Finite-Memory Strategies in Two-Player Infinite Games

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Abstract

We study infinite two-player win/lose games (A, B, W) where A, B are finite and $W \subseteq (A \times B)^\omega$. At each round Player 1 and Player 2 concurrently choose one action in A and B, respectively. Player 1 wins iff the generated sequence is in W. Each history $h \in (A \times B)^*$ induces a game (A, B, W_h) with $W_h := \{\rho \in (A \times B)^\omega \mid h\rho \in W\}$. We show the following: if W is in Δ^0_2 (for the usual topology), if the inclusion relation induces a well partial order on the W_h's, and if Player 1 has a winning strategy, then she has a finite-memory winning strategy. Our proof relies on inductive descriptions of set complexity, such as the Hausdorff difference hierarchy of the open sets.

Examples in Σ^0_2 and Π^0_2 show some tightness of our result. Our result can be translated to games on finite graphs: e.g. finite-memory determinacy of multi-energy games is a direct corollary, whereas it does not follow from recent general results on finite memory strategies.

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1 Introduction

Two-player win/lose games have been a useful tool in various areas of logic and computer science. The two-player win/lose games in this article consist of infinitely many rounds. At each round i, Player 1 and Player 2 concurrently choose one *action* each, i.e. a_i and b_i in their respective sets A and B. Player 1 wins and Player 2 loses if the play $(a_0, b_0)(a_1, b_1)\ldots$ belongs to a fixed $W \subseteq (A \times B)^\omega$. Otherwise Player 2 wins and Player 1 loses. We call W the *winning set* of Player 1, or the winning condition for Player 1. A *strategy* is a map that tells a player how to play after any finite *history* of actions played: a Player 1 (resp. 2) strategy is a map from $(A \times B)^*$ to A (resp. B). A strategy is *finite-memory* (FM) if the map can be implemented by a finite-state machine. Also, each history $h \in (A \times B)^*$ induces a game starting at h and taking the past into account, i.e. with winning set $W_h := \{\rho \in (A \times B)^\omega \mid h\rho \in W\}$.

For now we state a slightly weaker version of our main result: if A and B are finite, if the $(W_h)_{h\in(A\times B)^*}$ constitute a well partial order (wpo) for the inclusion, if $W \in \Delta^0_2$, i.e. in the usual cylinder topology W is a countable union of closed sets and a countable intersection of open sets, and if Player 1 has a winning strategy, then she has a finite-memory winning strategy. Of course, our result also applies to the turn-based version of such games.

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On the proof of the main result. The proof of our main result relies on descriptive set theory. The Hausdorff-Kuratowski theorem (see, e.g., [5]) states that each set in Δ^0_2 can be expressed as an ordinal difference of open sets, and conversely. In general, this implies that properties of sets in Δ^0_2 may be proved by induction over the countable ordinals. Accordingly, we prove our main result by induction on W, but the inductive step suggested by the Hausdorff-Kuratowski theorem does not suit us completely. Instead, we mix it with a folklore alternative way of describing Δ^0_2 by induction. After this mix, our base case consists of the open sets, the first inductive step consists of *union with a closed set*, and the second inductive step of *open union*.

To prove our result, the base case, where W is open, amounts to reachability games, and the wpo assumption is not needed. The case where W is closed is easy, and it includes the multi-energy games (where Player 1 keeps all energy levels positive). Just above these, the case where W is the union of an open set and a closed set is harder to prove. It includes disjunctions of a reachability condition and a multi-energy condition. We will present this harder case in details because it shows part of the complexity of the full result.

Yet another representation of Δ^0_2. Above, we mentioned two hierarchies that describe Δ^0_2. In addition, this paper (re-)proves the folklore result that Π^0_2 corresponds to Büchi winning conditions and Σ^0_2 to co-Büchi. If labeling each history with 0 or 1, the Büchi (co-Büchi) condition requires that infinitely (only finitely) many 1's be seen on a branch/play. The Δ^0_2 sets are therefore the sets that can be expressed both by Büchi and co-Büchi conditions. This is possible exactly if on every infinite play not both 0 and 1 occurs infinitely often. This corresponds to the play's crossing only finitely many layers in the previous paragraph.

From the four representations and the layer intuition, we can conclude that Δ^0_2 sets are infinite Boolean combinations of open sets, of course in a restricted sense.

Tightness of the result. The collection of the countable unions of closed sets is called Σ^0_2, and the collection of their complements, i.e. the countable intersections of open sets, is called Π^0_2. So $\Delta^0_2 = \Sigma^0_2 \cap \Pi^0_2$. In this article we provide one example of a winning set W in Σ^0_2 and one example in Π^0_2 that satisfy the wpo assumption but not the FM-strategy sufficiency. Hence tightness.

Note that without the wpo assumption, even Turing-computable strategies may not suffice to win for closed winning sets: take a non-computable binary sequence ρ and a game where Player 1 wins iff she plays ρ. She has a winning strategy, but no computable ones.

Connections with graph games. Our game $\langle A, B, W\rangle$ can be seen as a one-state concurrent graph game where the winning condition is defined via the actions rather than the visited states. Winning strategies (resp. FM strategies) coincide in both models.

Alternatively, we can unfold any concurrent graph game into an infinite tree game $\langle A, B, W\rangle$ whose nodes are the histories of pairs of actions. Winning strategies coincide in both models. Moreover, an FM strategy in the tree game is, up to isomorphism, also an FM strategy in the graph game. The converse may not hold since, informally, the player may observe the current state only in the graph model. Nevertheless for finite graphs, the observation of the state can be simulated by an additional finite memory, i.e. the graph itself. To sum up, any FM result in our tree games can be translated into an FM result in graph games. Almost conversely, FM results in finite-graph games can be obtained from our tree games, possibly with non-optimal memory.

Related works and applications. The two articles [7] and [2] provide abstract criteria to show finite-memory determinacy in finite-graph games: [7] by Boolean combination of complex FM winning conditions with simple winning conditions defined via regular languages; [2] by characterizing, for a fixed memory, the winning conditions that yield, in all finite-graph games, FM determinacy via this fixed memory. The FM determinacy of multi-energy games is a corollary of neither, but as mentioned above, it is a direct corollary of our result. (Note that this specific result was already proved in [10].)

More generally on a finite-graph game, consider the conjunction or disjunction of a multi-energy winning condition and a Boolean combination of reachability conditions. This is in Δ^0_2 (actually low at some finite level of the hierarchy), this induces a wpo, so if Player 1 has a winning strategy she has an FM one.

However, the FM determinacy of Büchi games in finite-graph games is not a corollary of our result, because the Büchi conditions may not be in Δ^0_2. We mention three things about this. First, this determinacy does not contradict our tightness results: in Π^0_2 or Σ^0_2, FM determinacy holds when the corresponding labeling is regular. Second, in future work we plan to seek a general theorem having both this determinacy and our main result as special cases. Third, in finite-graph games, many winning conditions that yield memoryless or FM determinacy can be simulated by finite games, and therefore clopen winning conditions, i.e. Δ^0_1 instead of our more general Δ^0_2. For instance, see [1], [8], [4]. This suggests that our work could be used to prove more FM sufficiency results by reduction of finite-graph games to tree games with wpo winning condition in Δ^0_2.

Structure of the article. Section 2 defines our games and finite-memory strategies; Section 3 presents the related descriptive set theory; Section 4 presents our main result; Section 5 discusses tightness of our main result; Section 6 mentions possible future work.

2 Setting and definitions

We study two-player games, which consist in a tuple (A, B, W): A is the *action set* for Player 1, B is the action set for Player 2 and $W \subseteq (A \times B)^\omega$ is the *winning set*. Here we only consider finitely branching games, where A and B are both finite. Such a game is played in the following way: at each round, each player chooses an action from their respective action set in a concurrent way, thus producing a pair of actions in $A \times B$. The game then continues for infinitely many rounds, generating a *play* which consists in an infinite word in $(A \times B)^\omega$. Player 1 then wins if the generated play belongs to the winning set W, while Player 2 wins if it does not. In the following we will focus on Player 1.

To describe the Players' behavior in such a game, we use the concepts of *histories* and *strategies*. A history is a finite word in $(A \times B)^*$ and represents the state of the game after finitely many rounds. We call $\mathcal{H}$ the set of histories. For a (finite or infinite) word w and $k \in \mathbb{N}$, we denote by w_k the k-th letter of w and by $w_{<k}$ the prefix of w of length k. A strategy $s : \mathcal{H} \to A$ for Player 1 is a function that maps histories to actions and represents a behavior for Player 1: in history h, she will play action $s(h)$. Given a strategy s, a history h and a word $\beta \in B^\omega$, we call $\mathrm{out}(h, s, \beta)$ the only play where both players first play h, then Player 1 plays according to s and Player 2 plays the actions of β in order. This play is defined inductively as follows:

- for $k \leq |h|$, $\mathrm{out}(h, s, \beta)_{<k} = h_{<k}$;
- for $k > |h|$, $\mathrm{out}(h, s, \beta)_{<k} = \mathrm{out}(h, s, \beta)_{<k-1}(s(\mathrm{out}(h, s, \beta)_{<k-1}), \beta_{k-|h|-1})$.

We say that a play ρ is *compatible* with a given strategy s if there exists $\beta \in B^\omega$ such that $\rho = \mathrm{out}(\varepsilon, s, \beta)$, where ε is the empty history. Similarly, a history is compatible with s if it is the finite prefix of a compatible play. We say that a strategy is *winning* if all the plays compatible with it belong to W: if Player 1 plays according to such a strategy, she is guaranteed to win. We can extend this concept to say that a strategy s is winning *from a history h* when for all $\beta \in B^\omega$ we have $\mathrm{out}(h, s, \beta) \in W$ (if after history h Player 1 starts playing according to s then she will win). We call a *winning history* a history from which there exists a winning strategy.

We call a *tree* any subset of $(A \times B)^*$ which is closed by prefix, and a branch any (finite or infinite) sequence of elements $e_0 = \varepsilon \sqsubset e_1 \sqsubset e_2 \sqsubset \ldots$ of a tree. In particular, all histories compatible with a given strategy form a tree, which we call the *strategic tree* induced by the strategy. We makes extensive use of Kőnig's lemma [6], which states that if a finitely branching tree has no infinite branch then it is a finite tree. Specifically, we often use the derived result that if some family in a tree intersects all infinte branches of the tree then it has a finite subset that also does.

Given a history $h \in \Gamma$, we say that an action $a \in A$ is non-losing for h if for any action $b \in B$, $h(a, b)$ is a winning history. Among all histories, we are particularly interested in the set of histories along which Player 1 has only played non-losing actions: we call Γ this particular set. Notice that Player 1 has a winning strategy from any history $h \in \Gamma$, but that playing only non-losing actions for Player 1 might be a losing strategy.

A history $h \in \mathcal{H}$ *induces* a winning set W_h defined as $W_h = \{\rho \in (A \times B)^\omega \mid h\rho \in W\}$. The set W_h contains all the infinite continuations ρ such that $h\rho$ is a winning play.

Recall that we introduced strategies for Player 1 as functions mapping histories to actions in A. Among these strategies, we are particularly interested in those that can be described as finite machines: we call them *finite-memory* strategies. Let us introduce first the concept of *finite-memory decision machines.* A finite-memory decision machine is a tuple (M, σ, μ, m_0) such that:

- M is a finite set (the memory);
- $\sigma : M \to A$ is the decision function;
- $\mu : M \times (A \times B) \to M$ is the memory update function;
- $m_0 \in M$ is the initial memory state.

Given a finite-memory decision machine (M, σ, μ, m_0), we extend μ by defining $\mu(m, h)$ for $h \in \mathcal{H}$ in the following inductive way:

- $\mu(m, \varepsilon) = m$
- for all h in $\mathcal{H}$ and $(a, b) \in A \times B$, $\mu(m, h(a, b)) = \mu(\mu(m, h), (a, b))$.

For readability's sake, in case $m = m_0$ and the context is clear, we write $\mu(h)$ for $\mu(m, h)$.

A finite-memory decision machine (M, σ, μ, m_0) induces a strategy s for Player 1 defined for all $h \in \mathcal{H}$ as $s(h) = \sigma(\mu(m_0, h))$. We say that a strategy s is a finite-memory strategy if it is induced by some finite-memory decision machine, and abusively write $s = (M, \sigma, \mu, m_0)$ (identifying the finite-memory decision machine with the strategy it induces) when it is the case.

We often describe a finite-memory decision machine by only defining μ for the action pairs that are compatible with σ (i.e. for $m \in M$ we only define $\mu(m, (a, b))$ when $a = \sigma(m)$). Such a partial machine can be easily extended to a complete one, and contains all the relevant information to decide on the winning aspect of the strategy (or rather, strategies, as many different extensions are possible) it induces as it describes all plays compatible with itself.

3 Descriptive set theory

3.1 Open sets and the Borel hierarchy

Given a set C and a finite word $w \in C^*$, we call *cylinder* of w the set $\mathrm{cyl}(w) = \{w\rho \mid \rho \in C^\omega\}$. This set contains all the infinite words that start with w. In concordance with the usual cylinder topology on C^ω, the cylinders serve as the basis for the open sets, in the sense that we define as an *open set* any set that can be written as an arbitrary union of cylinders. We say that a family of words $\mathcal{F}$ is a *generating family* for an open set O if we have $O = \cup_{f\in\mathcal{F}}\mathrm{cyl}(f)$, that is, O is the set of all plays that have at least one finite prefix in $\mathcal{F}$.

These open sets allow to define a Borel algebra on C^ω as the smallest σ-algebra that contains all open sets. More precisely, the Borel algebra is the smallest collection of sets that contains the open sets and is closed under both countable union and complement (for more information about Borel sets, see [5]). This collection of sets can be organized into what is called the Borel hierarchy, which is defined for countable ordinals in the following way:

- Σ^0_1 is the collection of all the open sets;
- for all countable ordinals θ, Π^0_θ is the collection of sets whose complements are in Σ^0_θ;
- for all countable ordinals θ, Σ^0_θ is the collection of sets that can be defined as a countable union of sets belonging to lower levels of the hierarchy;
- finally, for all countable ordinals θ, Δ^0_θ is the collection of sets that are in both Σ^0_θ and Π^0_θ.

To illustrate, let us detail the lowest levels of the hierarchy:

- as per the definition, Σ^0_1 is the collection of all the open sets;
- the sets in Π^0_1 are the sets whose complement is an open set, we call them the *closed* sets;
- Σ^0_2 contains the sets which can be written as a countable union of closed sets;
- Π^0_2 contains the sets which complement can be written as a countable union of closed sets: by properties of the complement, these are the sets that can be written as a countable intersection of open sets;
- finally, Δ^0_2 is the collection of sets that can be written both as a countable union of closed sets and as a countable intersection of open sets.

In the following we will focus on the collection of sets Δ^0_2.

3.2 The Hausdorff difference hierarchy

The Hausdorff difference hierarchy (see for instance [5]) provides us with a way of defining inductively all the sets in Δ^0_2. Formally, given an ordinal θ and an increasing sequence of open sets $(O_\eta)_{\eta<\theta}$, the set $D_\theta((O_\eta)_{\eta<\theta})$ is defined by:

$$\rho \in D_\theta((O_\eta)_{\eta<\theta}) \quad \Leftrightarrow \quad \begin{array}{l}\rho \in \cup_{\eta<\theta} O_\eta \text{ and the least } \eta \text{ such that } \rho \in O_\eta \\ \text{has parity opposite to that of } \theta.\end{array}$$

For any ordinal θ, we call $\mathcal{D}_\theta$ the collection of sets S such that there exists an increasing family of open sets $(O_\eta)_{\eta<\theta}$ such that $S = D_\theta((O_\eta)_{\eta<\theta})$. To illustrate, $\mathcal{D}_1$ is the collection of all the open sets, $\mathcal{D}_2$ is the collections of the sets that can be written as $O_1 \setminus O_0$ where O_1 and O_0 are two open sets (and hence contains the closed sets), $\mathcal{D}_3$ is the collection of the sets that can be written as $O_2 \setminus (O_1 \setminus O_0)$ where O_2, O_1 and O_0 are three open sets, etc.

The Hausdorff-Kuratowski theorem [5] then states that a set S belongs to Δ^0_2 if and only if there exists an ordinal θ such that $S \in \mathcal{D}_\theta$.

3.3 The fine Hausdorff hierarchy

In the spirit of the Hausdorff difference hierarchy, we propose another inductive way of defining the sets in Δ_2^0. This other hierarchy was already introduced in [9], and may have appeared earlier in the literature but to our knowledge has never been studied in similar depth. Most of the related results can however be considered folklore. First we introduce the concept of *open union*: we say that the union of a family of sets $(S_i)_{i\in I}$ is an open union if there exists a family of disjoint open sets $(O_i)_{i\in I}$ such that for all $i \in I$ have $S_i \subseteq O_i$. We denote such a union by $\Cup_{i\in I} S_i$.

We then define inductively collections of sets Λ_θ and K_θ, with θ a positive ordinal, in the following way:

- a set S is in Λ_1 if and only if it is an open set;
- a set S is in K_θ if and only if its complement is in Λ_θ;
- a set S is in Λ_θ with $\theta > 1$ if and only if there exists a family of sets $(S_i)_{i\in I}$ such that for each i there exists $\eta_i < \theta$ such that $S_i \in \Lambda_{\eta_i} \cup K_{\eta_i}$ and we have $S = \Cup_{i\in I} S_i$.

This definition is akin to the definition of the Borel hierarchy, with the exception that the union operation is replaced with an open union. We then prove the following theorem, which shows our hierarchy is a refinement of the Hausdorff difference hierarchy (and hence justifies its name):

▶ **Theorem 1** (folklore). *For all ordinals θ, we have $\mathcal{D}_\theta = \Lambda_\theta$.*

This theorem is naturally proven by induction on θ and requires intermediate results which help understand the nature of the two hierarchies. In particular, we have: (i) for all ordinals θ the collection $\mathcal{D}_\theta$ is closed under open union, (ii) for all ordinals θ the collection Λ_θ is closed by intersection with an open set, (iii) for all ordinals θ the two collections Λ_θ and K_θ are closed under intersection with a cylinder and (iv) for all limit ordinals θ the collection $\mathcal{D}_\theta$ is the collection of sets that can be written as the open unions of sets in $\cup_{\eta<\theta}\mathcal{D}_\eta$. One observation which proves pivotal for proving our main result on the existence of finite-memory strategies is that for all ordinals θ, all sets in K_θ can be written as the union of a closed set and a set that belongs to Λ_θ (if θ is a successor odinal, we can be even more precise as K_θ is the collection of all sets that can be written as the union of a closed set and a set in $\Lambda_{\theta-1}$). All the details surrounding these two views and the proof of theorem 1 can be found in [3]

3.4 The 0-1 eventually constant labelling

A third possible view of sets in Δ_2^0 is given via eventually constant labelling functions. We say that a labelling function $l : C^* \to \{0, 1\}$ is eventually constant if for all infinite words ρ in C^ω, the sequence $(l(\rho_{<n}))_{n\in\mathcal{N}}$ is eventually constant, which means that there exists a finite $k \in \mathbb{N}$ and $i \in \{0, 1\}$ such that for all $n \geq k$ we have $l(\rho_{<n}) = i$. We then call 1_l the set of infinite words $\rho \in C$ such that the set $\{n \mid l(\rho_{<n}) = 1\}$ is infinite. As we will see, the sets S that belong to Δ_2^0 are the sets such that there exists an eventually constant labelling function l such that $S = 1_l$.

3.5 Equivalence of representations

3.5.1 Representations of sets in Δ_2^0

As expressed by the following theorem, the Hausdorff difference hierarchy, fine Hausdorff hierarchy and eventually constant labelling functions actually define the same sets, which are exactly the sets that belong to Δ_2^0.

▶ **Theorem 2** (folklore). *Given a subset S of C^ω, the following propositions are equivalent:*
(1) $S \in \Delta_2^0$*;*
(2) *S belongs to the Hausdorff difference hierarchy;*
(3) *S belongs to the fine Hausdorff hierarchy;*
(4) *there exists an eventually constant labelling function l such that $S = 1_l$.*

The detailed proof can be found in [3], but we give some elements here: the Hausdorff-Kuratowski theorem [5] shows that (1) ⇔ (2), and Theorem 1 shows that (2) ⇔ (3). We prove that (3) ⇒ (4) by showing that the collection of sets of the form 1_l where l is an eventually constant labelling function is closed under both open union and complement, and finally prove that (4) ⇒ (1) by showing that all sets of the form 1_l where l is an eventually constant labelling function can be expressed both as countable intersection of open sets and countable union of closed sets.

3.5.2 Correspondence between Büchi/co-Büchi conditions and Π_2^0/Σ_2^0

Two much studied types of winning condition in computer science are the *Büchi* and *co-Büchi* conditions. Such winning conditions are given by a coloring function c that provides a color (elements in $\{0, 1\}$) for every history. In the case of a Büchi condition, a play is then winning if infinitely many of its prefixes are associated with the color 1 while in the case of a co-Büchi condition it is winning if finitely many of its prefixes are associated with the color 1 (Büchi and co-Büchi conditions are thus the complement of each other). As stated by the following lemma, whose proof can be found in [3], Büchi conditions actually describe the sets in Π_2^0:

▶ **Lemma 3.** *A subset S of C^ω belongs to Π_2^0 if and only if it can be expressed as a Büchi condition.*

A trivial corollary is that co-Büchi conditions describe the sets in Σ_2^0:

▶ **Corollary 4.** *W belongs to Σ_2^0 if and only if it can be expressed as a co-Büchi condition.*

4 On the existence of finite-memory winning strategies when the winning set belongs to the Hausdorff difference hierarchy

Our aim is to exhibit conditions on W that ensure Player 1 has a finite-memory winning strategy when some winning strategy exists.

Consider a game (A, B, W) where the winning set W belongs to Δ_2^0. We introduce a new hypothesis on the induced winning sets of this game: the set inclusion relation, denoted by $\subseteq$, induces a well partial order (wpo) on the winning sets induced by the histories in Γ. That is, for any sequence $(h_n)_{n\in\mathbb{N}}$ of histories in Γ, there exists $k < l$ such that $W_{h_k} \subseteq W_{h_l}$. A known property of well partial orders which we will use is that any set $S \subseteq \Gamma$ contains a finite subset M such that for all $h \in S$ there exists $m \in M$ such that $W_m \subseteq W_h$. The set of winning sets induced by the histories in M effectively functions as a finite set of under-approximations for the winning sets induced by the histories in S.

Such hypotheses might seem exotic and restrictive, but are effectively satisfied for well-studied classes of games, such as energy games or multi-energy games played on graphs (see for instance [10]), or games with a winning condition expressed as a boolean combination of reachability/safety conditions. Indeed, in the first case the induced winning sets are isomorphic to the cartesian product of the state space and $\mathbb{N}^k$, where k is the number of energy dimensions, while in the second case they are isomorphic to the cartesian product of the state space and the set of possible valuations.

Under these specific conditions, we prove that Player 1 always has a finite-memory winning strategy when she has a winning strategy:

▶ **Theorem 5.** *Assume that W belongs to Δ_2^0 and $\subseteq$ induces a well partial order on $\{W_h \mid h \in \Gamma\}$. If Player 1 has a winning strategy from ε, then she also has a finite-memory one.*

Given a set S in the Hausdorff difference hierarchy, the rank of S is the least ordinal θ such that $S \in \mathcal{D}_\theta$. We prove Theorem 5 by a transfinite induction on the rank of W.

First notice that the inclusion of induced winning sets has the nice property of being preserved by the addition of a suffix, which is formally expressed by the following lemma:

▶ **Lemma 6.** *If $W_h \subseteq W_{h'}$ then for all $(a,b) \in A \times B$ we have $W_{h(a,b)} \subseteq W_{h'(a,b)}$.*

▶ **Corollary 7.** *If $W_h \subseteq W_{h'}$ then all non-losing actions of h are also non-losing for h'.*

4.1 Proof for open sets

We begin by the case where W is an open set (W has rank 1), generated by a set $\mathcal{F}$ of histories.

▶ **Lemma 8.** *If W is an open set and $\varepsilon \in \Gamma$, then Player 1 has a finite-memory winning strategy from ε.*

Proof. Suppose that there exists a winning strategy s from ε. Then consider the strategic tree T induced by s, and consider the tree $T' = T \setminus \{h \in \mathcal{H} \mid \exists f \in \mathcal{F}, f \sqsubset h\}$, where $\sqsubset$ is the strict prefix relation. Since s is winning, there is no infinite branch in T'. By Kőnig's lemma, this means that T' is finite and by definition all maximal elements (with regards to $\sqsubseteq$, the prefix relation) of T' belong to $\mathcal{F}$. T' can then serve as the memory of a finite-memory winning strategy $s_f = (T', \sigma, \mu, m_0)$ defined by:

- for $t \in T' \setminus \mathcal{F}$, $\sigma(t) = s(t)$;
- for $t \in T' \cap \mathcal{F}$, we set $\sigma(t)$ as any action $a \in A$;
- for $t \in T'$ and $b \in B$, $\mu(t, (\sigma(t), b)) = t(\sigma(t), b)$ if $t \notin \mathcal{F}$ and $\mu(t, (\sigma(t), b)) = t$ if $t \in \mathcal{F}$;
- $m_0 = \varepsilon$.

The strategy s_f works by simply following s alongside the branches of T' until it reaches a history in $\mathcal{F}$, and is thus winning. ◀

4.2 Proof for closed sets

We now focus on the study of the case where the winning set W is a closed set. In that case, the plays ρ such that $\rho \in W$ are precisely the plays for which all finite prefixes h are such that $W_h \neq \emptyset$. As a consequence, any strategy playing non-losing actions for Player 1 is a winning strategy: such a strategy only generates histories h in Γ, and in particular $W_h \neq \emptyset$.

Furthermore, if $\subseteq$ induces a well partial order on the partial winning sets, then:

($*$) there exists a finite subset M of Γ such that for all $h \in \Gamma$ there exists $m \in M$ such that $W_m \subseteq W_h$;

($**$) any play ρ has two finite prefixes ρ_0 and ρ_1 such that $\rho_0 \sqsubset \rho_1$ and $W_{\rho_0} \subseteq W_{\rho_1}$.

These two observations are the basis for two different approaches to prove the next lemma.

▶ **Lemma 9.** *If W is a closed set, if $\subseteq$ induces a well partial order on $(W_h)_{h \in \Gamma}$ and if $\varepsilon \in \Gamma$, then Player 1 has a finite-memory winning strategy from ε.*

Proofs using the two approaches. Consider indeed a game (A, B, W) where W is a closed set, $\subseteq$ induces a well partial order on the partial winning sets associated with the winning histories and such that ε is a winning history. We consider a strategy s that is a winning strategy for Player 1.

The first approach, derived from observation $(*)$, consists in building a finite-memory strategy (M, σ, μ, m_0) with memory set M in the following way:

- for $m \in M$, we let $\sigma(m)$ be any non-losing action from m,
- for $m \in M$ and $b \in B$, since $\sigma(m)$ is non-losing we know that $m(\sigma(m), b) \in \Gamma$, which means that there exists $m' \in M$ such that $W_{m'} \subseteq W_{m(\sigma(m),b)}$; we then let $\mu(m, (\sigma(m), b)) = m'$;
- finally $m_0 \in M$ is chosen such that we have $W_{m_0} \subseteq W_\varepsilon$.

Informally, we have as our memory the set M which contains under-approximations for all winning sets induced by the histories of Γ. We use the transition function to maintain an under-approximation of the "real" induced winning set associated to the current history, and play according to this under-approximation. By Corollary 7, this ensures that we always play a non-losing action, which is enough to guarantee the win because W is a closed set.

The second approach is derived from observation $(**)$. Consider the winning strategy s and its associated strategic tree T_s. Along every infinite branch ρ of T_s, there exist two histories h, h' such that $h \sqsubset h'$ and $W_h \subseteq W_{h'}$. Consider then the tree T_s^f obtained by pruning T_s along these histories: $T_s^f = \{h' \in T_s \mid \forall h \in T_s, h \sqsubset h' \Rightarrow W_h \nsubseteq W_{h'}\}$. By Kőnig's lemma, T_s^f is a finite tree. We call $\mathcal{P}$ the set $\{h \in T_s \mid h \notin T_s^f \text{ and } h' \sqsubset h \Rightarrow h' \in T_s^f\}$ of the minimal elements (with regards to the prefix relation) of T_s that do not belong to T_s^f. We then build a finite-memory strategy (M', σ, μ, m_0) in the following way:

- $M' = T_s^f$
- for $m \in M'$, we let $\sigma(m) = s(m)$,
- for $m \in M'$ and $(a, b) \in A \times B$ such that $a = \sigma(m) = s(m)$,
  - if $m(a, b) \in T_s^f$ then we let $\mu(m, (a, b)) = m(a, b)$,
  - else by construction we have $m(a, b) \in \mathcal{P}$ and there exists $m' \in T_s^f$ such that $m' \sqsubset m(a, b)$ and $W_m \subseteq W_{m(a,b)}$; we then let $\mu(m, (a, b)) = m'$,
- finally $m_0 = \varepsilon$.

Informally, this approach consists in playing according to s until we reach a history whose induced winning set is bigger than one we already met. We then forget the current history and continue playing as if we were in the history with the smaller induced winning set. This second approach also ensures that the memory consists of an under-approximation of the "real" induced winning set, and hence by Corollary 7 it guarantees that the resulting strategy is non-losing, and thus winning since W is a closed set. ◀

4.3 Limitations to the above approaches

Until now, we have studied the lowest levels of the Hausdorff difference hierarchy, focusing on the cases where the winning sets belongs to Λ_1, the open sets, and K_1, the closed sets. We will explain later how to handle the case for Λ_2 and for now turn our attention to K_2, as it proves pivotal to the understanding of our method.

The sets in K_2 are the sets that can be written as the union of a closed set and an open set. Informally, this means that Player 1 can win in two different ways, by ensuring that either the generated play lies in the closed set or they reach a history which belongs to the generating family of the open set.

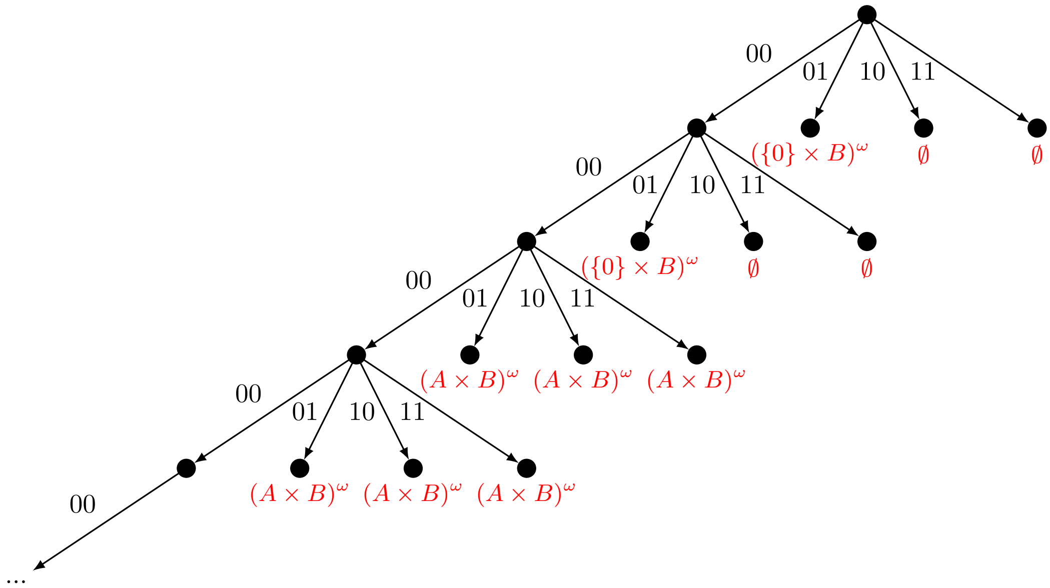

Figure 1 A game for which the naive algorithm does not work. For the sake of concision, the pairs of actions in $A \times B$ are written as two-letter words. In red is the partial winning set of the corresponding history.

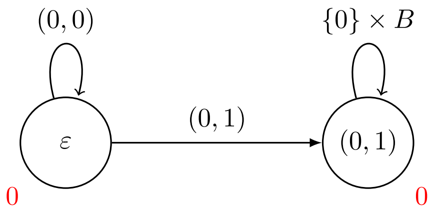

Figure 2 The finite-memory strategy generated by approaches $(*)$ and $(**)$ for the game represented in Figure 1. Each state is labeled by the history associated to it, and in red is the action associated to that state.

The first condition is akin to a safety objective (Player 1 manages to never go out of a certain region) while the second condition is akin to a reachability objective (Player 1 meets a certain given condition at a finite time and it suffices to ensure the win). As shown by the following example, the two simple approaches we detailed previously for closed sets do not suffice here:

▶ **Example 10.** Consider the game (A, B, W) with $A = B = \{0, 1\}$ and $W = (0, 0)^*(0, 1)(\{0\} \times B)^\omega + (0, 0)^2(0, 0)^*(\{1\} \times B + A \times \{1\})(A \times B)^\omega$. This game is described in Figure 1. In other words, Player 1 has two ways to win:

- either the players play $(0, 1)$ or $(0, 0)(0, 1)$ and Player 1 then only has to play action 0 forever;
- or the players play $(0, 0)$ twice and then one player plays action 1, reaching a point where all possible continuations are winning for Player 1.

As W can be expressed via a regular expression, it induces finitely many partial winning sets, which means that $\subseteq$ trivially induces a well partial order over said partial winning sets. Moreover, W can be expressed as the union of an open set and a closed set (the closed set corresponds to the first item above, while the open set corresponds to the second item), and hence belongs to the Hausdorff difference hierarchy (more precisely it belongs to K_2).

Moreover, one can easily check that $W_\varepsilon \subseteq W_{(0,0)}$. As a consequence, both approach $(*)$ and approach $(**)$ yield the finite-memory strategy described in Figure 2. This strategy is not winning for Player 1, as if Player 2 always plays action 0 it will generate the play $(0,0)^\omega$, which does not belong to W.

4.4 Proofs for sets in K_2

To better understand how the proof works in the general case, we propose here to study the basic case of sets in K_2. As we have seen previously, the two approaches that worked well for the case where the winning set is a closed set do not suffice in that case. Nevertheless, we prove the following result:

▶ **Theorem 11.** *If W is in K_2, if $\subseteq$ induces a well partial order on $(W_h)_{h\in\Gamma}$ and if $\varepsilon \in \Gamma$, then Player 1 has a finite-memory winning strategy from ε.*

Let us suppose then that W is in K_2: as already mentioned, W is the union of a closed set C and an open set O. We let $\mathcal{F}$ be the generating family of O and denote by $\mathrm{Pref}(C)$ the set of histories which have at least one continuation in C, that is $\mathrm{Pref}(C) = \{h \in \mathcal{H} \mid \exists\rho, h\rho \in C\}$.

As a preliminary observation, recall we already know how to handle the case when the winning set is open. The method also works well for the general case when Player 1 is able to reach O by herself (that is, she have a winning strategy for O). We also know that finding a finite-memory non-losing strategy for Player 1 is always possible (see for instance the method $(*)$ for the case where the winning set is closed). As a consequence, a simple method one would be tempted to try would be the following:

- follow some non-losing strategy as long as the current history belongs to $\mathrm{Pref}(C)$;
- as soon as we detect we have left $\mathrm{Pref}(C)$, play some finite-memory winning strategy to reach a history in $\mathcal{F}$ (this is possible because if we have played in a non-losing fashion so far and the current history does not belong to $\mathrm{Pref}(C)$, then the only way to win from there is to produce a play that belongs to O).

This method should produce a finite-memory winning strategy, however it relies on the assumption that one is able to detect whether or not the current history belongs to $\mathrm{Pref}(C)$ using only finite memory. This assumption does not rely on any solid ground, which makes this method incorrect. We propose another construction of a finite-memory winning strategy, which does not need to detect when the current history stops belonging to $\mathrm{Pref}(C)$, but which ensures that $\mathcal{F}$ will be reached if it were the case (despite not knowing it).

Consider indeed a history $\overline{h}$ in $\Gamma \cap \mathrm{Pref}(C)$ and a history h in Γ such that $h \notin \mathrm{Pref}(C)$ and $W_{\overline{h}} \subseteq W_h$. We call $TC(\overline{h}, h)$ the set $\{hl \mid \overline{h}l \in \mathrm{Pref}(C)\}$ consisting of the finite continuations from $\overline{h}$ that belong to $\mathrm{Pref}(C)$, but rooted in h. Notice that for all infinite continuations ρ such that $\overline{h}\rho \in C$, we have $\overline{h}\rho \in W$, which means that $h\rho \in W$ (since $W_{\overline{h}} \subseteq W_h$) and hence that $h\rho \in O$ since $h \notin \mathrm{Pref}(C)$ (which means that all winning continuations of h belong to O because they cannot belong to C). We thus know that $TC(\overline{h}, h)$ contains a family of histories $\mathcal{F}_{\overline{h},h}$ included in $\mathcal{F}$ (or in the case where h itself has a strict prefix in $\mathcal{F}$, we set $\mathcal{F}_{\overline{h},h} = \{h\}$) and such that all infinite branches of $TC(\overline{h}, h)$ have a finite prefix in $\mathcal{F}_{\overline{h},h}$, and by Kőnig's lemma we know this family is finite. We call $\mathrm{depth}(\overline{h}, h)$ the maximal length of the elements in $\mathcal{F}_{\overline{h},h}$. Intuitively, this means that, if the current history were h but Player 1 only knew of its under-approximation $\overline{h}$, she could ensure the win by following a play whose finite prefixes l were such that $\overline{h}l \in \mathrm{Pref}(C)$ for $\mathrm{depth}(\overline{h}, h)$ steps. This however still requires to compute the value of $\mathrm{depth}(\overline{h}, h)$ and hence to know of h. However, as stated by the following lemma, the value of $\mathrm{depth}(\overline{h}, h)$ for all eligible h is bounded.

▶ **Lemma 12.** *For all* $\overline{h}$ *in* $\mathrm{Pref}(C)$, $\{depth(\overline{h}, h) \mid h \in \Gamma, h \notin \mathrm{Pref}(C), W_{\overline{h}} \subseteq W_h\}$ *is bounded.*

The proof of this lemma can be found in appendix [3], and makes use of the well partial order hypothesis.

For $\overline{h}$ in $\Gamma \cap \mathrm{Pref}(C)$, we will then call $\mathrm{depth}(\overline{h})$ the upper bound of $\mathrm{depth}(\overline{h}, h)$ for h meeting the criteria described above. The idea is the following: if the current history is $h \in \Gamma$, but Player 1 only knows of its under-approximation $\overline{h}$, and then plays some finite continuation l of length $\mathrm{depth}(\overline{h})$ (which is independent of h) such that $\overline{h}l \in \mathrm{Pref}(C)$, then she ensured the win as hl has a finite prefix in $\mathcal{F}$. This is formally stated in the following lemma:

▶ **Lemma 13.** *Let* $\overline{h} \in \mathrm{Pref}(C)$ *and* $h \notin \mathrm{Pref}(C)$ *such that* $W_{\overline{h}} \subseteq W_h$. *Let* $\rho \in (A \times B)^\omega$ *such that for all finite prefixes* l *of* ρ *such that* $|l| \leq depth(\overline{h})$ *we have* $\overline{h}l \in \mathrm{Pref}(C)$. *Then* $h\rho \in O$.

Proof. Let l be the finite prefix of ρ of length $\mathrm{depth}(\overline{h})$. As $\mathrm{depth}(\overline{h}, h) \leq \mathrm{depth}(\overline{h})$ we know that hl has a prefix in $\mathcal{F}$, hence the result. ◀

Consider now a finite family $(h_i)_{i \in I}$ of histories in $\Gamma \setminus \mathrm{Pref}(C)$ such that for all $h \in \mathrm{Pref}(C)$ there exists $i \in I$ such that $W_{h_i} \subseteq W_h$. For all $i \in I$ there exists a finite-memory decision machine $(M_i, \sigma_i, \mu_i, m_i)$ associated with a of finite-memory strategy $(s_i)_{i \in I}$ such that s_i wins from h_i. As a consequence, for all $h \in \Gamma \setminus \mathrm{Pref}(C)$ there exists $i \in I$ such that s_i wins from h. Consider also a finite family $(\overline{h}_j)_{j \in J}$ of histories in $\Gamma \cap \mathrm{Pref}(C)$ indexed by $J \subseteq \mathbb{N}$ such that for all histories $\overline{h}$ in $\mathrm{Pref}(C) \cap \Gamma$ there exists $j \in J$ such that $W_{\overline{h}_j} \subseteq W_{\overline{h}}$. For all $j \in J$, let $T_j = \{\overline{h}_j l \mid |l| \leq \mathrm{depth}(\overline{h}_j)\}$. Up to renaming, we can suppose that the T_j's are disjoint from one another. We build our finite-memory winning strategy $s = (M, \sigma, \mu, m_0)$ in the following way:

- $M = \cup_{i \in I} M_i \cup \cup_{j \in J} T_j$;
- for $m \in M_i$ we let $\sigma(m) = \sigma_i(m)$;
- for $t \in T_j$ we let $\sigma(t)$ be any non-losing action from t;
- for $m \in M_i$ and $(a, b) \in A \times B$ we let $\mu(m, (a, b)) = \mu_i(m, (a, b))$;
- for $t \in T_j$ and $(a, b) \in A \times B$ such that $a = \sigma(t)$:
  - if $t(a, b) \in T_j$ then $\mu(t, (a, b)) = t(a, b)$;
  - else if $t(a, b) \in \Gamma \setminus \mathrm{Pref}(C)$ then there exists $i \in I$ such that s_i wins from $t(a, b)$: we let $\mu(t, (a, b)) = m_i$;
  - else if $t(a, b) \in \Gamma \cap \mathrm{Pref}(C)$ then there exists $j \in J$ such that $W_{\overline{h}_j} \subseteq W_{t(a,b)}$ and we let $\mu(h, (a, b)) = \overline{h}_j$;
- $m_0 = \overline{h}_j$ where $j \in J$ is such that $W_{\overline{h}_j} \subseteq W_\varepsilon$.

We prove this finite-memory strategy is winning for Player 1. To this end, let us first show that for all compatible histories h such that $\mu(h) \in T_j$ for some $j \in J$, the memory state $\mu(h)$ provides an under-approximation of the winning set induced by h:

▶ **Lemma 14.** *If* $\mu(h) \in T_j$ *for some* $j \in J$ *then we have* $W_{\mu(h)} \subseteq W_h$.

Proof. The proof is by induction on h. First we have $\mu(\varepsilon) = m_0$ and per the definition $W_{m_0} \subseteq W_\varepsilon$. Now consider $h \in \mathcal{H}$ such that $\mu(h) \in T_j$ for some $j \in J$ and $W_{\mu(h)} \subseteq W_h$. Let $(a, b) \in A \times B$ such that $\mu(h(a, b)) \in T_{j'}$ for some $j' \in J$. Then,

- if $\mu(h)(a, b) \in T_j$ then we have $\mu(h(a, b)) = \mu(h)(a, b)$ and the desired result follows by Lemma 6,

- else we must have $\mu(h)(a,b) \in \Gamma \cap \mathrm{Pref}(C)$ (else we would not have $\mu(h(a,b)) \in T_{j'}$) and $\mu(h(a,b)) = \overline{h}_{j'}$ with j' such that $W_{\overline{h}_{j'}} \subseteq W_{\mu(h)(a,b)}$, and $W_{\mu(h)(a,b)} \subseteq W_{h(a,b)}$ once again by Lemma 6, which ensures the result. ◀

As a consequence of Lemma 14, when h is such that $\mu(h) \in T_j$ for some $j \in J$ then we have $W_{\mu(h)} \subseteq W_h$. As a consequence, for all $(a,b) \in A \times B$ we have $W_{\mu(h)(a,b)} \subseteq W_{h(a,b)}$. This ensures that if $\mu(h)(a,b) \in \Gamma \setminus \mathrm{Pref}(C)$ and the strategy s_i wins from $\mu(h)(a,b)$ then s_i also wins from $h(a,b)$, which explains why our finite-memory strategy is winning. Formally, we have the following lemma:

▶ **Lemma 15.** *If $\rho \in (A \times B)^\omega$ is compatible with s and there exists $k \in \mathbb{N}$ and $i \in I$ such that $\mu(\rho_{\leq k}) = m_i$ then $\rho \in W$.*

Proof. Without loss of generality, we can suppose that k is the smallest integer such that $\mu(\rho_{\leq k}) \notin \cup_{j\in J} T_j$. We want to prove there exists a history h such that $W_h \subseteq W_{\rho_{\leq k}}$ and s_i is winning from h, as this will yield the desired result. Our candidate for h is $\mu(\rho_{\leq k-1})(a,b)$ where $(a,b) = \rho_k$.

- By Lemma 14 we know that $W_{\mu(\rho_{\leq k-1})} \subseteq W_{\rho_{\leq k-1}}$, and thus $W_{\mu(\rho_{\leq k-1})(a,b)} \subseteq W_{\rho_{\leq k-1}}$.
- Furthermore, we know per the definition of s that s_i is winning from $\mu(\rho_{\leq k-1}(a,b))$ since $\mu(\rho_{\leq k}) = m_i$.

Finally, a trivial induction shows that for any $\beta \in B^\omega$ we have $\mathrm{out}(\rho_{\leq k}, s, \beta) = \mathrm{out}(\rho_{\leq k}, s_i, \beta)$, and since s_i is winning from $\rho_{\leq k}$ we have $\rho \in W$. ◀

Finally, with the help of Lemma 14 and Lemma 15, we can prove the following, which concludes the proof of Theorem 11.

▶ **Lemma 16.** *s is winning from ε.*

Proof. Let us consider $\rho \in (A \times B)^\omega$ compatible with s. We want to show that $\rho \in W$. If there exists $k \in \mathbb{N}$ such that $\mu(\rho_{\leq k}) \in \cup_{i\in I} M_i$ then Lemma 15 suffices to conclude. Suppose then that for all $k \in \mathbb{N}$ we have $\mu(\rho_{\leq k}) \in \cup_{j\in J} T_j$. If $\rho \in C$ then obviously we have $\rho \in W$, so let us suppose that there exists $n_0 \in \mathbb{N}$ such that $\rho_{\leq n_0} \notin \mathrm{Pref}(C)$. Due to the construction of s, there exists some $n_1 \geq n_0$ and some $j \in J$ such that $\mu(\rho_{\leq n_1}) = \overline{h}_j$. By Lemma 14 we then have $W_{\mu(\rho_{\leq n_1})} \subseteq W_{\rho_{\leq n_1}}$. Notice that we also have $\rho_{\leq n_1} \notin \mathrm{Pref}(C)$. Finally, let l be the prefix of $\rho_{>n_1}$ of length $\mathrm{depth}(\overline{h}_j)$. By construction and since we do not have $\mu(\rho_{\leq k}) \in \cup_{i\in I} M_i$ for any k, for all $k \leq \mathrm{depth}(\overline{h}_j)$ we have $\mu(\rho_{\leq n_1+k}) = \overline{h}_j l_{\leq k} \in \mathrm{Pref}(C)$, and hence by application of Lemma 13 we can conclude that $\rho \in O$ and thus $\rho \in W$. ◀

We illustrate our method on the game described in Figure 1:

▶ **Example 17.** Consider once again the game represented in 1. We recall here that we have $A = B = \{0,1\}$ and $W = (0,0)^*(0,1)(\{0\} \times B)^\omega + (0,0)^2(0,0)^*(\{1\} \times B + A \times \{1\})(A \times B)^\omega$. We let $C = (0,1)(\{0\} \times B)^\omega + (0,0)(0,1)(\{0\} \times B)^\omega$ and $O = (0,0)^2(0,0)^*[(0,1) + (1,0) + (1,1)](A \times B)^\omega$, and we have $W = C \cup O$. One can check easily that C is a closed set and O is an open set generated by the family of histories $\mathcal{F} = (0,0)^2(0,0)^*[(0,1) + (1,0) + (1,1)]$.

The histories in $\mathrm{Pref}(C)$ are of three different types: ε, whose induced winning set is W, $(0,0)$, whose induced winning set is included in W, and all the other histories in $\mathrm{Pref}(C)$, whose induced winning set is $(\{0\} \times B)^\omega$. We choose two histories in $\mathrm{Pref}(C)$ that provide under-approximations for the open sets induced by all elements of $\mathrm{Pref}(C)$: ε and $(0,1)$. Except for histories which already have a prefix in $\mathcal{F}$, no history out of $\mathrm{Pref}(C)$ induces a winning set that includes $W_{(0,1)}$, which means that $\mathrm{depth}((0,1)) = 0$. However W_ε is included in all winning sets induced by the histories in $(0,0)^2(0,0)^*$. Since $(0,0)^2(0,0)^*(0,0)(0,1) \subseteq \mathcal{F}$,

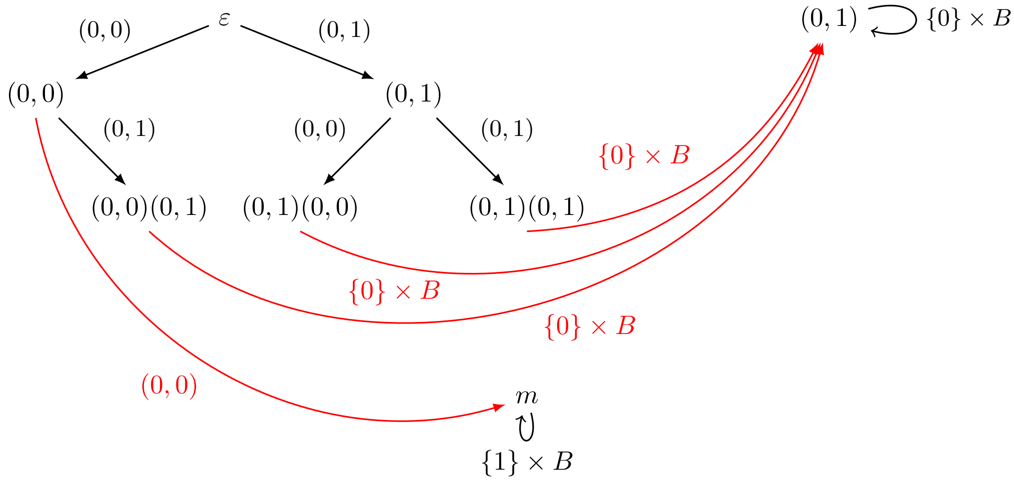

Figure 3 The finite-memory strategy for the game represented in Figure 1.

we have depth$(\varepsilon) = 2$. The only continuation of length 2 from ε that goes out of Pref(C) is $(0,0)(0,0)$, and the finite-memory strategy we chose that reaches O from $(0,0)(0,0)$ is one where Player 1 always plays action 1.

4.5 Proof for sets in Λ_θ

We study now the case where $W \in \Lambda_\theta$ for an ordinal $\theta > 1$ and Theorem 5 is true for all winning sets belonging to Λ_η or K_η for all $\eta < \theta$. As always, we suppose that Player 1 has a winning strategy from ε and we want to show she has a finite-memory winning strategy.

As $W \in \Lambda_\theta$, there exists a family of sets $(S_i)_{i\in I}$ such that $W = \biguplus_{i\in I} S_i$ and for each i there exists $\theta_i < \theta$ such that $S_i \in \Lambda_{\theta_i}$ or $S_i \in K_{\theta_i}$. Furthermore, there exists a disjoint family of open sets $(O_i)_{i\in I}$ such that for each $i \in I$ we have $S_i \subseteq O_i$. Finally, for each i the set O_i is generated by a family of histories $\mathcal{F}_i$.

Consider then a winning strategy s for Player 1 and let T_s be its induced strategic tree. Consider the tree $T'_s = T_s \setminus \{h \in \mathcal{H} \mid \exists i \in I, \exists f \in \mathcal{F}_i, f \sqsubset h\}$. Since s is winning, all infinite branches of T_s belongs to O_i for some i, and hence T'_s does not have any infinite branch. By Kőnig's lemma, this means that T'_s is a finite tree. Let us consider the set $\mathcal{L}$ of maximal elements (with regards to the prefix relation) in T'_s. By construction all histories h in $\mathcal{L}$ are such that there exists a unique (because the O_i's are disjoint from one another) $i \in I$ such that $h \in \mathcal{F}_i$. This means that the winning plays which have h as a prefix are included in S_i and hence for $h \in \mathcal{L}$ we have $W_h = h^{-1}(S_i \cap \mathrm{cyl}(h))$. Since both Λ_{θ_i} and K_{θ_i} are closed under intersection with a cylinder (see [3]) we know that $W_h \in \Lambda_{\theta_i}$ or $W_h \in K_{\theta_i}$ (depending on whether S_i belongs to Λ_{θ_i} or K_{θ_i}). Moreover, since all histories in $\mathcal{L}$ belong to T_s and s is winning we know that $\mathcal{L} \subseteq \Gamma$. This means that we have all the hypotheses we need to apply the induction hypothesis to any $l \in \mathcal{L}$ (namely, $l \in \Gamma$ and $W_l \in \Lambda_\eta$ or $W_l \in K_\eta$ for some $\eta < \theta$), and hence for all $l \in \mathcal{L}$ there exists a finite-memory strategy $s_l = (M_l, \sigma_l, \mu_l, m_l)$ that wins from l (up to renaming, we suppose that the M_l's are disjoint from one another and from T_s).

We will build a finite-memory strategy $s_f = (M, \sigma, \mu, m_0)$ for Player 1 in the following way:

- $M = T'_s \cup \uplus_{l \in \mathcal{L}} M_l$;
- $\sigma(m) = s(m)$ for $m \in T'_s$, and $\sigma(m) = \sigma_l(m)$ for $m \in M_l$;
- $\mu(m, (a, b)) = m(a, b)$ if $m \in T'_s$ and $m(a, b) \neq l$ for all $l \in \mathcal{L}$ (where $a = \sigma(m)$);
- $\mu(m, (a, b)) = m_l$ if $m \in T'_s$ and $m(a, b) = l$ for some $l \in \mathcal{L}$ (where $a = \sigma(m)$);
- $\mu(m, (a, b)) = \mu_l(m(a, b))$ if $m \in M_l$;
- $m_0 = \varepsilon$ if $\varepsilon \neq l$ for all $l \in \mathcal{L}$, and $m_0 = m_l$ if there exists $l \in \mathcal{L}$ such that $l = \varepsilon$.

The idea behind this construction is the following: we follow the winning strategy s until we reach some $l \in \mathcal{L}$, from which we know the strategy s_l is winning. It remains only to emulate s_l from this point onwards to guarantee the win. The formal proof that this strategy is winning can be found in [3].

4.6 Proof for sets in K_θ

The full induction step for K_θ is more complex than for K_2, in term of overall structure as well as detail-level technicalities. Especially, it involves nested inductions on the Hausdorff difference hierarchy. We do not present it here, but provide a detailed proof in appendix [3].

5 Tightness of the result

We explore the tightness of our result. In particular, the winning sets of the two games in Example 18 below are in Π^0_2 and Σ^0_2, respectively, just above Δ^0_2 in the Borel hierarchy; the two games satisfy the well partial order assumption and Player 1 has winning strategies, but no finite-memory winning strategies. Note that the example in Π^0_2 is harder to define and deal with than the one in Σ^0_2. (See details in [3].)

▶ **Example 18.** For the counter-example in Π^0_2, let A be a finite set of at least two elements and w a disjunctive sequence on $A \times \{0\}$. We define the labeling function $l : (A \times \{0\})^* \to \{0, 1\}$ in the following way: $l(h) = 1$ if and only if there exists h_0 and h' such that $h = h_0 h'$ and h' is the longest factor of h that is also a prefix of w. Let then W be the set defined by $\rho \in W$ if and only if infinitely many prefixes h of ρ are such that $l(h) = 1$ and consider the game $(A, \{0\}, W)$.

For the counter-example in Σ^0_2, let A be a finite set of at least two elements and w an irregular word (i.e. a word with infinitely many different suffixes) in $(A \times \{0\})^\omega$. Let $W = \{\rho \in (A \times \{0\})^\omega \mid \exists h \in (A \times \{0\})^*, \rho = h\rho_0$ where ρ_0 is a suffix of $w\}$. W is the set of sequences which have a suffix in common with w. Consider then the game $(A, \{0\}, W)$.

We also considered a relaxation of the well partial order assumption, but found a counter-example with a closed winning set. (See details in [3].)

Finally, we studied the following statement "given a two-player game (A, B, W) where W belongs to the Hausdorff difference hierarchy and such that $\subseteq$ induces a well partial order on the induced winning sets for Player 1, if Player 2 has a winning strategy for ε then is it the case that he also has a finite-memory winning strategy?". When W is a closed set (and hence its complement an open set), the answer is obviously yes, but we found a counter-example where $W \in \Lambda_2$.

6 Conclusion

To conclude, we have proven the existence of finite-memory winning strategies under certain conditions on the winning set for Player 1. These conditions are met for well studied games such as energy games [10] or games where the winning condition is a Boolean combination of reachability and safety objectives, which makes our result a generalization of known results on the topic. This result relies on descriptive set theory, in particular representation of sets in Δ^0_2.

We have also studied the tightness of our result. The results we currently have in this direction encourage us to think that our hypotheses are tight and that weakening them is no easy task. In the future, we want to extend these tightness results by exploring other possible hypotheses, as well as study infinitely branching games, when the action sets of the players are not finite.

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