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Research article

Non-Lipschitz heterogeneous reaction with a p-Laplacian operator

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Abstract: The intention along this work is to provide analytical approaches for a degenerate parabolic equation formulated with a p-Laplacian operator and heterogeneous non-Lipschitz reaction. Firstly, some results are discussed and presented in relation with uniqueness, existence and regularity of solutions. Due to the degenerate diffusivity induced by the p-Laplacian operator (specially when $\nabla u = 0$, or close zero), solutions are studied in a weak sense upon definition of an appropriate test function. The p-Laplacian operator is positive for positive solutions. This positivity condition is employed to show the regularity results along propagation. Afterwards, profiles of solutions are explored specially to characterize the propagating front that exhibits the property known as finite propagation speed. Finally, conditions are shown to the loss of compact support and, hence, to the existence of blow up phenomena in finite time.

Keywords: p-Laplace; non-Lipschitz; degenerate diffusion; heterogeneous reaction; blow-up; critical exponent
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1. Introduction and model description

The p-Laplacian operator for Parabolic Differential Equations started to be studied more than 50 years ago with the seminal research in [5], [6] and [7]. The p-Laplacian operator has been considered in applied sciences to model different phenomena. For instance in [23], the author presents different analytical and numerical approaches to fluid mechanics problems involving a p-Laplacian. The use of a p-Laplacian operator is ubiquitous and can be found in other problems of mathematical interest. This is the case of the Emden-Fowler equation discussed in [18] or the Einstein-Yang-Mills equation discussed in [16]. In addition, other interesting discussions hold, such as the existence and nonexistence of solutions in nonlinear elasticity [13], glaciology [15] and petroleum extraction [14].

The mathematical treatment of p-Laplacian operators is a source of research currently. In [26], the authors proposed an inverse power method for the computation of the first homogeneous eigenpair in the p(x)-Laplacian problem. Another interesting study can be consulted in [25] where the authors considered the existence of a global attractor together with the existence of a family of semicontinuously attractors in relation with the parameters introduced in the non-linear diffusion.

Furthermore, the p-Laplacian has been studied in bounded domains: In [33], the authors study the existence of solutions for the p-Laplacian homogeneous problem with non-linear boundary conditions based on topological and variational principles. In [35], the authors study resonance problems for an arbitrary eigenvalue showing existence of weak solutions with a variational approach.

The p-Laplacian has been faced with finite elements as well. In [34], the authors proposed the discrete formulation of a p-Laplacian based on the union of triangles to define the given domain. In addition, the authors in [36] developed a model based on p-Laplacian finite elements to simulate the sub-diffuse interface phenomena in phase transitions resulting in microstructure evolution.

Some problems related with degenerate diffusion to a Porous Medium Equations can be read in [9], [11] where concepts such as finite speed of propagation and support are introduced. Note that the formulation of problems with a non-linear diffusion is an important aspect in applied sciences. The p-Laplacian operator can be seen as particular kind of extensive diffusion within the wider category of non-linear diffusion operators. As set of examples, non-linear diffusion problems emerge in areas like biology through the classical Keller and Segel model [21] that is intended to study the cells chemotaxis. Note that more recent studies can be consulted in [1], [8], [29], [32]. Furthermore, in problems where porosity is relevant, the non-linear diffusion needs to be considered. For instance, in [28] a Darcy law is proposed based on a non-linear diffusion for nanofluids. Other applications of non-linear diffusion and fluids can be found to enhance the accuracy in modelling blood coagulation effects in vessels [4] or to model the peristaltic transport in a Jeffrey fluid [19].

In addition, the proposed problem is formulated with an exponent in the reaction, u^q. Note that De Pablo and Vázquez studied an exponent reaction to a non-linear diffusion in [10] and [9] to conclude that for $q \in (0, 1)$, there is not local in time blow up phenomena. Note that along the presented analysis, the heterogeneous reaction introduced will induce finite time blow-up. This result permits to generalize the scope treated in [10] and [9].

The proposed model to be analyzed can be justified from a biological perspective. In particular, it can be used to model the behaviour of an invasive population that has the intention to conquer a small area (for instance a castle site in the Middle Age). To this end, consider that the zone to be invaded is characterized as $|x| \to 0^+$. In this hostile zone, the growing rate on the invasive $u_t \sim 0$. Having $u_t \sim 0$ with sufficiently small initial data in the hostile area represents the slow movement of the invasive along such hostile zone. This can be identified with a finite propagation feature. Far from the hostile zone $|x| >> 1$, the growing rate is high. At the beginning the invasive will prosper and grow quickly (this is the intention for u^q, $0 < q < 1$ in the reaction) as the media is full of nutrients. The proper behaviour of the invasive far from the hostile zone make the invasive to send additional efforts to the hostile zone. Such invasive bahaviour is intended to be modelled by a diffusivity that depends on the spatial gradient in the invasive concentration. Indeed, the invasive will move to the hostile zone in areas where the overcrowd pressure is the highest, in other words, in areas of higher gradients. Then, the diffusivity is model as $D(u) = |\nabla u|^{p-2}$, $p > 2$. As a consequence of the presented discussions, the proposed problem P is as follows:

$$u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u) + |x|^\rho u^q, \\ p > 1, \; p \neq 2, \; 0 < q < 1, \\ u_0(x) > 0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \tag{1.1}$$

The methodology followed along this work can be applied similarly to the case of $q \geq 1$ although the non-Lipschitz approximation introduces some further general techniques. Our aim is based on showing the regularity of P with certain truncations to bound the heterogeneous and non-Lipschitz reaction. Afterwards, the original problem is recovered to show the uniform convergence of solutions. In addition, two solutions (minimal and maximal) are shown to exist when the non-Lipschitz predominates over the diffusion. Due to the diffusion degeneracy, the problem is analyzed within the weak formulation scope, although solutions are explored making use of precise solutions profiles afterwards. Further, the propagating support is characterized, the property of finite propagation shown and solutions are precisely determined along such support. Finally, a critical exponent q^* is explored to segregate the finite support evolution from the existence of a blow up phenomena in finite time (i.e. loss of compact support).

2. Materials and methods

Firstly, as supporting material, the following propositions are introduced based on the work in [27], but with certain differences to account for the functional spaces dealt along the presented study:

Proposition 1. *Admit the homogeneous p-Laplacian equation $u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad p > 2$ with a finite mass pulse of size N_δ at the origin $u(x,0) = N_\delta \delta(x), \quad N_\delta > 0$. Then the following Barenblatt solution holds ([3] and [27]):*

$$w(x,t) = t^{-\alpha_0} \left(C - k|\eta|^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}}, \tag{2.1}$$

with

$$\eta = x\, t^{-\alpha_0/d}, \quad \alpha_0 = \left(p - 2 + \frac{p}{d} \right)^{-1}, \quad k = \frac{p-2}{p} \left(\frac{\alpha_0}{d} \right)^{\frac{1}{p-1}}. \tag{2.2}$$

Note that $C = c(d,p) N^\beta$, such that $\beta = \frac{p(p-2)\alpha_0}{d(p-1)}$ and c can be determined with the mass preserving condition $\int w(x,t)dx = N_\delta$.

Proposition 2. *Given $u_0(x) \in L^1(\mathbb{R}^d)$, there exists a time $T(\|u_0\|_1)$ and a weak solution to the homogeneous p-Laplacian equation $u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ existing in $0 < t \leq T(\|u_0\|_1)$ such that:*

$$T(\|u_0\|_1) = C_0(d,p) \|u_0\|_1^{-(p-2)}, \tag{2.3}$$

In addition, the following holds for compact supported initial data:

$$\|u(\cdot,t)\|_1 \leq C_1(d,p) \|u_0\|_1, \tag{2.4}$$

$$|u(x,t)| \leq C_2(d,p) t^{-\alpha_0} R^{p(p-2)} \|u_0\|_1^{\frac{p\alpha_0}{d}}, \quad |x| \leq R, \tag{2.5}$$

$$|\nabla u(x,t)| \leq C_3(d,p) t^{\frac{-(d+1)\alpha_0}{d}} R^{\frac{2}{p-2}} \|u_0\|_1^{\frac{2\alpha_0}{d}}, \quad |x| \leq R. \tag{2.6}$$

Both $u(x,t)$ and $\nabla u(x,t)$ are Hölder continuous along their domain of existence.

Note that in [27], the authors studied a more general initial condition $u_0 \in L^1_{loc}$ for which a dedicated norm is introduced to control the growing condition of solutions. Note that along the presented analysis, it is the intention to study the solutions behaviour along compact spatial subset B_R with $R < \infty$. As a consequence, the norm introduced in Section 2 of [27] is equivalent to study the L^1 norm along compact subsets with finite R. Based on Proposition 2 results, the following Corollary applies:

Corollary 1. *Any solution to the homogeneous p-Laplacian equation with compact support initial data preserves the support as it expands. In addition, the following assessment holds to assess the time at which u stays at null.*

This Corollary can be further precised by standard operations in the expressions outlined in Proposition 2, indeed:

$$0 < C_2 t^{-\alpha_0} R^{p(p-2)} \|u_0\|_1^{\frac{p\alpha_0}{d}} \leq C_1 \|u_0\|_1 = C_1\, C_0^{\frac{1}{p-2}} T^{\frac{-1}{p-2}}. \tag{2.7}$$

As a result, u stays null for a time given by the interval:

$$0 < t \leq T = \left(C_1\, C_0^{\frac{1}{p-2}} C_2^{-1} R^{-p(p-2)} \|u_0\|_1^{-\frac{p\alpha_0}{d}} \right)^{\frac{1}{\alpha_1}}, \quad \alpha_1 = \frac{p/d}{(p-2)\left(p-2+\frac{p}{d}\right)}, \tag{2.8}$$

along the compact subset B_R with finite $R > 0$.

The asymptotic behaviour of solutions to (1.1) is firstly discussed. The aim is to show that any nonnegative weak solution to (1.1) behaves asymptotically as the fundamental solution given in Proposition 1 for an appropriate test function.

Before proceeding further, the following statement of result,in the form of proposition, is required. Note that the result is shown afterwards in Section 3.2.

Proposition 3. *Admit, ad hoc, that a particular L^1_{loc} solution to* (1.1) *is given as:*

$$u = |x|^{\frac{p}{1-q}} (1-q)^{\frac{1}{1-q}} t^{\frac{1}{1-q}}. \tag{2.9}$$

Further insights on the last proposition are shown in Section 3.2.

The methodology followed along the present work is based on introducing a general functional space together with integrable functions along compact subsets. Based on this approach, supported by appropriate test functions when required, assymptotic analysis, existence, uniqueness and comparison of solutions are introduced. Afterwards, solutions are explored making use of a radially symmetric selfsimilar solution and comparison argument with known compactly supported solutions.

3. Discussions

3.1. On existence, uniqueness and comparison

Firstly, consider the following theorem:

Theorem 1. *Admit that $u(x,t) \geq 0$ is a solution to* (1.1) *with $u_0(x) \in L^1$, then:*

$$\lim_{t\to\infty} |u(x,t) - w(x,t)| = 0, \tag{3.1}$$

uniformly in $\mathbb{R}^d$.

Proof. The degenerate diffusion and reaction dealt require to consider a weak formulation. To this end, admit the test function $\Sigma \in C^\infty([0,T] \times \mathbb{R}^d) \cap L^\infty([0,T]\,,\, W^{1,p}(\mathbb{R}^d))$ so that the following holds (see [34]):

$$\int_{\mathbb{R}^d} u(t)\,\Sigma(t) = \int_{\mathbb{R}^d} u(0)\,\Sigma(0) + \int_0^t \int_{\mathbb{R}^d} [u\,\Sigma_t - |\nabla u|^{p-2}\,\nabla u \cdot \nabla\Sigma + |x|^\rho u^q \Sigma] ds. \tag{3.2}$$

Similarly the fundamental solution, w, admits a similar weak formulation:

$$\int_{\mathbb{R}^d} w(t)\,\Sigma(t) = \int_{\mathbb{R}^d} w(0)\,\Sigma(0) + \int_0^t \int_{\mathbb{R}^d} [w\,\Sigma_t - |\nabla w|^{p-2}\,\nabla w \cdot \nabla\Sigma] ds. \tag{3.3}$$

Consider the following particular form of test function:

$$\Sigma(x,t) = \frac{e^{-j(t)}}{(1+|x|^2)^\sigma}, \tag{3.4}$$

where $j(t) \geq 0$ is Hölder continuous and $\sigma > 0$ shall be chosen to ensure asymptotic convergence for the involved integrals to appear later.

As w is a fundamental solution, the first variation vanishes asymptotically provided the chosen test function vanishes at infinity for certain values of σ, i.e.:

$$\int_{\mathbb{R}^d} |\nabla w|^{p-2}\,\nabla w \cdot \nabla\Sigma \leq \int_{\mathbb{R}^d} C_3^{p-1} t^{\frac{-(d+1)\alpha_0(p-1)}{d}} |x|^{2+\frac{2}{p-2}} \|u_0\|_1^{\frac{2\alpha_0(p-1)}{d}} |-\sigma| \frac{e^{-j(t)} 2|x|}{(1+|x|^2)^{\sigma+1}}. \tag{3.5}$$

The vanishing condition at infinity requires to consider values of σ (named as σ_0) complying with $\sigma_0 \geq \frac{p}{p-2}$.

Making (3.2) minus (3.3) and using the asymptotic result in (3.5):

$$\int_{\mathbb{R}^d} (u-w)(t)\,\Sigma(t) \leq \int_{\mathbb{R}^d} (u-w)(0)\,\Sigma(0) + \int_0^t \int_{\mathbb{R}^d} [(u-w)\,\Sigma_t - |\nabla u|^{p-2}\,\nabla u \cdot \nabla\Sigma + |x|^\rho u^q \Sigma] ds. \tag{3.6}$$

Now, according to Proposition 2 and Proposition 3, the following holds:

$$\int_0^t \int_{\mathbb{R}^d} |x|^\rho u^q \Sigma ds \leq \int_0^t \int_{\mathbb{R}^d} \left(|x|^\rho C_2^q t^{-q\alpha_0} |x|^{qp(p-2)} \|u\|_1^{qp\alpha_0/N} + |x|^{\frac{\rho q}{1-q}} (1-q)^{\frac{q}{1-q}} t^{\frac{q}{1-q}} \right) \frac{e^{-j(t)}}{(1+|x|^2)^\sigma} ds \tag{3.7}$$

Asymptotic convergence in the previous expression is given provided the σ exponent compensates both terms on the left hand integral. To this end, it suffices to consider the following value of σ named as σ_1:

$$\sigma_1 \geq 2\ \max\left\{ \frac{\rho + qp(p-2)+1}{2}, \frac{1+q(\rho-1)}{2(1-q)} \right\} \tag{3.8}$$

Additionally, based in Proposition 3 for a particular solution and in Proposition 2 to bound the homogeneous, the asymptotic convergence of $\int_0^t \int_{\mathbb{R}^d} |\nabla u|^{p-2}\,\nabla u \cdot \nabla\Sigma ds$ is given for:

$$\sigma_2 \geq 2\ \max\left\{ \frac{\rho + qp(p-2)+1}{2}, \frac{(\rho-1+q)(p-1)+q-1}{2} \right\} \tag{3.9}$$

Consequently, for convergence in (3.6), it suffices to take:

$$\sigma \geq \max\{\sigma_0, \sigma_1, \sigma_2\}. \tag{3.10}$$

Coming to (3.6) and considering the contraction condition in L_1 (see [2]):

$$\int_{\mathbb{R}^d} (u-w)(t)\,\Sigma(t) \leq \|u-w\|_1(t)\|\Sigma\|_\infty(t) \leq \|u_0-w_0\|_1\|\Sigma\|_\infty(t) + \int_0^t \varepsilon(s)\|u-w\|_1(s)ds, \tag{3.11}$$

where $\varepsilon(s) = \max_{\mathbb{R}^d}\{\Sigma_t\}$ that exists upon the smooth conditions defined for the test function.

Note that w_0 is a finite mass function, i.e. $\int w_0 = M$. Admit a sequence $\{u_{0,n}\}$ with compact support that approximates to u_0, then $0 \leq u_{0,n} \leq u_0$, $n \in \mathbb{N}$. Each element in the sequence is finite mass, $\int u_{0,n} = M_n$, with $M_n \nearrow M$.

Admit now the following scaling [27]:

$$T_\kappa u = \frac{1}{\kappa^{\alpha_0}} u\left(\frac{x}{\kappa^{\alpha_0/d}}, \frac{t}{\kappa}\right), \tag{3.12}$$

that permits to normalize, and hence, compare any evolution within same orders. Such normalization is applied to the sequence of $\{u_n\}$ resulting upon evolution of $\{u_{0,n}\}$.

$$\|T_\kappa(u_n) - T_\kappa(w)\|_1(t)\|\Sigma\|_\infty(t) \leq \|u_{0,n} - w_0\|_1\|\Sigma\|_\infty(t) + \int_0^t \varepsilon(s)\|T_\kappa(u_n) - T_\kappa(w)\|_1(s)ds. \tag{3.13}$$

Consider now:

$$\|u_{0,n} - w_0\|_1 = \int u_{0,n} - \int w_0 = M_n - M, \tag{3.14}$$

and integrating over mean values, the following holds:

$$(\|\Sigma\|_\infty(t) - t\,\varepsilon(\tau))\|T_\kappa(u_n) - T_\kappa(w)\|_1(t) \leq (M_n - M)\|\Sigma\|_\infty(t). \tag{3.15}$$

for an appropriate $0 < \tau \leq t$. Then, uniformly as $n \to \infty$, $M_n \nearrow M$ and hence $u_n \nearrow u$:

$$(\|\Sigma\|_\infty(t) - t\,\varepsilon(\tau))\|T_\kappa(u) - T_\kappa(w)\|_1(t) \leq 0. \tag{3.16}$$

In the asymptotic approach $1 << \kappa << t$, then:

$$T_\kappa(u) = \varepsilon(\kappa) u\left(\frac{x}{\kappa^{\alpha_0/d}}, \frac{t}{\kappa}\right), \quad T_\kappa w = \varepsilon(\kappa) w\left(\frac{x}{\kappa^{\alpha_0/d}}, \frac{t}{\kappa}\right), \tag{3.17}$$

where $|\varepsilon(\kappa)| << 1$, then (3.16):

$$u(x', t') = w(x', t'), \quad x' \to 0, \quad t' \to 0, \tag{3.18}$$

where $(x',\ t')$ are the re-scaled variables.

Similarly for $\kappa \to 0$:

$$u(x'', t'') = w(x'', t''), \quad x'' \to \infty, \quad t'' \to \infty, \tag{3.19}$$

showing the asymptotic behaviour of spatially distributed solutions to the fundamental one as stated in the theorem postulation.

□

The next intention is to study the bound properties of solutions. Previously, some regularity conditions shall be studied when the problem is formulated with a test function as defined in (3.4). Admit $R >> |x| >> 1$ so that the ball B_R is considered and the following problem (P^{Σ}) is defined accordingly in $[0, T] \times B_R$:

$$u\,\Sigma_t + |\nabla u|^{p-2}\,\nabla u \cdot \nabla\Sigma + |x|^{\rho} u^q \Sigma = 0. \tag{3.20}$$

The ball B_R defines a sufficiently smooth border ∂B_R such that the test function meets the zero flux condition:

$$\nabla\Sigma \cdot \pi = 0, \tag{3.21}$$

π normal unitary vector along ∂B_R. Note that initially any solution is bounded as $u_0(x) \in L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d)$. Then and to ensure that the problem P^{Σ} is tractable with available results, the following truncatures are introduced for $0 < \vartheta < \infty$:

$$u_{\vartheta} = \left\{ \begin{array}{c} u\,,\ \ |u| < \vartheta \\ \vartheta\,,\ \ |u| \geq \vartheta \end{array} \right\}, \quad |x|^{\rho}_{\vartheta} = \left\{ \begin{array}{c} |x|^{\rho}\,,\ \ |x| < \delta \\ \vartheta^{\rho}\,,\ \ |x| \geq \vartheta \end{array} \right\}. \tag{3.22}$$

In addition, admit the following Lipschitz approximation to the reaction term:

$$U_{\delta}(f) = \left\{ \begin{array}{c} \delta^{(q-1)} f\,,\ \ 0 \leq f < \ \delta \\ f^q\,,\ \ f \geq \delta \end{array} \right\}, \tag{3.23}$$

In the same manner a truncation can be defined for (3.23) leading to:

$$U_{\delta,\vartheta}(f_{\vartheta}) = \left\{ \begin{array}{c} \delta^{(q-1)} f_{\vartheta}\,,\ \ 0 \leq f_{\vartheta} < \ \delta \\ f^q_{\vartheta}\,,\ \ f_{\vartheta} \geq \delta \end{array} \right\}, \quad f_{\vartheta} = \left\{ \begin{array}{c} f\,,\ \ |f| < \vartheta \\ \vartheta\,,\ \ |f| \geq \vartheta \end{array} \right\}, \quad f^q_{\vartheta} = \left\{ \begin{array}{c} f^q\,,\ \ |f| < \vartheta^{\frac{1}{q}} \\ \vartheta^{\frac{1}{q}}\,,\ \ |f^q| \geq \vartheta^{\frac{1}{q}} \end{array} \right\}. \tag{3.24}$$

Note that the truncating variable ϑ is considered similarly for u and $|x|$, for the sake of simplicity. Although such procedure requires some further dimensional checks, it is sufficient for our purposes. As a consequence, the following truncated problem P^{Σ}_{ϑ} is proposed:

$$u_{\vartheta}\,\Sigma_t + |\nabla u_{\vartheta}|^{p-2}\,\nabla u_{\vartheta} \cdot \nabla\Sigma + |x|^{\rho}_{\vartheta} U_{\delta,\vartheta}\Sigma = 0. \tag{3.25}$$

with the same boundary like and initial conditions to P^{Σ}. Solutions $\Sigma(x, t)$ to problem P^{Σ}_{ϑ} exist and are unique. This can be shown based on already available results. Once a solution is positive in a certain region, a maximum principle holds for the p-Laplacian operator [27]. In addition the reaction term is monotone with regards to Σ and has regular bounded coefficients. Based on these two premises, existence and uniqueness of $\Sigma-$solutions can be shown by a monotonic approximation for positive operators in [24]. In addition, similar existence and uniqueness results in a weak sense follow from [12] and [27].

Theorem 2. *Any solution to problem P (1.1), $u(x, t)$, is bounded in $t \in (0, \infty)$ for compact subsets B_R, $0 < R < \infty$.*

Proof. Consider $\varepsilon \in \mathbb{R}^+$ so that the following cut off function (σ) is proposed [11]:

$$\begin{array}{ll} \sigma_{\varepsilon} \in C_0^{\infty}(\mathbb{R}^d), & 0 \leq \sigma_{\varepsilon} \leq 1, \\ \sigma_{\varepsilon} = 1, \quad x \in B_{R-\varepsilon}, & \sigma_{\varepsilon} = 0, \quad x \in \{\mathbb{R}^d - B_{R-\varepsilon}\}. \end{array} \tag{3.26}$$

So that:

$$|\nabla\sigma_\varepsilon| \le \frac{c_1}{\varepsilon}, \quad |\Delta\sigma_\varepsilon| \le \frac{c_1}{\varepsilon^2}. \tag{3.27}$$

Consider (3.25) so that the following holds in $\{(t,x) \in (0,\tau) \times B_R\}$ for $0 < \tau < t < \infty$:

$$\int_0^\tau \int_{B_R} u_\vartheta\, \Sigma_t \sigma_\varepsilon + \int_0^\tau \int_{B_R} |\nabla u_\vartheta|^{p-2}\, \nabla u_\vartheta \cdot \nabla\Sigma\, \sigma_\varepsilon + \int_0^\tau \int_{B_R} |x|_\vartheta^\rho u_\vartheta^q \Sigma\, \sigma_\varepsilon = 0 \tag{3.28}$$

Both Σ, Σ_t and σ_ε are continuous smooth functions with null support at infinity. Consequently, a solution is bounded provided the integral $\int_0^\tau \int_{B_R} u_\vartheta\, \Sigma_t \sigma_\varepsilon$ is finite. To this end, the following integrals are assessed making use of Proposition 2 and considering the fact that any solution converges asymptotically to the homogeneous for the value of σ given in Theorem 1. For $\vartheta >> 1$:

$$\begin{aligned}
\int_0^\tau \int_{B_R} |\nabla u_\vartheta|^{p-2}\, \nabla u_\vartheta \cdot \nabla\Sigma\, \sigma_\varepsilon &\le \int_0^t \int_{\mathbb{R}^d} C_3^2\, t^{-\frac{(d+1)\alpha_0(p-1)}{d}} R^{\frac{2(p-1)}{p-2}} \|u_0\|_1^{\frac{2\alpha_0}{d}(p-1)} \sigma \frac{e^{-j(t)} 2R}{(1+R^2)^\sigma}\, \sigma_\varepsilon \\
&\le \int_0^\tau \int_{B_R} C_3^2\, t^{-\frac{(d+1)\alpha_0(p-1)}{d}} R^{\frac{2(p-1)}{p-2}} \|u_0\|_1^{\frac{2\alpha_0}{d}(p-1)} \sigma \frac{e^{-j(t)} 2R}{(1+R^2)^\sigma}\, \frac{c_1}{\varepsilon} \\
&\le \int_0^\tau \int_{B_{R\to\infty}} 2c_1\, C_3^2\, t^{-\frac{(d+1)\alpha_0(p-1)}{d}} R^{\frac{2(p-1)}{p-2} - 2\sigma} \|u_0\|_1^{\frac{2\alpha_0}{d}(p-1)} \sigma e^{-j(t)}\,,
\end{aligned} \tag{3.29}$$

for ε close to R. The test function follows from (3.4), where $|\nabla\Sigma| \sim |x|^{1-2\sigma}$ for $|x| < R$. The balance of the previous integral along compact subsets in B_R requires to select:

$$\sigma \ge \frac{1}{2}\frac{2p-1}{p-2}. \tag{3.30}$$

Similarly:

$$\int_0^\tau \int_{B_R} |x|_\vartheta^\rho u_\vartheta^q \Sigma\, \sigma_\varepsilon \le \int_0^\tau \int_{B_R} R^\rho c_2^q t^{-q\alpha_0} R^{qp(p-2)} \|u_0\|_1^{qp\alpha_0/d} R^{-2\sigma}. \tag{3.31}$$

The balance along compact subsets requires to select:

$$\sigma \ge \frac{1}{2}(\rho + qp(p-2) + 1), \tag{3.32}$$

Then, it suffices to admit:

$$\sigma \ge \max\left\{\frac{1}{2}\frac{2p-1}{p-2},\ \frac{1}{2}(\rho + qp(p-2) + 1)\right\} \tag{3.33}$$

Returning to (3.28)

$$\int_0^\tau \int_{B_R} u_\vartheta\, \Sigma_t \sigma_\varepsilon \le \left|\int_0^\tau \Lambda_1(R)\, t^{-\frac{(d+1)\alpha_0(p-1)}{d}}\, e^{-j(t)} dt\right| + \left|\int_0^\tau \Lambda_2(R)\, t^{-q\alpha_0} dt\right|, \tag{3.34}$$

where $0 < \Lambda_i < \infty$, $\forall R$, $i \in \{1,2\}$, $|\Lambda_i| << 1$ for $R \to \infty$ and Λ_i are defined upon integration along the compact set B_R.

Note that the right hand integrals in (3.34) are finite for any $0 < t < \tau < \infty$, leading to the bound of the left hand integral. Then and for $\vartheta >> 1$, it is possible to conclude on the bound of any weak solution $u(x,t)$.

□

The next intention is to show existence and uniqueness of solutions through a monotone argument. To this end, admit the following definitions:

Definition 1. *Given $G \in C([0,T], L^1_{loc}(\mathbb{R}^d)) \cap L^1([0,T], W^{1,p-1}_{loc}(\mathbb{R}^d))$, the following operator is defined:*

$$LG = G_t - \nabla \cdot (|\nabla G|^{p-2} \nabla G) = G_t - L_s G \tag{3.35}$$

The p-Laplacian induces the following diffusivity:

$$D(G) = |\nabla G|^{p-2}. \tag{3.36}$$

The defined non-linear operator L is a monotone operator whenever $G > 0$, i.e. for positive solutions, in the same manner as the classical single linear laplacian Δ. This is a well known result (see [2], [30] and [17] for additional recent references). This parabolic property of L in zones where $\nabla G > 0$ enables the following definition:

Definition 2. *Consider the upper solution $\hat{u}$ and the lower $\tilde{u}$ such that:*

$$\hat{u}, \tilde{u} \in C([0,T], L^1_{loc}(\mathbb{R}^d)) \cap L^1([0,T], W^{1,p-1}_{loc}(\mathbb{R}^d)). \tag{3.37}$$

Then $\hat{u}$ is an upper solution if:

$$L\hat{u} \geq |x|^\rho \, \hat{u}^q. \tag{3.38}$$

In the same manner, $\tilde{u}$ is a lower solution if:

$$L\tilde{u} \leq |x|^\rho \, \tilde{u}^q. \tag{3.39}$$

Admit $\tilde{u}(x,0) \leq u(x,0) \leq \hat{u}(x,0)$, with

$$\hat{u}(x,0) = u(x,0) + \mu, \;\; \tilde{u}(x,0) = u(x,0) - \mu, \tag{3.40}$$

$$0 < \mu < \min_{x \in \mathbb{R}^d} \{u(x,0)\}. \tag{3.41}$$

Note that the same upper and lower solutions definition can be applied with the truncatures u_ϑ, $|x|^\rho_\vartheta$ and $U_{\delta,\vartheta}$ as defined in (3.22) and (3.24). In fact and from now on, the different arguments shall be done with the defined truncatures. Note that the original solution is recovered for $\vartheta >> 1$ along compact subsets and making $\delta \leftarrow 0$. Consequently, the following problem $P_{\vartheta,\delta}$, with the associated truncatures, holds:

$$u_{\vartheta,t} = \nabla \cdot (|\nabla u_\vartheta|^{p-2} \nabla u_\vartheta) + |x|^\rho_\vartheta U_{\delta,\vartheta}, \tag{3.42}$$

with $u_0(x) > 0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. The positivity of the p-Laplacian in compact subsets B_R, $R < \vartheta$ together with the regularity in the independent term upon truncation permits to state on the following a-priory assessment ([20] and Ch. 3 [31]), $u_\vartheta(x,t) \in L^1(B_R \times (0,T))$. Similarly, any upper or lower solution to problem $P_{\vartheta,\delta}$ as per Definition 2 satisfy $\hat{u}_\vartheta(x,t)$, $\tilde{u}_\vartheta(x,t) \in L^1(B_R \times (0,T))$.

Admit, now, the sequence $(u^{(i)}_\vartheta, U^{(i)}_{\delta,\vartheta})_{i=0,1,2,...}$:

$$u^{(i)}_{\vartheta,t} = \nabla \cdot (|\nabla u^{(i)}_\vartheta|^{p-2} \nabla u^{(i)}_\vartheta) + |x|^\rho_\vartheta U^{(i-1)}_{\delta,\vartheta}, \quad i = 0, 1, 2... \tag{3.43}$$

Given the Lipschitz and bounded regularity in the independent term together with the parabolic regularity along compact subsets, the existence and uniqueness to (3.43) follow from already known results (refer to Ch. 3 [31] and Theorems 2.1 and 2.2, Ch. 7 [24]).

In addition, given the purely increasing behaviour of the regular truncation $U_{\delta,\vartheta}$, it is possible to conclude on the order preserving property, with the exponent (i), of the introduced sequence $(u_\vartheta^{(i)}, U_{\delta,\vartheta}^{(i)})_{i=0,1,2,...}$ (Ch. 12 of [24]). Then and returning to the defined upper and lower solutions, the following holds::

$$\tilde{u}_\vartheta \leq \tilde{u}_\vartheta^{(i+1)} \leq \hat{u}_\vartheta^{(i+1)} \leq \hat{u}_\vartheta, \quad \tilde{U}_{\delta,\vartheta} \leq \tilde{U}_{\delta,\vartheta}^{(i+1)} \leq \hat{U}_{\delta,\vartheta}^{(i+1)} \leq \hat{U}_{\delta,\vartheta}. \tag{3.44}$$

In addition (Ch. 12 of [24]),

$$\hat{u}_\vartheta^{(i)} \geq \hat{u}_\vartheta^{(i+1)}; \;\; \tilde{u}_\vartheta^{(i)} \leq \tilde{u}_\vartheta^{(i+1)}; \;\; \hat{U}_{\delta,\vartheta}^{(i)} \geq \hat{U}_{\delta,\vartheta}^{(i+1)}; \;\; \tilde{U}_{\delta,\vartheta}^{(i)} \leq \tilde{U}_{\delta,\vartheta}^{(i+1)}. \tag{3.45}$$

Indeed, the order preserving with (i) upper solution sequence $(\hat{u}_\vartheta^{(i)}, \hat{U}_{\delta,\vartheta}^{(i)})$ is non-increasing while the sequence of lower solutions $(\tilde{u}_\vartheta^{(i)}, \tilde{U}_{\delta,\vartheta}^{(i)})$ is non-decreasing. As a consequence, the following holds:

Theorem 3. *Admit positive solutions along compact subsets under the operator parabolicity to $P_{\vartheta,\delta}$* (3.42)*, i.e. $u > 0$ in $B_R \times (0, T)$, $R < \vartheta$. Then, the sequence defined for the upper and lower solutions, $(\hat{u}_\vartheta^{(i)}, \hat{U}_{\delta,\vartheta}^{(i)})$ and $(\tilde{u}_\vartheta^{(i)}, \tilde{U}_{\delta,\vartheta}^{(i)})$ lead to regular monotonic convergence in the upper and lower solutions as per Definition 2.*

Proof. Both, the lower and upper solutions are considered in a weak sense. To this end, admit the test function defined in (3.4) along $B_R \times (0, \tau)$, $\;0 < \tau < t < \infty,\;$ so that the following holds:

$$\int_{B_R} \hat{u}_\vartheta^{(i)}(t)\,\Sigma(t) = \int_{B_R} (u_0 + \mu(i))\,\Sigma(0) + \int_0^\tau \int_{B_R} [\hat{u}_\vartheta^{(i)}\,\Sigma_t + L_s(\hat{u}_\vartheta^{(i)}) \cdot \nabla\Sigma + |x|_\vartheta^\rho \hat{U}_{\delta,\vartheta}^{(i)}\,\Sigma] ds, \tag{3.46}$$

$$\int_{B_R} \tilde{u}_\vartheta^{(i)}(t)\,\Sigma(t) = \int_{B_R} (u_0 - \mu(i))\,\Sigma(0) + \int_0^\tau \int_{B_R} [\tilde{u}_\vartheta^{(i)}\,\Sigma_t + L_s(\tilde{u}_\vartheta^{(i)}) \cdot \nabla\Sigma + |x|_\vartheta^\rho \tilde{U}_{\delta,\vartheta}^{(i)}\,\Sigma] ds, \tag{3.47}$$

such that $\lim_{i\to\infty} \mu = 0$. Then:

$$\begin{aligned} \int_{B_R} \lim_{i\to\infty} \hat{u}_\vartheta^{(i)}(t)\,\Sigma(t) = &\int_{B_R} (u_0 + \lim_{i\to\infty} \mu(i))\,\Sigma(0) \\ &+ \int_0^\tau \int_{B_R} [\lim_{i\to\infty} \hat{u}_\vartheta^{(i)}\,\Sigma_t + \lim_{i\to\infty} L_s(\hat{u}_\vartheta^{(i)}) \cdot \nabla\Sigma + \; |x|_\vartheta^\rho \lim_{i\to\infty} \hat{U}_{\delta,\vartheta}^{(i)}\,\Sigma] ds. \end{aligned} \tag{3.48}$$

Identically for $\tilde{u}_\vartheta^{(i)}$

$$\begin{aligned} \int_{B_R} \lim_{i\to\infty} \tilde{u}_\vartheta^{(i)}(t)\,\Sigma(t) = &\int_{B_R} (u_0 - \lim_{i\to\infty} \mu(i))\,\Sigma(0) \\ &+ \int_0^\tau \int_{B_R} [\lim_{i\to\infty} \tilde{u}_\vartheta^{(i)}\,\Sigma_t + \lim_{i\to\infty} L_s(\tilde{u}_\vartheta^{(i)}) \cdot \nabla\Sigma + \; |x|_\vartheta^\rho \lim_{i\to\infty} \tilde{U}_{\delta,\vartheta}^{(i)}\,\Sigma] ds. \end{aligned} \tag{3.49}$$

As shown in Theorem 1, any solution behaves asymptotically as the fundamental one. In addition, the operator L_s refers to the stationary solution for which the first variation vanishes (see [22]) provided the chosen test function vanishes at infinity:

$$\int_0^\tau \int_{\mathbb{R}^d} L_s(\tilde{u}_\vartheta^{(i)}) \cdot \nabla \Sigma = 0, \quad \int_0^\tau \int_{\mathbb{R}^d} L_s(\hat{u}_\vartheta^{(i)}) \cdot \nabla \Sigma = 0, \tag{3.50}$$

for solution converging to the homogeneous one under the scope of Theorem 1.

After assessing the difference:

$$\int_{B_{R\to\infty}} \lim_{i\to\infty}(\hat{u}_\vartheta^{(i)} - \tilde{u}_\vartheta^{(i)})(t)\,\Sigma(t) = \int_0^\tau \int_{B_{R\to\infty}} [\lim_{i\to\infty}(\hat{u}_\vartheta^{(i)} - \tilde{u}_\vartheta^{(i)})\,\Sigma_t + \\ + |x|_\vartheta^\rho \lim_{i\to\infty}(\hat{U}_{\delta,\vartheta}^{(i)} - \tilde{U}_{\delta,\vartheta}^{(i)})\,\Sigma]ds. \tag{3.51}$$

Note that both $\hat{U}_{\delta,\vartheta}^{(i)}$, $\tilde{U}_{\delta,\vartheta}^{(i)}$ are Lipschitz:

$$|\hat{U}_{\delta,\vartheta}^{(i)} - \tilde{U}_{\delta,\vartheta}^{(i)}| \le \delta^{(q-1)}|\hat{u}_\vartheta^{(i)} - \tilde{u}_\vartheta^{(i)}|, \tag{3.52}$$

being $\delta^{(q-1)}$ the Lipschitz constant. Making $\delta \leftarrow 0$ with the condition $R \to \infty < \vartheta$, the truncatures lead to the lower and upper solutions as per Definition 2. Then, (3.51) is re-written as:

$$\int_{B_{R\to\infty}} \lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})(t)\,\Sigma(t) \le \int_0^\tau \int_{B_{R\to\infty}} [\lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})\,\Sigma_t \\ + \delta^{q-1}\,|x|^\rho \lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})\,\Sigma]ds. \tag{3.53}$$

Now, for a sufficiently large t:

$$\lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})(t)\,\Sigma(t) \le [\lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})\,\Sigma_t + \delta^{q-1}\,|x|^\rho \lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})\,\Sigma]\,t, \tag{3.54}$$

$$\lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})(t)\,\Sigma(t) \le \lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})[\,\Sigma_t \\ + \delta^{q-1}\,|x|^\rho\,\Sigma]\,t \le \lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})C_6 t. \tag{3.55}$$

Note that the bounding constant C_6 holds as the test function is defined as per $\Sigma \in C^\infty([0,T] \times \mathbb{R}^d) \cap L^\infty([0,T]\,,\, W^{1,p}(\mathbb{R}^d))$. Then:

$$\lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})\,\Sigma \le \lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})C_6 t\,, \quad \lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})(\Sigma - C_6\,t) \le 0. \tag{3.56}$$

Note that for

$$\Sigma < C_6\,t \;\; \to \;\; \lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)})(\Sigma - C_6\,t) \ge 0, \tag{3.57}$$

then, compliance to (3.56) requires:

$$\lim_{i\to\infty}(\hat{u}^{(i)} - \tilde{u}^{(i)}) = 0, \tag{3.58}$$

Or equivalently, $\hat{u}^{(i)} \searrow \tilde{u}^{(i)}$ and $\tilde{u}^{(i)} \nearrow \hat{u}^{(i)}$, $i \to \infty$ as intended to show. In addition, each of the $\hat{u}^{(i)}$ and $\tilde{u}^{(i)}$ verifies the regularity conditions in (3.37).

□

Theorem 4. *Given positive initial data, the positivity of the p-Laplacian Operator (see Definition 1) and admitting the lower solutions (given by $(u_\vartheta, U_{\delta,\vartheta})$ along $B_R \times (0,T)$) are positive, then $(u_\vartheta, U_{\delta,\vartheta})$ coincide with the upper solutions, i.e. solutions are unique.*

Proof. Consider the upper set $(\hat{u}_\vartheta, \hat{U}_{\delta,\vartheta})$ with the initial data (3.40). Note that in this occasion, the lower set is defined with initial condition $u(x,0) = u_0(x)$ and the governing equation as per (3.43). Given a test function as defined in (3.4), considering the expression (3.43) for $(\hat{u}_\vartheta, \hat{U}_{\delta,\vartheta})$ and $(u_\vartheta, U_{\delta,\vartheta})$, the expression (3.5) for $R >> 1$ and making the difference:

$$\int_{B_{R\to\infty}} (\hat{u}_\vartheta - u_\vartheta)(t)\,\Sigma(t) \le \int_0^\tau \int_{B_{R\to\infty}} [(\hat{u}_\vartheta - u_\vartheta)\,\Sigma_t + |x|_\vartheta^\rho(\hat{U}_{\delta,\vartheta} - U_{\delta,\vartheta})\,\Sigma]ds. \tag{3.59}$$

Admit the particular expression for the test function:

$$\Sigma(|x|, s) = e^{K_1(T-s)}(1+|x|^2)^{-\sigma}, \tag{3.60}$$

for finite K_1 and σ both positive. Note that:

$$\Sigma_t = -K_1\Sigma(x,s), \tag{3.61}$$

hence:

$$\begin{aligned}(\hat{u}_\vartheta - u_\vartheta)\,\Sigma_t + |x|_\vartheta^\rho(\hat{U}_{\delta,\vartheta} - U_{\delta,\vartheta})\,\Sigma \\ \le -K_1(\hat{u}_\vartheta - u_\vartheta)\Sigma + |x|_\vartheta^\rho(\hat{U}_{\delta,\vartheta} - U_{\delta,\vartheta})\,\Sigma.\end{aligned} \tag{3.62}$$

Note that $K_1 > 0$ so that:

$$(-K_1)(\hat{u}_\vartheta - u_\vartheta)\Sigma \le 0. \tag{3.63}$$

Then, the expression (3.59) is rewritten as:

$$0 \le \int_{B_{R\to\infty}} (\hat{u}_\vartheta - u_\vartheta)(t)\,\Sigma(t) \le \int_0^\tau \int_{B_{R\to\infty}} |x|_\vartheta^\rho(\hat{U}_{\delta,\vartheta} - U_{\delta,\vartheta})\,\Sigma ds. \tag{3.64}$$

Considering the Lipschitz condition in (3.52) and making $\delta \leftarrow 0$, $\vartheta >> 1$ along compact subsets, the following holds:

$$0 \le \int_{B_{R\to\infty}} (\hat{u} - u)(t)\,\Sigma(t) \le \int_0^\tau \int_{B_{R\to\infty}} \delta^{q-1}\,|x|^\rho(\hat{u} - u)\,\Sigma ds. \le \delta^{q-1}R^{\rho+1} \int_0^\tau \int_{B_{R\to\infty}} (\hat{u} - u)\,\Sigma ds. \tag{3.65}$$

Making the time derivative:

$$\frac{d}{dt}\int_{B_{R\to\infty}} (\hat{u} - u)(t)\,\Sigma(t) \le \delta^{q-1}R^{\rho+1} \int_{B_{R\to\infty}} (\hat{u} - u)(t)\,\Sigma(t). \tag{3.66}$$

Upon standard resolution:

$$0 \le \int_{B_{R\to\infty}} (\hat{u} - u)(t)\,\Sigma(t) \le (\hat{u}_0 - u_0)\,e^{\delta^{q-1}R^{\rho+1}\,t}. \tag{3.67}$$

Note that $\hat{u}_0 \to u_0$ for a sufficiently small μ (see expression (3.40)), then the last inequality holds provided $\hat{u} \to u$ leading to show the theorem postulations. Such convergence of lower solutions to upper solutions permits to show uniqueness of solutions as stated.

□

The next intention consists on showing the order preserving property:

Lemma 1. *Admit two different initial conditions, such that* $u_{0,1}(x,0) \geq u_{0,2}(x,0)$*, then the resulting solutions upon them are order preserving, i.e.* $u_1(x,t) \geq u_2(x,t)$ *in* $B_R \times (0,T)$*.*

Proof. Considering the involved truncatures, the already discussed weak formulation and the expression (3.5) for $R >> 1$ the following holds (Note that the term $\sim$ refers to expression sufficiently close for our purposes):

$$\begin{aligned}\int_{B_{R\to\infty}} (u_{1,\vartheta}-u_{2,\vartheta})(t)\,\Sigma(t) &\sim \int_{B_{R\to\infty}} (u_{0,1}-u_{0,2})\,\Sigma(0) \\ &+ \int_0^\tau \int_{B_{R\to\infty}} [(u_{1,\vartheta}-u_{2,\vartheta})\,\Sigma_t + |x|_\vartheta^\rho (U_{1,\delta,\vartheta}-U_{2,\delta,\vartheta})\,\Sigma] ds,\end{aligned} \tag{3.68}$$

$$\begin{aligned}\int_{B_{R\to\infty}} (u_{1,\vartheta}-u_{2,\vartheta})(t)\,\Sigma(t) - \int_0^\tau \int_{B_{R\to\infty}} [(u_{1,\vartheta}-u_{2,\vartheta})\,\Sigma_t + |x|_\vartheta^\rho (U_{1,\delta,\vartheta}-U_{2,\delta,\vartheta})\,\Sigma] ds \\ \sim \int_{B_{R\to\infty}} (u_{0,1}-u_{0,2})\,\Sigma \geq 0,\end{aligned} \tag{3.69}$$

where Σ is as per (3.60). Note that $\Sigma_t = -K_1\Sigma$. Now, consider (3.69) together with the Gronwall's inequality, so that:

$$\begin{aligned}\int_{B_{R\to\infty}} (u_{1,\vartheta}-u_{2,\vartheta})(t)\,\Sigma(t) + \int_{B_{R\to\infty}} M_1(u_{1,\vartheta}-u_{2,\vartheta})K_1\Sigma - \int_{B_{R\to\infty}} M_2(u_{0,1}-u_{0,2})K_1\Sigma \\ - \int_{B_{R\to\infty}} M_3|x|_\vartheta^\rho (U_{1,\delta,\vartheta}-U_{2,\delta,\vartheta})\,\Sigma + \int_{B_{R\to\infty}} M_4|x|_\vartheta^\rho (u_{0,1}-u_{0,2})\,\Sigma \geq 0,\end{aligned} \tag{3.70}$$

where $M_i > 0$, $i = \{1,2,3,4\}$ refer to the Gronwall's constants. Asymptotically for $|x| >> 1$ and recovering the original solution for $\delta \leftarrow 0$ and $\vartheta >> 1$ along compact subsets:

$$\begin{aligned}\int_{B_{R\to\infty}} (1 + M_1K_1 - \delta^{q-1}M_3|x|^\rho)(u_1-u_2)(t)\,\Sigma(t) \\ \geq \int_{B_{R\to\infty}} (M_2K_1 - M_4|x|^\rho)(u_{0,1}-u_{0,2})\,\Sigma \geq 0,\end{aligned} \tag{3.71}$$

where the constant K_1 shall be selected for each compact integration satisfying $1 + M_1K_1 > \delta^{q-1}M_3R^\sigma$ and $M_2K_1 > M_5R^\sigma$, i.e.

$$K_1 \geq \max\left\{\frac{\delta^{q-1}M_3R^\sigma - 1}{M_1}, \frac{M_4R^\sigma}{M_2}\right\}. \tag{3.72}$$

In return to (3.71) and based on the monotonic property of integrals and the positivity of the test function:

$$u_{0,1}(x,0) \geq u_{0,2}(x,0) \quad \rightarrow \quad u_1(x,t) \geq u_2(x,t), \tag{3.73}$$

as intended to show.

□

3.2. Behaviour of solutions

The reaction term is predominant over the p-Laplacian diffusion whenever $0 < u << 1$ as $q \in (0, 1)$. In this case, the non-Lipschitz condition in the reaction may induce the loss of uniqueness. Then a minimal solution is given by the null state, $u = 0$, while a maximal solutions is shown to be:

$$u_{\kappa}^{M} = |x|^{\frac{\rho}{1-q}}(1-q)^{\frac{1}{1-q}}(t+\kappa)^{\frac{1}{1-q}} \tag{3.74}$$

where κ is a time translation to the left to ensure the maximality of solution and can be assessed as:

$$\kappa = ||u_0||_{\infty}^{1-q} R^{\rho}(1-q), \tag{3.75}$$

where R represent the spatial ball amplitude over compact subsets.

To show the expression (3.74) is a maximal solution, admit that such solution is of the generic form $u_{\kappa}^{M} = |x|^{\mu} N (t+\kappa)^{\beta}$. After replacement in (1.1) and considering the leading reaction, the following values for the involved parameters hold:

$$\mu = \frac{\rho}{1-q}, \quad \beta = \frac{1}{1-q}, \quad N = (1-q)^{\frac{1}{1-q}}. \tag{3.76}$$

In addition and considering the diffusion in (1.1) as well, the following holds:

$$|x|^{\mu} N\beta(t+\kappa)^{\beta-1} = \mu^{p-1} N^{p-1}(t+\kappa)^{\beta(p-1)}(\mu-1)(p-1)|x|^{(\mu-1)(p-1)-1} + |x|^{\rho+\mu q} N^{q}(t+\kappa)^{\beta q}. \tag{3.77}$$

Once any solution between the null and the maximal is slightly positive, reaction predominates if:

$$\rho + \mu q > (\mu-1)(p-1) - 1 \;\rightarrow\; (1-q)(\rho+p) + \rho(q+1-p) > 0. \tag{3.78}$$

Consequently, the following criteria holds depending on the values to the involved problem parameters:

- $(1-q)(\rho+p) + \rho(q+1-p) < 0$.

  The p-Laplacian operator predominates over the reaction. In this case, solutions behaves asymptotically as the fundamental given in Proposition 1 where a finite propagation support is given.
- $(1-q)(\rho+p) + \rho(q+1-p) \geq 0$.

  The reaction predominates over the diffusion and two solutions have been shown to exist. One maximal given by (3.74) and a minimal given by the null state. Any other solution exists between the maximal and the minimal provided the translation variable κ is selected in accordance with expression (3.75).

In the case of a predominant diffusion, the following result aims to characterize the propagating support along compact subsets B_R.

Theorem 5. *Consider a local positive solution to* (1.1), *i.e.* $u(x,t) > 0$ *in a given point* $(x_0, t_0) \in B_R \times (0, T]$ *then:*

- *the solution* u *preserves the positive condition, i.e.* $u(x_0, t) \geq C_p(x_0)(t - t_0)^{-\beta_0}$, $t > t_0$, *where* β_0 *coincides with* α_0 *(see Proposition 1).*

- *the solution support evolves as:*

$$|x| = \left(\frac{1}{C_{spt}}\right)^{\frac{q-1}{p-q-1}} t^{\frac{q-1}{(p-2+p/d)d(p-q-1)}}, \tag{3.79}$$

$$C_{spt} = \left(\frac{C_1}{\left(\frac{p-2}{p}\right)\left(\frac{1}{d(p-2-p/d)}\right)}\right)^{\frac{p-1}{p}}, \quad C_1 = (-\beta_0 + \gamma d)^{\frac{p-2}{(q-1)(p-1)}}. \tag{3.80}$$

In addition, $u(x,t) > 0$ for $t > t_0$ along the inner region given by:

$$|x - x_0| < \left(\frac{1}{C_{spt}}\right)^{\frac{q-1}{p-q-1}} (t - t_0)^{\frac{q-1}{(p-2+p/d)d(p-q-1)}}. \tag{3.81}$$

Proof. The propagation properties are characterized with a lower regular asymptotic approximation. To this end, the following truncation to the reaction is defined:

$$g_{\phi,m} = m^{\rho} \min\left[u^q, \phi^{q-1} u\right], \quad m > 0, \quad \phi > 0. \tag{3.82}$$

Note that the original reaction term holds for $m \to \infty$ and $\phi \leftarrow 0$. Based on the truncation principle the following problem is proposed:

$$u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u) + g_{\phi,m}. \tag{3.83}$$

Note that $g_{\phi,m}$ is a Lipschitz function, then solutions to (3.83) exist and uniqueness holds [24], [12] and [27].

In the search of radially symmetric selfsimilar solutions of the form:

$$H(x,t) = t^{-\beta_0} f(|x|t^{\gamma}), \quad \xi = |x|t^{\gamma}. \tag{3.84}$$

Upon substitution in (3.83):

$$-\beta_0 t^{-\beta_0-1} f + \gamma \xi t^{-\beta_0-1} f' = t^{-\beta_0(p-1)} |\nabla f|^{p-2} (p-2)\Delta f + \frac{d-1}{\xi} |\nabla f|^{p-2} |\nabla f| + g_{\phi,m}, \tag{3.85}$$

where:

$$g_{\phi,m}(f,t) = m^{\rho} \min\left[f^q, \phi^{q-1} t^{\beta_0(q-1)} f\right], \tag{3.86}$$

or equivalently:

$$g_{\phi,m}(f,t) = m^{\rho} \min\left[t^{-\beta_0 q} f^q, \phi^{q-1} t^{-\beta_0} f\right]. \tag{3.87}$$

Note that both terms defined in (3.86) and (3.87) have the same intersection along $f = t^{\beta_0}\phi$. For the sake of simplicity, the coming assessment are done with the linear term f (3.86). In addition and for a sufficiently large time (to be precisely characterized), the following holds:

$$g_{\phi,m}(f,t) \geq m^{\rho}\, k\, f \tag{3.88}$$

so that k is selected as $k = m^{-\rho}(-\beta_0 + \gamma d)$ for convenience in solving (3.85).

Note that the selfsimilar profile $f(\xi)$ is a solution to the elliptic problem (3.85) at each selected time (for simplification and without loss of generality consider $t = 1$), then a minimal solution profile is obtained to the equation:

$$-\beta_0 f + \gamma\xi f' = |\nabla f|^{p-2}(p-2)\Delta f + \frac{d-1}{\xi}|\nabla f|^{p-2}\,|\nabla f| + (-\beta_0 + \gamma d)f. \tag{3.89}$$

As shown in Theorem 1, any solution to (3.89) behaves asymptotically as the fundamental solution given in Proposition 1, then a solution to (3.89) is of the form:

$$f(\xi) = \left(C - K|\xi|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \tag{3.90}$$

Note that the constant α_0 in Proposition 1 shall be replaced by β_0 as used along the present theorem.

In addition, note that the stated solution in (3.90) is valid for a sufficiently large time, so that inequality (3.88) holds, then:

$$m^\rho \min\left[f^q, \phi^{q-1}t^{\beta_0(q-1)}f\right] \geq m^\rho\, k\, f, \quad \min\left[f^q, \phi^{q-1}t^{\beta_0(q-1)}f\right] \geq \, k\, f. \tag{3.91}$$

Considering the sub-linear part:

$$\phi^{q-1}t^{\beta_0(q-1)} \geq m^{-\rho}(-\beta_0 + \gamma d). \tag{3.92}$$

Now, admit $\phi = 1/m$ to jointly assess both terms when $m \to \infty$ along compact subsets. This permits to recover the original problem. Then, an explicit expression for t_m is obtained from (3.92) as:

$$t_m = (-\beta_0 + \gamma d)^{\frac{-1}{\beta_0(1-q)}} m^{1/\beta_0} m^{\frac{\rho}{\beta_0(1-q)}}. \tag{3.93}$$

such that (3.90) is a lower solutions provided $t \geq t_m$.

Returning to (3.90), a global evolution for the maximum is given by making $\xi = 0$, so that the solution departs from the positive C:

$$u(x,t) = C^{\frac{p-1}{p-2}} t^{-\beta_0}. \tag{3.94}$$

From (3.91), a particular value for C is obtained by making $\xi = 0$:

$$\min\left[C^{\frac{(q-1)(p-1)}{p-2}}, \phi^{q-1}t^{\beta_0(q-1)}\right] \geq \, m^{-\rho}(-\beta_0 + \gamma d). \tag{3.95}$$

Then:

$$C = m^{\frac{\rho(p-2)}{(q-1)(p-1)}}(-\beta_0 + \gamma d)^{\frac{p-2}{(q-1)(p-1)}} = C_1(\beta_0, \gamma, d, p, q)\, m^{\frac{\rho(p-2)}{(q-1)(p-1)}}. \tag{3.96}$$

And recovering the spatial variable $|x|$:

$$C(x) = C_1(\beta_0, \gamma, d, p, q)\, |x|^{\frac{\rho(p-2)}{(q-1)(p-1)}}. \tag{3.97}$$

Finally, returning to (3.94):

$$u_m(x,t) = |x|^{\frac{\rho}{1-q}}(-\beta_0 + \gamma d)^{\frac{1}{q-1}} t^{-\beta_0}. \tag{3.98}$$

This last expression provides the proof of the first part of the theorem considering that:

$$C_p(x) = |x|^{\frac{\rho}{1-q}}(-\beta_0 + \gamma d)^{\frac{1}{q-1}}. \tag{3.99}$$

The next intention is to show a precise evolution of the support. To this end, consider the expression (3.90) and make $f(\xi) = 0$, so that:

$$\xi = \left(\frac{C}{K}\right)^{\frac{p-1}{p}}, \quad \xi = |x|t^{\gamma}, \quad \xi_{spt} = \left(\frac{C_1}{\left(\frac{p-2}{p}\right)\left(\frac{1}{d(p-2-p/d)}\right)}\right)^{\frac{p-1}{p}} |x|^{\frac{p-2}{q-1}}. \tag{3.100}$$

Define:

$$C_{spt} = \left(\frac{C_1}{\left(\frac{p-2}{p}\right)\left(\frac{1}{d(p-2-p/d)}\right)}\right)^{\frac{p-1}{p}}. \tag{3.101}$$

Upon recovery of (x, t) variables in (3.100), the evolution of the support is obtained as:

$$|x| = \left(\frac{1}{C_{spt}}\right)^{\frac{q-1}{p-q-1}} t^{\frac{q-1}{(p-2+p/d)d(p-q-1)}}. \tag{3.102}$$

Note that given any local point (x_0, t_0), positivity of solutions is ensured in the inner region to the support, i.e.

$$u(x,t) > 0, \quad t > t_0 \quad |x - x_0| < \left(\frac{1}{C_{spt}}\right)^{\frac{q-1}{p-q-1}} (t - t_0)^{\frac{q-1}{(p-2+p/d)d(p-q-1)}}, \tag{3.103}$$

as intended to show.

□

Once the support has been characterized, the intention is to obtain solutions evolving along such support. In addition, the property known as finite propagation is shown making use of a maximal solution exhibiting such property.

Theorem 6. *Along the propagating support, the following holds:*

$$u(x,t) = (A\,t)^{\frac{3+q-3p+p(p-q)}{(2-p)(1-q)}} |x|^{\frac{\rho}{1-q}} \;\; in \;\; B_{R,\varepsilon} \times (0, T], \tag{3.104}$$

where $B_{R,\varepsilon} = [R - \varepsilon, R)$ and $A(\|u_0\|_\infty)$ is a suitable constant depending on initial data.

In addition, any solution exhibits finite propagation speed along the evolving support.

Proof. Firstly, the equation (1.1) is assessed with the p-Laplacian in polar coordinates. In addition, admit that any spacial derivative satisfies $\Delta u << |\nabla u|$. Then, the equation (1.1) is as per:

$$u_t = \frac{d-1}{|x|}|\nabla u|^{p-1} + |x|^{\rho} u^q. \tag{3.105}$$

The intention is to get solution profiles in the inner region along the leading propagating front. As a consequence $0 < u << 1$. In such infinitesimal approach, admit the following asymptotic separations of variables:

$$u(x,t) = m(t)\varphi(x) + n(t)\psi(x) + \dots, \qquad |n(t)| << |m(t)|, \quad t \to \infty. \tag{3.106}$$

where $\varphi(x)$, $\psi(x)$ are $C^{\infty}(B_R)$. Returning to (3.105):

$$m_t\varphi = \frac{d-1}{|x|}m^{p-1}\varphi_x^{p-1} + |x|^{\rho}m^q\varphi^q. \tag{3.107}$$

Solving by standard separation of variables:

$$m(t) = A^{\frac{1}{2-p}}t^{\frac{1}{2-p}}. \tag{3.108}$$

In addition and along compact subsets B_R

$$\frac{\frac{d-1}{R}\varphi_x^{p-1} + R^{\rho}m^{q-p+1}\varphi^q}{\varphi} = A. \tag{3.109}$$

Whenever $\varphi \to 0$ along compact subsets $\varphi_x^{p-1} << \varphi^q$, then the following asymptotic profile holds:

$$\varphi = |x|^{\frac{\rho}{1-q}}\, m^{\frac{q-p+1}{1-q}}. \tag{3.110}$$

Eventually and considering the asymptotic expansion in (3.106), the following solution is obtained:

$$u(x,t) = (A\,t)^{\frac{3+q-3p+p(p-q)}{(2-p)(1-q)}}|x|^{\frac{\rho}{1-q}}, \tag{3.111}$$

along compact subsets $B_R \times (0,T]$ and in the proximity of the support characterized in Theorem 5, i.e. along the strip $B_{R,\varepsilon} = [R-\varepsilon, R)$ as intended to show.

To complete the theorem, it is required to show the finite propagation speed along the support. To this end admit that the degeneracy on the diffusivity permits to define a maximal solution (close to the approximation followed in [9]) that shall be characterized and explored to exhibit the finite speed features. Indeed, any other solution below will preserve the maximal property related with finite speed. Admit that such maximal solution is given by:

$$\Upsilon(x,t) = \alpha_1\left(\beta_1 t + |x| - \frac{1}{h}\right)_+, \quad h \in \mathbb{N}, \tag{3.112}$$

where α_1 and β_1 are suitable constants, to be assessed, that ensure the maximal condition of Υ. Now, consider $0 \le \varsigma \le 1$ so that $\beta_1\tau = \frac{1}{2h}$. Then:

$$\Upsilon(x,t) \equiv 0 \quad |x| < \frac{1}{2h}, \quad 0 \le t \le \varsigma. \tag{3.113}$$

Note that solutions to (3.105) are bounded (see Theorem 2), then:

$$\Upsilon(x,t) \le \delta_1, \;\; x \in \mathbb{B}_{\mathbb{R}}, \;\; 0 \le t \le \varsigma\,, \;\delta_1(\|u_0\|_\infty). \tag{3.114}$$

$\Upsilon(x,t)$ is a maximal solution provided $\Upsilon(x,t) \ge u(x,t)$, then:

$$\alpha_1\left(\beta_1 t + |x| - \frac{1}{h}\right)_+ \ge \delta_1. \tag{3.115}$$

Admit that $|x| > \frac{1}{h}$, for instance $|x| = \frac{2}{h}$. Then in $t = 0$:

$$\alpha_1 \left(\frac{2}{h} - \frac{1}{h} \right)_+ \geq \delta_1, \quad \alpha_1 \geq h\, D_1. \tag{3.116}$$

Indeed $\Upsilon(x,t) \geq u(x,t)$ for $|x| = \frac{2}{h}$, $0 \leq t \leq \varsigma$. To determine β_1, impose that $\Upsilon(x,t)$ is a maximal solution in $0 < |x| < \frac{2}{h}$, $0 \leq t \leq \varsigma$:

$$\Upsilon_t \geq \frac{d-1}{|x|} |\nabla \Upsilon|^{p-1} + |x|^\rho \Upsilon^q. \tag{3.117}$$

Two asymptotic conditions arise for $|x| << 1$ and another for $|x| >> 1$ along the support with $0 < \Upsilon << 1$. Both conditions shall be assessed considering that:

$$\Upsilon_t = \alpha_1 \beta_1; \;\; \Upsilon_{|x|} = \alpha_1. \tag{3.118}$$

Then, for $x >> 1$ along compact subsets B_R:

$$\beta_1^{x>>1} \geq \frac{d-1}{R} \alpha_1^{p-1}. \tag{3.119}$$

For $|x| >> 1$ along the support with $0 < \Upsilon << 1$:

$$\alpha_1^{1-q} \geq R^\rho \left(\beta_1^{x<<1}\, t_{spt} + R - \frac{1}{h} \right)^q \tag{3.120}$$

where t_{spt} is defined as per the inverse of (3.79). Then:

$$\beta_1 = \max\{\beta_1^{x>>1}, \beta_1^{x<<1}\}. \tag{3.121}$$

Given α_1 and β_1 in accordance with (3.116) and (3.121) respectively, the solution $\Upsilon(x,t)$ is a local maximal solution.

$$\Upsilon(x,t) \geq u(x,t), \quad 0 < |x| < \frac{2}{h}, \quad 0 \leq t \leq \varsigma. \tag{3.122}$$

The maximal solution assessed is null along the support given by:

$$|x| = \frac{1}{h} - \beta_1 t, \quad h \in \mathbb{N}, \quad 0 \leq t \leq \varsigma < 1. \tag{3.123}$$

and β_1 as per (3.121).

□

The next intention is to provide a characterization of blow-up phenomena or lost of compact support. To this end the following theorem holds:

Theorem 7. *The following critical exponent q^* holds:*

$$q^* = sign_+ \left(1 - \frac{\rho(p-1)}{p+1} \right). \tag{3.124}$$

For

$$q > q^*, \tag{3.125}$$

finite time blow-up exists and, therefore, the loss of compact support along the domain. In addition, if:

$$q \leq q^*, \tag{3.126}$$

global solutions with compact support exist.

Proof. Admit the following self-similar structure:

$$G(x,t) = t^{-\alpha} f(\eta), \quad \eta = |x| t^{-\beta}. \tag{3.127}$$

where $d = 1$ for the sake of simplicity. Upon replacement into (1.1):

$$-\alpha t^{-\alpha-1} f - \beta \underbrace{|x| t^{-\beta}}_{\eta} t^{-\alpha-1} f_\eta = t^{-p(\alpha+\beta)-\beta} p f_\eta^{p-1} f_{\eta\eta} + \eta^\rho t^{\rho\beta-\alpha q} f^q. \tag{3.128}$$

Now, the leading exponent in the time variable leads to:

$$\begin{gathered} -\alpha - 1 = -p(\alpha+\beta) - \beta, \\ -\alpha - 1 = \rho\beta - \alpha q. \end{gathered} \tag{3.129}$$

Upon resolution, the following values for β and α read:

$$\begin{gathered} \alpha = \tfrac{\rho+p+1}{\rho(p-1)+(q-1)(p+1)}, \\ \\ \beta = \tfrac{q-p}{\rho(p-1)+(q-1)(p+1)}. \end{gathered} \tag{3.130}$$

Note that the glow-up or lost of support phenomena is given provided:

$$\rho(p-1) + (q-1)(p+1) > 0. \tag{3.131}$$

Upon operation in the last expression, the following critical exponent arises:

$$q^* = sign_+ \left(1 - \frac{\rho(p-1)}{p+1}\right), \tag{3.132}$$

with $q^* \in (0,1)$. For any $q > q^*$, solutions loss the propagating support and finite time blow up is given. Note that $sign_+$ provides zero whenever:

$$\left(1 - \frac{\rho(p-1)}{p+1}\right) < 0. \tag{3.133}$$

In addition, the complementary condition $q \leq q^*$ gives solutions with compact support along propagation in $B_R \times (0, T]$.

□

4. Conclusions

The presented analysis has provided assessments on regularity, existence and uniqueness to a class of p-Laplacian parabolic problem formulated with an heterogenous non-Lipschitz reaction. The propagating support has been characterized and solutions obtained in the proximity of such support based on self-similar structures. Finally, the existence of a critical exponent q^* has been shown to exist. Such critical exponent segregates the finite support from the case of blow up in finite time or loss of compact support.

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Conflict of interest

The author declares no conflict of interests

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