Maximally Satisfying Lower Quotas in the Hospitals/Residents Problem with Ties

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Abstract

Motivated by the serious problem that hospitals in rural areas suffer from a shortage of residents, we study the Hospitals/Residents model in which hospitals are associated with lower quotas and the objective is to satisfy them as much as possible. When preference lists are strict, the number of residents assigned to each hospital is the same in any stable matching because of the well-known rural hospitals theorem; thus there is no room for algorithmic interventions. However, when ties are introduced to preference lists, this will no longer apply because the number of residents may vary over stable matchings.

In this paper, we formulate an optimization problem to find a stable matching with the maximum total satisfaction ratio for lower quotas. We first investigate how the total satisfaction ratio varies over choices of stable matchings in four natural scenarios and provide the exact values of these maximum gaps. Subsequently, we propose a strategy-proof approximation algorithm for our problem; in one scenario it solves the problem optimally, and in the other three scenarios, which are NP-hard, it yields a better approximation factor than that of a naive tie-breaking method. Finally, we show inapproximability results for the above-mentioned three NP-hard scenarios.

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1 Introduction

The Hospitals/Residents model (HR), a many-to-one matching model, has been extensively studied since the seminal work of Gale and Shapley [11]. Its input consists of a set of residents and a set of hospitals. Each resident has a preference over hospitals; similarly, each hospital has a preference over residents. In addition, each hospital is associated with a positive integer called the upper quota, which specifies the maximum number of residents it can accept. In this model, stability is the central solution concept, which requires the nonexistence of a blocking pair, i.e., a resident–hospital pair that has an incentive to deviate jointly from the current matching. In the basic model, each agent (resident or hospital) is assumed to have a strict preference for possible partners. For this model, the resident-oriented Gale–Shapley algorithm (also known as the deferred acceptance mechanism) is known to find a stable matching. This algorithm has advantages from both computational and strategic viewpoints: it runs in linear time and is strategy-proof for residents.

In reality, people typically have indifference among possible partners. Accordingly, a stable matching model that allows *ties* in preference lists, denoted by *HRT* in the context of HR, was introduced [20]. For such a model, several definitions of stability are possible. Among them, *weak stability* provides a natural concept, in which agents have no incentive to move within the ties. It is known that if we break the ties of an instance I arbitrarily, any stable matching of the resultant instance is a weakly stable matching of I. Hence, the Gale–Shapley algorithm can still be used to obtain a weakly stable matching. In applications, typically, ties are broken randomly, or participants are forced to report strict preferences even if their true preferences have ties. Hereafter, "stability" in the presence of ties refers to "weak stability," unless stated otherwise.

It is commonly known that HR plays an important role not only in theory but also in practice; for example, in assigning students to high schools [1,2] and residents to hospitals [30]. In such applications, "imbalance" is one of the major problems. For example, hospitals in urban areas are generally more popular than those in rural areas; hence it is likely that the former are well-staffed whereas the latter suffer from a shortage of doctors. One possible solution to this problem is to introduce a *lower quota* of each hospital, which specifies the minimum number of residents required by a hospital, and obtain a stable matching that satisfies both the upper and lower quotas. However, such a matching may not exist in general [16, 28], and determining if such a stable matching exists in HRT is known to be NP-complete (which is an immediate consequence from page 276 of [29]).

In general, it is too pessimistic to assume that a shortage of residents would force hospitals to go out of operation. In some cases, the hospital simply has to reduce its service level according to how much its lower quota is satisfied. In this scenario, a hospital will wish to satisfy the lower quota as much as possible, if not completely. To formulate this situation, we introduce the following optimization problem, which we call *HRT to Maximally Satisfy Lower Quotas* (*HRT-MSLQ*). Specifically, let R and H be the sets of residents and hospitals, respectively. All members in R and H have complete preference lists that may contain ties. Each hospital h has an upper quota $u(h)$, the maximum number of residents it can accept. The stability of a matching is defined with respect to these preference lists and upper quotas, as in conventional HRT. In addition, each hospital h is associated with a lower quota $\ell(h)$, which specifies the minimum number of residents required to keep its service level. We assume that $\ell(h) \leq u(h) \leq |R|$ for each $h \in H$. For a stable matching M, let $M(h)$ be the set of residents assigned to h. The *satisfaction ratio*, $s_M(h)$, of hospital $h \in H$ (with respect to $\ell(h)$) is defined as $s_M(h) = \min\left\{1, \frac{|M(h)|}{\ell(h)}\right\}$. Here, we let $s_M(h) = 1$ if $\ell(h) = 0$, because the lower quota is automatically satisfied in this case. The satisfaction ratio reflects a situation in which hospital h's service level increases linearly with respect to the number of residents up to $\ell(h)$ but does not increase after that, even though h is still willing to accept $u(h) - \ell(h)$ more residents. These $u(h) - \ell(h)$ positions may be considered as "marginal seats," which do not affect the service level but provide hospitals with advantages, such as generous work shifts. Our HRT-MSLQ problem asks us to maximize the total satisfaction ratio over the family $\mathcal{M}$ of all stable matchings in the problem instance, i.e.,

$$\max_{M \in \mathcal{M}} \sum_{h \in H} s_M(h).$$

The following are some remarks on our problem: (1) To our best knowledge, almost all previous works on lower quotas have investigated cases with no ties and have assumed lower quotas to be hard constraints. Refer to the discussion at the end of this section. (2) Our assumption that all preference lists are complete is theoretically a fundamental scenario used to study the satisfaction ratio for lower quotas. Moreover, there exist several cases in which this assumption is valid [4, 14]. For example, according to Goto et al. [14], a complete list assumption is common in student–laboratory assignment in engineering departments of Japanese universities because it is mandatory that every student be assigned. (3) If preference lists contain no ties, the satisfaction ratio $s_M(h)$ is identical for any stable matching M because of the *rural hospitals theorem* [12, 30, 31]. Hence, there is no chance for algorithms to come into play if the stability is not relaxed. In our setting (i.e., with ties), the rural hospitals theorem implies that our task is essentially to find an optimal tie-breaking. However, it is unclear how to find such a tie-breaking.

Our Contributions. First, we study the goodness of any stable matching in terms of the total satisfaction ratios. For a problem instance I, let $\mathrm{OPT}(I)$ and $\mathrm{WST}(I)$, respectively, denote the maximum and minimum total satisfaction ratios of the stable matchings of I. For a family of problem instances $\mathcal{I}$, let $\Lambda(\mathcal{I}) = \max_{I \in \mathcal{I}} \frac{\mathrm{OPT}(I)}{\mathrm{WST}(I)}$ denote the maximum gap of the total satisfaction ratios. In this paper, we consider the following four fundamental scenarios of $\mathcal{I}$: (i) *general model*, which consists of all problem instances, (ii) *uniform model*, in which all hospitals have the same upper and lower quotas, (iii) *marriage model*, in which each hospital has an upper quota of 1 and a lower quota of either 0 or 1, and (iv) *R-side ML model*, in which all residents have identical preference lists. The exact values of $\Lambda(\mathcal{I})$ for all such fundamental scenarios are listed in the first row of Table 1, where $n = |R|$. In the uniform model, we write $\theta = \frac{u(h)}{\ell(h)}$ for the ratio of the upper and lower quotas, which is common to all hospitals. Further detailed analyses can be found in the full version [13].

Subsequently, we consider our problem algorithmically. Note that the aforementioned maximum gap corresponds to the worst-case approximation factor of the *arbitrarily tie-breaking Gale–Shapley algorithm*, which is frequently used in practice; this algorithm first breaks ties in the preference lists of agents arbitrarily and then applies the Gale–Shapley algorithm on the resulting preference lists. This correspondence easily follows from the rural hospitals theorem (see the full version [13] for the details).

In this paper, we show that there are two types of difficulties inherent in our problem HRT-MSLQ for all scenarios except (iv). Even for scenarios (i)–(iii), we show that (1) the problem is NP-hard and that (2) there is no algorithm that is strategy-proof for residents and always returns an optimal solution; see Section 6 and Appendix A.1.

We then consider strategy-proof approximation algorithms. We propose a strategy-proof algorithm Double Proposal, which is applicable in all above possible scenarios, whose approximation factor is substantially better than that of the arbitrary tie-breaking method. The approximation factors are listed in the second row of Table 1, where ϕ is a function defined by $\phi(1) = 1$, $\phi(2) = \frac{3}{2}$, and $\phi(n) = n(1 + \lfloor \frac{n}{2} \rfloor)/(n + \lfloor \frac{n}{2} \rfloor)$ for any $n \geq 3$. Note that $\frac{\theta^2+\theta-1}{2\theta-1} < \theta$ holds whenever $\theta > 1$. We also provide inapproximability results in the last row, where ϵ denotes an arbitrarily small positive constant.

Table 1 Maximum gap $\Lambda(\mathcal{I})$, approximation factor of Double Proposal, and inapproximability of HRT-MSLQ for four fundamental scenarios $\mathcal{I}$.

	General	Uniform	Marriage	R-side ML
Maximum gap $\Lambda(\mathcal{I})$ (i.e., Approx. factor of arbitrary tie-breaking GS)	$n+1$	θ	2	$n+1$
Approx. factor of Double Proposal	$\phi(n)$ ($\sim \frac{n+2}{3}$)	$\frac{\theta^2+\theta-1}{2\theta-1}$	1.5	1
Inapproximability	$n^{\frac{1}{4}-\epsilon}$	$\frac{3\theta+4}{2\theta+4} - \epsilon$	$\frac{9}{8} - \epsilon$	—

∗) Under P $\neq$ NP
†) Under the Unique Games Conjecture

Techniques. Our algorithm Double Proposal is based on the resident-oriented Gale–Shapley algorithm and is inspired by previous research on approximation algorithms [17, 25] for another NP-hard problem called MAX-SMTI. Unlike in the conventional Gale–Shapley algorithm, our algorithm allows each resident r to make proposals twice to each hospital. Among the hospitals in the top tie of the current preference list, r prefers hospitals to which r has not yet proposed to those which r has already proposed to once. When a hospital h receives a new proposal from r, hospital h may accept or reject it, and in the former case, h may reject a currently assigned resident to accommodate r. In contrast to the conventional Gale–Shapley algorithm, a rejection may occur even if h is not full. If at least $\ell(h)$ residents are currently assigned to h and at least one of them has not been rejected by h so far, then h rejects such a resident, regardless of its preference. This process can be considered as the algorithm dynamically finding a tie-breaking in r's preference list.

The main difficulty in our problem originates from the complicated form of our objective function $s(M) = \sum_{h \in H} \min\{1, \frac{|M(h)|}{\ell(h)}\}$. In particular, non-linearity of $s(M)$ makes the analysis of the approximation factor of Double Proposal considerably hard. We therefore introduce some new ideas and techniques to analyze the maximum gap Λ and approximation factor of our algorithm, which is one of the main novelties of this paper.

To estimate the approximation factor of the algorithm, we need to compare objective values of a stable matching M output by the algorithm and an (unknown) optimal stable matching N. A typical technique used to compare two matchings is to consider a graph of their union. In the marriage model, the connected components of the union are paths and cycles, both of which are easy to analyze; however, this is not the case in a general many-to-one matching model. For some problems, this approach still works via "cloning," which transforms an instance of HR into that of the marriage model by replacing each hospital h with an upper quota of $u(h)$ by $u(h)$ hospitals with an upper quota of 1. Unfortunately, however, in HRT-MSLQ there seems to be no simple way to transform the general model into the marriage model because of the non-linearity of the objective function.

In our analysis of the uniform model, the union graph of M and N may have a complex structure. We categorize hospitals using a procedure like breadth-first search starting from the set of hospitals h with the satisfaction ratio $s_N(h)$ larger than $s_M(h)$, which allows us to provide a tight bound on the approximation factor. For the general model, instead of using the union graph, we define two vectors that distribute the values $s(M)$ and $s(N)$ to the residents. By making use of the local optimality of M proven in Section 3, we compare such two vectors and give a tight bound on the approximation factor.

We finally remark that the improvement of Double Proposal over the maximum gap shows that our problem exhibits a different phenomenon from that of MAX-SMTI because the approximation factor of MAX-SMTI cannot be improved from a naive tie-breaking method if strategy-proofness is imposed [17].

Related Work. Recently, the Hospitals/Residents problems with lower quotas are quite popular in the literature; however, most of these studies are on settings without ties. The problems related to HRT-MSLQ can be classified into three models. The model by Hamada et al. [16], denoted by HR-LQ-2 in [28], is the closest to ours. The input of this model is the same as ours, but the hard and soft constraints are different from ours; their solution must satisfy both upper and lower quotas, the objective being to maximize the stability (e.g., to minimize the number of blocking pairs). Another model, introduced by Biró et al. [5] and denoted by HR-LQ-1 in [28], allows some hospitals to be closed; a closed hospital is not assigned any resident. They showed that it is NP-complete to determine the existence of a stable matching. This model was further studied by Boehmer and Heeger [6] from a parameterized complexity perspective. Huang [19] introduced the *classified stable matching* model, in which each hospital defines a family of subsets R of residents and each subset of R has an upper and lower quota. This model was extended by Fleiner and Kamiyama [9] to a many-to-many matching model where both sides have upper and lower quotas. Apart from these, several matching problems with lower quotas have been studied in the literature, whose solution concepts are different from stability [3, 10, 26, 27, 33].

Paper Organization. The rest of the paper is organized as follows. Section 2 formulates our problem HRT-MSLQ, and Section 3 describes our algorithm Double Proposal for HRT-MSLQ. Section 4 shows the strategy-proofness of Double Proposal. Section 5 is devoted to proving the maximum gaps Λ and approximation factors of algorithm Double Proposal for the several scenarios mentioned above. Finally, Section 6 provides hardness results such as NP-hardness and inapproximability for several scenarios. Because of space constraints, some proofs are omitted and included in the full version [13].

2 Problem Definition

Let $R = \{r_1, r_2, \ldots, r_n\}$ be a set of residents and $H = \{h_1, h_2, \ldots, h_m\}$ be a set of hospitals. Each hospital h has a lower quota $\ell(h)$ and an upper quota $u(h)$ such that $\ell(h) \leq u(h) \leq n$. We sometimes denote a hospital h's quota pair as $[\ell(h), u(h)]$ for simplicity. Each resident has a preference list over hospitals, which is complete and may contain ties. If a resident r prefers a hospital h_i to h_j, we write $h_i \succ_r h_j$. If r is indifferent between h_i and h_j (including the case that $h_i = h_j$), we write $h_i =_r h_j$. We use the notation $h_i \succeq_r h_j$ to signify that $h_i \succ_r h_j$ or $h_i =_r h_j$ holds. Similarly, each hospital has a preference list over residents and the same notations as above are used. In this paper, a preference list is denoted by one row, from left to right according to the preference order. When two or more agents are of equal preference, they are enclosed in parentheses. For example, "r_1: h_3 (h_2 h_4) h_1" is a preference list of resident r_1 such that h_3 is the top choice, h_2 and h_4 are the second choice with equal preference, and h_1 is the last choice.

An *assignment* is a subset of $R \times H$. For an assignment M and a resident r, let $M(r)$ be the set of hospitals h such that $(r, h) \in M$. Similarly, for a hospital h, let $M(h)$ be the set of residents r such that $(r, h) \in M$. An assignment M is called a *matching* if $|M(r)| \leq 1$ for each resident r and $|M(h)| \leq u(h)$ for each hospital h. For a matching M, a resident r

is called *matched* if $|M(r)| = 1$ and *unmatched* otherwise. If $(r, h) \in M$, we say that r is *assigned to* h and h is *assigned* r. We sometimes abuse notation $M(r)$ to denote the unique hospital where r is assigned. A hospital h is called *deficient* or *sufficient* if $|M(h)| < \ell(h)$ or $\ell(h) \leq |M(h)| \leq u(h)$, respectively. Additionally, a hospital h is called *full* if $|M(h)| = u(h)$ and *undersubscribed* otherwise.

A resident–hospital pair (r, h) is called a *blocking pair* for a matching M (or we say that (r, h) *blocks* M) if (i) r is either unmatched in M or prefers h to $M(r)$ and (ii) h is either undersubscribed in M or prefers r to at least one resident in $M(h)$. A matching is called *stable* if it admits no blocking pair. Recall that the satisfaction ratio of a hospital h (which is also called *the score* of h) in a matching M is defined by $s_M(h) = \min\{1, \frac{|M(h)|}{\ell(h)}\}$, where we define $s_M(h) = 1$ if $\ell(h) = 0$. The *total satisfaction ratio* (also called *the score*) of a matching M, is the sum of the scores of all hospitals, that is, $s(M) = \sum_{h \in H} s_M(h)$. The Hospitals/Residents problem with Ties to Maximally Satisfy Lower Quotas, denoted by *HRT-MSLQ*, is to find a stable matching M that maximizes the score $s(M)$. The optimal score of an instance I is denoted by $\mathrm{OPT}(I)$.

Note that if $|R| \geq \sum_{h \in H} u(h)$, then all hospitals are full in any stable matching (recall that preference lists are complete). Hence, all stable matchings have the same score $|H|$, and the problem is trivial. Therefore, throughout this paper, we assume $|R| < \sum_{h \in H} u(h)$. In this setting, all residents are matched in any stable matching as an unmatched resident forms a blocking pair with an undersubscribed hospital.

3 Algorithm

In this section, we present our algorithm Double Proposal for HRT-MSLQ along with a few of its basic properties. Its strategy-proofness and approximation factors for several models are presented in the following sections.

Our proposed algorithm Double Proposal is based on the resident-oriented Gale–Shapley algorithm but allows each resident r to make proposals twice to each hospital. Here, we explain the ideas underlying this modification.

Let us apply the ordinary resident-oriented Gale–Shapley algorithm to HRT-MSLQ, which starts with an empty matching $M := \emptyset$ and repeatedly updates M by a proposal-acceptance/rejection process. In each iteration, the algorithm takes a currently unassigned resident r and lets her propose to the hospital at the top of her current list. If the preference list of resident r contains ties, the proposal order of r depends on how to break the ties in her list. Hence, we need to define a priority rule for hospitals that are in a tie. Recall that our objective function is given by $s(M) = \sum_{h \in H} \min\{1, \frac{|M(h)|}{\ell(h)}\}$. This value immediately increases by $\frac{1}{\ell(h)}$ if r proposes to a deficient hospital h, whereas it does not increase if r proposes to a sufficient hospital h', although the latter may cause a rejection of some resident if h' is full. Therefore, a naive greedy approach is to let r first prioritize deficient hospitals over sufficient hospitals and then prioritize those with small lower quotas among deficient hospitals. This approach is useful for attaining a larger objective value for some instances; however, it is not enough to improve the approximation factor in the sense of worst case analysis, as a deficient hospital h in some iteration might become sufficient later and it might be better if r had made a proposal to a hospital other than h in the tie. Furthermore, this naive approach sacrifices strategy-proofness as demonstrated in Appendix A.2. This failure of strategy-proofness follows from the adaptivity of this tie-breaking rule, in the sense that the proposal order of each resident is affected by the other residents' behaviors.

In our algorithm Double Proposal, each resident can propose twice to each hospital. If the head of r's preference list is a tie when r makes a proposal, then the hospitals to which r has not yet proposed are prioritized. This idea was inspired by an algorithm of [17]. Recall that each hospital h has an upper quota $u(h)$ and a lower quota $\ell(h)$. In our algorithm, we use $\ell(h)$ as a dummy upper quota. Whenever $|M(h)| < \ell(h)$, a hospital h accepts any proposal. If h receives a new proposal from r when $|M(h)| \geq \ell(h)$, then h checks whether there is a resident in $M(h) \cup \{r\}$ who has not been rejected by h so far. If such a resident exists, h rejects that resident regardless of the preference of h. Otherwise, we apply the usual acceptance/rejection operation, i.e., h accepts r if $|M(h)| < u(h)$ and otherwise replaces r with the worst resident r' in $M(h)$. Roughly speaking, the first proposals are used to implement priority on deficient hospitals, and the second proposals are used to guarantee stability.

Formally, our algorithm Double Proposal is described in Algorithm 1. For convenience, in the preference list, a hospital h that is not included in any tie is regarded as a tie consisting of h only. We say that a resident is *rejected* by a hospital h if she is chosen as r' in Lines 12 or 17. To argue strategy-proofness, we need to make the algorithm deterministic. To this end, we remove arbitrariness using indices of agents as follows. If there are multiple hospitals (resp., residents) satisfying the condition to be chosen at Lines 5 or 7 (resp., at Lines 12 or 17), take the one with the smallest index (resp., with the largest index). Furthermore, when there are multiple unmatched residents at Line 3, take the one with the smallest index. In this paper, Double Proposal always refers to this deterministic version.

Algorithm 1 Double Proposal.

Input: An instance I where each $h \in H$ has quotas $[\ell(h), u(h)]$.
Output: A stable matching M.

1: $M := \emptyset$
2: **while** there is an unmatched resident **do**
3:   Let r be any unmatched resident and T be the top tie of r's list.
4:   **if** T contains a hospital to which r has not proposed yet **then**
5:     Let h be such a hospital with minimum $\ell(h)$.
6:   **else**
7:     Let h be a hospital with minimum $\ell(h)$ in T.
8:   **end if**
9:   **if** $|M(h)| < \ell(h)$ **then**
10:     Let $M := M \cup \{(r, h)\}$.
11:   **else if** there is a resident in $M(h) \cup \{r\}$ who has not been rejected by h **then**
12:     Let r' be such a resident (possibly $r' = r$).
13:     Let $M := (M \cup \{(r, h)\}) \setminus \{(r', h)\}$.
14:   **else if** $|M(h)| < u(h)$ **then**
15:     $M := M \cup \{(r, h)\}$.
16:   **else** {i.e., when $|M(h)| = u(h)$ and all residents in $M(h) \cup \{r\}$ have been rejected by h once}
17:     Let r' be any resident that is worst in $M(h) \cup \{r\}$ for h (possibly $r' = r$).
18:     Let $M := (M \cup \{(r, h)\}) \setminus \{(r', h)\}$.
19:     Delete h from r''s list.
20:   **end if**
21: **end while**
22: Output M and halt.

▶ **Lemma 1.** *Algorithm* Double Proposal *runs in linear time and outputs a stable matching.*

Proof. Clearly, the size of the input is $O(|R||H|)$. As each resident proposes to each hospital at most twice, the while loop is iterated at most $2|R||H|$ times. At Lines 5 and 7, a resident prefers hospitals with smaller $\ell(h)$, and hence we need to sort hospitals in each tie in an increasing order of the values of ℓ. Since $0 \leq \ell(h) \leq n$ for each $h \in H$, ℓ has only $|R|+1$ possible values. Therefore, the required sorting can be done in $O(|R||H|)$ time as a preprocessing step using a method like bucket sort. Thus, our algorithm runs in linear time.

Observe that a hospital h is deleted from r's list only if h is full. Additionally, once h becomes full, it remains so afterward. Since each resident has a complete preference list and $|R| < \sum_{h \in H} u(h)$, the preference list of each resident never becomes empty. Therefore, all residents are matched in the output M.

Suppose, to the contrary, that M is not stable, i.e., there is a pair (r, h) such that (i) r prefers h to $M(r)$ and (ii) h is either undersubscribed or prefers r to at least one resident in $M(h)$. By the algorithm, (i) implies that r is rejected by h twice. Just after the second rejection, h is full, and all residents in $M(h)$ have once been rejected by h and are no worse than r for h. Since $M(h)$ is monotonically improving for h, at the end of the algorithm h is still full and no resident in $M(h)$ is worse than r, which contradicts (ii). ◀

In addition to stability, the output of Double Proposal satisfies the following property, which plays a key role in the analysis of the approximation factors in Section 5.

▶ **Lemma 2.** *Let M be the output of* Double Proposal*, r be a resident, and h and h' be hospitals such that $h =_r h'$ and $M(r) = h$. Then, we have the following conditions:*

(i) *If $\ell(h) > \ell(h')$, then $|M(h')| \geq \ell(h')$.*

(ii) *If $|M(h)| > \ell(h)$, then $|M(h')| \geq \ell(h')$.*

Proof. (i) Since $h =_r h'$, $\ell(h) > \ell(h')$, and r is assigned to h in M, the definition of the algorithm (Lines 4, 5, and 7) implies that r proposed to h' and was rejected by h' before she proposes to h. Just after this rejection occurred, $|M(h')| \geq \ell(h')$ holds. Since $|M(h')|$ is monotonically increasing, we also have $|M(h')| \geq \ell(h')$ at the end.

(ii) Since $|M(h)| > \ell(h)$, the value of $|M(h)|$ changes from $\ell(h)$ to $\ell(h)+1$ at some moment of the algorithm. By Line 11 of the algorithm, at any point after this, $M(h)$ consists only of residents who have once been rejected by h. Since $M(r) = h$ for the output M, at some moment r must have made the second proposal to h. By Line 4 of the algorithm, $h =_r h'$ implies that r has been rejected by h' at least once, which implies that $|M(h')| \geq \ell(h')$ at this moment and also at the end. ◀

Lemma 2 states some local optimality of Double Proposal. Suppose that we reassign r from h to h'. Then, h may lose and h' may gain score, but Lemma 2 says that the objective value does not increase. To see this, note that if the objective value were to increase, h' must gain score and h would either not lose score or lose less score than h' would gain. The former and the latter are the "if" parts of (ii) and (i), respectively, and in either case the conclusion $|M(h')| \geq \ell(h')$ implies that h' cannot gain score by accepting one more resident.

4 Strategy-proofness

An algorithm is called *strategy-proof* for residents if it gives residents no incentive to misrepresent their preferences. The precise definition follows. An algorithm that always outputs a matching deterministically can be regarded as a mapping from instances of HRT-MSLQ into matchings. Let A be an algorithm. We denote by $A(I)$ the matching returned by A for an instance I. For any instance I, let $r \in R$ be any resident, who has a preference $\succeq_r$. Additionally, let I' be an instance of HRT-MSLQ which is obtained from I by replacing $\succeq_r$ with some other $\succeq'_r$. Furthermore, let $M \coloneqq A(I)$ and $M' \coloneqq A(I')$. Then, A is strategy-proof if $M(r) \succeq_r M'(r)$ holds regardless of the choices of I, r, and $\succeq'_r$.

In the setting without ties, it is known that the resident-oriented Gale–Shapley algorithm is strategy-proof for residents (even if preference lists are incomplete) [8,15,32]. Furthermore, it has been proved that no algorithm can be strategy-proof for both residents and hospitals [32]. As in many existing papers on two-sided matching, we use the term "strategy-proofness" to refer to strategy-proofness for residents.

Before proving the strategy-proofness of Double Proposal, we remark that the exact optimization and strategy-proofness are incompatible even if a computational issue is set aside. The following fact is demonstrated in Appendix A.1.

▶ **Proposition 3.** *There is no algorithm that is strategy-proof for residents and returns an optimal solution for any instance of HRT-MSLQ. The statement holds even for the uniform and marriage models.*

This proposition implies that, if we require strategy-proofness for an algorithm, then we should consider approximation even in the absence of computational constraints. Now, we show the strategy-proofness of our approximation algorithm.

▶ **Theorem 4.** *Algorithm* Double Proposal *is strategy-proof for residents.*

Proof. To establish the strategy-proofness, we show that an execution of Double Proposal for an instance I can be described as an application of the resident-oriented Gale–Shapley algorithm to an auxiliary instance I^*. The construction of I^* is based on the proof of Lemma 8 in [17]; however, we need nontrivial extensions.

Let R and H be the sets of residents and hospitals in I, respectively. An auxiliary instance I^* is an instance of the Hospitals/Residents problem that has neither lower quotas nor ties and allows incomplete lists. The set of residents in I^* is $R' \cup D$, where $R' = \{r'_1, r'_2, \dots, r'_n\}$ is a copy of R and $D = \{\, d_{j,p} \mid j = 1, 2, \dots, m,\ p = 1, 2, \dots, u(h_j) \,\}$ is a set of $\sum_{j=1}^{m} u(h_j)$ dummy residents. The set of hospitals in I^* is $H^\circ \cup H^\bullet$, where each of $H^\circ = \{h^\circ_1, h^\circ_2, \dots, h^\circ_m\}$ and $H^\bullet = \{h^\bullet_1, h^\bullet_2, \dots, h^\bullet_m\}$ is a copy of H. Each hospital $h^\circ_j \in H^\circ$ has an upper quota $u(h_j)$ while each $h^\bullet_j \in H^\bullet$ has an upper quota $\ell(h_j)$.

For each resident $r'_i \in R'$, her preference list is defined as follows. Consider any tie $(h_{j_1} h_{j_2} \cdots h_{j_k})$ in r_i's preference list. Let $j'_1\, j'_2 \cdots j'_k$ be a permutation of $j_1\, j_2 \cdots j_k$ such that $\ell(h_{j'_1}) \le \ell(h_{j'_2}) \le \cdots \le \ell(h_{j'_k})$, and for each j'_p, j'_q with $\ell(h_{j'_p}) = \ell(h_{j'_q})$, $p < q$ implies $j'_p < j'_q$. We replace the tie $(h_{j_1} h_{j_2} \cdots h_{j_k})$ with a strict order of $2k$ hospitals $h^\bullet_{j'_1} h^\bullet_{j'_2} \cdots h^\bullet_{j'_k} h^\circ_{j'_1} h^\circ_{j'_2} \cdots h^\circ_{j'_k}$. The preference list of r'_i is obtained by applying this operation to all ties in r_i's list, where a hospital not included in any tie is regarded as a tie of length one. The following is an example of the correspondence between the preference lists of r_i and r'_i:

$$r_i: \quad (\ h_2\ h_4\ h_5\ )\ h_3\ (\ h_1\ h_6\ ) \quad \text{where} \quad \ell(h_4) = \ell(h_5) < \ell(h_2) \text{ and } \ell(h_6) < \ell(h_1)$$
$$r'_i: \quad h^\bullet_4\ h^\bullet_5\ h^\bullet_2\ h^\circ_4\ h^\circ_5\ h^\circ_2\ h^\bullet_3\ h^\circ_3\ h^\bullet_6\ h^\bullet_1\ h^\circ_6\ h^\circ_1$$

For each $j = 1, 2, \dots, m$, the dummy residents $d_{j,p}$ $(p = 1, 2, \dots, u(h_j))$ have the same list:

$$d_{j,p}: \quad h^\circ_j \quad h^\bullet_j$$

For $j = 1, 2, \ldots, m$, let $P(h_j)$ be the preference list of h_j in I and let $Q(h_j)$ be the strict order on R' obtained by replacing residents r_i with r'_i and breaking ties so that residents in the same tie of $P(h_j)$ are ordered in ascending order of indices. The preference lists of hospitals h°_j and $h^\bullet_j$ are then defined as follows:

$$
\begin{array}{llll}
h^\circ_j : & \quad Q(h_i) & & d_{j,1}\ d_{j,2}\ \cdots\ d_{j,u(h_j)} \\
h^\bullet_j : & d_{j,1}\ d_{j,2}\ \cdots\ d_{j,u(h_j)} & & r'_1\ \ r'_2\ \ \cdots\ r'_n
\end{array}
$$

Let M be the output of Double Proposal applied to I. For each resident r_i, there are two cases: she has never been rejected by $M(r_i)$, and she had been rejected once by $M(r_i)$ and accepted upon her second proposal. Let M_1 be the set of pairs $(r_i, M(r_i))$ of the former case and M_2 be that of the latter. Note that $|M_1(h_j)| \le \ell(h_j)$ for any h_j. Define a matching M^* of I^* by

$$
\begin{aligned}
M^* = &\{\, (r'_i, h^\circ_j) \mid (r_i, h_j) \in M_2 \,\} \cup \{\, (r'_i, h^\bullet_j) \mid (r_i, h_j) \in M_1 \,\} \\
&\cup \{\, (d_{j,p}, h^\circ_j) \mid \ 1 \le p \le u(h_j) - |M_2(h_j)| \ \} \\
&\cup \{\, (d_{j,p}, h^\bullet_j) \mid \ u(h_j) - |M_2(h_j)| < p \le \min\{u(h_j) - |M(h_j)| + \ell(h_j),\ u(h_j)\} \ \}\,.
\end{aligned}
$$

Then, the following holds.

▶ **Lemma 5.** *M^* coincides with the output of the resident-oriented Gale–Shapley algorithm applied to the auxiliary instance I^*.*

We now complete the proof of the theorem.

Given an instance I, suppose that some resident r_i changes her preference list from $\succeq_{r_i}$ to some other $\succeq'_{r_i}$. Let J be the resultant instance. Define an auxiliary instance J^* from J in the manner described above. Let N be the output of Double Proposal for J and N^* be a matching defined from N as we defined M^* from M. By Lemma 5, the resident-oriented Gale–Shapley algorithm returns M^* and N^* for I^* and J^*, respectively. Note that all residents except r'_i have the same preference lists in I^* and J^* and so do all hospitals. Therefore, by the strategy-proofness of the Gale–Shapley algorithm, we have $M^*(r'_i) \succeq_{r'_i} N^*(r'_i)$. By the definitions of I^*, J^*, M^*, and N^*, we have $M(r_i) \succeq_{r_i} N(r_i)$, which means that r_i is no better off in N than in M with respect to her true preference $\succeq_{r_i}$. Thus, Double Proposal is strategy-proof for residents. ◀

5 Maximum Gaps and Approximation Factors of Double Proposal

In this section, we analyze the approximation factors of our algorithm, together with the maximum gaps Λ for the four models mentioned in Section 1. All results in this section are summarized in the first and second rows of Table 1 in Section 1.

For an instance I of HRT-MSLQ, let $\mathrm{OPT}(I)$ and $\mathrm{WST}(I)$ respectively denote the maximum and minimum scores over all stable matchings of I, and let $\mathrm{ALG}(I)$ be the score of the output of our algorithm Double Proposal. Then, $\mathrm{WST}(I)$ can be the score of the output of the algorithm that first breaks ties arbitrarily and then applies the Gale–Shapley algorithm for the resultant instance (see the full version [13]). Therefore, the maximum gap is equivalent to the approximation factor of such arbitrary tie-breaking GS algorithm.

For a model $\mathcal{I}$ (i.e., subfamily of problem instances of HRT-MSLQ), let

$$\Lambda(\mathcal{I}) = \max_{I \in \mathcal{I}} \frac{\mathrm{OPT}(I)}{\mathrm{WST}(I)} \quad \text{and} \quad \mathrm{APPROX}(\mathcal{I}) = \max_{I \in \mathcal{I}} \frac{\mathrm{OPT}(I)}{\mathrm{ALG}(I)}.$$

In subsequent subsections, we provide exact values of $\Lambda(\mathcal{I})$ and $\mathrm{APPROX}(\mathcal{I})$ for the four fundamental models. Recall our assumptions that preference lists are complete, $|R| < \sum_{h\in H} u(h)$, and $\ell(h) \le u(h) \le n$ for each $h \in H$.

5.1 General Model

Let $\mathcal{I}_{\mathrm{Gen}}$ denote the family of all instances of HRT-MSLQ, which we call the *general model*.

▶ **Proposition 6.** *The maximum gap for the general model satisfies $\Lambda(\mathcal{I}_{\mathrm{Gen}}) = n+1$. Moreover, this equality holds even if residents have a master list, and preference lists of hospitals contain no ties.*

We next obtain the value of $\mathrm{APPROX}(\mathcal{I}_{\mathrm{Gen}})$. Recall that ϕ is a function of $n = |R|$ defined by $\phi(1) = 1$, $\phi(2) = \frac{3}{2}$, and $\phi(n) = n(1 + \lfloor \frac{n}{2} \rfloor)/(n + \lfloor \frac{n}{2} \rfloor)$ for $n \ge 3$.

▶ **Theorem 7.** *The approximation factor of* Double Proposal *for the general model satisfies* $\mathrm{APPROX}(\mathcal{I}_{\mathrm{Gen}}) = \phi(n)$.

We provide a full proof in the full version of the paper [13]. Here, we present the ideas to show the inequality $\frac{\mathrm{OPT}(I)}{\mathrm{ALG}(I)} \le \phi(n)$ for any $I \in \mathcal{I}_{\mathrm{Gen}}$.

Proof sketch of Theorem 7. Let M be the output of the algorithm and N be an optimal stable matching. We define vectors p_M and p_N on R, which distribute the scores to residents. For each $h \in H$, among residents in $M(h)$, we set $p_M(r) = \frac{1}{\ell(h)}$ for $\min\{\ell(h), |M(h)|\}$ residents and $p_M(r) = 0$ for the remaining $|M(h)| - \min\{\ell(h), |M(h)|\}$ residents. Similarly, we define p_N from N. We write $p_M(A) \coloneqq \sum_{r\in A} p_M(r)$ for any $A \subseteq R$. By definition, $p_M(M(h)) = s_M(h)$ and $p_N(N(h)) = s_N(h)$ for each $h \in H$, and hence $s(M) = \sum_{h\in H} s_M(h) = p_M(R)$ and $s(N) = \sum_{h\in H} s_N(h) = p_N(R)$. Thus, $\frac{p_N(R)}{p_M(R)} = \frac{s(N)}{s(M)}$, which needs to be bounded.

Let $R' = \{r'_1, r'_2, \ldots, r'_n\}$ be a copy of R and identify p_N as a vector on R'. Consider a bipartite graph $G = (R, R'; E)$ whose edge set is $E \coloneqq \{\, (r_i, r'_j) \in R \times R' \mid p_M(r_i) \ge p_N(r'_j) \,\}$. For any matching $X \subseteq E$ in G, denote by $\partial(X) \subseteq R \cup R'$ the set of vertices covered by X. Then, $p_M(R \cap \partial(X)) \ge p_N(R' \cap \partial(X))$ holds since each edge $(r_i, r'_j) \in X \subseteq E$ satisfies $p_M(r_i) \ge p_N(r'_j)$. In addition, the value of $p_N(R' \setminus \partial(X)) - p_M(R \setminus \partial(X))$ is bounded from above by $|R \setminus \partial(X)| = |R| - |X| = n - |X|$ because $p_N(r') \le 1$ for any $r' \in R'$ and $p_M(r) \ge 0$ for any $r \in R$. Therefore, the existence of a matching $X \subseteq E$ with large $|X|$ helps us bound $\frac{p_N(R)}{p_M(R)}$. Indeed, the following claim plays a key role in our proof: $(\star)$ The graph G admits a matching $X \subseteq E$ with $|X| \ge \lceil \frac{n}{2} \rceil$.

In the proof in the full version [13], the required bound of $\frac{p_N(R)}{p_M(R)}$ is obtained using a stronger version of $(\star)$. Here we concentrate on showing $(\star)$. To this end, we divide R into

$$R_+ \coloneqq \{\, r \in R \mid M(r) \succ_r N(r) \,\},$$
$$R_- \coloneqq \{\, r \in R \mid N(r) \succ_r M(r) \text{ or } [M(r) =_r N(r),\ p_N(r) > p_M(r)] \,\}, \text{ and}$$
$$R_0 \coloneqq \{\, r \in R \mid M(r) =_r N(r),\ p_M(r) \ge p_N(r) \,\}.$$

Let R'_+, R'_-, R'_0 be the corresponding subsets of R'. We show the following two properties.

- There is an injection $\xi_+ \colon R_+ \to R'$ such that $p_M(r) = p_N(\xi_+(r))$ for every $r \in R_+$.
- There is an injection $\xi_- \colon R'_- \to R$ such that $p_N(r') = p_M(\xi_-(r'))$ for every $r' \in R'_-$.

We first define ξ_+. For each hospital h with $M(h) \cap R_+ \ne \emptyset$, there is $r \in M(h) \cap R_+$ with $h = M(r) \succ_r N(r)$. By the stability of N, hospital h is full in N. Then, we can define an injection $\xi_+^h \colon M(h) \cap R_+ \to N(h)$ so that $p_M(r) = p_N(\xi_+^h(r))$ for all $r \in M(h) \cap R_+$. By regarding $N(h)$ as a subset of R' and taking the direct sum of ξ_+^h for all hospitals h with $M(h) \cap R_+ \ne \emptyset$, we obtain a required injection $\xi_+ \colon R_+ \to R'$.

We next define ξ_-. For each hospital h' with $N(h') \cap R'_- \neq \emptyset$, any $r \in N(h') \cap R'_-$ satisfies either $h' = N(r) \succ_r M(r)$ or $[h' = N(r) =_r M(r),\ p_N(r) > p_M(r)]$. If some $r \in N(h') \cap R'_-$ satisfies the former, the stability of M implies that h' is full in M. If all $r \in N(h') \cap R'_-$ satisfy the latter, they all satisfy $0 \neq p_N(r) = \frac{1}{\ell(h')}$, and hence $|N(h') \cap R'_-| \leq \ell(h')$. Additionally, $p_N(r) > p_M(r)$ implies either $p_M(r) = 0$ or $\ell(h') < \ell(h)$, where $h \coloneqq M(r)$. Observe that $p_M(r) = 0$ implies $|M(h)| > \ell(h)$. By Lemma 2, each of $\ell(h') < \ell(h)$ and $|M(h)| > \ell(h)$ implies $|M(h')| \geq \ell(h') \geq |N(h') \cap R'_-|$. Then, in any case, we can define an injection $\xi_-^{h'} \colon N(h') \cap R'_- \to M(h')$ such that $p_N(r') = p_M(\xi_-^{h'}(r'))$ for all $r' \in N(h') \cap R'_-$. By taking the direct sum of $\xi_-^{h'}$ for all hospitals h' with $M(h') \cap R_- \neq \emptyset$, we obtain $\xi_- \colon R'_- \to R$.

Let $G^* = (R, R'; E^*)$ be a bipartite graph (possibly with multiple edges), where E^* is the disjoint union of E_+, E_-, and E_0, defined by

$$E_+ \coloneqq \{\, (r, \xi_+(r)) \mid r \in R_+ \,\}, \quad E_- \coloneqq \{\, (\xi_-(r'), r') \mid r \in R'_- \,\}, \text{ and}$$
$$E_0 \coloneqq \{\, (r, r') \mid r \in R_0 \text{ and } r' \text{ is the copy of } r \,\}.$$

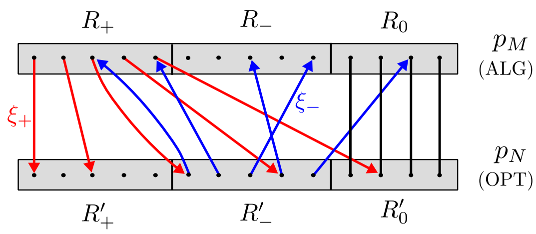

Figure 1 A graph $G^* = (R, R'; E^*)$.

See Fig. 1 for an example. By the definitions of ξ_+, ξ_-, and R_0, any edge (r, r') in E^* belongs to E, and hence any matching in G^* is also a matching in G. Since $\xi_+ \colon R_+ \to R'$ and $\xi_- \colon R'_- \to R$ are injections, we observe that every vertex in G^* is incident to at most two edges in E^*. Then, E^* is decomposed into paths and cycles, and hence E^* contains a matching of size at least $\lceil \frac{|E^*|}{2} \rceil$. Since $|E^*| = |R_+| + |R_-| + |R_0| = n$, this means that there exists a matching $X \subseteq E$ with $|X| \geq \lceil \frac{n}{2} \rceil$, as required. ◀

5.2 Uniform Model

Let $\mathcal{I}_{\text{Uniform}}$ denote the family of uniform problem instances of HRT-MSLQ, where an instance is called *uniform* if upper and lower quotas are uniform. In the rest of this subsection, we assume that ℓ and u are nonnegative integers to represent the common lower and upper quotas, respectively, and let $\theta \coloneqq \frac{u}{\ell}$ (≥ 1). We call $\mathcal{I}_{\text{Uniform}}$ the *uniform model*.

▶ **Proposition 8.** *The maximum gap for the uniform model satisfies* $\Lambda(\mathcal{I}_{\text{Uniform}}) = \theta$. *Moreover, this equality holds even if preference lists of hospitals contain no ties.*

▶ **Theorem 9.** *The approximation factor of* Double Proposal *for the uniform model satisfies* $\text{APPROX}(\mathcal{I}_{\text{uniform}}) = \frac{\theta^2+\theta-1}{2\theta-1}$.

Note that $\frac{\theta^2+\theta-1}{2\theta-1} < \theta$ whenever $\ell < u$ because $\theta - \frac{\theta^2+\theta-1}{2\theta-1} = \frac{(\theta-1)^2}{2\theta-1} > 0$. Here is the ideas to show that $\frac{\text{OPT}(I)}{\text{ALG}(I)} \leq \frac{\theta^2+\theta-1}{2\theta-1}$ holds for any $I \in \mathcal{I}_{\text{Uniform}}$.

Proof sketch of Theorem 9. Let M be the output of the algorithm and N be an optimal stable matching, and assume $s(M) < s(N)$. Consider a bipartite graph $(R, H; M \cup N)$, which may have multiple edges. Take an arbitrary connected component, and let R^* and H^* be the sets of residents and hospitals, respectively, contained in it. It is sufficient to bound $\frac{s_N(H^*)}{s_M(H^*)}$.

Let H_0 be the set of all hospitals in H^* having strictly larger scores in N than in M, i.e.,

$$H_0 \coloneqq \{\, h \in H^* \mid s_N(h) > s_M(h) \,\}.$$

Using this, we sequentially define

$$R_0 \coloneqq \{\, r \in R^* \mid N(r) \in H_0 \,\}, \quad H_1 \coloneqq \{\, h \in H^* \setminus H_0 \mid \exists r \in R_0 : M(r) = h \,\},$$
$$R_1 \coloneqq \{\, r \in R^* \mid N(r) \in H_1 \,\}, \quad H_2 \coloneqq H^* \setminus (H_0 \cup H_1), \quad \text{and} \quad R_2 \coloneqq R^* \setminus (R_0 \cup R_1).$$

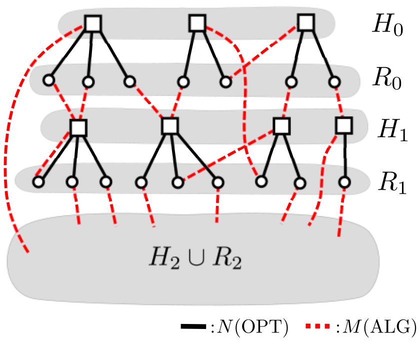

Figure 2 Example with $[\ell, u] = [2, 3]$.

See Fig. 2 for an example. We use scaled score functions $v_M \coloneqq \ell \cdot s_M$ and $v_N \coloneqq \ell \cdot s_N$ and write $v_M(A) = \sum_{h \in A} v_M(h)$ for any $A \subseteq H$. We bound $\frac{v_N(H^*)}{v_M(H^*)}$, which equals $\frac{s_N(H^*)}{s_M(H^*)}$. Note that the set of residents assigned to H^* is R^* in both M and N. The scores differ depending on how efficiently those residents are assigned. In this sense, we may think that a hospital h is assigned residents "efficiently" in M if $|M(h)| \leq \ell$ and is assigned "most redundantly" if $|M(h)| = u$. Since $v_M(h) = \min\{\ell, |M(h)|\}$, we have $v_M(h) = |M(h)|$ in the former case and $v_M(h) = \frac{1}{\theta} \cdot |M(h)|$ in the latter. We show that hospitals in H_1 provide us with advantage of M; any hospital in H_1 is assigned residents either efficiently in M or most redundantly in N.

For any $h \in H_0$, $s_M(h) < s_N(h)$ implies $|M(h)| < \ell$. Then, the stability of M implies $M(r) \succeq_r N(r)$ for any $r \in R_0$. Hence, the following $\{H_1^{\succ}, H_1^{=}\}$ defines a bipartition of H_1:

$$H_1^{\succ} \coloneqq \{\, h \in H_1 \mid \exists r \in M(h) \cap R_0 : h \succ_r N(r) \,\},$$
$$H_1^{=} \coloneqq \{\, h \in H_1 \mid \forall r \in M(h) \cap R_0 : h =_r N(r) \,\}.$$

For each $h \in H_1^{\succ}$, as some r satisfies $h \succ_r N(r)$, the stability of N implies that h is full, i.e., h is assigned residents most redundantly, in N. Note that any $h \in H_1^{\succ}$ satisfies $v_M(h) \geq v_N(h)$ because $h \notin H_0$, and hence $v_M(h) = v_N(h) = \ell$. Then, $|N(h)| = u = \theta \cdot v_N(h) = (\theta - 1) \cdot v_M(h) + v_N(h)$ for each $h \in H_1^{\succ}$. Additionally, for any $h \in H^*$, we have $|N(h)| \geq \min\{\ell, |N(h)|\} = v_N(h)$. Since $|R^*| = \sum_{h \in H^*} |N(h)|$, we have

$$|R^*| \geq (\theta - 1) \cdot v_M(H_1^{\succ}) + v_N(H_1^{\succ}) + v_N(H^* \setminus H_1^{\succ}) = (\theta - 1) \cdot v_M(H_1^{\succ}) + v_N(H^*).$$

For each $h \in H_1^{=}$, there is $r \in R_0$ with $M(r) = h =_r N(r)$. As $r \in R_0$, the hospital $h' := N(r)$ belongs to H_0, and hence $|M(h')| < \ell$. Then, Lemma 2(ii) implies $|M(h)| \leq \ell$, i.e., h is assigned residents efficiently in M. Note that any $h \in H_0$ satisfies $v_M(h) < v_N(h) \leq \ell$. Then, the number of residents assigned to $H_0 \cup H_1^{=}$ is $v_M(H_0 \cup H_1^{=})$. Additionally, the number of residents assigned to $H_1^{\succ} \cup H_2$ is at most $\theta \cdot v_M(H_1^{\succ} \cup H_2)$. Thus, we have

$$|R^*| \leq v_M(H_0 \cup H_1^{=}) + \theta \cdot v_M(H_1^{\succ} \cup H_2) = v_M(H^*) + (\theta - 1) \cdot v_M(H_1^{\succ} \cup H_2).$$

From these two estimations of $|R^*|$, we obtain $v_N(H^*) \leq (\theta - 1) \cdot v_M(H_2) + v_M(H^*)$, which gives us a relationship between $v_M(H^*)$ and $v_N(H^*)$. Combining this with other inequalities, we can obtain the required upper bound of $\frac{v_N(H^*)}{v_M(H^*)}$. ◀

5.3 Marriage Model

Let $\mathcal{I}_{\text{Marriage}}$ denote the family of instances of HRT-MSLQ, in which each hospital has an upper quota of 1. We call $\mathcal{I}_{\text{Marriage}}$ the *marriage model*. By definition, $[\ell(h), u(h)]$ in this model is either $[0, 1]$ or $[1, 1]$ for each $h \in H$. Since this is a one-to-one matching model, the union of two stable matchings can be partitioned into paths and cycles. By applying standard arguments used in other stable matching problems, we can obtain $\Lambda(\mathcal{I}_{\text{Marriage}}) = 2$ and $\text{APPROX}(\mathcal{I}_{\text{Marriage}}) = 1.5$.

As shown in Example 15 in Appendix A.1, there is no strategy-proof algorithm that can achieve an approximation factor better than 1.5 even in the marriage model. Therefore, we cannot improve this ratio without sacrificing strategy-proofness.

5.4 Resident-side Master List Model

Let $\mathcal{I}_{\text{R-ML}}$ denote the family of instances of HRT-MSLQ in which all residents have the same preference list. This is well studied in literature on stable matching [7, 21–23]. We call $\mathcal{I}_{\text{R-ML}}$ the *R-side ML model*. We have already shown in Proposition 6 that $\Lambda(\mathcal{I}_{\text{R-ML}}) = n + 1$. Our algorithm, however, solves this model exactly.

Note that this is not the case for the hospital-side master list model, which is NP-hard as shown in Theorem 14 below. This difference highlights the asymmetry of two sides in HRT-MSLQ.

6 Hardness Results

We obtain various hardness and inapproximability results for HRT-MSLQ. First, we show that HRT-MSLQ in the general model is inapproximable and that we cannot hope for a constant factor approximation.

▶ **Theorem 10.** *HRT-MSLQ is inapproximable within a ratio $n^{\frac{1}{4}-\epsilon}$ for any $\epsilon > 0$ unless P=NP.*

Proof. We show the theorem by way of a couple of reductions, one from the maximum independent set problem (*MAX-IS*) to the maximum 2-independent set problem (*MAX-2-IS*), and the other from MAX-2-IS to HRT-MSLQ.

For an undirected graph $G = (V, E)$, a subset $S \subseteq V$ is an *independent set* of G if no two vertices in S are adjacent. S is a *2-independent set* of G if the distance between any two vertices in S is at least 3. MAX-IS (resp. MAX-2-IS) asks to find an independent set (resp. 2-independent set) of maximum size. Let us denote by $\text{IS}(G)$ and $\text{IS}_2(G)$, respectively, the sizes of optimal solutions of MAX-IS and MAX-2-IS for G. We assume without loss of generality that input graphs are connected. It is known that, unless P=NP, there is no polynomial-time algorithm, given a graph $G_1 = (V_1, E_1)$, to distinguish between the two cases $\mathrm{IS}(G_1) \le |V_1|^{\epsilon_1}$ and $\mathrm{IS}(G_1) \ge |V_1|^{1-\epsilon_1}$, for any constant $\epsilon_1 > 0$ [34].

Now, we give the first reduction, which is based on the NP-hardness proof of the minimum maximal matching problem [18]. Let $G_1 = (V_1, E_1)$ be an instance of MAX-IS. We construct an instance $G_2 = (V_2, E_2)$ of MAX-2-IS as $V_2 = V_1 \cup E_1 \cup \{s\}$ and $E_2 = \{\, (v, e) \mid v \in V_1,\ e \in E_1, e \text{ is incident to } v \text{ in } G_1 \,\} \cup \{\, (s, e) \mid e \in E_1 \,\}$, where s is a new vertex not in $V_1 \cup E_1$. For any two vertices u and v in V_1, if their distance in G_1 is at least 2 then that in G_2 is at least 4. Hence, any independent set in G_1 is also a 2-independent set in G_2. Conversely, for any 2-independent set S in G_2, $S \cap V_1$ is independent in G_1 and $|S \cap (V_2 \setminus V_1)| \le 1$. These facts imply that $\mathrm{IS}_2(G_2)$ is either $\mathrm{IS}(G_1)$ or $\mathrm{IS}(G_1) + 1$. Since $|E_2| = 3|E_1| \le 3|V_1|^2$, distinguishing between $\mathrm{IS}_2(G_2) \le |E_2|^{\epsilon_2}$ and $\mathrm{IS}_2(G_2) \ge |E_2|^{1/2-\epsilon_2}$ for some constant $\epsilon_2 > 0$ would imply distinguishing between $\mathrm{IS}(G_1) \le |V_1|^{\epsilon_1}$ and $\mathrm{IS}(G_1) \ge |V_1|^{1-\epsilon_1}$ for some constant $\epsilon_1 > 0$, which in turn implies P=NP.

We then proceed to the second reduction. Let $G_2 = (V_2, E_2)$ be an instance of MAX-2-IS. Let $n_2 = |V_2|$, $m_2 = |E_2|$, $V_2 = \{v_1, v_2, \ldots, v_{n_2}\}$, and $E_2 = \{e_1, e_2, \ldots, e_{m_2}\}$. We construct an instance I of HRT-MSLQ as follows. For an integer p which will be determined later, define the set of residents of I as $R = \{\, r_{i,j} \mid 1 \le i \le n_2,\ 1 \le j \le p \,\}$, where $r_{i,j}$ corresponds to the jth copy of vertex $v_i \in V_2$. Next, define the set of hospitals of I as $H \cup Y$, where $H = \{\, h_k \mid 1 \le k \le m_2 \,\}$ and $Y = \{\, y_{i,j} \mid 1 \le i \le n_2,\ 1 \le j \le p \,\}$. The hospital h_k corresponds to the edge $e_k \in E_2$ and the hospital $y_{i,j}$ corresponds to the resident $r_{i,j}$.

We complete the reduction by giving preference lists and quotas in Fig. 3, where $1 \le i \le n_2$, $1 \le j \le p$, and $1 \le k \le m_2$. Here, $N(v_i) = \{\, h_k \mid e_k \text{ is incident to } v_i \text{ in } G_2 \,\}$ and "($N(v_i)$)" denotes the tie consisting of all hospitals in $N(v_i)$. Similarly, $N(e_k) = \{\, r_{i,j} \mid e_k \text{ is incident to } v_i \text{ in } G_2,\ 1 \le j \le p \,\}$ and "($N(e_k)$)" is the tie consisting of all residents in $N(e_k)$. The notation "$\cdots$" denotes an arbitrary strict order of all agents missing in the list.

$r_{i,j}$:	($N(v_i)$)	$y_{i,j}$	$\cdots$	h_k $[0,p]$:	($N(e_k)$)	$\cdots$
				$y_{i,j}$ $[1,1]$:	$r_{i,j}$	$\cdots$

Figure 3 Preference lists of residents and hospitals.

We will show that $\mathrm{OPT}(I) = m_2 + p \cdot \mathrm{IS}_2(G_2)$. To do so, we first see a useful property. Let $G_3 = (V_3, E_3)$ be the subdivision graph of G_2, i.e., $V_3 = V_2 \cup E_2$ and $E_3 = \{\, (v, e) \mid v \in V_2, e \in E_2, e \text{ is incident to } v \text{ in } G_2 \,\}$. Then, the family $\mathcal{I}_2(G_2)$ of 2-independent sets in G_2 is characterized as follows [18]:

$$\mathcal{I}_2(G_2) = \left\{ V_2 \setminus \bigcup_{e \in M} \{\text{endpoints of } e\} \;\middle|\; M \text{ is a maximal matching of } G_3 \right\}.$$

In other words, for a maximal matching M of G_3, if we remove all vertices matched in M from V_2, then the remaining vertices form a 2-independent set of G_2, and conversely, any 2-independent set of G_2 can be obtained in this manner for some maximal matching M of G_3.

Let S be an optimal solution of G_2 in MAX-2-IS, i.e., a 2-independent set of size $\mathrm{IS}_2(G_2)$. Let $\tilde{M}$ be a maximal matching of G_3 corresponding to S. We construct a matching M of I as $M = M_1 \cup M_2$, where $M_1 = \{\, (r_{i,j}, h_k) \mid (v_i, e_k) \in \tilde{M},\ 1 \le j \le p \,\}$ and $M_2 = \{\, (r_{i,j}, y_{i,j}) \mid v_i \in S,\ 1 \le j \le p \,\}$. It is not hard to see that each resident is matched by exactly one of M_1 and M_2 and that no hospital exceeds its upper quota.

We then show the stability of M. Each resident matched by M_1 is assigned to a first-choice hospital, so if there were a blocking pair, then it would be of the form $(r_{i,j}, h_k)$ where $M(r_{i,j}) = y_{i,j}$ and $h_k \in N(v_i)$. Then, v_i is unmatched in $\tilde{M}$. Additionally, all residents assigned to h_k (if any) are its first choice; hence, h_k must be undersubscribed in M. Then, e_k is unmatched in $\tilde{M}$. $h_k \in N(v_i)$ implies that there is an edge $(v_i, e_k) \in E_3$, so $\tilde{M} \cup \{(v_i, e_k)\}$ is a matching of G_3, contradicting the maximality of $\tilde{M}$. Hence, M is stable in I.

A hospital in H has a lower quota of 0, so it obtains a score of 1. The number of hospitals in Y that are assigned a resident is $|M_2| = p|S| = p \cdot \mathrm{IS}_2(G_2)$. Hence, $s(M) = m_2 + p \cdot \mathrm{IS}_2(G_2)$. Therefore, we have $\mathrm{OPT}(I) \geq s(M) = m_2 + p \cdot \mathrm{IS}_2(G_2)$.

Conversely, let M be an optimal solution for I, i.e., a stable matching of score $\mathrm{OPT}(I)$. Note that each $r_{i,j}$ is assigned to a hospital in $N(v_i) \cup \{y_{i,j}\}$ as otherwise $(r_{i,j}, y_{i,j})$ blocks M. We construct a bipartite multi-graph $G_M = (V_2, E_2; F)$ where $V_2 = \{v_1, v_2, \ldots, v_{n_2}\}$ and $E_2 = \{e_1, e_2, \ldots, e_{m_2}\}$ are identified as vertices and edges of G_2, respectively, and an edge $(v_i, e_k)_j \in F$ if and only if $(r_{i,j}, h_k) \in M$. Here, a subscript j of edge $(v_i, e_k)_j$ is introduced to distinguish the multiplicity of edge (v_i, e_k). The degree of each vertex of G_M is at most p, so by Kőnig's edge coloring theorem [24], G_M is p-edge colorable and each color class c induces a matching M_c $(1 \leq c \leq p)$ of G_M. Each M_c is a matching of G_3, and by the stability of M, we can show that it is in fact a maximal matching of G_3. Let M_* be a minimum cardinality one among them.

Define a subset S of V_2 by removing vertices that are matched in M_* from V_2. By the above observation, S is a 2-independent set of G_2. We will bound its size. Note that $s(M) = \mathrm{OPT}(I)$ and each hospital in H obtains the score of 1, so M assigns residents to $\mathrm{OPT}(I) - m_2$ hospitals in Y and each such hospital receives one resident. There are pn_2 residents in total, among which $\mathrm{OPT}(I) - m_2$ ones are assigned to hospitals in Y, so the remaining $pn_2 - (\mathrm{OPT}(I) - m_2)$ ones are assigned to hospitals in H. Thus F contains this number of edges and so $|M_*| \leq \frac{pn_2 - (\mathrm{OPT}(I) - m_2)}{p} = n_2 - \frac{\mathrm{OPT}(I) - m_2}{p}$. Since $|V_2| = n_2$ and exactly one endpoint of each edge in M_* belongs to V_2, we have that $|S| = |V_2| - |M_*| \geq \frac{\mathrm{OPT}(I) - m_2}{p}$. Therefore $\mathrm{IS}_2(G_2) \geq |S| \geq \frac{\mathrm{OPT}(I) - m_2}{p}$. Hence, we obtain $\mathrm{OPT}(I) = m_2 + p \cdot \mathrm{IS}_2(G_2)$ as desired. Now we let $p = m_2$, and have $\mathrm{OPT}(I) = m_2(1 + \mathrm{IS}_2(G_2))$.

Therefore distinguishing between $\mathrm{OPT}(I) \leq (m_2)^{1+\delta}$ and $\mathrm{OPT}(I) \geq (m_2)^{3/2-\delta}$ for some δ would distinguish between $\mathrm{IS}_2(G_2) \leq (m_2)^{\epsilon_2}$ and $\mathrm{IS}_2(G_2) \geq (m_2)^{1/2-\epsilon_2}$ for some constant $\epsilon_2 > 0$. Since $n = |R| = n_2 m_2 \leq (m_2)^2$, a polynomial-time $n^{1/4-\epsilon}$-approximation algorithm for HRT-MSLQ can distinguish between the above two cases for a constant $\delta < \epsilon/2$. Hence, the existence of such an algorithm implies P=NP. This completes the proof. ◀

We then show inapproximability results for the uniform model and the marriage model under the Unique Games Conjecture (UGC).

▶ **Theorem 11.** *Under UGC, HRT-MSLQ in the uniform model is not approximable within a ratio* $\frac{3\theta+3}{2\theta+4} - \epsilon$ *for any positive* ϵ.

▶ **Theorem 12.** *Under UGC, HRT-MSLQ in the marriage model is not approximable within a ratio* $\frac{9}{8} - \epsilon$ *for any positive* ϵ.

Furthermore, we give two examples showing that HRT-MSLQ is NP-hard even in very restrictive settings. The first is a marriage model for which ties appear in one side only.

▶ **Theorem 13.** *HRT-MSLQ in the marriage model is NP-hard even if there is a master preference list of hospitals and ties appear only in preference lists of residents or only in preference lists of hospitals.*

The other is a setting like the capacitated house allocation problem, where all hospitals are indifferent among residents.

▶ **Theorem 14.** *HRT-MSLQ in the uniform model is NP-hard even if all the hospitals quotas are* $[1, 2]$*, preferences lists of all residents are strict, and all hospitals are indifferent among all residents (i.e., there is a master list of hospitals consisting of a single tie).*

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A Examples

We give some examples that show the difficulty of implementing strategy-proof algorithms for HRT-MSLQ.

A.1 Incompatibility between Optimization and Strategy-proofness

Here, we provide two examples that show that solving HRT-MSLQ exactly is incompatible with strategy-proofness even if we ignore computational efficiency. This incompatibility holds even for restrictive models. The first example is an instance in the marriage model in which ties appear only in preference lists of hospitals. The second example is an instance in the uniform model in which ties appear only in preference lists of residents.

▶ **Example 15.** Consider the following instance I, consisting of two residents and three hospitals.

$$\begin{array}{llll} r_1: & h_1 & h_2 & h_3 \\ r_2: & h_1 & h_2 & h_3 \end{array} \qquad \begin{array}{lll} h_1\ [1,1]: & (r_1 & r_2) \\ h_2\ [1,1]: & (r_1 & r_2) \\ h_3\ [0,1]: & (r_1 & r_2) \end{array}$$

Then, I has two stable matchings $M_1 = \{(r_1, h_1), (r_2, h_2)\}$ and $M_2 = \{(r_1, h_2), (r_2, h_1)\}$, both of which have a score of 3. Let A be an algorithm that outputs a stable matching with a maximum score for any instance of HRT-MSLQ. Without loss of generality, suppose that A returns M_1. Let I' be obtained from I by replacing r_2's list with "$r_2 : h_1\ h_3\ h_2$." Then, the stable matchings for I' are $M_3 = \{(r_1, h_1), (r_2, h_3)\}$ and $M_4 = \{(r_1, h_2), (r_2, h_1)\}$, which have scores 2 and 3, respectively. Since A should return one with a maximum score, the output is M_4, in which r_2 is assigned to h_1 while she is assigned to h_2 in M_1. As $h_1 \succ_{r_3} h_2$ in her true preference, this is a successful manipulation for r_2, and A is not strategy-proof.

Example 15 shows that there is no strategy-proof algorithm for HRT-MSLQ that attains an approximation factor better than 1.5 even if there are no computational constraints.

▶ **Example 16.** Consider the following instance I, consisting of six residents and five hospitals, where the notation "$\cdots$" at the tail of lists denotes an arbitrary strict order of all agents missing in the list.

$$\begin{array}{lllll} r_1: & h_1 & \cdots & & \\ r_2: & h_3 & h_2 & h_1 & \cdots \\ r_3: & h_3 & \cdots & & \\ r_4: & (h_3 & h_4) & \cdots & \\ r_5: & h_4 & \cdots & & \\ r_6: & h_4 & h_5 & h_1 & \cdots \end{array} \qquad \begin{array}{lllll} h_1\ [1,2]: & r_1 & r_2 & r_6 & \cdots \\ h_2\ [1,2]: & r_2 & \cdots & & \\ h_3\ [1,2]: & r_3 & r_4 & r_2 & \cdots \\ h_4\ [1,2]: & r_5 & r_4 & r_6 & \cdots \\ h_5\ [1,2]: & r_6 & \cdots & & \end{array}$$

This instance I has two stable matchings

$$M_1 = \{(r_1,h_1),(r_2,h_2),(r_3,h_3),(r_4,h_3),(r_5,h_4),(r_6,h_4)\}, \text{ and}$$
$$M_2 = \{(r_1,h_1),(r_2,h_3),(r_3,h_3),(r_4,h_4),(r_5,h_4),(r_6,h_5)\},$$

both of which have a score of 4. Let A be an algorithm that outputs an optimal solution for any input. Then, A must output either M_1 or M_2.

Suppose that A outputs M_1. Let I' be an instance obtained by replacing r_2's preference list from "$r_2 : h_3\ h_2\ h_1 \cdots$" to "$r_2 : h_3\ h_1\ h_2 \cdots$." Then, the stable matchings I' admits are M_2 and $M_1' = \{(r_1,h_1),(r_2,h_1),(r_3,h_3),(r_4,h_3),(r_5,h_4),(r_6,h_4)\}$, whose score is 3. Hence, A must output M_2. As a result, r_2 is assigned to a better hospital h_3 than h_2, so this manipulation is successful.

If A outputs M_2, then r_6 can successfully manipulate the result by changing her list from "$r_6 : h_4\ h_5\ h_1 \cdots$" to "$r_6 : h_4\ h_1\ h_5 \cdots$." The instance obtained by this manipulation has two stable matchings M_1 and $M_2' = \{(r_1,h_1),(r_2,h_3),(r_3,h_3),(r_4,h_4),(r_5,h_4),(r_6,h_1)\}$, whose score is 3. Hence, A must output M_1 and r_6 is assigned to h_4, which is better than h_5.

A.2 Absence of Strategy-proofness in Adaptive Tie-breaking

We provide an example that demonstrates that introducing a greedy tie-breaking method into the resident-oriented Gale–Shapley algorithm in an adaptive manner destroys the strategy-proofness for residents.

▶ **Example 17.** Consider the following instance I (in the uniform model), consisting of five residents and three hospitals.

r_1:	h_1	h_2	h_3	h_1 $[1,2]$:	r_2	r_3	r_5	r_1	r_4
r_2:	$(h_1$	$h_2)$	h_3	h_2 $[1,2]$:	r_2	r_4	r_1	r_3	r_5
r_3:	h_1	h_2	h_3	h_3 $[1,2]$:	r_1	r_2	r_3	r_4	r_5
r_4:	h_2	h_1	h_3						
r_5:	h_1	h_3	h_2						

Consider an algorithm that is basically the resident-oriented Gale–Shapley algorithm and let each resident prioritize deficient hospitals over sufficient hospitals among the hospitals in the same tie. Its one possible execution is as follows. First, r_1 proposes to h_1 and is accepted. Next, as h_1 is sufficient while h_2 is deficient, r_2 proposes to h_2 and is accepted. If we apply the ordinary Gale–Shapley procedure afterward, then we obtain a matching $\{(r_1,h_3),(r_2,h_2),(r_3,h_1),(r_4,h_2),(r_5,h_1)\}$. Thus, r_1 is assigned to her third choice.

Let I' be an instance obtained by swapping h_1 and h_2 in r_1's preference list. If we run the same algorithm for I', then r_1 first proposes to h_2. Next, as h_2 is sufficient while h_1 is deficient, r_2 proposes to h_1 and is accepted. By applying the ordinary Gale–Shapley procedure afterward, we obtain $\{(r_1,h_2),(r_2,h_1),(r_3,h_1),(r_4,h_2),(r_5,h_3)\}$. Thus, r_1 is assigned to a hospital h_2, which is her second choice in her original list. Therefore, this manipulation is successful for r_1.