$. Furthermore, the test in Line 8 will never force exit, since we are going to put only $3\left\|\widehat{x}_{v}\right\|_{0} \cdot \log N \leq 3 b \log N$ nodes in the $S_{v}$ (there are $\left\|\widehat{x}_{v}\right\|_{0} \log N$ nodes with a non-trivial frequency cone, and we multiply by 3 to account for all their children, which could be inserted in $S_{v}$ ), and each such node can cause increment of the counter NodesExplored at most two times.

Lemma 11. (Running Time of the Promise Problem)
Consider an invocation of the algorithm PromiseSparseFT( $x, \widehat{\chi}, v, \operatorname{SideTree}, b)$. The sample complexity is upper bounded by

$$
\widetilde{O}\left(2^{\mid \text {SideTree } \mid} \cdot b^{3}\right)
$$

and the running time is upper bounded by

$$
\widetilde{O}\left(\|\widehat{\chi}\|_{0} \cdot b^{2}+2^{\mid \text {SideTree } \mid} \cdot b^{3}\right) .
$$

Proof. Note that due to the test in Line 8 , the number of executions of the while loop is $O(b \log N)$. Furthermore, since NodesExplored is at most $6 b \log N$, we have that $|S| \leq 6 b \log N$. By invoking Kraft averaging (Lemma 6) on the tree $S$, we obtain that the node $z$ picked in Line 11, satisfies $w_{S}(z) \leq \log (6 b \log N)$. This in turn gives

$$
2^{w_{\text {SUSIIETREEE }}(z)} \leq 6 \cdot 2^{\mid \text {SIDETREE } \mid} \cdot b \log N .
$$

Thus, every call to ZeroTest uses

$$
\widetilde{O}\left(2^{w_{S U S \mathrm{SidTreE}}(z)} \cdot\left|\mathrm{RIP}_{b}\right|\right)=\widetilde{O}\left(2^{|\operatorname{SideTreE}|} \cdot b \log N\right) \cdot \widetilde{O}(b)=\widetilde{O}\left(2^{|\operatorname{SideTreE}|} \cdot b^{2}\right)
$$

samples, and every call to EstimateFreq uses $O\left(2^{\mid \text {SideTree }} b \log N\right)$ samples. Thus, over all $O(b \log N)$ nodes we get the desired bound on the sample complexity.

Similarly, the running time is upper bounded by the time spent on calling Zerotest and the time spent on calling EstimateFreq. The first one can be controlled as

$$
\underbrace{O(b \log N)}_{\text {number of nodes }} \cdot \underbrace{\widetilde{O}\left(\left\|\widehat{\chi}_{v}\right\|_{0} \cdot\left|\mathrm{RIP}_{b}\right|+2^{w_{S U S \text { SidTREE }}(z)} \cdot\left|\mathrm{RIP}_{b}\right|\right)}_{\text {time spent on } \mathrm{ZEROTEST}}=
$$

which is within the time bound. The second one can be controlled as

$$
\underbrace{O(b \log N)}_{\text {number of nodes }} \cdot \underbrace{\widetilde{O}\left(\|\widehat{\chi}\|_{0}+2^{|\operatorname{SideTREE}|} b\right)}_{\text {time spent on EsTiMATEFREQ }},
$$

which is again within the time bound.
Lemma 12. (Invariant of the algorithm: signal containment) At all times, the frequency cones of nodes in Frontier contain $\operatorname{supp}(\widehat{x}-\widehat{\chi})$. Formally, $\operatorname{supp}(\widehat{x}-\widehat{\chi}) \subseteq \bigcup_{v \in \text { Frontier }}$ FreqCone $(v)$.

Proof. We prove the claim by induction. The base case is at the first step when Frontier $=\{$ root $\}$, in which case the claim is obvious. For the inductive step, consider node $v$ chosen with minimum weight from Frontier. If we follow the branch in Line 8 or the branch in Line 19, this means that $(\widehat{x}-\widehat{\chi})_{\text {FreqCone }(v)}=\{0\}^{n^{d}}$ (equivalently, $\widehat{\chi}_{v}=\widehat{x}_{v}$ ) in which case removal of $v$ from Frontier does not violate the invariant. If we follow the else branch in Line 23, $v$ is removed from Frontier and its two children are inserted, and the invariant clearly holds.

Lemma 13. (Frontier "heaviness" property) Any node v ever inserted in Frontier, apart from root, has a heavy parent, i.e. a parent $v^{\prime}$ such that $\left|\widehat{x}_{v^{\prime}}\right|>b$. Furthermore, the total number of distinct nodes inserted in Frontier during the execution of $\operatorname{SparseFT}-\operatorname{WarmUp}(x, k)$ is at most $3(k / b) \log N$.

Proof. Consider any node $v \neq$ root satisfying $\|\widehat{x}\|_{0} \leq b$ which is inserted in the Frontier during the execution of the algorithm, and let $v^{\prime}$ be its parent. Then, consider the calls PromiseS$\operatorname{ParseFT}\left(x, \widehat{\chi}, v_{\text {left }}, P_{\text {left }}, b\right)$ and PromiseSparseFT $\left(x, \widehat{\chi}, v_{\text {right }}, P_{\text {right }}, b\right)$, where $v_{\text {left }}, v_{\text {right }}$ are the children of $v$. By the fact that $\left\|\widehat{x}_{v_{\text {left }}}\right\|_{0},\left\|\widehat{x}_{v_{\text {right }}}\right\|_{0} \leq\left\|\widehat{x}_{v}\right\|_{0} \leq b$ and the invariant of Lemma 12 , the conditions that guarantee correct execution of PromiseSparseFT, i.e. Lemma 10, apply and whence the above two calls will correctly return $\widehat{x}_{v_{\text {left }}}$ and $\widehat{x}_{v_{\text {right }}}$. Thus, the call to ZeroTest $(x, \widehat{\chi}+$ $\left.\widehat{\chi_{\text {left }}}+\widehat{\chi_{\text {right }}}, v, \operatorname{SideTreE}_{v}, k\right)$ in Line 19 will succeed, and hence $v$ will be removed from the

Frontier. This immediately means that $v$ cannot be a parent of any element ever inserted the Frontier. In turn, since we picked any such $v$ that any node ever inserted in the Frontier must have a heavy parent.

Let us now prove the second part of the Lemma. Let $\mathcal{C}:=\left\{v \in T_{N}^{\text {full }}:\left\|\widehat{x}_{v}\right\|_{0}>b\right\}$ be the collection of heavy nodes in $T_{N}^{\text {full }}$, and define

$$
\partial \mathcal{C}:=\left\{v \in T_{N}^{\text {full }}: \nexists v^{\prime} \in C, v \text { is an ancestor of } v^{\prime}\right\}
$$

i.e. $\partial \mathcal{C}$ is the collection of heavy nodes which do not have any descendants in $C$ (the boundary of $C)$. Note that $|\partial \mathcal{C}|<(k / b)$, otherwise we would get more than $k$ coordinates in the $\operatorname{supp}(\widehat{x})$ by the disjointedness of the corresponding frequency cones, i.e. $|\partial \mathcal{C}| \cdot b<\sum_{v \in \partial \mathcal{C}}\left\|\widehat{x}_{v}\right\|_{0} \leq\|\widehat{x}\|_{0} \leq k$. By the argument in the first paragraph, the number of nodes which could be inserted in Frontier are the nodes on $\partial \mathcal{C}$, their children, and their ancestors. This gives that the total number of nodes ever inserted in the Frontier is upper bounded by $3(k / b) \log N$.

Lemma 14. (Bounding the size of the argument of PromiseSparseFT) At all times during the execution of SparseFT-WarmUP, the routine PromisePromiseSparseFT will be called with argument SideTree satisfying $\mid$ SideTree $\left\lvert\, \leq 1+\left(\frac{3 k \log N}{b}\right)\right.$.

Proof. By Lemma 13 we have that $\mid$ Frontier $\left\lvert\, \leq \frac{3 k \log N}{b}\right.$. Invoking Lemma 6, we obtain that the node $v$ in Line $\left[6\right.$ must satisfy $\left|\operatorname{SideTree}_{v}\right|=\left|w_{\text {Frontier }}(v)\right| \leq \frac{3 k \log N}{b}$. Thus, the calls to PromiseSparseFT in Lines 17, 18 will be called with the fourth argument being of size at most $1+\frac{3 k \log N}{b}$.

We are now in position to prove Theorem 7 .
Proof. We first prove correctness. Observe that when a node $v \in$ Frontier is picked in Line 6 , three things can happen.

1. ZeroTest $\left(v, \widehat{\chi}, v, \operatorname{SideTreE}_{v}, k\right)$ returns False in Line 8 then $\widehat{\chi}_{v}=\widehat{x}_{v}$, using invariant 12 and the fact that $\operatorname{SideTree}_{v}$ isolates $v$ from the rest of Frontier. Thus, $v$ will be removed from Frontier (this case covers also the scenario when $\widehat{x}_{v}=0$ ).
2. $\operatorname{ZeroTest}\left(v, \widehat{\chi}, v, \operatorname{SideTreE}_{v}, k\right)$ returns False in Line 8 but the call in Line 19 returns True: then $\widehat{\chi}_{v}=\widehat{x}_{v}$, using invariant 12 and the fact that $\operatorname{SideTree}_{v}$ isolates $v$ from the rest of Frontier. Thus, $v$ will be removed from Frontier.
3. ZeroTest returns False in both Lines 8, 19, then $v$ will be removed from Frontier, but its children $v_{\text {left }}, v_{\text {right }}$ will be added to it.

Using Lemma 13, we observe that the scenario in Bullet (3) can happen at most $3(k / b) \log N$ times. After that, only Bullets (1) and (2) can happen, in which case we get that everything from that point onwards is correct, reaching up to root. Upon termination, $\widehat{\chi}$ will equal $\widehat{x}$.

The sample complexity is a sum of the contribution from ZeroTest (Lines 8, 19), and PromiseSparseft. Every node in $v \in$ Frontier which is considered in Line 6 has weight at most $\log \left(\frac{3 k \log N}{b}\right)$ by Lemma 8.1. Thus, ZeroTest calls use

$$
\underbrace{O((k / b) \log N)}_{2^{w_{\text {FRONTIRR }}(v)}} \cdot \underbrace{\widetilde{O}(k)}_{\text {size of } \mathrm{RIP}_{k}}
$$

samples per node in Frontier, for a total of $\widetilde{O}\left(k^{3} / b^{2}\right)$ samples. There are also $O((k / b) \log N)$ calls to PromiseSparseFT, each one using $\widetilde{O}\left(\frac{k \log N}{b} \cdot b^{3}\right)=\widetilde{O}\left(k b^{2}\right)$ samples, by Lemma 8.1. Summing over all those $O((k / b) \log N)$ nodes, this yields sample complexity

$$
\widetilde{O}\left(\frac{k^{3}}{b^{2}}+b k^{2}\right) .
$$

To bound the running time, we shall the time spent on Zerotest and on PromiseSparseFT.
Consider one of the $O(k \log N / b)$ nodes $v$ in Frontier picked in Line 6. By Lemma 8.1 it holds $2^{\left|\operatorname{SideTree}_{v}\right|}=O\left(\frac{k \log N}{b}\right)$. The total time spent on calling ZeroTest on $v$ can be handled by Lemma 9 as

$$
\begin{aligned}
& O\left(\|\widehat{\chi}\|_{0} \cdot\left|\operatorname{RIP}_{k}\right|+2^{w_{\text {FROMTIER }}(v)} \cdot\left|\operatorname{RIP}_{k}\right|\right)= \\
& \widetilde{O}\left(\|\hat{\chi}\|_{0} \cdot k+\frac{k}{b} \cdot k\right)= \\
& \widetilde{O}\left(\|\hat{\chi}\|_{0} \cdot k+\frac{k}{b} \cdot k\right)
\end{aligned}
$$

By summing over all $O\left(\frac{k \log N}{b}\right)$ nodes ever inserted in Frontier we obtain that the total time spent on calling ZeroTest for the nodes in Frontier is at most

$$
\widetilde{O}\left(\|\widehat{\chi}\|_{0} \cdot \frac{k^{2}}{b}+\frac{k^{3}}{b^{2}}\right)
$$

Using Lemma 11, we can upper bound the time spent on calling PromiseProblem on the children of $v$ by

$$
\widetilde{O}\left(\|\widehat{\chi}\|_{0} \cdot b^{2}+\frac{k \log N}{b} \cdot b^{3}\right)=\widetilde{O}\left(\|\widehat{\chi}\|_{0} \cdot b^{2}+k \cdot b^{2}\right)=
$$

By summing over all $O\left(\frac{k \log N}{b}\right)$ nodes ever inserted in Frontier we obtain that the total time spent on calls to PromiseProblem is at most

$$
\widetilde{O}\left(\|\widehat{\chi}\|_{0} \cdot k b+k^{2} b\right) .
$$

Taking into account that $\|\widehat{\chi}\|_{0} \leq k$, we obtain that the total running time is

$$
\widetilde{O}\left(\|\widehat{\chi}\|_{0} \cdot \frac{k^{2}}{b}+\frac{k^{3}}{b^{2}}\right)+\widetilde{O}\left(\|\widehat{\chi}\|_{0} \cdot k b+k^{2} b\right)=\widetilde{O}\left(\frac{k^{3}}{b}+k^{2} b\right)
$$

By plugging in our choice of $b$, we obtain the desired bound.

### 8.2 The Almost Quadratic-Time Algorithm

This subsection is devoted to (finally) proving our first main result, namely Theorem 1.

Theorem 8 (Theorem 1, restated). The sparse Fourier transform problem with an exactly $k$ (Fourier sparse) signal $x:[n]^{d} \rightarrow \mathbb{C}$, i.e., $\|\widehat{x}\|_{0} \leq k$ can be solved in

$$
\left.m=\widetilde{O}\left(k^{2} \cdot 2^{O(\sqrt{\log k \cdot \log \log N})}\right)\right)
$$

time, deterministically.
For a parameter $\alpha$, and we shall call ExactSparseFT $\left(x,\{0\}^{n^{d}},\{\right.$ root $\left.\}, \varnothing, 0\right)$, the pseudocode of which is depicted in Algorithm 1. We also pick an absolute constant $C$ sufficiently larger than 1, which governs the threshold in NodesExplored.

Proof of Correctness. We shall prove that ExactSparseFT $(x, \widehat{\chi}, v, \operatorname{SideTree}, s)$ outputs $\widehat{x}_{v}$, if

1. $\left\|\widehat{x}_{v}\right\|_{0} \leq s$ (correct guess on the sparsity of $x$ in FreqCone $(v)$,
2. $\operatorname{supp}(\widehat{\chi}) \cap \operatorname{FreqCone}(v)=\varnothing$ (have not recovered anything in FreqCone $(v))$,
3. $\operatorname{supp}(\widehat{x}-\widehat{\chi}) \subseteq \operatorname{FreqCone}(v) \cup_{u \in \operatorname{SideTree}} \operatorname{Freq} \operatorname{Cone}(u)$ (signal isolation from every other frequencyactive node).

The second condition is not necessary for the analysis, but provides a conceptual simplification, since this will be the case for the recursive calls of our algorithm. The aforementioned conditions clearly hold for our call of interest, which is $\operatorname{ExactSparseFT}\left(x,\{0\}^{n^{d}},\{\operatorname{root}\}, \varnothing, 0\right)$, so the claim suffices for proving correctness. We shall perform induction on $\left\|\widehat{x}_{v}\right\|_{0}$.

Initially, observe that by the check in Line 39, whether or not the preconditions (1)-(3) above hold, the output ExactSparseFT $(x, \widehat{\chi}, v, \operatorname{SideTree}, s)$ will be at most $s$. In turn, by the check on NodesExplored in Line 39 we have that $\left\|\widehat{\chi_{\text {out }}}\right\|_{0} \leq(C \log N / \alpha) \cdot(\alpha s)=C s \log N$. In turn, this means that at all times during the execution of the call ExactSparseFT $(x, \widehat{\chi}, v, \operatorname{SideTree}, s)$ it holds $\left\|\left(\widehat{x}-\widehat{\chi}-\widehat{\chi_{\text {out }}}\right)\right\|_{0} \leq\|\widehat{x}-\widehat{\chi}\|_{0}+C \cdot s \log N \leq 2 C \cdot s \log N$. Thus, the calls to Zerotest in Lines 24 and 25 will always succeed. This means that for the children $z_{\text {left }}, z_{\text {right }}$ of $z$ picked in Line 11, we can always decide whether we have perfectly recovered $\widehat{x}_{z_{\text {left }}}\left(\right.$ respectively $\left.\widehat{x}_{z_{\text {right }}}\right)$. If $\left\|\widehat{x}_{z_{\text {left }}}\right\|_{0} \leq \alpha s$ then the preconditions for the call $\operatorname{ExactSparseFT}\left(x, \widehat{\chi}+\widehat{\chi_{o u t}}, z_{\text {left }}, P_{z} \cup\left\{z_{\text {right }}\right\}, a \cdot s\right)$ in Line 22 are satisfied, and hence by induction the algorithm will perfectly recover $\widehat{x}_{z_{\text {left }}}$. Similarly for $\widehat{x}_{z_{\text {right }}}$. Furthermore, by the discussion on the correctness of the ZeroTest calls, $z_{\text {left }}$ (resp. $z_{\text {right }}$ ) will be correctly removed from the tree in that case. By the call in Line 13 or the test in Line 27, $z$ will be removed from the set $S$ when both of its two children $z_{\text {left }}, z_{\text {right }}$ are removed, in which case we have perfectly recovered $\widehat{x}_{z_{\text {left }}}, \widehat{x}_{z_{\text {right }}}$, and hence $\widehat{x}_{z}$. Hence, a node $v$ will be removed from the set $S$ only when $\widehat{x}_{v}$ is completely recovered. Eventually, $\widehat{x}_{v}$ will be perfectly recovered, since at all times, either we explore by increasing $S$ (and we can explore a finite number of times), or we completely recover the signal in lying in a frequency cone (and this can of course happen a finite number of times). The only thing that could stop this process is the check NodesExplored $>\frac{C \log N}{\alpha}$ in Line 38 . Note that by an averaging argument there can be at most $\log N / \alpha$ nodes $u$ which are descendants of $v$, such that $\left\|\widehat{x}_{u}\right\|_{0} \geq \alpha s$. The nodes that could ever be inserted in $S$ are those nodes along with their children, giving at most $\frac{3 \log N}{\alpha}$ nodes in total, each one causing at most 2 increments of the counter NodesExplored. By setting $C \geq 6$, we ensure that NodesExplored will never become more than $\frac{C \log N}{\alpha}$, and hence the one part of the test in Line 38 will not force premature stop of the algorithm.

```
Algorithm 5 ExactSparseFT( \(x, \widehat{\chi}, v, \operatorname{SideTree}, s\) )
```



```
    SideTree, under the assumption that \(\left\|(\widehat{x}-\widehat{\chi})_{v}\right\|_{0} \leq s\).
    if \(s \leq 1 / \alpha\) then
        \(/ /\) We shall set the parameter \(\alpha:=2^{-\Theta(\sqrt{\log k \cdot \log \log n})}\), where \(k\) is the sparsity of the initial
    vector we want to recover.
        \(\widehat{\chi_{o u t}} \leftarrow \operatorname{PromiseSparseFT}(x, \widehat{\chi}, v, \operatorname{SideTreE}, k, 1 / \alpha)\)
        If \(\left\|\widehat{\chi_{\text {out }}}\right\|_{0} \leq \frac{1}{\alpha}\) return \(\widehat{\chi_{o u t}}\) else return \(\{0\}^{n^{d}}\)
    \(\widehat{\chi_{\text {out }}} \leftarrow\{0\}^{n^{d}}\)
    \(S \leftarrow\{v\}\)
    NodesExplored \(\leftarrow 0\)
    \(/ / S\) holds the descendants \(u\) of \(v\) (including itself), for which we guess that \(\left\|\left(\widehat{x}-\widehat{\chi}-\widehat{\chi_{\text {out }}}\right)_{v}\right\|_{0}<\alpha \cdot s\)
    repeat
        \(z \leftarrow\) leaf in \(S\) with the minimum weight
        NodesExplored \(\leftarrow\) NodesExplored +1
        if \(\operatorname{Zerotest}\left(x, \widehat{\chi}+\widehat{\chi_{o u t}}, z, S \cup \operatorname{SideTree}, s \log N\right)\) then
            Remove \(z\) from \(S\)
            Continue
        if \(z\) is a leaf in \(T_{N}^{\text {full }}\) then
            \(\widehat{\chi}_{\text {out }}\left(\boldsymbol{f}_{z}\right) \leftarrow \operatorname{EstimateFreq}\left(x, \widehat{\chi}+\widehat{\chi_{o u t}}, z, S \cup \operatorname{SidETREE}\right)\)
            Remove \(z\) from \(S\)
            Continue
        \(z_{\text {left }}, z_{\text {right }} \leftarrow\) left and right child of \(z\) in \(S\) respectively.
        \(P_{z} \leftarrow\) sub-path of \(z\) in SideTree \(\cup S\)
        \(\widehat{\chi_{\text {left }}} \leftarrow \operatorname{ExACTSPARSEFT}\left(x, \widehat{\chi}+\widehat{\chi_{o u t}}, z_{\text {left }}, P_{z} \cup\left\{z_{\text {right }}\right\}, a \cdot s\right)\)
        \(\widehat{\chi_{\text {right }}}=\operatorname{ExACTSPARSEFT}\left(x, \widehat{\chi}+\widehat{\chi_{o u t}}, z_{\text {right }}, P_{z} \cup\left\{z_{\text {left }}\right\}, a \cdot s\right)\)
        \(I s Z e r o_{l e f t} \leftarrow \operatorname{ZEROTEST}\left(x, \widehat{\chi}+\widehat{\chi_{\text {out }}}+\widehat{\chi_{\text {left }}}, z_{\text {left }}, S \cup \operatorname{SidETREE} \cup\left\{z_{\text {right }}\right\}, 2 C \cdot s \log N\right)\)
        \(I s Z e r o_{\text {right }} \leftarrow \operatorname{ZEROTEST}\left(x, \widehat{\chi}+\widehat{\chi_{\text {out }}}+\widehat{\chi_{\text {right }}}, z_{\text {right }}, S \cup \operatorname{SideTrEE} \cup\left\{z_{\text {left }}\right\}, 2 C \cdot s \log N\right)\)
        //Check whether residual signal was recovered correctly on children of \(z\)
        if \(I s Z e r O_{\text {left }}\) and \(I s Z e r o_{\text {right }}\) then
            Remove \(z\) from \(S\)
            \(\widehat{\chi_{o u t}} \leftarrow \widehat{\chi_{\text {out }}}+\widehat{\chi_{\text {left }}}+\widehat{\chi_{\text {right }}}\)
        if not \(I s Z e r o_{\text {left }}\) and IsZero \(_{\text {right }}\) then
            \(S \leftarrow S \cup\left\{z_{\text {left }}\right\}\)
            \(\widehat{\chi_{\text {out }}} \leftarrow \widehat{\chi_{\text {out }}}+\widehat{\chi_{\text {right }}}\)
        if IsZero \(_{\text {left }}\) and not IsZero \(_{\text {right }}\) then
            \(S \leftarrow S \cup\left\{z_{\text {right }}\right\}\)
            \(\widehat{\chi_{\text {out }}} \leftarrow \widehat{\chi_{\text {out }}}+\widehat{\chi_{\text {left }}}\)
        if not IsZero \(_{\text {left }}\) and not IsZero \(_{\text {right }}\) then
            \(S \leftarrow S \cup\left\{z_{\text {left }}, z_{\text {right }}\right\}\)
    until \(S=\varnothing\) or NodesExplored \(>\frac{C \log N}{\alpha}\)
    if \(S=\varnothing\) and \(\left\|\widehat{\chi}_{\text {out }}\right\|_{0} \leq s\) then Return \(\widehat{\chi_{o u t}}\)
    else Return \(\{0\}^{n^{d}}\)
```

The running time is a sum of the time spent ZeroTest and on the time spent on EstimateFreq. Furthermore, note that we can split the total running time to three components:
(type-I runtime) time needed to access $x$;
(type-II runtime) time needed to subtract $\widehat{\chi}$ from the measurements.
We note that the time needed to create the aliasing filters (time needed to compute $\widehat{G}$ in Line 3 of EstimateFreq) constitutes a lower order term, and we therefore ignore it. The type-I running time of the call $\operatorname{ZeroTest}(x, \widehat{\chi}, v, T, s)$ is $O\left(2^{w_{T}(v)} \cdot|\operatorname{RIP}|\right)=\widetilde{O}\left(2^{w_{T}(v)} \cdot s\right)$, the type-II running time of the same call is $O\left(\|\widehat{\chi}\|_{0} \cdot|\operatorname{RIP}|\right)=\widetilde{O}\left(\|\widehat{\chi}\|_{0} \cdot s\right)$. The type-I running time of the call $\operatorname{EstimateFreq}(x, \widehat{\chi}, u, S)$ is $O\left(2^{w_{S}(u)}\right)$, the type-II running time is $O\left(\|\widehat{\chi}\|_{0}\right)$. We shall bound each type separately, and pick $\alpha:=2^{-\Theta(\sqrt{\log k \cdot \log \log N})}$.

The following observations can be easily inferred by inspection of the algorithm. The first one follows by the usage of the counter NodesExplored and the condition in Line 38, while the second by the checks on $\left\|\widehat{\chi_{\text {out }}}\right\|_{0}$.
Observation I. In each call ExactSparseFT $(x, \widehat{\chi}, v, \operatorname{SideTree}, s)$, there can be at most $\frac{C \log N}{\alpha}$ nodes ever inserted in $S$. Also, there are at most $2 \frac{C \log N}{\alpha}$ recursive calls to ExactSparseFT.
Observation II. Each call ExactSparseFT $(x, \widehat{\chi}, v, \operatorname{SideTree}, s)$ always outputs a vector which has always sparsity at most $s$.
Observation III. In each execution of ExactSparseFT $(x, \widehat{\chi}, v, \operatorname{SideTree}, s)$ we have at all times $\left\|\widehat{\chi}+\widehat{\chi_{\text {out }}}\right\|_{0} \leq\|\widehat{\chi}\|_{0}+\widetilde{O}(s)$. Indeed, by Observation I and II we have that $\left\|\widehat{\chi_{\text {out }}}\right\|_{0} \leq(\alpha s) \cdot \frac{2 C \log N}{\alpha}=$ $\tilde{O}(s)$.
The following observation follows by Kraft averaging on $S$ (Lemma 6), which satisfies $|S| \leq \frac{C \log N}{\alpha}$ (due to the check on the variable NodesExplored).
Observation III. Consider the call ExactSparseFT( $x, \widehat{\chi}, v, \operatorname{SideTree}, s$ ), and some $z$ picked in Line 11 during the execution of the algorithm. It holds that

$$
2^{w_{\text {SUSSIDETREE }}(z)} \leq 2^{\mid \text {SideTreE } \mid} \cdot \frac{C \log N}{\alpha}
$$

Thus, for its children (if they exist), it holds that

$$
2^{w_{S \cup S i d e T R E E}\left(z_{\text {left }}\right)}, 2^{w_{S \cup \operatorname{SidETREE}}\left(z_{\text {right }}\right)} \leq 2^{|\operatorname{SidETREE}|+1} \cdot \frac{C \log N}{\alpha}
$$

Analysis of the time spent on accessing $x$ (type-I). We define $T_{1}[s, W, \ell]$ to be the type-I running time of $\operatorname{ExactSparseFT}(x, \widehat{\chi}, v, \operatorname{SideTree}, s)$, when $\operatorname{SideTree} \subseteq T_{N}^{\text {full }}$ is a sub-path of $v$ in $T_{N}^{\text {full }}, 2^{\mid \text {SideTree } \mid}=W$, and $\ell$ is the distance of $v$ to root. With the above notation, the time complexity of $\operatorname{ExactSparseFT}\left(x,\{0,1\}^{n^{d}}\right.$, $\left.\{\operatorname{root}\}, \varnothing, 0\right)$ is $T_{1}[k, 1,0]$.

By using Observations I-III, we obtain that the recursive calls in Line 22 and Line 23 take time at most $T_{1}\left[\alpha s, 2 W \cdot \frac{C \log N}{\alpha}, \ell+1\right]$, since the weights of $z_{\text {left }}, z_{\text {right }}$ are increased by a multiplicative $W \cdot \frac{C \log N}{\alpha}$ factor, the sparsity budget decreases by a multiplicative $\alpha$ factor, and the distance of $z_{\text {left }}, z_{\text {right }}$ to root $\in T_{N}^{\text {full }}$ increases by at least 1 .

The base case corresponds to $s \leq \frac{1}{\alpha}$ or $\ell=\log N$. We have

$$
T_{1}[s, W, \ell] \leq W s^{3} \cdot \operatorname{poly}(\log N) \quad(\star),
$$

by Lemma 11 and Lemma 8. In what follows, in order to simplify notation we define

$$
\begin{aligned}
s_{\beta} & :=\alpha^{\beta} s \text { (sparsity after } \beta \text { recursive calls), } \\
A & :=\frac{C \log N}{\alpha}
\end{aligned}
$$

For the other setting of parameters, we can write the recursive relation of the type-I running time as follows, using Lemma 9 and Observation II. In particular, we have

$$
T_{1}[s, W, \ell] \leq A \cdot\left(\widetilde{O}(W \cdot A \cdot s)+O(W)+2 \cdot T_{1}[\alpha s, 2 W \cdot A, \ell+1]\right)
$$

Let us explain the above relation. First of all, we have at most $A=\frac{C \log N}{\alpha}$ nodes ever inserted in $S$ by Observation I. For any such node $z$, we may make at most 2 calls to ZeroTest with sparsity budget $O(s \log N)$. By Observation III, we have $w_{S}(z) \leq W \cdot A$. The term $\widetilde{O}(W \cdot A \cdot s)$ follows by Lemma 9 and Observation III. The term $O(W)$ follows by the guarantee of Lemma 8 . For the $2 \cdot T_{1}[\alpha s, 2 W \cdot A, \ell+1]$ first note that there are most $2 A$ recursive calls (at most 2 children for each node ever inserted in $S$ ). Again by Observation III, we have that weight of the third argument in lines 22, 23 (i.e. $\left.w_{S}\left(z_{\text {left }}\right), w_{S}\left(z_{\text {right }}\right)\right)$ will be at most $2 \cdot W \cdot A$. This explains the above recursive relationship.

If we iterate the relation $\beta$ times, with $\ell+\beta \leq \log N$, we obtain the uncommon relationship

$$
T_{1}[s, W, \ell] \leq W \cdot s \cdot A^{\beta} \cdot \operatorname{poly}(\log N)^{\beta}+2^{\beta} A^{\beta} \cdot T_{1}\left[s_{\beta}, 2^{\beta} \cdot W \cdot A^{\beta}, \ell+\beta\right]
$$

For the minimum $\beta^{\star}$ such that $s_{\beta^{\star}} \leq \frac{1}{\alpha}$, i.e. for $\beta^{\star}=1+\left\lceil\frac{\log s}{\log (1 / \alpha)}\right\rceil=\frac{\log s}{\log (1 / \alpha)}+O(1)$, we have

$$
\begin{aligned}
A^{\beta^{\star}} & =2^{\beta^{\star} \cdot \log A} \\
& =2^{\left(\frac{\log s}{\log (1 / \alpha)}+O(1)\right) \cdot(\log (1 / \alpha)+\log \log N+O(1))} \\
& =\widetilde{O}\left(2^{\log s+\frac{\log s \cdot \log \log N}{\log (1 / \alpha)}+\frac{\log s}{\log (1 / \alpha)}+O(\log (1 / \alpha)}\right) \\
& =\widetilde{O}\left(s \cdot 2^{\frac{\log s \cdot \log \log N}{\log (1 / \alpha)}+\frac{\log s}{\log (1 / \alpha)}+O(\log (1 / \alpha)}\right) \\
& =\widetilde{O}\left(s \cdot 2^{O(\sqrt{\log k \cdot \log \log N})}\right)
\end{aligned}
$$

by our choice of $\alpha:=2^{-\Theta(\sqrt{\log k \cdot \log \log N})}$ and the fact that $s \leq k$. We also have the crude bound

$$
2^{\beta^{\star}}=2^{\frac{\log s}{\log (1 / \alpha)}+O(1)}=2^{O(\sqrt{\log k \cdot \log \log n})}
$$

We also have that

$$
\begin{aligned}
& (\operatorname{poly}(\log N))^{\beta^{\star}}=2^{\beta^{\star} \cdot O(\log \log N)}= \\
& 2^{\left(\frac{\log s}{\log (1 / \alpha)}+O(1)\right) \cdot O(\log \log N)}= \\
& \widetilde{O}\left(2^{O(\sqrt{\log k \cdot \log \log N)})}\right.
\end{aligned}
$$

Thus, getting back to $(\star)$ we obtain

$$
T_{1}\left[s_{\beta^{\star}}, 2^{\beta^{\star}} W \cdot A^{\beta^{\star}}, \ell\right]=\widetilde{O}\left(2^{\beta^{\star}} W \cdot A^{\beta^{\star}} \cdot 2^{O(\log (1 / \alpha))}\right)=\widetilde{O}\left(W \cdot s 2^{O(\sqrt{\log k \cdot \log \log N})}\right)
$$

Lastly, we get back to $(\dagger)$ for $\beta=\beta^{\star}$ to obtain

$$
\begin{aligned}
& T_{1}[s, W, \ell]=\widetilde{O}\left(W \cdot s \cdot s \cdot 2^{O(\sqrt{\log k \cdot \log \log N})}\right) \cdot(\operatorname{poly}(\log N))^{\beta} \\
& +\widetilde{O}\left(s \cdot 2^{O(\sqrt{\log s \cdot \log \log N})}\right) \cdot \widetilde{O}\left(W \cdot s 2^{O(\sqrt{\log s \cdot \log \log N})}\right) \\
& =\widetilde{O}\left(W \cdot s^{2} \cdot 2^{O(\sqrt{\log k \cdot \log \log N})}\right)
\end{aligned}
$$

Since we call ExactSparseFT $\left(x,\{0\}^{n^{d}},\{\operatorname{root}\}, \varnothing, 0\right)$, we have that $s=k, W=1, \ell=0$, and hence we can upper bound the type-I running time by $\widetilde{O}\left(k^{2} \cdot 2^{O(\sqrt{\log k \cdot \log \log N)})}\right.$, as desired.

Analysis of the time spent on subtracting $\widehat{\chi}$ from the buckets (type-II). We define $T_{2}[s, \ell]$ to be the type-II running time spent on the call ExactSparseFT $(x, \widehat{\chi}, v, \operatorname{SideTree}, s)$, when SideTree $\subseteq T_{N}^{\text {full }}$ is a sub-path of $v$ in $T_{N}^{\text {full }}$, and $\ell$ is the distance of $v$ to root $\in T_{N}^{\text {full }}$. With the above notation, the time complexity of $\operatorname{ExactSparseFT}\left(x,\{0,1\}^{n^{d}},\{\operatorname{root}\}, \varnothing, 0\right)$, i.e. which is the call of the algorithm for Theorem 1, is $T_{2}[k, 0]$. We also assume that $\|\widehat{\chi}\|_{0}=\widetilde{O}(k)$ at all times, which is clearly the case for our algorithm.

For $s \leq \frac{1}{\alpha}$ we have $T_{2}[s, \ell] \leq k \cdot\left(\frac{1}{\alpha}\right)^{2} \cdot \operatorname{poly}(\log N)(\star \star)$ by Lemma 11 and Observation III. For $\ell=\log N$ we have $T_{2}[s, \ell]=\widetilde{O}(k)$ by Lemma 8 and Observation III. Recall that

$$
\begin{aligned}
& s_{\beta}:=\alpha^{\beta} s \text { (sparsity after } \beta \text { recursive calls), } \\
& A:=\frac{c \log N}{\alpha}
\end{aligned}
$$

Now, we may write down the recursive relationship as

$$
T_{2}[s, \ell] \leq A \cdot\left(\widetilde{O}(k+s)+2 \cdot T_{2}[\alpha s, \ell+1]\right)
$$

We explain the recursive relationship. As before there are at most $A$ nodes ever inserted in $S$ (Observation I). For each such node the $\widetilde{O}(k+s)$ follows from ZeroTest follows by Lemma 9 , or by Lemma 8 . The $2 \cdot T_{2}[\alpha s, \ell+1]$ component follows by the calls in line 22, 23, follows by Iterating the relation $\beta$ times, with $\ell+\beta \leq \log N$, we obtain the interesting relationship

$$
T_{2}[s, \ell]=\widetilde{O}(k) \cdot O\left(2^{\beta} A^{\beta}\right)+2^{\beta} A^{\beta} \cdot T_{2}\left[s_{\beta}, \ell+\beta\right](\ddagger)
$$

Consider the minimum $\beta^{\star}$ such that $s_{\beta^{\star}} \leq \frac{1}{\alpha}$, i.e. for $\beta^{\star}=1+\left\lceil\frac{\log s}{\log (1 / \alpha)}\right\rceil=\frac{\log s}{\log (1 / \alpha)}+O(1)$. Recall that $A^{\beta^{\star}}=s \cdot 2^{O(\sqrt{\log k \cdot \log \log n})}, 2^{\beta^{\star}}=2^{O(\sqrt{\log k \cdot \log \log N})},\left(\operatorname{poly}(\log N)^{\beta^{\star}}=2^{O(\sqrt{\log k \cdot \log \log n})}\right.$ by our choice of $\alpha$. Given the above, we may now plug $(\star \star)$ in ( $\ddagger$ ) to obtain

$$
T_{2}[s, \ell]=\widetilde{O}\left(k \cdot s \cdot 2^{O(\sqrt{\log k \cdot \log \log n})}+s \cdot 2^{O(\sqrt{\log k \cdot \log \log n})} \cdot k\left(\frac{1}{\alpha}\right)^{2}\right)
$$

By our choice of $\alpha:=2^{-\Theta(\sqrt{\log k \cdot \log \log N})}$ we obtain

$$
T_{2}[s, \ell]=\widetilde{O}\left(s k 2^{O(\sqrt{\log s \log \log k})}\right) .
$$

Thus, the total type-II running time is $T_{2}[k, 0]=\widetilde{O}\left(k^{2} \cdot 2^{O(\sqrt{\log k \log \log N})}\right)$.
Putting together the contribution from type-I and type-II running times, we obtain the desired result.

## 9 Lower Bound on Non-Equispaced Fourier Transform

The main result of this section is the following theorem.
Theorem 9. (Detailed version of Theorem (2) For every $c>0$ larger than an absolute constant and every $\delta>0$ there exists $c^{\prime}>0$ and $\delta^{\prime}>0$ such that if for all $\epsilon \in(0,1 / 2)$, for all $N$ a power of two and all $k \leq 2^{c^{\prime}(\log N)^{1 / 3}}$ there exists an algorithm that solves the 1-dimensional non-equispaced Fourier Transform problem on universe size $N$, sparsity $k$ in time $k^{2-\delta^{\prime}} \operatorname{poly}(\log (N / \epsilon))$, then there exists an algorithm which solves $\mathrm{OV}_{k, d}$ with $d=c \log k$ in time $k^{2-\delta}$.

As also mentioned in the abstract of this paper, this answers one of the subproblems of Problem 21 from IITK Workshop on Algorithms for Data Streams, Kanpur 2006. Additionally, the following proof facilities gives also the lower bound on sparse multipoint evaluation, i.e. Theorem 3.

Proof. Given an Orthogonal Vectors instance, we shall appropriately construct a non-equispaced Fourier transform instance, such that an algorithm for the non-equispaced Fourier transform with strongly subquadratic running time in $k$ implies a strongly subquadratic time algorithm for the Orthogonal Vectors problem.

Let $A=\left\{a_{0}, \ldots, a_{k-1}\right\}, B=\left\{b_{0}, \ldots, b_{k-1}\right\} \subseteq\{0,1\}^{d}$ be the input to an $\mathrm{OV}_{k, d}$ instance with $d=c \log k$. We denote by $a_{j}(r)$ the $r$-th coordinate of vector $a_{j} \in A$. We first pick sufficiently large integers $N, M, q$ that are powers of 2 such that $M=k d^{C_{1} d}, q=C_{2} d$, and $N=M^{2 d q}$, where $C_{1}, C_{2}$ are sufficiently large absolute constants.

Next, we define for $j \in[k]$ :

$$
t_{j}:=\sum_{r \in[d]} a_{j}(r) \cdot M^{r q}, \quad f_{j}:=\sum_{r \in[d]} b_{j}(r) \cdot \frac{N}{M^{r q+1}},
$$

and set $F=\left\{f_{0}, \ldots, f_{k-1}\right\}, T=\left\{t_{0}, \ldots, t_{k-1}\right\}$. Furthermore, we define vector $x \in \mathbb{C}^{N}$ such that $x_{t}=1$ if $t \in T$, and 0 otherwise, and we pick $\epsilon=\frac{1}{N}$. Thus, to transform our initial $\mathrm{OV}_{k, d}$ instance to an instance of non-equispaced Fourier transform, we show that from additive $\epsilon\|\widehat{x}\|_{2}$-approximations of $\widehat{x}_{f_{0}}, \ldots, \widehat{x}_{f_{k-1}}$ we can infer whether $(A, B)$ contains a pair of orthogonal vectors. It then follows that an algorithm for non-equispaced Fourier transform running in time $k^{2-\delta^{\prime}} \operatorname{poly}(\log (N / \epsilon))$ would imply a strongly subquadratic time algorithm for Orthogonal Vectors.

Our first claim postulates that $\widehat{x}_{f_{j}}$ corresponds to summing up $\exp \left(-2 \pi i \cdot \frac{1}{M}\left\langle a_{\ell}, b_{j}\right\rangle\right)$ for all $\ell \in[k]$, up to error terms in the exponent.
Claim 1. For every $j \in[k]$ it holds that

$$
\widehat{x}_{f_{j}}=\sum_{\ell \in[k]} \exp \left(-2 \pi i \cdot\left(\frac{1}{M}\left\langle a_{\ell}, b_{j}\right\rangle+\xi_{\ell, j}\right)\right),
$$

for a real number $\xi_{\ell, j}$ satisfying

$$
\left|\xi_{\ell, j}\right| \leq\binom{ d}{2} M^{-q-1}
$$

Proof. Fix $j \in[k]$ and note that

$$
\begin{aligned}
& \widehat{x}_{f_{j}}=\sum_{t \in T} \exp \left(-2 \pi i \frac{f_{j} t}{N}\right) \\
& =\sum_{\ell \in[k]} \exp \left(-\frac{2 \pi i}{N} \cdot\left(\sum_{r^{\prime} \in[d]} a_{\ell}(r) \cdot M^{r q}\right) \cdot\left(\sum_{r \in[d]} b_{j}\left(r^{\prime}\right) \cdot \frac{N}{M^{r^{\prime} q+1}}\right)\right) \\
& =\sum_{\ell \in[k]} \exp \left(-2 \pi i \cdot \sum_{\left(r, r^{\prime}\right) \in[d] \times[d]} a_{\ell}(r) b_{j}\left(r^{\prime}\right) \cdot M^{\left(r-r^{\prime}\right) q-1}\right) \\
& =\sum_{\ell \in[k]} \prod_{\left(r, r^{\prime}\right) \in[d] \times[d]} \exp \left(-2 \pi i \cdot a_{\ell}(r) b_{j}\left(r^{\prime}\right) \cdot M^{\left(r-r^{\prime}\right) q-1}\right)
\end{aligned}
$$

We now investigate the exponents of the complex exponentials, namely $a_{\ell}(r) b_{j}\left(r^{\prime}\right) \cdot M^{\left(r-r^{\prime}\right) q-1}$ for $\ell \in[k]$ and $\left(r, r^{\prime}\right) \in[d] \times[d]$. In particular, we find that:

1. For any pair $\left(r, r^{\prime}\right)$ with $r>r^{\prime}$, we have $\left(r-r^{\prime}\right) q-1 \geq 0$, meaning that the corresponding exponent is an integer multiple of $2 \pi i$. In turn, the corresponding term in the product contributes 1 , so it can be ignored.
2. For any pair $\left(r, r^{\prime}\right)$ with $r<r^{\prime}$ we have $\left(r-r^{\prime}\right) q-1 \leq-q-1$. For a fixed $\ell$, there are $\binom{d}{2}$ such products, and hence their total contribution to the exponent of the $\ell$-th summand is at most $\binom{d}{2} M^{-q-1}$ (in absolute value).
3. The pairs $\left(r, r^{\prime}\right)$ with $r=r^{\prime}$ contribute to the exponent of the $\ell$-th summand the term $-2 \pi i \cdot M^{-1} \sum_{r \in[d]} a_{\ell}(r) b_{j}(r)=-2 \pi i \cdot M^{-1}\left\langle a_{\ell}, b_{j}\right\rangle$.

Putting everything together we arrive at the proof of the claim.
In the remainder of this proof we write

$$
V_{j, h}:=\sum_{\ell \in[k]}\left\langle a_{\ell}, b_{j}\right\rangle^{h} .
$$

Next, we perform a series expansion and error analysis on the exponential function to obtain:
Claim 2. For every $j \in[k]$ it holds that

$$
\widehat{x}_{f_{j}}=\xi_{j}^{\prime}+\sum_{h \geq 0}\left(-\frac{2 \pi i}{M}\right)^{h} \frac{1}{h!} \cdot V_{j, h},
$$

for a complex number $\xi_{j}^{\prime}$ satisfying

$$
\left|\xi_{j}^{\prime}\right| \leq M^{-q}
$$

Proof. Let $a, b$ be real numbers. Starting from the basic fact $|\exp (-2 \pi i b)-1| \leq 2 \pi|b|$, we obtain $\exp (-2 \pi i(a+b))=\exp (-2 \pi i a)+\exp (-2 \pi i a)(\exp (-2 \pi i b)-1)=\exp (-2 \pi i a)+\xi_{a, b}^{\prime}$ with $\left|\xi_{a, b}^{\prime}\right| \leq$ $2 \pi|b|$. In particular, with notation as in Claim 1, we have

$$
\exp \left(-2 \pi i \cdot\left(\frac{1}{M}\left\langle a_{\ell}, b_{j}\right\rangle+\xi_{\ell, j}\right)\right)=\exp \left(-2 \pi i \cdot \frac{1}{M}\left\langle a_{\ell}, b_{j}\right\rangle\right)+\xi_{\ell, j}^{\prime},
$$

with $\left|\xi_{\ell, j}^{\prime}\right| \leq 2 \pi\left|\xi_{\ell, j}\right| \leq 2 \pi\binom{d}{2} M^{-q-1} \leq M^{-q}$.

Summing over all $\ell \in[k]$ now yields

$$
\widehat{x}_{f_{j}}=\sum_{\ell \in[k]} \exp \left(-2 \pi i \cdot\left(\frac{1}{M}\left\langle a_{\ell}, b_{j}\right\rangle+\xi_{\ell, j}\right)\right)=\xi_{j}^{\prime}+\sum_{\ell \in[k]} \exp \left(-2 \pi i \cdot \frac{1}{M}\left\langle a_{\ell}, b_{j}\right\rangle\right),
$$

with $\left|\xi_{j}^{\prime}\right| \leq 2 \pi k\binom{d}{2} M^{-q-1}$. Using that $M=k d^{C_{1} d}$ for a sufficiently large constant $C_{1}>0$, we obtain $\left|\xi_{j}^{\prime}\right| \leq M^{-q}$.

Finally, we use the series expansion of $\exp ($.$) to obtain$

$$
\widehat{x}_{f_{j}}=\xi_{j}^{\prime}+\sum_{h \geq 0}\left(-\frac{2 \pi i}{M}\right)^{h} \frac{1}{h!} \cdot \sum_{\ell \in[k]}\left\langle a_{\ell}, b_{j}\right\rangle^{h} .
$$

We now show that in our expression for $\widehat{x}_{f_{j}}$ the summands $\left(-\frac{2 \pi i}{M}\right)^{h} \frac{1}{h!} \cdot V_{j, h}$ lie sufficiently far apart, so that each summand can be reconstructed from an approximation of $\widehat{x}_{f_{j}}$.
Claim 3. Let $j \in[k], H \in[d]$, and let $\widetilde{x}_{f_{j}}$ be an additive $\epsilon\|\widehat{x}\|_{2}$ approximation of $\widehat{x}_{f_{j}}$. Then

$$
\widetilde{x}_{f_{j}}-\sum_{h=0}^{H-1}\left(-\frac{2 \pi i}{M}\right)^{h} \frac{1}{h!} \cdot V_{j, h}=\left(-\frac{2 \pi i}{M}\right)^{H} \frac{1}{H!} \cdot\left(V_{j, H}+\xi_{j, H}^{\prime \prime}\right),
$$

for a complex number $\xi_{j, H}^{\prime \prime}$ satisfying

$$
\left|\xi_{j, H}^{\prime \prime}\right|<1 / 3
$$

Proof. Note that by Parseval's identity, we have $\|\widehat{x}\|_{2}=\sqrt{N} \cdot\|x\|_{2}=\sqrt{N \cdot k}$. Therefore, $\left|\widetilde{x}_{f_{j}}-\widehat{x}_{f_{j}}\right| \leq$ $\epsilon\|\widehat{x}\|_{2} \leq \epsilon \sqrt{N \cdot k} \leq \sqrt{k / N}$ as $\epsilon=1 / N$. Since $N=M^{2 d q}$ and $M \geq k$, we obtain $\left|\widetilde{x}_{f_{j}}-\widehat{x}_{f_{j}}\right| \leq$ $M^{-(q-1)}$.

Note that

$$
\begin{aligned}
\left|\sum_{h>H}\left(-\frac{2 \pi i}{M}\right)^{h} \frac{1}{h!} \cdot V_{j, h}\right| & \leq \sum_{h>H}\left(\frac{2 \pi}{M}\right)^{h} \frac{1}{h!} \cdot \sum_{\ell \in[k]}\left\langle a_{\ell}, b_{j}\right\rangle^{h} \\
& \leq\left(\frac{2 \pi}{M}\right)^{H} \frac{1}{H!} \cdot \sum_{h>H}\left(\frac{2 \pi}{M}\right)^{h-H} \cdot k \cdot d^{h} \\
& =\left(\frac{2 \pi}{M}\right)^{H} \frac{1}{H!} \cdot k d^{H} \cdot \sum_{h>H}\left(\frac{2 \pi d}{M}\right)^{h-H} .
\end{aligned}
$$

Since $M$ is sufficiently larger than $d$, the latter sum can be bounded by $\frac{4 \pi d}{M}$, and hence

$$
\begin{equation*}
\left|\sum_{h>H}\left(-\frac{2 \pi i}{M}\right)^{h} \frac{1}{h!} \cdot V_{j, h}\right| \leq\left(\frac{2 \pi}{M}\right)^{H} \frac{1}{H!} \cdot \frac{4 \pi k d^{H+1}}{M} \leq \frac{1}{10} \cdot\left(\frac{2 \pi}{M}\right)^{H} \frac{1}{H!}, \tag{2}
\end{equation*}
$$

using the fact that $M=k d^{C_{1} d}$ for a sufficiently large constant $C_{1}>0$ and $H \in[d]$.
We now, using Claim 2, decompose:

$$
\begin{aligned}
& \widetilde{x}_{f_{j}}-\sum_{h=0}^{H-1}\left(-\frac{2 \pi i}{M}\right)^{h} \frac{1}{h!} \cdot V_{j, h} \\
& =\left(\widetilde{x}_{f_{j}}-\widehat{x}_{f_{j}}\right)+\left(\widehat{x}_{f_{j}}-\sum_{h=0}^{H-1}\left(-\frac{2 \pi i}{M}\right)^{h} \frac{1}{h!} \cdot V_{j, h}\right) \\
& =\left(\widetilde{x}_{f_{j}}-\widehat{x}_{f_{j}}\right)+\xi_{j}^{\prime}+\left(-\frac{2 \pi i}{M}\right)^{H} \frac{1}{H!} \cdot V_{j, H}+\sum_{h>H}\left(-\frac{2 \pi i}{M}\right)^{h} \frac{1}{h!} \cdot V_{j, h} .
\end{aligned}
$$

Recall that $\left|\widetilde{x}_{f_{j}}-\widehat{x}_{f_{j}}\right| \leq M^{-(q-1)}$ and $\left|\xi_{j}^{\prime}\right| \leq M^{-q}$. We use $H \in[d]$ and our choice of $M=k d^{C_{1} d}$ and $q=C_{2} d$ for sufficiently large constants $C_{1}, C_{2}>0$ to conclude that $M^{-(q-1)} \leq \frac{1}{10} \cdot\left(\frac{2 \pi}{M}\right)^{H} \frac{1}{H!}$. Together with inequality (2), this gives

$$
\widetilde{x}_{f_{j}}-\sum_{h=0}^{H-1}\left(-\frac{2 \pi i}{M}\right)^{h} \frac{1}{h!} \cdot V_{j, h}=\left(-\frac{2 \pi i}{M}\right)^{H} \frac{1}{H!} \cdot\left(V_{j, H}+\xi_{j, H}^{\prime \prime}\right),
$$

for a complex number $\xi_{j, H}^{\prime \prime}$ with $\left|\xi_{j, H}^{\prime \prime}\right|<1 / 3$.
Repeatedly applying the above claim allows us to reconstruct the numbers $V_{j, 0}, \ldots, V_{j, d}$ :
Claim 4. Fix $j \in[k]$. Let $\epsilon=\frac{1}{N}$. Given an additive $\epsilon\|\widehat{x}\|_{2}=\sqrt{k / N}$ approximation to $\widehat{x}_{f_{j}}$ we can infer the exact values of

$$
V_{j, h}:=\sum_{\ell \in[k]}\left\langle a_{\ell}, b_{j}\right\rangle^{h},
$$

for any $h \in[d]$, in time $\operatorname{poly}(d, \log k)$.
Proof. Suppose that we have already computed the sums $V_{j, h}$ for all $0 \leq h<H$. Then we know the left hand side of Claim 3. Since $\left|\xi_{j, H}^{\prime \prime}\right|<1 / 3$, there is a unique integer $V_{j, H}=\sum_{\ell \in[k]}\left\langle a_{\ell}, b_{j}\right\rangle^{H}$ that satisfies the equation in Claim 3. Hence, we can infer $V_{j, H}$. Therefore, we can iteratively compute $V_{j, 0}, V_{j, 1}, \ldots, V_{j, d-1}$.

Note that when evaluating expressions of the form $\left(-\frac{2 \pi i}{M}\right)^{h} \frac{1}{h!}$, we can compute them up to precision $\epsilon$ in time poly $(d, \log k)$, since it suffices to perform arithmetic on numbers with poly $(d, \log k)$ digits. This yields another additive error in the same order of magnitude as in the proof of Claim3. The same error analysis therefore shows that this precision is sufficient to compute the exact integers $V_{j, h}$.

The above claim postulates that we can infer the values $V_{j, h}$ for $h \in[d]$. We next show that these values allow us to determine whether there exists a pair of orthogonal vectors.

Claim 5. Given the values $V_{h}:=V_{j, h}$ for all $h \in[d]$ and some fixed $j$, we can find out whether there exists an $\ell$ such that $\left\langle a_{\ell}, b_{j}\right\rangle=0$, in time $\operatorname{poly}(d, \log k)$.

Proof. This relies on the observation that we can write $V_{h}$ as

$$
V_{h}=\sum_{r=0}^{d-1} Z_{r} \cdot r^{h}
$$

for

$$
Z_{r}:=\left|\left\{\ell \in[k] \mid\left\langle a_{\ell}, b_{j}\right\rangle=r\right\}\right| .
$$

In other words, the values $V_{h}$ are obtained from the values $Z_{r}$ by multiplication with a Vandermonde matrix. Since this $d \times d$ matrix is invertible and all elements of this matrix and $V_{h}$ are of value at most $k \cdot d^{d}$, we can infer the values $Z_{r}$ from the values $V_{h}$ in poly $(d, \log k)$ time. Indeed, we can compute the inverse of this Vandermonde matrix multiplied by its determinant (so that the resulting matrix contains integer entries) using poly ( $d$ ) operations on integers with poly ( $d, \log k$ ) digits (each such operation takes poly $(d, \log k)$ time). Multiplying the vector of $V_{h}$ 's by this matrix yields $Z_{r}$ 's multiplied by the determinant of the Vandermonde matrix, which can be computed and canceled using poly $(d, \log k)$ operations by manipulating large integers with poly $(d, \log k)$ number of digits. This yields the value $Z_{0}=\left|\left\{\ell \mid\left\langle a_{\ell}, b_{j}\right\rangle=0\right\}\right|$ and thus allows us to decide whether $b_{j} \in B$ is orthogonal to some vector in $A$.

Using Claims 4 and 5 over all Fourier evaluations $\left\{\widehat{x}_{f}\right\}_{f \in F}$ we can determine in time $k$. $\operatorname{poly}(d, \log k)$ whether whether $(A, B)$ contains an orthogonal pair. Thus, for $\delta \in(0,1 / 2)$ an algorithm for non-equispaced Fourier transform running in time $k^{2-\delta^{\prime}} \operatorname{poly}(\log (N / \epsilon))$ for $\epsilon=1 / N$, would imply the existence of a $k^{2-\delta^{\prime}} \operatorname{poly}(d, \log k)$ time algorithm for $\mathrm{OV}_{k, d}$, since $\log (N / \epsilon)=$ $2 \log N=O\left(d^{2}\right) \log M=\operatorname{poly}(d, \log k)$ for any choice of constants $C_{1}, C_{2}>0$. For any constant $c>0$, if dimension $d=c \log k$, this running time can be bounded by $O\left(k^{2-\delta}\right)$ as long as $\delta^{\prime} \geq 2 \delta$, contradicting the Orthogonal Vectors Hypothesis (Conjecture 1). Finally, it remains to note that since $d=c \log k$ and

$$
N=M^{2 d q}=\left(k d^{C_{1} d}\right)^{2 d q}=(c \log k)^{C_{1} C_{2} c^{3} \log ^{3} k},
$$

we have that $2^{c^{\prime}(\log N / \log \log N)^{1 / 3}} \leq k \leq 2^{c^{\prime \prime}(\log N)^{1 / 3}}$ as long as $c^{\prime}$ is sufficiently small as a function of $c, C_{1}, C_{2}$, and $c^{\prime \prime}$ is sufficiently large as required.

## 10 Robust analysis of adaptive aliasing filters

This section is devoted to our technical innovation regarding adaptive aliasing filters. This a delicate analysis of how the filters act on an arbitrary vector. Such a robustification will be useful in order to control the amount of energy a measurement receives from the elements outside of the head. The absence of the properties derived in this section constitutes the restriction that has driven the "exactly $k$-sparse" assumption in KVZ19.

### 10.1 One-dimensional case

We first develop the appropriate machinery for the one-dimensional case. Generalizing the idea to higher dimensions can be done using tensoring, as we shall show in the next subsection. We first present a standalone computation of the Gram matrix of adaptive aliasing filters corresponding to a specific tree $T \subseteq T_{n}^{\text {full }}$.

Lemma 15. (Gram Matrix of adaptive aliasing filters) Consider a tree $T \subseteq T_{n}^{\text {full }}$, and two distinct leaves $v, v^{\prime}$ of $T$. Let $G_{v}$ (resp. $\left.G_{v^{\prime}}\right)$ be the $(v, T)$-isolating (resp. $\left(v^{\prime}, T\right)$-isolating) filter, as per (1). Then,

1. (diagonal terms) the energy of the filter corresponding to $v$ is proportional to $2^{-w_{T}(v)}$. In particular,

$$
\left\|\widehat{G}_{v}\right\|_{2}^{2}:=\sum_{\xi \in[n]}\left|\widehat{G}_{v}(\xi)\right|^{2}=\frac{n}{2^{w_{T}(v)}}
$$

2. (cross terms) the adaptive aliasing filters corresponding to $v$ and $v^{\prime}$ are orthogonal, i.e.

$$
\left\langle\widehat{G}_{v}, \widehat{G}_{v^{\prime}}\right\rangle:=\sum_{\xi \in[n]} \widehat{G}_{v}(\xi) \cdot \widehat{\widehat{G}}_{v^{\prime}}(\xi)=0 .
$$

Proof. We prove each bullet separately. Both bullets follow by symmetry considerations: cancellations that occur either by the fact that roots of unity cancel across a poset of a group, or by the sign change happening to specific complex exponentials at branching points of the tree $T$. The first one uses Kraft's equality.

Proof of Bullet 1. Let $f:=f_{v}$ and $f^{\prime}:=f_{v^{\prime}}$ denote the labels of $v$ and $v^{\prime}$, respectively. By (1), we have

$$
\begin{aligned}
\left|\widehat{G}_{v}(\xi)\right|^{2} & =4^{-w_{T}(v)} \cdot \prod_{\ell \in \operatorname{Anc}(v, T)}\left(1+e^{2 \pi i \frac{\xi-f}{2^{\ell+1}}}\right) \cdot\left(1+e^{-2 \pi i \frac{\xi-f}{2^{\ell+1}}}\right) \\
& =4^{-w_{T}(v)} \cdot \prod_{\ell \in \operatorname{Anc}(v, T)}\left(2+e^{2 \pi i \frac{\xi-f}{2^{\ell+1}}}+e^{-2 \pi i \frac{\xi-f}{2^{\ell+1}}}\right) \\
& =4^{-w_{T}(v)} \cdot \sum_{\substack{ \\
\hline \subseteq \operatorname{Anc}(v, T) \\
S \cap T=\varnothing}} 2^{|\operatorname{Anc}(v, T)|-|S \cup T|} \cdot e^{2 \pi i(\xi-f) \cdot\left(\sum_{\ell \in S} \frac{1}{\left.2^{\ell+1}-\sum_{\ell \in T} \frac{1}{2^{\ell+1}}\right)}\right.} \\
& =4^{-w_{T}(v)} \cdot\left(2^{w_{T}(v)}+\sum_{\substack{S, T \subseteq \operatorname{Anc}(v, T) \\
S \cap T=\varnothing, S \cup T \neq \varnothing}} 2^{w_{T}(v)-|S \cup T|} \cdot e^{2 \pi i(\xi-f) \cdot\left(\sum_{\ell \in S} \frac{1}{\left.2^{\ell+1}-\sum_{\ell \in T} \frac{1}{2^{\ell+1}}\right)}\right)}\right.
\end{aligned}
$$

Note that the expression $\operatorname{expr}_{S, T}=\sum_{\ell \in S} \frac{1}{2^{2^{++1}}}-\sum_{\ell \in T} \frac{1}{2^{\ell+1}}$ inside the complex exponential can be 0 if and only if $S=T$, which is precluded by the fact that $S \cap T=\varnothing, S \cup T \neq \varnothing$. Thus, this gives rise to the exponential $e^{2 \pi i(\xi-f) \cdot \operatorname{expr}, \mathrm{T}}$, which cancels out when summing over all $\xi$. Hence, we obtain that

$$
\sum_{\xi \in[n]}\left|\widehat{G}_{v}(\xi)\right|^{2}=\sum_{\xi \in[n]} 4^{-w_{T}(v)} \cdot\left(2^{w_{T}(v)}+0\right)=\frac{n}{2^{w_{T}(v)}}
$$

Proof of Bullet 2. By (1), we have that

$$
\begin{aligned}
&\left\langle\widehat{G}_{v}, \widehat{G}_{v^{\prime}}\right\rangle=\sum_{\xi \in[n]} \widehat{G}_{v}(\xi) \cdot \overline{\widehat{G}_{v^{\prime}}(\xi)} \\
&=\sum_{\xi \in[n]}\left(\frac{1}{2^{w_{T}(v)}} \prod_{\ell \in \operatorname{Anc}(v, T)}\left(1+e^{2 \pi i \frac{\xi-f}{2^{\ell+1}}}\right)\right) \cdot\left(\frac{1}{2^{w_{T}\left(v^{\prime}\right)}} \prod_{\ell \in \operatorname{Anc}\left(v^{\prime}, T\right)}\left(1+e^{-2 \pi i \frac{\xi-f^{\prime}}{2^{\ell+1}}}\right)\right) \\
&=2^{-w_{T}(v)-w_{T}\left(v^{\prime}\right)} \cdot \sum_{\xi \in[n]} \sum_{\substack{S \in \operatorname{Anc}(v, T) \\
S^{\prime} \subseteq \operatorname{Anc}\left(v^{\prime}, T\right)}} e^{2 \pi i(\xi-f) \cdot \sum_{\ell \in S} \frac{1}{2^{\ell+1}}-2 \pi i\left(\xi-f^{\prime}\right) \cdot \sum_{\ell \in S^{\prime}} \frac{1}{2^{\ell+1}}} \\
&=2^{-w_{T}(v)-w_{T}\left(v^{\prime}\right)} \cdot \sum_{\substack{S \subseteq \operatorname{Anc}(v, T)}} \sum_{\xi \in[n]}^{S^{\prime} \subseteq \operatorname{Anc}\left(v^{\prime}, T\right)} \\
& e^{2 \pi i(\xi-f) \cdot \sum_{\ell \in S} \frac{1}{2^{\ell+1}}-2 \pi i\left(\xi-f^{\prime}\right) \cdot \sum_{\ell \in S^{\prime}} \frac{1}{2^{\ell+1}}} \\
&:=2^{-w_{T}(v)-w_{T}\left(v^{\prime}\right)} \cdot(A+B),
\end{aligned}
$$

where $A$ is sum of the terms that satisfy $S \neq S^{\prime}$, and $B$ is sum of terms satisfying $S=S^{\prime}$. We will show that $A=B=0$ separately. The equality $A=0$ holds by a summation over all $\xi$ and the fact that roots of unity cancel across a poset of a subgroup, whereas the equality $B=0$ by a symmetry argument which exploits the sign change in the lowest common ancestor of $v$ and $v^{\prime}$.

Computing $A$. We will prove that if $S \neq S^{\prime}$ then

$$
\sum_{\xi \in[n]} e^{2 \pi i(\xi-f) \cdot \sum_{\ell \in S} \frac{1}{2^{\ell+1}}-2 \pi i\left(\xi-f^{\prime}\right) \cdot \sum_{\ell \in S^{\prime}} \frac{1}{2^{\ell+1}}=0, ~, ~}
$$

which suffices to establish $A=0$. Note that

$$
\begin{array}{r}
e^{2 \pi i(\xi-f) \cdot \sum_{\ell \in S} \frac{1}{2^{\ell+1}}-2 \pi i\left(\xi-f^{\prime}\right) \cdot \sum_{\ell \in S^{\prime}} \frac{1}{2^{\ell+1}}}= \\
e^{2 \pi i \xi \cdot\left(\sum_{\ell \in S} \frac{1}{2^{\ell+1}}-\sum_{\ell \in S^{\prime}} \frac{1}{2^{\ell+1}}\right)} \cdot g,
\end{array}
$$

where $g=e^{2 \pi i f^{\prime} \cdot \sum_{\ell \in S^{\prime}} \frac{1}{2^{\ell+1}}-2 \pi i f \cdot \sum_{\ell \in S} \frac{1}{2^{\ell+1}}}$ does not depend on $\xi$. Summing over all $\xi \in[n]$ and taking into account that $\sum_{\ell \in S} \frac{1}{2^{\ell+1}}-\sum_{\ell \in S^{\prime}} \frac{1}{2^{\ell+1}} \neq 0$ by the fact that $S \neq S^{\prime}$, yields the desired result (the summation can also be viewed a summation of the roots of unity over $\frac{n}{2^{\max \left\{S \Delta S^{\prime}\right\}}}$ copies of a poset of an additive subgroup of size $2^{\max \left\{S \triangle S^{\prime}\right\}}$, where $\triangle$ denotes symmetric difference of sets).

Computing $B$. This quantity contains only terms corresponding to $S=S^{\prime}$. Note that in this case $S \subseteq \operatorname{Anc}(v, T) \cap \operatorname{Anc}\left(v^{\prime}, T\right)$, and we have

$$
\begin{array}{r}
B=\sum_{S \subseteq \operatorname{Anc}(v, T) \cap \operatorname{Anc}\left(v^{\prime}, T\right)} \sum_{\xi \in[n]} e^{2 \pi i\left(f^{\prime}-f\right) \cdot \sum_{\ell \in S} \frac{1}{2^{\ell+1}}}= \\
n \cdot \sum_{S \subseteq \operatorname{Anc}(v, T) \cap \operatorname{Anc}\left(v^{\prime}, T\right)} e^{2 \pi i\left(f^{\prime}-f\right) \cdot \sum_{\ell \in S} \frac{1}{2^{\ell+1}}} .
\end{array}
$$

Let $u$ be the lowest common ancestor of $v, v^{\prime}$ in tree $T$, i.e. the node on which the paths from the root to those two nodes split. Partition the powerset of $\operatorname{Anc}(v, T) \cap \operatorname{Anc}\left(v^{\prime}, T\right)$ to pair $\left(S, S \cup\left\{l_{T}(u)\right\}\right)$, where $l_{T}(u) \notin S$. We shall prove that

$$
e^{2 \pi i\left(f^{\prime}-f\right) \cdot \sum_{\ell \in S} \frac{1}{2^{\ell+1}}}+e^{2 \pi i\left(f^{\prime}-f\right) \cdot \sum_{\ell \in S \cup\left\{l_{T}(u)\right\}} \frac{1}{2^{\ell+1}}}=0 .
$$

Indeed, by definition of $u$ we have that $\left(f^{\prime}-f\right) \equiv 2^{l_{T}(u)} \bmod 2^{l_{T}(u)+1}$, which in turn gives that $e^{2 \pi i\left(f^{\prime}-f\right) \cdot \frac{1}{2^{l} T^{(u)+1}}}=e^{2 \pi i \frac{l^{l} T^{(u)}}{2^{l} T^{(u)+1}}}=e^{\pi i}=-1$. This gives

$$
\begin{aligned}
& e^{2 \pi i\left(f^{\prime}-f\right) \cdot \sum_{\ell \in S} \frac{1}{2^{\ell+1}}}+e^{2 \pi i\left(f^{\prime}-f\right) \cdot \sum_{\ell \in S \cup\left\{l_{T}(u)\right\}} \frac{1}{2^{\ell+1}}}= \\
& e^{2 \pi i\left(f^{\prime}-f\right) \cdot \sum_{\ell \in S} \frac{1}{2^{\ell+1}}} \cdot\left(1+e^{\left.2 \pi i\left(f^{\prime}-f\right) \cdot \frac{1}{2^{l} T^{(u)+1}}\right)=0 .}\right.
\end{aligned}
$$

Thus, we conclude that $B=0$, which finishes the proof of this Lemma.
The next lemma proves that for any tree $T$, the sum of squared values of adaptive aliasing filters corresponding to all leaves of $T$ is equal to 1 at every frequency. The $(v, T)$-isolating filters for different leaves $v$ of $T$ can have very different behaviors and shapes in the Fourier domain, nevertheless, these filters collectively act as an isometry in the sense that the sum of their squared values is 1 everywhere in the Fourier domain.

Lemma 16. (Total contribution of adaptive aliasing filters to one frequency) Consider a tree $T \subseteq$ $T_{n}^{\text {full. }}$. For every leaf $v$ of $T$, let $G_{v}$ denote the $(v, T)$-isolating filter as per (11), then it holds that

$$
\forall \xi \in[n]: \sum_{v \in \operatorname{LEAVES}(T)}\left|G_{v}(\xi)\right|_{2}^{2}=1 .
$$

Proof. Fix $\xi \in[n]$. By (11), we have

$$
\begin{aligned}
& \quad \sum_{v \in \operatorname{LEAVES}(T)}\left|\widehat{G_{v}}(\xi)\right|^{2}= \\
& \quad \sum_{v \in \operatorname{LEAVES}(T)} 4^{-w_{T}(v)} \cdot \prod_{\ell \in \operatorname{Anc}(v, T)}\left|1+e^{2 \pi i\left(\xi-f_{v}\right) / 2^{\ell+1}}\right|^{2}= \\
& \sum_{v \in \operatorname{LEAVES}(T)} 4^{-w_{T}(v)} \cdot \prod_{\ell \in \operatorname{Anc}(v, T)}\left(2+e^{2 \pi i\left(\xi-f_{v}\right) / 2^{\ell+1}}+e^{-2 \pi i\left(\xi-f_{v}\right) / 2^{\ell+1}}\right)= \\
& \sum_{v \in \operatorname{LEAVES}(T)} 2^{-w_{T}(v)} \cdot \prod_{\ell \in \operatorname{Anc}(v, T)}\left(1+\cos \left(2 \pi\left(\xi-f_{v}\right) / 2^{\ell+1}\right)\right)= \\
& \sum_{v \in \operatorname{LEAVES}(T)} 2^{-w_{T}(v)} \sum_{S \subseteq \operatorname{Anc}(v, T)} \prod_{\ell \in S} \cos \left(2 \pi \frac{\xi-f_{v}}{2^{\ell+1}}\right) .
\end{aligned}
$$

Thus, it suffices to prove that for all $\xi \in[n]$

$$
\begin{equation*}
\sum_{v \in \operatorname{LEAVES}(T)} 2^{-w_{T}(v)} \sum_{S \subseteq \operatorname{Anc}(v, T)} \prod_{\ell \in S} \cos \left(2 \pi \frac{\xi-f_{v}}{2^{\ell+1}}\right)=1 \tag{3}
\end{equation*}
$$

We will implicitly interchange the summation between $v$ and $S$ in (3) and carefully group terms together so that most of them cancel out, due to the sign change in each branching point. In particular, fix a branching point, i.e. a node $u \in T$ with two children. We will estimate the contribution of all sets $S$ such that $\max (S)=l_{T}(u)$ in (3). Let $u_{l}$ be the left child of $u$ in $T$, and let $u_{r}$ be the right child of $u$ in $T$. Note that,

$$
\forall f \in \operatorname{FreqCone}_{T}\left(u_{l}\right), f^{\prime} \in \operatorname{FreqCone}_{T}\left(u_{r}\right): f-f^{\prime} \equiv 2^{l_{T}(u)} \bmod 2^{l_{T}(u)+1}
$$

In turn, this implies that for any $\xi \in[n]$ and any two $f, f^{\prime}$ as above we have: $(\xi-f) \equiv\left(\xi-f^{\prime}\right)+2^{l_{T}(u)}$ $\bmod 2^{l_{T}(u)+1}$, which gives the desired change in the branching point:

$$
\cos \left(2 \pi \frac{\xi-f}{2^{l_{T}(u)+1}}\right)=-\cos \left(2 \pi \frac{\xi-f^{\prime}}{2^{l_{T}(u)+1}}\right) .
$$

Thus, if we let $T_{r}$ and $T_{l}$ denote the subtrees of $T$ rooted at $u_{r}$ and $u_{l}$, respectively, then the total contribution of a set $S$ that satisfies $\max (S)=l_{T}(u)$ and $S \subseteq \operatorname{Anc}(v, T)$ for some leaf $v$ of $T$ to (3) can be expressed as

$$
\begin{aligned}
& \prod_{\ell \in S \backslash\left\{l_{T}(u)\right\}} \cos \left(2 \pi \frac{\xi-f_{u}}{2^{\ell+1}}\right) \cdot\left(\sum_{v \in T_{r}} \frac{1}{2^{w_{T}(v)}}-\sum_{v \in T_{l}} \frac{1}{2^{w_{T}(v)}}\right) \\
& \quad=\prod_{\ell \in S \backslash\{\mathrm{br}\}} \cos \left(2 \pi \frac{\xi-f_{u}}{2^{\ell+1}}\right) \cdot 2^{-w_{T}(u)}\left(\sum_{v \in T_{r}} \frac{1}{2^{w_{T_{r}}(v)}}-\sum_{v \in T_{l}} \frac{1}{2^{w_{T_{l}}(v)}}\right)=0 .
\end{aligned}
$$

The latter holds since $\sum_{v \in T_{r}} \frac{1}{2^{w} T_{r}(v)}=1$ by Kraft's equality; similarly $\sum_{v \in T_{l}} \frac{1}{2^{w} T_{l}(v)}=1$.
Thus, we will get cancellation of the contribution of all non-empty sets $S$ by summing over all branching points. On the other hand, the contribution of the empty set $S=\varnothing$ is exactly $2^{-w_{T}(v)}$, for each leaf $v$. The sum of all those contributions is 1 , again by Kraft's equality, giving the lemma.

### 10.2 Extension to $d$ dimensions

We are now ready to proceed with the generalization of the robustness properties of the adaptive aliasing filters given in Sectior 10.1 to high dimensions. The following lemma states that the isolating filters constructed in Lemma 5, collectively for all leaves, preserve (in particular, do not increase) the energy of a signal.
 $(v, T)$-isolating filter constructed in Lemma 5, then for every $\boldsymbol{\xi} \in[n]^{d}$,

$$
\sum_{v \in \operatorname{LEAVES}(T)}\left|\widehat{G}_{v}(\boldsymbol{\xi})\right|^{2}=1
$$

Proof. The proof is by induction on the dimension $d$.
Base of induction: Lemma 16 precisely proves the inductive claim for $d=1$.
Inductive step: Suppose that the inductive hypothesis holds for $d-1$ dimensional isolating filters. Given this inductive hypothesis, we want to prove that the inductive claim holds for $d$ dimensional filters. Let $T$ be a subtree of $T_{N}^{f u l l}$, where $N=n^{d}$. For every leaf $v$ of tree $T$, let $v_{0}, v_{1}, \cdots v_{l}$ denote the path from root to $v$ where $v_{0}$ is the root and $v_{l}=v$. We let $p_{v}$ denote a vertex in $T$, defined as

$$
p_{v}:=\left\{\begin{array}{ll}
v_{\log _{2} n} & \text { if } l_{T}(v) \geq \log _{2} n \\
v & \text { otherwise }
\end{array} .\right.
$$

Now, we construct the tree $T^{*}$ by making a copy of the tree $T$ and then removing every node which is at distance more than $\log _{2} n$ from the root. Let the nodes of $T^{*}$ be labeled by projecting the labels of $T$ to their first coordinate as follows,

$$
\text { for every node } u \in T^{*}: f_{u}=f_{1} \text {, where }\left(f_{1}, f_{2}, \cdots f_{d}\right) \text { is the label of } u \text { in } T \text {. }
$$

One can easily verify that the set $P:=\left\{p_{v}: v \in \operatorname{LEAVES}(T)\right\}$ specifies the set Leaves $\left(T^{*}\right)$. For every $u \in P$ let $H_{u}$ be a $\left(u, T^{*}\right)$-isolating filter, constructed as in Lemma 5 .

Moreover, for every leaf $u \in P$ we define $T_{u}$ to be a copy of the subtree of $T$ which is rooted at $u$. We label the nodes of the tree $T_{u}$ by projecting the labels of $T$ to their last $d-1$ coordintates as follows,

$$
\text { for every node } z \in T_{u}: \boldsymbol{f}_{z}=\left(f_{2}, f_{3}, \cdots f_{d}\right) \text {, where }\left(f_{1}, f_{2}, \cdots f_{d}\right) \text { is the label of } u \text { in } T \text {. }
$$

For every leaf $v$ of $T$, let $\widehat{Q}_{v}$ be the Fourier domain $\left(v, T_{p_{v}}\right)$-isolating filter constructed in Lemma 5 . Note that in case $p_{v}=v$, the tree $T_{p_{v}}$ will be empty and by convention we define our $\left(v, T_{p_{v}}\right)$ isolating filter to be $\widehat{Q}_{v} \equiv 1$. Therefore, using these definitions, for every leaf $v \in \operatorname{LEAVES}(T)$, the $(v, T)$-isolating filter $\widehat{G}_{v}$ constructed in Lemma 5 satisfies

$$
\widehat{G}_{v}(\boldsymbol{\xi}) \equiv H_{p_{v}}\left(\xi_{1}\right) \cdot Q_{v}\left(\xi_{2}, \xi_{3}, \ldots \xi_{d}\right)
$$

for every $\boldsymbol{\xi}=\left(\xi_{1}, \xi_{2}, \ldots, \xi_{d}\right) \in[n]^{d}$. Hence, we can write

$$
\begin{aligned}
\sum_{v \in \operatorname{LEAVES}(T)}\left|\widehat{G}_{v}(\boldsymbol{\xi})\right|^{2} & =\sum_{v \in \operatorname{LEAVES}(T)}\left|H_{p_{v}}\left(\xi_{1}\right) \cdot Q_{v}\left(\xi_{2}, \xi_{3}, \ldots \xi_{d}\right)\right|^{2} \\
& =\sum_{u \in P} \sum_{v \in \operatorname{LEAVES}(T)}^{\text {s.t. } p_{v}=u} \\
& \left|H_{u}\left(\xi_{1}\right)\right|^{2} \cdot\left|Q_{v}\left(\xi_{2}, \xi_{3}, \ldots \xi_{d}\right)\right|^{2} \\
& =\sum_{u \in P}\left|H_{u}\left(\xi_{1}\right)\right|^{2} \sum_{\substack{v \in \operatorname{LEAVES}(T) \\
\text { s.t. } p_{v}=u}}\left|Q_{v}\left(\xi_{2}, \xi_{3}, \ldots \xi_{d}\right)\right|^{2} .
\end{aligned}
$$

We proceed by proving that for every $u \in P, \sum_{\substack{v \in \operatorname{LEAAES}(T) \\ \text { s.t. } p_{v}(u)}}\left|Q_{v}\left(\xi_{2}, \xi_{3}, \ldots \xi_{d}\right)\right|^{2}=1$. Recall that for every leaf $v \in \operatorname{leaves}(T), Q_{v}$ is a $\left(v, T_{p_{v}}\right)$-isolating filter, constructed in Lemma 5. Therefore, for every leaf $v$ of $T$ such that $p_{v}=u, Q_{v}$ is indeed a $\left(v, T_{u}\right)$-isolating filter as per the construction of Lemma 5. Hence,

$$
\sum_{\substack{v \in \operatorname{LEAVES}(T) \\ \text { s.t. } p_{v}=u}}\left|Q_{v}\left(\xi_{2}, \xi_{3}, \ldots \xi_{d}\right)\right|^{2}=\sum_{v \in \operatorname{LEAVES}\left(T_{u}\right)}\left|Q_{v}\left(\xi_{2}, \xi_{3}, \ldots \xi_{d}\right)\right|^{2}
$$

Now we can invoke the inductive hypothesis because $T_{u}$ is a subtree of $T_{N^{\prime}}^{f u l l}$ where $N^{\prime}=n^{d-1}$. therefore,

$$
\sum_{\substack{v \in \operatorname{LEAVES}(T) \\ \text { s.t. } p_{v}=u}}\left|Q_{v}\left(\xi_{2}, \xi_{3}, \ldots \xi_{d}\right)\right|^{2}=\sum_{v \in \operatorname{LEAVES}\left(T_{u}\right)}\left|Q_{v}\left(\xi_{2}, \xi_{3}, \ldots \xi_{d}\right)\right|^{2}=1
$$

Consequently, we have,

$$
\sum_{v \in \operatorname{LEAVES}(T)}\left|\widehat{G}_{v}(\boldsymbol{\xi})\right|^{2}=\sum_{u \in P}\left|H_{u}\left(\xi_{1}\right)\right|^{2}=\sum_{u \in \operatorname{LEAVES}\left(T^{*}\right)}\left|H_{u}\left(\xi_{1}\right)\right|^{2}=1,
$$

where the last equality follows because $H_{u}$ is a $\left(u, T^{*}\right)$-isolating filter as per the construction of Lemma 4 and hence by Lemma $16, \sum_{u \in \operatorname{LEAVES}\left(T^{*}\right)}\left|H_{u}\left(\xi_{1}\right)\right|^{2}=1$. This completes the inductive proof and ergo the Lemma.

We readily find that the following corollary of the above lemma holds,
Corollary 1. The Fourier domain isolating filter $\widehat{G}$ constructed in Lemma 5 satisfies $\|\widehat{G}\|_{\infty} \leq 1$.

## 11 Robust Sparse Fourier Transform I

The section is devoted to proving our first result on robust Sparse Fourier transforms, which illustrates techniques II to IV and partially technique I. We first remind the reader about the high SNR regime we consider.
$k$-High SNR Regime. A vector $x:[n]^{d} \rightarrow \mathbb{C}$ satisfies the $k$-high SNR assumption, if there exists vectors $w, \eta:[n]^{d} \rightarrow \mathbb{C}$ such that i) $\widehat{x}=\widehat{w}+\widehat{\eta}$, ii) $\operatorname{supp}(\widehat{w}) \cap \operatorname{supp}(\widehat{\eta})=\varnothing$, iii) $|\operatorname{supp}(\widehat{w})| \leq k$ and iv) $\left|\widehat{w}_{f}\right| \geq 3 \cdot\|\widehat{\eta}\|_{2}$, for every $f \in \operatorname{supp}(\widehat{w})$. In the rest of this section we prove the following main theorem.

Theorem 10 (Robust Sparse Fourier Transform). Given oracle access to $x:[n]^{d} \rightarrow \mathbb{C}$ with $x=w+\eta$ in $k$-high SNR model and parameter $\epsilon>0$, we can find using

$$
m=\widetilde{O}\left(k^{7 / 3}+\frac{k^{2}}{\epsilon}\right)
$$

samples from $x$ and in $\widetilde{O}\left(\frac{k^{3}}{\epsilon}\right)$ time a signal $\widehat{\chi}$ such that

$$
\|\widehat{\chi}-\widehat{x}\|_{2}^{2} \leq(1+\epsilon) \cdot\|\widehat{\eta}\|_{2}^{2}
$$

with high probability in $N$.
For every tree $T$ and node $v \in T$, we let $\widehat{x}_{v}$ be the vector $\widehat{x}_{\text {FreqCone }(v)}$, i.e. signal $\widehat{x}$ supported on frequencies in the frequency cone of $v$ and zeroed out everywhere else. At all times, for every $v \in T$, our algorithm maintains a signal $\widehat{\chi}_{v}:[n]^{d} \rightarrow \mathbb{C}$ that is supported on FreqCone $_{T}(v)$. This signal will serve as our estimate for $\widehat{w}_{v}$. Initially, all these vectors are going to be $\{0\}^{n^{d}}$. The execution of our algorithm ensures that we can always keep sparse representations of those vectors. Parameters and variables $n, d$ and $N=n^{d}$ are treated as global.

Furthermore, for any signal $y:[n]^{d} \rightarrow \mathbb{C}$ and parameter $\mu \geq 0$ we define

$$
\begin{equation*}
\operatorname{HEAD}_{\mu}(y):=\left\{j \in[n]^{d}:\left|y_{j}\right| \geq 3 \mu\right\} . \tag{4}
\end{equation*}
$$

Under this notation, we are interested in recovering the set $\operatorname{HEAD}_{\|\hat{\eta}\|_{2}}(\widehat{x})$, as well as obtain accurate estimations for the values of $\widehat{x}$ on frequencies in set $\operatorname{HEAD}_{\|\widehat{\eta}\|_{2}}(\widehat{x})$. Using the notion of $\operatorname{HEAD}_{\mu}(y)$, one can see that a signal $x$ is in the $k$-high SNR regime iff there exists a $\mu>0$ such that $\left|\operatorname{HEAD}_{\mu}(\widehat{x})\right| \leq k$ and $\mu \geq\left\|\widehat{x}-\widehat{x}_{\text {HEAD }_{\mu}(\widehat{x})}\right\|_{2}$.

At all times, we keep a set Est, corresponding to the coordinates in $\operatorname{supp}(\widehat{w})$ that we have estimated. We define $L_{v}:=\operatorname{FreqCone}_{T}(v) \cap(\operatorname{supp}(\widehat{w}) \backslash \mathrm{Est})$, which corresponds to the unestimated coordinates in the support of $w$ that lie in the frequency cone of $v$.

Our main algorithm consists of an outer loop that we call RobustSparsefft and an inner loop that we call RobustPromiseSFT. Our algorithm also makes use of an auxiliary primitive for estimating the values of located frequencies as well as a primitive for testing whether a signal is "heavy" (meaning that it contains a head element). In the rest of this section we first give the primitives Estimate and HeavyTest together with the guarantee on their performance. Then we present the main algorithm and prove its performance. The HeavyTest routine is analogous to ZeroTesT from Section 6. However, the RIP property alone does not suffice (and hence we cannot pick a deterministic collection of samples). Instead, we use a random collection of samples, which suffices for upper bounding the contribution of the tail while simultaneously satisfying RIP.

### 11.1 Computational Primitives for the Robust Setting

In this subsection we give some of the primitives that will be used in our algorithms. The proof of correctness of these primitives is postponed to subsection 11.3 .

The very first primitive we present is HeavyTest, see Algorithm 6. This primitive performs a test on the signal to detect whether a given frequency cone contains heavy elements or not.

```
Algorithm 6 Test whether \(v\) is a frequency-active node, i.e. \(\left\|(\widehat{x-\chi})_{v}\right\|_{2}>2\|\widehat{\eta}\|_{2}\)
    procedure \(\operatorname{HEAvyTest}(x, \widehat{\chi}, T, v, m, \theta)\)
        \(\boldsymbol{f} \leftarrow \boldsymbol{f}_{v}\)
        \(\left(G_{v}, \widehat{G}_{v}\right) \leftarrow \operatorname{MultiDimFilter}(T, v, n)\)
        \(/ /(v, T)\)-isolating filters as per Lemma 5
        for \(z=1\) to \(32 \log N\) do
            \(\operatorname{RIP}_{m}^{z} \leftarrow\) Multiset of \(m\) i.i.d. uniform samples from \([n]^{d}\)
            \(h_{\Delta}^{z} \leftarrow \sum_{\boldsymbol{\xi} \in[n]^{d}}\left(e^{2 \pi i \frac{\boldsymbol{\xi}^{\top} \Delta}{n}} \cdot \widehat{\chi}(\boldsymbol{\xi}) \cdot \widehat{G}_{v}(\boldsymbol{\xi})\right)\) for every \(\Delta \in \operatorname{RIP}_{m}^{z}\)
            \(H^{z} \leftarrow \frac{1}{\left|\operatorname{RIP}_{m}^{z}\right|} \sum_{\Delta \in \mathrm{RIP}_{m}^{z}}\left|N \cdot \sum_{\boldsymbol{j} \in[n]^{d}} G_{v}(\Delta-\boldsymbol{j}) \cdot x(\boldsymbol{j})-h_{\Delta}^{z}\right|^{2}\)
        if \(\operatorname{MEDIAN}_{z \in[32 \log N]}\left\{H^{z}\right\} \leq \theta\) then
            \(/ / \theta=5\|\widehat{\eta}\|_{2}^{2}\).
            return False
        else
            return True
```

Lemma 18 (HeavyTest guarantee). Consider signals $x, \widehat{\chi}:[n]^{d} \rightarrow \mathbb{C}$ and an arbitrary subtree $T$ of $T_{N}^{\text {full }}$. For an arbitrary leaf $v$ of $T$, let $\widehat{y}:=(\widehat{x}-\widehat{\chi}) \cdot \widehat{G}_{v}$, where $\widehat{G}_{v}$ be the Fourier domain $(v, T)$-isolating filter constructed in Lemma 5. Then the following statements hold, for any $\theta>0$ :

- If there exists a set $S \subseteq[n]^{d}$ such that $\left\|\widehat{y}_{S}\right\|_{2}^{2}>\frac{11 \theta}{10}$, then $\operatorname{HEAVYTEST}(x, \widehat{\chi}, T, v, m, \theta)$ (Algorithm (6) outputs True with probability $1-\frac{1}{N^{16}}$, provided that $m$ is a large enough integer satisfying

$$
m=\Omega\left(|S| \cdot \frac{\|\widehat{y}\|_{2}^{2}}{\left\|\widehat{y}_{S}\right\|_{2}^{2}} \cdot \log ^{2}|S| \log N\right)
$$

- If $\|\widehat{y}\|_{2}^{2} \leq \theta / 5$, then HeavyTest outputs False with probability $1-\frac{1}{N^{5}}$.
- The sample complexity of this procedure is $\widetilde{O}\left(2^{w_{T}(v)} \cdot m\right)$.
- The runtime of the HeavyTest procedure is $\widetilde{O}\left(\|\widehat{\chi}\|_{0} \cdot m+2^{w_{T}(v)} \cdot m\right)$.

Next, we present the second auxiliary primitive Estimate in Algorithm 6 ,
Lemma 19 (Estimate guarantee). Consider signals signals $x, \widehat{\chi}:[n]^{d} \rightarrow \mathbb{C}$, a subtree $T$ of $T_{N}^{\text {full }}$, and an integer parameter $m$. For a subset $S \subseteq \operatorname{leaves}(T)$, the procedure $\operatorname{Estimate}(x, \widehat{\chi}, T, S, m)$ (see Algorithm $(7)$ outputs $\left\{\widehat{H}_{v}\right\}_{v \in S}$ such that

$$
\operatorname{Pr}\left[\sum_{v \in S}\left|\widehat{H}_{v}-(\widehat{x-\chi})\left(\boldsymbol{f}_{v}\right)\right|^{2} \leq \frac{16}{m} \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)}|(\widehat{x-\chi})(\boldsymbol{\xi})|^{2}\right] \geq 1-\frac{|S|}{N^{8}} .
$$

```
Algorithm 7 For \(S \subseteq T\), estimates \((\widehat{x}-\widehat{\chi})_{S}\) by isolating \(S\) from every node in \(T\).
    procedure Estimate \((x, \widehat{\chi}, T, S, m)\)
        for \(v \in S\) do
            \(\boldsymbol{f} \leftarrow \boldsymbol{f}_{v}\)
            \(\left(G_{v}, \widehat{G}_{v}\right) \leftarrow \operatorname{MultiDimFilter}(T, v, n)\)
            \(/ /(v, T)\)-isolating filters as per Lemma 5
            for \(z=1\) to \(16 \log N\) do
                \(\operatorname{RIP}_{m}^{z} \leftarrow\) Multiset of \(B\) i.i.d. uniform samples from \([n]^{d}\)
                \(h_{v}^{z} \leftarrow \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} e^{-2 \pi i \frac{f^{\top} \Delta}{n}} \sum_{\boldsymbol{\xi} \in[n]^{d}} e^{2 \pi i \frac{\boldsymbol{\xi}^{\top} \Delta}{n}} \cdot \widehat{\chi}(\boldsymbol{\xi}) \cdot \widehat{G}_{v}(\boldsymbol{\xi})\)
            \(H_{v}^{z} \leftarrow \frac{1}{\left|\operatorname{RIP}_{m}^{z}\right|}\left(N \cdot \sum_{\Delta \in \operatorname{RIP}_{m}^{z}}\left(e^{-2 \pi i \frac{f^{\top} \Delta}{n}} \sum_{\boldsymbol{j} \in[n]^{d}} G_{v}(\Delta-\boldsymbol{j}) \cdot x(\boldsymbol{j})\right)-h_{v}^{z}\right)\)
            \(\widehat{H}_{v} \leftarrow \operatorname{MEDIAN}_{z \in[16 \log N]}\left\{H_{v}^{z}\right\}\)
            //Median of real and imaginary parts separately
        return \(\left\{\widehat{H}_{v}\right\}_{v \in S}\)
```

The sample complexity of this procedure is $\widetilde{O}\left(m \cdot \sum_{v \in S} 2^{w_{T}(v)}\right)$ and the runtime of the procedure is $\widetilde{O}\left(m \cdot \sum_{v \in S} 2^{w_{T}(v)}+|S| \cdot m \cdot\|\widehat{\chi}\|_{0}\right)$.

Lastly, we need the following primitive whose objecive is to find a subset of identified leaves that are cheap to estimate on average.

Claim 6 (ExtractCheapSubset guarantee). For every subtree $T$ of $T_{N}^{\text {full }}$ and every subset $S \subseteq$ Leaves $(T)$ that satisfies $\sum_{u \in S} 2^{-w_{T}(u)} \geq \frac{1}{2}$, the primitive ExtractCheapSubset $(T, S)$ (see bottom of Algorithm 9) outputs a non-empty subset $L \subseteq S$ such that

$$
|L| \cdot(8+4 \log |S|) \geq \max _{v \in L} 2^{w_{T}(v)}
$$

### 11.2 Main Algorithm

In this subsection we present our main sparse FFT algorithm. The algorithms consists of an outer loop and an inner loop. The outer loop, called RobustSparseFT, always maintains a vector $\widehat{\chi}$ a tree Frontier such that

$$
\operatorname{HEAD}_{\mu}(\widehat{x}-\widehat{\chi}) \subseteq \cup_{u \in \operatorname{Frontier}} \operatorname{Freq} \operatorname{Cone}(u) .
$$

At every point in time, we explore the frequency cones of the low-weight Frontier by running the RobustPromiseSFT algorithm. For the pseudocodes of the routines RobustPromiseSFT and RobustSparseFT, see Algorithms 8 and 9 , respectively.

Overview of RobustPromiseSFT (Algorithm8): Consider an invocation of RobustPromis$\operatorname{ESFT}\left(x, \widehat{\chi}_{i n}\right.$, SideTree, $\left.v, b, k, \mu\right)$. Suppose that $\widehat{y}:=\widehat{x}-\widehat{\chi}_{i n}$ is a signal in the $k$-high SNR regime, i.e., $\widehat{y}$ has $k$ heavy frequencies and the value of each such heavy frequency is at least 3 times higher than the tail's norm. More formally, let HEAD $\subseteq[n]^{d}$ denote the set of heavy (head) frequencies of $\widehat{y}$ and suppose that $|\operatorname{HEAD}| \leq k$, and the tail norm of $\widehat{y}$ satisfies $\left\|\widehat{y}-\widehat{y}_{\text {HEAD }}\right\|_{2} \leq \mu$ and additionally suppose that $|\widehat{y}(\boldsymbol{f})| \geq 3 \mu$ for every $\boldsymbol{f} \in$ head. If SideTree fully captures the heavy frequencies of $\widehat{y}$, i.e., HEAD $\subseteq \operatorname{supp}(\operatorname{SideTreE})$, and the number of heavy frequencies in frequency cone of node $v$


Figure 5: Illustration of an instance of RobustPromiseSFT (Algorithm 8). This procedure takes in a tree SideTree (shown with thin edges) together with a leaf $v \in$ leaves(SideTree) and adaptively explores/constructs the subtree $T$ rooted at $v$ to find all heavy frequencies that lie in FreqCone Sidetree $(v)$. If head denotes the set of heavy frequencies, then the algorithm finds head $\cap \operatorname{FreqCone}_{\text {Sidetree }}(v)$ by exploring $T$. Once the identity of a leaf is fully revealed, the algorithm adds that leaf to the set Marked. When the number of marked leaves grows to the point where marked frequencies can be estimated cheaply, our algorithm estimates them all in a batch, subtracts off the estimated signal, and removes all corresponding leaves from $T$.
is bounded by $b$, i.e., $\mid \operatorname{Head} \cap \operatorname{Freq}^{\operatorname{Cone}}$ Sidetree $(v) \mid \leq b$, then RobustPromiseSFT finds a signal $\widehat{\chi}_{v}$ such that $\operatorname{supp}\left(\widehat{\chi}_{v}\right)=\operatorname{HEAD} \cap \operatorname{FreqCone}_{\text {SideTree }}(v):=S$ and $\left\|\widehat{y}_{S}-\widehat{\chi}_{v}\right\|_{2}^{2} \leq \frac{\mu^{2}}{20}$. An example of the input tree SideTree is illustrated in Figure 5 with thin solid black edges. Additionally, one can see node $v$ which is a leaf of SideTree in this figure.

Algorithm 8 recovers heavy frequencies in the subree of $v$, i.e., $S=\operatorname{HEAD}^{\text {FFreqCone }}{ }_{\text {Sidetree }}(v)$, by iteratively exploring the subtree of SideTree rooted at $v$, which we denote by $T$, and simultaneously updating $\widehat{\chi}_{v}$. We show an example of subtree $T$ at some iteration of our algorithm in Figure 5 with thick solid edges. Our algorithm, in all iterations, maintains a subtree $T$ such that the frequency cone of each of its leaves contain at least one head element, i.e.,

$$
\begin{equation*}
\text { for every } u \in \operatorname{LEAVES}(T): \operatorname{Freq}^{\operatorname{Cone}} \mathrm{SideTreevt}(u) \cap \operatorname{HEAD} \neq \varnothing . \tag{5}
\end{equation*}
$$

We demonstrate, in Figure 5, the leaves that correspond to set $S=$ Head $\cap \operatorname{FreqCone}_{\text {Sidetree }(v)}$ via leaves at bottom level of the subtree rooted at $v$. One can easily verify (5) in this figure by noting that the frequency cone of each leaf of $T$ contains at least one element from the set HEAD. Additionally, at every iteration of the algorithm, the union of all frequency cones of subtree $T$ captures all heavy frequencies that are not recovered yet, i.e.,

$$
\begin{equation*}
S \backslash \operatorname{supp}\left(\widehat{\chi}_{v}\right) \subseteq \operatorname{supp}(\operatorname{SidETREE} \cup T) . \tag{6}
\end{equation*}
$$

In Figure 5, we show the set of fully recovered leaves (frequencies), i.e., $\operatorname{supp}\left(\widehat{\chi}_{v}\right)$, using red thin dashed subtrees. These frequencies are subtracted from the residual signal $\widehat{y}-\widehat{\chi}_{v}$ and their corresponding leaves are removed from subtree $T$, as well. One can verify that condition 6 holds in the example depicted in Figure 5. Moreover, the estimated value of every frequency that is recovered so far, is accurate up to an average error of $\frac{\mu}{\sqrt{20 b}}$. More precisely, in every iteration of the algorithm the following property is maintained,

$$
\begin{equation*}
\frac{\sum_{\boldsymbol{f} \in \operatorname{supp}\left(\widehat{\chi}_{v}\right)}\left|\widehat{y}(\boldsymbol{f})-\widehat{\chi}_{v}(\boldsymbol{f})\right|^{2}}{\left|\operatorname{supp}\left(\widehat{\chi}_{v}\right)\right|} \leq \frac{\mu^{2}}{20 b} . \tag{7}
\end{equation*}
$$

At the start of the procedure, subtree $T$ is initialized to be the leaf $v$, i.e., $T=\{v\}$. Moreover, we initialize $\widehat{\chi}_{v} \equiv 0$. Trivially, these initial values satisfy (5), (6), and (7). The algorithm also keeps a subset of leaves denoted by Marked that contains the leaves of $T$ that are fully identified, that is the set of leaves that are at the bottom level and hence there is no ambiguity in their frequency content. Initially Marked is empty. We show the set of marked leaves in Figure 5 using blue squares. The algorithm operates by picking the unmarked leaf of $T$ that has the smallest weight. Then the algorithm explores the children of this node by running HeavyTest on them to detect if any heavy frequencies lie in their frequency cone. If a child passes the HeavyTest the algorithm updates tree $T$ by adding that child to $T$. As soon as a leaf of $T$ gets to the bottom level and becomes a leaf of $T_{N}^{\text {full }}$, the algorithm marks it, i.e., adds that leaf to the Marked set. It can be seen in Figure 5 that all marked leaves are at the bottom level of the tree. The marked leaves need not be explored any further because they are at the bottom level and their frequency content is fully identified. These operations ensure that the invariants (5), (6), and (7) are maintained.

Once the size of set Marked grows sufficiently, the algorithm estimates the values of the marked frequencies. More precisely, at some point, the size of MARKED will be comparable to the maximum weight of the leaves it contains, and when this happens, the values of all marked frequencies can be estimated cheaply. Hence, when Marked is a cheap to estimate set of leaves, our algorithm esimates those frequencies in a batch up to an average error of $\frac{\mu}{20 b}$, updates $\widehat{\chi}_{v}$ accordingly and removes all estimated (Marked) leaves from $T$. This ensures that invariants (5), (6), and (7) are maintained. The estimated leaves are illustrated in Figure 5 using red thin dashed subtrees. We also demontrate the subtrees of $T$ that contain HEAD element and are yet to be explored by our algorithm using gray cones and dashed edges in Figure 5. The gray cone means that there are heavy elements in that frequency cone that need to be identified as that node has not reached the bottom level yet.

Finally, the algorithm keeps tabs on the runtime it spends and ensures that even if the input signal does not satisfy the preconditions for successful recovery, in particular if |HEAD $\cap$ $\operatorname{FreqCone}_{\text {Sidetree }}(v) \mid>b$, the runtime stays bounded. Additionally, the algorithm performs a quality control by running a HeavyTest on the residual and if the recovered signal is not correct due to violation of some preconditions, it reflects this in its output.

Overview of Algorithm 9: Consider an invocation of RobustSparseFT $(x, k, \epsilon, \mu)$. Suppose that $\widehat{x}$ is a signal in the $k$-high SNR regime, i.e., $\widehat{x}$ has $k$ heavy frequencies and the value of each such heavy frequency is at least 3 times higher than the tail's norm. More formally, let HEAD $\subseteq[n]^{d}$ denote the set of heavy (head) frequencies of $\widehat{x}$ and suppose that $\mid$ HEAD $\mid \leq k$, and the tail norm of $\widehat{x}$ satisfies $\left\|\widehat{x}-\widehat{x}_{\text {HEAD }}\right\|_{2} \leq \mu$ and additionally suppose that $|\widehat{x}(\boldsymbol{f})| \geq 3 \mu$ for every $\boldsymbol{f} \in$ HEAD. The primitive RobustSparseFT finds a signal $\widehat{\chi}$ such that $\|\widehat{x}-\widehat{\chi}\|_{2}^{2} \leq(1+\epsilon) \mu^{2}$.

Algorithm 9 recovers heavy frequencies of the input signal $\widehat{x}$, i.e., HEAD, by iteratively exploring the tree that captures the heavy frequencies, which we denote by Frontier, and simultaneously

```
Algorithm 8 The Inner Loop of Sparse FFT Algorithm
    procedure RobustPromiseSFT( \(x, \widehat{\chi}_{i n}\), SideTree, \(v, b, k, \mu\) )
        // \(\mu\) : upper bound on tail norm \(\|\eta\|_{2}\)
        \(\widehat{\chi}_{\text {out }} \leftarrow\{0\}^{n^{d}} \quad \triangleright\) Sparse vector to approximate \(\left(\widehat{x}-\widehat{\chi}_{\text {in }}\right)_{\text {FreqCone }_{\text {SIIETREE }}(v)}\)
        MARKED \(\leftarrow \varnothing \quad \triangleright\) Set of marked nodes to be estimated later
        Let \(T\) denote the subtree of SideTree rooted at \(v\) - i.e., \(T \leftarrow\{v\}\)
        repeat
            if \(|\operatorname{leaves}(T)|+\left\|\widehat{\chi}_{v}\right\|_{0}>b\) then
                return (False, \(\{0\}^{n^{d}}\) ) \(\triangleright\) Exit because budget of \(v\) is wrong
            if MARKED \(\neq \varnothing\) and \(\frac{\mid \text { MARKED }{ }^{\max _{u \in \operatorname{MARKED}} 2^{2}(u)}}{2} \geq \frac{1}{4+2 \log b}\) then
                //The set of marked frequencies that are cheap to estimate on average
                \(\left\{\widehat{H}_{u}\right\}_{u \in \text { Marked }} \leftarrow \operatorname{Estimate}\left(x, \widehat{\chi}_{\text {in }}+\widehat{\chi}_{\text {out }}\right.\), SideTree \(\cup T\), Marked,\(\left.\frac{368 b}{\mid \text { Marked } \mid}\right)\)
                for \(u \in\) MARKED do
                    \(\widehat{\chi}_{\text {out }}\left(\boldsymbol{f}_{u}\right) \leftarrow \widehat{H}_{u}\)
                    Remove node \(u\) from \(T\)
                Marked \(\leftarrow \varnothing\)
                    continue
            \(z \leftarrow \operatorname{argmin}_{u \in \operatorname{LEAVES}(T) \backslash \operatorname{Marked}} w_{T}(u)\)
            //Find the minimum weight unmarked leaf in \(T\)
            if \(z \in \operatorname{LEAVES}\left(T_{N}^{\text {full }}\right)\) then
                //Frequency \(\boldsymbol{f}_{z}\) and leaf \(z\) are fully identified
                    Marked \(\leftarrow \operatorname{Marked} \cup\{z\}\)
            else
                \(z_{\text {left }}:=\) left child of \(z\) and \(z_{\text {right }}:=\) right child of \(z\)
                \(T^{\prime} \leftarrow T \cup\left\{z_{\text {left }}, z_{\text {right }}\right\} \quad \triangleright\) Explore children of \(z\)
                \(\operatorname{Heavy}_{\ell} \leftarrow \operatorname{HeavyTest}\left(x, \widehat{\chi}_{i n}+\widehat{\chi}_{v}, \operatorname{SideTree} \cup T^{\prime}, z_{\text {left }}, O\left(b \log ^{3} N\right), 6 \mu^{2}\right)\)
                \(\operatorname{Heavy}_{r} \leftarrow \operatorname{HeavyTest}\left(x, \widehat{\chi}_{i n}+\widehat{\chi}_{v}, \operatorname{SideTree} \cup T^{\prime}, z_{\text {right }}, O\left(b \log ^{3} N\right), 6 \mu^{2}\right)\)
                if Heavy \({ }_{\ell}\) then
                        Add \(z_{\text {left }}\) as the left child of \(z\) to tree \(T\)
                    if \(\mathrm{Heavy}_{r}\) then
                    Add \(z_{\text {right }}\) as the right child of \(z\) to tree \(T\)
                if \(z \neq v\) and both Heavy \({ }_{\ell}\) and \(\mathrm{Heavy}_{r}\) are False then
                    return (False, \(\{0\}^{n^{d}}\) ) \(\quad\) Exit because budget of \(v\) is wrong
        until \(T\) has no leaves besides \(v\)
        if HeavyTest \(\left(x, \widehat{\chi}_{i n}+\widehat{\chi}_{v}\right.\), SideTree, \(\left.v, O\left(k \log ^{3} N\right), 6 \mu^{2}\right)\) then
            \(/ /\) The number of heavy coordinates in \(\mathrm{FreqCone}_{\text {Sidetree }}(v)\) is more than \(b\)
            return (False, \(\{0\}^{n^{d}}\) )
        else
            return (True, \(\widehat{\chi}_{\text {out }}\) )
```

updating the proxy signal $\widehat{\chi}$. At the begining of the procedure, tree Frontier only consists of a root and will be dynamically changing throughout the execution of our algorithm. Moreover, $\widehat{\chi}$ is initially zero. The algorithm also maintains a subset of leaves denoted by Marked that contains the leaves of Frontier that are fully identified, that is the set of leaves that are at the bottom
level and hence there is no ambiguity in their frequency content (there is exactly one element in frequency cone of marked leaves). Tree Frontier, in all iterations of our algorithm, maintains the invariant that the frequency cone of each of its leaves contain at least one head element and furthermore the frequency cone of each of its unmarked leaves contain at least $b+1$ head element, where $b=k^{1 / 3}$, i.e.,

Additionally, at every iteration of the algorithm, the union of all frequency cones of tree Frontier captures all heavy frequencies that are not recovered yet, i.e.,

$$
\begin{equation*}
\operatorname{HEAD} \backslash \operatorname{supp}(\widehat{\chi}) \subseteq \operatorname{supp}(\text { Frontier }) . \tag{9}
\end{equation*}
$$

The set of fully recovered leaves (frequencies), i.e., $\operatorname{supp}\left(\widehat{\chi}_{v}\right)$, are subtracted from the residual signal $\widehat{x}-\widehat{\chi}$ by our algorithm and their corresponding leaves get removed from Frontier, as well. Moreover, the estimated value of every frequency that is recovered so far, is accurate up to an average error of $\sqrt{\frac{\epsilon}{k}} \cdot \mu$. More precisely, in every iteration of the algorithm the following property is maintained,

$$
\begin{equation*}
\frac{\sum_{\boldsymbol{f} \in \operatorname{supp}(\hat{\chi})}|\widehat{x}(\boldsymbol{f})-\widehat{\chi}(\boldsymbol{f})|^{2}}{|\operatorname{supp}(\widehat{\chi})|} \leq \frac{\epsilon}{k} \cdot \mu^{2} \tag{10}
\end{equation*}
$$

At the start of the procedure, Frontier is initialized to only contain a root, i.e., Frontier $=$ \{root\}. Moreover, we initialize $\widehat{\chi} \equiv 0$. Trivially, these initial values satisfy (8), (9), and (10). Also the set of fully identified leaves Marked is initially empty. The algorithm explores Frontier by picking the unmarked leaf that has the smallest weight, let us call it $v$. Then the algorithm explores the children of this node by running RobustPromiseSFT on them to recover the heavy frequencies that lie in their frequency cone. We denote by $v_{\text {left }}$ and $v_{\text {right }}$ the left and right children of $v$. Let us consider exploration of the left child $v_{\text {left }}$, the right child is exactly the same. If the number of heavy frequencies in the frequency cone of $v_{\text {left }}$ is bounded by $b=k^{1 / 3}$, i.e., $\mid$ HEAD $\cap$ FreqCone $_{\text {Frontieru }\left\{v_{\text {left }}, v_{\text {right }}\right\}}\left(v_{\text {left }}\right) \mid \leq b$, then RobustPromiseSFT recovers every frequency in the set HEAD $\cap$ FreqCone $_{\text {Frontieru }\left\{v_{\text {left }}, v_{\text {right }}\right\}}\left(v_{\text {left }}\right)$ up to average error $\frac{\mu}{\sqrt{20 b}}$. Note that this everage estimation error is not sufficient for achieving the invariant (10), hence, instead of directly using the values that RobustPromiseSFT recovered and update $\widehat{\chi}$ at the newly recovered heavy frequencies, our algorithm adds the leaves corresponding to the recovered set of frequencies, i.e., head $\cap$ FreqCone $_{\text {Frontieru }\left\{v_{\text {left }}, v_{\text {right }}\right\}}\left(v_{\text {left }}\right)$, at the bottom level of Frontier and marks them as fully identified (adds them to Marked). For achieving maximum efficinecy we employ a new lazy estimation scheme, that is, the estimation of values of marked leaves is delayed until there is a large number of marked leaves and thus there exists a subset of them that is cheap to estimate. On the other hand, if the number of head elements in frequency cone of $v_{\text {left }}$ is more than $b$ then RobuSTPromisesFT detects this and notifies our algorithms about it and our algorithm adds node $v_{\text {left }}$ to Frontier. These operations ensure that the invariants (8), (9), and (10) are maintained.

Once the size of set Marked grows sufficiently such that it contains a subset that is cheap to estimate, our algorithm estimates the values of the cheap frequencies. More precisely, at some point, Marked will contains a non-empty subset Cheap such that the values of all frequencies in Cheap can be estimated cheaply and subsequently, our algorithm esimates those frequencies in a batch up to an average error of $\sqrt{\frac{\epsilon}{k}} \cdot \mu$, updates $\widehat{\chi}$ accordingly and removes all estimated (Cheap) leaves from Frontier and Marked. This ensures that invariants (8), (9), and (10) are maintained.

```
Algorithm 9 Robust High-dimensional Sparse FFT Algorithm
    procedure RobustSparseFT \((x, k, \epsilon, \mu)\)
        \(/ / \mu\) is an upper bound on tail norm \(\|\eta\|_{2}\)
        Frontier \(\leftarrow\{\) root \(\}, \boldsymbol{f}_{\text {root }} \leftarrow 0\)
        \(b \leftarrow\left\lceil k^{1 / 3}\right\rceil\)
        \(\widehat{\chi} \leftarrow\{0\}^{n^{d}}\)
        Marked \(\leftarrow \varnothing \quad \triangleright\) Set of fully identified leaves (frequencies)
        repeat
            if \(\sum_{u \in \text { Marked }} 2^{-w_{\text {Fronter }}(u)} \geq \frac{1}{2}\) then
                        Cheap \(\leftarrow\) ExtractCheapSubset (Frontier, Marked)
                //Lazy estimation: We extract from the batch of marked leaves a subset that is cheap
    to estimate on average
            \(\left\{\widehat{H}_{u}\right\}_{u \in \text { Cheap }} \leftarrow \operatorname{Estimate}\left(x, \widehat{\chi}\right.\), Frontier, Cheap, \(\left.\frac{32 k}{\epsilon \cdot \mid \text { Cheap } \mid}\right)\)
                for \(u \in\) CHEAP do
                \(\widehat{\chi}\left(\boldsymbol{f}_{u}\right) \leftarrow \widehat{H}_{u}\)
                Remove node \(u\) from tree Frontier
                    Marked \(\leftarrow\) Marked \(\backslash\) Cheap
                    continue
                \(v \leftarrow \operatorname{argmin}_{u \in \text { Leaves (Frontier) }} \operatorname{Marked} w_{\text {Frontier }}(u)\)
                //pick the minimum weight leaf in Frontier which is not in Marked
                \(v_{\text {left }} \leftarrow\) left child of \(v\) and \(v_{\text {right }} \leftarrow\) right child of \(v\)
                \(T \leftarrow\) Frontier \(\cup\left\{v_{\text {left }}, v_{\text {right }}\right\}\)
                \(\left(\operatorname{IsCorR}_{\text {left }}, \widehat{\chi}_{\text {left }}\right) \leftarrow \operatorname{RobustPromiseSFT}\left(x, \widehat{\chi}, T, v_{\text {left }}, b, k, \mu\right)\)
                \(\left(\operatorname{IsCorR}_{\text {right }}, \widehat{\chi}_{\text {right }}\right) \leftarrow \operatorname{RobustPromiseSFT}\left(x, \widehat{\chi}, T, v_{\text {right }}, b, k, \mu\right)\)
                if \(\mathrm{IsCorR}_{\text {left }}\) then
                    \(\forall \boldsymbol{f} \in \operatorname{supp}\left(\widehat{\chi}_{\text {left }}\right)\), add the unique leaf corresponding to \(\boldsymbol{f}\) to Frontier and Marked
                else
                    Add \(v_{\text {left }}\) to Frontier
                if IsCorR \(_{\text {right }}\) then
                    \(\forall \boldsymbol{f} \in \operatorname{supp}\left(\widehat{\chi}_{\text {right }}\right)\), add the unique leaf corresponding to \(\boldsymbol{f}\) to Frontier and Marked
                else
                    Add \(v_{\text {right }}\) to Frontier
                if \(\mathrm{IsCorR}_{\text {left }}\) and \(\mathrm{IsCorR}_{\text {right }}\) then
                    Remove \(v\) from Frontier
        until Frontier has no leaves besides root
        return \(\widehat{\chi}\)
    procedure ExtractCheapSubset \((T, S)\)
        \(L \leftarrow \varnothing\)
        while \(|L| \cdot(8+4 \log |S|)<\max _{v \in L} 2^{w_{T}(v)}\) do
            \(L \leftarrow L \cup\left\{\operatorname{argmin}_{u \in S \backslash L} w_{T}(u)\right\}\)
        Return \(L\)
```

Analysis of RobustPromiseSFT. First we analyze the runtime and sample complexity of primitive RobustPromiseSFT in the following lemma.

Lemma 20 (RobustPromiseSFT - Time and Sample Complexity). Consider an invocation of

RobustPromiseSFT ( $x, \widehat{\chi}_{i n}$, SideTree, $v, b, \mu$ ), where SideTree is a subtree of $T_{N}^{\text {full }}, v$ is some leaf of $T, k$ and $b$ are integers with $k>b, \mu \geq 0$, and $x, \widehat{\chi}_{\text {in }}:[n]^{d} \rightarrow \mathbb{C}$. Then

- The running time of primitive is bounded by

$$
\widetilde{O}\left(\left\|\widehat{\chi}_{i n}\right\|_{0} \cdot\left(b^{2}+k\right)+b k+2^{w_{\text {SIIETREE }}(v)} \cdot\left(b^{3}+k\right)\right) .
$$

- The number of accesses it makes on $x$ is always bounded by

$$
\widetilde{O}\left(2^{w_{\mathrm{SIDETREE}}(v)} \cdot\left(b^{3}+k\right)\right) .
$$

Furthermore, the output signal $\widehat{\chi}_{v}$ always satisfies $\left\|\widehat{\chi}_{v}\right\|_{0} \leq b$ and $\operatorname{supp}\left(\widehat{\chi}_{v}\right) \subseteq \operatorname{Freq}^{\operatorname{Cone}} \mathrm{SideTreE}(v)$.
Proof. First we prove that Algorithm 8 terminates after a bounded number of iterations. In order to bound the number of iterations of RobustPromiseSFT, we use a potential function argument. Let $\widehat{\chi}_{v}^{(t)}$ denote the signal $\widehat{\chi}_{v}$ at the end of iteration $t$ of the algorithm. Furthermore, let $T^{(t)}$ denote the subtree $T$ at the end of $t^{t h}$ iteration. Additionally, let MARKED ${ }^{(t)}$ and Identified ${ }^{(t)}$ denote the set Marked (defined in Algorithm 8) at the end of iteration $t$.

We prove that the algorithm always terminates after $O(b \cdot \log N)$ iterations. We prove this by contradiction. For any integer $t$, define the following potential function

$$
\phi_{t}:=\left|\operatorname{MarkeD}^{(t)}\right|+2 \log N \cdot\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0}+\sum_{u \in \operatorname{LEAVES}\left(T^{(t)}\right)} l_{T^{(t)}}(u) .
$$

Towards contradiction, suppose that Algorithm 8 does not terminate after $4 b \log N$ iterations. We show that the above potential function increases by at least 1 at every iteration $2 \leq t \leq 4 b \log N$, i.e., $\phi_{t} \geq \phi_{t-1}+1$. This is enough to conclude the termination of the algorithm because the ifstatement in line 7 ensures that $\left|\operatorname{Marked}^{(t)}\right| \leq\left|\operatorname{Leaves}\left(T^{(t)}\right)\right| \leq b$ and also $\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0} \leq b$, thus, $\phi_{t}=O(b \log N)$ for any $t$, which proves that algorithm terminates after $O(b \log N)$ iterations.

At any given iteration $t$ of the algorithm, there are 3 possibilities that can happen. We show that if any of these possibilities happen, then the potential function $\phi_{t}$ increases by at least 1 .

Case 1 - the if-statement in line 9 of Algorithm 8 is True. In this case, the algorithm constructs $T^{(t)}$ by removing all leaves that are in the set MARKED ${ }^{(t-1)}$ from tree $T^{(t-1)}$ and leaving the rest of the tree unchanged. Furthermore, the algorithm sets MARKED ${ }^{(t)} \leftarrow \varnothing$. By construction, the level of the leaves that are in Marked ${ }^{(t-1)}$ is at most $\log N$, thus

$$
\sum_{u \in \operatorname{LEAVES}\left(T^{(t)}\right)} l_{T^{(t)}}(u) \geq \sum_{u \in \operatorname{LEAVES}\left(T^{(t-1)}\right)} l_{T^{(t-1)}}(u)-\log N \cdot\left|\operatorname{MARKED}^{(t-1)}\right|
$$

Additionally, in this case, the algorithm computes $\left\{\widehat{H}_{u}\right\}_{u \in \operatorname{MARKED}}{ }^{(t-1)}$ by running the procedure Estimate in line 11 and then updates $\widehat{\chi}_{v}^{(t)}\left(\boldsymbol{f}_{u}\right) \leftarrow \widehat{H}_{u}$ for every $u \in \operatorname{Marked}^{(t-1)}$ and $\widehat{\chi}_{v}^{(t)}(\boldsymbol{\xi})=\widehat{\chi}_{v}^{(t-1)}(\boldsymbol{\xi})$ at every other frequency $\boldsymbol{\xi}$. Therefore, $\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0}=\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0}+\left|\operatorname{MARKED}^{(t-1)}\right|$. Also, $\left|\operatorname{Marked}^{(t)}\right|=0$. Hence,

$$
\phi_{t}-\phi_{t-1} \geq(\log N-1) \cdot\left|\operatorname{MARKED}^{(t-1)}\right| \geq 1
$$

where the inequality above holds because the if-statement in line 9 of the algorithm is True, ensuring that $\operatorname{Marked}^{(t-1)} \neq \varnothing$.

Case 2 - the if-statement in line 9 is False and if-statement in line 19 is True. In this case, in line 21, the algorithm updates Marked by adding the leaf $z$ to this set, i.e., Marked ${ }^{(t)} \leftarrow$ $\operatorname{Marked}^{(t-1)} \cup\{z\}$. Additionally, tree $T$ and signal $\widehat{\chi}_{v}$ stay unchanged, i.e., $\widehat{\chi}_{v}^{(t)}=\widehat{\chi}_{v}^{(t-1)}$ and $T^{(t)}=T^{(t-1)}$. Therefore, in this case, $\phi_{t+1}-\phi_{t}=1$.

Case 3 - both if-statements in lines 9 and 19 are False. In this case, either the algorithm terminates by the if-statement in line 31, which is exactly what we have assumed towards a contradiction that did not happen, or $\sum_{u \in \operatorname{LEAVES}\left(T^{(t)}\right)} l_{T^{(t)}}(u) \geq \sum_{u \in \operatorname{LEAVES}\left(T^{(t-1)}\right)} l_{T^{(t-1)}}(u)+1$, while $\left|\operatorname{MARKED}^{(t)}\right|=\left|\operatorname{MARKED}^{(t-1)}\right|$ and $\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0}=\left\|\widehat{\chi}_{v}^{(t-1)}\right\|_{0}$ (since we assumed $t \geq 2$ and hence $z \neq v$ ). Thus, $\phi_{t+1}-\phi_{t} \geq 1$.

So far we have showed that at every iteration, under the cases 1, 2, and 3, the potential function $\phi_{t}$ increases by at least one. Now we show that, at every iteration, exactly one of these three cases happens and hence the algorithm never stalls. For the sake of contradiction suppose that at iteration $t$, the algorithm stalls. For this to happen, we must have that all leaves of $T^{(t-1)}$ are in the set Marked ${ }^{(t-1)}$. By the if-statement in line 7 of Algorithm 8, we are guaranteed that $\mid$ Marked $^{(t-1)} \mid \leq b$. Therefore, by Lemma 7 , there must exist a subset $\varnothing \neq L \subset$ MARKED $^{(t-1)}$ such that $|L| \geq \frac{1}{4+2 \log b} \cdot \max _{u \in L} 2^{w_{T} T^{(t-1)}(u)}$. Hence, it follows from the way our algorithm explores the nodes of the tree in an increasing order of weights, that there must exist some $t^{\prime}<t$ such that $\varnothing \neq$ Marked ${ }^{\left(t^{\prime}-1\right)} \subseteq$ Marked $^{(t-1)}$ such that the if-statement in line 9 becomes True on Marked ${ }^{\left(t^{\prime}-1\right)}$. Therefore, case 1 must have happened at iteration $t^{\prime}$, resulting in emptying the set of identified frequencies, i.e., Marked ${ }^{\left(t^{\prime}\right)} \leftarrow \varnothing$. This would have resulted in Marked ${ }^{\left(t^{\prime}-1\right)} \nsubseteq$ Marked $^{(t-1)}$ which is the contradiction we wanted. Therefore the algorithm never stalls and always exactly one of case 1, 2, and 3 happen.

We proved that $\phi_{t}$ must increase by at least 1 at every iteration. Since $\phi_{1} \geq 0$ and we assumed that the algorithm did not terminate after $q=4 b \log N$ iterations, this potential will have a value of at least $4 b \log N-1$ :

$$
\phi_{q} \geq 4 b \log N-1, \text { where } q=4 b \log N .
$$

On the other hand, since the if-statement in line 7 ensures that the number of leaves of $T^{(t)}$ is always bounded by $b-\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0}$, the sum $\sum_{u \in \operatorname{LEAVES}\left(T^{(t)}\right)} l_{T^{(t)}}(u)$ is always bounded by $\left(b-\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0}\right) \cdot \log N$. Also, the size of the set $\operatorname{Marked}^{(t)}$, which is a subset of $\operatorname{LEAVES}\left(T^{(t)}\right)$, is always bounded by $b-\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0}$. This means that we must have $\phi_{q} \leq b \cdot(\log N+1)+(\log N-1) \cdot\left\|\widehat{\chi}_{v}^{(q)}\right\|_{0}$. The ifstatement in line 7 also ensures that $\left\|\widehat{\chi}_{v}^{(q)}\right\|_{0} \leq b$ which implies that $\phi_{q} \leq 2 b \cdot \log N$ which contradicts $\phi_{q} \geq 4 b \cdot \log N-1$. This proves that the number of iterations of the algorithm must be bounded by $O(b \cdot \log N)$, guaranteeing termination of RobustSparseFT. The termination quarantee along with the way our algorithm constructs $\widehat{\chi}_{v}$ and the if-staement in line 7, imply that the output signal $\widehat{\chi}_{v}$ always satisfies $\left\|\widehat{\chi}_{v}\right\|_{0} \leq b$ and $\operatorname{supp}\left(\widehat{\chi}_{v}\right) \subseteq \operatorname{FreqCone}_{\text {SideTree }(v) \text {. Now we bound the running }}$ time and sample complexity of the algorithm.

Sample Complexity and Runtime: First recall that we proved $\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0} \leq b$ for every iteration $t$. Additionally, the weight of the node $z$ at every iteration of the algorithm is bounded by $w_{T^{(t)}}(z) \leq$ $\log (2 b)$. To see this, note that if at some iteration $t$, the set of identified frequencies (or leaves) that our algorithm keeps, $\operatorname{Marked}^{(t)}$, is such that there exists a leaf $u \in \operatorname{Marked}^{(t)}$ with $w_{T^{(t)}}(u)>$ $\log (2 b)$, then by Lemma 7. MARKED ${ }^{(t)}$ contains a non-empty subset that is cheap to estimate. Thus, at some iteration $t^{\prime}<t$, where $\varnothing \neq \operatorname{Marked}^{\left(t^{\prime}\right)} \subset$ Marked $^{(t)}$ holds, it must have been the case
that the if-statement in line 9 became True on Marked ${ }^{\left(t^{\prime}\right)}$. If this happened, our algorithm would have estimated Marked ${ }^{\left(t^{\prime}\right)}$ at iteration $t^{\prime}$ and so we would have MARKED ${ }^{\left(t^{\prime}\right)} \cap \operatorname{MARKED}^{(t)}=\varnothing$ which is a contradiction.

Given the above inequalities, by Lemma 18, time and sample complexities of every invocation of HeavyTest in lines 25 and 26 of Algorithm 8 are bounded by $\widetilde{O}\left(\left\|\widehat{\chi}_{i n}\right\|_{0} \cdot b+2^{w_{\text {Sidetree }}(v)} \cdot b^{2}\right)$ and $\widetilde{O}\left(2^{w_{\text {SIDETREE }}(v)} \cdot b^{2}\right)$, respectively. Also, since $\left\|\widehat{\chi}_{v}\right\|_{0} \leq b$, the runtime and sample complexity of the HeavyTest in line 34 of the algorithm are bounded by $\widetilde{O}\left(\left\|\widehat{\chi}_{i n}\right\|_{0} \cdot k+b k+2^{w_{\text {SideTree }}(v)} \cdot k\right)$ and $\widetilde{O}\left(2^{w_{\text {Siletree }}(v)} \cdot k\right)$, respectively. Thus, total sample and time complexity of all invocations of HeavyTest throughout the execution of our algorithm are bounded by $\widetilde{O}\left(2^{w_{\text {Sidetree }}(v)} \cdot\left(b^{3}+k\right)\right)$ and $\widetilde{O}\left(\left\|\widehat{\chi}_{i n}\right\|_{0} \cdot\left(b^{2}+k\right)+b k+2^{w_{\text {SIDETREE }}(v)} \cdot\left(b^{3}+k\right)\right)$, respectively

Additionally, by Lemma 19, the sample and time complexity of every invocation of EstiMATE in line 11 of our algorithm are bounded by $\widetilde{O}\left(\frac{b \cdot 2^{w} \text { wimetreb }^{(v)}}{\left|\operatorname{MARKED}^{(t-1)}\right|} \cdot \sum_{u \in \operatorname{MARKED}^{(t-1)}} 2^{w_{T^{(t-1)}}(u)}\right)$ and $\widetilde{O}\left(\frac{b \cdot 2^{w} \text { Sinetrre }^{(v)}}{\left|\operatorname{MARKED}^{(t-1)}\right|} \cdot \sum_{u \in \operatorname{MARKED}^{(t-1)}} 2^{w_{T^{(t-1)}}(u)}+b \cdot\|\widehat{\chi}\|_{0}\right)$, respectively. Because we run Estimate only when the if-statement in line 9 holds true, the runtime and sample complexity of Estimate can be further upper bounded by $\widetilde{O}\left(\left|\operatorname{MARKED}^{(t-1)}\right| \cdot b \cdot 2^{w_{\text {SIDETREE }}(v)}+b \cdot\|\widehat{\chi}\|_{0}\right)$ and $\widetilde{O}\left(\left|\operatorname{Marked}^{(t-1)}\right| \cdot b \cdot 2^{w_{\text {Sidetree }}(v)}\right)$, respectively. Using the fact that

$$
\sum_{t: \text { if-statement in line } 9 \text { is True }}\left|\operatorname{MARKED}^{(t-1)}\right|=\left\|\widehat{\chi}_{v}\right\|_{0} \leq b,
$$

the total runtime and sample complexity of all invocations of Estimate in all iterations can be upper bounded by $\widetilde{O}\left(2^{w_{\text {SIDETREE }}(v)} \cdot b^{2}+b^{2} \cdot\|\widehat{\chi}\|_{0}\right)$ and $\widetilde{O}\left(2^{w_{\text {SIDETREE }}(v)} \cdot b^{2}\right)$, respectively. Therefore, by adding up the above contributions we can upper bound the total runtime and sample complexity by $\widetilde{O}\left(\left\|\widehat{\chi}_{i n}\right\|_{0} \cdot\left(b^{2}+k\right)+b k+2^{w_{\text {SIDETREE }}(v)} \cdot\left(b^{3}+k\right)\right)$ and $\widetilde{O}\left(2^{w_{\text {SIDETREE }}}(v) \cdot\left(b^{3}+k\right)\right)$ which completes the proof of the lemma.

We are now in a position to present the main invariant of primitive RobustPromiseSFT.
Lemma 21 (RobustPromiseSFT - Invariants). Consider the preconditions of Lemma 20. Let $\widehat{y}:=\widehat{x}-\widehat{\chi}_{\text {in }}$ and $S:=\operatorname{FreqCone}_{\text {SideTree }}(v) \cap \operatorname{HEAD}_{\mu}(\widehat{y})$, where $\operatorname{HEAD}_{\mu}(\cdot)$ is defined as per (4). If i) $\operatorname{HEAD}_{\mu}(\widehat{y}) \subseteq \operatorname{supp}(\operatorname{SidETrEE})$, ii) $\left\|\widehat{y}-\widehat{y}_{\operatorname{HEAD}_{\mu}(\widehat{y})}\right\|_{2}^{2} \leq \frac{11 \mu^{2}}{10}$, and iii) $|S| \leq k$, then with probability at least $1-\frac{1}{N^{4}}$, the output (Budget, $\widehat{\chi}_{v}$ ) of Algorithm 8 satisfies the following,

1. If $|S| \leq b$ then Budget $=\operatorname{True}, \operatorname{supp}\left(\widehat{\chi}_{v}\right) \subseteq S$, and $\left\|\widehat{y}_{S}-\widehat{\chi}_{v}\right\|_{2}^{2} \leq \frac{\mu^{2}}{20}$;
2. If $|S|>b$ then Budget $=$ False and $\widehat{\chi}_{v} \equiv\{0\}^{n^{d}}$.

Proof. We first analyze the algorithm under the assumption that the primitives HeavyTest and Estimate are replaced with more powerful primitives that succeeds deterministically. Hence, we assume that HeavyTest correctly tests the "heavy" hypothesis on its input signal with probability 1 and also Estimate achieves the estimation guarantee of Lemma 19 deterministrically. With these assumptions in place, we prove that the lemma holds deterministically (with probability 1 ). We then establish a coupling between this idealized execution and the actual execution of our algorithm, leading to our result.

We prove the first statement of lemma by induction on the Repeat-Until loop of the algorithm. Let $\widehat{\chi}_{v}^{(t)}$ denote the signal $\widehat{\chi}_{v}$ at the end of iteration $t$ of the algorithm. Furthermore, let $T^{(t)}$ denote the subtree $T$ at the end of $t^{t h}$ iteration. Additionally, let Marked ${ }^{(t)}$ denote the set Marked (defined in Algorithm 8) at the end of iteration $t$. We prove that if the precondition of statement 1 (that is $|S| \leq b$ ) together with i, ii and iii hold, then at every iteration $t=0,1,2, \ldots$ of Algorithm 8 , the following properties are maintained,
$P_{1}(t) S \backslash \operatorname{supp}\left(\hat{\chi}_{v}^{(t)}\right) \subseteq \operatorname{supp}\left(T^{(t)}\right):=\bigcup_{u \in \operatorname{LEAVES}\left(T^{(t)}\right)} \operatorname{Freq}^{\operatorname{Cone}}{\operatorname{SideTreEU} T^{(t)}}(u) ;$
$P_{2}(t)$ For every leaf $u \neq v$ of subtree $T^{(t)}, \operatorname{HEAD}_{\mu}(\widehat{y}) \cap \operatorname{FreqCone}_{\operatorname{SideTreev~}^{(t)}}(u) \neq \varnothing$;
$P_{3}(t)\left\|\widehat{y}_{S^{(t)}}-\widehat{\chi}_{v}^{(t)}\right\|_{2}^{2} \leq \frac{\left|S^{(t)}\right|}{20 b} \cdot \mu^{2}$, where $S^{(t)}:=\operatorname{supp}\left(\widehat{\chi}_{v}^{(t)}\right)$;
$P_{4}(t) S^{(t)} \subseteq S$ and $S^{(t)} \cap\left(\bigcup_{\substack{u \in \operatorname{Leaves}\left(T^{(t)}\right) \\ u \neq v}} \operatorname{FreqCone}_{\operatorname{SideTreEuT}^{(t)}(u)}\right)=\varnothing ;$
The base of induction corresponds to the zeroth iteration $(t=0)$, at which point $T^{(0)}=\{v\}$ is a subtree of SideTree that solely consists of node $v$. Moreover, $\widehat{\chi}_{v}^{(0)} \equiv 0$. Thus, statement $P_{1}(0)$ trivially holds by definition of set $S$. The statement $P_{2}(0)$ holds since there exists no leaf $u \neq v$ in $T^{(0)}$. Statements $P_{3}(0)$ and $P_{4}(0)$ hold because of the fact that $\widehat{\chi}_{v}^{(0)} \equiv 0$.

We now prove the inductive step by assuming that the inductive hypothesis, $P(t-1)$ is satisfied for some iteration $t-1$ of Algorithm 8, and then proving that $P(t)$ holds. First, we remark that if inductive hypotheses $P_{2}(t-1)$ and $P_{4}(t-1)$ hold true, then by the precondition of statement 1 of the lemma (that is $|S| \leq b$ ) the if-statement in line 7 of Algorithm 8 is False and hence lines 7 and 8 of the algorithm can be ignored in our analysis. We proceed to prove the induction by considering the three cases that can happen in iteration $t$ :

Case 1 - the if-statement in line 9 of Algorithm 8 is True. In this case, the algorithm computes $\left\{\widehat{H}_{u}\right\}_{u \in \operatorname{Marked}^{(t-1)}}$ by running the procedure Estimate in line 11 and then updates $\widehat{\chi}_{v}^{(t)}\left(\boldsymbol{f}_{u}\right) \leftarrow \widehat{H}_{u}$ for every $u \in \operatorname{MARKED}^{(t-1)}$ and $\widehat{\chi}_{v}^{(t)}(\boldsymbol{\xi})=\widehat{\chi}_{v}^{(t-1)}(\boldsymbol{\xi})$ at every other frequency $\boldsymbol{\xi}$. Therefore, if we let $L:=\left\{\boldsymbol{f}_{u}: u \in \operatorname{MARKED}^{(t-1)}\right\}$, then $S^{(t)} \backslash S^{(t-1)}=L$, by inductive hypothesis $P_{4}(t-1)$. By $P_{3}(t-1)$ along with Lemma 19 (its deterministic version that succeeds with probability 1), we find that

$$
\begin{align*}
\left\|\left(\widehat{\chi}_{v}^{(t)}-\widehat{y}\right)_{S^{(t)}}\right\|_{2}^{2} & =\left\|\left(\widehat{\chi}_{v}^{(t)}-\widehat{y}\right)_{S^{(t-1)}}\right\|_{2}^{2}+\left\|\left(\widehat{\chi}_{v}^{(t)}-\widehat{y}\right)_{S^{(t)} \backslash S^{(t-1)}}\right\|_{2}^{2} \\
& =\left\|\left(\widehat{\chi}_{v}^{(t-1)}-\widehat{y}\right)_{S^{(t-1)}}\right\|_{2}^{2}+\|\left(\left(\hat{\chi}_{v}^{(t)}-\widehat{y}\right)_{L} \|_{2}^{2}\right. \\
& \leq \frac{\left|S^{(t-1)}\right|}{20 b} \mu^{2}+\frac{|L|}{23 b} \sum_{\boldsymbol{\xi} \in[n] d \backslash \operatorname{supp}\left(\text { SideTREEUT } T^{(t-1)}\right)}\left|\left(\widehat{y}-\widehat{\chi}_{v}^{(t-1)}\right)(\boldsymbol{\xi})\right|^{2} . \tag{11}
\end{align*}
$$

Now we bound the second term above,

$$
\leq \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{HEAD}_{\mu}(\widehat{y})}|\widehat{y}(\boldsymbol{\xi})|^{2}+\left\|\widehat{y}_{S^{(t-1)}}-\widehat{\chi}_{v}^{(t-1)}\right\|_{2}^{2} \quad \text { (by } P_{1}(t-1) \text {, precondition i and definition of } S \text { ) }
$$

$$
\leq \frac{23}{20} \cdot \mu^{2} \quad\left(\text { by } P_{3}(t-1) \text { and } P_{4}(t-1) \text { and precondition }|S| \leq b\right)
$$

Therefore, by plugging the above bound back to (11) we find that,

$$
\left\|\left(\widehat{\chi}_{v}^{(t)}-\widehat{y}\right)_{S^{(t)}}\right\|_{2}^{2} \leq \frac{\left|S^{(t-1)}\right|}{20 b} \cdot \mu^{2}+\frac{|L|}{23 b} \cdot\left(\frac{23}{20} \mu^{2}\right)=\frac{\left|S^{(t)}\right|}{20 b} \cdot \mu^{2},
$$

which proves the inductive claim $P_{3}(t)$. Moreover, $P_{2}(t-1)$ implies that $L \subseteq S$. Thus, the fact $S^{(t)}=S^{(t-1)} \cup L$ together with inductive hypothesis $P_{4}(t-1)$ as well as the construction of $T^{(t)}\left(T^{(t)}\right.$ is constructed by removing leaves of MARKED ${ }^{(t-1)}$ from tree $\left.T^{(t-1)}\right)$, imply $P_{4}(t)$. The construction of $T^{(t)}$ together with the fact that $\left|\operatorname{FreqCone}_{\operatorname{SideTreeut}}{ }^{(t-1)}(u)\right|=1$ for every $u \in \operatorname{Marked}^{(t-1)}$ give $P_{1}(t)$ and $P_{2}(t)$.

We now consider the other two cases. Let $z \in \operatorname{LEAVES}\left(T^{(t-1)}\right)$ be the smallest weight leaf chosen by the algorithm in line 17 .

Case 2 - the if-statement in line 9 is False and if-statement in line 19 is True. In this case, in line 21, the algorithm updates Marked by adding the leaf $z$ to this set, i.e., Marked ${ }^{(t)} \leftarrow$ Marked ${ }^{(t-1)} \cup\{z\}$. Additionally, in this case the tree $T$ and signal $\widehat{\chi}_{v}$ stay unchanged, i.e., $\widehat{\chi}_{v}^{(t)}=\widehat{\chi}_{v}^{(t-1)}$ and $T^{(t)}=T^{(t-1)}$. Therefore, $P_{1}(t), P_{2}(t), P_{3}(t)$, and $P_{4}(t)$ all trivially hold because of the inductive hypothesis $P(t-1)$.

Case 3 - both if-statements in lines 9 and 19 are False. In this case, the algorithm constructs tree $T^{\prime}$ by adding leaves $z_{\text {right }}$ and $z_{\text {left }}$ to tree $T^{(t-1)}$ as right and left children of $z$ in line 24 Then we compute Heavy $y_{\ell}$ and $\mathrm{Heavy}_{r}$ in lines 25 and 26 by running the primitive HeavyTest with inputs $\left(x, \widehat{\chi}_{v}^{(t-1)}+\widehat{\chi}_{i n}, \operatorname{SidETREE} \cup T^{\prime}, z_{\text {left }}, O\left(b \log ^{3} N\right), 6 \mu^{2}\right)$ and $\left(x, \widehat{\chi}_{v}^{(t-1)}+\widehat{\chi}_{i n}, \operatorname{SideTreE} \cup T^{\prime}, z_{\text {right }}, O\left(b \log ^{3}\right.\right.$ respectively. There are two possibilities that can happen to each of Heavy ${ }_{\ell}$ and Heavy $r_{r}$. In the following we focus on analyzing Heavy $\boldsymbol{H}_{\ell}$, but $\mathrm{Heavy}_{r}$ can be analyzed exactly the same way.

Possibility 1) FreqCone $_{\text {SideTreevt }^{\prime}}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y})=\varnothing$. Note that, by construction of $T^{\prime}$ we have

$$
\begin{aligned}
& \sum_{\boldsymbol{\xi} \in[n] \backslash \backslash \operatorname{supp}\left(\operatorname{SideTRebuT} T^{(t-1)}\right)}\left|\left(\hat{y}-\widehat{\chi}_{v}^{(t-1)}\right)(\boldsymbol{\xi})\right|^{2} \\
& =\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(\operatorname{SidETREE})}|\widehat{y}(\boldsymbol{\xi})|^{2}+\sum_{\boldsymbol{\xi} \in \operatorname{Freq}^{\operatorname{Cone}} \mathrm{Simetares}^{(v) \backslash \operatorname{supp}\left(T^{(t-1)}\right)}}\left|\left(\widehat{y}-\widehat{\chi}_{v}^{(t-1)}\right)(\boldsymbol{\xi})\right|^{2} \\
& =\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(\text { Sidetree })}|\widehat{y}(\boldsymbol{\xi})|^{2} \\
& +\sum_{\boldsymbol{\xi} \in \text { FreqCone }_{\text {SineTRez }}(v) \backslash\left(\operatorname{supp}\left(T^{(t-1)}\right) \cup S^{(t-1)}\right)}|\widehat{y}(\boldsymbol{\xi})|^{2}+\left\|\widehat{y}_{S^{(t-1)}}-\widehat{\chi}_{v}^{(t-1)}\right\|_{2}^{2} \\
& =\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash\left(\operatorname{supp}\left(\operatorname{SidETREEU} T^{(t-1)}\right) \cup S^{(t-1)}\right)}|\widehat{y}(\boldsymbol{\xi})|^{2}+\left\|\widehat{y}_{S^{(t-1)}}-\widehat{\chi}_{v}^{(t-1)}\right\|_{2}^{2}
\end{aligned}
$$

Hence, by inductive hypothesis $P_{4}(t-1)$ we have,

$$
\begin{aligned}
& \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}\left(\operatorname{SidETREEU} T^{\prime}\right)}\left|\left(\widehat{y}-\hat{\chi}_{v}^{(t-1)}\right)(\xi)\right|^{2} \\
& =\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(\text { SideTREE })}|\widehat{y}(\boldsymbol{\xi})|^{2} \\
& +\sum_{\boldsymbol{\xi} \in \text { FreqCone }_{\text {Sider TeiE }}(v) \backslash \operatorname{supp}\left(T^{(t-1)}\right)}\left|\left(\widehat{y}-\widehat{\chi}_{v}^{(t-1)}\right)(\boldsymbol{\xi})\right|^{2} \\
& =\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(\text { Sidetree })}|\widehat{y}(\boldsymbol{\xi})|^{2} \\
& +\sum_{\boldsymbol{\xi} \in \text { FreqCone }_{\text {Sinstraes }}(v) \backslash\left(\operatorname{supp}\left(T^{(t-1)}\right) \cup S^{(t-1)}\right)}|\widehat{y}(\boldsymbol{\xi})|^{2}+\left\|\widehat{y}_{S^{(t-1)}}-\widehat{\chi}_{v}^{(t-1)}\right\|_{2}^{2} \\
& \leq \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash\left(\operatorname{supp}\left(\operatorname{SideTREE} U^{\prime}\right) \cup S^{(t-1)}\right)}|\widehat{y}(\boldsymbol{\xi})|^{2}+\frac{\mu^{2}}{20},
\end{aligned}
$$

where the last inequality above follows by inductive hypotheses $P_{3}(t-1)$ and $P_{4}(t-1)$ and precondition $|S| \leq b$. Therefore, if $\widehat{G}_{\ell}$ is a $\left(z_{\text {left }}\right.$, SideTree $\left.\cup T^{\prime}\right)$-isolating filter as per the construction in Lemma 5, then by Corollary 1 along with the above inequality, we have

$$
\begin{aligned}
& \leq\left\|\widehat{y}_{\text {FreqCone }_{\text {Sidetrareut }}\left(z_{\text {left }}\right)}\right\|_{2}^{2}+\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash\left(\operatorname{supp}\left(\text { SideTRebuT }^{\prime}\right) \cup S^{(t-1)}\right)}|\widehat{y}(\boldsymbol{\xi})|^{2}+\frac{\mu^{2}}{20} \\
& \leq \sum_{\boldsymbol{\xi} \in[n] d \backslash \operatorname{HEAD}_{\mu}(\widehat{y})}|\widehat{y}(\boldsymbol{\xi})|^{2}+\frac{\mu^{2}}{20} \\
& \leq \frac{23}{20} \cdot \mu^{2}
\end{aligned}
$$

where the third line above follows from the assumption that FreqCone SideTreeut $\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y})=$ $\varnothing$, inductive hypothesis $P_{1}(t-1)$, precondition i of the lemma together with the definition of set $S$. This proves that the precondition of the second claim of Lemma 18 holds and therefore by invoking this lemma (the deterministic version of it that succeeds with probability 1), we have that Heavy $\ell_{\ell}$ in line 25 of the algorithm is False. Using a similar argument, if FreqCone Sidetreeut $^{\prime}\left(z_{\text {right }}\right) \cap$ $\operatorname{HEAD}_{\mu}(\widehat{y})=\varnothing$, then Heavy ${ }_{r}$ is False.

Possibility 2) Suppose that $\operatorname{FreqCone}_{\text {SideTreeut }}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y}) \neq \varnothing$. If filter $\widehat{G}_{\ell}$ is a $\left(z_{\text {left }}, \operatorname{SideTree} \cup T^{\prime}\right)$-isolating filter constructed in Lemma 5, then by Corollary 1 along with inductive hypothesis $P_{4}(t-1)$,

$$
\begin{aligned}
& \left\|\left(\left(\widehat{y}-\widehat{\chi}_{v}^{(t-1)}\right) \cdot \widehat{G}_{\ell}\right)_{[n]^{d} \backslash S}\right\|_{2}^{2}=\left\|\left(\widehat{y} \cdot \widehat{G}_{\ell}\right)_{[n]^{d} \backslash S}\right\|_{2}^{2} \\
& \leq \| \widehat{y}_{\text {FreqCone }}^{\text {SimeFrraru }^{\prime}\left(z_{\text {left }}\right) \backslash S \|_{2}^{2}}+\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash(\operatorname{supp}(\text { SideTreEUT }) \cup S)}|\widehat{y}(\boldsymbol{\xi})|^{2} \\
& \leq\left\|\widehat{y}-\widehat{y}_{\text {HEAD }_{\mu}(\widehat{y})}\right\|_{2}^{2} \leq \frac{11}{10} \cdot \mu^{2} . \quad \text { (precondition ii) }
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
\left\|\left(\left(\widehat{y}-\widehat{\chi}_{v}^{(t-1)}\right) \cdot \widehat{G}_{\ell}\right)_{S}\right\|_{2}^{2} & \geq\left\|\left(\widehat{y}-\widehat{\chi}_{v}^{(t-1)}\right)_{\text {FreqCone }_{\text {SideTrebuT }}\left(z_{\text {left }}\right) \cap S}\right\|_{2}^{2} \\
& =\left\|\widehat{y}_{\text {FreqCone }_{\text {SideTrebuT }}\left(z_{\text {left }}\right) \cap S}\right\|_{2}^{2} \geq 9 \mu^{2},
\end{aligned}
$$

which follows by the assumption FreqCone Sidetreeut $\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y}) \neq \varnothing$ along with the definition of $S$ and $\operatorname{HEAD}_{\mu}(\cdot)$. Hence, by the above inequalities and the precondition $|S| \leq b$, we can invoke Lemma 18 to conclude that Heavy $y_{\ell}$ in line 25 of the algorithm is True. Using a similar argument, if FreqCone Sidetreeut $^{\prime}\left(z_{\text {right }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y}) \neq \varnothing$ then Heavy ${ }_{r}$ is True.

Based on the above arguments, according to the values of $\mathrm{Heavy}_{\ell}$ and Heavy ${ }_{r}$, there are various cases that can happen. First, it cannot happen that Heavy ${ }_{\ell}$ and $\mathrm{Heavy}_{r}$ are both False unless $z=v$, by the inductive hypothesis $P(t-1)$. If Heavy ${ }_{\ell}=\mathrm{Heavy}_{r}=$ False and $z=v$, the algorithm returns $\widehat{\chi}_{v}^{(t)} \equiv\{0\}^{n^{d}}$ which satisfies all properties in $P(t)$. The second case corresponds to Heavy ${ }_{\ell}=$ False and Heavy ${ }_{r}=$ True. In this case, tree $T^{(t)}$ is obtained from $T^{(t-1)}$ by adding $z_{\text {right }}$ as the right child of $z$. Therefore, by inductive hypothesis $P(t-1)$, all properties in $P(t)$ immediately hold. One can show that $P(t)$ holds in the case of Heavy ${ }_{\ell}=$ True and $\mathrm{Heavy}_{r}=$ False in exactly the same fashion. Finally, if both of $\mathrm{Heavy}_{\ell}$ and $\mathrm{Heavy}_{r}$ are True, then tree $T^{(t)}$ is obtained by adding leaves $z_{\text {right }}$ and $z_{\text {left }}$ as right and left children of $z$ to tree $T^{(t-1)}$. It follows straightforwardly from the inductive hypothesis $P(t-1)$ that $P(t)$ holds.

So far we have showed that under cases $\mathbf{1}, \mathbf{2}$, and $\mathbf{3}$, the property $P(t)$ is maintained. Recall that in the proof of Lemma 20 we showed that, at every iteration, exactly one of these three cases happen and hence the algorithm never stalls. This completess the induction and proves that properties $P(t)$ are maintained throughout the execution of Algorithm 8, assuming that preconditions i, ii, and iii of the lemma along with the precondition $|S| \leq b$ hold.

In Lemma 20 we showed that Algorithm 8 must terminate after some $q$ iterations. When the algorithm terminates, the condition of the Repeat-Until loop in line 33 of the algorithm must be True. Thus, when the algorithm terminates, at $q^{\text {th }}$ iteration, there is no leaf in subtree $T_{v}^{(q)}$ besides $v$ and as a consequence the set $\operatorname{Marked}^{(q)}$ must be empty. This, together with $P_{1}(q)$ imply that the signal $\widehat{\chi}_{v}^{(q)}$ satisfies,

$$
\operatorname{supp}\left(\widehat{\chi}_{v}^{(q)}\right)=S=\operatorname{Freq}_{\operatorname{Cone}}^{\text {SIDETREE }}(v) \cap \operatorname{HEAD}_{\mu}(\widehat{y}) .
$$

Moreover, $P_{3}(q)$ together with precondition $|S| \leq b$ imply that

$$
\left\|\widehat{y}_{S}-\widehat{\chi}_{v}^{(q)}\right\|_{2}^{2} \leq \frac{|S|}{20 b} \cdot \mu^{2} \leq \frac{\mu^{2}}{20} .
$$

Now we analyze the if-statement in line 34 of the algorithm. The above equalities and inequalities on $\widehat{\chi}_{v}^{(q)}$ imply that,

$$
\begin{aligned}
\left\|\left(\widehat{y}-\widehat{\chi}_{v}^{(q)}\right)_{\operatorname{FreqCone}_{\text {SideTreE }}(v)}\right\|_{2}^{2} & =\left\|\widehat{y}_{\operatorname{FreqCone}_{\text {SideTREE }}(v) \backslash S}\right\|_{2}^{2}+\left\|\left(\widehat{y}-\widehat{\chi}_{v}^{(q)}\right)_{S}\right\|_{2}^{2} \\
& \leq\left\|\widehat{y}_{\operatorname{FreqCone}_{\text {SideTreE }}(v) \backslash \operatorname{HEAD} \mu}(\widehat{y})\right\|_{2}^{2}+\frac{\mu^{2}}{20} .
\end{aligned}
$$

Therefore, if $\widehat{G}_{v}$ is a Fourier domain $(v, \operatorname{SideTree})$-isolating filter constructed in Lemma 5 , then
by Corollary 1 along with the above inequality, we have

$$
\begin{aligned}
& \left\|\left(\widehat{y}-\widehat{\chi}_{v}^{(q)}\right) \cdot \widehat{G}_{v}\right\|_{2}^{2} \leq \sum_{\boldsymbol{\xi} \in[n] d \backslash \operatorname{supp}(\operatorname{SidETREE})}|\widehat{y}(\boldsymbol{\xi})|^{2}+\left\|\left(\widehat{y}-\widehat{\chi}_{v}^{(q)}\right)_{\text {FreqCones }_{\text {Sine Teree }}(v)}\right\|_{2}^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \leq\left\|\widehat{y}-\widehat{y}_{\mathrm{HEAD}_{\mu}(\widehat{y})}\right\|_{2}^{2}+\frac{\mu^{2}}{20} \leq \frac{23}{20} \cdot \mu^{2} .
\end{aligned}
$$

Thus, the preconditions of the second claim of Lemma 18 hold. So, we can invoke this lemma to conclude that the if-statement in line 34 of the algorithm is False and hence the algorithm outputs (True, $\widehat{\chi}_{v}^{(q)}$ ). This proves statement 1 of the lemma.

Now we prove the second statement of lemma. Suppose that preconditions i, ii, iii along with the precondition of statement 2 (that is $|S|>b$ ) hold. Lemma 20 proved that the signal $\hat{\chi}_{v}$
 Consequently, if $\widehat{G}_{v}$ is a Fourier domain ( $v$, SideTree)-isolating filter constructed in Lemma 5, then by definition of isolating filters we have

$$
\left\|\left(\left(\widehat{y}-\widehat{\chi}_{v}\right) \cdot \widehat{G}_{v}\right)_{S \cup \operatorname{supp}\left(\widehat{\chi}_{v}\right)}\right\|_{2}^{2} \geq\left\|\left(\widehat{y}-\widehat{\chi}_{v}\right)_{S \cup \operatorname{supp}\left(\hat{\chi}_{v}\right)}\right\|_{2}^{2} \geq\left\|\widehat{y}_{S \backslash \operatorname{supp}\left(\widehat{\chi}_{v}\right)}\right\|_{2}^{2} \geq 9 \mu^{2},
$$

which follows from the definition of $S$ and $\operatorname{HEAD}_{\mu}(\cdot)$. On the other hand,

$$
\begin{aligned}
\left\|\left(\left(\widehat{y}-\widehat{\chi}_{v}\right) \cdot \widehat{G}_{v}\right)_{[n]^{d} \backslash\left(S \cup \operatorname{supp}\left(\widehat{\chi}_{v}\right)\right)}\right\|_{2}^{2} & =\left\|\left(\widehat{y} \cdot \widehat{G}_{v}\right)_{[n]^{d} \backslash\left(S \cup \operatorname{supp}\left(\widehat{\chi}_{v}\right)\right)}\right\|_{2}^{2} \\
& \leq\left\|\left(\widehat{y} \cdot \widehat{G}_{v}\right)_{[n]^{d} \backslash S}\right\|_{2}^{2} \\
& \leq\left\|\widehat{y}_{\text {FreqCone }_{\text {SIDETREE }}(v) \backslash S}\right\|_{2}^{2}+\sum_{\boldsymbol{\xi} \in[n]]^{2} \backslash \operatorname{supp}(\text { SideTREE })}|\widehat{y}(\boldsymbol{\xi})|^{2} \\
& \leq\left\|\widehat{y}-\widehat{y}_{\operatorname{HEAD}_{\mu}(\widehat{y})}\right\|_{2}^{2} \leq \frac{11}{10} \cdot \mu^{2} . \quad \text { (precondition ii) }
\end{aligned}
$$

Additionally note that $\left|S \cup \operatorname{supp}\left(\widehat{\chi}_{v}\right)\right| \leq k+b \leq 2 k$ by preconditions of the lemma and property of $\operatorname{supp}\left(\widehat{\chi}_{v}\right)$ that we have proved. Hence, by invoking the first claim of Lemma 18, the if-statement in line 34 of the algorithm is True and hence the algorithm outputs (False, $\{0\}^{n^{d}}$ ). This proves statement 2 of the lemma.

Finally, observe that throughout this analysis we have assumed that Lemma 18 holds with probability 1 for all the invocations of HeavyTest by our algorithm. Moreover, we assumend that Estimate successfully works with probability 1. In reality, we have to take the fact that these primitives are randomized into acount of our analysis.

The first source of randomness is the fact that HeavyTest only succeeds with some high probability. In fact, Lemma 18 tells us that every invocation of HEAVyTest succeeds with probability at least $1-1 / N^{5}$. Our analysis in proof of Lemma 20 shows that RobustPromiseSFT makes at most $O(b \log N)$ calls to HeavyTest. Therefore, by a union bound, the overall failure probability of all invocations of HeavyTest is bounded by $O\left(\frac{b \log N}{N^{5}}\right)$.

The second source of randomness is the fact that Estimate only succeeds with some high probability. Lemma 19 tells us that every invocation of Estimate on a set Marked, succeeds
with probability $1-|\operatorname{Marked}| / N^{8}$. Therefore if the algorithm invokes Estimate at iterations $t_{1}, t_{2}, \ldots$, then, by union bound, the total failure probability of all invocations of this primitive will be bounded by $\sum_{i} \frac{\left|\operatorname{Marked}\left(t_{i}\right)\right|}{N^{8}}=\frac{\left|\operatorname{supp}\left(\hat{\chi}_{v}\right)\right|}{N^{8}} \leq \frac{b}{N^{8}}$.

Finally, by another application of union bound, the overall failure probability of Algorithm 8 , is bounded by $\frac{1}{N^{4}}$. This proves that the lemma holds.

Analysis of RobustSparseFT. Now we present the invariants of RobustSparseFT.
Lemma 22 (Invariant of RobustSparseFT: Signal Containment and Energy Control). For every integer $t \geq 0$, let $\widehat{\chi}^{(t)}$ and Marked $^{(t)}$ denote the signal $\widehat{\chi}$ and the set Marked at the end of iteration $t$ of Algorithm 9, respectively. Furthermore, let Frontier ${ }^{(t)}$ denote the tree Frontier at the end of $t^{\text {th }}$ iteration and let $\mathrm{EST}^{(t)}$ denote the set of "estimated frequencies" so far, i.e., $\mathrm{EST}^{(t)}:=$ $\operatorname{supp}\left(\widehat{\chi}^{(t)}\right)$. Additionaly, for every leaf $v$ of $\operatorname{Frontier}^{(t)}$, let $L_{v}^{(t)}$ denote the "unestimated" frequencies in support of $\widehat{x}$ that lie in frequency cone of $v$, i.e., $L_{v}^{(t)}:=\operatorname{FreqCone}_{\mathrm{Frontier}^{(t)}}(v) \cap \operatorname{HEAD}_{\mu}(\widehat{x})$, where $\operatorname{HEAD}_{\mu}(\cdot)$ is defined as per (4). If $\left|\operatorname{HEAD}_{\mu}(\widehat{x})\right| \leq k$ and $\left\|\widehat{x}-\widehat{x}_{\operatorname{HEAD}_{\mu}(\widehat{x})}\right\|_{2} \leq \mu$, then for every non-negative integer $t$ the following properties are maintained at the end of $t^{\text {th }}$ iteration of Algorithm 9, with probability at least $1-\frac{4 t}{N^{4}}$,
$P_{1}(t) \operatorname{HEAD}_{\mu}(\widehat{x}) \backslash \operatorname{Est}^{(t)} \subseteq \operatorname{supp}\left(\operatorname{Frontier}^{(t)}\right) ;$
$P_{2}(t)$ For every leaf $u \neq$ root of tree $\operatorname{Frontier}^{(t)},\left|L_{u}^{(t)}\right| \geq 1$. Additionally, if $u \notin \operatorname{Marked}^{(t)}$, then $\left|L_{u}^{(t)}\right|>b ;$
$P_{3}(t)\left\|\widehat{x}_{\mathrm{EST}^{(t)}}-\widehat{\chi}^{(t)}\right\|_{2}^{2} \leq \epsilon \cdot \frac{\left|\mathrm{EsT}^{(t)}\right|}{k} \cdot \mu^{2}$;
$P_{4}(t) \operatorname{EsT}^{(t)} \subseteq \operatorname{HEAD}_{\mu}(\widehat{x})$ and $\operatorname{EsT}^{(t)} \cap \operatorname{supp}\left(\operatorname{Frontier}^{(t)}\right)=\varnothing ;$
$P_{5}(t)$ In every iteration $t>1$, if the if-statement in line 8 of Algorithm 9 is False, then the following potential function decreases by at least b. Additionally, when the if-statement in line 8 is True, the potential decreases by at least $\log N$. Furthermore, the potential does not increase at iteration $t=1$.

$$
\phi_{t}:=\sum_{u \in \operatorname{LEAVES}\left(\text { Frontier }^{(t)}\right)}\left(2 \log N-l_{\text {Frontier }^{(t)}}(u)\right) \cdot\left|L_{u}^{(t)}\right| ;
$$

Proof. The proof is by induction on the Repeat-Until loop of the algorithm. The base of induction corresponds to the zeroth iteration $(t=0)$, at which point Frontier ${ }^{(0)}=\{\operatorname{root}\}$ is a tree that solely consists of a root and has no other leaves. Moreover, $\widehat{\chi}^{(0)} \equiv 0$. The statement $P_{1}(t)$ trivially holds because FreqCone Frontier $^{(0)}(r)=[n]^{d}$. The statement $P_{2}(t)$ holds since there exists no leaf $u \neq$ root in Frontier ${ }^{(0)}$. The statements $P_{3}(t)$ and $P_{4}(t)$ hold because of the facts $\widehat{\chi}^{(0)} \equiv 0$ and $\mathrm{EsT}^{(0)}=\varnothing$.

We now prove the inductive step by assuming that the inductive hypotheses, i.e property $P(t-1)$ is satisfied for some iteration $t-1$ of Algorithm 9 with probability a least $1-\frac{4(t-1)}{N^{4}}$, and then proving that property $P(t)$ holds at the end of iteration $t$ with probabiliy at least $1-\frac{4 t}{N^{4}}$. We also show that the value of the quantity $\phi_{t}$ defined in $P_{5}(t)$, satisfies $\phi_{t}-\phi_{t-1} \leq-b$ if the if-statement in line 8 of the algorithm is False in iteration $t>1$ and $\phi_{t}-\phi_{t-1} \leq-\log N$ if the if-statement in line 8 is True in iteration $t$ and also $\phi_{1}-\phi_{0} \leq 0$. At any given iteration $t$ of
the algorithm, there are two possibilities that can happen. We proceed to prove the induction by considering any of the two possibilities:

Case 1 - the if-statement in line 8 of Algorithm 9 is True. In this case, we have that $\sum_{u \in \text { Marked }^{(t-1)}} 2^{-w_{\text {Frontira }}(t-1)(u)} \geq \frac{1}{2}$. As a result, by Claim 6, the set Cheap $\subseteq$ MARKED $^{(t-1)}$ that the algorithm computes in line 9 by running the primitive ExtractCheapSubset satisfies the property that $\mid$ Cheap $\mid \cdot\left(8+4 \log \left|\operatorname{Marked}^{(t-1)}\right|\right) \geq \max _{u \in \text { Chear }} 2^{w_{\text {Frontire }}(t-1)}{ }^{(u)}$. Clearly Cheap $\neq$ $\varnothing$, by Claim 6. Then the algorithm computes $\left\{\widehat{H}_{u}\right\}_{u \in \text { Cheap }}$ by running the procedure Estimate in line 11 and then updates $\widehat{\chi}^{(t)}\left(\boldsymbol{f}_{u}\right) \leftarrow \widehat{H}_{u}$ for every $u \in$ CHEAP and $\widehat{\chi}^{(t)}(\boldsymbol{\xi})=\widehat{\chi}^{(t-1)}(\boldsymbol{\xi})$ at every other frequency $\boldsymbol{\xi}$. Therefore, if we let $L:=\left\{\boldsymbol{f}_{u}: u \in \mathrm{CHEAP}\right\}$, then $\mathrm{EsT}^{(t)} \backslash \mathrm{EST}^{(t-1)}=L$, by inductive hypothesis $P_{4}(t-1)$. By $P_{3}(t-1)$ along with Lemma 19, we find that with probability at least $1-\frac{\mid \text { Cheap } \mid}{N^{8}} \geq 1-\frac{1}{N^{7}}$ the following holds,

$$
\begin{align*}
\left\|\widehat{\chi}^{(t)}-\widehat{x}_{\mathrm{EST}^{(t)}}\right\|_{2}^{2} & =\left\|\widehat{\chi}^{(t-1)}-\widehat{x}_{\mathrm{EST}^{(t-1)}}\right\|_{2}^{2}+\left\|\left(\widehat{\chi}^{(t)}-\widehat{x}\right)_{L}\right\|_{2}^{2} \\
& \leq \frac{\epsilon\left|\operatorname{EST}^{(t-1)}\right|}{k} \mu^{2}+\frac{\epsilon|L|}{2 k} \sum_{\boldsymbol{\xi} \in[n] \backslash \backslash \operatorname{supp}\left(\mathrm{FrONTIER}^{(t-1)}\right)}\left|\left(\widehat{\chi}^{(t-1)}-\widehat{x}\right)(\boldsymbol{\xi})\right|^{2} . \tag{12}
\end{align*}
$$

Now we bound the second term above,

$$
\begin{aligned}
& \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}\left(\text { Frontier }^{(t-1)}\right)}\left|\left(\widehat{x}-\widehat{\chi}^{(t-1)}\right)(\boldsymbol{\xi})\right|^{2} \\
& =\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash\left(\operatorname{supp}\left(\text { Frontier }^{(t-1)}\right) \cup \text { ESTr }^{(t-1)}\right)}|\widehat{x}(\boldsymbol{\xi})|^{2}+\left\|\widehat{x}_{\mathrm{EST}^{(t-1)}}-\widehat{\chi}^{(t-1)}\right\|_{2}^{2} \\
& \leq \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{HEAD}_{\mu}(\widehat{x})}|\widehat{x}(\boldsymbol{\xi})|^{2}+\left\|\widehat{x}_{\mathrm{EST}^{(t-1)}}-\widehat{\chi}^{(t-1)}\right\|_{2}^{2} \\
& \left.\leq 2 \mu^{2} \quad \quad \text { (by } P_{3}(t-1) \text { and } P_{4}(t-1), \text { preconditions of lemma and } \epsilon \leq 1\right) .
\end{aligned}
$$

Therefore, by plugging the above bound back to (12) we find that,

$$
\left\|\widehat{\chi}^{(t)}-\widehat{x}_{\mathrm{EST}^{(t)}}\right\|_{2}^{2} \leq \epsilon \cdot \frac{\left|\operatorname{EsT}^{(t-1)}\right|}{k} \cdot \mu^{2}+\epsilon \cdot \frac{|L|}{2 k} \cdot\left(2 \mu^{2}\right)=\epsilon \cdot \frac{\left|\operatorname{EsT}^{(t)}\right|}{k} \cdot \mu^{2},
$$

which proves the inductive claim $P_{3}(t)$. Moreover, $P_{2}(t-1)$ implies that $L \subseteq \operatorname{HEAD}_{\mu}(\widehat{x})$. Thus, the fact that $\mathrm{EsT}^{(t)}=\mathrm{EsT}^{(t-1)} \cup L$ together with inductive hypothesis $P_{4}(t-1)$ as well as the construction of Frontier (Frontier ${ }^{(t)}$ is constructed by removing leaves of Cheap from tree Frontier ${ }^{(t-1)}$ ), imply $P_{4}(t)$. The construction of Frontier ${ }^{(t)}$ together with the fact that $\mid$ FreqCone $_{\text {Frontier }^{(t-1)}}(u) \mid=1$ for every $u \in$ Cheap give $P_{1}(t)$ and $P_{2}(t)$. Additionally, we have,

$$
\begin{aligned}
\phi_{t}-\phi_{t-1} & =-\sum_{u \in \text { ChEAP }}\left(2 \log N-l_{\text {Frontier } \left.^{(t-1)}(u)\right) \cdot\left|L_{u}^{(t-1)}\right|}\right. \\
& =-\sum_{u \in \text { ChEAP }} \log N \cdot\left|L_{u}^{(t-1)}\right| \\
& =-\sum_{u \in \text { CHEAP }} \log N \leq-\log N,
\end{aligned}
$$

where the last inequality follows from the fact that ChEAP $\neq \varnothing$. This proves $P_{5}(t)$.

Case 2 - the if-statement in line 8 is False. Let $v \in \operatorname{LEAVES}\left(\right.$ FRONTIER $\left.^{(t-1)}\right) \backslash$ MARKED $^{(t-1)}$ be the smallest weight leaf chosen by the algorithm in line 17. The algorithm constructs tree $T$ by adding leaves $v_{\text {right }}$ and $v_{\text {left }}$ to tree $\mathrm{FRONTIER}^{(t-1)}$ as right and left children of $v$, in line 20. Then, the algorithm runs RobustPromisesFT with inputs $\left(x, \widehat{\chi}^{(t-1)}, T, v_{\text {left }}, b, k, \mu\right)$ and $\left(x, \widehat{\chi}^{(t-1)}, T, v_{\text {right }}, b, k, \mu\right)$ in lines 21 and 22 respectively. In the following we focus on analyzing $\left(I_{s C o R R}^{\text {left }}\right.$, $\left.\widehat{\chi}_{\text {left }}\right)$ but ( $\mathrm{ISCORR}_{\text {right }}, \widehat{\chi}_{\text {right }}$ ) can be analyzed exactly the same way. There are two possibilities that can happen:

Possibility 1) $\mid$ FreqCone $_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{x}) \mid \leq b$. In this case, the inductive hypothesis $P_{4}(t-$ 1) implies that $\left|\mathrm{EST}^{(t-1)}\right| \leq k$ and hence inductive hypothesis $P_{3}(t-1)$ along with the assumption $\epsilon \leq \frac{1}{10}$ gives

$$
\begin{equation*}
\left\|\widehat{x}_{\mathrm{EST}^{(t-1)}}-\widehat{\chi}^{(t-1)}\right\|_{2}^{2} \leq \epsilon \mu^{2} \leq \frac{\mu^{2}}{10} \tag{13}
\end{equation*}
$$

hence, $\operatorname{HEAD}_{\mu}\left(\widehat{x}-\widehat{\chi}^{(t-1)}\right)=\operatorname{HEAD}_{\mu}(\widehat{x}) \backslash \operatorname{EST}^{(t-1)}$. Consequently, if we let $\widehat{y}:=\widehat{x}-\widehat{\chi}^{(t-1)}$, then: i) $\operatorname{HEAD}_{\mu}(\widehat{y}) \subseteq \operatorname{supp}\left(T^{\prime}\right)$, by $P_{1}(t-1)$, ii) $\left\|\widehat{y}-\widehat{y}_{\operatorname{HEAD} \mu}(\widehat{y})\right\|_{2}^{2} \leq \frac{11 \mu^{2}}{10}$, by precondition of the lemma along with $(13)$, and iii) $\left|\operatorname{Freq}^{13} \operatorname{Cone}_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y})\right| \leq b$, by the assumption that $\mid$ FreqCone $_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{x}) \mid \leq b$. Therefore, all preconditions of the first statement of Lemma 21 hold, and thus, by invoking this lemma we have that, with probability at least $1-\frac{1}{N^{4}}, \operatorname{IsCorR}_{\text {left }}=$ True, and $\operatorname{supp}\left(\widehat{\chi}_{\text {left }}\right) \subseteq \operatorname{FreqCone}_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y})$, and $\left\|\widehat{y}_{\text {FreqCone }_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD} \mu(\widehat{y})}-\widehat{\chi}_{\text {left }}\right\|_{2}^{2} \leq \frac{\mu^{2}}{20}$. This together with inductive hypothesis $P_{4}(t-1)$ imply that, with probability at least $1-\frac{1}{N^{4}}$, $\operatorname{ISCoRR}_{\text {left }}=\operatorname{True}$ and $\operatorname{supp}\left(\widehat{\chi}_{\text {left }}\right)=\operatorname{FreqCone}_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{x})$.

So, the if-statement in line 23 of the algorithm is True and consequently the algorithm adds all leaves that correspond to frequencies in $\operatorname{FreqCone}_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{x})$ to $\operatorname{Frontier}^{(t-1)}$ and also updates

$$
\operatorname{Marked}^{(t)} \leftarrow \operatorname{Marked}^{(t-1)} \cup\left\{u \in \operatorname{LEAVES}(\text { Frontier }): \boldsymbol{f}_{u} \in \operatorname{FreqCone}_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{x})\right\}
$$

By a similar argument, if $\mid$ Freq $_{\text {Cone }}^{T}\left(v_{\text {right }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{x}) \mid \leq b$, then, with probability at least $1-\frac{1}{N^{4}}$, the algorithm adds all leaves corresponding to frequencies in FreqCone $T_{T}\left(v_{\text {right }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{x})$ to Frontier $^{(t-1)}$ and updates

$$
\operatorname{MARKED}^{(t)} \leftarrow \operatorname{MARKED}^{(t-1)} \cup\left\{u \in \operatorname{LEAVES}(\operatorname{FRONTIER}): \boldsymbol{f}_{u} \in \operatorname{FreqCone}_{T}\left(v_{\text {right }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{x})\right\}
$$

Possibility 2) $\left|\operatorname{FreqCone}_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{x})\right|>b$. Same as in possibility 1, the inductive hypothesis $P_{4}(t-1)$ implies that $\left|\mathrm{EST}^{(t-1)}\right| \leq k$ and hence inductive hypothesis $P_{3}(t-1)$ along with the assumption $\epsilon \leq \frac{1}{10}$ gives $(13)$. Hence, $\operatorname{HEAD}_{\mu}\left(\widehat{x}-\widehat{\chi}^{(t-1)}\right)=\operatorname{HEAD}_{\mu}(\widehat{x}) \backslash \operatorname{EsT}^{(t-1)}$. Consequently, if we let $\widehat{y}:=\widehat{x}-\widehat{\chi}^{(t-1)}$, then it holds that: i) $\operatorname{HEAD}_{\mu}(\widehat{y}) \subseteq \operatorname{supp}(T)$, by $P_{1}(t-1)$, ii) $\left\|\widehat{y}-\widehat{y}_{\text {HEAD }_{\mu}(\widehat{y})}\right\|_{2}^{2} \leq$ $\frac{11 \mu^{2}}{10}$, by precondition of the lemma along with (13), and iii) $\mid$ FreqCone $_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y}) \mid \leq$ $\left|\operatorname{HEAD}_{\mu}(\widehat{x})\right| \leq k$, by precondition of the lemma. Additionally, by $P_{4}(t-1)$, we find that

$$
\left|\operatorname{FreqCone}_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y})\right|=\left|\operatorname{FreqCone}_{T}\left(v_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{x})\right|>b
$$

Therefore, all preconditions of the second statement of Lemma 21 hold, and thus, by invoking this lemma we have that, with probability at least $1-\frac{1}{N^{4}}, \operatorname{IsCorR}_{\text {left }}=$ False, and $\widehat{\chi}_{\text {left }} \equiv 0$. So, the if-statement in line 23 of the algorithm is False and consequently the algorithm adds leaf $v_{\text {left }}$ as the left child of $v$ to tree $\operatorname{FRONTIER}^{(t-1)}$. By a similar argument, if $\mid$ FreqCone $_{T}\left(v_{\text {right }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{x}) \mid>$ $b$, then, with probability $1-\frac{1}{N^{4}}$, the algorithm adds leaf $v_{\text {right }}$ as the left child of $v$ to tree Frontier ${ }^{(t-1)}$.

Based on the above arguments, according to the values of $\mathrm{ISCORR}_{\text {left }}$ and $\mathrm{ISCORR}_{\text {right }}$, there are various cases that can happen. From the way tree FRONTIER $^{(t)}$ and set MARKED ${ }^{(t)}$ are obtained
from Frontier ${ }^{(t-1)}$ and $\operatorname{Marked}^{(t-1)}$, it follows that in any case, the first 4 properties of $P(t)$ are maintained with probability at least $1-\frac{2}{N^{4}}$. Furthermore, the way tree $T^{(t)}$ is constructed implies that,

$$
\sum_{u \in \operatorname{LEAVEs}\left(\text { Frontier }_{v}^{(t)}\right)}\left|L_{u}^{(t)}\right|=\left|L_{v}^{(t-1)}\right| .
$$

Therefore, for every $t>1$, by inductive hypothesis $P_{2}(t-1)$, the change in potential is bounded as follows,

$$
\begin{aligned}
\phi_{t}-\phi_{t-1} & =\sum_{u \in \operatorname{LEAVES}\left(\text { Frontier }_{v}^{(t)}\right)}\left(2 \log N-l_{\text {Frontier } \left.^{(t)}(u)\right) \cdot\left|L_{u}^{(t)}\right|-\left(2 \log N-l_{\text {Frontier } \left.^{(t-1)}(v)\right)} \cdot\left|L_{v}^{(t-1)}\right|\right.}\right. \\
& \leq-\left|L_{v}^{(t-1)}\right|<-b .
\end{aligned}
$$

Moreover, if $t=1$ then the change in potential satisfies $\phi_{1}-\phi_{0} \leq-\left|L_{v}^{(t-1)}\right| \leq 0$ (because in this case $v=$ root). This proves the inductive claim $P_{5}(t)$.

We have proved that for every $t$, if the inductive hypothesis $P(t-1)$ is satisfied then the property $P(t)$ is maintained with probability at least $1-\frac{2}{N^{4}}-\frac{1}{N^{7}} \geq 1-\frac{4}{N^{4}}$. Therefore, using the inductive hypothesis that $\operatorname{Pr}[P(t-1)] \geq 1-\frac{4(t-1)}{N^{4}}$, by using union bound we find that

$$
\operatorname{Pr}[P(t)] \geq \operatorname{Pr}[P(t) \mid P(t-1)] \cdot \operatorname{Pr}[P(t-1)] \geq 1-\frac{4 t}{N^{4}}
$$

This complets the proof of the lemma.
Now we are in a position to prove the main result of this section.
Proof of Theorem 10. The proof basically follows by invoking Lemma 22 and then analyzing the runtime and sample complexity of Algorithm 9. If we let $\mu:=\|\eta\|_{2}$ then because $x$ is a signal in the $k$-high SNR regime, we have that $\left|\operatorname{HEAD}_{\mu}(\widehat{x})\right| \leq k$ and $\left\|\widehat{x}-\widehat{x}_{\operatorname{HEAD}_{\mu}(\widehat{x})}\right\|_{2} \leq \mu$. Therefore, if we run the procedure RobustSparseFT (Algorithm 9) with inputs $(x, k, \epsilon, \mu)$, then the preconditions of Lemma 22 hold and hence by invoking this lemma we conclude that all the invariants $P_{1}(t)$ through $P_{5}(t)$, defined in Lemma 22, hold throughout the execution of Algorithm 9 for every non-negative integer $t$.

Using this, we first prove the termination of the algorithm. Let $q=O\left(k+\frac{k \log N}{b}\right)$ be some large enough integer. We show that the algorithm must terminate in $q$ iterations. Note that the probability that the properties $P(t)$ hold for all iterations $t \in\{0,1, \ldots q\}$ of algorithm RobustSparseFFT is at least $1-\frac{4(q+1)}{N^{4}} \geq 1-\frac{1}{N^{3}}$, by Lemma 22 . From now on, we condition on the event corresponding to $P(t)$ holding for all iterations $t \in\{0,1, \ldots q\}$, which holds with probability at least $1-\frac{1}{N^{3}}$. Conditioned on this event we prove that the algorithm terminates in less than $q$ iterations.

Note that, the potential function $\phi_{t}$ defined in $P_{5}(t)$ is non-negative for every $t$. Moreover, at the zeroth iteration of the algorithm $T^{(0)}=\{\operatorname{root}\}$ and hence $L_{\text {root }}^{(0)}=\operatorname{HEAD}_{\mu}(\widehat{x})$, thus

$$
\phi_{0} \leq 2 k \log N
$$

Therefore, it follows from $P_{5}(t)$ that Algorithm 9 must terminate in at most $q=O\left(k+\frac{k \log N}{b}\right)$ iterations.

When the algorithm terminates, the condition of the Repeat-Until loop in line 33 of the algorithm must be True. Thus, when the algorithm terminates, there is no leaf in tree $T^{(q)}$ besides the root. Cosequently, by invariants $P_{1}(q)$ and $P_{3}(q)$, the output of the algorithm satisfies, $\operatorname{HEAD}_{\mu}(\widehat{x}) \subseteq$ $\operatorname{supp}(\widehat{\chi})$ and $\left\|\widehat{x}_{\mathrm{EsT}}-\widehat{\chi}\right\|_{2}^{2} \leq \frac{\epsilon|\mathrm{EsT}|}{k} \cdot \mu^{2}$, where $\operatorname{Est}=\operatorname{supp}(\widehat{\chi})$. Using the invariant $P_{4}(q)$, the latter can be Further upper bounded as $\left\|\widehat{x}_{\text {EST }}-\widehat{\chi}\right\|_{2}^{2} \leq \epsilon \cdot \mu^{2}$. This together with the $k$-high SNR assumption of the theorem gives the approximation guarantee of the theorem $\|\widehat{x}-\widehat{\chi}\|_{2}^{2} \leq$ $(1+\epsilon) \cdot\|\eta\|_{2}^{2}$.

Runtime and Sample Complexity. The expensive components of the algorithm are primitive Estimate in line 11 and primitive RobustPromiseSFT in lines 21 and 22 of the algorithm. We first bound the time and sample complexity of invoking Estimate in line 11. We remark that, at any iteration $t$, the algorithm runs primitive Estimate only if case 1 that we mentioned earlier in the proof happens. Therefore, in this case, the set $\varnothing \neq \operatorname{ChEAP}^{(t)} \subseteq \operatorname{MARKED}^{(t-1)}$ that our algorithm computes in line 9 by running the primitive ExtractCheapSubset satisfies the property that $\left|\operatorname{Cheap}^{(t)}\right| \cdot\left(8+4 \log \left|\operatorname{Marked}^{(t-1)}\right|\right) \geq \max _{u \in \operatorname{Cheap~}^{(t)}} 2^{w_{\text {Fronter }}(t-1)}(u)$. By $P_{2}(t-1)$, and


Thus, by Lemma 19, the runtime and sample complexity of every invocation of Estimate in line 11 of our algorithm are bounded by $\widetilde{O}\left(\frac{k}{\epsilon \cdot \mid \text { Cheap }^{(t)} \mid} \sum_{u \in \text { Cheap }^{(t)}} 2^{w_{\text {Frontier }}(t-1)}(u)+\frac{k}{\epsilon}\left\|\widehat{\chi}^{(t-1)}\right\|_{0}\right)$ and $\widetilde{O}\left(\frac{k}{\epsilon \cdot\left|\operatorname{Cheap}^{(t)}\right|} \sum_{u \in \operatorname{Cheap}^{(t)}} 2^{w_{\text {Frontifer }}(t-1)}(u)\right)$, respectively. Using $P_{4}(t-1)$, the runtime and sample complexity of Estimate can be further upper bounded by $\widetilde{O}\left(\frac{k}{\epsilon} \cdot\left|\operatorname{CHEAP}^{(t)}\right|+\frac{k^{2}}{\epsilon}\right)$ and $\widetilde{O}\left(\left.\frac{k}{\epsilon} \cdot \right\rvert\,\right.$ Cheap $\left.^{(t)} \mid\right)$, respectively. By property $P_{5}(t)$ we find that the total number of iterations in which case 1 happens, and hence number of times we run Estimate in line 11 of the algorithm, is bounded by $O(k)$. Using this together with the fact that $\sum_{t: \text { if-statement in line } \square \text { is True }}\left|\operatorname{CHEAP}^{(t)}\right|=$ $\|\widehat{\chi}\|_{0} \leq k$, the total runtime and sample complexity of all invocations of Estimate in all iterations can be upper bounded by $\widetilde{O}\left(\frac{k^{3}}{\epsilon}\right)$ and $\widetilde{O}\left(\frac{k^{2}}{\epsilon}\right)$, respectively.

Now we bound the runtime and sample complexity of invoking RobustPromiseSFT in lines 21 and 22 of the algorithm. Note that at any iteration $t$, the algorithm runs RobustPromiseSFT in lines 21 and 22 only if case 2 that we mentioned earlier in the proof happens. Since we pick leaf $v$ in line 17 of the algorithm with smallest weight, and since the number of leaves that are not in the set $\operatorname{Marked}^{(t-1)}$ are bounded by $\frac{k}{b}$ (by invariant $P_{2}(t-1)$ ), we have $w_{T^{(t-1)}}(v) \leq \log \frac{k}{b}$. Also note that $\left\|\widehat{\chi}^{(t-1)}\right\|_{0} \leq k$ by invariant $P_{4}(t-1)$ and the $k$-high SNR assumption.

Therefore, by Lemma 20, the runtime and sample complexity of each invokation of RobustPromiseSFT by our algorithm are bounded by $\widetilde{O}\left(k \cdot\left(b^{2}+k\right)+\frac{k}{b} \cdot\left(b^{3}+k\right)\right)$ and $\widetilde{O}\left(\frac{k}{b} \cdot\left(b^{3}+k\right)\right)$. By property $P_{5}(t)$ we find that the total number of iterations in which case 2 happens, and hence the number of times we run RobustPromiseSFT in lines 21 and 22 of the algorithm, is bounded by $O\left(\frac{k \log N}{b}\right)$. Therefore, by using $b \approx k^{1 / 3}$, we find that the total runtime and sample complexity of all invocations of RobustPromiseSFT are bounded by $\widetilde{O}\left(k^{8 / 3}\right)$ and $\widetilde{O}\left(k^{7 / 3}\right)$, respectively. Hence, the total time and sample complexity of the algorithm are bounded by $\widetilde{O}\left(\frac{k^{3}}{\epsilon}\right)$ and $\widetilde{O}\left(k^{7 / 3}+\frac{k^{2}}{\epsilon}\right)$, respectively.

### 11.3 Proving the Correctness of our Computational Primitives

In this subsection, we shall prove Lemmas 18, 19, and Claim 6. We proceed by proving them in the aforementioned order.
Proof of Lemma 18:
By convolution-multiplication theorem, $h_{\Delta}^{z}$ computed in line 8 of Algorithm 6 satisfies $h_{\Delta}^{z}=$ $N \cdot\left(\chi \star G_{v}\right)(\Delta)$, and thus

$$
\begin{aligned}
H^{z} & =\frac{1}{\left|\operatorname{RIP}_{m}^{z}\right|} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}}\left|N \cdot \sum_{j \in[n]^{d}} G_{v}(\Delta-\boldsymbol{j}) \cdot x(\boldsymbol{j})-h_{\Delta}^{z}\right|^{2} \\
& =\frac{N^{2}}{\left|\operatorname{RIP}_{m}^{z}\right|} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}}\left|\left((x-\chi) \star G_{v}\right)(\Delta)\right|^{2} .
\end{aligned}
$$

Therefore, by the convolution-multiplication duality and using the definition $\widehat{y}:=(\widehat{x}-\widehat{\chi}) \cdot \widehat{G}_{v}$, if we let $y$ be the inverse Fourier transform of $\widehat{y}$, we find that for every $z \in[32 \log N]$,

$$
H^{z}=\frac{N^{2}}{\left|\operatorname{RIP}_{m}^{z}\right|} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}}|y(\Delta)|^{2}
$$

We first prove the first claim of the Lemma. Let us write $\widehat{y}=\widehat{y}_{S}+\widehat{y}_{\bar{S}}$, where $\widehat{y}_{S} \in \mathbb{C}^{n^{d}}$ is defined as $\widehat{y}_{S}(\boldsymbol{f}):=\widehat{y}(\boldsymbol{f}) \cdot \mathbb{1}_{\{\boldsymbol{f} \in S\}}$ and $\widehat{y}_{\bar{S}} \in \mathbb{C}^{n^{d}}$ is defined as $\widehat{y}_{\bar{S}}(\boldsymbol{f}):=\widehat{y}(\boldsymbol{f}) \cdot \mathbb{1}_{\{\boldsymbol{f} \notin S\}}$. By the assumption of lemma $\left\|\widehat{y}_{S}\right\|_{2}^{2}>\frac{11 \theta}{10}$. Let $y_{S}$ and $y_{\bar{S}}$ denote the inverse Fourier transform of $\widehat{y}_{S}$ and $\widehat{y}_{\bar{S}}$ respectively. We have $y=y_{S}+y_{\bar{S}}$. Thus we find that,

$$
\begin{aligned}
\frac{1}{\left|\operatorname{RIP}_{m}^{z}\right|} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}}|y(\Delta)|^{2} & =\frac{1}{m} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}}\left|y_{S}(\Delta)+y_{\bar{S}}(\Delta)\right|^{2} \\
& =\frac{1}{m} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}}\left|y_{S}(\Delta)\right|^{2}+\left|y_{\bar{S}}(\Delta)\right|^{2}+2 \Re\left\{y_{S}(\Delta)^{*} \cdot y_{\bar{S}}(\Delta)\right\} \\
& \geq \frac{1}{m} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}}\left|y_{S}(\Delta)\right|^{2}+2 \Re\left\{y_{S}(\Delta)^{*} \cdot y_{\bar{S}}(\Delta)\right\}
\end{aligned}
$$

First note that since $\widehat{y}_{S}$ is $|S|$-sparse and because we assumed $m=\Omega\left(|S| \log ^{2}|S| \log N\right)$ and because $\Delta$ 's are i.i.d. uniform samples from $[n]^{d}$, by Theorem 5 ,

$$
\begin{equation*}
\operatorname{Pr}\left[\frac{1}{m} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}}\left|y_{S}(\Delta)\right|^{2} \geq 0.99 \cdot \frac{\left\|\widehat{y}_{S}\right\|_{2}^{2}}{N^{2}}\right] \geq 1-\frac{1}{N^{2}} \tag{14}
\end{equation*}
$$

Now it suffices to bound the term $\frac{1}{m} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} 2 \Re\left\{y_{S}(\Delta)^{*} \cdot y_{\bar{S}}(\Delta)\right\}$. First, note that

$$
\begin{aligned}
\mathbb{E}\left[\frac{1}{m} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} 2 \Re\left\{y_{S}(\Delta)^{*} \cdot y_{\bar{S}}(\Delta)\right\}\right] & =\frac{1}{m} \sum_{\Delta \in \mathrm{RIP}_{m}^{z}} \mathbb{E}\left[y_{S}(\Delta)^{*} \cdot y_{\bar{S}}(\Delta)\right]+\mathbb{E}\left[y_{S}(\Delta) \cdot y_{\bar{S}}(\Delta)^{*}\right] \\
& =\frac{1}{m} \sum_{\Delta \in \mathrm{RIP}_{m}^{z}} \frac{1}{N}\left\langle y_{S}, y_{\bar{S}}\right\rangle+\frac{1}{N}\left\langle y_{\bar{S}}, y_{S}\right\rangle \\
& =\frac{1}{m} \sum_{\Delta \in \mathrm{RIP}_{m}^{z}} \frac{1}{N^{2}}\left\langle\widehat{y}_{S}, \widehat{y}_{\bar{S}}\right\rangle+\frac{1}{N^{2}}\left\langle\widehat{y}_{\bar{S}}, \widehat{y}_{S}\right\rangle \\
& =0,
\end{aligned}
$$

where the last line follows because the support of $\widehat{y}_{\bar{S}}$ and $\widehat{y}_{S}$ are disjoint. We proceed by bounding the second moment of the quantity $\frac{1}{m} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} 2 \Re\left\{y_{S}(\Delta)^{*} \cdot y_{\bar{S}}(\Delta)\right\}$ as follows,

$$
\begin{aligned}
\mathbb{E}\left[\left|\frac{1}{m} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} 2 \Re\left\{y_{S}(\Delta)^{*} \cdot y_{\bar{S}}(\Delta)\right\}\right|^{2}\right] & \leq \mathbb{E}\left[\left|\frac{2}{m} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} y_{S}(\Delta)^{*} \cdot y_{\bar{S}}(\Delta)\right|^{2}\right] \\
& =\frac{4}{m} \mathbb{E}\left[\left|y_{S}(\Delta)^{*} \cdot y_{\bar{S}}(\Delta)\right|^{2}\right] \quad \text { (By independence of } \Delta \prime \text { s) } \\
& \leq \frac{4}{m} \mathbb{E}\left[\left\|y_{S}\right\|_{\infty}^{2}\left|y_{\bar{S}}(\Delta)\right|^{2}\right] \\
& =\frac{4}{m}\left\|y_{S}\right\|_{\infty}^{2} \mathbb{E}\left[\left|y_{\bar{S}}(\Delta)\right|^{2}\right] \\
& =\frac{4}{m}\left\|y_{S}\right\|_{\infty}^{2} \frac{\left\|\widehat{y}_{\bar{S}}\right\|_{2}^{2}}{N^{2}}
\end{aligned}
$$

By Chebyshev's inequality we have the following,

$$
\begin{aligned}
\operatorname{Pr}\left[\left|\frac{1}{m} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} 2 \Re\left\{y_{S}(\Delta)^{*} \cdot y_{\bar{S}}(\Delta)\right\}\right| \geq 1 / 20 \cdot \frac{\left\|\widehat{y}_{S}\right\|_{2}^{2}}{N^{2}}\right] & \leq \frac{1600 N^{2}\left\|y_{S}\right\|_{\infty}^{2}\left\|\widehat{y}_{\bar{S}}\right\|_{2}^{2}}{m\left\|\widehat{y}_{S}\right\|_{2}^{4}} \\
& \leq \frac{1600\left\|\widehat{y}_{S}\right\|_{1}^{2}\left\|\widehat{y}_{\widehat{S}_{S}}\right\|_{2}^{2}}{m\left\|\widehat{y}_{S}\right\|_{2}^{4}} \\
& \leq \frac{1600|S| \cdot\left\|\widehat{y}_{S}\right\|_{2}^{2}\left\|\widehat{y}_{\bar{S}}\right\|_{2}^{2}}{m\left\|\widehat{y}_{S}\right\|_{2}^{4}} \quad \text { (Cauchy-Schwarz) } \\
& =\frac{1600|S| \cdot\left\|\hat{y}_{S}\right\|_{2}^{2}}{m\left\|\widehat{y}_{S}\right\|_{2}^{2}} .
\end{aligned}
$$

Therefore because we assumed that $m=\Omega\left(|S| \frac{\|\widehat{y}\|_{2}^{2}}{\left\|\widehat{y_{S}}\right\|_{2}^{2}}\right)$, the following holds,

$$
\operatorname{Pr}\left[\left|\frac{1}{m} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} 2 \Re\left\{y_{S}(\Delta)^{*} \cdot y_{\bar{S}}(\Delta)\right\}\right| \geq 1 / 20 \cdot \frac{\left\|\widehat{y}_{S}\right\|_{2}^{2}}{N^{2}}\right] \leq 1 / 10 .
$$

Combining the above inequality with (14) using union bound gives,

$$
\operatorname{Pr}\left[H^{z} \leq 0.94 \cdot\left\|\widehat{y}_{S}\right\|_{2}^{2}\right] \leq 1 / 8 .
$$

Since in line 11 of the algorithm we compare $\operatorname{MEDIAN}_{z \in[32 \log N]}\left\{H^{z}\right\}$ to $\theta$, using the fact that $\left\|\widehat{y}_{S}\right\|_{2}^{2}>\frac{11 \theta}{10}$, we have the following,

$$
\begin{aligned}
\operatorname{Pr}[\text { HeavyTest }=\text { False }] & \leq \operatorname{Pr}\left[\operatorname{MEDIAN}_{z \in[32 \log N]}\left\{H^{z}\right\} \leq 10 / 11 \cdot\left\|\widehat{y}_{S}\right\|_{2}^{2}\right] \\
& \leq\binom{ 32 \log N}{16 \log N} \frac{1}{8^{16 \log N}} \\
& \leq \frac{2^{32 \log N}}{8^{16 \log N}}=\frac{1}{N^{16}} .
\end{aligned}
$$

This completes the proof of the first claim.
The proof of the second claim of the lemma is more straightforward. The expected value of $H^{z}$ is,

$$
\mathbb{E}\left[H^{z}\right]=\frac{N^{2}}{\left|\operatorname{RIP}_{m}^{z}\right|} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} \mathbb{E}\left[|y(\Delta)|^{2}\right]=\|\widehat{y}\|_{2}^{2}
$$

Therefore by Markov's inequality we find that for every $z \in[32 \log N]$,

$$
\operatorname{Pr}\left[H^{z} \geq 5\|\widehat{y}\|_{2}^{2}\right] \leq 1 / 5 .
$$

The assumption of the lemma in this case is that $\|\widehat{y}\|_{2}^{2} \leq \theta / 5$, thus we have,

$$
\begin{aligned}
\operatorname{Pr}[\text { HeavyTest }=\text { True }] & \leq \operatorname{Pr}\left[\operatorname{Median}_{z \in[32 \log N]}\left\{H_{f}^{z}\right\}>5 \cdot\left\|\widehat{y}_{S}\right\|_{2}^{2}\right] \\
& \leq\binom{ 32 \log N}{16 \log N} \frac{1}{5^{16 \log N}} \\
& \leq \frac{2^{32 \log N}}{5^{16 \log N}}=\frac{1}{N^{5}} .
\end{aligned}
$$

This completes the proof of the second claim of the lemma.
Sample Complexity and Runtime: Computing the filters $\left(G_{v}, \widehat{G}_{v}\right)$ uses $O\left(2^{w_{T}(v)}+\log N\right)$ runtime, by Lemma 5. Given filter $\widehat{G}_{v}$, computing the quantities $h_{\Delta}^{z}$ for all $\Delta$ and $z$ in line 8 of the algorithm uses $O\left(\|\widehat{\chi}\|_{0} \cdot \sum_{z}\left|\operatorname{RIP}_{m}^{z}\right|\right)=O\left(\|\widehat{\chi}\|_{0} \cdot m \log N\right)$ time. Given filter $G_{v}$ with $\left|\operatorname{supp}\left(G_{v}\right)\right|=2^{w_{T}(v)}$, computing the quantity $H^{z}$ for all $z$ requires $O\left(2^{w_{T}(v)} \cdot \sum_{z}\left|\operatorname{RIP}_{m}^{z}\right|\right)=$ $O\left(2^{w_{T}(v)} \cdot m \log N\right)$ accesses to the signal $x$ and $O\left(2^{w_{T}(v)} \cdot m \log N\right)$ runtime. Therefore, the total sample complexity of the algorithm is $O\left(2^{w_{T}(v)} \cdot m \log N\right)$ and the total runtime of the algorithm is $O\left(2^{w_{T}(v)} \cdot m \log N+\|\widehat{\chi}\|_{0} \cdot m \log N\right)$

Proof of Lemma 19; Note that the algorithm constructs $(v, T)$-isolating filters ( $G_{v}, \widehat{G}_{v}$ ) for every leaf $v \in S$. By Lemma 5, constructing filters $G_{v}$ and $\widehat{G}_{v}$ takes time $O\left(2^{w_{T}(v)}+\log N\right)$. Moreover, Lemma 5 tells us that filter $G_{v}$ has support size $\left|\operatorname{supp}\left(G_{v}\right)\right|=2^{w_{T}(v)}$ and $\widehat{G}_{v}$ can be accessed at any frequency using $O(\log N)$ operations.
Therefore, for every fixed $v \in S$, computing $h_{v}^{z}=\sum_{\Delta \in \operatorname{RIP}_{m}^{z}} e^{-2 \pi i \frac{f^{\top} \Delta}{n}} \sum_{\boldsymbol{\xi} \in[n] d} e^{2 \pi i \frac{\xi^{T} \Delta}{n}} \cdot \widehat{\chi}_{\xi} \cdot \widehat{G}_{v}(\boldsymbol{\xi})$ in line 9 of Algorithm 7 can be done in total time $O\left(\left|\operatorname{RIP}_{m}^{z}\right| \log N \cdot\|\widehat{\chi}\|_{0}\right)=O\left(B \log N \cdot\|\widehat{\chi}\|_{0}\right)$ for all $z$. By convolution-multiplication duality theorem, $h_{v}^{z}$ satisfies $h_{v}^{z}=N \cdot \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} e^{-2 \pi i \frac{f^{\top} \Delta}{n}}\left(\chi \star G_{v}\right)(\Delta)$, and thus, for every leaf $v \in S$ :

$$
\begin{aligned}
H_{v}^{z} & =\frac{1}{\left|\operatorname{RIP}_{m}^{z}\right|} \cdot\left(N \cdot \sum_{\Delta \in \operatorname{RIP}_{m}^{z}}\left(e^{-2 \pi i \frac{f^{\top} \Delta}{n}} \sum_{j \in[n]^{d}} G_{v}(\Delta-\boldsymbol{j}) \cdot x(\boldsymbol{j})\right)-h_{v}^{z}\right) \\
& =\frac{N}{\left|\operatorname{RIP}_{m}^{z}\right|} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} e^{-2 \pi i \frac{f^{\top} \Delta}{n}}\left((x-\chi) \star G_{v}\right)(\Delta) .
\end{aligned}
$$

To simplify the notation, let us use $y_{v}:=(x-\chi) \star G_{v}$. Because $G_{v}$ is $(v, T)$-isolating, by Definition 7 , we have that $\widehat{y}_{v}(\boldsymbol{\xi})=0$ for every $\boldsymbol{\xi} \in \bigcup_{\substack{u \in \operatorname{LEAvEs}(T) \\ u \neq v}} \operatorname{Freq}^{\operatorname{Cone}}{ }_{T}(u)$ and also $\widehat{y}_{v}(\boldsymbol{f})=(\widehat{x-\chi})(\boldsymbol{f})$, where $f:=\boldsymbol{f}_{v}$ is the frequency label of the leaf $v$. Using these facts together with the above equality and the assumption of the lemma on $\operatorname{IsIdentified~}(T, v)=$ True, we can write,

$$
\begin{aligned}
H_{v}^{z} & =\frac{N}{\left|\operatorname{RIP}_{m}^{z}\right|} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} e^{-2 \pi i \frac{f^{\top} \Delta}{n}} y_{v}(\Delta) \\
& =\widehat{y}_{v}(\boldsymbol{f})+\frac{1}{\left|\operatorname{RIP}_{m}^{z}\right|} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)} e^{2 \pi i \frac{(\boldsymbol{\xi}-f)^{\top} \Delta}{n}} \cdot \widehat{y}_{v}(\boldsymbol{\xi}) .
\end{aligned}
$$

We continue by computing the expectation of the above quantity. Since $\boldsymbol{f} \in \operatorname{FreqCone}_{T}(v), \boldsymbol{\xi}-\boldsymbol{f} \neq$ 0 for every $\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)$, which in turn implies that,

$$
\mathbb{E}\left[H_{v}^{z}\right]=\widehat{y}_{v}(\boldsymbol{f})+\frac{1}{\left|\operatorname{RIP}_{m}^{z}\right|} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)} \mathbb{E}_{\Delta}\left[e^{2 \pi i \frac{(\xi-f)^{\top} \Delta}{n}}\right] \widehat{y}_{v}(\boldsymbol{\xi})=\widehat{y}_{v}(\boldsymbol{f}) .
$$

In the above expectation we used the fact that $\Delta$ is distributed uniformly on $[n]^{d}$. Next we compute the second moment of $H_{v}^{z}$. We have,

$$
\begin{aligned}
\mathbb{E}\left[\left|H_{v}^{z}-\widehat{y}_{v}(\boldsymbol{f})\right|^{2}\right] & =\frac{1}{\left|\operatorname{RIP}_{m}^{z}\right|^{2}} \sum_{\Delta \in \operatorname{RIP}_{m}^{z}} \mathbb{E}\left[\left|\sum_{\mid \boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)} e^{2 \pi i \frac{(\boldsymbol{\xi}-\boldsymbol{f})^{\top} \Delta}{n}} \widehat{y}_{v}(\boldsymbol{\xi})\right|^{2}\right] \quad \text { (by independence of } \Delta \prime \mathrm{s} \text { ) } \\
& =\frac{1}{\left|\operatorname{RIP}_{m}^{z}\right|} \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)}\left|\widehat{y}_{v}(\boldsymbol{\xi})\right|^{2} \quad\left(\text { since } \Delta \text { is uniform over }[n]^{d} \text { and } \boldsymbol{\xi}-\boldsymbol{f} \neq 0\right. \text { ) } \\
& =\frac{1}{B} \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)}\left|(\widehat{x-\chi})(\boldsymbol{\xi}) \cdot \widehat{G}_{v}(\boldsymbol{\xi})\right|^{2} .
\end{aligned} \quad \text { (by definition of } y \text { ) }
$$

In the final line above we used the fact that the multiset RIP $_{m}^{z}$ defined in Algorithm 7 has size $m$. Therefore, Markov's inequality implies that for every $z \in[16 \log N]$,

$$
\operatorname{Pr}\left[\left|H_{v}^{z}-\widehat{y}_{v}(\boldsymbol{f})\right|^{2} \geq \frac{8}{m} \cdot \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)}\left|(\widehat{x-\chi})(\boldsymbol{\xi}) \cdot \widehat{G}_{v}(\boldsymbol{\xi})\right|^{2}\right] \leq \frac{1}{8}
$$

Since in line 11 of Algorithm 7 we set $\widehat{H}_{v}=\operatorname{MEDIAN}_{z \in[16 \log N]}\left\{H_{v}^{z}\right\}$, where the median of real and imaginary parts are computed separately, we find that

$$
\begin{aligned}
\operatorname{Pr}\left[\left|\widehat{H}_{v}-\widehat{y}_{v}(\boldsymbol{f})\right|^{2} \geq \frac{16}{m} \cdot \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)}\left|(\widehat{x-\chi})(\boldsymbol{\xi}) \cdot \widehat{G}_{v}(\boldsymbol{\xi})\right|^{2}\right] & \leq\binom{ 16 \log N}{8 \log N} \frac{1}{8^{8 \log N}} \\
& \leq \frac{2^{16 \log N}}{8^{8 \log N}}=\frac{1}{N^{8}}
\end{aligned}
$$

By recalling that $\widehat{y}_{v}(\boldsymbol{f})=(\widehat{x-\chi})\left(\boldsymbol{f}_{v}\right)$ for every $v \in S$ and applying union bound we find that,

$$
\begin{equation*}
\operatorname{Pr}\left[\sum_{v \in S}\left|\widehat{H}_{v}-(\widehat{x-\chi})\left(\boldsymbol{f}_{v}\right)\right|^{2} \geq \frac{16}{m} \cdot \sum_{v \in S} \sum_{\boldsymbol{\xi} \in[n] d \backslash \operatorname{supp}(T)}\left|(\widehat{x-\chi})(\boldsymbol{\xi}) \cdot \widehat{G}_{v}(\boldsymbol{\xi})\right|^{2}\right] \leq \frac{|S|}{N^{8}} . \tag{15}
\end{equation*}
$$

In the last step, we bound the quantity $\sum_{v \in S} \sum_{\boldsymbol{\xi} \in[n] d \backslash \operatorname{supp}(T)}\left|(\widehat{x-\chi})(\boldsymbol{\xi}) \cdot \widehat{G}_{v}(\boldsymbol{\xi})\right|^{2}$ as follows,

$$
\begin{aligned}
\sum_{v \in S} \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)}\left|(\widehat{x-\chi})(\boldsymbol{\xi}) \cdot \widehat{G}_{v}(\boldsymbol{\xi})\right|^{2} & =\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)}|(\widehat{x-\chi})(\boldsymbol{\xi})|^{2} \cdot \sum_{v \in S}\left|\widehat{G}_{v}(\boldsymbol{\xi})\right|^{2} \\
& \leq \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)}|(\widehat{x-\chi})(\boldsymbol{\xi})|^{2} \cdot \sum_{v \in \operatorname{LEAvEs}(T)}\left|\widehat{G}_{v}(\boldsymbol{\xi})\right|^{2} \\
& =\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(T)}|(\widehat{x-\chi})(\boldsymbol{\xi})|^{2}, \quad \text { (By Lemma 17) }
\end{aligned}
$$

hence, plugging the above bound into (15) gives,

$$
\operatorname{Pr}\left[\sum_{v \in S}\left|\widehat{H}_{v}-(\widehat{x-\chi})\left(\boldsymbol{f}_{v}\right)\right|^{2} \geq \frac{16}{m} \cdot \sum_{\boldsymbol{\xi} \in[n] d \backslash \operatorname{supp}(T)}|(\widehat{x-\chi})(\boldsymbol{\xi})|^{2}\right] \leq \frac{|S|}{N^{8}} .
$$

Lastly, we prove the correctness of ExtractCheapSubset, and in particular Claim 6 .
Proof of Claim 6; First let $S^{\prime}:=\left\{u \in S: 2^{w_{T}(u)} \leq 4|S|\right\}$. It easily follows that $\sum_{u \in S^{\prime}} 2^{-w_{T}(u)} \geq$ $\frac{1}{4}$. For every $j=0,1, \ldots\lfloor\log (4|S|)\rfloor$, let $L_{j}$ denote the subset of $S^{\prime}$ defined as $L_{j}:=\{u: u \in$ $\left.S^{\prime}, w_{T}(u)=j\right\}$. We can write,

$$
\sum_{u \in S^{\prime}} 2^{-w_{T}(u)}=\sum_{j=0}^{\lfloor\log (4|S|)\rfloor} \frac{\left|L_{j}\right|}{2^{j}}
$$

Therefore, by the fact that $\sum_{u \in S^{\prime}} 2^{-w_{T}(u)} \geq \frac{1}{4}$, we have that there must exist an integer $j \in$ $\{0,1, \ldots\lfloor\log (4|S|)\rfloor\}$ such that $\frac{\left|L_{j}\right|}{2^{j}} \geq \frac{1}{4\lfloor\log (4|S|)\rfloor}$. Hence, there must exist a set $L \subseteq S$ such that $|L|$. $(8+4 \log |S|) \geq \max _{v \in L} 2^{w_{T}(v)}$. The primitive ExtractCheapSubset finds this set $L$ efficiently.

## 12 Robust Sparse Fourier Transform II

In this section we present an algorithm that can compute a $1+\epsilon$ approximation to the Fourier transform of a singnal in the $k$-high SNR regime using a sample complexity that is nearly quadratic in $k$ and a runtime that is cubic in $k$, fully making use of techniques I-IV.

Formally we prove the following theorem,
Theorem 4 (Robust Sparse Fourier Transform with Near-quadratic Sample Complexity). Given oracle access to $x:[n]^{d} \rightarrow \mathbb{C}$ in the $k$-high $S N R$ model and parameter $\epsilon>0$, we can solve the $\ell_{2} / \ell_{2}$ Sparse Fourier Transform problem with high probability in $N$ using

$$
m=\widetilde{O}\left(\frac{k^{2}}{\epsilon}+k^{2} \cdot 2^{\Theta(\sqrt{\log k \cdot \log \log N})}\right)
$$

samples from $x$ and $\widetilde{O}\left(\frac{k^{3}}{\epsilon}\right)$ running time.
We first present a recursive procedure in Algorithm 10 that is the main computational component of achieving the abovementioned theorem for a constant value of $\epsilon=\frac{1}{20}$. Any sparse $\widehat{\chi}$ that satisfies the approximation guarantee of Theorem 4 for constant $\epsilon$, by the $k$-high SNR assumption, must recover all the head elements of $\widehat{x}$ correctly. Once we have the set of heavy frequencies of $\widehat{x}$ we can estimate the head vlaues to a higher $\epsilon$ precision for arbitrarily small $\epsilon$ using a simple algorithm. We present the procedure that achieves such $1+\epsilon$ approximation and thus achieves the guarantee of Theorem 4 in Algorithm 11. We demonstrate the execution of primitive RecursiveRobustSFT (Algorithm 10) in Figure 6 .

Overview of RecursiveRobustSFT (Algorithm 10): Consider an invocation of Recur$\operatorname{siveRobustSFT}\left(x, \widehat{\chi}_{i n}\right.$, Frontier, $\left.v, k, \alpha, \mu\right)$. Suppose that $\widehat{y}:=\widehat{x}-\widehat{\chi}_{i n}$ is a signal in the high SNR regime, i.e., the value of each heavy frequency of signal $\widehat{y}$ is at least 3 times higher than the tail's norm. More formally, let HEAD $\subseteq[n]^{d}$ denote the set of heavy (head) frequencies of $\widehat{y}$ and suppose that the tail norm of $\widehat{y}$ satisfies $\left\|\widehat{y}-\widehat{y}_{\text {HEAD }}\right\|_{2} \leq \mu$ and additionally suppose that $|\widehat{y}(\boldsymbol{f})| \geq 3 \mu$ for every $\boldsymbol{f} \in$ HEAD. If Frontier fully captures the heavy frequencies of $\widehat{y}$, i.e., HEAD $\subseteq \operatorname{supp}($ Frontier $)$, and the number of heavy frequencies in frequency cone of node $v$ is bounded by $k$, i.e., $\mid$ head $\cap \operatorname{FreqCone~}_{\text {Frontier }}(v) \mid \leq k$, then RecursiveRobustSFT finds a signal $\widehat{\chi}_{v}$ such that $\operatorname{supp}\left(\widehat{\chi}_{v}\right)=\operatorname{HEAD} \cap \operatorname{FreqCone}_{\text {Frontier }}(v):=S$ and $\left\|\widehat{y}_{S}-\widehat{\chi}_{v}\right\|_{2}^{2} \leq \frac{\mu^{2}}{40 \log _{1}^{2} k}$. An example of the input tree Frontier is illustrated in Figure 6 with thin solid black edges. Additionally, one can see node $v$ which is a leaf of Frontier in this figure.

Algorithm 10 recovers heavy frequencies of signal $\widehat{y}$ that lie in the subree of $v$, i.e., set $S=$ head $\cap \operatorname{FreqCone}_{\text {Frontier }}(v)$, by iteratively exploring the subtree of Frontier rooted at $v$, which we denote by $T$, and simultaneously updating the proxy signal $\widehat{\chi}_{v}$. We show an example of subtree $T$ at some iteration of our algorithm in Figure 6 with thick solid edges. The algorithm also maintains a subset of leaves denoted by Marked that contains the leaves of Frontier that are fully identified, that is the set of leaves that are at the bottom level and hence there is no ambiguity in their frequency content (there is exactly one element in frequency cone of marked leaves). We show the set of marked leaves in Figure 6 using blue squares. Subtree $T$, in all iterations of our algorithm, maintains the invariant that the frequency cone of each of its leaves contain at least one head element and furthermore the frequency cone of each of its unmarked leaves contain at least $b+1$ head element, where $b=\alpha k$, i.e.,

$$
\left|\operatorname{FreqCone}_{\text {Frontier } \cup T}(u) \cap \operatorname{HEAd}\right| \geq\left\{\begin{array}{ll}
1 & \text { for every } u \in \operatorname{Marked}  \tag{16}\\
b+1 & \text { for every } u \in \operatorname{LEAVES}(T) \backslash \operatorname{Marked}
\end{array} .\right.
$$



Figure 6: Illustration of an instance of RecursiveRobustSFT (Algorithm 10). This procedure takes in a tree Frontier (shown with thin edges) together with a leaf $v \in$ leaves(Frontier) and adaptively explores/constructs the subtree $T$ rooted at $v$ to find all heavy frequencies that lie in FreqCone $_{\text {Frontier }}(v)$. If head denotes the set of heavy frequencies, then the algorithm finds head $\cap$ $\operatorname{FreqCone}_{\text {Sidetree }}(v)$ by exploring $T$. Once the identity of a leaf is fully revealed, the algorithm adds that leaf to the set Marked. When the number of marked leaves grows to the point where there exists a subset of marked frequencies that can be estimated cheaply, our algorithm estimates the Cheap subset in a batch, subtracts off the estimated signal, and removes all corresponding leaves from $T$ and Marked.

We demonstrate, in Figure 6, the leaves that correspond to set $S=$ head $\cap \operatorname{FreqCone}_{\text {Frontier }}(v)$ via leaves at bottom level of the subtree rooted at $v$. Assuming that for the example shown in this figure $b=\alpha k=2$, one can easily verify (16) by noting that the frequency cone of each leaf of $T$ contains at least one element from the set HEAD and frequency cones of unmarked leaves contain at least two element of head. Additionally, at every iteration of the algorithm, the union of all frequency cones of subtree $T$ captures all heavy frequencies that are not recovered yet, i.e.,

$$
\begin{equation*}
S \backslash \operatorname{supp}\left(\widehat{\chi}_{v}\right) \subseteq \operatorname{supp}(\text { Frontier } \cup T) . \tag{17}
\end{equation*}
$$

In Figure 6, we show the set of fully recovered leaves (frequencies), i.e., $\operatorname{supp}\left(\widehat{\chi}_{v}\right)$, using red thin dashed subtrees. These frequencies are subtracted from the residual signal $\widehat{y}-\widehat{\chi}_{v}$ and their corresponding leaves are removed from subtree $T$, as well. One can verify that condition 17 holds in the example depicted in Figure 6. Moreover, the estimated value of every frequency that is recovered so far, is accurate up to an average error of $\frac{\mu}{\sqrt{40 k} \cdot \log _{\frac{1}{\alpha}} k}$. More precisely, in every iteration of the
algorithm the following property is maintained,

$$
\begin{equation*}
\frac{\sum_{\boldsymbol{f} \in \operatorname{supp}\left(\hat{\chi}_{v}\right)}\left|\widehat{y}(\boldsymbol{f})-\widehat{\chi}_{v}(\boldsymbol{f})\right|^{2}}{\left|\operatorname{supp}\left(\widehat{\chi}_{v}\right)\right|} \leq \frac{\mu^{2}}{40 k \cdot \log _{\frac{1}{\alpha}}^{2} k} . \tag{18}
\end{equation*}
$$

At the begining of the procedure, subtree $T$ is initialized to be the leaf $v$, i.e., $T=\{v\}$, and will be dynamically changing throughout the execution of our algorithm. Moreover, we initialize $\widehat{\chi}_{v} \equiv 0$. Trivially, these initial values satisfy (16), (17), and (18).

The algorithm operates by picking the unmarked leaf of $T$ that has the smallest weight. Then the algorithm explores the children of this node by recursively running RecursiveRobustSFT on them with a reduced budget to recover the heavy frequencies that lie in their frequency cones. To be more precise, let us call the unmarked leaf of $T$ that has the smallest weight $z$. We denote by $z_{\text {left }}$ and $z_{\text {right }}$ the left and right children of $z$. Let us consider exploration of the left child $z_{\text {left }}$, the right child is exactly the same. If the number of heavy frequencies in the frequency cone of $z_{\text {left }}$ is bounded by $b=\alpha k$, i.e., $\mid$ HEAD $\cap$ FreqCone $_{\text {Frontieru }\left\{z_{\text {left }}, z_{\text {right }}\right\}}\left(z_{\text {left }}\right) \mid \leq b$, then Recursiver$\operatorname{ObuStSFT}\left(x, \widehat{\chi}_{i n}+\widehat{\chi}_{v}\right.$, Frontier $\left.\cup T \cup\left\{z_{\text {left }}, z_{\text {right }}\right\}, z_{\text {left }}, b, \alpha, \mu\right)$ recovers every frequency in the set HEAD $\cap$ FreqCone ${\left.\text { Frontieru } u z_{\text {left }}, z_{\text {right }}\right\}}\left(z_{\text {left }}\right)$ up to an average error of $\frac{\mu}{\sqrt{40 b \cdot \log _{\frac{1}{\alpha}} b}}$. Note that this everage estimation error is not sufficient for achieving the invariant (18), hence, instead of directly using the values that the recursive call of RecursiveRobustSFT recovered to update $\widehat{\chi}_{v}$ at the newly recovered heavy frequencies, our algorithm adds the leaves corresponding to the recovered set of frequencies, i.e., HEAD $\cap \operatorname{FreqCone}_{\left.\text {Frontieru } u v_{\text {left }}, v_{\text {right }}\right\}}\left(v_{\text {left }}\right)$, at the bottom level of $T$ and marks them as fully identified (adds them to Marked). It can be seen in Figure 6 that all marked leaves are at the bottom level of the tree. For achieving maximum efficinecy we employ a new lazy estimation scheme, that is, the estimation of values of marked leaves is delayed until there is a large number of marked leaves and thus there exists a subset of them that is cheap to estimate. On the other hand, if the number of head elements in frequency cone of $z_{\text {left }}$ is more than $b$ then our algorithm detects this and subsequently adds node $z_{\text {left }}$ to $T$. These operations ensure that the invariants (16), 17), and (18) are maintained.

Once the size of set Marked grows sufficiently such that it contains a subset that is cheap to estimate, our algorithm estimates the values of the cheap frequencies. More precisely, at some point, Marked will contains a non-empty subset Cheap such that the values of all frequencies in Cheap can be estimated cheaply and subsequently, our algorithm esimates those frequencies in a batch up to an average error of $O\left(\frac{\mu}{\sqrt{k} \cdot \log N}\right)$, updates $\widehat{\chi}$ accordingly and removes all estimated (Cheap) leaves from Frontier and Marked. This ensures that invariants (16), (17), and (18) are maintained. The estimated leaves are illustrated in Figure 6 using red thin dashed subtrees. We also demontrate the subtrees of $T$ that contain HEAD element and are yet to be explored by our algorithm using gray cones and dashed edges in Figure 6. The gray cone means that there are heavy elements in that frequency cone that need to be identified as that node has not reached the bottom level yet.

Finally, the algorithm keeps tabs on the runtime it spends and ensures that even if the input signal does not satisfy the preconditions for successful recovery, in particular if |HEAD $\cap$ FreqCone $_{\text {Frontier }}(v) \mid>k$, the runtime stays bounded. Additionally, the algorithm performs a quality control by running HeavyTest on the residual and if the recovered signal is not correct due to violation of some preconditions, it will be reflected in the output of our algorithm.

Analysis of RecursiveRobustSparseFT. Frirst we analyze the runtime and sample complexity of RecursiveRobustSparseFT in the following lemma.

```
Algorithm 10 A Recursive Robust High-dimensional Sparse FFT Algorithm
    procedure RecursiveRobustSFT( \(x, \widehat{\chi}_{i n}\), Frontier, \(v, k, \alpha, \mu\) )
        // \(\mu\) : upper bound on tail norm \(\|\eta\|_{2}\)
        if \(k \leq \frac{1}{\alpha}\) then return PromiseSparseFT \(\left(x, \widehat{\chi}_{i n}\right.\), Frontier, \(\left.v, k,\left\lceil\frac{k}{\alpha}\right\rceil, \mu\right)\)
        Let \(T\) denote the subtree of Frontier rooted at \(v\) - i.e. \(T \leftarrow\{v\}\)
        \(\widehat{\chi}_{v} \leftarrow\{0\}^{n^{d}} \quad \triangleright\) Sparse vector to approximate \(\left(\widehat{x}-\widehat{\chi}_{i n}\right)_{\text {FreqCone }_{\text {Frontirg }}(v)}\)
        \(b \leftarrow\lceil\alpha k\rceil\), Marked \(\leftarrow \varnothing \quad \triangleright\) Marked: set of fully identified leaves (frequencies)
        repeat
            if \((b+1) \cdot\left|\operatorname{leaves}\left(T_{v}\right) \backslash \operatorname{Marked}\right|+|\operatorname{Marked}|+\left\|\widehat{\chi}_{v}\right\|_{0}>k\) then
                return (False, \(\{0\}^{n^{d}}\) ) \(\triangleright\) Exit because budget of \(v\) is wrong
            if \(\sum_{u \in \text { Marked }} 2^{-w_{T}(u)} \geq \frac{1}{2}\) then
                    Cheap \(\leftarrow \operatorname{FindCheapToEstimate~(~} T\), Marked)
                    //Lazy estimation: We extract from the batch of marked leaves a subset that is cheap
    to estimate on average
                            \(\left\{\widehat{H}_{u}\right\}_{u \in \text { Cheap }} \leftarrow \operatorname{Estimate}\left(x, \widehat{\chi}_{i n}+\widehat{\chi}_{v}\right.\), Frontier \(\cup T\), Cheap, \(\left.\frac{736 k \cdot \log ^{2} N}{\mid \text { Cheap } \mid}\right)\)
                    for \(u \in\) CHEAP do
                    \(\widehat{\chi}_{v}\left(\boldsymbol{f}_{u}\right) \leftarrow \widehat{H}_{u}\)
                Remove node \(u\) from subtree \(T\)
                    Marked \(\leftarrow\) Marked \(\backslash\) Cheap
                    continue
            \(z \leftarrow \operatorname{argmin}_{u \in \operatorname{LEAVES}(T) \backslash \operatorname{MarkEd}} w_{T}(u)\)
            //pick the minimum weight leaf in subtree \(T\) which is not in Marked
            \(z_{\text {left }}:=\) left child of \(z\) and \(z_{\text {right }}:=\) right child of \(z\)
            \(T^{\prime} \leftarrow T \cup\left\{z_{\text {left }}, z_{\text {right }}\right\} \quad \triangleright\) Explore children of \(z\)
            \(\left(\operatorname{IsCorR}_{\text {left }}, \widehat{\chi}_{\text {left }}\right) \leftarrow \operatorname{RECURSIVEROBUSTSFT}\left(x, \widehat{\chi}_{i n}+\widehat{\chi}_{v}\right.\), Frontier \(\left.\cup T^{\prime}, z_{\text {left }}, b, \alpha, \mu\right)\)
            \(\left(\right.\) ISCorR \(\left._{\text {right }}, \widehat{\chi}_{\text {right }}\right) \leftarrow \operatorname{RECURSIVERobuStSFT}\left(x, \widehat{\chi}_{i n}+\widehat{\chi}_{v}\right.\), Frontier \(\left.\cup T^{\prime}, z_{\text {right }}, b, \alpha, \mu\right)\)
            if IsCorR \(_{\text {left }}\) and \(\operatorname{IsCorR}_{\text {right }}\) and \(z \neq v\) and \(\left\|\widehat{\chi}_{\text {left }}\right\|_{0}+\left\|\widehat{\chi}_{\text {right }}\right\|_{0} \leq b\) then
                return (False, \(\{0\}^{n^{d}}\) ) \(\triangleright\) Exit because budget of \(v\) is wrong
            if \(\mathrm{IsCorR}_{\text {left }}\) then
                    \(\forall \boldsymbol{f} \in \operatorname{supp}\left(\widehat{\chi}_{\text {left }}\right)\), add the unique leaf corresponding to \(\boldsymbol{f}\) to subtree \(T\) and Marked
        else
                            Add \(z_{\text {left }}\) to subtree \(T\)
        if \(\mathrm{IsCorR}_{\text {right }}\) then
            \(\forall \boldsymbol{f} \in \operatorname{supp}\left(\widehat{\chi}_{\mathrm{right}}\right)\), add the unique leaf corresponding to \(\boldsymbol{f}\) to subtree \(T\) and Marked
        else
            Add \(z_{\text {right }}\) to subtree \(T\)
        until \(T\) has no leaves besides \(v\)
        if HeavyTest \(\left(x, \widehat{\chi}_{i n}+\widehat{\chi}_{v}\right.\), Frontier, \(\left.v, O\left(\frac{k}{\alpha} \log ^{3} N\right), 6 \mu^{2}\right)\) then
            \(/ /\) The number of heavy coordinates in \(\mathrm{FreqCone}_{\text {Frontier }}(v)\) is more than \(k\)
            return (False, \(\{0\}^{n^{d}}\) )
        else
            return (True, \(\widehat{\chi}_{v}\) )
```

```
Algorithm 11 Robust High-dimensional Sparse FFT with \(\widetilde{O}\left(k^{3}\right)\) Time and \(\widetilde{O}\left(k^{2+o(1)}\right)\) Samples
    procedure RobustSFT \((x, k, \epsilon, \mu)\)
        \(\alpha \leftarrow 2^{-\sqrt{\log k \cdot \log (2 \log N)}}\)
        (IsCorr, \(\widehat{\chi}) \leftarrow \operatorname{RECURSIVERobustSFT}\left(x,\{0\}^{n^{d}},\{\right.\) root \(\}\), root, \(\left.k, \alpha, \mu\right)\)
        Let \(T\) be the splitting tree corresponding to the \(\operatorname{set} \operatorname{supp}(\widehat{\chi})\)
        \(\widehat{\chi}_{\epsilon} \leftarrow\{0\}^{n^{d}}\)
        while tree \(T\) has a leaf besides its root do
            Cheap \(\leftarrow\) FindCheapToEstimate \((T\), Leaves \((T))\)
            \(/ /\) The set of frequencies that are cheap to estimate on average
            \(\left\{\widehat{H}_{u}\right\}_{u \in \text { Cheap }} \leftarrow \operatorname{Estimate}\left(x, \widehat{\chi}_{\epsilon}, T\right.\), Cheap, \(\left.\frac{32 k}{\epsilon \cdot \mid \text { Cheap } \mid}\right)\)
            for \(u \in\) Cheap do
                \(\widehat{\chi}_{\epsilon}\left(\boldsymbol{f}_{u}\right) \leftarrow \widehat{H}_{u}\)
                Remove node \(u\) from tree \(T\)
        return \(\widehat{\chi}_{\epsilon}\)
```

Lemma 23 (RecursiveRobustSFT - Time and Sample Complexity). For every subtree Frontier of $T_{N}^{\text {full }}$, every leaf $v$ of Frontier, positive integer $k$, every $\alpha=o\left(\frac{1}{\log N}\right)$ and $\mu \geq 0$, and every signals $x, \widehat{\chi}_{i n}:[n]^{d} \rightarrow \mathbb{C}$, consider an invocation of primitive RECURSIVERobustSFT (Algorithm 10 ) with inputs ( $x, \widehat{\chi}_{i n}$, Frontier, $v, k, \alpha, \mu$ ). Then,

- The running time of primitive is bounded by

$$
\widetilde{O}\left(\left(\frac{k^{2}}{\alpha} \cdot 2^{w_{\text {FRONTIRR }}(v)}+\frac{k}{\alpha} \cdot\left\|\widehat{\chi}_{i n}\right\|_{0}\right) \cdot(2 \log N)^{\log _{\frac{1}{\alpha}} k}+k^{2} \cdot\left\|\widehat{\chi}_{i n}\right\|_{0}+k^{3}\right) .
$$

- The number of accesses it makes on $x$ is always bounded by

$$
\widetilde{O}\left(\frac{k^{2}}{\alpha} \cdot 2^{w_{\text {FRONTIER }}(v)} \cdot(2 \log N)^{\log _{\frac{1}{\alpha}} k}\right) .
$$

Moreover, the output signal $\widehat{\chi}_{v}$ always satisfies $\operatorname{supp}\left(\widehat{\chi}_{v}\right) \subseteq \operatorname{Freq}^{\operatorname{Cone}}{ }_{\text {Frontier }}(v)$ and $\left\|\widehat{\chi}_{v}\right\|_{0} \leq k$.
Proof. The proof is by induction on parameter $k$. The base of induction corresponds to $k \leq \frac{1}{\alpha}$. For every $k \leq \frac{1}{\alpha}$, Algorithm 10 simply runs PromiseSparseFT( $x, \widehat{\chi}_{i n}$, Frontier, $v, k,\left\lceil\frac{k}{\alpha}\right\rceil, \mu$ ) in line 3. Therefore, by Lemmat the runtime and sample complexity of our algorithm are bounded by $\widetilde{O}\left(\frac{k}{\alpha} \cdot\left\|\widehat{\chi}_{i n}\right\|_{0}+\frac{k^{2}}{\alpha} \cdot 2^{w_{\text {Froмtire }}(v)}\right)$ and $\widetilde{O}\left(\frac{k^{2}}{\alpha} \cdot 2^{w_{\text {Froктев }}(v)}\right)$, respectively. Moreover, by Lemma 20 , the output signal $\widehat{\chi}_{v}$ satisfies $\left\|\widehat{\chi}_{v}\right\|_{0} \leq k$ as well as $\operatorname{supp}\left(\widehat{\chi}_{v}\right) \subseteq \operatorname{FreqCone}_{\text {Frontier }}(v)$. This proves that the inductive hypothesis holds for every integer $k \leq \frac{1}{\alpha}$, hence the base of induction holds.

To prove the inductive step, suppose that the lemma holds for every $k \leq m-1$ for some integer $m \geq\left\lfloor\frac{1}{\alpha}\right\rfloor+1$. Assuming the inductive hypothesis, we prove that the lemma holds for $k=m$. First, we prove that Algorithm 10 terminates after a bounded number of iterations. For the purpose of having a tight analysis of the runtime and sample complexity, we need to have tight upper bounds on the number of times our algorithm invokes primitive Estimate in line 13 as well as the number of times our algorithm recursively calls itself in lines 23 and 24 . First, we show that the number of iterations in which the if-staement in line 10 is True, and hence the number of times we invoke Estimate in line 13, is bounded by $O(k)$. The reason is, everytime the if-staement in line 10
becomes True the sparsity of $\widehat{\chi}_{v}$, i.e., $\left\|\widehat{\chi}_{v}\right\|_{0}$, increases by $\mid$ CHEAP $\mid \geq 1$, because the if-staement in line 10 ensures that preconditions of Claim 6 hold, hence, by invoking this claim, Cheap $\neq \varnothing$. On the other hand, we can see from the way our algorithm operates that the sparity of $\widehat{\chi}_{v}$ does not decrease in any of the iterations of our algorithm. Therefore, because the if-statement in line 8 of the algorithm makes sure that $\left\|\widehat{\chi}_{v}\right\|_{0}$ does not exceed $k$, we conclude that the total number of iterations in which the if-statement in line 10 is True is bounded by $O(k)$. Hence, the number of times our algorithm calls Estimate in line 13 is $O(k)$.

In order to bound the number of iterations of our algorithm in which the if-statement in line 10 is False, we use a potential function. Let $\hat{\chi}_{v}^{(t)}$ denote the signal $\widehat{\chi}_{v}$ at the end of iteration $t$ of the algorithm. Furthermore, let $T^{(t)}$ denote the subtree $T$ at the end of $t^{t h}$ iteration. Additionally, let $\mathrm{Marked}^{(t)}$ denote the set Marked (defined in Algorithm 10) at the end of iteration $t$. We prove that the number of iterations in which the if-statement in line 10 of our algorithm is False is bounded by $O\left(\frac{\log N}{\alpha}\right)$ using the following potential function, defined for non-negative integer $t$ :

$$
\phi_{t}:=(\log N+1) \cdot\left|\operatorname{MARKED}^{(t)}\right|+2 \log N \cdot\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0}+b \cdot \sum_{u \in \operatorname{LEAVES}\left(T^{(t)}\right) \backslash \operatorname{MARKED}} l_{T^{(t)}}(u) .
$$

We prove that assuming the algorithm does not terminate in $q$ iterations, for some integer $q$, then in every positive iteration $t \leq q$, if the if-statement in line 10 of Algorithm 10 is False, then the above potential function increases by at least $b$, i.e., $\phi_{t} \geq \phi_{t-1}+b$. Additionally, when the if-statement in line 10 is True, the potential increases by at least $\log N-1$, i.e., $\phi_{t} \geq \phi_{t-1}+\log N-1$. We show that at any given iteration $t$ of the algorithm the potential function $\phi_{t}$ increases in the abovementioned fashion.

Case 1 - the if-statement in line 10 of Algorithm 10 is True. In this case, we have that $\sum_{u \in \operatorname{MARKED}^{(t-1)}} 2^{-w_{T^{(t-1)}}(u)} \geq \frac{1}{2}$. As a result, by Claim 6 , the set Cheap ${ }^{(t)} \subseteq \operatorname{MARKED}^{(t-1)}$ that the algorithm computes in line 11 by running the primitive FindCheapToEstimate is nonempty. Then, the algorithm constructs $T^{(t)}$ by removing all leaves that are in the set ChEAP ${ }^{(t)}$ from tree $T^{(t-1)}$ and leaving the rest of the tree unchanged. Furthermore, the algorithm updates the set Marked ${ }^{(t)}$ by subtracting Cheap ${ }^{(t)}$ from Marked ${ }^{(t-1)}$. Additionally, in this case, the algorithm computes $\left\{\widehat{H}_{u}\right\}_{u \in \text { Cheap }}{ }^{(t)}$ by running the procedure Estimate in line 13 and then updates $\widehat{\chi}_{v}^{(t)}\left(\boldsymbol{f}_{u}\right) \leftarrow \widehat{H}_{u}$ for every $u \in \operatorname{CHEAP}^{(t)}$ and $\widehat{\chi}_{v}^{(t)}(\boldsymbol{\xi})=\widehat{\chi}_{v}^{(t-1)}(\boldsymbol{\xi})$ at every other frequency $\boldsymbol{\xi}$. Therefore, $\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0}=\left\|\widehat{\chi}_{v}^{(t)}\right\|_{0}+\left|\operatorname{CHEAP}^{(t)}\right|$. Thus,

$$
\phi_{t}-\phi_{t-1}=(\log N-1) \cdot\left|\operatorname{CHEAP}^{(t)}\right| \geq \log N-1,
$$

where the inequality follows from $\operatorname{ChEAP}^{(t)} \neq \varnothing$. This proves the potential increase that we wanted.
Case 2 - the if-statement in line 10 is False. In this case, either the algorithm terminates by the if-statement in line 25, which contradicts with our assumption that the algorithm does not terminate after $q \geq t$ iterations, or the following holds,

$$
\begin{aligned}
& \left|\operatorname{MARKED}^{(t)}\right|+b \cdot \sum_{u \in \operatorname{LEAVES}\left(T^{(t)}\right) \backslash \operatorname{MARKED}}{ }^{(t)} l_{T^{(t)}(u)} \\
& \geq\left|\operatorname{MARKED}^{(t-1)}\right|+b \cdot \sum_{u \in \operatorname{LEAVES}\left(T^{(t-1)}\right) \backslash \operatorname{MARKED}^{(t-1)}} l_{T^{(t-1)}}(u)+b,
\end{aligned}
$$

while $\left|\operatorname{MARKED}^{(t)}\right| \geq\left|\operatorname{MARKED}^{(t-1)}\right|$ and $\left\|\hat{\chi}_{v}^{(t)}\right\|_{0}=\left\|\hat{\chi}_{v}^{(t-1)}\right\|_{0}$. Thus, in this case, $\phi_{t+1}-\phi_{t} \geq b$ which is the potential increase that we wanted to prove.

So far, we proved that $\phi_{t}$ must increase by at least $\log N-1$ at every iteration of the algorithm. Moreover, at every iteration of the algorithm where the if-statement in line 10 is False the potential increases by at least $b$. Also, the potential function $\phi_{t}$ is non-negative for every $t$. On the other hand, the if-statement in line 8 ensures that at any iteration $t \leq q$ it must hold that $\phi_{t} \leq 2 k \log N$. Therefore, the potential increse that we proved implies that Algorithm 10 must terminate after at most $q=2 k \log N$ iterations, where only in $\frac{2 \log N}{\alpha}$ of the iterations the if-statement in line 10 can be False. Therefore, the total number of times our algorithm recursively invokes itself in lines 23 and 24 is bounded by $\frac{2 \log N}{\alpha}$.

Now that we have the termination quarantee, we can use the fact that our algorithm constructs $\widehat{\chi}_{v}$ by exclusively estimating the values of frequencies that lie in FreqCone Frontier $^{(v)}$ ) in line 13 , one can see that the output signal $\widehat{\chi}_{v}$ always satisfies $\operatorname{supp}\left(\widehat{\chi}_{v}\right) \subseteq \operatorname{FreqCone}_{\text {Frontier }}(v)$. Additionally, the if-staement in line 8 , ensures that $\left\|\widehat{\chi}_{v}\right\|_{0} \leq k$. Now we bound the running time and sample complexity of the algorithm.

Sample Complexity and Runtime: The expensive components of the algorithm are primitive Estimate in line 13, the recursive call of RecursiveRobustSFT in lines 23 and 24, and invocation of HeavyTest in line 36 of the algorithm.

We first bound the time and sample complexity of invoking Estimate in line 13 . We remark that, at any iteration $t$, the algorithm runs primitive Estimate only if case 1 that we mentioned earlier in the proof happens. Therefore, by Claim 6 , the set $\varnothing \neq \operatorname{CHEAP}^{(t)} \subseteq$ MARKED $^{(t-1)}$ that our algorithm computes in line 11 by running the primitive FindCheapToEstimate satisfies the property that $\left|\operatorname{ChEAP}^{(t)}\right| \cdot\left(8+4 \log \left|\operatorname{MARKED}^{(t-1)}\right|\right) \geq \max _{u \in \operatorname{ChEAP}^{(t)}} 2^{w} T^{(t-1)}(u)$. By the if-statement in line 8 of the algorithm, this implies that $\mid$ Cheap $^{(t)} \mid \cdot(8+4 \log k) \geq \max _{u \in \text { Cheap }^{(t)}} 2^{w_{T} T^{(t-1)}}{ }^{(u)}$. Thus, by Lemma 19, the time and sample complexity of every invocation of Estimate in line 13 of our algorithm are bounded by

$$
\widetilde{O}\left(\frac{k}{\left|\operatorname{CHEAP}^{(t)}\right|} \sum_{u \in \operatorname{Cheap}^{(t)}} 2^{w_{\text {Frontirat }}{ }^{(t-1)}(u)}+k \cdot\left\|\widehat{\chi}_{v}^{(t-1)}+\widehat{\chi}_{i n}\right\|_{0}\right)
$$

and $\widetilde{O}\left(\frac{k}{\mid \text { Cheap }^{(t)} \mid} \sum_{u \in \text { Cheap }^{(t)}} 2^{w_{\text {Frontierut }}{ }^{(t-1)}(u)}\right)$, respectively. Using the fact that $\left\|\widehat{\chi}_{v}^{(t-1)}\right\|_{0} \leq k$, these time and sample complexities are further upper bounded by

$$
\widetilde{O}\left(k \cdot\left(2^{w_{\text {FRомтіев }}(v)} \cdot\left|\operatorname{CHEAP}^{(t)}\right|+\left\|\widehat{\chi}_{i n}\right\|_{0}\right)+k^{2}\right)
$$

and $\widetilde{O}\left(k \cdot 2^{w_{\text {Frontír }}(v)} \cdot\left|\operatorname{ChEAP}^{(t)}\right|\right)$, respectively. We proved that the total number of times we run Estimate in line 13 of the algorithm, is bounded by $O(k)$. Using this together with the fact that $\sum_{t: \text { if-statement in line } 10 \text { is True }}\left|\operatorname{CHEAP}^{(t)}\right|=\left\|\widehat{\chi}_{v}\right\|_{0} \leq k$, the total runtime and sample complexity of all invocations of Estimate in all iterations can be upper bounded by $\widetilde{O}\left(k^{3}+k^{2}\left(\left\|\widehat{\chi}_{i n}\right\|_{0}+2^{w_{\text {Froмтев }}(v)}\right)\right)$ and $\widetilde{O}\left(k^{2} \cdot 2^{w_{\text {FRONTIER }}(v)}\right)$, respectively.

Now we bound the runtime and sample complexity of invoking RecursiveRobustSFT in lines 23 and 24 of the algorithm. Note that at any iteration $t$, our algorithm recursively calls RECURSIVEROBUSTSFT only if case 2 that we mentioned earlier in the proof occurs. As we showed, the total number of times that this happens is bounded by $\frac{2 \log N}{\alpha}$. Since, in line 19 of
the algorithm, we pick leaf $z$ with the smallest weight, and since the number of leaves of subtree $T^{(t-1)}$ that are not in the set MARKED ${ }^{(t-1)}$ are bounded by $\frac{k}{b+1}$ (ensured by the if-statement in line 80, we have $w_{\text {Frontierut }}\left(z_{\text {left }}\right)=w_{\text {Frontierut }}\left(z_{\text {right }}\right) \leq w_{\text {Frontier }}(v)+\log \frac{k}{b+1}+1$. Also note that $\left\|\widehat{\chi}_{v}^{(t-1)}\right\|_{0} \leq k$, ensured by the if-statement in line 8 . Therefore, by the inductive hypothesis, the time and sample complexities of each recursive invocation of RECURSIVERobuSTSFT by our algorithm are bounded by

$$
\widetilde{O}\left(\left(\frac{b^{2} \cdot 2^{w_{\text {Frontier }}(v)}}{\alpha^{2}}+\frac{b}{\alpha} \cdot\left\|\widehat{\chi}_{i n}\right\|_{0}\right) \cdot(2 \log N)^{\log _{\frac{1}{\alpha}} b}+b^{2} \cdot\left\|\widehat{\chi}_{i n}\right\|_{0}+k b^{2}\right)
$$

and $\widetilde{O}\left(\frac{b^{2}}{\alpha^{2}} \cdot 2^{w_{\text {FRontier }}(v)} \cdot(2 \log N)^{\log _{\frac{1}{\alpha}} b}\right)$. We proved that the total number of iterations in which case 2 happens, and hence the number of times we run RecursiveRobustSFT in lines 23 and 24 of the algorithm, is bounded by $\frac{2 \log N}{\alpha}$. Therefore, the total time and sample complexity of all invocations of PromiseSparseFT in lines 23 and 24 are bounded by

$$
\widetilde{O}\left(\left(\frac{k^{2}}{\alpha} \cdot 2^{w_{\text {РRомтাев }}(v)}+\frac{k}{\alpha} \cdot\left\|\widehat{\chi}_{i n}\right\|_{0}\right) \cdot(2 \log N)^{\log _{\frac{1}{\alpha}} k}+\alpha k^{2} \cdot\left\|\widehat{\chi}_{i n}\right\|_{0}+\alpha k^{3}\right)
$$

and $\widetilde{O}\left(\frac{k^{2}}{\alpha} \cdot 2^{w_{\text {Frontire }}(v)} \cdot(2 \log N)^{\log _{\frac{1}{\alpha}} k}\right)$, respectively.
Finally, we bound the time and sample complexity of invoking HeavyTest in line 36 of our algorithm. Since $\left\|\widehat{\chi}_{v}\right\|_{0} \leq k$, by Lemma 18, the time and sample complexity of the HeavyTest in line 36 are bounded by $\widetilde{O}\left(\left\|\widehat{\chi}_{i n}\right\|_{0} \cdot \frac{k}{\alpha}+\frac{k^{2}}{\alpha}+2^{w_{\text {Frontifr }}(v)} \cdot \frac{k}{\alpha}\right)$ and $\widetilde{O}\left(2^{w_{\text {Frontike }}(v)} \cdot \frac{k}{\alpha}\right)$, respectively. Hence, we find that the total time and sample complexity of our algorithm are bounded by

$$
\widetilde{O}\left(\left(\frac{k^{2} \cdot 2^{w_{\text {FRoNTIER }}(v)}}{\alpha}+\frac{k}{\alpha} \cdot\left\|\widehat{\chi}_{i n}\right\|_{0}\right) \cdot(2 \log N)^{\log _{\frac{1}{\alpha}} k}+k^{2} \cdot\left\|\widehat{\chi}_{i n}\right\|_{0}+k^{3}\right)
$$

and $\widetilde{O}\left(\frac{k^{2}}{\alpha} \cdot 2^{w_{\text {FRомтাев }}(v)} \cdot(2 \log N)^{\log _{\frac{1}{\alpha}} k}\right)$, respectively. This proves the inductive step of the proof and consequently completes the proof of our lemma.

Now we are in a position to present the main invariant of primitive RecursiveRobustSFT.
Lemma 24 (RecursiveRobustSFT - Invariants). Consider the preconditions of Lemma 23. Let $\widehat{y}:=\widehat{x}-\widehat{\chi}_{\text {in }}$ and $S:=\operatorname{FreqCone}_{T}(v) \cap \operatorname{HEAD}_{\mu}(\widehat{y})$, where $\operatorname{HEAD}_{\mu}(\cdot)$ is defined as per (4). If $i$ ) $\operatorname{HEAD}_{\mu}(\widehat{y}) \subseteq \operatorname{supp}($ FRONTIER $)$, ii) $\left\|\widehat{y}-\widehat{y}_{\operatorname{HEAD}_{\mu}(\widehat{y})}\right\|_{2}^{2} \leq \frac{21 \mu^{2}}{20}+\frac{\mu^{2}}{20 \log _{\frac{1}{\alpha}}(k / \alpha)}$, and iii) $|S| \leq \frac{k}{\alpha}$, then with probability at least $1-O\left(\left(\frac{2 \log N}{\alpha}\right)^{\log _{\frac{1}{\alpha}} k} \cdot N^{-4}\right)$, the output (Budget, $\widehat{\chi}_{v}$ ) of Algorithm 10 satisfies the following,

1. If $|S| \leq k$ then Budget $=$ True, $\operatorname{supp}\left(\widehat{\chi}_{v}\right) \subseteq S$, and $\left\|\widehat{y}_{S}-\widehat{\chi}_{v}\right\|_{2}^{2} \leq \frac{\mu^{2}}{40 \log _{1 / \alpha}^{2} k}$;
2. If $|S|>k$ then Budget $=$ False and $\widehat{\chi}_{v} \equiv\{0\}^{n^{d}}$.

Proof. The proof is by induction on parameter $k$. The base of induction corresponds to $k \leq \frac{1}{\alpha}$. For every $k \leq \frac{1}{\alpha}$, Algorithm 10 simply runs PromiseSparseFT $\left(x, \widehat{\chi}_{i n}\right.$, Frontier, $\left.v, k,\left\lceil\frac{k}{\alpha}\right\rceil, \mu\right)$ in line 3. Therefore, by Lemma 21 , the claims of the lemma hold with probability at least $1-\frac{1}{N^{4}}$. This
proves that the inductive hypothesis holds for every integer $k \leq \frac{1}{\alpha}$, hence the base of induction holds.

To prove the inductive step, suppose that the lemma holds for every $k \leq m-1$ for some integer $m \geq\left\lfloor\frac{1}{\alpha}\right\rfloor+1$. Assuming the inductive hypothesis, we prove that the lemma holds for $k=m$. To prove the inductive claim, we first analyze the algorithm under the assumption that the primitives HeavyTest and Estimate are replaced with more powerful primitives that succeeds deterministically. Hence, we assume that HeavyTest correctly tests the "heavy" hypothesis on its input signal with probability 1 and also Estimate achieves the estimation guarantee of Lemma 19 deterministrically. Moreover, we assume that our inductive invocation of RecursiveRobustsFT in lines 23 and 24 of the algorithm succeed deterministically, hence, we assume that the inductive hypothesis (the lemma) holds with probability 1 . With these assumptions in place, we prove that the lemma holds deterministically (with probability 1 ). We then establish a coupling between this idealized execution and the actual execution of our algorithm, leading to our result.

We prove the first statement of lemma by (another) induction on the Repeat-Until loop of the algorithm. Note that we are proving the inductive step of an inductive proof using another induction (two nested inductions). The first (outer) induction was on the integer $k$ and the second (inner) induction is on the iteration number $t$ of the Repeat-Until loop of our algorithm. Let $\widehat{\chi}_{v}^{(t)}$ denote the signal $\widehat{\chi}_{v}$ at the end of iteration $t$ of the algorithm. Furthermore, let Frontier ${ }^{(t)}$ denote the subtree $T$ at the end of $t^{t h}$ iteration. Also, let Marked ${ }^{(t)}$ denote the set Marked (defined in Algorithm 10) at the end of iteration $t$. Additionaly, for every leaf $u$ of subtree $T^{(t)}$, let $L_{u}^{(t)}$ denote the "unestimated" frequencies in support of $\widehat{y}$ that lie in frequency cone of $u$, i.e., $L_{u}^{(t)}:=\operatorname{FreqCone}_{\mathrm{Frontierut}^{(t)}}(u) \cap \operatorname{HEAD}_{\mu}(\widehat{y})$ We prove that if preconditions i, ii and iii together with the presondition of statement 1 (that is $|S| \leq k$ ), hold, then at every iteration $t=0,1,2, \ldots$ of Algorithm 10, the following properties are maintained,
$P_{1}(t) S \backslash \operatorname{supp}\left(\widehat{\chi}_{v}^{(t)}\right) \subseteq \operatorname{supp}\left(T^{(t)}\right):=\bigcup_{u \in \operatorname{Leaves}\left(T^{(t)}\right)} \operatorname{FreqCone}_{\operatorname{Frontierut~}^{(t)}(u) ;}$
$P_{2}(t)$ For every leaf $u \neq v$ of subtree $T^{(t)},\left|L_{u}^{(t)}\right| \geq 1$. Additionally, if $u \notin \operatorname{MARKED}^{(t)}$, then $\left|L_{u}^{(t)}\right|>b ;$
$P_{3}(t)\left\|\widehat{y}_{S^{(t)}}-\widehat{\chi}_{v}^{(t)}\right\|_{2}^{2} \leq \frac{\left|S^{(t)}\right|}{40 k \cdot \log _{1 / \alpha}^{2} k} \cdot \mu^{2}$, where $S^{(t)}:=\operatorname{supp}\left(\widehat{\chi}_{v}^{(t)}\right) ;$
$P_{4}(t) S^{(t)} \subseteq S$ and $S^{(t)} \cap\left(\bigcup_{\substack{u \in \operatorname{Leaves}\left(T^{(t)}\right) \\ u \neq v}} \operatorname{FreqCone}_{\text {Frontierut }^{(t)}(u)}\right)=\varnothing ;$
The base of induction corresponds to the zeroth iteration ( $t=0$ ), at which point $T^{(0)}$ is a subtree that solely consists of node $v$ and has no other leaves. Moreover, $\widehat{\chi}_{v}^{(0)} \equiv 0$. Thus, statement $P_{1}(0)$ trivially holds by definition of set $S$. The statement $P_{2}(0)$ holds since there exists no leaf $u \neq v$ in $T^{(0)}$. The statements $P_{3}(0)$ and $P_{4}(0)$ hold because of the fact $\widehat{\chi}_{v}^{(0)} \equiv 0$.

We now prove the inductive step by assuming that the inductive hypothesis, $P(t-1)$ is satisfied for some iteration $t-1$ of Algorithm 10, and then proving that $P(t)$ holds. First, we remark that if inductive hypotheses $P_{2}(t-1)$ and $P_{4}(t-1)$ hold true, then by the precondition of statement 1 of the lemma (that is $|S| \leq k$ ) the if-statement in line 8 of Algorithm 10 is False and hence lines 8 and 9 of the algorithm can be ignored in our analysis. We proceed to prove the induction by considering the two cases that can happen in every iteration $t$ of the algorithm:

Case 1 - the if-statement in line 10 of Algorithm 10 is True. In this case, we have that $\sum_{u \in \operatorname{Marked}^{(t-1)}} 2^{-w_{T^{(t-1)}}(u)} \geq \frac{1}{2}$. As a result, by Claim 6, the set Cheap $\subseteq$ Marked $^{(t-1)}$ that the algorithm computes in line 11 by running the primitive FindCheap ToEstimate satisfies the property that $\mid$ ChEAP $\mid \cdot\left(8+4 \log \left|\operatorname{MARKED}^{(t-1)}\right|\right) \geq \max _{u \in \operatorname{ChEAP}} 2^{w_{T^{(t-1)}}(u)}$. Clearly ChEAP $\neq \varnothing$, by Claim 6. Then the algorithm computes $\left\{\widehat{H}_{u}\right\}_{u \in \text { Cheap }}$ by running the procedure Estimate in line 13 and then updates $\widehat{\chi}^{(t)}\left(\boldsymbol{f}_{u}\right) \leftarrow \widehat{H}_{u}$ for every $u \in$ CHEAP and $\widehat{\chi}^{(t)}(\boldsymbol{\xi})=\widehat{\chi}^{(t-1)}(\boldsymbol{\xi})$ at every other frequency $\boldsymbol{\xi}$. Therefore, if we let $L:=\left\{\boldsymbol{f}_{u}: u \in\right.$ CHEAP $\}$, then $S^{(t)} \backslash S^{(t-1)}=L$, by inductive hypothesis $P_{4}(t-1)$. By $P_{3}(t-1)$ along with Lemma 19 (its deterministic version that succeeds with probability 1 ), we find that

$$
\begin{align*}
\left\|\widehat{\chi}_{v}^{(t)}-\widehat{y}_{S^{(t)}}\right\|_{2}^{2} & =\left\|\left(\widehat{\chi}_{v}^{(t)}-\widehat{y}\right)_{S^{(t-1)}}\right\|_{2}^{2}+\left\|\left(\widehat{\chi}_{v}^{(t)}-\widehat{y}\right)_{S^{(t)} \backslash S^{(t-1)}}\right\|_{2}^{2} \\
& =\left\|\widehat{\chi}_{v}^{(t-1)}-\widehat{y}_{S^{(t-1)}}\right\|_{2}^{2}+\left\|\left(\widehat{\chi}_{v}^{(t)}-\widehat{y}\right)_{L}\right\|_{2}^{2} \\
& \leq \frac{\left|S^{(t-1)}\right| \cdot \mu^{2}}{40 k \log _{1 / \alpha}^{2} k}+\frac{|L|}{46 k \log ^{2} N} \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}\left(\operatorname{FrontieruT}^{(t-1)}\right)}\left|\left(\widehat{y}-\widehat{\chi}_{v}^{(t-1)}\right)(\boldsymbol{\xi})\right|^{2} . \tag{19}
\end{align*}
$$

Now we bound the second term above,

$$
\begin{aligned}
& \left.\sum_{\xi \in[n]^{d} \backslash \operatorname{supp}(\text { FrontieruT }}{ }^{(t-1)}\right)\left|\left(\widehat{y}-\widehat{\chi}_{v}^{(t-1)}\right)(\boldsymbol{\xi})\right|^{2}
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(\text { Frontier })}|\widehat{y}(\boldsymbol{\xi})|^{2} \\
& +\sum_{\boldsymbol{\xi} \in \text { FreqCone }_{\text {Froxrres }}(v) \backslash\left(\operatorname{supp}\left(T^{(t-1)}\right) \cup S^{(t-1)}\right)}|\widehat{y}(\boldsymbol{\xi})|^{2}+\left\|\widehat{y}_{S^{(t-1)}}-\widehat{\chi}_{v}^{(t-1)}\right\|_{2}^{2} \\
& =\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash\left(\operatorname{supp}\left(\text { Frontieru } T^{(t-1)}\right) \cup S^{(t-1)}\right)}|\widehat{y}(\boldsymbol{\xi})|^{2}+\left\|\widehat{y}_{S^{(t-1)}}-\widehat{\chi}_{v}^{(t-1)}\right\|_{2}^{2} \\
& \leq \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{HEAD}_{\mu}(\widehat{y})}|\widehat{y}(\boldsymbol{\xi})|^{2}+\left\|\widehat{y}_{S^{(t-1)}}-\widehat{\chi}_{v}^{(t-1)}\right\|_{2}^{2} \quad \text { (by } P_{1}(t-1) \text {, precondition i and definition of } S \text { ) } \\
& \leq \frac{21 \mu^{2}}{20}+\frac{\mu^{2}}{20 \log _{\frac{1}{\alpha}}(k / \alpha)}+\frac{\mu^{2}}{40 \log _{\frac{1}{\alpha}}^{2} k} \quad\left(\text { by } P_{3}(t-1) \text { and } P_{4}(t-1) \text { and precondition }|S| \leq b\right) \\
& \leq \frac{23 \mu^{2}}{20} .
\end{aligned}
$$

Therefore, by plugging the above bound back to (19) we find that,

$$
\left\|\widehat{\chi}_{v}^{(t)}-\widehat{y}_{S^{(t)}}\right\|_{2}^{2} \leq \frac{\left|S^{(t-1)}\right|}{40 k \log _{\frac{1}{\alpha}}^{2} k} \cdot \mu^{2}+\frac{|L|}{46 k \log ^{2} N} \cdot\left(\frac{23}{20} \mu^{2}\right) \leq \frac{\left|S^{(t)}\right|}{40 k \log _{\frac{1}{\alpha}}^{2} k} \cdot \mu^{2}
$$

which proves the inductive claim $P_{3}(t)$.
Moreover, in this case, the algorithm constructs $T^{(t)}$ by removing all leaves that are in the set Cheap from tree $T^{(t-1)}$ and leaving the rest of the tree unchanged. Furthermore, the algorithm updates the set Marked ${ }^{(t)}$ by subtracting Cheap from Marked ${ }^{(t-1)}$. Note that, $P_{2}(t-1)$ implies
that $L \subseteq S$. Thus, the fact $S^{(t)}=S^{(t-1)} \cup L$ together with inductive hypothesis $P_{4}(t-1)$ as well as the construction of $T^{(t)}$, imply $P_{4}(t)$. The construction of $T^{(t)}$ together with the fact that $\mid$ FreqCone $_{\text {Frontierut }}{ }^{(t-1)}(u) \mid=1$ for every $u \in \operatorname{Marked~}^{(t-1)}$ give $P_{1}(t)$ and $P_{2}(t)$.

Case 2 - the if-statement in line 10 is False. Let $z \in \operatorname{LEAVES}\left(T^{(t-1)}\right) \backslash$ MARKED $^{(t-1)}$ be the smallest weight leaf chosen by the algorithm in line 19. In this case, the algorithm constructs tree $T^{\prime}$ by adding leaves $z_{\text {right }}$ and $z_{\text {left }}$ to tree $T^{(t-1)}$ as right and left children of $z$ in line 22 Then, the algorithm runs RECURSIVEROBUSTSFT with inputs $\left(x, \widehat{\chi}_{i n}+\widehat{\chi}_{v}^{(t-1)}, T^{\prime}, z_{\text {left }}, b, \alpha, \mu\right)$ and $\left(x, \widehat{\chi}_{i n}+\widehat{\chi}_{v}^{(t-1)}, T^{\prime}, z_{\text {right }}, b, \alpha, \mu\right)$ in lines 23 and 24 respectively. Now we analyze the output of the recursive invocation of RECURSIVEROBUSTSFT in lines 23 and 24 . In the following we focus on analyzing ( $\mathrm{ISCORR}_{\text {left }}, \widehat{\chi}_{\text {left }}$ ) but ( $\mathrm{IsCorR}_{\text {right }}, \widehat{\chi}_{\text {right }}$ ) can be analyzed exactly the same way. There are two possibilities that can happen:

Possibility 1) $\mid$ FreqCone $_{\text {Frontierut }^{\prime}}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y}) \mid \leq b$. In this case, the inductive hypothesis $P_{4}(t-1)$ implies that $\left|S^{(t-1)}\right| \leq k$ and hence inductive hypothesis $P_{3}(t-1)$ gives

$$
\begin{equation*}
\left\|\widehat{y}_{S^{(t-1)}}-\widehat{\chi}_{v}^{(t-1)}\right\|_{2}^{2} \leq \frac{\mu^{2}}{40 \log _{1 / \alpha}^{2} k}, \tag{20}
\end{equation*}
$$

hence, $\operatorname{HEAD}_{\mu}\left(\widehat{y}-\widehat{\chi}_{v}^{(t-1)}\right)=\operatorname{HEAD}_{\mu}(\widehat{y}) \backslash S^{(t-1)}$. Consequently, if we let $\widehat{g}:=\widehat{y}-\widehat{\chi}_{v}^{(t-1)}$, then: i) $\operatorname{HEAD}_{\mu}(\widehat{g}) \subseteq \operatorname{supp}\left(\right.$ Frontier $\left.\cup T^{\prime}\right)$, by (20) along with $P_{1}(t-1)$, ii) $\left\|\widehat{g}-\widehat{g}_{\operatorname{HEAD}_{\mu}(\widehat{g})}\right\|_{2}^{2} \leq \frac{21 \mu^{2}}{20}+$ $\frac{\mu^{2}}{20 \log _{\frac{1}{\alpha}(b / \alpha)}}$, by precondition of the lemma along with 20 , and iii)

$$
\left|\operatorname{Freq}^{\operatorname{Cone}}{ }_{\text {FrontieruT }}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{g})\right| \leq b
$$

by assumption $\mid \operatorname{Freq}^{\operatorname{Cone}}$ Frontierut $\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y}) \mid \leq b$. Therefore, all preconditions of the first statement of Lemma 24 hold. Since we invoke primitive RecursiveRobustSFT with sparsity $b \leq m-1$, by our inducive hypothesis that Lemma 24 holds for any sparsity parameter $k \leq m-1$, we can invoke this lemma (a deterministic version of it that succeeds with probability 1) and conclude that, $\operatorname{IsCorR}_{\text {left }}=\operatorname{True}$, and $\operatorname{supp}\left(\widehat{\chi}_{\text {left }}\right) \subseteq \operatorname{FreqCone}_{\text {Frontierut }^{\prime}}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{g})$, and $\| \widehat{g}_{\text {FreqCone }_{\text {Frontierut }}}\left(z_{\text {left }}\right)$ nhead $_{\mu}(\widehat{g})-\widehat{\chi}_{\text {left }} \|_{2}^{2} \leq \frac{\mu^{2}}{40 \log _{1 / \alpha}^{2} b} \leq \frac{\mu^{2}}{10}$. This together with inductive hypothesis $P_{4}(t-1)$ imply that, $\operatorname{supp}\left(\widehat{\chi}_{\text {left }}\right)=\operatorname{FreqCone}_{\text {Frontierut }}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y})$.

So, if $\left|\operatorname{FreqCone}_{\mathrm{Frontierut}^{\prime}}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y})\right| \leq b$, then the algorithm adds all leaves that correspond to frequencies in FreqCone Frontierut $^{\prime}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y})$ to tree $T^{(t-1)}$ as well as set $\operatorname{Marked}^{(t-1)}$. By a similar argument, if $\left|\operatorname{FreqCone}_{\text {Frontierut }}{ }^{\prime}\left(z_{\text {right }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y})\right| \leq b$, then the algorithm adds all leaves corresponding to frequencies in FreqCone Frontierut $^{\prime}\left(z_{\text {right }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y})$ to tree $T^{(t-1)}$ and set MARKED ${ }^{(t-1)}$.

Possibility 2) $\mid$ FreqCone $_{\mathrm{Frontierut}^{\prime}}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y}) \mid>b$. Same as in possibility 1, the inductive hypothesis $P_{4}(t-1)$ implies that $\left|S^{(t-1)}\right| \leq k$, hence, inductive hypothesis $P_{3}(t-1)$ gives (20). Hence, $\operatorname{HEAD}_{\mu}\left(\widehat{y}-\widehat{\chi}_{v}^{(t-1)}\right)=\operatorname{HEAD}_{\mu}(\widehat{y}) \backslash S^{(t-1)}$. Consequently, if we let $\widehat{g}:=\widehat{y}-\widehat{\chi}_{v}^{(t-1)}$, then we find that i) $\operatorname{HEAD}_{\mu}(\widehat{g}) \subseteq \operatorname{supp}\left(\right.$ Frontier $\left.\cup T^{\prime}\right)$, by $P_{1}(t-1)$, ii) $\left\|\widehat{g}-\widehat{g}_{\operatorname{HEAD}_{\mu}(\widehat{g})}\right\|_{2}^{2} \leq \frac{21 \mu^{2}}{20}+$ $\frac{\mu^{2}}{20 \log _{\frac{1}{\alpha}(b / \alpha)}}$, by precondition of the lemma along with 20 , and iii)

$$
\mid \text { FreqCone }_{\text {Frontierut }^{\prime}}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{g})|\leq|S| \leq k,
$$

by precondition of statement 1 of the lemma. Additionally, by $P_{4}(t-1)$, we find that

$$
\left|\operatorname{FreqCone}_{\text {Frontierut }^{\prime}}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{g})\right|=\left|\operatorname{FreqCone}_{\text {Frontierut }^{\prime}}\left(z_{\text {left }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y})\right|>b .
$$

Since we invoke primitive RecursiveRobustSFT with sparsity $b \leq m-1$, by our inducive hypothesis that Lemma 24 holds for any sparsity parameter $k \leq m-1$, we can invoke this lemma (a deterministic version of it that succeeds with probability 1) and conclude that, $\mathrm{IsCoRR}_{\text {left }}=$ False, and $\widehat{\chi}_{\text {left }} \equiv 0$.

We remark that since
the inductive hypothesis $P_{2}(t-1)$ along with the above arguments imply that the if-statement in line 25 of our algorithm cannot be True and hence in the rest of our analysis we can ignore lines 25 and 26 of the algorithm. Furthermore, in this case the algorithm adds leaf $z_{\text {left }}$ as the left child of $v$ to tree $T^{(t-1)}$. By a similar argument, if $\mid$ FreqCone $_{\text {Frontierut }^{\prime}}\left(z_{\text {right }}\right) \cap \operatorname{HEAD}_{\mu}(\widehat{y}) \mid>b$, then the algorithm adds leaf $z_{\text {right }}$ as the left child of $v$ to tree $T^{(t-1)}$.

Based on the above arguments, according to the values of $\mathrm{IsCorR}_{\text {left }}$ and $\mathrm{IsCorR}_{\text {right }}$, there are various cases that can happen. From the way tree $T^{(t)}$ and set Marked ${ }^{(t)}$ are obtained from $T^{(t-1)}$ and Marked ${ }^{(t-1)}$, it follows that in any case all 4 properties of $P(t)$ are maintained. We have proved that for every $t$, if the inductive hypothesis $P(t-1)$ is satisfied then the property $P(t)$ is maintained. This completess the induction (i.e., the inner induction, recall that we have nested inductions) and proves that properties $P(t)$ is maintained throughout the execution of Algorithm 10, assuming that preconditions i, ii, and iii of the lemma along with the precondition $|S| \leq k$ of statement 1 of the lemma hold.

Lemma 23 proves that Algorithm 10 must terminate after some $q$ iterations. When the algorithm terminates, the condition of the Repeat-Until loop in line 35 of the algorithm must be True. Thus, when the algorithm terminates, at $q^{\text {th }}$ iteration, there is no leaf in subtree $T^{(q)}$ besides $v$ and as a consequence the set Marked ${ }^{(q)}$ must be empty. This, together with $P_{1}(q)$ imply that the signal $\widehat{\chi}_{v}^{(q)}$ satisfies,

$$
\operatorname{supp}\left(\widehat{\chi}_{v}^{(q)}\right)=S=\operatorname{FreqCone}_{\text {Frontier }}(v) \cap \operatorname{HEAD}_{\mu}(\widehat{y}) .
$$

Moreover, $P_{3}(q)$ together with precondition $|S| \leq k$ imply that

$$
\left\|\widehat{y}_{S}-\widehat{\chi}_{v}^{(q)}\right\|_{2}^{2} \leq \frac{|S|}{40 k \log _{1 / \alpha}^{2} k} \cdot \mu^{2} \leq \frac{\mu^{2}}{40 \log _{1 / \alpha}^{2} k} .
$$

Now we analyze the if-statement in line 36 of the algorithm. The above equalities and inequalities on $\widehat{\chi}_{v}^{(q)}$ imply that,

$$
\begin{aligned}
\left\|\left(\widehat{y}-\widehat{\chi}_{v}^{(q)}\right)_{\text {FreqCone }_{\text {Frontier }}(v)}\right\|_{2}^{2} & =\left\|\widehat{y}_{\text {FreqCone }_{\text {Frontier }}(v) \backslash S}\right\|_{2}^{2}+\left\|\left(\widehat{y}-\widehat{\chi}_{v}^{(q)}\right)_{S}\right\|_{2}^{2} \\
& \leq\left\|\widehat{\text { FreqCone }}_{\text {Frontire }(v) \backslash \operatorname{HEAD}_{\mu}(\widehat{y})}\right\|_{2}^{2}+\frac{\mu^{2}}{40} .
\end{aligned}
$$

Therefore, if $\widehat{G}_{v}$ is a Fourier domain ( $v$, Frontier)-isolating filter constructed in Lemma 5 , then by Corollary 1 along with the above inequality, we have

$$
\begin{aligned}
\left\|\left(\widehat{y}-\widehat{\chi}_{v}^{(q)}\right) \cdot \widehat{G}_{v}\right\|_{2}^{2} & \leq \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(\text { Frontier })}|\widehat{y}(\boldsymbol{\xi})|^{2}+\left\|\left(\widehat{y}-\widehat{\chi}_{v}^{(q)}\right)_{\text {FreqCone }_{\text {Frontier }}(v)}\right\|_{2}^{2} \\
& \leq \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(\text { Frontier })}|\widehat{y}(\boldsymbol{\xi})|^{2}+\left\|\widehat{y}_{\text {FreqCone }_{\text {Frontier }}(v) \backslash \operatorname{HEAD} \mu}(\widehat{y})\right\|_{2}^{2}+\frac{\mu^{2}}{40} \\
& \leq\left\|\widehat{y}-\widehat{y}_{\text {HEAD }_{\mu}(\widehat{y})}\right\|_{2}^{2}+\frac{\mu^{2}}{40} \leq \frac{11}{10} \cdot \mu^{2} .
\end{aligned}
$$

Thus, the preconditions of the second claim of Lemma 18 hold. So, we can invoke this lemma to conclude that the if-statement in line 36 of the algorithm is False and hence the algorithm outputs (True, $\widehat{\chi}_{v}^{(q)}$ ). This completes the inductive proof of statement 1 of the lemma.

Now we proceed with the inductive step towards proving the second statement of lemma. Suppose that preconditions i, ii, iii along with the precondition of statement 2 (that is $|S|>k$ ) hold. Lemma 23 proved that the signal $\widehat{\chi}_{v}$ always satisfies $\operatorname{supp}\left(\widehat{\chi}_{v}\right) \subseteq \operatorname{FreqCone}_{\text {Frontier }}(v)$ and $\left\|\widehat{\chi}_{v}\right\|_{0} \leq k$. Therefore, $S \backslash \operatorname{supp}\left(\widehat{\chi}_{v}\right) \neq \varnothing$. Consequently, if $\widehat{G}_{v}$ is a Fourier domain ( $v$, Frontier)isolating filter constructed in Lemma 5, then by definition of isolating filters we have

$$
\left\|\left(\left(\widehat{y}-\widehat{\chi}_{v}\right) \cdot \widehat{G}_{v}\right)_{S \cup \operatorname{supp}\left(\hat{\chi}_{v}\right)}\right\|_{2}^{2} \geq\left\|\left(\widehat{y}-\widehat{\chi}_{v}\right)_{S \cup \operatorname{supp}\left(\widehat{x}_{v}\right)}\right\|_{2}^{2} \geq\left\|\widehat{y}_{S \backslash \operatorname{supp}\left(\widehat{\chi}_{v}\right)}\right\|_{2}^{2} \geq 9 \mu^{2}
$$

which follows from the definition of $S$ and $\operatorname{HEAD}_{\mu}(\cdot)$. On the other hand,

$$
\begin{aligned}
\left\|\left(\left(\widehat{y}-\widehat{\chi}_{v}\right) \cdot \widehat{G}_{\ell}\right)_{[n]^{d} \backslash\left(S \cup \operatorname{supp}\left(\widehat{\chi}_{v}\right)\right)}\right\|_{2}^{2}= & \left\|\left(\widehat{y} \cdot \widehat{G}_{\ell}\right)_{[n]^{d} \backslash\left(S \cup \operatorname{supp}\left(\widehat{\chi}_{v}\right)\right)}\right\|_{2}^{2} \\
\leq & \left\|\left(\widehat{y} \cdot \widehat{G}_{\ell}\right)_{[n]^{d} \backslash S}\right\|_{2}^{2} \\
\leq & \left\|\widehat{y}_{\text {FreqCone }_{\text {Frontire }}(v) \backslash S}\right\|_{2}^{2} \\
& +\sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}(\operatorname{FronTIER})}|\widehat{y}(\boldsymbol{\xi})|^{2} \\
\leq & \left\|\widehat{y}-\widehat{y}_{\text {HEAD } \mu}(\widehat{y})\right\|_{2}^{2} \leq \frac{11}{10} \cdot \mu^{2} . \quad \quad \text { (precondition ii) }
\end{aligned}
$$

Additionally note that $\left|S \cup \operatorname{supp}\left(\widehat{\chi}_{v}\right)\right| \leq k / \alpha+k \leq 2 k / \alpha$ by preconditions of the lemma and property of $\operatorname{supp}\left(\widehat{\chi}_{v}\right)$ that we have proved. Hence, by invoking the first claim of Lemma 18 , the if-statement in line 36 of the algorithm is True and hence the algorithm outputs (False, $\{0\}^{n^{d}}$ ). This proves statement 2 of the lemma.

Finally, observe that throughout this analysis we have assumed that Lemma 18 holds with probability 1 for all the invocations of HeavyTest by our algorithm. Moreover, we assumend that Estimate successfully works with probability 1 . Also we assumed that the inductive hypothesis (that is Lemma 24 for sparsity parameters $k \leq m-1$ ) holds deterministically. In reality, we have to take the fact that these primitives are randomized into acount of our analysis.

The first source of randomness is the fact that HeavyTest only succeeds with some high probability. In fact, Lemma 18 tells us that every invocation of HEAVYTEST succeeds with probability at least $1-1 / N^{5}$.

The second source of randomness is the fact that Estimate only succeeds with some high probability. Lemma 19 tells us that every invocation of Estimate on a set Cheap, succeeds with probability $1-\frac{\text { Cheap| }}{N^{8}} \geq 1-\frac{1}{N^{7}}$. Since, our analysis in proof of Lemma 23 shows that RecursiveRobustSFT makes at most $k$ recursive calls to Estimate, by a union bound, the overall failure probability of all invocations of this primitive will be bounded by $\frac{k}{N^{7}}$.

The third and last source of randomness in our algorithm is the recursive invocations of ReCURSIVEROBUSTSFT in lines 23 and 24 of our algorithm. By the inductive hypothesis (statement of Lemma 24, the invocation of this primitive succeeds with probability $1-O\left(\left(\frac{2 \log N}{\alpha}\right)^{\log _{1 / \alpha} b} \cdot N^{-4}\right)$. Our analysis in proof of Lemma 23 shows that RecursiveRobustSFT makes at most $\frac{2 \log N}{\alpha}$ recur-
sive calls to RecursiveRobustSFT. Therefore, by a union bound, the overall failure probability of all invocations of RecursiveRobustSFT is bounded by $O\left(\left(\frac{2 \log N}{\alpha}\right)^{\log _{1 / \alpha} k} \cdot N^{-4}\right)$.

Finally, by another application of union bound, the overall failure probability of Algorithm 10 , is bounded by $O\left(\left(\frac{2 \log N}{\alpha}\right)^{\log _{1 / \alpha} k} \cdot N^{-4}\right)$. This completes the proof of the lemma.

Now we are ready to present our main robust sparse Fourier transform algorithm that achieves the guarantee of Theorem 4 for any $\epsilon$ using a number of samples that is near quadratic in $k$ and a runtime that is cubic and prove the main result of this section.
Proof of Theorem 4: The procedure that achieves the guarantees of the theorem is presented in Algorithm 11. The correctness proof basically follows by invoking Lemma 24 and the runtime and sample complexity follows from Lemma 23. If we let $\mu:=\|\eta\|_{2}$ then because $x$ is a signal in the $k$-high SNR regime, we have that $\left|\operatorname{HEAD}_{\mu}(\widehat{x})\right| \leq k$ and $\left\|\widehat{x}-\widehat{x}_{\text {HEAD }_{\mu}(\widehat{x})}\right\|_{2} \leq \mu$. Therefore, the signal $\widehat{\chi}$ that we computed in line 3 of Algorithm 11 by running procedure RECURSIVEROBUSTSFT (Algorithm 10 , with inputs $\left(x,\{0\}^{n^{d}},\{\right.$ root $\}$, root, $\left.k, \alpha, \mu\right)$, then all preconditions of Lemma 24 hold and hence by invoking the first statement of this lemma we conclude that, with probability at least $1-\frac{1}{2 N^{3}}, \widehat{\chi}$ satisfies the following properties:

$$
\|\widehat{x}-\widehat{\chi}\|_{2}^{2} \leq \frac{\mu^{2}}{40} \quad \text { and } \quad \operatorname{supp}(\widehat{\chi}) \subseteq \operatorname{HEAD}_{\mu}(\widehat{x}) .
$$

This together with the $k$-high SNR assumption imply that, with probability at least $1-\frac{1}{2 N^{3}}$, $\operatorname{supp}(\widehat{\chi})=\operatorname{HEAD}_{\mu}(\widehat{x})$. Therefore, tree $T$ that we construct in line 4 of Algorithm 11 is in fact the spliting tree of the set $\operatorname{HEAD}_{\mu}(\widehat{x})$, that is, $\operatorname{supp}(T)=\operatorname{HEAD}_{\mu}(\widehat{x})$ and $|\operatorname{LEAVES}(T)|=\left|\operatorname{HEAD}_{\mu}(\widehat{x})\right|$.

In the rest of the correctness proof we condition on the event that tree $T$ is the spliting tree of the set $\operatorname{HEAD}_{\mu}(\widehat{x})$ and analyze the evolution of singal $\widehat{\chi}_{\epsilon}$ and tree $T$ in every iteration $t=0,1,2, \ldots$ of the while loop in Algorithm 11. Let $\widehat{\chi}_{\epsilon}^{(t)}$ denote the signal $\widehat{\chi}_{\epsilon}$ at the end of iteration $t$, and let $T^{(t)}$ denote the tree $T$ at the end of iteration $t$. In every iteration $t$, Algorithm 11 computes a subset Cheap ${ }^{(t)}$ of leaves of the tree $T^{(t-1)}$ by running the primitive FindCheapToEstimate in line 7 of the algorithm. By Claim 6 , the set Cheap ${ }^{(t)} \subseteq$ LEAVES $\left(T^{(t-1)}\right)$ satisfies the property that $\left|\operatorname{Cheap}^{(t)}\right| \cdot(8+4 \log k) \geq \max _{u \in \text { Cheap }^{(t)}} 2^{w_{T}(t-1)}(u)$. Clearly Cheap ${ }^{(t)} \neq \varnothing$, by Claim 6. Then the algorithm computes $\left\{\widehat{H}_{u}\right\}_{u \in \text { Cheap }^{(t)}}$ by running the procedure Estimate in line 9 and then updates $\widehat{\chi}_{\epsilon}^{(t)}\left(\boldsymbol{f}_{u}\right) \leftarrow \widehat{H}_{u}$ for every $u \in \operatorname{CHEAP}^{(t)}$ and $\widehat{\chi}_{\epsilon}^{(t)}(\boldsymbol{\xi})=\widehat{\chi}_{\epsilon}^{(t-1)}(\boldsymbol{\xi})$ at every other frequency $\boldsymbol{\xi}$. Moreover, the algorithm updates the tree $T^{(t)}$ by removing every leaf that is in the set ChEAP from tree $T^{(t-1)}$. Hence, one can readily see that since at each iteration of the while loop, tree $T$ looses at least one of its leaves, the algorithm terminates after at most $\left|\operatorname{leaves}\left(T^{(0)}\right)\right|=k$ iterations, since initially the number of leaves of $T^{(0)}$ equals $\left|\operatorname{HEAD}_{\mu}(\widehat{x})\right|=k$.

If we denote by $S^{(t)}$ the set $\operatorname{supp}\left(\widehat{\chi}_{\epsilon}^{(t)}\right)$ for every $t$, then we claim that the following holds,

$$
\operatorname{Pr}\left[\left\|\widehat{x}_{S^{(t)}}-\widehat{\chi}_{\epsilon}^{(t)}\right\|_{2}^{2} \leq \frac{\epsilon\left|S^{(t)}\right|}{k} \cdot \mu^{2}\right] \geq 1-\frac{\left|S^{(t)}\right|}{N^{8}}
$$

We prove the above claim by induction on iteration number $t$ of the while loop of our algorithm. One can see that the base of induction trivially holds for $t=0$ because $\widehat{\chi}_{\epsilon}^{(0)} \equiv 0$. To prove the
inductive step, suppose that the inductive hypothesis holds for $t-1$, that is,

$$
\operatorname{Pr}\left[\left\|\widehat{x}_{S^{(t-1)}}-\widehat{\chi}_{\epsilon}^{(t-1)}\right\|_{2}^{2} \leq \frac{\epsilon\left|S^{(t-1)}\right|}{k} \cdot \mu^{2}\right] \geq 1-\frac{\left|S^{(t-1)}\right|}{N^{8}} .
$$

If we let $L:=\left\{f_{u}: u \in \operatorname{ChEAP}^{(t)}\right\}$, then one can see from the way our algorithm updates signal $\widehat{\chi}_{\epsilon}^{(t)}$ and tree $T^{(t)}$ that $S^{(t)} \backslash S^{(t-1)}=L$ for every iteration $t$. Furthermore, by Lemma 19 and union bound, we find that with probability at least $1-\frac{\left|S^{(t-1)}\right|}{N^{8}}-\frac{\mid \text { Cheap }^{(t)} \mid}{N^{8}}=1-\frac{\left|S^{(t-1)}\right|}{N^{8}}$ the following holds

$$
\begin{align*}
\left\|\widehat{x}_{S^{(t)}}-\widehat{\chi}_{\epsilon}^{(t)}\right\|_{2}^{2} & =\left\|\left(\widehat{x}-\widehat{\chi}_{\epsilon}^{(t)}\right)_{S^{(t-1)}}\right\|_{2}^{2}+\left\|\left(\widehat{x}-\widehat{\chi}_{\epsilon}^{(t)}\right)_{S^{(t)} \backslash S^{(t-1)}}\right\|_{2}^{2} \\
& =\left\|\widehat{x}_{S^{(t-1)}}-\widehat{\chi}_{\epsilon}^{(t-1)}\right\|_{2}^{2}+\left\|\left(\widehat{x}-\widehat{\chi}_{\epsilon}^{(t)}\right)_{L}\right\|_{2}^{2} \\
& \leq \frac{\epsilon\left|S^{(t-1)}\right| \mu^{2}}{k}+\frac{\epsilon|L|}{2 k} \sum_{\boldsymbol{\xi} \in[n] d \backslash \operatorname{supp}\left(T^{(t-1)}\right)}\left|\left(\widehat{x}-\widehat{\chi}_{\epsilon}^{(t-1)}\right)(\boldsymbol{\xi})\right|^{2} . \tag{21}
\end{align*}
$$

Now we bound the second term above,

$$
\begin{aligned}
& \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{supp}\left(T^{(t-1)}\right)}\left|\left(\widehat{x}-\widehat{\chi}_{\epsilon}^{(t-1)}\right)(\boldsymbol{\xi})\right|^{2} \\
&= \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash\left(\operatorname{supp}\left(T^{(t-1)}\right) \cup S^{(t-1)}\right)}|\widehat{x}(\boldsymbol{\xi})|^{2}+\left\|\widehat{x}_{S^{(t-1)}}-\widehat{\chi}_{\epsilon}^{(t-1)}\right\|_{2}^{2} \\
& \leq \sum_{\boldsymbol{\xi} \in[n]^{d} \backslash \operatorname{HEAD}_{\mu}(\widehat{x})}|\widehat{x}(\boldsymbol{\xi})|^{2}+\left\|\widehat{x}_{S^{(t-1)}}-\widehat{\chi}_{\epsilon}^{(t-1)}\right\|_{2}^{2} \quad\left(T \text { was initially the splitting tree of } \operatorname{HEAD}_{\mu}(\widehat{x})\right) \\
& \leq 2 \mu^{2} \quad \text { (by the inductive hypothesis). }
\end{aligned}
$$

Therefore, by plugging the above bound back to (21) we find that,

$$
\operatorname{Pr}\left[\left\|\widehat{x}_{S^{(t)}}-\widehat{\chi}_{\epsilon}^{(t)}\right\|_{2}^{2} \leq \frac{\epsilon\left|S^{(t)}\right|}{k} \cdot \mu^{2}\right] \geq 1-\frac{\left|S^{(t)}\right|}{N^{8}},
$$

which proves the inductive claim. Therefore, by another application of union bound, with probability at least $1-\frac{1}{N^{3}}$, the output of the algorithm $\widehat{\chi}_{\epsilon}$ satisfies $\left\|\widehat{x}-\widehat{\chi}_{\epsilon}\right\|_{2}^{2} \leq(1+\epsilon) \cdot \mu^{2}$. This proves the correctness of Algorithm 11 .

Runtime and Sample Complexity. By Lemma 23, the running time and sample complexity of invoking primitive RecursiverobustSFT in line 3 of the algorithm are bounded by $\widetilde{O}\left(k^{3}\right)$ and $\widetilde{O}\left(k^{2} \cdot 2^{2 \sqrt{\log k \cdot \log (2 \log N)}}\right)$, respectively. Additionally, by Lemma 19 , the runtime and sample complexity of every invocation of Estimate in line 9 of our algorithm are bounded by $\widetilde{O}\left(\frac{k}{\epsilon\left|\operatorname{ChEAP}^{(t)}\right|} \sum_{u \in \operatorname{ChEAP}^{(t)}} 2^{w} T^{(t-1)}(u)+\frac{k}{\epsilon} \cdot\left\|\widehat{\chi}_{\epsilon}^{(t-1)}\right\|_{0}\right)$ and $\widetilde{O}\left(\frac{k}{\epsilon\left|\operatorname{ChEAP}^{(t)}\right|} \sum_{u \in \operatorname{ChEAP}^{(t)}} 2^{w_{T^{(t-1)}}(u)}\right)$, respectively. Using the fact that $\left|\operatorname{CHEAP}^{(t)}\right| \cdot(8+4 \log k) \geq \max _{u \in \text { Cheap }^{(t)}} 2^{w_{T}{ }^{(t-1)}(u)}$ together with $\left\|\widehat{\chi}_{\epsilon}^{(t-1)}\right\|_{0} \leq k$, these time and sample complexities are further upper bounded by $\widetilde{O}\left(\frac{k \mid \text { Cheap }^{(t)} \mid}{\epsilon}+\frac{k^{2}}{\epsilon}\right)$ and $\widetilde{O}\left(\frac{k}{\epsilon} \cdot\left|\operatorname{CHEAP}^{(t)}\right|\right)$, respectively. We proved that the total number of iterations, and hence number of times we run Estimate in line 9 of the algorithm, is bounded by $k$. Using this together with
the fact that $\sum_{t}\left|\operatorname{CHEAP}^{(t)}\right|=\left\|\widehat{\chi}_{\epsilon}\right\|_{0}=\left|\operatorname{HEAD}_{\mu}(\widehat{x})\right| \leq k$, the total runtime and sample complexity of all invocations of Estimate in all iterations can be upper bounded by $\widetilde{O}\left(\frac{k^{3}}{\epsilon}\right)$ and $\widetilde{O}\left(\frac{k^{2}}{\epsilon}\right)$, respectively. Therefore the total time and sample complexities of our algorithm are bounded by $\widetilde{O}\left(\frac{k^{3}}{\epsilon}\right)$ and $\widetilde{O}\left(\frac{k^{2}}{\epsilon}+k^{2} \cdot 2^{2 \sqrt{\log k \cdot \log (2 \log N)}}\right)$, respectively.

## 13 Experiments

In this section, we empirically show that our FFT backtracking algorithm for high dimensional sparse signals is extremely fast and can compete with highly optimized software packages such as the FFTW [Fri99, FJ]. Our experiments mainly focus on our Algorithm 4 which exploits only one level of FFT backtracking and runs in $\widetilde{O}\left(k^{2.5}\right)$ time (see Theorem 7). One of the baselines that we compare our algorithm to is the vanilla FFT tree pruning of [KVZ19], in order to demonstrate the speed gained by our backtracking technique. Furthermore, we compare our method against the SFFT 2.0 HIKP12b, HIKP], which is optimized for 1-dimensional signals, and show that our method's performance for small sparsity $k$ is comparable to that of the SFFT 2.0 even in dimension one.

In a subset of our experiments, we exploit a technique introduced in $\mathrm{GHI}^{+} 13$ to speed up the high-dimensional Sparse FFT algorithms. This method works as follows. By fixing one of the coordinates of a $d$-dimensional signal we get a $(d-1)$-dimensional signal whose Fourier transform corresponds to projecting (aliasing) the Fourier transform of the original signal along the coordinate that was fixed in time domain. Thus we can effectively project the Fourier spectrum into a (d-1)dimensional plane by computing a ( $d-1$ )-dimensional FFT. Using a small number of measurements (projections with different values of the fixed coordinate) we can figure out which frequencies are projected without collision and recover them. We use this trick to recover the frequencies that get isolated under the projection and then run our algorithm on the residual signal. Since the residual signal is likely to have a smaller sparsity than the original one, this projection technique can speed up our Sparse FFT algorithms.

Sparse signal classes: In our experiments, we benchmark all methods on the following classes of $k$-sparse signals:

1. Random support with overtones: The Fourier spectrum of this signal class is the superposition of a set of random frequencies and a set of overtones of these frequencies. Specifically, the support of this $\operatorname{signal}$ is $\operatorname{supp}(\widehat{x})=S_{\text {Random }} \cup S_{\text {OVERtone }}$, which are defined as follows,

$$
\begin{aligned}
& S_{\text {RANDOM }}:=\left\{\boldsymbol{f}_{1}, \boldsymbol{f}_{2}, \ldots \boldsymbol{f}_{k /(d+1)} \sim \text { i.i.d. } \operatorname{UnIF}\left(\mathbb{Z}_{n}^{d}\right)\right\}, \\
& S_{\text {OVERTONE }}:=\left\{\boldsymbol{f}+(n / 2) \cdot \mathbf{e}_{i}: \forall \boldsymbol{f} \in S_{\text {RANDOM }}, i \in[d]\right\},
\end{aligned}
$$

where $\mathbf{e}_{i}$ is the standard basis vector along coordinate $i$ in dimension $d$. Note that every $f \in S_{\text {random }}$ will collide with at least one overtone under projection along any coordinate, thus, $S_{\text {Random }}$ cannot be recovered using the projection trick. We added the overtones precisely for this reason, i.e., to ensure that the projection trick does no recover the signal entirely and there will be something left for the Sparse FFT to recover.
2. Randomly shifted $d$-dimensional Dirac Comb: The Fourier support of a Dirac Comb (without shift) is the following,

$$
S_{\mathrm{COMB}}:=\left\{\left(i_{1} \cdot \frac{n}{k^{1 / d}}, i_{2} \cdot \frac{n}{k^{1 / d}}, \ldots i_{d} \cdot \frac{n}{k^{1 / d}}\right): i_{1}, i_{2}, \ldots i_{d} \in\left[k^{1 / d}\right]\right\} .
$$



Figure 7: The runtime of recovering: (a) superposition of a $k / 2$-sparse signal with random support and a 3D Dirac Comb of sparsity $k / 2$, (b) a randomly shifted 3D Dirac Comb with sparsity $k$, and (c) mixture of two randomly shifted 3D Dirac Combs of sparsities $k / 2$.

We generate a random frequency shift $\tilde{\boldsymbol{f}} \sim \operatorname{UnIF}\left(\mathbb{Z}_{n}^{d}\right)$ and a random phase shift $\tilde{\boldsymbol{t}} \sim \operatorname{UNIF}\left(\mathbb{Z}_{n}^{d}\right)$ then define the $k$-sparse $\widehat{x}$ as,

$$
\widehat{x}_{\boldsymbol{f}}:=\sum_{\boldsymbol{j} \in S_{\text {Coмß }}} e^{2 \pi i \frac{f^{\top} \tilde{t}}{n}} \cdot \mathbb{1}_{\{\boldsymbol{f}=\boldsymbol{j}+\tilde{\boldsymbol{f}}\}} .
$$

Note that the projection trick will not help at all on this signal and thus it is a good test case for the Sparse FFT algorithms. Additionally, this signal in time domain is also a randomly shifted Dirac Comb with sparsity $N / k$ and thus distinguishing it from zero with constant probability would require $\Omega(k)$ samples. This makes the Dirac Comb a hard test case for our tree exploration algorithms which heavily rely on the ZEROTEST primitive to distinguish a sparse signal from a zero signal.
3. Superposition of a $k / 2$-sparse signal with random support and a $d$-dimensional Dirac Comb of sparsity $k / 2$ : This signal is a mixture of instances defined in (1) and (2)
4. Superposition of two randomly shifted $d$-dimensional Dirac Combs of sparsity $k / 2$ : This signal is a mixture of two independent instances of the randomly shifted Dirac Comb defined in (2).

Reproducibility. All the codes used to produce our experimental results are publicly available at this link: https://bitbucket.org/michaelkapralov/sfft-experiments/src/master/

### 13.1 FFT Backtracking vs Vanilla FFT Tree Pruning

We first show that our backtracking technique highly improves the runtime of FFT tree pruning and compare our Algorithm 4 against the vanilla tree exploration of Kapralov et al. [KVZ19] as a baseline. We run both algorithms on a variety of sparse signals of size $N=2^{21}$ in dimension $d=3$. We tune the parameters of both algorithms to achieve success probabilities of higher than $90 \%$ over 100 independent trials with different random seeds. Projection recovery [GHI ${ }^{+}$13] is turned off for both algorithms to fairly demonstrate the effect of our backtracking technique. In Figure 7, we benchmark our methods on 3 different classes of $k$-sparse signals and observe that our Backtracked Sparse FFT algorithm consistently achieves a faster runtime and also scales slower as a function of sparsity $k$ compared to the Vanialla Sparse FFT Tree Pruning of [KVZ19].


Figure 8: The runtime of recovering various signal classes with sparsity $k=32$. We consider two variants of our Backtracked Sparse FFT: (a) purely Algorithm 4 with no prefiltering or projection tricks, (b) enhanced version of Algorithm 4 which first applies the projection trick.

### 13.2 Sparse FFT Backtracking vs FFTW

Next we compare our Algorithm 4 against the highly optimized FFTW 3.3.9 software package and show that our algorithm outperforms FFTW by a large margin when the signal size $N$ is large. We run both algorithms on a variety of signals of sparsity $k=32$ in dimension $d=3$. As in previous set of experiments, the parameters of our algorithm is tuned to succeed in over $90 \%$ of instances. In Figure 8 , we benchmark our method and the FFTW on 4 different classes of $k$-sparse signals and observe that in all cases the runtime of our Backtracked Sparse FFT algorithm scales very weakly with signal size $N$, particularly, our runtime grows far slower than that of FFTW. Consequently our algorithm is orders of magnitude faster than FFTW for any $N \geq 2^{18}$.


Figure 9: The runtime of recovering: (a) $k$-sparse signal with random support and (b) a randomly shifted Dirac Comb with sparsity $k$.

### 13.3 Comparison to SFFT 2.0 in Dimension One

Finally, in this set of experiments we compare our Algorithm 4 against the SFFT software package HIKP which is highly optimized for 1-dimensional sparse signals and show that we can achieve comparable performance even in dimension one. We run both algorithms on two classes of signals with sparsity $k=32$ in dimension $d=1$. We remark that the runtime of SFFT, which is implemented based on HIKP12b, will certainly scale badly in high dimensions due to filter support increasing. However, since there is no optimized code available for SFFT in high dimensions, we feel that it is more informative to compare our optimized code to their optimized code in 1D rather than have a weak extension of their approach as a benchmark.

The SFFT package includes two versions: 1.0 and 2.0. The difference is that SFFT 2.0 adds a Comb prefiltering heuristic to improve the runtime. The idea of this heuristic is to apply the aliasing filter, which is very efficient and has no leakage, to restrict the locations of the large coefficients according to their values mod some number $B=O(k)$. The heuristic, in a preprocessing stage, subsamples the signal at rate $1 / B$ and then takes the FFT of the subsampled signal.

In Figure 9 , we benchmark our method and SFFT (1.0 and 2.0) on 2 different classes of $k$-sparse signals and observe that the runtime of our Backtracked Sparse FFT algorithm is comparable to that of SFFT. In Fig. 9a we run the algorithms on a signal with random Fourier support and observe that SFFT 2.0 runs slightly faster. Since the support is random, the heuristic trick used in SFFT 2.0 can recover a large portion of the frequencies and thus SFFT 2.0 owes much of its speed to the heuristic trick. On the other hand, in Fig. 9b, we run the algorithms on a randomly shifted Dirac Comb and observe that our method outperforms SFFT 1.0. Note that since the Comb prefiltering heuristic used in SFFT 2.0 completely fails on a Dirac Comb input, we used SFFT 1.0 in this experiment instead. This result demonstrates that for signals with small sparsity $k$, our algorithm can run even faster than SFFT when the input's support is a multiplicative subgroup of $\mathbb{Z}_{n}$, such as the Dirac Comb.

## 14 Acknowledgements

Michael Kapralov and Amir Zandieh have received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No. 759471) for the project SUBLINEAR. Amir Zandieh was supported by the Swiss NSF grant No. P2ELP2_195140. Karl Bringmann and Vasileios Nakos have received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No. 850979) for the project TIPEA.

## References

[ABDN18] Amir Abboud, Karl Bringmann, Holger Dell, and Jesper Nederlof. More consequences of falsifying SETH and the orthogonal vectors conjecture. In Ilias Diakonikolas, David Kempe, and Monika Henzinger, editors, Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, Los Angeles, CA, USA, June 25-29, 2018, pages 253-266. ACM, 2018.
[AGS03] A Akavia, S Goldwasser, and S Safra. Proving hard-core predicates using list decoding. In 44th Annual IEEE Symposium on Foundations of Computer Science, 2003. Proceedings., pages 146-157. IEEE, 2003.
[Aka10] Adi Akavia. Deterministic sparse fourier approximation via fooling arithmetic progressions. In COLT, pages 381-393, 2010.
[AWW14] Amir Abboud, Virginia Vassilevska Williams, and Oren Weimann. Consequences of faster alignment of sequences. In International Colloquium on Automata, Languages, and Programming, pages 39-51. Springer, 2014.
[AZKK19] Andisheh Amrollahi, Amir Zandieh, Michael Kapralov, and Andreas Krause. Efficiently Learning Fourier Sparse Set Functions. Advances In Neural Information Processing Systems 32 (Nips 2019), 32(CONF), 2019.
$\left[\mathrm{BCG}^{+} 12\right]$ Petros Boufounos, Volkan Cevher, Anna C Gilbert, Yi Li, and Martin J Strauss. What's the Frequency, Kenneth?: Sublinear Fourier Sampling Off the Grid. In Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, pages 61-72. Springer, 2012.
[ $\left.\mathrm{BFJ}^{+} 94\right]$ Avrim Blum, Merrick Furst, Jeffrey Jackson, Michael Kearns, Yishay Mansour, and Steven Rudich. Weakly learning DNF and characterizing statistical query learning using Fourier analysis. In Proceedings of the twenty-sixth annual ACM symposium on Theory of computing, pages 253-262, 1994.
[BM96] Sonali Bagchi and Sanjit K Mitra. The nonuniform discrete fourier transform and its applications in filter design. i. 1-d. IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing, 43(6):422-433, 1996.
[BM12] Sonali Bagchi and Sanjit K Mitra. The nonuniform discrete Fourier transform and its applications in signal processing, volume 463. Springer Science \& Business Media, 2012.
[BOT88] Michael Ben-Or and Prasoon Tiwari. A deterministic algorithm for sparse multivariate polynomial interpolation. In Proceedings of the twentieth annual ACM symposium on Theory of computing, pages 301-309, 1988.
[Bou14] Jean Bourgain. An improved estimate in the restricted isometry problem. In Geometric aspects of functional analysis, pages 65-70. Springer, 2014.
[Can] E. Candes. Lecture 11 from the course Applied Fourier Analysis and Elements of Modern Signal Processing, Winter 2016. https://statweb.stanford.edu/~candes/ teaching/math262/Lectures/Lecture11.pdf.
[CGV13] Mahdi Cheraghchi, Venkatesan Guruswami, and Ameya Velingker. Restricted isometry of Fourier matrices and list decodability of random linear codes. SIAM Journal on Computing, 42(5):1888-1914, 2013.
[CI17] Mahdi Cheraghchi and Piotr Indyk. Nearly optimal deterministic algorithm for sparse Walsh-Hadamard transform. ACM Transactions on Algorithms (TALG), 13(3):1-36, 2017.
[CKPS16] Xue Chen, Daniel M Kane, Eric Price, and Zhao Song. Fourier-sparse interpolation without a frequency gap. In 2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS), pages 741-750. IEEE, 2016.
[CKSZ17] Volkan Cevher, Michael Kapralov, Jonathan Scarlett, and Amir Zandieh. An adaptive sublinear-time block sparse Fourier transform. In Proceedings of the 49 th Annual ACM SIGACT Symposium on Theory of Computing, pages 702-715, 2017.
[CRT06] E. Candes, J. Romberg, and T. Tao. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. IEEE Transactions on Information Theory, 52:489-509, 2006.
[CT06] Emmanuel J Candes and Terence Tao. Near-optimal signal recovery from random projections: Universal encoding strategies? IEEE transactions on information theory, 52(12):5406-5425, 2006.
[Don06] D. Donoho. Compressed sensing. IEEE Transactions on Information Theory, 52(4):1289-1306, 2006.
[DR93] Alok Dutt and Vladimir Rokhlin. Fast Fourier transforms for nonequispaced data. SIAM Journal on Scientific computing, 14(6):1368-1393, 1993.
[FJ] Matteo Frigo and Steven G. Johnson. FFTW: C subroutine library for computing the discrete fourier transform (DFT). https://www.fftw.org/.
[FR13] Simon Foucart and Holger Rauhut. A Mathematical Introduction to Compressive Sensing. Springer, 2013.
[Fri99] Matteo Frigo. A fast Fourier transform compiler. In Proceedings of the ACM SIGPLAN 1999 conference on Programming language design and implementation, pages 169-180, 1999.
[FS03] Jeffrey A Fessler and Bradley P Sutton. Nonuniform fast fourier transforms using min-max interpolation. IEEE transactions on signal processing, 51(2):560-574, 2003.
[GGI ${ }^{+} 02$ ] Anna C Gilbert, Sudipto Guha, Piotr Indyk, Shanmugavelayutham Muthukrishnan, and Martin Strauss. Near-optimal sparse Fourier representations via sampling. In Proceedings of the thiry-fourth annual ACM symposium on Theory of computing, pages 152-161, 2002.
$\left[\mathrm{GHI}^{+}\right.$13] Badih Ghazi, Haitham Hassanieh, Piotr Indyk, Dina Katabi, Eric Price, and Lixin Shi. Sample-optimal average-case sparse Fourier transform in two dimensions. In 2013 51st Annual Allerton Conference on Communication, Control, and Computing (Allerton), pages 1258-1265. IEEE, 2013.
[GIKW19] Jiawei Gao, Russell Impagliazzo, Antonina Kolokolova, and Ryan Williams. Completeness for first-order properties on sparse structures with algorithmic applications. ACM Trans. Algorithms, 15(2):23:1-23:35, 2019.
[GL89] Oded Goldreich and Leonid A Levin. A hard-core predicate for all one-way functions. In Proceedings of the twenty-first annual ACM symposium on Theory of computing, pages 25-32, 1989.
[GL04] Leslie Greengard and June-Yub Lee. Accelerating the nonuniform fast fourier transform. SIAM review, 46(3):443-454, 2004.
[GMS05] Anna C Gilbert, Shan Muthukrishnan, and Martin Strauss. Improved time bounds for near-optimal sparse Fourier representations. In Wavelets XI, volume 5914, page 59141A. International Society for Optics and Photonics, 2005.
[GR87] Leslie Greengard and Vladimir Rokhlin. A fast algorithm for particle simulations. Journal of computational physics, 73(2):325-348, 1987.
[HIKP] Haitham Hassanieh, Piotr Indyk, Dina Katabi, and Eric Price. Sparse Fast Fourier Transform code (SFFT 1.0 and 2.0). https://groups.csail.mit.edu/netmit/sFFT/ code.html.
[HIKP12a] Haitham Hassanieh, Piotr Indyk, Dina Katabi, and Eric Price. Nearly optimal sparse Fourier transform. In Proceedings of the forty-fourth annual ACM symposium on Theory of computing, pages 563-578. ACM, 2012.
[HIKP12b] Haitham Hassanieh, Piotr Indyk, Dina Katabi, and Eric Price. Simple and practical algorithm for sparse Fourier transform. In Proceedings of the twenty-third annual ACMSIAM symposium on Discrete Algorithms, pages 1183-1194. SIAM, 2012.
[HR16] Ishay Haviv and Oded Regev. The restricted isometry property of subsampled Fourier matrices. In 27th Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, pages 288-297. Association for Computing Machinery, 2016.
[HR17] Ishay Haviv and Oded Regev. The restricted isometry property of subsampled fourier matrices. In Geometric aspects of functional analysis, pages 163-179. Springer, 2017.
[IGS07] M. A. Iwen, A. Gilbert, and M. Strauss. Empirical Evaluation of a Sub-Linear Time Sparse DFT Algorithm. Communications in Mathematical Sciences, 5, 2007.
[IK14] Piotr Indyk and Michael Kapralov. Sample-optimal Fourier sampling in any constant dimension. In 2014 IEEE 55th Annual Symposium on Foundations of Computer Science, pages 514-523. IEEE, 2014.
[IKP14] Piotr Indyk, Michael Kapralov, and Eric Price. (Nearly) Sample-optimal sparse Fourier transform. In Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 480-499. SIAM, 2014.
[Iwe10] Mark A Iwen. Combinatorial sublinear-time Fourier algorithms. Foundations of Computational Mathematics, 10(3):303-338, 2010.
[JENR15] Nagaraj Thenkarai Janakiraman, Santosh K. Emmadi, Krishna R. Narayanan, and Kannan Ramchandran. Exploring connections between sparse fourier transform computation and decoding of product codes. In 53rd Annual Allerton Conference on Communication, Control, and Computing, Allerton 2015, Allerton Park $\mathcal{B}$ Retreat Center, Monticello, IL, USA, September 29-October 2, 2015, pages 1366-1373. IEEE, 2015.
[JLS20] Yaonan Jin, Daogao Liu, and Zhao Song. A robust multi-dimensional sparse Fourier transform in the continuous setting. arXiv preprint arXiv:2005.06156, 2020.
[Kap16] Michael Kapralov. Sparse Fourier transform in any constant dimension with nearlyoptimal sample complexity in sublinear time. In Proceedings of the forty-eighth annual ACM symposium on Theory of Computing, pages 264-277, 2016.
[Kap17] Michael Kapralov. Sample efficient estimation and recovery in sparse FFT via isolation on average. In Foundations of Computer Science (FOCS), 2017 IEEE 58th Annual Symposium on, pages 651-662. Ieee, 2017.
[KKP09] Jens Keiner, Stefan Kunis, and Daniel Potts. Using nfft 3-a software library for various nonequispaced fast fourier transforms. ACM Transactions on Mathematical Software (TOMS), 36(4):1-30, 2009.
[KM93] Eyal Kushilevitz and Yishay Mansour. Learning decision trees using the fourier spectrum. SIAM Journal on Computing, 22(6):1331-1348, 1993.
[KVZ19] Michael Kapralov, Ameya Velingker, and Amir Zandieh. Dimension-independent sparse Fourier transform. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 2709-2728. SIAM, 2019.
[KY11] Krzysztof Kazimierczuk and Vladislav YU. Accelerated nmr spectroscopy by using compressed sensing. Angewandte Chemie International Edition, 2011.
[LDSP08] Michael Lustig, David L Donoho, Juan M Santos, and John M Pauly. Compressed sensing MRI. IEEE signal processing magazine, 25(2):72-82, 2008.
[LMN93] N. Linial, Y. Mansour, and N. Nisan. Constant depth circuits, Fourier transform, and learnability. Journal of the ACM (JACM), 1993.
[Man94] Y. Mansour. Learning Boolean Functions via the Fourier Transform. Theoretical Advances in Neural Computation and Learning, 1994.
[Man95] Yishay Mansour. Randomized interpolation and approximation of sparse polynomials. SIAM Journal on Computing, 24(2):357-368, 1995.
[Mat] Non-uniform fast fourier transforms in matlab. https://www.mathworks.com/help/ matlab/ref/double.nufft.html.
[MZIC17] Sami Merhi, Ruochuan Zhang, Mark A Iwen, and Andrew Christlieb. A New Class of Fully Discrete Sparse Fourier Transforms: Faster Stable Implementations with Guarantees. Journal of Fourier Analysis and Applications, pages 1-34, 2017.
[NSW19] Vasileios Nakos, Zhao Song, and Zhengyu Wang. (nearly) sample-optimal sparse fourier transform in any dimension; ripless and filterless. In 2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS), pages 1568-1577. IEEE, 2019.
[OHR19] Frank Ong, Reinhard Heckel, and Kannan Ramchandran. A fast and robust paradigm for fourier compressed sensing based on coded sampling. In IEEE International Conference on Acoustics, Speech and Signal Processing, ICASSP 2019, Brighton, United Kingdom, May 12-17, 2019, pages 5117-5121. IEEE, 2019.
[OPR15] Frank Ong, Sameer Pawar, and Kannan Ramchandran. Fast and efficient sparse 2d discrete fourier transform using sparse-graph codes. CoRR, abs/1509.05849, 2015.
[PR13] Sameer Pawar and Kannan Ramchandran. Computing a k-sparse n-length discrete fourier transform using at most 4 k samples and o ( $\mathrm{k} \log \mathrm{k}$ ) complexity. In 2013 IEEE International Symposium on Information Theory, pages 464-468. IEEE, 2013.
[PR14] Sameer Pawar and Kannan Ramchandran. A robust R-FFAST framework for computing a k-sparse n -length DFT in o (k $\log \mathrm{n}$ ) sample complexity using sparse-graph codes. In 2014 IEEE International Symposium on Information Theory, pages 18521856. IEEE, 2014.
[PS15] Eric Price and Zhao Song. A robust sparse Fourier transform in the continuous setting. In 2015 IEEE 56th Annual Symposium on Foundations of Computer Science, pages 583-600. IEEE, 2015.
[PST01] Daniel Potts, Gabriele Steidl, and Manfred Tasche. Fast fourier transforms for nonequispaced data: A tutorial. In Modern sampling theory, pages 247-270. Springer, 2001.
[Sau18] Tomas Sauer. Prony's method: an old trick for new problems. 2018.
[Uma19] Chris Umans. Fast generalized dfts for all finite groups. In 2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS), pages 793-805. IEEE, 2019.
[Wil05] Ryan Williams. A new algorithm for optimal 2-constraint satisfaction and its implications. Theoretical Computer Science, 348(2-3):357-365, 2005.
[Wol67] J Wolf. Decoding of bose-chaudhuri-hocquenghem codes and prony's method for curve fitting (corresp.). IEEE Transactions on Information Theory, 13(4):608-608, 1967.


[^0]:    *The work was done while this author was a summer intern at EPFL

[^1]:    ${ }^{1}$ This is the case with the groups of interest in the Sparse FT literature. Furthermore, these are the groups on which the FFT algorithm of Cooley and Tukey operates. For general finite groups $G$, the fastest FT algorithm runs in time almost $|G|^{\omega / 2}$ Uma19, where $\omega$ is the matrix multiplication exponent.

[^2]:    ${ }^{2}$ The constant 3 is arbitrary, and can be driven down to $(1+\zeta)$, for any $\zeta>0$.

[^3]:    ${ }^{3}$ For a tree $T$ and a set $S \subseteq \operatorname{LEAVES}(T)$ we shall refer to the quantity $\sum_{v \in S} 2^{-w_{T}(v)}$ as the Kraft mass occupied by $S$ in $T$, or just the Kraft mass of $S$ if it is clear from context.

[^4]:    ${ }^{4}$ In fact, this is an oversimplification of our approach (as well as slightly inaccurate), but for the sake of discussion let us assume that this is the case.

[^5]:    ${ }^{5}$ The $2^{w_{T}(v)} \cdot \mid$ RIP $_{s} \mid$ correspond to the number of accesses on $x$, and $\|\widehat{\chi}\|_{0} \cdot \mid$ RIP $_{s} \mid$ corresponds to the time needed to subtract $\widehat{\chi}$ from the measurements. Lemma 7 in KVZ19 has an additional third component, which corresponds to the time needed to prepare the isolating filter $\widehat{G}$. It is not hard to see that this third component can always be bounded by $O(\log N)$, and hence can be safely ignored.

