High Quality Consistent Digital Curved Rays via Vector Field Rounding

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Abstract

We consider the *consistent digital rays* (CDR) of curved rays, which approximates a set of curved rays emanating from the origin by the set of rooted paths (called *digital rays*) of a spanning tree of a grid graph. Previously, a construction algorithm of CDR for *diffused families* of curved rays to attain an $O(\sqrt{n \log n})$ bound for the distance between digital ray and the corresponding ray is known [11]. In this paper, we give a description of the problem as a rounding problem of the vector field generated from the ray family, and investigate the relation of the quality of CDR and the discrepancy of the range space generated from gradient curves of rays. Consequently, we show the existence of a CDR with an $O(\log^{1.5} n)$ distance bound for any diffused family of curved rays.

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1 Introduction

Digital pictures and graphic displays are modeled by using a digital plane consisting of pixels in the square region $[0, n] \times [0, n]$. A *pixel* often means the unit square that is a cell of the integer grid, but it is represented by the grid point at its lower-left corner, and the unit square is called *pixel square* if necessary in this paper. In the digital plane, geometric objects are represented by sets of pixels. In such a pixel-based representation, geometric computation (e.g. the intersection computation) can be done pixel-wise using the pixel buffers equipped in GPU. Thus, the pixel-based representation of digital objects would lead to an additional methodology for geometric computation.

However, conversion of geometric objects into digital objects is a nontrivial problem [14], and it may cause several inconsistencies of computation. In particular, the digital objects representing basic objects in Euclidean geometry do not always satisfy Euclidean axioms. The first two Euclidean axioms are the properties on line segments: (1) we can draw a line segment between any given two points, and (2) we can extend a line segment straightly and continuously to a line. Also, it is implied that the line segment between two points is unique, and it is a subset of any longer line segment going through them. As a consequence, a nonempty intersection of two line segments must be either a point or a line segment (the second case happens if the line segments are on the same line). These axioms are also considered in non-Euclidean geometries, where line segments are replaced by geodesic curves.

A naive digital line segment representing the line segment pq between two pixels p and q is the set of pixels corresponding to pixel squares intersecting the real line segment pq. However, the axioms do not hold for this definition of digital line segments. As a consequence, as shown in Figure 1, the intersection of a pair of such digital line segments may have more than one connected components in the 4-neighbor topology of the digital plane, which may cause inconsistency in computation. It is a curious and important issue in mathematics and computer science to investigate a digital representation of a family of geometric objects such that they satisfy discrete counterparts of the Euclidean axioms.

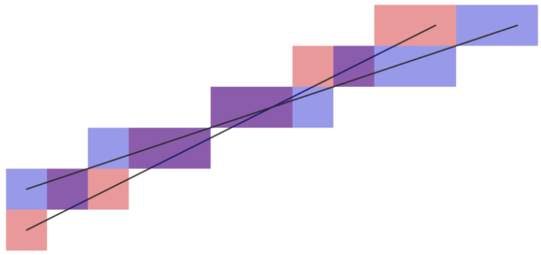

Figure 1 The intersection (purple pixel squares) of naive digital line segments may be disconnected.

The concept of *consistent digital rays* gives a model of digitization of a family of rays in the first quadrant[11, 12], which enables us to investigate the theoretical limit of digitization quantitatively by using the discrepancy theory [5, 16]. Here, a ray is a nondecreasing curve in the first quadrant emanating from the origin, and a pair of rays in the family do not intersect each other except at the origin (a concrete definition is given in Section 3).

Consider the triangular region Δ defined by $\{(x, y) \mid x \geq 0, y \geq 0, x+y \leq n\}$ in the plane, and the integer grid $G = \{(i, j) \mid i, j \in \{0, 1, \ldots, n\}, i + j \leq n\}$ in the region.

Each element of G is called a *pixel* (corresponding to the pixel in a digital picture). A pixel is called a *boundary pixel* if it lies on the off-diagonal boundary $x + y = n$ of Δ. The directed grid graph structure $\mathcal{G} = (G, E(\mathcal{G}))$ corresponding to the four-neighbor topology is given such that we have directed edges from $(i, j) \in G$ to $(i+1, j)$ and $(i, j+1)$ if $i+j \leq n-1$.

A *digital ray* is a directed path in $\mathcal{G}$ from the origin o to a pixel p. A digital ray is identified with the set of pixels on it, and regarded as a subset of G. Let us consider a family $\Pi = \{\Pi(p) \mid p \in G\}$ of digital rays. The family is called *consistent* if the following three properties hold:

1. *Uniqueness property*: For each $p \in G$, there exists a unique digital ray $\Pi(p)$ from the origin o to p in the family. We define $\Pi(o) = \{o\}$.
2. *Subsegment property*: If $q \in \Pi(p)$, then $\Pi(q) \subseteq \Pi(p)$.
3. *Prolongation property*: For each $\Pi(p)$, there is a (not necessarily unique) boundary pixel r such that $\Pi(p) \subseteq \Pi(r)$.

These properties are considered as the digital counterparts of the Euclidean axioms modified for the family of all halflines (called linear rays) emanating from the origin in the first quadrant.[1]

It is observed that the union of edge sets of paths in a consistent family of digital rays forms a (directed) spanning tree $\mathcal{T}$ of $\mathcal{G}$ rooted at o such that all leaves are boundary pixels (this condition corresponds to the prolongation property). The tree $\mathcal{T}$ is identified with the family Π of digital rays, and both of them are called CDR (*Consistent Digital Rays*). See the pictures (a) and (b) of Figure 2 for examples of CDR.

[1] The *shortest-path property* given in [12, 11, 7] is omitted by defining $\mathcal{G}$ as a directed graph in this paper.

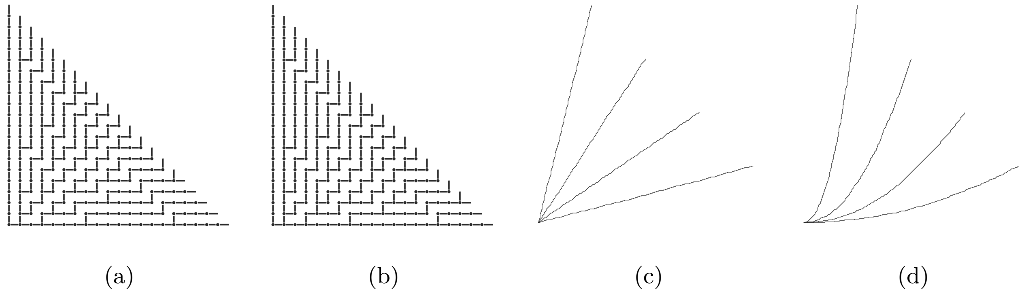

Figure 2 CDR for linear rays and parabolic rays in the triangular region of a 20×20 grid, and sampled linear and parabola digital rays in a 400×400 square grid.

Given a family of rays, it is desired to find a CDR approximating rays simultaneously. The quality of the approximation is measured by the largest distance between the digital ray $\Pi(p)$ and the corresponding ray $C(p)$ going through p over all $p \in G$. The Hausdorff distance is a popular distance between geometric objects, and considered in the previous works.

Historically, the theory started with how to realize digital straightness[14] to find a digitization of lines and line segments. Luby [15] first gave a construction of a CDR, where each $\Pi(p)$ simulates a linear ray within Hausdorff distance $O(\log n)$, and showed that the bound is asymptotically tight using geometric discrepancy. The construction was re-discovered by Chun et al. [12] in the formulation shown above. Christ et al. [10] gave a construction of consistent digital line segments where the lines need not go through the origin. There are works on variations and the high-dimensional generalizations [7, 8, 9].

The theory is extended by Chun et al. to families of curved rays [11]. A typical example is the family of parabolas $y = ax^2$ for $a \geq 0$. In Figure 2, the combinatorial difference between two CDRs (a) and (b) can be observed. The difference leads to the visual difference of digital rays illustrated in Figure 2, where it can be seen that the digital rays in (b) approximate parabolas as shown in (d) extended to a sufficiently large grid, while (a) approximates linear rays as shown in (c). A construction method of CDR for a wide class of families of curved rays called *diffused ray families* (its definition is given in Section 3.3) is given in [11]. However, the usage of discrepancy theory is limited because of difficulty to handle curved rays, and the attained distance bound is $O(\sqrt{n}\log n)$.

In this paper, we give a novel description of the problem as a rounding problem of a vector field, and regard the problem as a variant of the linear discrepancy problem. Intuitively, the rays are considered as geodesic curves for the vector field, and the *rounding* of the vector field naturally leads to a CDR. Then, in order to solve this variant of discrepancy problem, we apply the transference theory from the combinatorial discrepancy to the geometric discrepancy, and generate a tailor-made low-discrepancy pseudo-random sequence for the given family $\mathcal{F}$.

This enables us to prove the existence of a CDR with an $O(\log^{1.5} n)$ upper bound for the distance between rays and their corresponding digital rays for any diffused ray family. Although the above proof uses a non-constructive method in discrepancy theory, a CDR with a slightly weaker $O(\log^2 n)$ distance bound is computed in polynomial time.

2 Preliminaries on discrepancy theory

We introduce the definitions of three kinds of discrepancies used in this paper.

2.1 Range space and geometric Discrepancy

Consider a family $\mathcal{A}$ of subregions of $R = [0, n] \times [0, 1]$ and a set P of n points in R. The pair $(P, \mathcal{A})$ forms a *range space*. Let $\mathrm{vol}(A)$ be the area of $A \in \mathcal{A}$. We define

$$
\begin{aligned}
D(P, A) &= |\mathrm{vol}(A) - |P \cap A|| \quad \text{for} \quad A \in \mathcal{A}, \\
D(P, \mathcal{A}) &= \sup_{A \in \mathcal{A}} D(P, A), \text{ and} \\
D(n, \mathcal{A}) &= \inf_{|P|=n} D(P, \mathcal{A}).
\end{aligned}
$$

$D(P, \mathcal{A})$ and $D(n, \mathcal{A})$ are called the *geometric discrepancies* of the range space $(P, \mathcal{A})$ and the region family $\mathcal{A}$, respectively. See [16] for the geometric discrepancy theory. [2]

2.2 Combinatorial Discrepancy

For a finite set X, a family $\mathcal{S} \subseteq 2^X$ is called a *set system* on X. It generates a *hypergraph* $H = (X, \mathcal{S})$. A hypergraph coloring (bi-coloring) of H is a mapping $\chi : X \to \{-1, +1\}$, and we define $\chi(S) = \sum_{x \in S} \chi(x)$ for $S \in \mathcal{S}$. The *combinatorial discrepancy* is a measure of the balance of the coloring defined as follows:

$$
\begin{aligned}
\mathrm{disc}(\chi, \mathcal{S}) &= \max_{S \in \mathcal{S}} |\chi(S)|, \\
\mathrm{disc}(\mathcal{S}) &= \min_{\chi} \mathrm{disc}(\chi, \mathcal{S}).
\end{aligned}
$$

Given a range space $(P, \mathcal{A})$, $\mathcal{A}|_P = \{P \cap A \mid A \in \mathcal{A}\}$ is a set system on P, and we can consider its combinatorial discrepancy $\mathrm{disc}(\mathcal{A}|_P)$. We define the combinatorial discrepancy of the region family $\mathcal{A}$ by $\mathrm{disc}(n, \mathcal{A}) = \max_{|P|=n} \mathrm{disc}(\mathcal{A}|_P)$.

The combinatorial discrepancy of a range space and the geometric discrepancy are strongly related via *transference principle* (Theorem 14).

2.3 Linear discrepancy

Given a hypergraph $H = (X, \mathcal{S})$ and a real valued function $w : X \to [-1, 1]$ called *weight function*, we consider a function $\chi : X \to \{-1, 1\}$ called a *rounding* of w. For each $S \in \mathcal{S}$, $w(S)$ and $\chi(S)$ are the summations of the values of w and χ over S, respectively. The *linear discrepancy* of the rounding χ is

$$
\mathrm{lindisc}(w, \chi) = \max_{S \in \mathcal{S}} |\chi(S) - w(S)|.
$$

$\min_\chi \mathrm{lindisc}(w, \chi)$ and $\max_w \min_\chi \mathrm{lindisc}(w, \chi)$ are called the linear discrepancy of w and H, respectively. The combinatorial discrepancy $\mathrm{disc}(\mathcal{S})$ is equivalent to the linear discrepancy of the weight function $w \equiv 0$.

[2] The geometric discrepancy is defined more generally in [16] for range spaces in $[0, 1]^d$ instead of $[0, n] \times [0, 1]$.

3 Consistent Digital Rays

3.1 The structure of consistent digital rays

As mentioned in the introduction, a CDR is regarded as a rooted directed spanning tree $\mathcal{T}$ of the grid graph $\mathcal{G}$ on the triangular grid G, such that $\mathcal{T}$ has no leaf in the interior of Δ. Let $\ell(z)$ be the off-diagonal line defined by $x+y=z$. $L(k)=\{(x,y)\in G \mid x+y=k\}=\ell(k)\cap G$ is a *level set* of G for a natural number $k\le n$. By definition, all leaves of $\mathcal{T}$ are in $L(n)$.

Each non-root pixel has exactly one incoming edge of $\mathcal{T}$. Also, as illustrated in Figure 3, there is a unique pixel (named branching pixel) in $L(k)$ with two outgoing edges for $k\neq n$, since $|L(k+1)|=|L(k)|+1$ and there is no leaf vertex in $L(k)$. Accordingly, there exists a point (not necessarily a pixel) $p\in\ell(k+1)$ such that all incoming edges to the pixels on the left (resp. right) of p are vertical (resp. horizontal). Such a point is called a *split point*, which partitions the incoming edges to each level into vertical and horizontal ones.

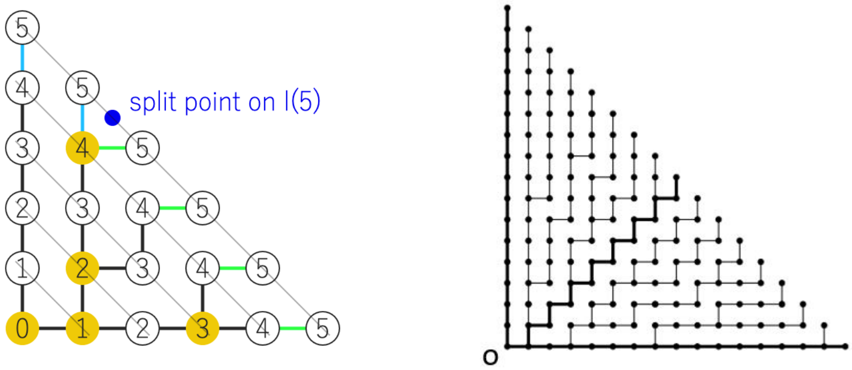

Figure 3 The branching pixels (colored yellow) and a split point are illustrated in the left picture, which shows the first five levels of the CDR in the right picture.

3.2 Off-diagonal distance between rays

A non-decreasing curve segment in Δ emanating from the origin is called a *partial ray*. We slightly abuse the notation so that a rooted path in $\mathcal{G}$ is also a partial ray, which consists of horizontal and vertical segments corresponding to its edges. We say that a partial ray terminates on $\ell(t)$ if it ends at a point on $\ell(t)$. A partial ray is called a *ray* if it terminates on the off-diagonal boundary $\ell(n)$ of Δ.

Given a partial ray C crossing $\ell(z)$, let $q_C(z)=(x_C(z),y_C(z))$ be the unique intersection point of C and $\ell(z)$. We define the discrete off-diagonal-wise L_∞ distance (*off-diagonal distance* in short) using $x_C(z)$ as follows:

Given partial rays C and C' both terminating on $\ell(m)$ for a natural number $m\le n$, their off-diagonal distance is defined by

$$d_o(C,C')=\max_{k=1,2,\ldots,m}|x_C(k)-x_{C'}(k)|.$$

In other words, we measure the distance between two partial rays by the maximum horizontal distance (the vertical distance is the same) between their intersection points with $\ell(k)$ over natural numbers $k\le m$. In particular, we can consider the off-diagonal distance $d_o(\Gamma,C)$ between a rooted path Γ in $\mathcal{G}$ and a partial ray C terminating at the same pixel.

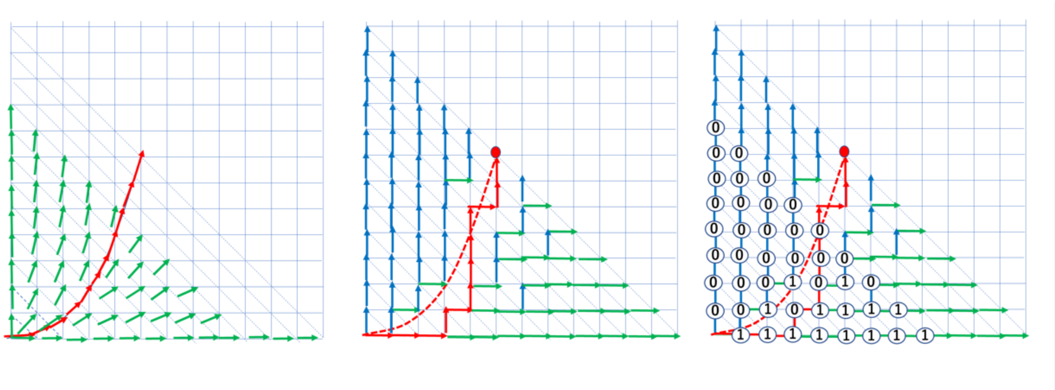

Figure 4 The vector field of the gradient vectors (left), a CDR approximating it (center), and its corresponding rounding χ (right, shown up to $L(8)$.)

The off-diagonal distance is a discrete variant of the L_∞-Hausdorff distance (i.e., the Hausdorff distance based on the L_∞ distance), which equals $\sup_{0<z\le m} |x_C(z) - x_{C'}(z)|$ for partial rays. It is observed that the Hausdorff distance (i.e., the Hausdorff distance based on the Euclidean distance) between C and C' is at most $\sqrt{2}(d_o(C, C') + 1)$, and at least $d_o(C, C')$ (see textbooks or [12] for the definition of the Hausdorff distance). Thus, we use the off-diagonal distance in our analysis, since its asymptotic bound gives that of the Hausdorff distance.

3.3 CDR as rounding of a vector field

A family $\mathcal{F}$ of rays is called a *ray family* if for each point $p = (x, y) \in \Delta \setminus \{o\}$ there exists a unique ray $C(p)$ of $\mathcal{F}$ going through it. We denote the partial ray that is the part of $C(p)$ terminating at p by $\tilde{C}(p)$.

A ray family $\mathcal{F}$ is called *smooth* if each ray in $\mathcal{F}$ is differentiable.

Let us focus on a smooth ray family $\mathcal{F}$. We give a description of CDR as a *rounding* problem of a vector field induced from $\mathcal{F}$ to a discrete vector field on pixels (see Figure 4).

For $p = (x_p, y_p)$ for $x_p > 0$, suppose that the ray $C(p)$ is given by a function $y = f_p(x)$ in a neighbourhood of p. The slope of $C(p)$ at p is given by $f'_p(x_p)$ using the derivative of f. Since the slope is nonnegative, we can write $f'_p(x_p) = \frac{1-\alpha_p}{\alpha_p}$ uniquely by using a real number $0 < \alpha_p \le 1$. It defines the *gradient vector* $\mathbf{V}_p = (\alpha_p, 1 - \alpha_p)$ to give the direction of the curve $C(p)$ at p normalized with respect to the L_1 norm. We set $\mathbf{V}_p = (0, 1)$ if $x_p = 0$ and $y_p > 0$. We do not define a gradient vector at $o = (0, 0)$. This defines a vector field $\mathbf{V} : \Delta \setminus \{o\} \to \mathbb{R}^2$ on the triangular region.[3]

As illustrated in the center picture of Figure 4, the CDR problem can be regarded as the problem to find an assignment of either $(1, 0)$ or $(0, 1)$ to each pixel of $G \setminus \{o\}$ such that the unit vector indicates the kind (horizontal or vertical) of the incoming edge of $\mathcal{T}$ to the pixel. If the CDR approximates the ray family $\mathcal{F}$, the assignment should approximate the vector field $\mathbf{V}$.

[3] If a potential function Φ to present gradient vectors as $(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y})$ is given, the rays are considered as geodesic paths in the potential field.

Each vector $\mathbf{V}_p$ is uniquely determined by $\alpha_p \in [0,1]$, and the vector field is converted to a $[0,1]$-valued function w defined by $w(p) = \alpha_p$. We call w the *gradient weight* of the vector field $\mathbf{V}$ in this paper. The vectors $(1,0)$ and $(0,1)$ are converted to 1 and 0 by this transformation.

Therefore, the CDR problem is converted to the problem to compute an assignment $\chi : G \setminus \{o\} \to \{0,1\}$ from the gradient weight (see the right picture of Figure 4). This is analogous to the linear discrepancy problem, if we scale the range of the weight from $[-1,1]$ to $[0,1]$. Thus, we call χ a *rounding* of w.

By definition, the off-diagonal distance between the digital ray $\mathit{\Gamma} = \mathit{\Pi}(p)$ and the partial ray $C = \tilde{C}(p)$ towards $p \in L(m)$ is $d_o(\mathit{\Gamma}, C) = \max_{k=1,2,\ldots m} |x_{\mathit{\Gamma}}(k) - x_C(k)|$, where $x_{\mathit{\Gamma}}(k)$ is the x-coordinate value of the pixel $q_{\mathit{\Gamma}}(k) = \mathit{\Gamma} \cap L(k)$. The following lemma relates the gradient weight and the rounding to the off-diagonal distance.

▶ **Lemma 1.** *$x_C(k) = \int_0^k w(q_C(z))dz$, and $x_{\mathit{\Gamma}}(k) = \sum_{i=1}^k \chi(q_{\mathit{\Gamma}}(i))$.*

Proof. If a ray goes through a point $q = (x,y)$ on $\ell(z)$ and reaches a point $(x+dx, y+dy)$ on $\ell(x+dz)$ for an infinitesimally small dz, then $dx = \alpha_q dz = w(q)dz$ by the definition of the gradient vector. If C is the ray, $q = q_C(z) = (x_C(z), y_C(z))$. Thus, $x_C(k) = \int_0^k w(q_C(z))dz$.

The x-value of a pixel $q = q_{\mathit{\Gamma}}(k)$ on a path $\mathit{\Gamma}$ is the number of horizontal edges up to the pixel, which is the prefix sum of χ over the path $\mathit{\Gamma}$ up to the level $L(k)$, and hence $x_{\mathit{\Gamma}}(k) = \sum_{i=1}^k \chi(q_{\mathit{\Gamma}}(i))$. ◀

A function f on Δ is called *off-diagonal monotone* if it is non-decreasing on each off-diagonal line $\ell(z)$. That is, $f(p) \geq f(q)$ if $x_p \geq x_q$ and $p, q \in \ell(z)$. It is called *strongly off-diagonal monotone* if it is increasing on each off-diagonal line.

The function χ corresponding to a spanning tree of G if $\chi(0,k) = 0$ and $\chi(k,0) = 1$ for $1 \leq k \leq n$ (i.e., the edges of $\mathcal{T}$ are vertical on the y-axis and horizontal on the x-axis). However, the spanning tree might have leaves in the interior of G (such a spanning tree is called a *weak CDR* in [7]). The spanning tree becomes a CDR if and only if χ is off-diagonal monotone, which is equivalent to the fact that there is a split point in each level.

We call a smooth ray family $\mathcal{F}$ *diffused* if the gradient weight w of its corresponding vector field is strongly off-diagonal monotone and continuous on each $\ell(z)$. This definition of the diffused ray family is equivalent to the one given in [11].

From now on, we focus on a CDR of a diffused family of rays, and regard it as the problem of seeking for a rounding χ minimizing the off-diagonal distance. The difference of this rounding problem from the ordinary linear discrepancy problem is as follows:

1. The set system is $\{\mathit{\Pi}(p) \mid p \in G\}$, which depends on $\mathcal{T}$, and hence on the choice of χ.
2. The rounding must preserve the off-diagonal monotonicity.
3. We must relate the off-diagonal distance to the discrepancy.

We apply the discrepancy theory to this vector field rounding problem.

4 Construction of CDR for diffused ray families

4.1 Construction algorithm of CDR via level-wise threshold rounding

We give a construction algorithm named *θ-threshold rounding algorithm* of a CDR approximating given diffused ray family $\mathcal{F}$ by using a $(0,1]$-valued sequence $\theta : \{1,2,\ldots n\} \to (0,1]$.

▶ **Definition 2.** *Given a gradient weight w and a $(0,1]$-valued sequence θ, the θ-threshold rounding χ of w is defined by the following:*

For $q \in L(k)$ $(k = 1,2,\ldots,n)$, $\chi(q) = 1$ if and only if $w(q) \geq \theta(k)$.

The construction algorithm is very simple: Given a diffused ray family $\mathcal{F}$, we consider its gradient weight w, compute its θ-threshold rounding, and obtain the corresponding CDR.

▶ **Example 3.** Consider the linear ray family $\mathcal{F} = \{C_a^{\text{lin}} : y = ax \mid a \in [0, \infty]\}$, where C_∞^{lin} is the line $x = 0$. The derivative of $y = ax$ is a, which is equal to $\frac{y}{x}$. Hence, the slope of $C(p)$ at $p = (x, y)$ is $\frac{y}{x}$, and the vector field $\mathbf{V}$ is defined by $\mathbf{V}_p = (\frac{x}{x+y}, \frac{y}{x+y})$, and $w(p) = \frac{x}{x+y}$. If $p = (kt, k(1-t)) \in L(k)$, $w(p) = t$. Thus, $\chi(p) = 1$ if and only if $t \geq \theta(k)$.

▶ **Example 4.** Consider the parabola family $\mathcal{F} = \{C_a^{\text{para}} : y = ax^2 \mid a \in [0, \infty]\}$, where C_∞^{para} is the line $x = 0$. The derivative of $y = ax^2$ is $y' = 2ax = \frac{2y}{x}$, and hence the slope of $C(p)$ at $p = (x, y)$ is $\frac{2y}{x}$. Thus, the vector field $\mathbf{V}$ is defined by $\mathbf{V}_p = (\frac{x}{x+2y}, \frac{2y}{x+2y})$, and $w(p) = \frac{x}{x+2y}$. If $p = (kt, k(1-t)) \in L(k)$, $w(p) = \frac{t}{2-t}$. Thus, $\chi(p) = 1$ if and only if $\frac{t}{2-t} \geq \theta(k)$.

The model of geometric computation to discuss the complexity and some more examples are given in the appendix.

4.2 Discrepancy that bounds the off-diagonal distance

The θ-threshold rounding algorithm is equivalent to the algorithm given in [11], where θ is fixed to be a random sequence or a known low-discrepancy sequence independently of choice of $\mathcal{F}$. In contrast to it, we seek for a tailor-made sequence θ to fit each ray family $\mathcal{F}$.

A ray $C \in \mathcal{F}$ defines its *gradient curve* $\varphi_C : \{(z, w(q_C(z))) \mid 0 < z \leq n\}$ in the (z, w) plane. Consider the family $\mathcal{F}^* = \{\varphi_C \mid C \in \mathcal{F}\}$ of gradient curves.

Given a curve $\varphi : w = f(z)$ in $\mathcal{F}^*$, let

$$\begin{aligned} R^-(\varphi, (a, b]) &= \{(z, w) \mid a < z \leq b,\ 0 \leq w < f(z)\} \text{ and} \\ R^+(\varphi, (a, b]) &= \{(z, w) \mid a < z \leq b,\ f(z) < w \leq 1\} \quad \text{for} \quad 0 \leq a \leq b \leq n. \end{aligned}$$

In other words, $R^-(\varphi, (a, b])$ (resp. $R^+(\varphi, (a, b])$) is the subregion of $(0, n] \times [0, 1]$ below (resp. above) φ and bounded by two vertical lines $z = a$ and $z = b$. We define the family of regions

$$\mathcal{A}_{\mathcal{F}^*} = \{R^\epsilon(\varphi, (a, b]) \mid 0 \leq a \leq b \leq n,\ \varphi \in \mathcal{F}^*,\ \epsilon \in \{+, -\}\} \cup \{(a, b] \times [0, 1] \mid 0 \leq a < b \leq n\}.$$

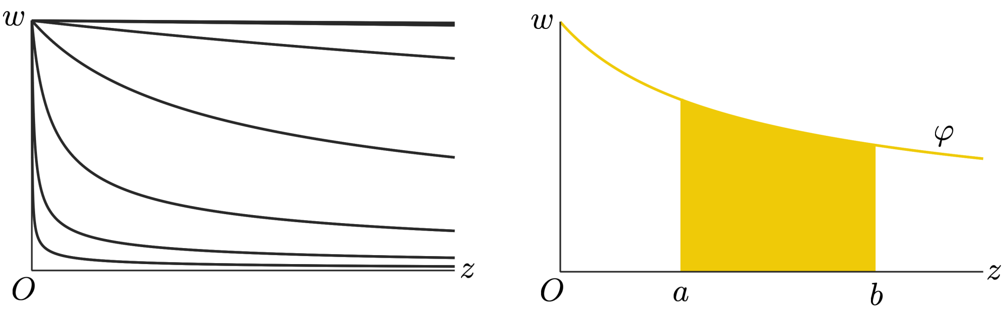

Figure 5 The gradient curves of parabola rays and a region $R^-(\varphi, (a, b])$ in $\mathcal{A}_{\mathcal{F}^*}$.

▶ **Example 5.** For the linear ray $C : y = ax$ $(a \geq 0)$, $w(q_c(z)) = \frac{1}{1+a}$, and hence φ_C is the horizontal line defined by $w = \frac{1}{1+a}$. Thus, $\mathcal{A}_{\mathcal{F}^*}$ is the family of axis parallel rectangles.

▶ **Example 6.** For the parabola ray $C : y = ax^2$ $(a \geq 0)$, $q_C(z) = (\frac{-1+\sqrt{1+4az}}{2a}, z - \frac{-1+\sqrt{1+4az}}{2a})$, and $w(q_C(z)) = \frac{1}{\sqrt{1+4az}}$. The curve φ_C is defined by $w = \frac{1}{\sqrt{1+4az}}$. The gradient curves and a region in $\mathcal{A}_{\mathcal{F}^*}$ for the family $\mathcal{F}$ of parabola rays are illustrated in Figure 5.

Let us fix the $(0, 1]$-valued sequence θ, and focus on the rounding χ and corresponding CDR Π constructed by the θ-threshold rounding algorithm.

The point set $S(\theta) = \{s_i = (i, \theta(i)) \mid 1 \le i \le n\}$ is called the θ-*Hammersley point set*, or Hammersley point set if θ is implicitly given[4].

The following lemma shows a relation between the positions of points of $S(\theta)$ in the arrangement of gradient curves and the assignment of χ-values of pixels in the arrangement of rays.

▶ **Lemma 7.** *If $s_k = (k, \theta(k)) \in S(\theta)$ is below (resp. above) the gradient curve φ_C, $\chi(p) = 1$ (resp. $\chi(p) = 0$) for all pixels $p \in L(k)$ lying on the right (resp. left) of C.*

Proof. We assume s_k is below φ_C (the other case is analogous). Hence, $\theta(k) < w(q_C(k))$.

Because of the continuity and strong monotonicity of w on $\ell(k)$, there is a unique point $u \in \ell(k)$ satisfying $w(u) = \theta(k)$. The point u becomes a split point because of the definition of the θ-threshold rounding.

By the assumption, $w(u) = \theta(k) < w(q_C(k))$, and the strong monotonicity of w implies that u is on the left of C. Thus, each pixel $p \in L(k)$ on the right of C is also on the right of u, and thus $\chi(p) = 1$ because of the definition of the split points. ◀

We consider the geometric discrepancy $D(S(\theta), \mathcal{A}_{\mathcal{F}^*})$, and the following theorem tells the explicit relation of the discrepancy and the off-diagonal discrepancy.

▶ **Theorem 8.** *Suppose that $D(S(\theta), \mathcal{A}_{\mathcal{F}^*}) \le \delta(n)$ for a function δ. Then, $d_o(\Pi(p), \tilde{C}(p)) \le \delta(n) + 1$ for each pixel p in G.*

Proof. Let $S = S(\theta)$, $\Gamma = \Pi(p)$, and $C = \tilde{C}(p)$. Without loss of generality, we assume $p \in L(n)$. We assume the off-diagonal distance $\max_{1 \le k \le n} |x_\Gamma(k) - x_C(k)|$ between Γ and C is d, and derive $d \le \delta(n) + 1$ to prove the theorem.

From the assumption, there exists k_0 such that $|x_\Gamma(k_0) - x_C(k_0)| = d$. Thus, either $x_\Gamma(k_0) = x_C(k_0) + d$ or $x_\Gamma(k_0) = x_C(k_0) - d$, and we focus on the former case, since the latter case can be handled analogously.

Consider the first index $m > k_0$ such that $x_\Gamma(m) \le x_C(m)$. In other words, m is the first index after k_0 such that the pixel of Γ in the level $L(m)$ comes on the left of (or on) C. Such m exists because both Γ and C reach p. Thus,

$$x_\Gamma(m) - x_C(m) \le 0 = x_\Gamma(k_0) - x_C(k_0) - d \tag{1}$$

Consider $R = R^-(\varphi_C, (k_0, m-1]) \in \mathcal{A}_{\mathcal{F}^*}$, which is the region below φ_C and $k_0 < z \le m-1$. Let $N(S, R)$ be the number of points of S in R.

The path Γ is on the right of C in the range $k_0 \le z \le m-1$, and it is derived from Lemma 7 that $\chi(q_\Gamma(k)) = 1$ if $s_k \in R$. Thus, we have the following:

$$\sum_{k=k_0+1}^{m-1} \chi(q_\Gamma(k)) \ge \sum_{k: s_k \in R} \chi(q_\Gamma(k)) = \sum_{k: s_k \in R} 1 = N(S, R). \tag{2}$$

From Lemma 1, $x_\Gamma(j) = \sum_{k=1}^{j} \chi(q_\Gamma(k))$, and hence combined with (2),

$$x_\Gamma(m-1) - x_\Gamma(k_0) = \sum_{k=k_0+1}^{m-1} \chi(q_\Gamma(k)) \ge N(S, R). \tag{3}$$

[4] The original 2-dimensional Hammersley point set uses the van der Corput sequence as θ, but the notation is abused to allow to use a general θ.

By the definitions of R and φ_C,

$$\mathrm{vol}(R) = \int_{k_0}^{m-1} \varphi_C(z)dz = \int_{k_0}^{m-1} w(q_C(z))dz.$$

On the other hand, from Lemma 1,

$$\int_{k_0}^{m-1} w(q_C(z))dz = x_C(m-1) - x_C(k_0).$$

Thus,

$$\mathrm{vol}(R) = x_C(m-1) - x_C(k_0).$$

Since the geometric discrepancy $D(S, \mathcal{A}_{\mathcal{F}^*})$ is bounded by $\delta(n)$,

$$N(S, R) \quad \geq \quad \mathrm{vol}(R) - \delta(n) = x_C(m-1) - x_C(k_0) - \delta(n).$$

Thus, combined with (3),

$$x_\Gamma(m-1) - x_\Gamma(k_0) \geq N(S, R) \geq x_C(m-1) - x_C(k_0) - \delta(n),$$

and hence

$$x_\Gamma(m-1) - x_C(m-1) + \delta(n) \geq x_\Gamma(k_0) - x_C(k_0). \tag{4}$$

From (1) and (4), we have

$$x_\Gamma(m-1) - x_C(m-1) + \delta(n) \geq x_\Gamma(m) - x_C(m) + d.$$

Equivalently,

$$x_C(m) - x_C(m-1) + \delta(n) \geq x_\Gamma(m) - x_\Gamma(m-1) + d.$$

Since the x-value of a ray increases by at most one if the ray proceeds one level, $x_C(m) - x_C(m-1) \leq 1$ and $x_\Gamma(m) - x_\Gamma(m-1) \geq 0$. Hence, we have

$$1 + \delta(n) \geq d.$$

This is what we desire to obtain. ◀

5 Construction of the tailor-made low-discrepancy sequence

We give an upper bound of $D(n, \mathcal{A}_{\mathcal{F}^*})$ using the transference principle that derives an upper bound of the geometric discrepancy from that of the combinatorial discrepancy. Then, we construct θ such that $S(\theta) = \{(i, \theta(i)) \mid i = 1, 2, \dots, n\}$ attains this discrepancy asymptotically.

5.1 Combinatorial property of the range space of gradient curves

▶ **Lemma 9.** *Given a diffused family $\mathcal{F}$, for any point $v = (z_0, w_0)$ in the rectangle $(0, n] \times [0, 1]$, there exists a unique gradient curve φ_C going through v.*

Proof. The range of w on $\ell(z_0)$ is $[0, 1]$ since $\mathcal{F}$ contains x-axis and y-axis. Because of the strong off-diagonal monotonicity and the continuity of w, there exists a point $q \in \ell(z_0)$ such that $w(q) = w_0$. Because of the definition of a ray family, there exists a unique ray $C \in \mathcal{F}$ going through q, and $w_C(q) = w(q)$. Thus, φ_C is the unique gradient curve going through v. ◀

▶ **Corollary 10.** *For a diffused family $\mathcal{F}$, each pair of gradient curves in $\mathcal{F}^*$ do not intersect each other in the domain $0 < z \leq n$.*

▶ **Definition 11** (Pseudo-rectangles). *Given a family $\mathcal{C}$ of x-monotone curves in $(0, n] \times [0, 1]$ such that each pair of curves do not intersect each other, a region bounded by a pair of curves and two vertical lines is called a pseudo-rectangle associated with $\mathcal{C}$. A (possibly infinite) set of such pseudo-rectangles is called a family of pseudo-rectangles associated with $\mathcal{C}$*

The following lemma follows the definition of $\mathcal{A}_{\mathcal{F}^*}$, Definition 11, and Corollary 10. See Figure 5 to get intuition.

▶ **Lemma 12.** *For a diffused ray family $\mathcal{F}$, $\mathcal{A}_{\mathcal{F}^*}$ is a family of pseudo-rectangles associated with $\mathcal{F}^*$.*

5.2 Discrepancies for the pseudo-rectangles

The Hammersley point set using the van der Corput sequence (van der Corput-Hammersley point set) is known to give an $O(\log n)$ bound for the geometric discrepancy for the family of axis-parallel rectangles (see [16]). However, it is known that its discrepancy becomes $\Omega(\sqrt{n})$ if we consider a rotated rectangle (Exercise 3, Section 2.1 of [16]), and hence the $O(\log n)$ bound cannot be applied to pseudo-rectangles. It seems difficult to directly convert the $O(\log n)$ bound of geometric discrepancy for rectangles to the one for pseudo-rectangles.

Fortunately, the combinatorial structure for the hypergraph of the range space of the pseudo-rectangles is the same as that of axis-parallel rectangles.

The problem to investigate the combinatorial discrepancy $\mathrm{disc}(n, \mathcal{R})$ for the family $\mathcal{R}$ of axis-parallel rectangles is called *Tusnády's problem*. An $O(\log^4 n)$ bound [4] was given by Beck, and it was improved by Bohus to $O(\log^3 n)$ as an application of *k-permutation problem* [6]. The current best bound is $O(\log^{1.5} n)$ given by Nikolov [17], although it is not constructive. The construction given by Bansal and Garg [2, 3] has an $O(\log^2 n)$ discrepancy, and their algorithm runs in polynomial time using the semi-definite programming as a subroutine.

Because the combinatorial discrepancy only depends on the combinatorial properties of the range space, all these bounds hold for the combinatorial discrepancy of a range space of pseudo-rectangles. Thus, we obtain the following theorem from Lemma 12.

▶ **Theorem 13.** *$disc(n, \mathcal{A}_{\mathcal{F}^*}) = O(\log^{1.5} n)$, and a set P of n points attaining $disc(\mathcal{A}_{\mathcal{F}^*}|_P) = O(\log^2 n)$ can be computed in polynomial time.*

It is known that an upper bound of the combinatorial discrepancy for range spaces can be converted to that of the geometric discrepancy as shown in the following theorem named *Transference Principle* or *Transference Lemma* (Proposition 1.8 of [16]):

▶ **Theorem 14** (Transference Principle). *Let $\mathcal{A}$ be a range space. If $D(n, \mathcal{A}) = o(n)$ and $\mathrm{disc}(n, \mathcal{A}) = O(f(n))$ for a function satisfying $f(2n) \leq (2 - \delta)f(n)$ for all n and fixed $\delta > 0$, then $D(n, \mathcal{A}) = O(f(n))$.*

The assumptions on $f(n)$ and the condition that $D(n, \mathcal{A}) = o(n)$ hold for the range space of pseudo-rectangles. Therefore, an upper bound of the combinatorial discrepancy is transferred to that of geometric discrepancy for the pseudo-rectangles. (A more general result is given by Aistleitner, Bilyk and Nikolov [1].) The transference is given in a constructive fashion such that a point set P giving the geometric discrepancy bound can be obtained in polynomial time in n if the coloring attaining the combinatorial discrepancy can be done in polynomial time. Thus, we have the following:

▶ **Theorem 15.** *$D(n, \mathcal{A}_{\mathcal{F}^*}) = O(\log^{1.5} n)$, and a set P of n points attaining $D(P, \mathcal{A}_{\mathcal{F}^*}) = O(\log^2 n)$ can be computed in polynomial time.*

Note: After the submission of this paper, Dutta [13] claimed an improved $O(\log^{7/4} n)$ combinatorial discrepancy for the Tusnády's problem with polynomial time construction. Accordingly, the corresponding $O(\log^2 n)$ bounds in Theorem 13, Theorem 15 and Theorem 17 is improved to $O(\log^{7/4} n)$ once the claim is confirmed.

5.3 Arraying a point set to obtain a uniform number sequence

We have shown that there exists a point set in $[0, n] \times [0, 1]$ attaining the $O(\log^{1.5} n)$ geometric discrepancy for the region family $\mathcal{A}_{\mathcal{F}^*}$. However, we need $\theta(i) \in [0, 1]$ such that its Hammersley point set $S(\theta) = \{s_i = (i, \theta(i)) \mid 1 \leq i \leq n\}$ forms a low-discrepancy point set to attain the discrepancy bound. We claim that any low-discrepancy point set for $\mathcal{A}_{\mathcal{F}^*}$ can be arrayed to become a Hammersley point set without losing the low-discrepancy property.

▶ **Lemma 16** (Arraying lemma). *If we have a set P of n points with $\delta(n)$ geometric discrepancy for $\mathcal{A}_{\mathcal{F}^*}$, we can construct a Hammersley point set P' with $O(\delta(n))$ geometric discrepancy.*

Proof. Consider the sorted list $p_1, p_2, \ldots, p_n$ of P in the abscissas in the (z, w) plane. Let C_i be the unique gradient curve in $\mathcal{F}^*$ going through p_i, and p'_i be the point on C_i with the abscissa i. In other words, each point p_i is moved along C_i to the position of the abscissa i. Now, we have the point set P'. Consider a region R bounded by a gradient curve $C \in \mathcal{F}^*$ and two vertical lines. Since each point is moved along a curve and no pair of curves intersect, a point p'_i is below C if and only if p_i is below C.

Consider the numbers $N(P, \ell)$ and $N(P', \ell)$ of points in P and P' to the left of a vertical line $\ell : z = a$, respectively. Since the points of P' are arrayed, $N(P', \ell) = \lfloor a \rfloor$. Since $D(P, \mathcal{A}_{\mathcal{F}^*}) \leq \delta(n)$ and $(0, a] \times [0, 1] \in \mathcal{A}_{\mathcal{F}^*}$ has the area a, $|N(P, \ell) - a| \leq \delta(n)$. Thus, $|N(P, \ell) - N(P', \ell)| \leq \delta(n) + 1$. Therefore, at most $\delta(n) + 1$ points of P move crossing ℓ, since the move of points keeps the sorting order. Thus, at most $2(\delta(n) + 1)$ points move crossing two vertical boundaries of R. Therefore, the discrepancy of P' is at most $3\delta(n) + 2 = O(\delta(n))$. Given P', the sequence θ such that $P' = S(\theta)$ is automatically obtained. ◀

Thus, θ is constructed as desired, and we obtain our main result shown below. Note that the asymptotic distance bounds hold for both of the off-diagonal and Hausdorff distances.

▶ **Theorem 17.** *For a diffused ray family $\mathcal{F}$, there exists a CDR with an $O(\log^{1.5} n)$ distance bound between a partial ray towards a pixel and its digital ray. A CDR with an $O(\log^2 n)$ distance bound can be computed in polynomial time in n.*

Proof. Immediate from Theorem 8, Theorem 15 and Lemma 16. ◀

6 Digital pseudoline arrangement

A family of curves is called a *pseudoline arrangement* if each pair of curves intersect at most once to each other. The consistent digital pseudoline arrangement is defined by Chun et al. [11].

One important class of the consistent digital pseudoline arrangement is given as a union of translated copies of a CDR $\mathcal{T}$. A translated copy $\mathcal{T}(s)$ is obtained by translating $\mathcal{T}$ so that the origin is translated to $(s, -s)$ for an integer s.

The union $\cup_{-k \le s \le k} \mathcal{T}(s)$ represents the set of digital rays emanating from $2k+1$ grid points on the off-diagonal line $x+y=0$. The union is called a family of *shifted digital rays*.

▶ **Example 18.** If we consider shifted digital rays using the CDR of linear rays given in Example 3, we can generate digital line segments for a line segments (with nonnegative slopes) between pixels in G as segments of shifted linear rays. This is a different construction of digital line segments from [10].

▶ **Example 19.** If we consider shifted digital rays using the CDR for parabola rays given in Example 4, we have an approximation of the family of parabolas with the vertical axes and peaks on the off-diagonal line $x+y=0$.

We can immediately apply our construction to improve the distance bound of shifted digital rays for a general diffused ray family to $O(\log^{1.5} n)$.

Another class of consistent digital pseudoline arrangements discussed in [11] is the digitized homogeneous polynomial family approximating the family $\{C_{j,a} \mid y = ax^j$ for $a > 0$ and $j \in \{1, 2, \ldots, k\}\}$ for an integer k. We can apply our formulation to construct a union of CDR for it, but unfortunately, we have technical difficulty to generalize the Arraying Lemma (Lemma 16) to guarantee an improved distance bound.

7 Concluding remarks

The distance bound $O(\log^{1.5} n)$ is near to the known $\Omega(\log n)$ lower bound, but it is curious whether we can improve it to $O(\log n)$. Moreover, if we remove the off-diagonal monotonicity condition on χ, we have a weak CDR. It is known that the distance bound for a weak CDR is reduced to $O(1)$ for the family of linear rays [7]. It is curious to investigate the weak CDR for a general ray family.

Developing a practical algorithm for computing theoretically guaranteed CDR is also an important problem. Although the θ-threshold rounding algorithm is very simple, the sequence θ attaining the $O(\log^{1.5} n)$ distance bound is not constructed explicitly. The one with $O(\log^2 n)$ distance bound has a polynomial time construction. However, we need to deal with hypergraphs on vertex sets with nearly n^2 vertices and polylogarithmic vertex degrees if we apply the transference principle. Moreover, the coloring of the hypergraph to attain the combinatorial discrepancy in [2, 3] uses the semi-definite programming(SDP). Therefore, the algorithm is not much efficient for practical use. It is desired to give an efficient construction of CDR for a given family of curved rays with theoretically near optimal distance bound.

There are $n!$ different CDRs in G corresponding to the ways to locate the branching pixel of each $L(k)$. Thus, it is implied that the infinite set of all diffused families of rays is mapped to $n!$ CDRs, and the inverse image of a CDR $\mathcal{T}$ is a class of families of rays within $O(\log^{1.5} n)$ distance from the set of paths of $\mathcal{T}$. It is curious to extend this observation to more general geometric objects in the plane.

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A Appendix

A.1 Geometric Primitives

Although the existence of the CDR is given mathematically by using abstract properties of the ray family, the θ-threshold rounding algorithm needs computation of the weight w and the sequence θ. Therefore, necessary primitive geometric operations (which is called *geometric primitives*) must be executed using information of the ray family.

Given $s = (z_s, w_s)$ and $t = (z_t, w_t)$ in the (z, w)-space, we say s is *higher* than t with respect to $\mathcal{F}^*$ if there exists a gradient curve (called *separating curve*) $\varphi_C : \{(z, w(q_C(z))) \mid 0 < z \leq n\}$ such that $w(q_C(z_s)) < w_s$ and $w(q_C(z_t)) \geq w_t$. If s and t are on the same gradient curve, we say they have the same height.

The following two geometric primitives are necessary for the algorithm.

1. Given $p \in \Delta$, compute the weight $w(p)$ with a sufficient precision so that necessary comparisons in the algorithm can be done properly.
2. Given s and t in the (z, w)-plane, decide which is higher (or they have the same height) with respect to $\mathcal{F}^*$.

A given set of points in the (z, w)-plane can be sorted with respect to the height by using the second primitive. This enables to identify a range space of pseudo-rectangles to that of axis-parallel rectangles combinatorially.

We assume that each geometric primitive can be done in polynomial time in n in order to guarantee the polynomial time complexity for computing a CDR.

The computation of the weight $w(p)$ needs locally differentiable representations of rays, and the computation of $q_C(z)$ needs solution of equations as shown in the examples given below.

A.2 Examples

In the following examples, the geometric primitives need numerical computation such as solution of non-algebraic equations.

▶ **Example 20.** Consider an increasing differentiable function $f(x)$ such that $f(0) = 0$. Then, the family $\mathcal{F} = \{C_a : y = af(x) \mid a \in [0, \infty]\}$ is a smooth ray family, if we consider C_∞ as the vertical line $x = 0$. Given any $p = (x_0, y_0) \in \Delta$ for $x_0 > 0$, C_{a_0} for $a_0 = \frac{y_0}{f(x_0)}$ is the unique ray going through p, and $w(p) = \frac{1}{1+a_0 f'(x_0)} = \frac{f(x_0)}{f(x_0)+y_0 f'(x_0)}$. The family $\mathcal{F}$ is diffused if $f(x)$ is a concave function.

For example, the sigmoid function $\sigma(x) = \frac{1-e^{-x}}{1+e^{-x}}$ is a concave function for $x \geq 0$. Thus, the family $\mathcal{F} = \{C_a^{\text{sig}} : y = a\sigma(x) \mid a \in [0, \infty]\}$ is a diffused family. The derivative at $p = (x, y)$ is $y' = 2a\frac{e^{-x}}{(1+e^{-x})^2}$, which equals $\frac{2ye^{-x}}{1-e^{-2x}}$ and hence $w(p) = \frac{1-e^{-2x}}{1-e^{-2x}+2ye^{-x}}$.

The x-coordinate value of the point $q_C(z)$ for $C = C_a^{\text{sig}}$ is the root of the equation $x + a\frac{1-e^{-x}}{1+e^{-x}} = z$, and it does not have an explicit analytic expression. Thus, the geometric primitive concerning $\varphi_C(z)$ requires substantial numerical computation.

▶ **Example 21.** For $0 < a \leq 1$, define the function $F_a(x) = (1 - a)n \ \sin \frac{\pi x}{2na}$ for $0 \leq x \leq na$. We define $C_a^{\sin} : y = F_a(x)$ for $a > 0$, and $C_0^{\sin}$ is defined to be the y-axis. Then, $\mathcal{F} = \{C_a^{\sin} \mid a \in [0, 1]\}$ is a family of (increasing segments of) sine curves in Δ. It is a diffused ray family, and we can apply our algorithm. The weight $w(p)$ for $p = (x, y)$ is not explicitly expressed by using elementary functions of x and y, and should be computed numerically.

A.3 Preliminary implementation and experiment

We give a preliminary experimental report of an implementation of the proposed θ-threshold rounding algorithm. The homogeneous polynomial ray families $\mathcal{F}_j = \{y = ax^j \mid a \in [0, \infty]\}$ for $j = 2, \ldots, 6$ are considered as the ray families in the experiment. We varied the grid size n in the range $n = 2^k$ $(1 \leq k \leq 14)$ to see the dependency of the maximum distance error on n.

As stated in the concluding remarks, it seems to be difficult to implement the SDP method with the theoretical $O(\log^2 n)$ combinatorial discrepancy bound so that it gives a practically good solution. Moreover, $n^{1/4} < \log^2 n$ if $n < 2^{44}$ (the base of logarithm is 2), and the SDP method needs $O(n^3)$ time, which is not feasible for a large n.

Hence, for the preliminary implementation, we have given a more casual method to attain an $O(n^{1/4})$ discrepancy. The hypergraph coloring is done by using the low-stabbing matching based on the k-d tree data structure on the point set described in [16]. This gives a randomized algorithm to attain an $O(n^{1/4})$ expected bound for the combinatorial discrepancy. To transfer this discrepancy bound, we apply the transference principle procedure given in [16] starting with $n^{1.5} = 2^{3k/2}$ grid points.

We measured the maximum Hausdorff distance between rays and digital rays as shown in Figure 6. The chart shows the tendency of increase of the error is about $2n^{1/4}$ to support the theory. From the chart, we can observe that the distance error is almost independent of the choice of the ray family. If $n = 14$, the grid size (width) n is 16384, and the maximum distance error is about $25 < \frac{n}{640}$ pixels, which is 0.05 inch in the 32 inch display.

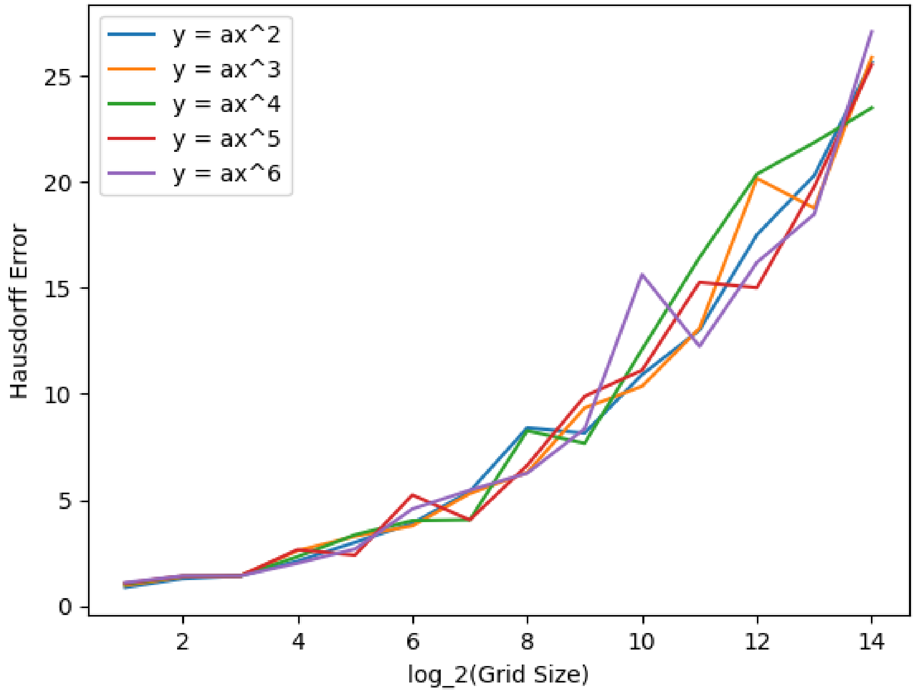

Figure 6 Distance error of the proposed algorithm.

The most expensive routine in the experiment is the measurement of the Hausdorff distance, which needs $O(n^2)$ time (in proportional to the number of pixels of the grid) if we want to measure exactly.

We compared our method with the θ-threshold rounding using the van der Corput sequence as θ, which is equivalent (although the description is different) to the previous method of Chun et al. [11]. Figure 7 shows that the distance error is about half in the van der Corput method compared with ours.

The chart shows that van der Corput method is experimentally better by a factor of approximately 2 than the low-stabbing matching method for the families considered in our experiment in the range $n \leq 2^{14}$.

This implies that although the van der Corput-Hammersley point set P gives only $O(\sqrt{n \log n})$ theoretical discrepancy bound of $D(P, \mathcal{A}_{\mathcal{F}^*})$ in the worst case, the discrepancy is practically better for most of curve families.

Therefore, although the results using the transference from the combinatorial discrepancy is theoretically better, it might be advantageous to use the van der Corput sequence (or its variants) to construct CDRs in practice.

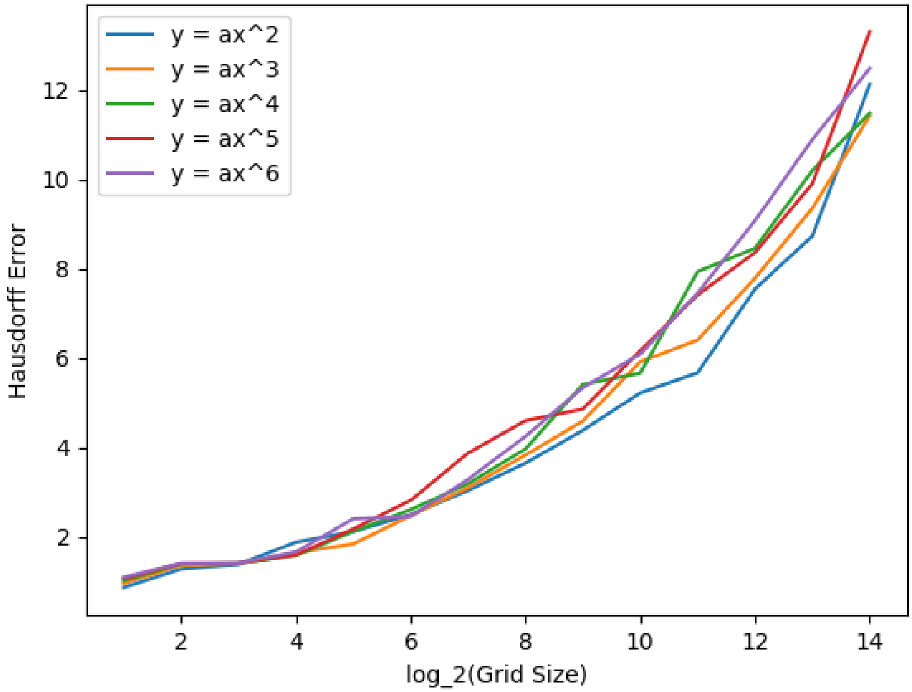

Figure 7 Distance error of the CDR using the van der Corput sequence.