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A general family of distributions, namely Kumaraswamy generalized family of (Kw-G) distribution, is considered for estimation of the unknown parameters and reliability function based on record data from Kw-G distribution. The maximum likelihood estimators (MLEs) are derived for unknown parameters and reliability function, along with its confidence intervals. A Bayesian study is carried out under symmetric and asymmetric loss functions in order to find the Bayes estimators for unknown parameters and reliability function. Future record values are predicted using Bayesian approach and non Bayesian approach, based on numerical examples and a monte carlo simulation.

Keywords: Kumaraswamy generalized distribution, reliability, interval estimation, Bayesian estimation, prediction

Introduction

Kumaraswamy (1980) defined a density function of double bounded random processes for handling of the problems occur in the hydrological field. This distribution did not gain much attention until Jones (2009) studied it thoroughly and gave some important remarks on this distribution with the Beta distribution. For example, both the distributions are unimodal, increasing, decreasing or constant depending on the values of the parameters. For quantile estimation, Kumaraswamy (Kw) distribution gives better mathematical tractability over the beta distribution whereas in calculation of moments or moment generating function Beta distribution provides flexible calculation than Kw distribution. So, it is better to say that, in most of the cases Kw distribution can easily be considered as an alternate to the Beta distribution but not always.

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The Kw distribution has received attention of the several authors (see Lemonte, 2011; Nadar, et al., 2013; and Kizilaslan and Nadar, 2016). Although Kw distribution overcomes most of the issues of the Beta distribution but still, having only finite range on (0,1), sometimes, it is not very useful to model the practical situations. Cordeiro and de Castro (2011) proposed a generalization of the Kw distribution (Kumaraswamy generalized (Kw-G) distribution) by introducing a baseline distribution function $G(x)$ in the existing density. Also, for $\theta = 1$, Kw-G distribution reduces to the Proportional reverse hazard rate model (Gupta and Gupta, 2007). The cumulative distribution function (cdf) of the Kw-G distribution is

$$F(x) = 1 - \left\{1 - G(x)^{\alpha}\right\}^{\theta}, \quad \alpha, \theta > 0, x \in \mathbb{R}, \tag{1}$$

and the corresponding probability density function (pdf) is

$$f(x) = \alpha\theta g(x) G(x)^{\alpha-1} \left\{1 - G(x)^{\alpha}\right\}^{\theta-1}, \quad \alpha, \theta > 0, x \in \mathbb{R}. \tag{2}$$

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with distribution function $F(x)$ and probability density function $f(x)$, then X_j is an upper record if $X_j > X_i$ for $j > i$. Clearly, X_1 is the first upper record. Similarly, we can define lower record values. Let $R_1, R_2, \ldots, R_n$ be n upper records and let $r_1, r_2, \ldots, r_n$ denote the observed values of $R_1, R_2, \ldots, R_n$ respectively. Then the joint density of upper records $\mathbf{R} = (R_1, R_2, \ldots, R_n)$ is given by

$$f_{\mathbf{R}}(\mathbf{r}) = \prod_{i=1}^{n-1} \left(\frac{f(r_i)}{1 - F(r_i)} \right) f(r_n), \quad -\infty < r_1 < r_2 < \ldots < r_{n-1} < r_n < \infty, \tag{3}$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ is observed value of $\mathbf{R}$. Record values are one of the widely accepted and applied theory in statistics because it relates to many real life problems such as: extreme rainfalls at a particular place, extreme weather conditions, highest stock prizes etc. Chandler (1952) was the first to study the mathematical properties of the record by defining record statistics as the model of the successive extremes in a sequence of i.i.d. random variables. For thorough understanding of records, readers can look through the following books Ahsanullah (1995) and Arnold, et al.

(1998). Various authors have proposed statistical structure of the records statistics on numerous distributions e.g., Kumar (2015), Khan and Arshad (2016), Asgharzadeh et al. (2017), Ahsanullah and Nevzorov (2017), Anwar et al. (2019) Arshad and Baklizi (2018), Arshad & Jamal (2019a), Arshad & Jamal (2019b), Hassan et al. (2020), Tripathi et al. (2021) and Azhad et al. (2021).

Classical Estimation

Maximum Likelihood Estimation

Using (1), (2) and (3), the likelihood function based on the upper records **R**, observed from Kw-G distribution is given by

$$L(\alpha,\theta)=(\alpha\theta)^{n}\left(\prod_{i=1}^{n-1}\frac{g(r_i)G(r_i)^{\alpha-1}}{\left(1-G(r_i)^{\alpha}\right)}\right)g(r_n)\left(G(r_n)\right)^{\alpha-1}\left(1-G(r_n)^{\alpha}\right)^{\theta-1}. \tag{4}$$

Taking log on both sides,

$$\begin{aligned}\ln L(\alpha,\theta)=n\ln(\alpha)+n\ln(\theta)+\sum_{i=1}^{n=1}\ln\left(\frac{g(r_i)G(r_i)^{\alpha-1}}{\left(1-G(r_i)\right)}\right)\\ +\ln g(r_n)+(\alpha-1)\ln G(r_n)+(\theta-1)\ln\left(1-G(r_n)^{\alpha}\right).\end{aligned} \tag{5}$$

Differentiating (5) with respect to α and θ and equating to 0, we get

$$n+\theta\ln\left(1-G(r_n)^{\alpha}\right)=0. \tag{6}$$

$$` \frac{n}{\alpha}+\sum_{i=1}^{n}\left(\frac{\ln G(r_i)}{1-G(r_i)^{\alpha}}\right)-\theta\frac{G(r_n)^{\alpha}\ln G(r_n)}{1-G(r_n)^{\alpha}}=0. \tag{7}$$

The solution of the nonlinear equations (6) and (7) gives the MLE $(\hat{\alpha},\hat{\theta})$ of (α,θ). Because of the nonlinear nature of these equations, it is very cumbersome to obtain the values of unknown parameters explicitly. So, we will apply numerical computation techniques such as Newton Raphson method to obtain the MLEs of the parameters (see Numerical Study, below). The corresponding MLE of the reliability function $\mathbb{R}(t)$ is obtained, after replacing α and θ with their respective MLEs in $\mathbb{R}(t)$. The MLE of the reliability function $\mathbb{R}(t)$ is given by

$$\hat{\mathbb{R}}(t)=\left(1-G(t)^{\hat{\alpha}}\right)^{\hat{\theta}}.$$

Interval Estimation

Exact and generalized confidence intervals are obtained for unknown parameter α and θ respectively. We know $R_1, R_2, \ldots, R_n$ be the first n upper records generated from Kw-G distributions given in (1). Clearly, the quantity $\bar{F}(R_i)$ has uniform distribution $U(0,1)$ and let $Y_1 = -\ln\bar{F}(R_i)$ and $Y_i = \ln\bar{F}(R_{i-1})-\ln\bar{F}(R_i)$ for each $i = 2, 3, \ldots, n$. It is easy to verify that Y_i\`s are i.i.d. exponential random variables (see Lemma 1 of Wang et al., 2015). Define

$$U_i=\left(\frac{Z_i}{Z_{i+1}}\right)^i, i=1,2,3,\ldots,n-1, \text{ and } U_n=Z_n,$$

where $Z_i = Y_1 + Y_2 + \ldots + Y_i = -\ln\bar{F}(R_i)$. The random variable $U_i(i = 1,2,\ldots,n - 1)$ has $U(0,1)$ distribution and U_n has gamma distribution with parameters $(n,1)$ (see Wang et al., 2010). Since $U_1, U_2, \ldots, U_{n-1}$ are independent $U(0,1)$, it follows that $-\ln(U_1), -\ln(U_2), \ldots, -\ln(U_{n-1})$ are independent $Exp(0,1)$. Clearly, $-\sum_{i=1}^{n-1}\ln(U_i) \sim \text{Gamma}(n - 1,1)$. Therefore,

$$\mathbb{K}_1(R,\alpha)=-2\sum_{i=1}^{n-1}\ln(U_i)=2\sum_{i=1}^{n-1}\ln\left(\frac{\ln\left(1-G(R_n)^{\alpha}\right)}{\ln\left(1-G(R_i)^{\alpha}\right)}\right)\sim\chi^2(2n-2), \tag{8}$$

where $\chi^2(\lambda)$ denotes the χ^2 distribution with λ degrees of freedom. Now we will observe the behavior of $\mathbb{K}_1(R,\alpha)$ with respect to $\alpha \in (0,\infty)$ for fixed $\mathbf{R}$. For $i = 1, 2, 3, \ldots, n - 1$, define

$$Q_i(\alpha) = \frac{\ln\left(1 - G(R_n)^{\alpha}\right)}{\ln\left(1 - G(R_i)^{\alpha}\right)}.$$

Now differentiating $Q_i(\alpha)$ with respect to α, we get

$$Q_i'(\alpha) = Q_i(\alpha)$$
$$= \left(\frac{G(R_i)^{\alpha} \ln\left(G(R_i)\right)}{\left(1 - G(R_i)^{\alpha}\right) \ln\left(1 - G(R_i)^{\alpha}\right)} - \frac{G(R_n)^{\alpha} \ln\left(G(R_n)\right)}{\left(1 - G(R_n)^{\alpha}\right) \ln\left(1 - G(R_n)^{\alpha}\right)} \right)$$

It is easily seen that $Q_i(\alpha) > 0$, for each $i = 1, 2, 3, \ldots, n-1$ and as $R_n > R_i$, then $Q'_i(\alpha) > 0$. Thus, for $i = 1, 2, \ldots, (n-1)$, $Q_i(\alpha)$ is an increasing function in α and $\ln(Q_i(\alpha))$ is also an increasing function in α. Therefore, $\mathbb{K}_1(R,\alpha)$ is an increasing function of α for a fixed $\mathbf{R}$. Hence, the exact confidence interval of α with confidence coefficient $(1-\beta)$ is given by

$$(\mathbb{K}_1^*(\mathbf{R}, \chi^2_{\beta/2}(2n-2)), \mathbb{K}_1^*(\mathbf{R}, \chi^2_{1-\beta/2}(2n-2))) \tag{9}$$

where $\chi^2_p(\lambda)$ denotes the p^{th} percentile of the χ^2 distribution with λ degrees of freedom and $\mathbb{K}^*_1(s)$ denotes the solution of the equation $\mathbb{K}_1(r,x) = s$ for fixed $\mathbf{R}$.

Consider Weerahandi (2004) to derive a generalized confidence interval for the parameter θ. $U_n = -\ln \bar{F}(R_n)$ follows gamma distribution with parameters $(n,1)$, which implies $V = 2U_n = -2\theta\ln(1 - G(R_n)^{\alpha})$ has a χ^2 distribution with $2n$ degrees of freedom. Therefore,

$$\theta = \frac{V}{-2\ln\left(1 - G(R_n)^{\alpha}\right)}. \tag{10}$$

Let W be a χ^2-distributed random variable with $2n-2$ degrees of freedom. $\mathbb{K}^*_1(\mathbf{R},W)$ is a unique solution of the equation $\mathbb{K}_1(\mathbf{R},x) = W$ for fixed $\mathbf{R}$. Substitute $\mathbb{K}^*_1(\mathbf{R},W)$ at place of α in equation (10). Therefore, the generalized pivotal quantity for θ is given by

$$\mathbb{K}_2 = -\frac{V}{2\ln\left(1-G\left(R_n\right)^{\mathbb{K}_1^*(\mathbf{R},W)}\right)} \tag{11}$$

$$= \frac{\theta \ln\left[1-G\left(R_n\right)^{\mathbb{K}_1^*(\mathbf{R},W)}\right]}{\ln\left(1-G\left(R_n\right)^{\mathbb{K}_1^*(\mathbf{R},W)}\right)}, \tag{12}$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ is the observed value of $\mathbf{R} = (R_1, R_2, \ldots, R_n)$. From (11) it does not contain any unknown parameters. It is also evident from (12) $\mathbb{K}_2$ reduces to θ when $\mathbf{R} = \mathbf{r}$. Thus, $\mathbb{K}_2$ is a generalized pivotal quantity for θ. The generalized confidence interval of θ with confidence coefficient $(1-\beta)$ can be obtained from the following simulation algorithm and is denoted by $[\mathbb{K}_{2,\beta/2}, \mathbb{K}_{2,(1-\beta/2)}]$.

1. Generate V from χ^2 distribution with $2n$ degrees of freedom.
2. Generate W from χ^2 distribution with $(2n-2)$ degrees of freedom and use generated value of W to obtain the unique solution $\mathbb{K}_1^*(\mathbf{R},W)$ of $\mathbb{K}_1^*(\mathbf{R},\alpha) = W$, for fixed $\mathbf{R}$.
3. Calculate the value of $\mathbb{K}_2$ by using the generated value of V obtained from step (1) and the solution $\mathbb{K}_1^*(\mathbf{R},W)$ obtained from step (2), in equation (11).
4. Repeat the above steps $m(\geq 10{,}000)$ times to obtain the $m(\geq 10{,}000)$ values of $\mathbb{K}_2$.
5. Calculate the $\beta/2$ and $(1-\beta/2)$ percentiles from the m generated values of $\mathbb{K}_2$ as $\mathbb{K}_{2,\beta/2}$ and $\mathbb{K}_{2,(1-\beta/2)}$ respectively.

Bayesian Estimation

Consider the estimation of parameters and reliability function of Kw-G distribution from Bayesian point of view. An important factor in Bayesian estimation is the loss function $L(\delta, \lambda)$, where δ denotes the decision rule (estimator) based on the data and λ is the unknown parameter. Consider two types of loss functions, symmetric and asymmetric loss functions. Square error loss function (SEL) is considered as the symmetric loss function and Linear exponential loss function (Linex) (see

Varian, 1975, and Zellner, 1986), entropy loss function (see James and Stein, 1961) are considered as the asymmetric loss functions.

Loss Function	Mathematical Form	Bayes Estimator
Linex	$e^{c(\delta-\lambda)} - c(\delta-\lambda) - 1$	$-(1/c)\ln(E(e^{-c\lambda}))$
Entropy	$(\delta/\lambda) - \ln(\delta/\lambda) - 1$	$(E(\lambda^{-1}))^{-1}$
Squared Error	$(\delta-\lambda)^2$	$E(\lambda)$

where $c \neq 0$ is the parameter of Linex loss function and expectation is taken over posterior distribution of λ given data. The form of the Kw-G distribution is complex and therefore a tractable continuous joint prior distribution for both parameters α and θ is difficult to obtain. Thus, to choose a joint prior that incorporates uncertainty about both parameters, we use the method proposed by Soland (1969). Assume that the parameter α is restricted to a finite number of values $\alpha_1, \alpha_2, \ldots, \alpha_k$ with prior probabilities $p_1, p_2, \ldots, p_k$, respectively, i.e., the prior distribution for α is given by

$$\pi\left(\alpha_j\right) = P\left(\alpha = \alpha_j\right) = p_j, \quad j = 1,2,\ldots,k. \tag{13}$$

Further, assume the conditional prior distribution for $\theta | \alpha_j$ has gamma distribution with parameters a_j and b_j for $j = 1, 2, \ldots, k$, i.e.,

$$\pi\left(\theta | \alpha_j\right) = \frac{b_j^{a_j} \theta^{a_j - 1} e^{-\theta b_j}}{\Gamma\left(a_j\right)}, \quad \theta > 0, a_j > 0, b_j > 0. \tag{14}$$

It follows from (13) and (14) that the joint prior distribution of (α,θ) is given by

$$\pi\left(\theta,\alpha_j\right) = \frac{p_j b_j^{a_j} \theta^{a_j - 1} e^{-\theta b_j}}{\Gamma\left(a_j\right)}, \quad \theta > 0, a_j > 0, b_j > 0, 0 \le p_j \le 1. \tag{15}$$

Using (4) and (15), the joint posterior density of θ and α is given by

$$\pi\left(\theta,\alpha_j | \mathbf{r}\right) = p_j \frac{b_j^{a_j} w_j \alpha_j^n}{\Gamma a_j Q} \theta^{n+a_j-1} e^{-\theta K\left(r_n,\alpha_j\right)} \frac{\left(G\left(r_n\right)\right)^{\alpha_j - 1}}{\left(1-\left(G\left(r_n\right)\right)^{\alpha_j}\right)}, \tag{16}$$
$$a_j > 0, \theta > 0, j = 1,2,\ldots,n,$$

where

$$w_j = \prod_{i=1}^{n-1} \frac{g(r_i)\left(G(r_i)\right)^{\alpha_j - 1}}{\left(1 - G(r_i)^{\alpha_j}\right)}, Q = \sum_{j=1}^{k} \frac{p_j b_j^{a_j} w_j \alpha_j^n \Gamma(n + a_j)}{\Gamma a_j \left[K(r_n, \alpha_j)\right]^{n+a_j}} \frac{\left(G(r_n)\right)^{\alpha_j - 1}}{\left(1 - \left(G(r_n)\right)^{\alpha_j}\right)},$$

and

$$K(r_n, \alpha_j) = b_j - \ln\left(1 - \left(G(r_n)\right)^{\alpha_j}\right), \quad j = 1, 2, \ldots, k.$$

Using (4) and (14) the conditional posterior density of $\theta | \alpha_j$ is given by

$$\pi(\theta | \alpha_j; \mathbf{r}) = \frac{\left[K(r_n, \alpha_j)\right]^{n+a_j}}{\Gamma(n + a_j)} \theta^{(n+a_j-1)} e^{-\theta K(r_n, \alpha_j)}, \quad \theta > 0, \tag{17}$$

and the marginal posterior density of α_j is

$$\begin{aligned} P_j &= \int_0^\infty \pi(\theta, \alpha_j | \mathbf{r}) d\theta \\ &= p_j \frac{b_j^{a_j} w_j \alpha_j^n \Gamma(n + a_j)}{\Gamma a_j \left[K(r_n, \alpha_j)\right]^{n+a_j}} \frac{\left(G(r_n)\right)^{\alpha_j - 1}}{Q\left(1 - \left(G(r_n)\right)^{\alpha_j}\right)}, \quad j = 1, 2, \ldots, k, \end{aligned} \tag{18}$$

Obtain the Bayes estimator under different loss functions. For Bayes estimators under Linex loss function for θ, α and $\mathbb{R}(t)$, we have

$$\theta_{BL}^* = -\frac{1}{c} \ln\left(\sum_{j=1}^{k} P_j \int_0^\infty e^{-c\theta} \pi(\theta | \alpha_j; \mathbf{r}) d\theta\right) = -\frac{1}{c} \ln\left(\sum_{j=1}^{k} P_j \left(1 + \frac{c}{K(r_n, \alpha_j)}\right)^{-(n+a_j)}\right),$$

$$\alpha_{BL}^* = -\frac{1}{c} \ln\left(E\left(e^{-c\alpha}\right)\right) = -\frac{1}{c} \ln\left(\sum_{j=1}^{k} P_j e^{-c\alpha_j}\right)$$

and

$$\mathbb{R}\left(t\right)_{BL}^{*}=-\frac{1}{c}\ln\left(\sum_{j=1}^{k}P_{j}\int_{0}^{\infty}\pi\left(\theta\,|\,\alpha;\mathbf{r}\right)d\theta\right)$$
$$=-\frac{1}{c}\ln\left(\sum_{j=1}^{k}\sum_{i=0}^{\infty}\frac{\left(-c\right)^{i}P_{j}}{i!}\left(1-\frac{i\ln\left(1-\left(G\left(t\right)\right)^{\alpha_{j}}\right)}{K\left(r_{n},\alpha_{j}\right)}\right)^{-\left(n+a_{j}\right)}\right).$$

The Bayes estimators under entropy error loss function for θ, α and $R(t)$ are given by

$$\theta_{BE}^{*}=\left(\int_{0}^{\infty}\frac{\pi\left(\theta\,|\,\alpha_{j};\mathbf{r}\right)\pi\left(\alpha_{j}\,|\,\mathbf{r}\right)}{\theta}d\theta\right)^{-1}=\left(\sum_{j=1}^{k}P_{j}\frac{K\left(r_{n},\alpha_{j}\right)}{\left(n+a_{j}-1\right)}\right)^{-1},$$

$$\alpha_{BE}^{*}=\left(\int_{0}^{\infty}\sum_{j=1}^{k}P_{j}\frac{1}{\alpha_{j}}\pi\left(\theta\,|\,\alpha_{j};\mathbf{r}\right)d\theta\right)^{-1}=\left(\sum_{j=1}^{k}\frac{P_{j}}{\alpha_{j}}\right)^{-1},$$

and

$$\mathbb{R}^{*}\left(t\right)_{BE}=\left(\sum_{j=1}^{k}P_{j}\int_{0}^{\infty}\left[1-F\left(t;\alpha_{j},\theta\right)\right]^{-1}\pi\left(\theta\,|\,\alpha_{j};\mathbf{r}\right)d\alpha\right)^{-1}$$
$$=\left(\int_{0}^{\infty}\sum_{j=1}^{k}P_{j}\frac{\pi\left(\theta\,|\,\alpha_{j};\mathbf{r}\right)}{\left(1-G\left(t\right)^{\alpha_{j}}\right)^{\theta}}d\theta\right)^{-1}$$
$$=\left(\sum_{j=1}^{k}P_{j}\left(1+\frac{\ln\left(1-G\left(t\right)^{\alpha_{j}}\right)}{K\left(r_{n},\alpha_{j}\right)}\right)^{-\left(n+a_{j}\right)}\right)^{-1}.$$

The Bayes estimators under squared error loss function for θ, α and $R(t)$ are given by

$$\theta_{BS}^{*}=\int_{0}^{\infty}\theta\sum_{j=1}^{k}P_{j}\pi\left(\theta\,|\,\alpha_{j};\mathbf{r}\right)d\theta=\sum_{j=1}^{k}P_{j}\frac{n+a_{j}}{K\left(r_{n},\alpha_{j}\right)},$$

$$\alpha_{BS}^{*}=\int_{0}^{\infty}\sum_{j=1}^{k}\alpha_{j}P_{j}\pi\left(\theta\,|\,\alpha_{j};\mathbf{r}\right)d\theta=\sum_{j=1}^{k}P_{j}\alpha_{j}$$

and

$$\begin{aligned}\mathbb{R}\left(t\right)_{BS}^{*}&=\int_{0}^{\infty}\sum_{j=1}^{k}\left[1-F\left(t;\theta,\alpha_{j}\right)\right]P_{j}\pi\left(\theta\,|\,\alpha_{j};\mathbf{r}\right)d\theta\\&=\sum_{j=1}^{k}P_{j}\left(1-\frac{\ln\left(1-G\left(t\right)^{\alpha_{j}}\right)}{K\left(r_{n},\alpha_{j}\right)}\right)^{-\left(n+a_{j}\right)}.\end{aligned}$$

Prediction For Future Records

Consider the problem of prediction of future s^{th} upper record value $R_s\,(s > n)$.

Non-Bayesian Prediction

A Non-Bayesian approach is presented to predict the R_s upper record value from a sequence of observed upper record up to n. The joint predictive likelihood function of $R_s = r_s$, α and θ is given by (Basak and Balakrishnan, 2003)

$$\begin{aligned}L\left(r_{s},\alpha,\theta;\mathbf{r}\right)=&\frac{\left[H\left(r_{s},\alpha,\theta;\mathbf{r}\right)-H\left(r_{n},\alpha,\theta;\mathbf{r}\right)\right]^{s-n-1}}{\Gamma\left(s-n\right)}\\&\times f\left(r_{s},\alpha,\theta\right)\prod_{i=1}^{n}\frac{f\left(r_{i};\alpha,\theta\right)}{1-F\left(r_{i};\alpha,\theta\right)}\end{aligned}\tag{19}$$

From (19), (1) and (2), we get the predictive likelihood function of Kw-G distribution as

$$L\left(r_s,\alpha,\theta;\mathbf{r}\right)=\frac{\alpha^{n+1}\theta^s}{\Gamma\left(s-n\right)}\left(\ln\left(1-G\left(r_n\right)^\alpha\right)-\ln\left(1-G\left(r_s\right)^\alpha\right)\right)^{s-n-1}$$
$$\times g\left(r_s\right)\left[1-G\left(r_s\right)^\alpha\right]^{\theta-1}\prod_{i=1}^{n}\left(\frac{g\left(r_i\right)}{1-G\left(r_i\right)^\alpha}\right) \tag{20}$$

After taking log on both sides in (20) and differentiating with respect to α, θ and r_s and then equating them to 0,

$$\frac{s}{\theta}+\ln\left(1-G\left(r_s\right)^\alpha\right)=0 \tag{21}$$

$$\frac{n+1}{\alpha}+\frac{\alpha\left(s-n-1\right)}{\ln\left(1-G\left(r_n\right)^\alpha\right)-\ln\left(1-G\left(r_s\right)^\alpha\right)}\left\{\frac{g\left(r_s\right)G\left(r_s\right)^{\alpha-1}}{1-G\left(r_s\right)^\alpha}-\frac{g\left(r_n\right)G\left(r_n\right)^{\alpha-1}}{1-G\left(r_n\right)^\alpha}\right\}$$
$$-\frac{\alpha\left(\theta-1\right)g\left(r_s\right)G\left(r_s\right)^{\alpha-1}}{1-G\left(r_s\right)^\alpha}+\alpha\sum_{i=1}^{n}\frac{g\left(r_i\right)G\left(r_i\right)^{\alpha-1}}{1-G\left(r_i\right)^\alpha}=0 \tag{22}$$

$$\frac{\alpha\left(s-n-1\right)g\left(r_s\right)G\left(r_s\right)^{\alpha-1}}{\left(\ln\left(1-G\left(r_n\right)^\alpha\right)-\ln\left(1-G\left(r_s\right)^\alpha\right)\right)}+\frac{g'\left(r_s\right)}{g\left(r_s\right)}-\frac{\alpha\left(\theta-1\right)g\left(r_s\right)G\left(r_s\right)^{\alpha-1}}{1-G\left(r_s\right)^\alpha}=0 \tag{23}$$

Reduce the above system of three equations into two equations by substituting the value of θ obtained from (21) into (22) and (23). The simplified form of the above system of equation is

$$\frac{n+1}{\alpha}+\frac{\alpha\left(s-n-1\right)}{\ln\left(1-G\left(r_n\right)^\alpha\right)-\ln\left(1-G\left(r_s\right)^\alpha\right)}\left\{\frac{g\left(r_s\right)G\left(r_s\right)^{\alpha-1}}{1-G\left(r_s\right)^\alpha}-\frac{g\left(r_n\right)G\left(r_n\right)^{\alpha-1}}{1-G\left(r_n\right)^\alpha}\right\}$$
$$-\frac{\alpha\left(\left(-s/\ln\left[1-G\left(r_s\right)^\alpha\right]\right)-1\right)g\left(r_s\right)G\left(r_s\right)^{\alpha-1}}{1-G\left(r_s\right)^\alpha}+\alpha\sum_{i=1}^{n}\frac{g\left(r_i\right)G\left(r_i\right)^{\alpha-1}}{1-G\left(r_i\right)^\alpha}=0 \tag{24}$$

$$\frac{\alpha(s-n-1)g(r_s)G(r_s)^{\alpha-1}}{\left(\ln\left(1-G(r_n)^{\alpha}\right)-\ln\left(1-G(r_s)^{\alpha}\right)\right)} + \frac{g'(r_s)}{g(r_s)} - \frac{\alpha\left(\left(-s/\ln\left[1-G(r_s)^{\alpha}\right]\right)-1\right)g(r_s)G(r_s)^{\alpha-1}}{1-G(r_s)^{\alpha}} = 0 \tag{25}$$

After solving (24) and (25), we obtain the value of future upper record r_s. This value is the point estimate of the future record.

Bayesian Prediction

Use the Bayesian approach to predict the future upper record R_s. The conditional distribution of R_s given R_n is obtained by using Markovian property (see Arnold et al., 1998).

$$f_{R_s|R_n}(r_s \mid r_n;\alpha,\theta) = \frac{\left[R(r_s)-R(r_n)\right]^{s-n-1}}{\Gamma(s-n)} \frac{f(r_s;\alpha,\theta)}{1-F(r_n;\alpha,\theta)}, \quad -\infty < r_s < r_n < \infty,$$

where $R(\cdot) = -\ln(1-F(\cdot))$. For Kw-G distribution, the conditional distribution of $R_s|R_n$ is

$$f_{R_s|R_n}(r_s \mid r_n;\alpha,\theta) = \frac{\alpha\theta^{s-n}}{\Gamma(s-n)} \times \left(\ln\left(\frac{1-G(r_n)^{\alpha}}{1-\left(G(r_s)\right)^{\alpha}}\right)\right)^{s-n-1} \left(\frac{1-\left(G(r_s)\right)^{\alpha}}{1-G(r_n)^{\alpha}}\right) \frac{g(r_s)\left(G(r_s)\right)^{\alpha-1}}{1-\left(G(r_s)\right)^{\alpha}} \tag{26}$$

From (26), the Bayes predictive density of $R_s = r_s$ given $R_n = r_n$ is given by

$$f(r_s \mid r_n) = \sum_{j=1}^{k} \int_0^{\infty} f_{R_s|R_n}(r_s \mid r_n;\alpha,\theta) P_j \pi(\theta \mid \alpha_j;\mathbf{r})\, d\theta. \tag{27}$$

Using equation (17) in (27),

$$
\begin{aligned}
f\left(r_s \mid r_n\right) = {} & \sum_{j=1}^{k} P_j \int_0^\infty \frac{\alpha_j \theta^{s-n}}{\Gamma(s-n)} \left(\ln\left(\frac{1-\left(G\left(r_n\right)\right)^{\alpha_j}}{1-\left(G\left(r_s\right)\right)^{\alpha_j}} \right) \right)^{s-n-1} \\
& \times \frac{g\left(r_s\right)\left(G\left(r_s\right)\right)^{\alpha_j-1}\left(1-\left(G\left(r_s\right)\right)^{\alpha_j}\right)^{\theta-1}}{\left(1-\left(G\left(r_n\right)\right)^{\alpha_j}\right)^{\theta}} \\
& \times \frac{\left[K\left(r_n,\alpha_j\right)\right]^{n+a_j}}{\Gamma\left(n+a_j\right)} \theta^{\left(n+a_j-1\right)} e^{-\theta K\left(r_n,\alpha_j\right)} d\theta \\
= {} & \sum_{j=1}^{k} \frac{P_j \left[K\left(r_n,\alpha_j\right)\right]^{n+a_j} \alpha_j}{B\left(n+a_j, s-n\right)} \\
& \times \left(\ln\left(\frac{1-\left(G\left(r_n\right)\right)^{\alpha_j}}{1-\left(G\left(r_s\right)\right)^{\alpha_j}} \right) \right)^{s-n-1} \left(\frac{1}{1-\left(G\left(r_s\right)\right)^{\alpha_j}} \right) \\
& \times \frac{g\left(r_s\right)\left(G\left(r_s\right)\right)^{\alpha_j-1}}{\left(b_j - \ln\left(1-\left(G\left(r_s\right)\right)^{a_j}\right)\right)^{s+a_j}}
\end{aligned}
\tag{28}
$$

where $B(a,b)$ is the complete beta function. Now we find the lower and upper 100 $(1-\alpha)$% prediction bounds for R_s. First, we find the predictive survival function $P(R_s \geq d|r_n)$, for any positive constant d

$$
P\left(R_s \geq d \mid r_n\right) = 1 - \int_{r_n}^{d} f\left(r_s \mid r_n\right) dr_s = 1 - \sum_{j=1}^{k} P_j \left(1 - \frac{IB\left(n+a_j, s-n, \zeta_j\right)}{B\left(n+a_j, s-n\right)} \right)
$$

where $\zeta_j = \dfrac{b_j - \ln\left(1-\left(G\left(r_n\right)\right)^{\alpha_j}\right)}{b_j - \ln\left(1-\left(G(d)\right)^{\alpha_j}\right)}$, and $IB(n+a_j, s-n, \chi)$ is the incomplete beta function defined as

$$IB(a,b,\chi)=\int_0^{\chi}u^{a-1}(1-u)^{b-1}\,du.$$

Let $L(r_n)$ and $U(r_n)$ be two constants such that

$$P\left[R_s>L(r_n)\,|\,r_n\right]=1-\frac{\tau}{2}\quad\text{and}\quad P\left[R_s>U(r_n)\,|\,r_n\right]=\frac{\tau}{2}. \tag{29}$$

Using (29), we obtain two sided $100\,(1-\tau)\%$ predictive bounds for R_s as $(L(r_n), U(r_n))$, i.e.,

$$P\left[L(r_n)<R_s<U(r_n)\right]=1-\tau.$$

Consider a special case when $s = n + 1$, which is of our interest practically because after getting n records, the next record $n + 1$ is needed. The predictive survival function of R_{n+1} is given as

$$P\left(R_{n+1}\geq d\,|\,r_n\right)=1-\sum_{j=1}^{k}P_j\left(1-\zeta_j^{n+a_j}\right)$$

Assume the case when $\alpha = 1$ (WLOG). For this case, predictive survival function can be written as

$$P\left(R_{n+1}\geq d\,|\,r_n\right)=\zeta^{n+a}. \tag{30}$$

From (29) and (30) we have lower and upper limits as

$$L(r_n)=G^{-1}\left(1-Exp\left\{b-\left(b-\ln\left(1-G(r_n)\right)\right)\left(1-\frac{\tau}{2}\right)^{\frac{-1}{n+a}}\right\}\right)$$

$$U(r_n)=G^{-1}\left(1-Exp\left\{b-\left(b-\ln\left(1-G(r_n)\right)\right)\left(\frac{\tau}{2}\right)^{\frac{-1}{n+a}}\right\}\right)$$

A Numerical Study

A numerical approach is applied to show the applicability of the results, obtained in this article. For this purpose, we consider a special case of Kw-G distribution by taking exponential distribution, i.e, $G(x) = 1 - e^{-x}$, as the baseline distribution function. So, the probability density function of Kumaraswamy exponential generalized (KwExp-G) distribution is

$$f(x) = \alpha\theta e^{-x}\left(1-e^{-x}\right)^{\alpha-1}\left\{1-\left(1-e^{-x}\right)^{\alpha}\right\}^{\theta-1}, \quad \alpha,\theta > 0, x > 0.$$

The corresponding distribution function is

$$F(x) = 1-\left\{1-\left(1-e^{-x}\right)^{\alpha}\right\}^{\theta}, \quad \alpha,\theta > 0, x > 0.$$

and the corresponding reliability function is

$$R(t) = \left\{1-\left(1-e^{-t}\right)^{\alpha}\right\}^{\theta}, \quad \alpha,\theta > 0, x > 0.$$

For the simulation purpose, we use R software (R Core Team, 2015).

Example 1 (Simulated data) Generate a random sample of upper record values of size $n = 8$ from KwExp-G distribution for $\alpha = 2$ and $\theta = 3$ as

0.38109.0.53398.1.04678.1.05722.1.27061.1.42030.1.87770.2.64394.

The MLE of α and θ with the help of Newton-Raphson method and using the equations (6) and (7), are $\hat{\alpha} = 1.54$ and $\hat{\theta} = 3.58$. Assume that the parameter α constitutes 10 finite discrete values as

1.6,1.7,1.8,1.9,2.0,2.1,2.2,2.3,2.4,2.5.

with equal probability 0.1. For the Bayes estimators, first calculate the hyper-parameters (a_j, b_j) for each $\alpha_j, j = 1, 2, \ldots, 10$. A nonparametric approach

$\tilde{R}\,(t_i = R_i) = (n - i + 0.625)/(n + 0.25)$, $i = 1, 2, 3, \ldots, n$ is used to estimate the a_j and b_j for any two different values of the reliability function $R(t_1)$ and $R(t_2)$ (see Martz and Waller, 1982). $\tilde{R}\,(\cdot)$ is the expected value of the reliability function for given $\alpha = \alpha_j$, i.e,

$$\begin{aligned} E_{\theta|\alpha_j}\left[R(t) \mid \alpha = \alpha_j\right] &= \int_0^\infty \left(1 - \left(1 - e^{-t}\right)^{\alpha_j}\right)^\theta \frac{b_j^{a_j} \theta^{a_j - 1} e^{-\theta b_j}}{\Gamma(a_j)} d\theta \\ &= \left(1 - \frac{\ln\left(1 - \left(1 - e^{-t}\right)^{\alpha_j}\right)}{b_j}\right)^{-a_j} \end{aligned} \tag{31}$$

Consider $\tilde{R}\,(1.05722) = 0.5606061$ and $\tilde{R}\,(1.4203) = 0.3181818$, for the solution of (a_j, b_j) for each $j = 1, 2, \ldots, 10$. These two values are substituted into (31), where a_j and b_j are solved numerically for each α_j, $j = 1, 2, \ldots, 10$, using the Newton-Raphson method. After that, with the help of calculated values of (a_j, b_j), posterior probabilities are calculated for each a_j, and presented in Table 1. The MLEs, Bayes estimators and reliability estimators (for different $t = 0.1, 0.3, 0.5$) are also calculated and presented in Table 2 and Table 3. Table 4 gives the confidence intervals of α, θ and also confidence bounds for the predicted value of the future upper record.

Table 1. Prior Information and Posterior Probabilities

j	**1**	**2**	**3**	**4**	**5**
α	1.6	1.7	1.8	1.9	2
p	0.1	0.1	0.1	0.1	0.1
a	4.0465112	2.0976982	7.9980718	0.8036931	3.7078846
b	1.947399	1.296337	2.667335	0.649253	1.682558
P	0.12634223	0.07664693	0.20789971	0.04588925	0.10586853
j	**6**	**7**	**8**	**9**	**10**
θ	2.1	2.2	2.3	2.4	2.5
p	0.1	0.1	0.1	0.1	0.1
a	5.3189091	3.2290364	1.6731381	4.5032895	2.5067792
b	1.99729	1.45264	0.96907	1.707029	1.20156
P	0.13647303	0.08937128	0.05299478	0.09980681	0.05870744

Table 2. Estimates of α and θ

	$(\star)_{ML}$	$(\star)_{BS}$	$(\star)_{BE}$	$(\star)_{BL}$				
				$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
α	1.5400	1.9970	1.9594	1.9594	2.0162	1.9782	1.9599	1.9424
θ	3.5800	3.8248	3.5403	4.5407	4.1330	3.5757	3.3666	3.1867

Table 3. Estimates of Reliability for different t

t	$(\star)_{ML}$	$(\star)_{BS}$	$(\star)_{BE}$	$(\star)_{BL}$			
				$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
1	0.9711	0.9654	0.9649	0.9600	0.9598	0.9596	0.9595
3	0.8062	0.7891	0.7804	0.7614	0.7578	0.7559	0.7541
5	0.5985	0.5777	0.5530	0.5319	0.5254	0.5222	0.5190

Table 4. Interval Estimates of unknown quantity and Prediction

	90%	95%	97%	99%
α	[0.1097, 2.5781]	[0.0577, 3.0000]	[0.0350, 3.2972]	[0.1080, 3.9060]
θ	[1.2365, 5.4395]	[1.0358, 6.2426]	[1.1019, 6.7173]	[0.7969, 8.1656]
R_9	[2.6655, 4.1139]	[2.6546, 4.5230]	[2.6503, 4.8452]	[2.6460, 5.5467]

Example 2 (Real Data) Nelson (1972) described the results of a life test experiment in which specimens of a type of electrical insulating fluid were subjected to a constant voltage stress. The length of time until each specimen failed, or "broke down," was observed. The experiment was tested at voltages ranging from 26 to 38 kilovolts (KV). Here, we consider the breakdown time of specimens at only 38 KV. The observed data is

$$0.47, 0.73, 1.40, 0.74, 0.39, 1.13, 0.09, 2.38.$$

From Kolmogorov-Smirnov (Ks) test, the observed data fits our KwExp-G distribution smoothly, with parameters' values $\alpha = 1.60$ and $\theta = 1.59$ and the Ks-Statistics for the fitted data is 0.25 with p-value 0.98. The upper record values from the observed data are $R = (0.47, 0.73, 1.40, 2.38)$. The estimated values of the parameters based on records are

Table 5. Estimates of α and θ

	$(\star)_{ML}$	$(\star)_{BS}$	$(\star)_{BE}$	$(\star)_{BL}$				
				$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
α	1.5800	1.5777	1.5141	1.6075	1.5879	1.5475	1.5275	1.5074
θ	2.0500	1.8978	1.4775	2.6164	2.1762	1.6995	1.5484	1.4281

Table 6. Estimates of Reliability for different t

t	$(\star)_{UMVUE}$	$(\star)_{ML}$	$(\star)_{BS}$	$(\star)_{BE}$	$(\star)_{BL}$			
					$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
1	0.8792	0.9818	0.9453	0.9458	0.9449	0.9449	0.9444	0.9439
3	0.6675	0.8838	0.7819	0.7851	0.7781	0.7787	0.7754	0.7719
5	0.4929	0.7525	0.6207	0.6259	0.6141	0.6141	0.6086	0.6018

Table 7. Interval Estimates

	90%	95%	97%	99%
α	[0.9700, 6.6790]	[0.6944, 7.4772]	[0.5395, 8.0339]	[0.3045, 9.1651]
θ	[0.3382, 3.7110]	[0.2486, 4.3604]	[0.2768, 4.8490]	[0.1505, 6.6982]
R_4	[1.4324, 3.8704]	[1.4159, 4.6491]	[1.4095, 5.2859]	[1.4031, 6.8624]

Given in Tables 5 and 6 are the various estimates of the parameters and reliability function respectively, and in Table 7 the interval estimates of the parameters. Table 7 shows that the prediction method is well developed as known value of R_4 is contained in the intervals. Proceeding in the same manner, the predicted next time of breakdown (upper record) R_5 of the specimens used in experiments given in Table 8.

Table 8. Prediction of next record breakdown time

	90%	95%	97%	99%
R_5	[2.4151, 4.9456]	[2.3973, 5.7192]	[2.3903, 6.3423]	[2.3854, 7.8526]

In order to compare the performances of the various estimators, use the concept of the mean square error and estimated risk:

1. Samples of upper records with different sample sizes (n) are generated from the KwExp-G distribution for various values of the unknown parameters.

2. The values of hyper-parameters and also all related posterior probabilities are calculated.

3. Estimates of $\hat{\alpha}$, $\hat{\theta}$ and reliability function are obtained.

4. Above steps are repeated *m* times to evaluate the estimated risks and MSEs of estimates using

$$ER(\delta)=\frac{1}{m}\sum_{i=1}^{m}L(\delta_i,\lambda) \quad \text{and} \quad MSE(\delta)=\frac{1}{m}\sum_{i=1}^{m}(\delta_i-\lambda)^2, \lambda\in\Theta.$$

From Table 9, observe the Bayes estimates of asymmetric loss functions are better performer than Bayes estimators of symmetric loss function. Also, the sample size of upper records increases, the MSEs are getting smaller, and similarly the same behavior of the unknown parameters and reliability function in Tables 10 and 11.

Table 9. MSEs of the Bayes estimates of α and θ

(α,θ) = (1.5,2)

n	$MSE(\alpha)_{BS}$	$MSE(\alpha)_{BE}$	$MSE(\alpha)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	0.2926	0.2518	1.3860	1.0507	0.6142	0.4703	0.3590
8	0.2815	0.2421	1.2667	0.9554	0.5523	0.4201	0.3184
10	0.2862	0.2466	0.9915	0.7433	0.4210	0.3156	0.2351

n	$MSE(\theta)_{BS}$	$MSE(\theta)_{BE}$	$MSE(\theta)_{BL}$				
			$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$	$c = 2.0$
6	0.2121	0.1023	0.5244	0.3354	0.1334	0.0858	0.0604
8	0.1833	0.0924	0.4616	0.2919	0.1159	0.0772	0.0588
10	0.1197	0.0707	0.3119	0.1924	0.0786	0.0597	0.0567

(α,θ) = (1.8,2)

n	$MSE(\alpha)_{BS}$	$MSE(\alpha)_{BE}$	$MSE(\alpha)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	0.0601	0.0428	0.7492	0.5174	0.2405	0.1603	0.1051
8	0.0578	0.0412	0.7070	0.4857	0.2243	0.1498	0.0993
10	0.0561	0.0400	0.7029	0.4840	0.2240	0.1401	0.0990

n	$MSE(\theta)_{BS}$	$MSE(\theta)_{BE}$	$MSE(\theta)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	0.1862	0.0860	0.4739	0.2994	0.1147	0.0726	0.0511
8	0.1766	0.0881	0.4471	0.2821	0.1113	0.0741	0.0568
10	0.1475	0.0760	0.3841	0.2388	0.0903	0.0641	0.0533

(α,θ) = (2,2)

n	$MSE(\alpha)_{BS}$	$MSE(\alpha)_{BE}$	$MSE(\alpha)_{BL}$				
			$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$	$c = 2.0$
6	0.0051	0.0034	0.4590	0.2892	0.1102	0.0697	0.0498
8	0.0050	0.0032	0.4468	0.2813	0.1100	0.0726	0.0552
10	0.0040	0.0030	0.3200	0.1986	0.0824	0.0627	0.0589

n	$MSE(\theta)_{BS}$	$MSE(\theta)_{BE}$	$MSE(\theta)_{BL}$				
			$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$	$c = 2.0$
6	0.1793	0.8269	0.4590	0.2892	0.1102	0.0697	0.0498
8	0.1755	0.8640	0.4468	0.2813	0.1100	0.0726	0.0552
10	0.1244	0.0740	0.3200	0.1986	0.0824	0.0627	0.0589

Table 10. Estimated risks for Bayes estimates of α and θ

$(\alpha,\theta) = (1.5,2)$

n	$ER(\alpha)_{BS}$	$ER(\alpha)_{BE}$	$ER(\alpha)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	0.2961	0.2547	1.2736	0.9644	0.5605	0.4272	0.3242
8	0.2942	0.2535	1.1826	0.8916	0.5130	0.3887	0.2931
10	0.2875	0.2479	1.1475	0.8628	0.4939	0.3731	0.2805

n	$ER(\theta)_{BS}$	$ER(\theta)_{BE}$	$ER(\theta)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	0.1789	0.0848	0.4562	0.2873	0.1110	0.0714	0.0522
8	0.1589	0.0797	0.4090	0.2559	0.0998	0.0672	0.0535
10	0.1516	0.0785	0.3924	0.2445	0.0959	0.0660	0.0545

$(\alpha,\theta) = (1.8,2)$

n	$ER(\alpha)_{BS}$	$ER(\alpha)_{BE}$	$ER(\alpha)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	0.0601	0.0428	0.6975	0.4798	0.2204	0.1458	0.0950
8	0.0592	0.0422	0.6380	0.4356	0.1984	0.1318	0.0875
10	0.0490	0.0389	0.3721	0.2448	0.1023	0.0666	0.0463

n	$ER(\theta)_{BS}$	$ER(\theta)_{BE}$	$ER(\theta)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	0.1681	0.0780	0.4339	0.2722	0.1032	0.0659	0.0483
8	0.1553	0.0817	0.3966	0.2486	0.0991	0.0687	0.0568
10	0.0783	0.0635	0.2039	0.1234	0.0576	0.0540	0.0463

$(\alpha,\theta) = (2,2)$

n	$ER(\alpha)_{BS}$	$ER(\alpha)_{BE}$	$ER(\alpha)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	0.0049	0.0030	0.4523	0.2867	0.1133	0.0748	0.0563
8	0.0040	0.0030	0.4094	0.2560	0.0992	0.0661	0.0521
10	0.0030	0.0019	0.3898	0.2422	0.0900	0.0643	0.0519

n	$ER(\theta)_{BS}$	$ER(\theta)_{BE}$	$ER(\theta)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	0.1826	0.0893	0.4589	0.2910	0.1148	0.0753	0.0561
8	0.1164	0.0637	0.4094	0.2560	0.0992	0.0663	0.0525
10	0.0899	0.0543	0.3832	0.2422	0.0940	0.0530	0.0511

Table 11. MSEs and Estimated risks (parenthesis) of Bayes estimates of $R(t)(\times 10^{-3})$

(α,θ,t) = (1.5,2,0.1)

n	$(\star)_{BS}$	$(\star)_{BE}$	$(\star)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	1.8158	1.8053	1.2560	1.2519	1.2438	1.2397	1.2356
	(1.8158)	(0.9872)	(0.6205)	(0.1556)	(0.1564)	(0.6273)	(1.4152)
8	1.5189	1.5057	1.2574	1.2519	1.2545	1.2457	1.2418
	(1.5189)	(0.8253)	(0.6212)	(0.1557)	(0.1566)	(0.6284)	(1.4179)
10	1.2619	1.2460	1.2564	1.2511	1.2498	1.2211	1.2388
	(1.2619)	(0.8253)	(0.6212)	(0.1557)	(0.1566)	(0.6284)	(1.4179)

(α,θ,t) = (1.8,2,0.1)

n	$(\star)_{BS}$	$(\star)_{BE}$	$(\star)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	0.1922	0.1891	0.0561	0.0555	0.0542	0.0537	0.0530
	(0.1922)	(0.0993)	(0.0280)	(0.0069)	(0.0068)	(0.0269)	(0.0599)
8	0.1126	0.1096	0.0610	0.0611	0.0540	0.0532	0.0510
	(0.1126)	(0.0651)	(0.0346)	(0.0086)	(0.0080)	(0.0335)	(0.0540)
10	0.0610	0.0589	0.0620	0.0615	0.0605	0.0600	0.0594
	(0.0610)	(0.0310)	(0.0309)	(0.0077)	(0.0076)	(0.0301)	(0.0672)

(α,θ,t) = (2,2,0.1)

n	$(\star)_{BS}$	$(\star)_{BE}$	$(\star)_{BL}$				
			$c = -1.0$	$c = -0.5$	$c = 0.5$	$c = 1.0$	$c = 1.5$
6	0.0149	0.0145	0.0370	0.0377	0.0390	0.0397	0.0404
	(0.0149)	(0.0075)	(0.0186)	(0.0047)	(0.0049)	(0.0198)	(0.0453)
8	0.0134	0.0140	0.0359	0.0364	0.0376	0.0382	0.0388
	(0.0134)	(0.0073)	(0.0179)	(0.0045)	(0.0046)	(0.0190)	(0.0435)
10	0.0610	0.0589	0.0620	0.0615	0.0605	0.0600	0.0594
	(0.0610)	(0.0310)	(0.0309)	(0.0077)	(0.0076)	(0.0301)	(0.0672)

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