The Two Dimensional Liquid Crystal Droplet Problem with Tangential Boundary Condition

Zhiyuan Geng · **Fanghua Lin**

Abstract This paper studies a shape optimization problem which reduces to a nonlocal free boundary problem involving perimeter. It is motivated by a study of liquid crystal droplets with a tangential anchoring boundary condition and a volume constraint. We establish in 2D the existence of an optimal shape that has two cusps on the boundary. We also prove the boundary of the droplet is a chord-arc curve with its normal vector field in the VMO space, and its arc-length parametrization belongs to the Sobolev space $H^{3/2}$. In fact, the boundary curves of such droplets closely resemble the so-called Weil-Petersson class of planar curves. In addition, the asymptotic behavior of the optimal shape when the volume becomes extremely large or small is also studied.

Keywords Nonlocal free boundary problem · Liquid crystal droplet · Tangential condition · Weil-Petersson curve

1 Introduction

1.1 Background

Liquid crystal droplets are of great interest from both the theory and applications. They are important in the studies of topological defects in the bulk or on the surface of liquid crystals; and they are useful in understandings of anisotropic surface energies and variety anchoring

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conditions. Determining the shape of the droplets and the associated equilibrium configurations of the liquid crystals leads to a shape optimization problem that, in some cases, becomes a nonlocal free boundary problem.

In fact, we are particularly interested in the elongated droplets known as *tactoids*, which usually possess a characteristic eye shape. After a quick examination, one finds the boundary anchoring condition for the molecular orientation to achieve such a desired shape needs to be a tangential anchoring , i.e. the director is orthogonal to the normal of the droplet boundary.

Mathematically, the most commonly used continuum theory to describe nematic liquid crystals is the Oseen-Frank theory, proposed by Oseen [35] in 1933 and Frank [15] in 1958. In the Oseen-Frank theory, the local state of the liquid crystal is described by a $\mathbb{S}^1$- or $\mathbb{S}^2$- valued vector n that represents the mean local orientation of molecule's optical axis. Let Ω be the region occupied by a nematic liquid crystal droplet, the Oseen-Frank bulk energy associated with the director field is the functional

$$E_{OF}(n,\Omega) = \int_\Omega w(n.\nabla n)\,dx, \tag{1.1}$$

where

$$\begin{aligned} w(n,\nabla n) =& k_1(\operatorname{div} n)^2 + k_2(n\cdot \operatorname{curl} n)^2 + k_3|n\times \operatorname{curl} n|^2 \\ &+ (k_2+k_4)\left(\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2\right). \end{aligned} \tag{1.2}$$

We shall consider the one-constant approximation, i.e., $k_1 = k_2 = k_3 = 1$ and $k_4 = 0$, the energy functional (1.1) reduces to

$$E_{OF}(n,\Omega) = \int_\Omega |\nabla n|^2\,dx, \tag{1.3}$$

which is the energy functional for harmonic maps. Harmonic maps have been extensively studied in the past several decades, which are much better understood. Interested readers can refer to [30] and the references therein.

Liquid crystal droplets are often either dispersed in an polymeric medium or surrounded by another fluid such as water. There is an interfacial energy which will play an essential role in determining the optimal shapes. Following [16] and [35], the surface energy may be written as

$$E_s(\Omega,n) = \int_{\partial\Omega} f(n\cdot\nu)\,dA \tag{1.4}$$

where ν is the outer normal on $\partial\Omega$ and for simplicity, f is assumed to have the form (see [10])

$$f(\rho) = \mu(1+\lambda\rho^2), \tag{1.5}$$

for some $\mu > 0$ and $-1 < \lambda < \infty$. Thus the total energy for a liquid crystal droplet configuration is given by:

$$E(\Omega,n) = E_{OF}(\Omega,n) + E_s(\Omega,n)$$

As both the shape of Ω and the director n are varying, determining the stable configuration leads to the following free boundary problem:

Problem 1.1 Find a pair (Ω,n), that minimizes the functional

$$E(\Omega,n)=\int_{\Omega} w(n,\nabla n)\,dx+\int_{\partial\Omega} f(n\cdot\nu)dA. \tag{1.6}$$

subject to the constraint $\mathrm{vol}(\Omega)=V$.

Here vol denotes Lebesgue measure and V is a positive constant.

Problem 1.1 draws great attention from both physicists and mathematicians. There are many research works on Problem 1.1 with physical experiments, numerical simulations and formal analysis, see for example [9,26,37,36,32,39,42,27,34]. On the other hand, rigorous theoretical treatment of this problem is more challenging because of the difficulty of determining the shape and the director at the same time. One way to overcome such difficulty is to assume the droplet have a simple geometry, such as a disk, an ellipse or a intersection region of two disks, see e.g. [24,42,44]. In these works, the shape of the droplet is either fixed, or determined by only one or two parameters (such as the eccentricity of an ellipse). And the minimization often involves finding the best shape parameter and the director field under various boundary conditions and different Oseen-Frank elastic constants. Another way is to presume the configuration of the director field (such as a constant vector field), and then find the best shape that minimizes the surface energy alone, subject to the fixed volume constraint, see e.g. [39,43]. These two methods are useful to partially justify the phenomena observed in experiments but are not satisfactory from a mathematical point of view.

A more rigorous study of Problem 1.1 was conducted by the second author and Poon in [29]. Under the key assumption that all admissible domains are convex, they establish the existence and partial regularity of Problem 1.1 (see [29, Theorem 2.4]). The convexity assumption on the shape of droplets, on one hand, makes the problem more accessible mathematically; and on the other hand, it does match many experimental observed liquid crystals droplets which are of shapes of ellipsoids (balls) and cigars. In this connection, they also studied the cases when the surface energy favors the normal boundary anchoring condition or the tangential boundary anchoring condition. When $\lambda>0$ and μ is very large, for any minimizing pair $\{\Omega,n\}$ of Problem 1.1, $n\cdot\nu$ has to be close to zero almost everywhere on $\partial\Omega$. This leads to the so-called tangential anchoring boundary condition: $n\cdot\nu=0$ on $\partial\Omega$. Problem 1.1 is then reformulated as follows:

Problem 1.2 (Problem B in [29]) Find a pair (Ω,n) that minimizes

$$\int_{\Omega} w(n,\nabla n)\,dx+\mu\,\mathrm{Area}(\partial\Omega)$$

and such that (i) $\mathrm{vol}(\Omega)=V$ and (ii) $n\cdot\nu=0$ on $\partial\Omega$.

When $-1<\lambda<0$ and μ is very large, $n\cdot\nu$ needs to be close to 1 on $\partial\Omega$. In this case one gets the normal anchoring boundary condition: $n\cdot\nu=1$ on $\partial\Omega$, which leads to the following problem:

Problem 1.3 (Problem C in [29]) Find a pair (Ω,n) that minimizes

$$\int_{\Omega} w(n,\nabla n)\,dx+\mu\,\mathrm{Area}(\partial\Omega)$$

and such that (i) $\mathrm{vol}(\Omega)=V$ and (ii) $n\cdot\nu=1$ on $\partial\Omega$.

It is proven in [29] that there are minimizers among convex domains Ω for both Problem 1.2 and Problem 1.3. Moreover, the only solution to Problem 1.3 (up to a Euclidean motion) is $(B_R, \frac{x}{|x|})$, such that $|B_R| = V$.

Li & Wang recently extend the previous result in which they replace the convexity assumption by a notion of M-uniform domains, see [28]. It is worth noting that the Problems 1.1, 1.2 and 1.3 thus presented were all derived from a phenomenological theory, see [35] and [10]. In a recent work [31], it is shown that one can rigorously establish these model problems from a general theory of Ericksen (for liquid crystals with variable degree of orientations [11]) or from the de Gennes-Landau model of liquid crystals [13] in suitable physical regimes.

From our experience, one likely can establish a general existence and partial regularity theory for Problems 1.1, 1.2 and 1.3 without the convexity assumption on the shape of Ω. However, one also expects such a theory will not be able to tell certain particular shapes and configurations (that are observed in experiments and numerical simulations) are minimizers and it is usually difficult to construct solutions for the Euler-Lagrange equation. In particular, one likely will not be able to deduce that *tactoids*, balls, cigars and apples shaped droplets are minimizers. The latter are in fact commonly observed in experiments and of interest to many researchers.

In this article, we will concentrate on the two dimensional case of Problem 1.2, where the tangential anchoring boundary condition and the fixed volume constraint are presumed. The minimizer is expected to have a spindle shape, which is known as *tactoids*, and a *bipolar* director field. Here the bipolar director field refers to an axially symmetric configuration with tangential anchoring boundary condition, such that two boojums are located at opposite ends of the axis. If one investigate thin liquid crystals samples in experiments, the region of nematic liquid crystals will form a planar domain (tactoid) whose boundary consists of two curves that meet at two singular points and form angles or cusps. For more experimental evidences and numerical simulations of tactoids with such bipolar director configurations, the readers are referred to [9,26,37,36,38,39,42] for more details. These works also manifest the significance of tactoids as an object of study.

There are several works that focus on the rigorous mathematical analysis of tactoids with tangential anchoring of the director on the surface. Shen *et al.* [40] discussed such bipolar configurations of droplet in the fixed spherical domain case as well as the free boundary case. For the latter, they introduce a relaxed energy to establish the existence of critical points and some stability results. Recently, a model problem based on highly disparate elastic constants is proposed by Golovaty, Novack, Sternberg and Venkatraman in [20] to understand corners and cusps that form on the nematic-isotropic interface. They prove some Γ-convergence results (when some elastic constant ε goes to 0) and study the role played by the boundary tangency requirement and the elastic anisotropy on the formation of interfacial singularities.

In this work, we investigate the planar tactoids by solving Problem 1.2. What distinguishes our work from the previous work of Lin & Poon is that we drop the convexity assumption on the domain Ω. Instead, we only assume a symmetry assumption with respect to the x-axis as well as some basic technical assumptions (see conditions (i)–(v) in Section 1.2) for the boundary of the droplet. Note that in most of experiments, it is observed that liquid crystal droplets possess the axial symmetry. We first prove some geometric properties of the free boundary. The main property is that away from two cusps, the boundary curve Γ is a vanishing chord-arc curve and the boundary normal vector ν is in VMO. Furthermore, we notice that our curve Γ has many similar properties with the so-called Weil-Petersson curve (see Section 2.3). As a consequence, the arc-length parameterization of the curve is in the Sobolev space $H^{3/2}$. Then using these properties, we demonstrate the existence of a global minimizer with two cusps on the boundary, which verifies the shape of tactoids. In fact, we show that two cusps must appear at the intersection points of the interface and the x-axis for all global minimizers. Moreover, an Euler-Lagrange equation is also derived under extra regularity assumptions on Γ. Finally, we study the asymptotic shape of the nematic drop when the volume tends to be very large or very small. Note that due to a very strong non-local character of this problem, currently we are not able to show that ν is continuous on the boundary. We hope to prove higher regularity results in the future.

1.2 Mathematical Formulation

Now we give the precise formulation of the model problem. Note that what we have in mind is the tactoid that forms two cusps on the boundary. Set $\Omega \subset \mathbb{R}^2$ as the simply-connected region which is a domain enclosed by a Jordan curve with finite length. We denote by $n \in \mathbb{S}^1$ the unit vector that represents the director of liquid crystal. The Oseen-Frank bulk energy is given by (1.3). Then the variational problem is

Problem 1.4 (2D case of Problem 1.2) Find a pair $\{\Omega, n\}$ that minimizes

$$\int_{\Omega} |\nabla n|^2 dx + \mathrm{Per}(\partial\Omega)$$

such that $\mathrm{vol}(\Omega) = V$ and $n \cdot \nu = 0$ on $\partial\Omega$. Here Per means the perimeter.

Here we want to point out that this formulation already implies that the boundary of the minimizer Ω cannot be smooth everywhere. We can explain it in this way: if $\partial\Omega$ is a closed smooth curve and the boundary tangential vector is continuous, then the topological degree of tangential vector is at least one and therefore there is no finite Dirichlet energy extension of $n|_{\partial\Omega}$ inside the 2D domain Ω (see [4, p. xiii]). Now we refine this problem by adding more constraints and then introduce the final version of the problem that we will study.

First we assume Ω is symmetric with respect to x-axis. And therefore we only consider half of the domain located in the upper-half plane. Let Γ be a rectifiable curve that satisfies following conditions:

(i) $\Gamma = \{(x(t), y(t)) : x, y \in AC([0, l(\Gamma)])\}$, where $l(\Gamma)$ is the length of Γ.
(ii) $(x(0), y(0)) = (-a, 0)$, $(x(1), y(1)) = (a, 0)$ for some $a > 0$.
(iii) $\mathscr{H}^1(\Gamma \cap \{(x, 0) : x \in \mathbb{R}\}) = 0$.
(iv) $x'(t) \geq 0,\ y(t) \geq 0,\ (x(t), y(t)) \neq (x(s), y(s))$ for $s \neq t$.
(v) $\sqrt{|x'(t)|^2 + |y'(t)|^2} = 1$ almost everywhere.

Note that here condition (i) and (v) mean that we parameterize Γ by unit length; condition (ii) implies two endpoints of Γ belong to x-axis; condition (iv) tells that Γ does not touch itself and will always "go from left to right". Now we define Ω_Γ as the region enclosed by Γ and x-axis. Note that so far Ω_Γ might not be a simply connected region since $\Gamma(t)$ may touch x-axis at some other point between two endpoints. However, we will show later in Lemma 2.1 that for a minimizer, Ω_Γ has to be simply connected.

The boundary condition for director n on $\partial\Omega_\Gamma = \{(x,0) : x \in [-a,a]\} \cap \Gamma$ is given by

$$n(x,y) = (1,0) \text{ on } \{(x,0) : x \in [-a,a]\},$$
$$n(x(t),y(t)) = (x'(t),y'(t)) \text{ on } (x(t),y(t)) \in \Gamma.$$

Here the condition $n(x,y) = (1,0)$ on $\{(x,0) : x \in [-a,a]\}$ is due to the axial symmetry assumption with respect to the x-axis.

Note that in 2D, for $n \in H^1(\Omega_\Gamma, \mathbb{S}^1)$, we can write it as $(n_1,n_2) = (\cos\Theta, \sin\Theta)$ for some $\Theta \in H^1(\Omega_\Gamma, \mathbb{R})$ (see [6, Theorem 1] for the existence of such lifting). Then by the chain rule we have

$$|\nabla n|^2 = \sin^2\Theta |\nabla\Theta|^2 + \cos^2\Theta |\nabla\Theta|^2 = |\nabla\Theta|^2.$$

We will work with this angle function Θ. Then the corresponding boundary condition for Θ is

$$\begin{aligned} &\Theta(x,y) = 0 \quad \text{ on } \{(x,0) : x \in [-a,a]\} \\ &\Theta(x(t),y(t)) = \arcsin y'(t) \quad \text{ on } (x(t),y(t)) \in \Gamma. \end{aligned} \tag{1.7}$$

Here arcsin is defined on $[-1,1]$ and maps to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Now we are ready to define the following admissible set for Γ:

$$\mathscr{G}_V := \{\Gamma \text{ satisfies condition (i–v), and } \Theta\big|_{\partial\Omega_\Gamma} \text{ has a harmonic extension } \Theta \text{ defined in } \overline{\Omega_\Gamma}$$
$$\text{such that } \int_{\Omega_\Gamma} |\nabla\Theta|^2 \, dxdy < \infty \text{ and } |\Omega_\Gamma| = V\}$$

Here V is a positive constant representing the volume of Ω_Γ. Figure 1 shows our assumptions on Γ, Ω_Γ and the tangential anchoring condition for Θ.

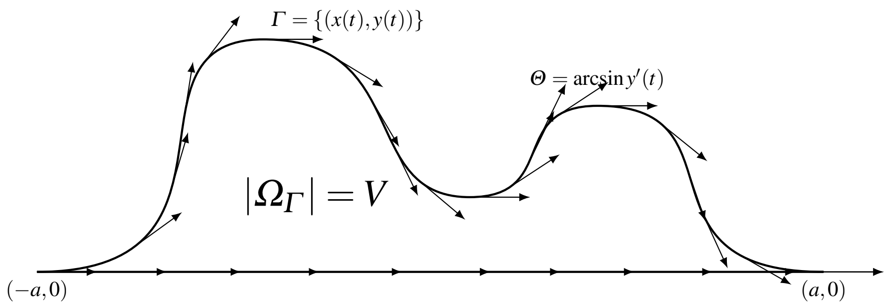


Fig. 1: Curve Γ, domain Ω_Γ and Θ

To this end, we consider the following variational problem

Problem 1.5 (Final Problem) Find $\Gamma \in \mathscr{G}_V$ that minimizes the following functional

$$E(\Gamma) = \int_{\Omega_\Gamma} |\nabla\Theta|^2 \, dxdy + l(\Gamma), \tag{1.8}$$

where Θ is determined by Γ in the following way

$$\begin{cases} \Delta\Theta = 0, & \text{in } \Omega_\Gamma, \\ \Theta\big|_{\partial\Omega_\Gamma} & \text{is defined as in (1.7).} \end{cases}$$

We will study the existence and properties of global energy minimizers of Problem 1.5 in the rest of the article. In Section 2 we prove various geometric properties of Γ and Ω_Γ. When the energy $E(\Gamma)$ is finite (not necessarily a minimizer), we show that Γ is a vanishing chord-arc curve and $\nu \in$ VMO on Γ. Moreover, the arc-length parametrization belongs to $H^{3/2}(0,l)$. As a consequence, the function Θ defined on $\bar{\Omega}_\Gamma$ can be extended to a H^1 function on $\mathbb{R}^2$ according to the classical theory on the relationship of quasidisks and Sobolev extention domains. The existence of a global minimizer for Problem 1.5 is established in Section 3. The proof relies heavily on the properties proved in Section 2. We also show that for all the global minimizers, Γ and x-axis will form two cusps near two intersection points. Under the assumption that Γ can be written as the graph of a sufficiently regular function f, the Euler-Lagrange equation for Γ is also derived. Finally in Section 4 we study asymptotic profiles of Γ when the volume V tends to be very large or small. We would like to point out that this article just represents an initial investigation of Problem 1.5, and there are many open problems to be studied in the future.

2 Geometric properties of Γ and Ω_Γ

2.1 Sobolev extension domain

We assume $V = 1$ throughout this section. And if there exists an energy minimizer for Problem 1.5, we denote it by Γ_m. We further write the corresponding Ω_{Γ_m} and Θ function as Ω_m and Θ_m. We start with the observation that

Claim $\mathscr{G}_1$ is not empty. There is at least one smooth curve $\Gamma \in \mathscr{G}_1$.

Actually we can find a smooth curve $\Gamma_0 \in \mathscr{G}_1$ by directly constructing a curve Γ_0. Let Γ_0 be the graph of function $f_0(x) = \frac{\cos x+1}{2\pi}$, $x \in [-\pi,\pi]$. By definition $\Omega_{\Gamma_0} = \{(x,y) : -\pi \le x \le \pi, 0 \le y \le f_0(x)\}$ and we set $\Theta_0(x,y) = -\frac{2\pi y}{\cos x+1} \arctan \frac{\sin x}{2\pi}$ for $(x,y) \in \Omega_{\Gamma_0}$. It is straightforward to check that Γ_0 satisfies the condition (i–v), $|\Omega_{\Gamma_0}| = 1$, and Θ_0 satisfies the boundary condition

(1.7). Then we compute the energy directly

$$
\begin{aligned}
E(\Gamma_0) &= \int_{-\pi}^{\pi} \sqrt{1+\left(\frac{df_0}{dx}\right)^2}\,dx + \int_{-\pi}^{\pi}\int_0^{\frac{\cos x+1}{2\pi}} |\partial_y \Theta_0|^2 + |\partial_x \Theta_0|^2\,dydx \\
&= \int_{-\pi}^{\pi} \left\{ \sqrt{1+\frac{\sin^2 x}{4\pi^2}} + \frac{2\pi\left|\arctan\left(\frac{\sin x}{2\pi}\right)\right|^2}{\cos x+1} + \frac{\left|\frac{\cos x(\cos x+1)}{2\pi(1+\frac{\sin^2 x}{4\pi^2})} + \arctan\left(\frac{\sin x}{2\pi}\right)\cdot \sin x\right|^2}{6\pi(\cos x+1)} \right\} dx \\
&\leq C,
\end{aligned}
$$

where C is a constant. To see the integral is finite, we only need to examine the integrand when $\cos x+1 \to 0$, i.e. $x \to \pm\pi$. Assume $x = \pi - \alpha$ for $0 < \alpha << 1$, then $\cos x + 1 \approx \frac{1}{2}\alpha^2$, $|\arctan(\frac{\sin x}{2\pi})| \approx \frac{\alpha}{2\pi}$ and $\sin x \approx \alpha$. It follows that the integrand is uniformly bounded on $x \in (-\pi,\pi)$, which implies the finiteness of the integral.

Therefore we have verified that $\Gamma_0 \in \mathscr{G}_1$. And if Problem 1.5 admits a global minimizer Γ_m, then we get the following upper bound for the energy infimum:

$$
M := E(\Gamma_0) \geq E(\Gamma_m)
$$

The next lemma tells us that the minimizing curve Γ_m, if exists, will not touch x-axis besides two endpoints, which means Ω_m is simply connected.

Lemma 2.1 *If Γ_m is the global minimizer of $E(\Gamma)$ among all $\Gamma \in \mathscr{G}_1$ and it is parametrized by arc length as in condition (i–v), then for any $t \in (0, l(\Gamma_m))$, we have $y(t) > 0$.*

Proof We prove by contradiction. Assume $y(t_0) = 0$ for some $t_0 \in (0, l(G(m)))$, then the point $(x(t_0), y(t_0))$ cuts Γ_m into two parts, which are denoted by Γ_1 and Γ_2 respectively. We call the domain enclosed by Γ_i and $x-$axis as Ω_i for $i = 1,2$. Let $\alpha := |\Omega_1|$. We can further assume $\alpha \in (0,1)$ because if $\alpha = 0$ or 1, then one of Γ_i will coincide with the $x-$axis which contradicts with the fact that Γ_m is a minimizer. To see this, assume Γ_1 coincides with the x-axis, then we can remove Γ_1 from Γ_m and find that the remaining curve Γ_2 still belongs to $\mathscr{G}_1$ and $E(\Gamma_2)$ is strictly less than $E(\Gamma_m)$. Now we set

$$
\begin{aligned}
&\tilde{\Gamma}_1 = \frac{1}{\sqrt{\alpha}}\Gamma_1, \quad \tilde{\Omega}_1 = \frac{1}{\sqrt{\alpha}}\Omega_1, \quad \tilde{\Theta}_1\left(\frac{x}{\sqrt{\alpha}}, \frac{y}{\sqrt{\alpha}}\right) = \Theta_m(x,y) \text{ for } (x,y) \in \Omega_1 \\
&\tilde{\Gamma}_2 = \frac{1}{\sqrt{1-\alpha}}\Gamma_2, \quad \tilde{\Omega}_2 = \frac{1}{\sqrt{1-\alpha}}\Omega_2, \quad \tilde{\Theta}_2\left(\frac{x}{\sqrt{1-\alpha}}, \frac{y}{\sqrt{1-\alpha}}\right) = \Theta_m(x,y) \text{ for } (x,y) \in \Omega_2
\end{aligned}
$$

We can check that for $i = 1,2$, $(\tilde{\Gamma}_i, \tilde{\Omega}_i, \tilde{\Theta}_i)$ are energy competitors (after some horizontal translations) for $(\Gamma_m, \Omega_m, \Theta_m)$. By basic scaling property we get

$$
\begin{aligned}
&l(\tilde{\Gamma}_1) = \frac{1}{\sqrt{\alpha}} l(\Gamma_1), \quad l(\tilde{\Gamma}_2) = \frac{1}{\sqrt{1-\alpha}} l(\Gamma_2), \\
&\int_{\tilde{\Omega}_i} |\nabla \tilde{\Theta}_i|^2 = \int_{\Omega_i} |\nabla \Theta|^2 \text{ for } i = 1,2
\end{aligned}
$$

The minimizing property yields

$$\frac{1}{\sqrt{\alpha}} l(\Gamma_1) + \int_{\Omega_1} |\nabla\Theta|^2 \geq l(\Gamma_1) + l(\Gamma_2) + \int_{\Omega_1} |\nabla\Theta|^2 + \int_{\Omega_2} |\nabla\Theta|^2,$$
$$\frac{1}{\sqrt{1-\alpha}} l(\Gamma_2) + \int_{\Omega_2} |\nabla\Theta|^2 \geq l(\Gamma_1) + l(\Gamma_2) + \int_{\Omega_1} |\nabla\Theta|^2 + \int_{\Omega_2} |\nabla\Theta|^2.$$

Combining these two inequalities we arrive at

$$l(\Gamma_1) \geq \frac{\sqrt{\alpha}}{1-\sqrt{\alpha}} l(\Gamma_2) \geq \frac{\sqrt{\alpha}}{1-\sqrt{\alpha}} \cdot \frac{\sqrt{1-\alpha}}{1-\sqrt{1-\alpha}} l(\Gamma_1)$$
$$\Rightarrow \sqrt{\alpha(1-\alpha)} \leq (1-\sqrt{\alpha})(1-\sqrt{1-\alpha}) \Rightarrow \alpha = 0 \text{ or } 1,$$

which yields a contradiction. □

We define the following "two point condition":

Definition. A rectifiable curve Γ is said to satisfy the "two point condition" if and only if there is a constant $C > 0$ such that for any $z_1, z_2 \in \Gamma$, it holds that $\operatorname{diam} \gamma_{z_1 z_2} \leq C|z_1 - z_2|$, where $\gamma_{z_1 z_2}$ denotes the arc of Γ between z_1, z_2 (if there are two arcs, take the one with smaller length).

Now we want to prove some geometric properties of $\Gamma \in \mathscr{G}_1$ (not necessarily a minimizer). The next statement says that for any three points on Γ, they are supposed to satisfy a reversed triangle inequality, with a constant depending only on $E(\Gamma)$.

Lemma 2.2 *If $\Gamma \in \mathscr{G}_1$ and $E(\Gamma) \leq M$, then there exists a constant $C = C(M)$ such that for any three points $z_1 = (x(t_1), y(t_1))$, $z_2 = (x(t_2), y(t_2))$ and $z_3 = (x(t_3), y(t_3))$ on Γ such that $t_1 < t_2 < t_3$, it holds that*

$$\max\{\operatorname{dist}(z_1,z_2), \operatorname{dist}(z_2,z_3)\} \leq C \operatorname{dist}(z_1,z_3). \tag{2.9}$$

Moreover, Γ satisfies the "two point condition".

Proof By definition, the "two point condition" follows directly from (2.9). So we just prove (2.9). Assume C is a large enough number (say larger than 100) which will be determined later. We simply write $(x(t_i), y(t_i))$ as (x_i, y_i) for $i = 1,2,3$. Then for the value of y_2, there are three cases:

1. $\max\{y_1, y_3\} \geq y_2 \geq \min\{y_1, y_3\}$,
2. $y_2 > \max\{y_1, y_3\}$,
3. $\min\{y_1, y_3\} > y_2$.

The inequality (2.9) for the first case is trivial, because by simple geometry we can get

$$\max\{\operatorname{dist}(z_1,z_2), \operatorname{dist}(z_2,z_3)\} \leq \operatorname{dist}(z_1,z_3).$$

Now we study the second case, and assume (2.9) is false. By triangle inequality, we have

$$\min\{\operatorname{dist}(z_1,z_2), \operatorname{dist}(z_2,z_3)\} \geq (C-1)\operatorname{dist}(z_1,z_3).$$

Therefore it holds that

$$\min\{y_2 - y_1, y_2 - y_3\} \geq (C-2)|x_1 - x_3|.$$

For convenience we assume $y_2 = \max_{t\in(t_1,t_3)} y(t)$. Otherwise we can take z_2 to be the point with the maximum value of y on Γ between z_1 and z_3. Note that such choice will not violate any of the above estimates.

We set the curve Γ between z_1, z_2 and z_2, z_3 as Γ_1, Γ_2, written as $\Gamma_1 := \Gamma_{z_1z_2}$, $\Gamma_2 := \Gamma_{z_2z_3}$. Set $l_i := l(\Gamma_i)$ for $i = 1,2$. Also we reparameterize Γ_1 and Γ_2 as following

$$\begin{aligned}\Gamma_1 := \{&(x(s),y(s)) : s\in[0,l_1],\, (x(0),y(0)) = (x_2,y_2),\, (x(l_1),y(l_1)) = (x_1,y_1),\\ &x'(s)\le 0,\, |x'(s)|^2+|y'(s)|^2 = 1\ a.e.\},\\ \Gamma_2 := \{&(x(s),y(s)) : s\in[0,l_2],\, (x(0),y(0)) = (x_2,y_2),\, (x(l_2),y(l_2)) = (x_3,y_3),\\ &x'(s)\ge 0,\, |x'(s)|^2+|y'(s)|^2 = 1\ a.e.\}\end{aligned}$$

Note that for such reparametrization, Γ_1 starts at z_2 and ends at z_1, while Γ_2 starts at z_2 and ends at z_3. Also we have $\Theta(x(s),y(s)) = -\arcsin y'(s)$ on Γ_1 and $\Theta(x(s),y(s)) = \arcsin y'(s)$ on Γ_2.

We first look at Γ_1. Set

$$\begin{aligned}\tilde r(s) &:= \sqrt{|x(s)-x_2|^2+|y(s)-y_2|^2},\ s\in[0,l_1],\\ I(r) &:= \{s\in[0,l_1] : \tilde r(s) = r\}\end{aligned}$$

For $0<r<|z_1-z_2|$, the circle $\{|z-z_2| = r\}$ will intersect with Γ_1 and therefore I_r is not empty. By definition we have

$$\int_0^{l_1} y'(s)\,ds = y_1 - y_2, \tag{2.10}$$

$$\int_0^{l_1} x'(s)\,ds = x_1 - x_2. \tag{2.11}$$

For $\tilde r(s)$, we can estimate its derivative by

$$|\tilde r'(s)| = \left|\frac{(x(s)-x_2)\cdot x'(s) + (y(s)-y_2)\cdot y'(s)}{r(s)}\right| \le \sqrt{|x'(s)|^2+|y'(s)|^2} = 1.$$

Then by coarea formula, we have

$$l_1 \ge \int_0^{l_1} |\tilde r'(s)|\,ds = \int_0^{|z_1-z_2|} \mathscr{H}^0(I_r)\,dr$$

This tells us that for almost every $r\in[0,|z_1-z_2|]$, $\mathscr{H}^0(I_r)$ is finite. Note that $\mathscr{H}^0$ is just the counting measure, and we will simply write it as $|I_r|$. Denote by A the subset of $[0,l_1]$ such that for any $s\in A$, $\tilde r'(s) = 0$. Again co-area formula gives

$$0 = \int_A |\tilde r'(s)|\,ds = \int_0^{|z_1-z_2|} \mathscr{H}^0(A\cap I_r)\,dr$$

So $A\cap I_r = \varnothing$ for a.e. $r\in[0,|z_1-z_2|]$. We define

$$R_0 := \{r\in[0,|z_1-z_2|] : |I_r| \text{ is finite, and } |\tilde r'(s)|>0 \text{ for any } s\in I_r\}.$$

We have $m([0,|z_1-z_2|]\backslash R_0)=0$. For any $r\in R_0$, we pick a representative from I_r in the following way:

$$s^r=\min\{s:s\in I_r\}.$$

We define the following two subsets:

$$\begin{aligned}R_1 &:= \{r\in[2[z_1-z_3],|z_1-z_2|]\cap R_0 : x'(s^r)\le -1/2\},\\ R_2 &:= ([2|z_1-z_3|,|z_1-z_2|]\cap R_0)\backslash R_1.\end{aligned}$$

Note that R_1 corresponds to the part of curve on Γ_1 where is not "too vertical". Using co-area formula again, we get

$$\begin{aligned}\frac{1}{2}m(R_1) &\le \left|\int_{R_1}\frac{dx(s^r)}{ds}\cdot\left|\frac{dr(s^r)}{ds}\right|^{-1}dr\right|\\ &\le |\int_{\tilde{r}^{-1}(R_1)}x'(s)\,ds|\\ &\le |x_1-x_2|.\end{aligned}$$

As a consequence, we get

$$m(R_2)\ge|z_1-z_2|-2|z_1-z_3|-2|x_1-x_2|\ge|z_1-z_2|-4|z_1-z_3|. \tag{2.12}$$

Now we make the following observation:

For any $r\in R_2$, $y'(s^r)<-\frac{\sqrt{3}}{2}$.

This is a consequence of definition of R_2 and s^r. Since $r\in R_2$, we have

$$\begin{gathered}y'(s^r)>\frac{\sqrt{3}}{2}\text{ or }y'(s^r)<-\frac{\sqrt{3}}{2},\\ y(s^r)<y_2-\sqrt{3}|z_1-z_3|,\quad x_2-|z_1-z_3|\le x(s^r)<x_2,\\ |y(s^r)-y_2|>\sqrt{3}|x(s^r)-x_2|.\end{gathered}$$

We also have that $\frac{d\tilde{r}(s^r)}{ds}>0$ because $(x(s^r),y(s^r))$ is the first point that Γ_1 touches $\{|z-z_2|=r\}$. If $y'(s^r)>\frac{\sqrt{3}}{2}$, then

$$\tilde{r}'(s^r)=\frac{x'(s^r)(x(s^r)-x_2)+y'(s^r)(y(s^r)-y_2)}{r}<0,$$

which yields a contradiction. Therefore we have verified the observation.

Now we deal with Γ_2 in the same way with several minor modifications. We can show that there exists a R_3 such that

$$\begin{gathered}R_3\subset[2|z_1-z_3|,|z_2-z_3|],\quad m(R_3)\ge|z_2-z_3|-4|z_1-z_3|,\\ \text{and }\ \forall r\in R_3,\ y'(s_r)<-\frac{\sqrt{3}}{2}.\end{gathered}$$

Here s_r is the point that Γ_2 first touches $\{|z-z_2|=r\}$.

We are now ready to derive a contradiction. Denoting $R := R_2 \cap R_3$, then we have

$$\begin{aligned} R &\subset \{r : 2|z_1 - z_3| \le r \le \min\{|z_1 - z_2|, |z_2 - z_3|\}\},\\ m(R) &\ge \min\{|z_1 - z_2|, |z_2 - z_3|\} - 8|z_1 - z_3|. \end{aligned}$$

For any $r \in R$, Γ_1 first intersects $\{|z - z_2| = r\}$ at $z_1(r) := (x(s^r), y(s^r))$ and Γ_2 first intersects with $\{|z - z_2| = r\}$ at $z_2(r) := (x(s_r), y(s_r))$. The arc $\widehat{z_1(r)z_2(r)}$ is contained in Ω_Γ because of the definitions of s^r, s_r. Moreover, $\Theta = -\arcsin y'(s^r) > \frac{\pi}{3}$ at $z_1(r)$ and $\Theta = \arcsin y'(s_r) < -\frac{\pi}{3}$ at $z_2(r)$. Then we are ready to estimate the Dirichlet energy of Θ in Ω_Γ,

$$\begin{aligned} \int_{\Omega_\Gamma} |\nabla\Theta|^2\, dxdy &\ge \int_0^{C|z_1-z_3|} dr \int_{\{|z-z_2|=r\}\cap\Omega_\Gamma} |\nabla\Theta(z_2 + re^{i\theta})|^2 r\, d\theta \\ &\ge \int_0^{C|z_1-z_3|} dr \int_{\{\theta: z_2+re^{i\theta}\in\Omega_\Gamma\}} \left|\frac{\partial\Theta(z_2+re^{i\theta})}{\partial\theta}\right|^2 \frac{1}{r}\, d\theta \\ &\ge \int_{r\in R} \frac{|\Theta(z_1(r)) - \Theta(z_2(r))|^2}{\pi r}\, dr \\ &\ge \int_{8|z_1-z_3|}^{(C-1)|z_1-z_3|} \frac{4\pi}{9}\frac{1}{r}\, dr \\ &\ge \log\frac{C-1}{8}. \end{aligned} \tag{2.13}$$

Here from the second line to the third line we use the Cauchy-Schwarz inequality to estimate the integral with respect to θ.

Now by choosing C satisfying $\log\frac{C-1}{8} \ge 2M$, we arrive at a contradiction with the energy bound. Thus we proved (2.9) for case (2).

For case (3) when $\min\{y_1, y_3\} \ge y_2$, the proof follows similar arguments. Assume $y_2 = \min_{t\in[t_1,t_3]} y(t)$. We still call the curve between z_1, z_2 and z_2, z_3 as Γ_1, Γ_2 and reparametrize them as before. And $\tilde{r}(s)$, $I(r)$, R_0, s^r, s_r, $z_1(r)$, $z_2(r)$ are all defined in the same way. Recall that $z_1(r) := (x(s^r), y(s^r)) \in \Gamma_1$ and $z_2(r) := (x(s_r), y(s_r)) \in \Gamma_2$. Similarly, we can find $R_2 \subset [2|z_1 - z_3|, |z_1 - z_2|]$ such that for any $r \in R_2$, $\Theta(z_1(r)) < -\frac{\pi}{3}$. Also there exists $R_3 \subset [2|z_1 - z_3|, |z_1 - z_3|]$ such that for $r \in R_3$, $\Theta(z_2(r)) > \frac{\pi}{3}$.

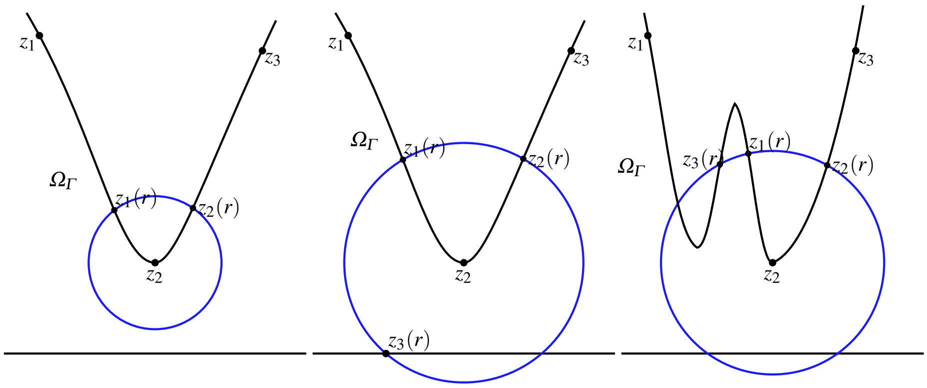
Now we claim that for any $r \in R := R_2 \cap R_3$, it holds that

$$\int_{\{|z-z_2|=r\}\cap\Omega_\Gamma} |\nabla\Theta(z_1 + re^{i\theta})|^2 r\, d\Theta \ge \frac{C_1}{r}. \tag{2.14}$$

Here C_1 is a constant that can be chosen as $\frac{\pi}{18}$. This is the place where case (3) differs from case (2), because in case (2) the set $\{|z - z_2| = r\} \cap \Omega_\Gamma$ is just the arc $\widehat{z_1(r)z_2(r)}$. However in case (3), $\{|z - z_2| = r\} \cap \Omega_\Gamma$ is more complicated. We prove the claim by discussing following three situations (see Figure 2):

1. $\{|z - z_2| = r\}$ only intersects with Γ at $z_1(r), z_2(r)$ and doesn't intersect with x-axis. Since $\Theta(z_1(r)) < -\frac{\pi}{3}$ and $\Theta(z_2(r)) > \frac{\pi}{3}$, we have

$$\int_{\{|z-z_2|=r\}\cap\Omega_\Gamma} |\nabla\Theta(z_2 + re^{i\theta})|^2 \ge \frac{1}{r}\int_{z_2+re^{i\theta}\in\Omega_\Gamma} |\frac{\partial\Theta}{\partial\theta}|^2 d\theta \ge \frac{C_1}{r}.$$



(a) Situation (1) (b) Situation (2) (c) Situation (3)

Fig. 2: Three different situations of I_r and $\{|z-z_2|=r\}\cap\Omega_\Gamma$, when $\min\{y_1,y_3\}>y_2$

2. $\{|z-z_2|=r\}$ only intersects with Γ at $z_1(r), z_2(r)$ and also intersects with x-axis at $z_3(r)$. Without loss of generality we can assume the arc $\widehat{z_1(r)z_3(r)}$ is contained in Ω_f. Then since $\Theta=0$ on x-axis, we have $|\Theta(z_1(r))-\Theta(z_3(r))|\geq\frac{\pi}{3}$, then we can verify (2.14) by the same calculation.
3. $\{|z-z_2|=r\}$ intersects with Γ at more than two points. Let $z_3(r)=(x_3(r),y_3(r))$ be another point of intersection besides $z_1(r), z_2(r)$. Without loss of generality we assume $x_3(r)<x_1(r)$. In our construction we make sure that $r'(s^r)>0$ at $z_1(r)$. And we can assume $\widehat{z_1(r)z_3(r)}\subset\overline{\Omega}_f$. Therefore we have $\Theta(z_3(r))\geq 0$ because at $z_3(r)$, $\Gamma(t)$ is "leaving" the disk $\{|z-z_2|\leq r\}$ as t increases. This implies that $|\Theta(z_1(r))-\Theta(z_3(r))|\geq\frac{\pi}{3}$ Then (2.14) follows immediately in the same way.

With (2.14), we can repeat the computation in (2.13) and finally verifies (2.9) for case (3). This completes our proof of Lemma2.2. □

It is proved by Ahlfors in [1] that a Jordan curve is a quasicircle if and only if it satisfies the "two point condition". A quasicircle is the image of the unit circle $\mathbb{T}$ under a quasiconformal mapping of the complex plane onto itself. And in 2D a quasidisk (domain enclosed by a quasicircle) is equivalent to a Sobolev extension domain, see [22]. However in our problem, $\partial\Omega_\Gamma=\Gamma\cap\{(x,0):x\in[-a,a]\}$ is not a quasicircle because near two endpoints $(-a,0)$ and $(a,0)$ the "two point condition" will be violated. The next lemma says that even though we cannot directly use the property of a quasidisk, we can still extend Θ to the whole plane with a uniform control on its H^1-norm.

Lemma 2.3 *(Extension domain) Assume $\Gamma\in\mathscr{G}_1$ and $E(\Gamma)\leq M$, then there exists a constant C_2 that depends on M such that $\Theta|_{\overline{\Omega}_\Gamma}$ can be extended to the whole plane with a norm control*

$$\|\Theta\|_{H^1(\mathbb{R}^2)}\leq C_2\|\Theta\|_{H^1(\Omega_\Gamma)}.$$

Proof The idea of the proof is to glue a rectangle together with Ω_Γ to make the combined domain a quasidisk. Assume the intersection points of Γ and x-axis are $(-a,0)$ and $(a,0)$, we set

$$D_1 := \{(x,y) : -a < x < a,\ -a < y \le 0\}, \quad D_\Gamma = \Omega_\Gamma \cup D_1.$$

We claim that D_Γ is a quasidisk. For any two points $z_1, z_2 \in \partial D_\Gamma$, if $z_1, z_2 \in \Gamma$, then Lemma 2.2 says they satisfy the "two point condition". If $z_1, z_2 \in \partial D_1 \cap \partial D_\Gamma$, they automatically satisfy the "two point condition" because a rectangle is a quasidisk. So we are left with the case $z_1 \in \Gamma$, $z_2 \in \partial D_1 \cap \partial D_\Gamma$. In such case, there are two situations:

1. $z_2 \in \{(x,-a) : -a \le x \le a\}$. Then $|z_1 - z_2| \ge a$ and Lemma 2.2 implies that $\mathrm{diam}(D_\Gamma) \le (C+2)a$ where C is the constant in (2.9). So the "two point condition" holds for this situation.
2. $z_2 \in \{(-a,y) : -a \le y \le 0\} \cup \{(a,y) : -a \le y \le 0\}$. Without loss of generality we assume $z_2 = (-a,y)$ and set $z_0 := (-a,0)$. Let γ be the arc of Γ between z_0 and z_1, then we get

   $$|z_1 - z_2| \ge \max\{|y|, |z_1 - z_0|\}, \quad \mathrm{diam}(\overline{z_2 z_0} \cup \gamma) \le |y| + \mathrm{diam}(\gamma),$$

   where $\overline{z_2 z_0}$ is the line segment between z_0 and z_2. One can easily shows that $\mathrm{diam}(\overline{z_2 z_0} \cup \gamma) \le (C+1)|z_1 - z_2|$, since $\mathrm{diam}\,\gamma \le C|z_1 - z_0|$ by (2.9). The "two point condition" is verified.

Therefore by Ahlfors' result, we prove the claim. Next we can trivially extend Θ to D_Γ by letting $\Theta(x,y) \equiv 0$ for $(x,y) \in D_1$, because Θ vanishes on $\{(x,0) : -a \le x \le a\}$. Obviously, $\|\Theta\|_{H^1(D_\Gamma)} = \|\Theta\|_{H^1(\Omega_\Gamma)}$. Moreover, we can further extend Θ to the whole plane $\mathbb{R}^2$, since a 2D domain is a quasidisk if and only if it is a Sobolev extension domain. Also from the above derivation and the proof of Lemma 2.2, the constant C in the "two point condition" for ∂D_Γ only depends on M. By [22, Theorem C & Theorem 4] and [33, Theorem 2.24] we can show that the norm of the extension operator only depends on M, i.e. there is a $C_2 := C_2(M)$ such that

$$\|\Theta\|_{H^1(\mathbb{R}^2)} \le C_2 \|\Theta\|_{H^1(\Omega_\Gamma)}.$$

This completes our proof of Lemma 2.3. □

2.2 Γ is a chord-arc curve

We will give more geometric properties of Γ by showing that it is a chord-arc curve, which means the length of the chord is comparable with the length of the arc (see [21] for a detailed discussion on chord-arc curves).

Proposition 2.4 *Let $\Gamma \in \mathscr{G}_1$ and $E(\Gamma) \le M$. There exists a constant $C_3(M)$ such that for any two points $z_1, z_2 \in \Gamma$ and the arc $\gamma := \Gamma_{z_1 z_2}$, we have*

$$l(\gamma) < C_3 |z_1 - z_2|.$$

In other words, Γ is a chord-arc curve.

Proof First assume C_3 is a very large number that will be determined later. We prove by contradiction. Suppose $\mathscr{H}_1(\gamma) = c|z_1 - z_2|$ for some $c > C_3$, our goal is to show that the Dirichelt energy "generated" by this part of boundary will be very large, which contradicts to the uniform bounds of Dirichlet energy ($E(\Gamma) \le M$). The basic idea can be roughly stated as following: if the length of curve is way too long compared with the chord length, then there will be lots of fluctuations of the curve, which will lead to large energy. The co-area formula will be used repeatedly in the proof.

Since 2D Dirichlet energy is scaling invariant, we simply let $|z_1 - z_2| = 1$, and reparametrize the curve γ in the following way:

$$\gamma = \{ (x(t), y(t)) : x, y \in \mathrm{AC}[0,c]; x'(t) \geq 0; z_1 = (x(0), y(0)) = (0,0), \\ z_2 = (x(c), y(c)) = (a, \pm\sqrt{1-a^2}) \text{ for some } a \in (0,1]; |x'(t)|^2 + |y'(t)|^2 = 1 \text{ a.e.} \}.$$

According to Lemma 2.2, we have $|y(t)| \leq C$ for $t \in [0,c]$, where C is the constant in (2.9). So there exists $Y_1 \leq 0, Y_2 \geq 0$ such that $|Y_i| \leq C$ for $i = 1,2$ and $\min y(t) = Y_1$, $\max y(t) = Y_2$. Furthermore we have

$$\Gamma \subset \{0 \leq x \leq a, Y_1 \leq y \leq Y_2\} =: Q. \tag{2.15}$$

We have the following upper bound for the energy

$$\int_{\Omega_\gamma} |\nabla \Theta|^2 \, dxdy \leq M, \text{ where } \Omega_\gamma := Q \cap \{(x,y) \text{ below } \gamma\}.$$

Note that by Lemma 2.3, we can extend the domain of Θ to all of Q such that

$$\int_Q |\nabla \Theta|^2 \, dxdy \leq C_2 M =: C_4.$$

Here this constant C_4 only depends on M. Also we make the following definitions:

$$\begin{aligned}
&T_s := \{t \in [0,c] : y(t) = s\}, \qquad \forall s \in [Y_1, Y_2] \\
&U := \{s \in [Y_1, Y_2] : |T_s| \text{ is infinite}\}, \quad W := \{t \in [0,c] : y(t) \in U\}. \\
&S := \{s \in [Y_1, Y_2] : |T_s| = 1\}, \qquad A := \{t \in [0,c] : y(t) \in S\}. \\
&V := [Y_1, Y_2] \backslash (S \cup U), \qquad B := [0,c] \backslash (A \cup W).
\end{aligned}$$

Here $|\cdot|$ denotes the cardinality of a set.

Now we have set up all the assumptions and are ready to derive a contradiction. First we point out several elementary observations:

(i) The following estimate holds:

$$\int_0^c |y'(t)|^2 \, dt = c - \int_0^c |x'(t)|^2 \, dt \geq c - 1, \tag{2.16}$$

where we have used $x'(t) \geq 0$ and $\int_0^c x'(t) \leq 1$.

(ii) If $Y_2 > 1$, then for any $s \in [1, Y_2)$, we have $|T_s| \geq 2$ by mean value theorem for continuous function. Similarly, if $Y_1 < -1$, for any $s \in (Y_1, -1]$, it holds that $|T_s| \geq 2$. In other words, we have

$$S \subset [-1,1] \cap \{Y_1, Y_2\}.$$

(iii) We can estimate the measure of A by

$$\begin{aligned} m(A) &= \int_A \sqrt{|x'(t)|^2 + |y'(t)|^2}\, dt \\ &\leq \int_A |x'(t)|\, dt + \int_A |y'(t)|\, dt \\ &\leq 1 + \int_S \mathscr{H}^0(T_s)\, ds \text{ (by coarea formula)} \\ &= 1 + |S| \leq 3. \end{aligned} \tag{2.17}$$

(iv) By co-area formula one can easily check that

$$m(U) = 0, \quad \int_W |y'(t)|\, dt = 0$$

For any $s \in V$, by definition we have $2 \leq T_s < \infty$, we want to derive a lower bound for the following quantity:

$$E(s) := \int_0^a |d_x\Theta(x,s)|^2\, dx.$$

Assume $T_s = \{t_1, \dots, t_n\}$ for some $n \geq 2$, and by definition we have

$$\sin(\Theta(x(t_i), s)) = y'(t_i), \text{ for } i = 1, \dots, n.$$

An easy observation is that for each two adjacent points, say t_i and t_{i+1},

$$y'(t_i) \cdot y'(t_{i+1}) \leq 0.$$

We deduce that

$$|\Theta(x(t_i), s) - \Theta(x(t_{i+1}), s)| \geq |\sin(\Theta(x(t_i), s)) - \sin(\Theta(x(t_{i+1}), s)| = |y'(t_i)| + |y'(t_{i+1})|.$$

Then we estimate $E(s)$ as following

$$\begin{aligned} E(s) &= \int_0^a |d_x(\Theta(x,s))|^2\, dx \\ &\geq \sum_{i=1}^{n-1} \int_{x(t_i)}^{x(t_{i+1})} |d_x(\Theta(x,s))|^2\, dx \\ &\geq \sum_{i=1}^{n-1} \int_{x(t_i)}^{x(t_{i+1})} \left| \frac{\Theta(x(t_{i+1}), s) - \Theta(x(t_i), s)}{x(t_{i+1}) - x(t_i)} \right|^2 dx \\ &\geq \sum_{i=1}^{n-1} \frac{(|y'(t_i)| + |y'(t_{i+1})|)^2}{x(t_{i+1}) - x(t_i)} \\ \text{(Cauchy-Schwarz)} &\geq \frac{(\sum_{i=1}^{n} |y'(t_i)|)^2}{x(t_n) - x(t_1)} \\ &\geq \sum_{t \in T_s} |y'(t)|^2. \end{aligned} \tag{2.18}$$

Using coarea formula, we get

$$\int_B |y'(t)|^3\,dt = \int_U \left(\sum_{t\in T_s} |y'(t)|^2 \right) ds \tag{2.19}$$

Then by estimating the Dirichlet energy inside Q using (2.18) and (2.19), we have that

$$\begin{aligned} C_4 &\geq \int_Q |\nabla\Theta(x,y)|^2\,dxdy \\ &\geq \int_V E(s)\,ds \\ &\geq \int_V \left(\sum_{t\in T_s} |y'(t)|^2 \right) ds \\ &= \int_B |y'(t)|^3\,dt \end{aligned} \tag{2.20}$$

Hölder inequality further implies that

$$\left(\int_B |y'(t)|^3\,dt \right) \geq \left(\int_B |y'(t)|^2\,dt \right)^{3/2} \cdot m(B)^{-1/2}. \tag{2.21}$$

By (2.16) and (2.17), we have

$$\int_B |y'(t)|^2\,dt \geq c-1-\int_A |y'(t)|^2\,dt - \int_W |y'(t)|^2\,dt \geq c-4. \tag{2.22}$$

As a result, combining (2.20), (2.21) and (2.22) leads to

$$C_4 \geq \frac{(c-4)^{3/2}}{c^{1/2}},$$

which yields a contradiction if we choose the constant C_3 to be large enough at first (recall that c is a real number larger than C_3). Now that since C_4 only depends on M, C_3 also only depends on M. This completes our proof of Proposition 2.4. □

Actually, we can examine the chord-arc property of Γ more closely and prove that it is indeed a vanishing chord-arc (also called "approximately smooth") curve, which is the following lemma.

Proposition 2.5 *Let $\Gamma \in \mathscr{G}_1$ and $E(\Gamma) \leq M$. For any $\varepsilon > 0$, there exists a $r = r(\varepsilon, \Gamma)$ such that for any two points $z_1, z_2 \in \Gamma$ that satisfy $|z_1 - z_2| \leq r$, we have*

$$l(\gamma) \leq (1+\varepsilon)|z_1 - z_2|,$$

where $\gamma = \Gamma_{z_1 z_2}$. That is to say, Γ is a vanishing chord-arc (approximately smooth) curve.

Proof The technique here will be very similar to the proof of Proposition 2.4. We will only present our main ingredients and omit some computational details. Take any $z_1, z_2 \in \Gamma$ such that $|z_1 - z_2| = r$. By Lemma 2.2, $\gamma = \Gamma_{z_1 z_2}$ must be contained in a rectangular domain $Q_{z_1 z_2}$ with width r and length $2Cr$ (see (2.15) for the existence of such rectangle). Let Θ be the angle function determined by Γ, we again extend Θ to $\mathbb{R}^2$ such that $\int_{\mathbb{R}^2} |\nabla\Theta|^2\, dx < C_2 M$. Then for any $\delta > 0$, there exists a $\sigma > 0$ such that

$$\int_E |\nabla\Theta|^2\, dx \le \delta, \quad \text{whenever } |E| < \sigma.$$

Therefore as $r \to 0$, the Dirichlet energy of Θ inside $Q_{z_1 z_2}$ will go to zero. The convergence rate doesn't depend on the choice of z_1, z_2 but only depends on their distance r. As a consequence, in order to prove the lemma, we only need to prove the following statement: for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$\int_{Q_{z_1 z_2}} |\nabla\Theta|^2\, dx \ge C(\varepsilon), \text{ whenever } l(\gamma) \ge (1+\varepsilon)|z_1 - z_2| \tag{2.23}$$

Now we fix $\varepsilon > 0$. By scaling invariant property, we assume without loss of generality that $|z_1 - z_2| = 1$ and $l(\gamma) = 1 + \varepsilon$, $z_1 = (0,0)$, $z_2 = (\cos\alpha, \sin\alpha)$ for some $\alpha \in (0, \frac{\pi}{2})$. Note that here $|\alpha| \ne \frac{\pi}{2}$, otherwise γ would be the line segment orthogonal to $x-$axis and $l(\gamma) = 1$. We parameterize γ as

$$\gamma := \{(x,y) : (x(0), y(0)) = z_1,\ (x(1+\varepsilon), y(1+\varepsilon)) = z_2,\ |x'(t)|^2 + |y'(t)|^2 = 1, \text{ a.e.}\}$$

Set

$$h(t) := \cos\alpha \cdot y(t) - \sin\alpha \cdot x(t), \quad g(t) := \sin\alpha \cdot y(t) + \cos\alpha \cdot x(t).$$

We have

$$\int_0^{1+\varepsilon} x'(t)\, dt = \cos\alpha, \quad \int_0^{1+\varepsilon} y'(t)\, dt = \sin\alpha, \quad h(0) = h(1+\varepsilon) = 0. \tag{2.24}$$

Set $h_{max} = \max_{0 \le t \le 1+\varepsilon} h(t)$ and $h_{min} = \min_{0 \le t \le 1+\varepsilon} h(t)$, and define

$$T_s := \{t \in [0, 1+\varepsilon] : h(t) = s\} \text{ for } s \in [h_{min}, h_{max}],$$
$$B := \{t \in [0, 1+\varepsilon] : 2 \le T_{h(t)} < \infty\}.$$

Obviously for any $s \in (h_{min}, h_{max})$ we have $|T_s| \ge 2$. Also we should deduct the subset of $[h_{min}, h_{max}]$ such that $|T_s|$ is infinite (see the definition of set U, W in the proof of Proposition 2.4). But from the argument in the proof of Proposition 2.4 we know it is a measure zero set and won't affect our computation, so we may simply assume $2 \le |T_s| < \infty$ for any $s \in (h_{min}, h_{max})$.

We discuss in two cases.

Case 1 If $g'(t) = \sin\alpha \cdot y'(t) + \cos\alpha \cdot x'(t) \ge 0$ for a.e. $t \in [0, 1+\varepsilon]$. We firstly define

$$\partial_g \Theta := \sin\alpha\, \partial_y \Theta + \cos\alpha\, \partial_x \Theta, \quad \partial_h \Theta := \cos\alpha\, \partial_y \Theta - \sin\alpha\, \partial_x \Theta.$$

We calculate in the same way as (2.18), (2.19) and (2.20) (the only difference is we replace $y(t)$ with $h(t)$) and obtain

$$\int_{Q_{z_1z_2}} |\nabla\Theta|^2\,dxdy \geq \int_{Q_{z_1z_2}} |\partial_g\Theta|^2\,dxdy \geq C\int_B |h'(t)|^3\,dt \tag{2.25}$$

Here C is a positive constant only depend on M. Also, from (2.24) and the assumption $g'(t)\geq 0$ we have

$$\begin{aligned}&\int_0^{1+\varepsilon} |\sin\alpha\cdot y'(t)+\cos\alpha\cdot x'(t)|^2\,dt \\ =&\int_0^{1+\varepsilon} |\sin\alpha\cdot y'(t)+\cos\alpha\cdot x'(t)|\cdot(\sin\alpha\cdot y'(t)+\cos\alpha\cdot x'(t))\,dt \\ \leq&\int_0^{1+\varepsilon} (\sin\alpha\cdot y'(t)+\cos\alpha\cdot x'(t))\,dt = 1\end{aligned} \tag{2.26}$$

and

$$\int_0^{1+\varepsilon} |\sin\alpha\cdot y'(t)+\cos\alpha\cdot x'(t)|^2+|\cos\alpha\cdot y'(t)-\sin\alpha\cdot x'(t)|^2\,dt = 1+\varepsilon$$

The above two inequalities imply

$$\int_0^{1+\varepsilon} |h'(t)|^2\,dt \geq \varepsilon.$$

By co-area formula we know that the set where $h(t)=h_{min}$ or h_{max} contributes nothing in the above integral, so we have

$$\int_B |h'(t)|^2\,dt \geq \varepsilon. \tag{2.27}$$

Then we combine (2.25), (2.27) and Hölder inequality to conclude that

$$\int_{Q_{z_1z_2}} |\nabla\Theta|^2\,dxdy \geq C\frac{\varepsilon^{3/2}}{(1+\varepsilon)^{1/2}}. \tag{2.28}$$

Case 2 Assume $g'(t)\geq 0$ doesn't hold almost everywhere, then we may lose the estimate (2.26). If we still have $\int_0^{1+\varepsilon}|g'(t)|^2 \leq 1$, then all the estimates in Case 1 still hold and there is nothing to prove. So we assume

$$\int_0^{1+\varepsilon} |g'(t)|^2\,dt = 1+\delta \quad \text{for some } 0<\delta\leq\varepsilon. \tag{2.29}$$

Then

$$\int_B |h'(t)|^2\,dt = \varepsilon-\delta. \tag{2.30}$$

Then the same computation leads to

$$\int_{Q_{z_1z_2}} \left|\partial_g\Theta\right|^2\,dxdy \geq C\frac{(\varepsilon-\delta)^{3/2}}{(1+\varepsilon)^{1/2}}. \tag{2.31}$$

Now we set

$$
\begin{aligned}
&g_{max} := \max_{0\le t\le 1+\varepsilon} g(t), \quad g_{min} := \min_{0\le t\le 1+\varepsilon} g(t),\\
&T_s^2 := \{t\in[0,1+\varepsilon] : g(t)=s\} \text{ for } s\in[g_{min},g_{max}],\\
&B_2 := \{t\in[0,1+\varepsilon] : 2\le |T^2_{g(t)}|<\infty\},\\
&A_2 := \{t\in[0,1+\varepsilon] : g'(t)<0\}.
\end{aligned}
$$

Again we can ignore the set where $|T^2_{g(t)}|=\infty$ since it may lead to more complicated notations but won't affect any of our estimates. So we assume for every $t\in[0,1+\varepsilon]$, we have $|T^2_{g(t)}|<\infty$. Then simple geometry tells us that $A_2\subset B_2$. Also, since $\int_0^{1+\varepsilon} g'(t)\,dt=1$ and $\int_0^{1+\varepsilon}|g'(t)|\ge 1+\delta$, it holds that

$$
\int_{B_2}|g'(t)|\,dt \ge \int_{A_2}|g'(t)|\,dt \ge \frac{\delta}{2}. \tag{2.32}
$$

Similar techniques in (2.18), (2.19) and (2.20) imply that

$$
\int_{Q_{z_1z_2}}|\partial_h\Theta|^2\,dxdy \ge C\int_{B_2}|g'(t)|^3\,dt \ge C\frac{(\delta/2)^3}{(1+\varepsilon)^2}. \text{ (Hölder inequality)} \tag{2.33}
$$

where C is a constant that only depends on M. We can combine this with (2.31) to get

$$
\begin{aligned}
&\int_{Q_{z_1z_2}}|\nabla\Theta|^2\,dxdy\\
\ge&\int_{Q_{z_1z_2}}|\partial_g\Theta|^2\,dxdy+\int_{Q_{z_1z_2}}|\partial_h\Theta|^2\,dxdy\\
\ge&C\left(\frac{(\varepsilon-\delta)^{3/2}}{(1+\varepsilon)^{1/2}}+\frac{(\delta/2)^3}{(1+\varepsilon)^2}\right)\\
\ge&C(\varepsilon,M)
\end{aligned}
$$

This implies (2.23) and completes our proof of Proposition 2.5. □

Corollary 2.6 *The normal vector ν along the curve Γ belongs to* VMO *(vanishing mean oscillation space), i.e.*

$$
\lim_{r\to 0}\left(\frac{1}{l(B(x,r)\cap\Gamma)}\int_{B(x,r)\cap\Gamma}|\nu-\nu_{B(x,r)}|\,dl\right)=0 \;\textit{ uniformly for } x\in\Gamma,
$$

where

$$
\nu_{B(x,r)}=\frac{1}{l(B(x,r)\cap\Gamma)}\int_{B(x,r)\cap\Gamma}\nu\,dl
$$

Proof We already have that Γ is a vanishing chord-arc curve. Then we can direct apply results of Kenig & Toro [25, Theorem 2.1] to conclude that $\nu\in$ VMO. □

2.3 Weil-Petersson curve and $H^{3/2}$ characterization

Recall that a quasicircle is the image of the unit circle $\mathbb{T}$ under a quasiconformal map f of $\mathbb{R}^2$, i.e., a homeomorphism of the plane that is conformal outside the unit disk $\mathbb{D}$, whose dilatation $\mu = f_{\bar{z}}/f_z$ satisfies $\|\mu\|_{L^\infty(\mathbb{D})} < 1$. The collection of planar quasicircles corresponds to the universal Teichmüller space $T(1)$ and the metric is defined in terms of $\|\mu\|_\infty$. We refer to [18,19] for an introduction to the uniform Teichmüller space and more details. In [41] Takhtajan and Teo defined a Weil-Petersson metric on the universal Teichmüler space $T(1)$ that makes it into a Hilbert manifold. A Weil-Petersson curve is the image of $\mathbb{T}$ under a quasiconformal map f on the plane, and satisfies $|\mu| \in L^2(dA_\rho)$, where $dA_\rho = \frac{4}{(1-|z|^2)^2} d^2z$ is the hyperbolic metric on $\mathbb{D}$ (see [8,41]). Another characterization for the Weil-Petersson curve is in terms of conformal mapping $f : \mathbb{D} \to \Omega$, where Ω is the domain bounded by Γ. Γ is a Weil-Petersson curve if and only if $(\log f')' \in L^2(\mathbb{D})$ (see [41, Chapter 2]).

In our problem, the curves in $\mathscr{G}_V$ strongly resemble the Weil-Petersson curves in the following way. For a Weil-Petersson curve Γ, let $f : \mathbb{D} \to \Omega$ be a conformal mapping. We focus on the boundary map $f : \mathbb{T} \to \Gamma$. Since $\log f'$ is in the Dirichlet space, we have that $\arg f'(z)$ (arg means the argument of a complex number), as a function on $\mathbb{T}$, has an extention with finite Dirichlet energy inside $\mathbb{D}$. One can check that for any $a \in \mathbb{T}$, it holds $\arg f'(a) = \arg \nu_\Gamma(f(a)) - 2\pi a$, where $\nu_\Gamma(f(a))$ is the outer normal vector of Γ at the point $f(a)$. Thus $\arg \nu_\Gamma(b) - 2\pi f^{-1}(b)$, as a function of $b \in \Gamma$, has a finite Dirichlet energy inside Ω. Note that in our definition of $\mathscr{G}_V$, we require the $\theta = \arg \nu_\Gamma - \frac{\pi}{2}$ on the curve Γ, and it has a finite energy extension inside Ω_Γ. Such characterization is very similar to the Weil-Petersson curve. The difference is that in our problem Γ is not a closed curve and the domain Ω_Γ is not a quasidisk.

In a recent work [5], Christopher Bishop gives 26 equivalent characterizations of the Weil-Petersson class. In particular, he shows that a curve Γ is Weil-Petersson if and only if it has arclength parametrization in $H^{3/2}(\mathbb{T})$, has finite Möbius energy or can be well approximated by polygons in some precise sense. Another equivalent characterization is that Weil-Peterssson curve has local curvature that is square integrable over all locations and scales, where local curvatures are measured using various quantities such as Peter Jone's β-numbers, conformal welding and Menger curvature. We will show that some of these function theoretic and geometric characterizations can be generalized to our curve $\Gamma \in \mathscr{G}_V$, which greatly deepen our understanding of the class $\mathscr{G}_V$. The proofs will follow Bishop's paper [5] closely, with some necessary modifications.

Given $\Gamma \in \mathscr{G}_V$, we denote the length of Γ by l. Let $z(t) = (x(t), y(t)) : [0,l] \to \Gamma$ be the arc-length parametrization of Γ, i.e. $\sqrt{x'(t)^2 + y'(t)^2} = 1$ almost everywhere. Then Γ has the following properties.

Proposition 2.7 *1. The arc-length parametrization $z(t) : [0,l] \to \Gamma$ is in the Sobolev space $H^{3/2}([0,l])$.*

2. Let ν be the normal vector, it holds that

$$\int_\Gamma \int_\Gamma \left(\frac{|\nu(z) - \nu(w)|}{|z-w|} \right)^2 |dz|\,|dw| < \infty,$$

where $|dz|$, $|dw|$ denote the element of arc length in the line integral on Γ.

Proof (1) As in the proof of Lemma 2.3, we construct a quasidisk D_Γ which is the combination of Ω_Γ and a rectangle with length $2a$ and width a

$$D_\Gamma = \Omega_\Gamma \cup \{(x,y) : -a < x < a,\ -a < y \le 0\}.$$

The length of ∂D_Γ is $l+4a$. We denote the arc-length between $z \in \Gamma$ and $(-a,0)$ as $l(z)$. Define the function $\phi : \mathbb{T} \to \mathbb{T}$ as

$$\phi(\theta) = \begin{cases} \arg\, \nu(z((l+4a)\theta)), & 0 \le \theta \le \frac{l}{l+4a}, \\ \frac{\pi}{2}, & \frac{l}{l+4a} < \theta < 1. \end{cases} \tag{2.34}$$

One can easily check that in order to show $z \in H^{3/2}([0,l])$, it suffices to show $\phi \in H^{1/2}(\mathbb{T})$. We also define an orientation preserving arclength parameterization $w : \mathbb{T} \to \partial D_\Gamma$, such that $|w'| = l+4a$, $w(0) = (-a,0)$ and $w(\theta) = z((l+4a)\theta)$ for $\theta \in [0, \frac{l}{l+4a}]$. Since D_Γ is a quasidisk, we can find a map f that is conformal in D_Γ and can be extended to a quasi-conformal mapping in the entire plane. Then on the boundary, f maps $\mathbb{T}$ to the quasicircle ∂D_Γ.

Let $\phi_f := \phi \circ w^{-1} \circ f$. By the definition of $\mathscr{G}_V$, one has $\phi_f \in H^{1/2}(\mathbb{T})$. The rest of the proof is exactly the same as that of [5, Lemma 8.1]. The idea is to show $f^{-1} \circ w$ is a quasisymmetric map by definition, and then use the arguments by Beurling and Ahlfors [3] that $H^{1/2}$ is invariant under composition with a quasisymmetric homeomorphism of $\mathbb{T}$.

(2) By the $H^{3/2}$ characterization, we know $\int_0^l \int_0^l \left| \frac{z'(t)-z'(s)}{t-s} \right|^2 dtds < \infty$. Since Γ is chord-arc, $\frac{|z(t)-z(s)|}{|s-t|} \in [\frac{1}{C}, 1]$ for some constant C. We have

$$\begin{aligned} \int_\Gamma \int_\Gamma \left(\frac{|\nu(z) - \nu(w)|}{|z-w|} \right)^2 |dz|\,|dw| &= \int_0^l \int_0^l \left(\frac{|z'(t) - z'(s)|}{|z(t) - z(s)|} \right)^2 dsdt \\ &\simeq \int_0^l \int_0^l \left| \frac{z'(t) - z'(s)}{t-s} \right|^2 dsdt < \infty. \end{aligned}$$

□

Remark 2.8 A direct consequence of the $H^{3/2}$ characterization is Γ has finite Möbius energy, i.e.

$$M\ddot{o}b(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|z-w|^2} - \frac{1}{l(z,w)^2} \right) |dz|\,|dw| < \infty. \tag{2.35}$$

Here $l(z,w)$ is the length of Γ between z and w. For the proof, one can refer to the proof of [5, Lemma 9.1]. The brief idea is to show that $M\ddot{o}b(\Gamma) \simeq \int_\Gamma \int_\Gamma \frac{\int_{\gamma_{z,w}} \int_{\gamma_{z,w}} |\nu(x)-\nu(y)|^2 |dx|\,|dy|}{|z-w|^4} |dz|\,|dw|$ and then change the order of integration. Furthermore, since Γ is chord-arc, it holds that

$$\frac{1}{|z-w|^2} - \frac{1}{l(z,w)^2} = \frac{(l(z,w) + |z-w|)(l(z,w) - |z-w|)}{|z-w|^2 l(z,w)^2} \simeq \frac{l(z,w) - |z-w|}{|z-w|^3}.$$

Then (2.35) implies that $\int_\Gamma \int_\Gamma \frac{l(z,w)-|z-w|}{|z-w|^3} |dz|\,|dw| < \infty$.

Remark 2.9 Other characterizations of Weil-Petersson curve in [5] include approximation by polygons in a precise sense and the square integrability of β-numbers. The arguments also work for curves in $\mathscr{G}_V$ and we state them here without proof. Again we consider the arc-length parameterization $z(t) : [0,l] \to \Gamma$. For each n, let $z_j^n = z\left(\frac{jl}{2^n}\right)$ for $j = 0,1,\dots,2^n$. Then it is obvious that $\{z_j^n\}$ divides Γ into 2^n intervals with equal length. Let Γ_n be the curve that consists of all the line segments $z_j^n z_{j+1}^n$ for $j = 0,\dots,2^n-1$. Then one has

$$\sum_{n=1}^{\infty} 2^n [l(\Gamma) - l(\Gamma_n)] < \infty.$$

Recall one of the equivalent definitions of Peter Jone's β-number (see [23]): given a curve Γ, $x \in \mathbb{R}^2$ and $t > 0$,

$$\beta_\Gamma(x,t) := \inf_L \sup_{z \in B(x,t)\cap\Gamma} \frac{\operatorname{dist}(z,L)}{t},$$

where the infimum is over all lines hitting $B(x,t)$. Then for $\Gamma \in \mathscr{G}_V$, it satisfies

$$\int_\Gamma \int_0^\infty \beta_\Gamma^2(x,t) \frac{dt\,dx}{t^2} < \infty.$$

3 Existence of minimizers

The primary goal of this section is to establish the existence for Problem 1.5. Before we state the theorem, we need to clarify some basic settings. Throughout this section we assume the volume $V = 1$. We will consider $\Gamma \in \mathscr{G}_1$ such that $E_\Gamma \le M$ for some constant $M > 0$ and Γ will only intersects with $x-$axis at two endpoints. As a consequence, Γ has all the geometric properties that we have shown in Section 2 (Lemma 2.2, Lemma 2.3, Proposition 2.4, Proposition 2.5 and Proposition 2.7).

Also we need to discuss different notions of boundary since we will perform integration by parts in Ω_Γ. In geometric measure theory, there are three different kinds of boundary for a set E of finite perimeter: topological boundary ∂E, measure-theoretical boundary $\partial^e E$ and reduced boundary $\partial^* E$. We refer to [14] for detailed definitions of these notions. It is well-known that $\partial^* E \subset \partial^e E \subset \partial E$. For our domain Ω_Γ, it is obvious that $\partial\Omega_\Gamma = \Gamma \cup \{(x,0), -a \le x \le a\}$. By Lemma 2.2, one can verify that for any $z \in \Gamma\backslash\{(-a,0),(a,0)\}$, we have

$$\liminf_{r\to 0} \frac{|\Omega_\Gamma \cap B(z,r)|}{\pi r^2} > 0, \quad \limsup_{r\to 0} \frac{|\Omega_\Gamma \cap B(z,r)|}{\pi r^2} < 1$$

This means $z \in \partial^e\Omega_\Gamma$, and therefore $\partial\Omega_\Gamma\backslash\partial^e\Omega_\Gamma \subset \{(-a,0),(a,0)\}$. As for the relation between measure-theoretical boundary and reduced boundary, a well-known result by Federer says that $\mathscr{H}^1(\partial^e E\backslash\partial^* E) = 0$ (see for instance [12, Lemma 1 in Section 5.8]). So in the proof below, we don't distinguish these different notions of boundary when we write boundary integral.

Theorem 3.1 *There exists a $\Gamma \in \mathscr{G}_1$ minimizing the functional $E(\Gamma)$ defined by* (1.8).

Proof Let $\{\Gamma_i\}_{i=1}^{\infty}$ be a minimizing sequence in $\mathscr{G}_1$.

$$\lim_{i\to\infty} E(\Gamma_i) = M_0 := \inf_{\Gamma\in\mathscr{G}_1} E(\Gamma).$$

Let Ω_i, Θ_i denote the corresponding $\Omega_{\Gamma_i}, \Theta_{\Gamma_i}$ respectively. For each i, we set $(\pm a_i, 0)$ as the two endpoints of Γ_i. Because $l(\Gamma_i) \leq M$ for every i, we have $a_i < \frac{M}{2}$. Also by Lemma 2.2 and the fact that $|\Omega_i| = 1$ we have $a_i \geq \frac{2}{C}$ where C is the constant in (2.9). Now we summarize all the properties (independent of i) we need for $\{(\Gamma_i, \Omega_i, \Theta_i, a_i)\}$ before taking a limit.

(a) $\frac{2}{C} \leq a_i \leq \frac{M}{2}$, $l(\Gamma_i) \leq M$.
(b) $\overline{\Omega}_i \subset B(0,2M)$, $|\Omega_i| = 1$ and $\partial\Omega_i = \Gamma_i \cup \{(x,0) : -a_i \leq x \leq a_i\}$.
(c) Γ_i can be parameterized by $(x_i(t), y_i(t))$ such that

$$\begin{gathered} x_i(0) = -a_i,\ x_i(l(\Gamma_i)) = a_i,\ y_i(0) = y_i(l(\Gamma_i)) = 0, \\ x_i'(t) \geq 0,\ y_i(t) \geq 0,\ |x_i'(t)|^2 + |y_i'(t)|^2 = 1,\ a.e. \\ \sqrt{|x_i(t+s) - x_i(t)|^2 + |y_i(t+s) - y_i(t)|^2} \geq \frac{s}{C_3} \text{ for } t,s>0,\ t+s<l(\Gamma_i), \end{gathered}$$

where C_3 is the constant in Proposition 2.4.
(d) Θ_i can be extended to a H^1 function on $B(0,2M)$ such that $\|\Theta_i\|_{H^1(B(0,2M))} \leq C_5$ for some universal constant C_5.

Then we claim that there is a subsequence, still denoted by $\{(\Gamma_i, \Omega_i, \Theta_i, a_i)\}$ that converges in the following sense:

1. $\chi_{\Omega_i} \to \chi_\Omega$ weakly in $BV(B(0,2M))$ and strongly in $L^1(B(0,2M))$, for some Ω which is a set of finite perimeter in $B(0,2M)$ with volume 1.
2. $\Theta_i \to \Theta$ weakly in $H^1(B(0,2M))$ and strongly in $L^2(B(0,2M))$ for some $\Theta \in H^1(B(0,1))$.
3. $l(\Gamma_i) \to l$, $a_i \to a$ for some constant $l > 0, a > 0$.
4. $\Gamma_i \to \Gamma$ in Hausdorff distance for some chord-arc curve Γ. Γ can be parameterized by $x(t), y(t)$ such that

$$\begin{aligned} &x(0) = -a,\ x(l) = a,\ y(0) = y(l) = 0, \\ &x'(t) \geq 0,\ y(t) \geq 0,\ |x'(t)|^2 + |y'(t)|^2 \leq 1,\ a.e. \\ &\sqrt{|x(t+s) - x(t)|^2 + |y(t+s) - y(t)|^2} \geq \frac{s}{C_3} \text{ for } t,s>0,\ t+s<l \end{aligned}$$

5. $\partial^e\Omega \subset \Gamma \cup \{(x,0) : -a \leq x \leq a\}$, $\nu \cdot (\cos\Theta, \sin\Theta) = 0$ a.e. on $\partial^*\Omega$.

Proof (Proof of the convergence claim) (1), (2), (3) are straightforward to check. (4) is a direct consequence of Arzela-Ascoli lemma and Property (c) that we list before, we omitted the detail of the proof. So we only prove (5). First we show $\partial^e\Omega \subset \Gamma \cup \{(x,0) : -a \leq x \leq a\}$. Assume there exists a point $z \in \partial^e\Omega$ such that $z \notin \Gamma \cup \{(x,0) : -a \leq x \leq a\}$. Then by convergence properties (3) and (4), there exists a $r_0 > 0$ and $n \in \mathbb{N}$ such that

$$B(z, r_0) \cap \partial\Omega_i = \varnothing,\ \forall i \geq n.$$

For any $i \geq n$, we have

$$\frac{|\Omega_i \cap B(z,r)|}{|B(z,r)|} = 0 \text{ or } 1,\ \text{for any } r \leq r_0$$

However, by convergence property (1), we have

$$\frac{|\Omega \cap B(z,r)|}{|B(z,r)|} = \lim_{i\to\infty} \frac{|\Omega_i \cap B(z,r)|}{|B(z,r)|} = 0 \text{ or } 1, \text{ for any } r \le r_0,$$

which contradicts with our assumption $z \in \partial^e \Omega$. Therefore we have proved $\partial^e \Omega \subset \Gamma \cup \{(x,0) : -a \le x \le a\}$.

Now we set

$$\Omega_{in} := \text{ the domain enclosed by } \Gamma \text{ and x-axis}, \quad \Omega_{out} := \mathbb{R}^2 \setminus (\Gamma \cup \{(x,0) : -a \le x \le a\} \cup \Omega_{in})$$

By similar density argument one can show $\Omega_{in} \subset \Omega^0$ and $\Omega_{out} \subset \Omega^1$, where Ω^t is defined as $\{z : \lim_{r\to 0} \frac{|\Omega\cap B(z,r)|}{\pi r^2} = t\}$. After a modification of a measure zero set, we can simply identify Ω as Ω_{in}. Now we are left to show the second part of (5), which says the tangential anchoring boundary condition still holds for the limit domain. Let ϕ be an arbitrary C^∞ function in $\mathbb{R}^2$, we define

$$n_i := (\cos\Theta_i, \sin\Theta_i), \quad n := (\cos\Theta, \sin\Theta), \quad \nu_i = \text{normal vector on } \partial\Omega_i.$$

Note that here all n_i and n are defined on the larger domain $B(0,2M)$. We first deduce that

$$\lim_{i\to\infty} \int_{\Omega_i} \operatorname{div}(\phi n_i)\,dx = \int_{\Omega} \operatorname{div}(\phi n)\,dx. \tag{3.36}$$

In fact,

$$\begin{aligned} &\Big|\int_{\Omega_i} \operatorname{div}(\phi n_i)\,dx - \int_{\Omega} \operatorname{div}(\phi n)\,dx\Big| \\ \le &\Big|\int_{\Omega} \operatorname{div}(\phi n - \phi n_i)\,dx\Big| + \Big|\int_{\Omega\Delta\Omega_i} |\operatorname{div}(\phi n_i)|\,dx\Big| \end{aligned}$$

As $i \to \infty$, the first term goes to zero because n_i converges to n weakly in H^1; the second term goes to zero since Ω_i converges to Ω in L^1 and n_i are uniformly bounded in H^1. Also we have that the following Gauss-Green formula holds

$$\int_{\Omega_i} \operatorname{div}(\phi n_i)\,dx = \int_{\partial^*\Omega_i} \phi n_i \cdot \nu_i \, d\mathscr{H}^1, \quad \int_{\Omega} \operatorname{div}(\phi n)\,dx = \int_{\partial^*\Omega} \phi n \cdot \nu \, d\mathscr{H}^1 \tag{3.37}$$

We want to point out that (3.37) is not trivial here since $\partial\Omega_i$ is not in C^1. However in our problem, it is valid because all Ω_i and Ω are Sobolev extension domains and one can define the trace of H^1 function on the reduced boundary. We refer to [28, Proposition 3.4.4] or [2, Theorem 3.84] for more details. We may now combine (3.37) with (3.36) and get that

$$0 = \lim_{i\to\infty} \int_{\partial\Omega_i} \phi n_i \cdot \nu_i \, d\mathscr{H}^1 = \int_{\partial\Omega} \phi n \cdot \nu \, d\mathscr{H}^1.$$

Thus the tangential anchoring boundary condition is proved for Ω and n.

On the boundaries $\partial\Omega_i$ or $\partial\Omega$, we define the tangent vector τ_i, or correspondingly τ, by rotating the normal vector ν_i or ν by $\frac{\pi}{2}$ clockwise. One can check that

$$n_i = \tau_i \text{ on } \Gamma_i \backslash \{(x,0) : x \in [-a_i, a_i]\}, \quad n_i = -\tau_i \text{ on } \{(x,0) : x \in [-a_i, a_i]\} \backslash \Gamma_i, \quad \forall i \in \mathbb{N}$$

Using similar arguments in the proof of tangential anchoring condition above again, we can show that

$$\begin{aligned} &n = \tau \quad \text{a.e. on } \Gamma \backslash \{(x,0) : x \in [-a,a]\}, \\ &n = -\tau = (1,0) \quad \text{a.e. on } \{(x,0) : x \in [-a,a]\} \backslash \Gamma, \end{aligned} \tag{3.38}$$

The idea is to carefully choose a cut-off function ϕ and calculate $\int_{\partial^*\Omega} \phi n \cdot \tau \, d\mathscr{H}^1$ using Gauss-Green formula. To be more specific, for any point $x \in \Gamma \backslash \{(x,0) : x \in [-a,a]\}$, and any small radius r, we take a non-negative smooth function ϕ such that $\phi \equiv 1$ in $B_r(x)$ and $\phi \equiv 0$ in $\mathbb{R}^2 \backslash B_{2r}(x)$. Then by calculating $\int_{\partial^*\Omega} \phi n \cdot \tau \, d\mathscr{H}^1$ as in (3.37) and letting x and r vary, one can obtain that $n \cdot \tau \geq 0$ (further implies $n = \tau$ because $n \cdot \nu = 0$) almost everywhere on $\Gamma \backslash \{(x,0) : x \in [-a,a]\}$. The points on x-axis can be treated in the same way and we omit the details here. Note that (3.38) is equivalent to our original boundary condition (1.7), therefore we have verified that $\Gamma \in \mathscr{G}_1$ and Ω, Θ are just the corresponding Ω_Γ, Θ_Γ.

Finally, by convergence result (1–5) and lower semi-continuity we conclude that

$$\int_\Omega |\nabla \Theta|^2 + l(\Gamma) \leq \liminf_{i \to \infty} E(\Gamma_i) = M_0$$

And by Lemma 2.1, Γ won't touch x-axis besides two endpoints. So (Γ, Ω, Θ) is a minimizer of Problem 1.5. The proof is complete. □

Next we want to study the behavior of $\Gamma \in \mathscr{G}_1$ near $(-a,0)$ and $(a,0)$. The following lemma indicates that Γ and $x-$axis form approximately cusps near two ends. Note that here we don't assume Γ is a minimizer.

Lemma 3.2 *Let $\Gamma \in \mathscr{G}_1$ satisfy $E(\Gamma) \leq M$. Γ only intersects with $x-$axis at $z_1 = (-a,0)$ and $z_2 = (a,0)$. For any $k > 0$, there exists a constant r that depends on k and Γ such that*

$$\begin{aligned} &\text{If } z = (x,y) \in \Gamma \cap B(z_1, r), \text{ then } \frac{y}{x+a} \leq k, \\ &\text{If } z = (x,y) \in \Gamma \cap B(z_2, r), \text{ then } \frac{y}{a-x} \leq k. \end{aligned}$$

Remark 3.3 This lemma implies that as $z \in \Gamma$ approaches z_1(or z_2), the angle between the ray $z - z_1$(or $z - z_2$) and $x-$axis converges to 0.

Proof (Proof of Lemma 3.2) Without loss of generality, we only prove the lemma near $z_1 = (-a,0)$. We argue by contradiction. Assume the Lemma is false, there would be a constant $k > 0$, a sequence of radiuses $\{r_i\}_{i=1}^\infty$ and a sequence of points $\{(x_i, y_i)\}_{i=1}^\infty \subset \Gamma$ such that

$$r_i \to 0, \quad \sqrt{(x_i+a)^2 + y_i^2} = r_i, \quad \frac{y_i}{x_i+a} = k_i \geq k.$$

By Lemma 2.3 we can extend Θ from Ω_Γ to the whole $\mathbb{R}^2$ such that $\|\Theta\|_{H^1(\mathbb{R}^2)} \leq C$. For every i, we introduce the following rescaled functions:

$$\Gamma_i := \{\frac{z-z_1}{r_i} : z \in \Gamma \cap B(z_1, r_i)\},$$
$$\Omega_i := \{\frac{z-z_1}{r_i} : z \in \Omega_\Gamma \cap B(z_1, r_i)\},$$
$$\Theta_i(z) := \Theta(z_1 + r_i z) \text{ for } z \in B(0,1)$$

One can easily check the following properties hold

(a) $l(G_i) \leq C_3$ for the constant C_3 from Proposition 2.4.
(b) $\Omega_i \subset B(0,1)$, $|\Omega_i| \geq \frac{\arctan k}{2} - \frac{k}{2(1+k^2)} =: C_6$
(c) Γ_i can be parameterized by $(x_i(t), y_i(t))$ such that

$$x_i(0) = 0,\ x_i(l(\Gamma_i)) = \frac{1}{\sqrt{1+k_i^2}},\ y_i(0) = 0,\ y(l(\Gamma_i)) = \frac{k_i}{\sqrt{1+k_i^2}},$$
$$x_i'(t) \geq 0,\ y_i(t) \geq 0,\ |x_i'(t)|^2 + |y_i'(t)|^2 = 1,\ a.e.$$
$$\sqrt{|x_i(t+s) - x_i(t)|^2 + |y_i(t+s) - y_i(t)|^2} \geq \frac{s}{C_3} \text{ for } t,s > 0,\ t+s < l(\Gamma_i)$$

(d) $\{\Theta_i\}_{i=1}^\infty$ is uniformly bounded in $H^1(B(0,1))$ and we have

$$\lim_{i\to\infty} \int_{B(0,1)} |\nabla\Theta_i|^2\, dx = 0$$

Passing if necessary to a subsequence, we get

1. $\chi_{\Omega_i} \to \chi_\Omega$ weakly in $BV(B(0,1))$ and strongly in $L^1(B(0,1))$, for some Ω with the volume lower bound $|\Omega| \geq C_6$.
2. $\Theta_i \to \Theta$ weakly in $H^1(B(0,1))$ and strongly in $L^2(B(0,1))$ for some $\Theta \in H^1(B(0,1))$.
3. $l(\Gamma_i) \to l$ for a constant $l > 0$.
4. $k_i \to k_0$ for some $k_0 \in [k, \infty]$.
5. $\Gamma_i \to \Gamma^*$ in the sense of Hausdorff distance for some chord-arc curve Γ^*. Γ^* can be parameterized by $(x(t), y(t))$ such that

$$(x(0), y(0)) = (0,0),$$
$$(x(l), y(l)) = \begin{cases} (\frac{1}{\sqrt{1+k_0^2}}, \frac{k_0}{\sqrt{1+k_0^2}}), & k_0 \neq +\infty, \\ (0,1), & k_0 = +\infty. \end{cases}$$
$$x'(t) \geq 0,\ y(t) \geq 0,\ \frac{1}{C_3} \leq |x'(t)|^2 + |y'(t)|^2 \leq 1,\ a.e.$$

6. $\partial^e\Omega \subset \Gamma^* \cup \{(x,0) : 0 \leq x \leq 1\} \cup \partial B(0,1), \quad \nu \cdot (\cos\Theta, \sin\Theta) = 0$ a.e. on $\partial^*\Omega$.

The proof of the above convergence property is the same as Theorem 3.1. By lower semi-continuity and weak convergence of Θ_i in $H^1(B(0,1))$ we have

$$\int_{B(0,1)} |\nabla\Theta|^2\,dx \le \lim_{i\to\infty}\int_{B(0,1)} |\nabla\Theta_i|^2\,dx = 0$$

Therefore Θ is a constant function, which contradicts with (5) since the normal vector of $\partial\Omega$ obviously cannot be orthogonal to a constant vector by simple geometry. □

As for the regularity of Γ away from two endpoints, Proposition 2.5 and Proposition 2.7 tells that ν belongs to VMO and $H^{1/2}([0,l])$. Unfortunately this is the best regularity result we have now. Here we give the following natural open problems:

Problem 3.4 Is Γ a C^∞ curve, or at least C^1?

Problem 3.5 Can one write Γ as a curve of function $f(x)$, such that $|\frac{df}{dx}|\le C$ for some constant $C<\infty$?

The difficulty in answering these questions is due to the strong non-local character of the tangential anchoring boundary condition. It prevents us from modifying Γ locally to obtain an energy competitor and then deduce decay of some energy quantities. Therefore some new ideas and methods are needed in order to utilize the minimality. We now assume Γ is the graph of function $f\in C^3([-a,a])$, and we compute the Euler-Lagrange equation that f should satisfy.

Let $\Gamma=\{(x,f(x)):x\in[-a,a]\}$ such that

$$f(-a)=f(a)=0,\quad f(x)>0 \text{ for } x\in(-a,a),\quad \text{and } \int_{-a}^{a} f(x)\,dx=1.$$

We write Ω_Γ as Ω_f. Then we consider the perturbation of $f(x)$ and Ω_f

$$f_t(x)=f(x)+tg(x),\ \Omega_{f_t}(x)=\{(x,y):x\in[-a,a],\ y\in[0,f(x)+tg(x)]\},$$

where $g\in C_0^\infty([-a,a])$. We denote the domain variation by $\Phi(t,x)$ such that

$$\Phi(t,\Omega_f)=\Omega_{f_t},\ \Phi(t,(x,y))=(x,y+\frac{y}{f(x)}tg(x))$$

By this definition, we can see that $\Phi(t)$ satisfies that

$$\Phi(0)=I,\quad \xi(x,y):=\Phi'(0)=\lim_{t\to0+}\frac{1}{t}(\Phi(t)-I)=(0,\frac{y}{f(x)}g(x)).$$

$\Theta(t,z)$ solves the equation

$$\begin{cases}-\Delta\Theta(t,z)=0 & \text{in } \Omega_{f_t},\\ \Theta(t)=\arctan(f'+tg') & \text{on } \{(x,f_t(x)):x\in[-a,a]\},\\ \Theta(t)=0 & \text{on } \{(x,0):x\in[-a,a]\}.\end{cases}\tag{3.39}$$

Here $z=(x,y)\in\mathbb{R}^2$ and f', g' denote the derivatives of functions f,g with respect to x. The functional becomes

$$F(f_t)=\int_{-a}^{a}\sqrt{1+|f_t'(x)|^2}\,dx+\int_{-a}^{a}\int_0^{f_t(x)}|\nabla\Theta(t,z)|^2\,dxdy=:F_1(f_t)+F_2(f_t).$$

Suppose $t \to \Theta(t)$ has good differentiability properties(denote by Θ' its derivative at 0). We can differentiate (3.39) inside Ω_f and at the boundary, by differentiating the following identity:

$$\text{For } z = (x, f(x)), \quad \Theta(t, \Phi(t,z)) = \arctan(f' + tg').$$

We obtain

$$\begin{aligned} &-\Delta\Theta' = 0 \text{ in } \Omega_f \\ &\Theta'(x, f(x)) + \nabla\Theta(x, f(x)) \cdot \xi = \frac{g'(x)}{1 + |f'(x)|^2} \text{ on } \Gamma \\ &\Theta' = 0 \text{ on } \{(x,0) : x \in [-a,a]\}. \end{aligned}$$

Therefore, Θ' is the harmonic function with a Dirichlet boundary condition(depending on f, g). We compute the derivative of $F(f_t)$ at $t = 0$. For the first part, we easily obtain

$$\frac{d}{dt}F_1 = \int_{-1}^{1} \frac{f'g'}{\sqrt{1+|f'|^2}} dx.$$

For the second part, we have

$$\begin{aligned} \frac{d}{dt}F_2 &= \int_{\Omega_f} \left\{ 2\nabla\Theta' \cdot \nabla\Theta + \operatorname{div}(|\nabla\Theta|^2 \xi) \right\} dxdy \\ &= \int_{\Gamma} \left\{ 2\frac{\partial\Theta}{\partial\nu} \left(\frac{g'}{1+|f'|^2} - \nabla\Theta \cdot \xi \right) + |\nabla\Theta|^2 (\xi \cdot \nu) \right\} d\mathscr{H}^1 \\ &= \int_{-a}^{a} \left\{ 2\frac{\partial\Theta}{\partial\nu} \frac{g'}{\sqrt{1+|f'(x)|^2}} - 2(\nabla\Theta \cdot \nu) \cdot (\nabla\Theta \cdot \xi)\sqrt{1+|f'(x)|^2} + g(x)|\nabla\Theta|^2 \right\} dx \end{aligned}$$

where ν is the normal vector on Γ. Here we have used the boundary condition of Θ' and integration by parts. Also we have used the following formula

$$\xi(z) = (0, g(x)), \quad \nu(z) = \frac{(-f'(x), 1)}{\sqrt{1+|f'(x)|^2}} \quad \text{for } z = (x, f(x)) \in \partial\Omega_f.$$

Let $\frac{d}{dt}F(f_t) = 0$ and take into account the volume constraint, we obtain the following Euler-Lagrange equation

$$\lambda = -\frac{d}{dx}\left(\frac{f'}{\sqrt{1+|f'|^2}} \right) - 2\frac{d}{dx}\left(\frac{\partial\Theta}{\partial\nu} \frac{1}{\sqrt{1+|f'(x)|^2}} \right) - \frac{\partial\Theta}{\partial\nu}\frac{\partial\Theta}{\partial y}\sqrt{1+|f'|^2} + |\nabla\Theta|^2 \quad (3.40)$$

where λ is the Lagrange multiplier and the derivative of Θ takes value at $(x, f(x))$. Note that (3.40) is complicated and contains some highly non-local terms, such as the Dirichlet-to-Neumann map $\frac{\partial\Theta}{\partial\nu}$. The first part of the equation is the minimal surface equation while the rest comes from the Dirichlet energy with tangential anchoring condition and domain variation. It will be very interesting to study the well-posedness of (3.40) and we believe that the key of solving the regularity problem of Γ is to understand this equation.

4 Large volume limit and small volume limit

In this section we study the behavior of the minimizer as the volume V tends to be extremely large or small. A naive idea is to analyze the functional (1.8) from a scaling point of view. The curve length term is of dimension one while the Dirichlet energy term is of dimension zero. Therefore, when the volume is very large, the first term will be the dominating term and the minimizer is expected to be close to a semicircle (minimizes length of graph under fixed volume constraint). On the other hand, when the volume is very small, the domain is energy preferable to be very thin to avoid large elastic energy. We will present more rigorous analysis in the rest of this section.

4.1 Large volume limit

Since we are only interested in the shape of Γ, we will modify Problem 1.5 and restrict $a = 1$. First we make the following notations:

$$\mathscr{G}_V^a := \{\Gamma \in \mathscr{G}_V : \Gamma \text{ only intersects with } x\text{-axis at } (a,0), (-a,0)\}, \quad \mathscr{G}^a := \bigcup_{V>0} \mathscr{G}_V^a.$$

Then we can write Problem 1.5 as

$$\min_{a>0} \min_{\Gamma \in \mathscr{G}_V^a} \left\{ \int_{\Omega_\Gamma} |\nabla \Theta|^2 \, dxdy + l(\Gamma) \right\}.$$

Let $\bar{x} = \frac{x}{a}$, $\bar{y} = \frac{y}{a}$, $\bar{\Theta}(\bar{x}, \bar{y}) = \Theta(a\bar{x}, a\bar{y})$, the minimization problem becomes

$$\min_{a>0} \min_{\Gamma \in \mathscr{G}^1_{V/a^2}} \left\{ \int_{\Omega_\Gamma} |\nabla \Theta|^2 \, dxdy + a \cdot l(\Gamma) \right\}.$$

Setting

$$\tilde{a} = \frac{a}{\sqrt{V}}, \tag{4.41}$$

which leads to

$$\min_{\tilde{a}>0} \min_{\Gamma \in \mathscr{G}^1_{1/\tilde{a}^2}} \left\{ \int_{\Omega_\Gamma} |\nabla \Theta|^2 \, dxdy + \tilde{a}\sqrt{V} \cdot l(\Gamma) \right\}. \tag{4.42}$$

This is equivalent to

$$\min_{\Gamma \in \mathscr{G}^1} \left\{ \int_{\Omega_\Gamma} |\nabla \Theta|^2 \, dxdy + \sqrt{V} \frac{l(\Gamma)}{\sqrt{|\Omega_\Gamma|}} \right\}. \tag{4.43}$$

When $V \gg 1$, we consider the following functional for $\Gamma \in \mathscr{G}^1$:

$$E_V(\Gamma) = \frac{1}{\sqrt{V}} \int_{\Omega_\Gamma} |\nabla \Theta|^2 \, dxdy + \frac{l(\Gamma)}{\sqrt{|\Omega_\Gamma|}}$$

We denote by Γ_V the minimizer of functional $E_V(\Gamma)$. As $V \to +\infty$, one expects that Γ_V will "converge" in some proper sense to $\Gamma^* := \{(x, \sqrt{1-x^2}) : x \in [-1,1]\}$, which is well known to minimize the following functional

$$F(\Gamma) = \frac{l(\Gamma)}{\sqrt{|\Omega_\Gamma|}} \text{ for } \Gamma \in \mathscr{G}^1.$$

We have the following lemma:

Lemma 4.1 $\lim_{V\to\infty} E_V(\Gamma_V) = \sqrt{2\pi} = F(\Gamma^*)$

Proof We borrow the idea of "adding two cusps" from [20]. We modify Γ^* near $x=-1$ and $x=1$ by adding two cusps. For $\varepsilon << 1$, we define a function $\bar{f}^\varepsilon$ as follows

$$\bar{f}^\varepsilon(x) = \begin{cases} \sqrt{1-x^2}, & |x| \le \sqrt{1-\varepsilon^2}, \\ \frac{\varepsilon}{1-\varepsilon} - \sqrt{(\frac{\varepsilon}{1-\varepsilon})^2 - (x + \sqrt{\frac{1+\varepsilon}{1-\varepsilon}})^2}, & x \in (-\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}, -\sqrt{1-\varepsilon^2}), \\ \frac{\varepsilon}{1-\varepsilon} - \sqrt{(\frac{\varepsilon}{1-\varepsilon})^2 - (-x + \sqrt{\frac{1+\varepsilon}{1-\varepsilon}})^2}, & x \in (\sqrt{1-\varepsilon^2}, \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}). \end{cases}$$

Note that here we change the graph near two endpoints of Γ^* into two circular arcs to make sure the derivative of $\bar{f}^\varepsilon$ vanishes near two end points (See Figure 3).

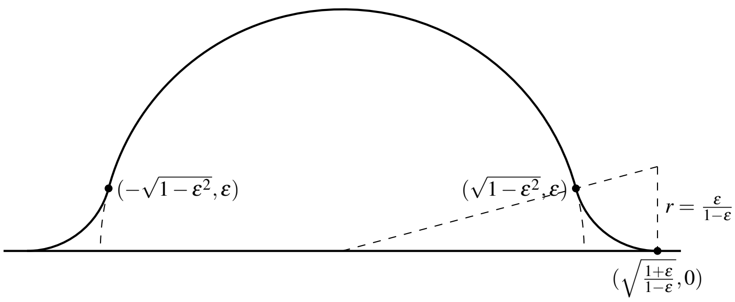


Fig. 3: The graph of function $\bar{f}^\varepsilon$

Then we set for any $\varepsilon \in (0, \frac{1}{4})$,

$$\begin{aligned} f^\varepsilon(x) &= \bar{f}^\varepsilon(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}x), \\ \Gamma^\varepsilon &= \{(x, f^\varepsilon x) : x \in [-1,1]\}, \\ \Theta^\varepsilon(x,y) &= \arctan \frac{d}{dx} f^\varepsilon(x) \quad \text{for } (x,y) \in \Gamma^\varepsilon \\ \Theta^\varepsilon(x,y) &= \frac{\arctan \frac{d}{dx} f^\varepsilon(x)}{f^\varepsilon(x)} \cdot y \quad \text{for } (x,y) \in \Omega_{\Gamma^\varepsilon} \end{aligned}$$

It is straightforward to check that Γ^ε satisfies the following property:

1. $\Gamma^\varepsilon \in \mathscr{G}^1$,
2. $\lim\limits_{\varepsilon\to 0} F(\Gamma^\varepsilon) = \sqrt{2\pi}$,
3. $\int_{\Omega_{\Gamma^\varepsilon}} |\nabla\Theta^\varepsilon|^2\, dydx = O(\frac{1}{\varepsilon})$.

For $V \gg 1$, we set $\varepsilon = V^{-\frac{1}{4}}$. Then $\frac{1}{\sqrt{V}} \int_{\Omega_{\Gamma^\varepsilon}} |\nabla\Theta^\varepsilon|^2\, dydx = O(V^{-\frac{1}{4}})$. This implies

$$\varlimsup_{V\to\infty} E_V(\Gamma_V) \le \lim_{V\to\infty} E_V(\Gamma^\varepsilon) = \sqrt{2\pi} \le \varliminf_{V\to\infty} E_V(\Gamma_V).$$

□

Remark 4.2 If $\lim\limits_{v\to\infty} F(\Gamma_V) = \sqrt{2\pi}$, then

$$\lim_{v\to\infty} |\Omega_{\Gamma_V} \Delta \Omega_{\Gamma^*}| = 0, \quad \lim_{v\to\infty} \mathrm{d}_{\mathscr{H}}(\Gamma_V, \Gamma^*) = 0,$$

where $\mathrm{d}_{\mathscr{H}}$ is the Hausdorff distance. This is an easy consequence of the stability of isoperimetric inequality (see [17]).

4.2 Small volume limit

First we prove the following lemma which provides a rough estimate for the Dirichlet energy when the volume of droplet is sufficiently small.

Lemma 4.3 *Take $\varepsilon \ll 1$, there exist constants c and C which are independent of ε, such that for any $\Gamma \in \mathscr{G}^1_\varepsilon$, it holds that*

$$c\,\varepsilon \le \int_{\Omega_\Gamma} |\nabla\Theta|^2\, dxdy \le C\,\varepsilon.$$

Proof Given $\Gamma \in \mathscr{G}^1_\varepsilon$, Θ is the corresponding angle function. We first estimate the lower bound of energy. We set

$$\tilde{\Gamma} = \{(x,y) : (x,\varepsilon y) \in \Gamma\}, \quad \tilde{\Theta}(x,y) = \arctan\left(\frac{\tan\Theta(x,\varepsilon y)}{\varepsilon}\right) \text{ for } (x,y) \in \Omega_{\tilde{\Gamma}}.$$

Then we can check that $|\Omega_{\tilde{\Gamma}}| = 1$ and Θ satisfies the boundary condition (1.7) corresponding to $\tilde{\Gamma}$. Thus there exists a constant c such that

$$\int_{\Omega_{\tilde{\Gamma}}} |\partial_y\tilde{\Theta}|^2\, dxdy \ge c \tag{4.44}$$

Otherwise one can use the similar argument in Section 3 to get a contradiction. On the other hand, by definition of $\tilde{\Gamma}$ and $\tilde{\Theta}$ we have

$$\begin{aligned}\int_{\Omega_{\tilde{\Gamma}}} |\partial_y\tilde{\Theta}|^2\, dxdy &= \int_{\Omega_\Gamma} \frac{|\partial_y\Theta|^2}{\varepsilon\cdot\left(1+\left|\frac{\tan\Theta(x,y)}{\varepsilon}\right|^2\right)^2\cdot|\cos\Theta(x,y)|^4}\, dxdy \\ &\le \frac{1}{\varepsilon}\int_{\Omega_\Gamma} |\partial_y\Theta|^2\, dxdy.\end{aligned} \tag{4.45}$$

Therefore

$$\int_{\Omega_\Gamma} |\nabla\Theta|^2 \, dxdy \geq c\varepsilon,$$

by (4.44) and (4.45). Meanwhile, we can construct a $\Gamma \in \mathscr{G}^1_\varepsilon$ such that $\int_{\Omega_\Gamma} |\nabla\Theta|^2 \, dxdy \leq C\varepsilon$ for some larger constant C. Set

$$\begin{aligned}\Gamma =& \{(x, f(x)) : x \in [-1,1], f(x) = \frac{\varepsilon}{2}(\cos x + 1)\}\\ \Theta(x,y) =& \frac{\arctan f'(x)}{f(x)} y, \quad \text{for } x \in (-1,1),\, y \in [0, f(x)].\end{aligned}$$

We can directly verify that $\Gamma \in \mathscr{G}^1_\varepsilon$ and $\int_{\Omega_\Gamma} |\nabla\Theta|^2 \, dxdy \leq C\varepsilon$ for some constant C independent of ε. This proves Lemma 4.3. □

Remark 4.4 Now we consider the minimization problem (4.42) with $V = \varepsilon^2 \ll 1$. We can determine the appropriate order of $\tilde{a}$ which is defined in (4.41). Assume $\tilde{a} \sim O(\varepsilon^{-\alpha})$ for some $\alpha \in \mathbb{R}$. Then the second term (surface energy term) is of order $\varepsilon^{1-\alpha}$. For the Dirichlet energy term since $\Gamma \in \mathscr{G}^1_{\varepsilon^{2\alpha}}$, by Lemma 4.3 we know it is of order $\varepsilon^{2\alpha}$. Matching these two terms gives $\alpha = \frac{1}{3}$. According to the deduction of (4.42) we know that if we don't fix two endpoints of Γ, then the energy-minimizing droplet with volume ε^2 will be a elongated drop with length of the order $\varepsilon^{\frac{2}{3}}$ and the total energy is of order $\varepsilon^{\frac{2}{3}}$.

Next we study the asymptotic shape of the rescaled droplet. For such purpose, we add some extra regularity assumption on the curve Γ. Consider a subset of $\mathscr{G}^1$, denoted by $\tilde{\mathscr{G}}^1$, which consists of all the curves in $\mathscr{G}^1$ that are graphs of H^2_0 functions,

$$\tilde{\mathscr{G}}^1 := \{\Gamma \in \mathscr{G}^1,\, \Gamma = \{(x, f(x))\},$$

where f satisfies

$$f \in H^2_0([-1,1]),\ f'(\pm 1) = 0,\ f(x) > 0 \text{ on } (-1,1). \tag{4.46}$$

Given $\varepsilon \ll 1$, we define a transformation operator $\mathscr{T}_\varepsilon$, which compresses $\Gamma \in \tilde{\mathscr{G}}^1$ in the vertical direction:

$$\mathscr{T}_\varepsilon(\Gamma) = \{(x, \varepsilon^{\frac{2}{3}} f(x)) :\ x \in [-1,1]\}, \quad \Gamma = \{(x, f(x))\}.$$

Now after taking $V = \varepsilon^2$ in (4.43) and multiplying $\varepsilon^{-\frac{2}{3}}$, we obtain the functional

$$\begin{aligned}E_\varepsilon(f) &= E_\varepsilon(\Gamma)\\ &= \varepsilon^{-\frac{2}{3}} \int_{\Omega_{\mathscr{T}_\varepsilon(\Gamma)}} |\nabla\Theta_{\mathscr{T}_\varepsilon(\Gamma)}|^2 \, dxdy + \varepsilon^{\frac{1}{3}} \frac{l(\mathscr{T}_\varepsilon(\Gamma))}{\sqrt{\Omega_{\mathscr{T}_\varepsilon(\Gamma)}}}\\ &= \varepsilon^{-\frac{2}{3}} \int_{\Omega_{\mathscr{T}_\varepsilon \Gamma}} |\nabla\Theta_{\mathscr{T}_\varepsilon(\Gamma)}|^2 \, dxdy + \frac{l(\mathscr{T}_\varepsilon(\Gamma))}{\sqrt{\Omega_\Gamma}}\\ &= \varepsilon^{-\frac{2}{3}} \int_{-1}^{1} \int_0^{\varepsilon^{\frac{2}{3}} f(x)} \{|\partial_x \Theta_{\mathscr{T}_\varepsilon(\Gamma)}|^2 + |\partial_y \Theta_{\mathscr{T}_\varepsilon(\Gamma)}|^2\} \, dydx + \frac{\int_{-1}^1 \sqrt{1 + \varepsilon^{\frac{4}{3}} |f'(x)|^2} \, dx}{\sqrt{\int_{-1}^1 f(x) \, dx}}.\end{aligned} \tag{4.47}$$

Then for a sequence of positive numbers $\varepsilon \to 0$, we consider the sequence of functionals on $H_0^2([-1,1])$

$$E_\varepsilon(f) := \begin{cases} E_\varepsilon(f) \text{ defined in (4.47)}, & \text{if } \Gamma = \{x, f(x)\} \in \mathscr{G}^1, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.48}$$

And we also define the candidate functional $E_0(f)$ for Γ-convergence,

$$E_0(f) := \int_{-1}^{1} \frac{|f'(x)|^2}{f(x)}\,dx + \frac{2}{\sqrt{\int_{-1}^{1} f(x)\,dx}}, \quad f \in H_0^2([-1,1]).$$

For the definition and general properties of Γ-convergence, we refer to the book [7]. We have the following result:

Proposition 4.5 *As $\varepsilon \to 0$, the sequence $\{E_\varepsilon\}$ Γ-converges to E_0 in the H^2 topology.*

Proof First we prove the lower semi-continuity condition, i.e. for any $g \in C_0^1[-1,1]$ and for any sequence $\{g_\varepsilon\}$ in $C_0^1[-1,1]$,

$$g_\varepsilon \to g \text{ in } H^2[-1,1] \text{ implies } \liminf_{\varepsilon \to 0} E_\varepsilon(g_\varepsilon) \geq E_0(g). \tag{4.49}$$

The case $\liminf_{\varepsilon\to 0} E_\varepsilon(g_\varepsilon) = +\infty$ is trivial. We therefore assume that $\liminf_{\varepsilon\to 0} E_\varepsilon(g_\varepsilon) = C < +\infty$. And by the C^1 convergence of g_ε, we also suppose that $|g_\varepsilon'(x)| \leq c$ for some constant c holds for any $\varepsilon > 0$ and $x \in [-1,1]$. Now we examine the first term of $E_\varepsilon(g_\varepsilon)$ more closely

$$\begin{aligned}
&\varepsilon^{-\frac{2}{3}} \int_{-1}^{1} \int_{0}^{\varepsilon^{\frac{2}{3}} g_\varepsilon(x)} \{|\partial_x \Theta_{\mathscr{T}_\varepsilon(\Gamma)}|^2 + |\partial_y \Theta_{\mathscr{T}_\varepsilon(\Gamma)}|^2\}\,dydx \\
>&\varepsilon^{-\frac{2}{3}} \int_{-1}^{1} \int_{0}^{\varepsilon^{\frac{2}{3}} g_\varepsilon(x)} \{|\partial_y \Theta_{\mathscr{T}_\varepsilon(\Gamma)}|^2\}\,dydx \\
\geq&\varepsilon^{-\frac{2}{3}} \int_{-1}^{1} \left\{ \frac{|\Theta_{\mathscr{T}_\varepsilon(\Gamma)}(x, \varepsilon^{\frac{2}{3}} g_\varepsilon(x))|^2}{\varepsilon^{\frac{2}{3}} g_\varepsilon(x)} \right\} dx \\
=&\varepsilon^{-\frac{2}{3}} \int_{-1}^{1} \left\{ \frac{|\arctan(\varepsilon^{\frac{2}{3}} g_\varepsilon'(x))|^2}{\varepsilon^{\frac{2}{3}} g_\varepsilon(x)} \right\} dx
\end{aligned}$$

Since $|g_\varepsilon'(x)| \leq c$, we have that for any $\sigma > 0$, there exists $\varepsilon_\sigma > 0$ such that for any $\varepsilon < \varepsilon_\sigma$, $|\arctan(\varepsilon^{\frac{2}{3}} g_\varepsilon'(x))| \geq (1-\sigma)|\varepsilon^{\frac{2}{3}} g_\varepsilon'(x)|$. And therefore we have

$$\begin{aligned}
&\varepsilon^{-\frac{2}{3}} \int_{-1}^{1} \left\{ \frac{|\arctan(\varepsilon^{\frac{2}{3}} g_\varepsilon'(x))|^2}{\varepsilon^{\frac{2}{3}} g_\varepsilon(x)} \right\} dx \\
\geq&\varepsilon^{-\frac{2}{3}} (1-\sigma)^2 \int_{-1}^{1} \left| \frac{\varepsilon^{\frac{4}{3}} |g_\varepsilon'(x)|^2}{\varepsilon^{\frac{2}{3}} g_\varepsilon(x)} \right| dx = (1-\sigma)^2 \int_{-1}^{1} \frac{|g_\varepsilon'(x)|^2}{g_\varepsilon(x)}\,dx, \quad \text{when } \varepsilon < \varepsilon_\sigma
\end{aligned}$$

We obtain

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0} E_\varepsilon(g_\varepsilon)\\
=&\liminf_{\varepsilon\to 0}\left\{\varepsilon^{-\frac{2}{3}}\int_{-1}^{1}\int_{0}^{\varepsilon^{\frac{2}{3}}g_\varepsilon(x)}\{|\partial_x\Theta_{\mathscr{T}_\varepsilon(\Gamma)}|^2+|\partial_y\Theta_{\mathscr{T}_\varepsilon(\Gamma)}|^2\}\,dydx+\frac{\int_{-1}^{1}\sqrt{1+\varepsilon^{\frac{4}{3}}|g_\varepsilon'(x)|^2}\,dx}{\sqrt{\int_{-1}^{1}g_\varepsilon(x)\,dx}}\right\}\\
\geq&\liminf_{\varepsilon\to 0}\left\{\int_{-1}^{1}\frac{|g_\varepsilon'(x)|^2}{g_\varepsilon(x)}\,dx+\frac{2}{\sqrt{\int_{-1}^{1}g_\varepsilon(x)\,dx}}\right\}\geq E_0(g)
\end{aligned}
$$

Here in the last step we used the C^1 convergence of g_ε and Fatou's lemma. This gives the proof of the lower semi-continuity (4.49).

The second part of proving Gamma-convergence is to find a recovery sequence for each f satisfying (4.46). We can simply take $f_\varepsilon = f$ for any $\varepsilon > 0$. By the same argument in the proof of the lower semi-continuity, we have

$$\lim_{\varepsilon\to 0} E_\varepsilon(f) \geq E_0(f)$$

On the other hand, take $\Theta_\varepsilon(x,y) = \frac{y}{\varepsilon^{\frac{2}{3}}f(x)}\arctan(\varepsilon^{\frac{2}{3}}f'(x))$ for (x,y) satisfying $-1\leq x\leq 1$, $0\leq y\leq \varepsilon^{\frac{2}{3}}f(x)$. It holds that

$$
\begin{aligned}
&\varepsilon^{-\frac{2}{3}}\int_{-1}^{1}\int_{0}^{\varepsilon^{\frac{2}{3}}f(x)}|\partial_x\Theta_\varepsilon|^2\,dydx\\
=&\int_{-1}^{1}\frac{\varepsilon^{\frac{4}{3}}}{3}f(x)^3\left|\frac{f''}{f(1+\varepsilon^{\frac{4}{3}}|f'|^2)}-\frac{f'\arctan(\varepsilon^{\frac{2}{3}}f')}{\varepsilon^{\frac{2}{3}}f^2}\right|^2 dx\\
\sim& O(\varepsilon^{\frac{4}{3}}).
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon^{-\frac{2}{3}}\int_{-1}^{1}\int_{0}^{\varepsilon^{\frac{2}{3}}f(x)}|\partial_y\Theta_\varepsilon|^2\,dydx\\
=&\int_{-1}^{1}\frac{|\arctan(\varepsilon^{\frac{2}{3}}f')|^2}{\varepsilon^{\frac{4}{3}}f}\,dx\sim O(1).
\end{aligned}
$$

After comparing the above two identities, we conclude that

$$
\begin{aligned}
&\varepsilon^{-\frac{2}{3}}\int_{-1}^{1}\int_{0}^{\varepsilon^{\frac{2}{3}}f(x)}|\nabla\Theta_\varepsilon|^2\,dydx+\frac{\int_{-1}^{1}\sqrt{1+\varepsilon^{\frac{4}{3}}|f'(x)|^2}\,dx}{\sqrt{\int_{-1}^{1}f(x)\,dx}}\\
=&(1+o(1))\int_{-1}^{1}\frac{|f'|^2}{f}\,dx+\frac{2}{\sqrt{\int_{-1}^{1}f(x)\,dx}}+o(1)=E_0(f)+o(1)
\end{aligned}
$$

Therefore we obtain $\lim_{\varepsilon\to 0}E_\varepsilon(f)=E_0(f)$ for any $f\in H_0^2([-1,1])$. The proof is complete. □

Proposition 4.5 inspires us to study the following minimization problem

$$\min_{g\in H_0^2([-1,1])}\left\{\frac{2}{\sqrt{\int_{-1}^1 g(x)\,dx}}+\int_{-1}^1\frac{|g'(x)|^2}{g(x)}\,dx\right\} \tag{4.50}$$

Let $g=h^2$, the problem becomes

$$\min_{h^2\in H_0^2[-1,1]}\left\{\frac{2}{\sqrt{\int_{-1}^1 h(x)^2\,dx}}+4\int_{-1}^1|h'(x)|^2\,dx\right\} \tag{4.51}$$

The Euler Lagrange equation is

$$h''(x)=-\frac{h(x)}{4(\int_{-1}^1 h^2\,dx)^{\frac{3}{2}}},\quad h\in H_0^2[-1,1].$$

This ODE can be solved explicitly,

$$h(x)=\pi^{-\frac{2}{3}}\cos\frac{\pi}{2}x$$

and therefore

$$g(x)=\pi^{-\frac{4}{3}}\left(\frac{1+\cos\pi x}{2}\right)$$

is the minimizer for the minimization problem (4.50). Using the above Γ-convergence result, we conclude that when the volume $V=\varepsilon^2<<1$, the approximated profile of Γ is

$$\left\{\left(x,\varepsilon^{\frac{4}{3}}\pi^{-\frac{2}{3}}\left(\frac{1+\cos(\varepsilon^{-\frac{2}{3}}\pi^{\frac{1}{3}}x)}{2}\right)\right):x\in[-\varepsilon^{\frac{2}{3}}\pi^{\frac{2}{3}},\varepsilon^{\frac{2}{3}}\pi^{\frac{2}{3}}]\right\}.$$

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