Parameters in Banach spaces and orthogonality

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Abstract

In Banach spaces, plenty of parameters have been considered: they are often defined by using pairs of vectors. Rarely they are defined by considering pairs of vectors which are orthogonal in the sense of Birkhoff and James; in that case the study is often not easy. In fact, it can be difficult to identify pairs of orthogonal vectors; so to calculate the value of these parameters, to compare them with the other parameters, to see if they have some stability with respect to changes of the norm. In this paper we shall do this for a couple of new parameters.

Keywords: Orthogonal vectors, James constant, uniform non squareness.
MSC[2020]: 46B20 46B99 46C15

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1 Introduction

Consider a real, Banach space X; we shall denote by S_X (or simply by S, if no confusion can arise) its unit sphere. As known, it is possible to consider in X several different notions of orthogonality. The most popular and used seems to be the one suggested by Birkhoff and James, that we shall consider here. We say that x is orthogonal to y, and we write $x \perp y$, if $||x|| \leq ||x + ty||$ for every $t \in \mathbb{R}$.
Considering parameters defined by using orthogonal pairs of vectors is not usual (and simple). Among the few attempts done in this direction, we recall that for example I. Serb considered the "orthogonal version" of a modulus of smoothness, indicating only "weak" results (see [6] and the references therein). We give a simple example showing that parameters defined by using orthogonal pairs can hardly be "stable".
Consider as X the space $\mathbb{R}^2$ with the maximum norm: for the vectors $x = (1, 1)$ and $y = (-1, 0)$ we have $x \perp y$. Now we "slightly" change the norm: for $x = (x_1, x_2)$ we set $||x|| = (|x_1|^p + |x_2|^p)^{1/p}$, with p "large"; then $||x||_p$ is near to 1 but on S_X only $y' = (1, -1)$ is such that $x \perp \pm y'$ and only $x' = (0, 1)$ ($\in S_X$) is such that $\pm x' \perp y$.
In this paper we consider "orthogonal versions" of two deeply studied parameters. We prove several facts, giving also new characterizations of uniformly non square spaces; we show by examples that our parameters have different behaviors with respect to the classical ones.

2 Old and new parameters

The following parameters received much attention during the last decades (see for example [3]) and are still studied in deep:

$$J(X) = \sup\{\min\{||x - y||, ||x + y||\} : x, y \in S_X\} \text{ (James constant)}$$

$$g(X) = \inf\{\max\{||x - y||, ||x + y||\} : x, y \in S_X\} \text{ (Schäffer constant).}$$

As known, we always have: $1 \leq g(X) \leq \sqrt{2} \leq J(X) \leq 2$; $g(X)J(X) = 2$. Also $g(X) = J(X) = \sqrt{2}$ in Hilbert spaces. Recall the definition of uniformly non square spaces, (UNS) for short.
The space X is (UNS) when there exists $\epsilon > 0$ such that for $x, y \in S_X$ either $||x - y|| < 2 - \epsilon$ or $||x + y|| < 2 - \epsilon$. Clearly:

$$X \text{ is } (UNS) \Longleftrightarrow J(X) = 2 \Longleftrightarrow g(X) = 1. \tag{2.1}$$

We define now:

$$J_\perp(X) = \sup\{\min\{||x - y||, ||x + y||\} : x, y \in S_X,\ x \perp y\}$$

;

$$g_\perp(X) = \inf\{\max\{||x - y||, ||x + y||\} : x, y \in S_X,\ x \perp y\}$$

. Of course $g(X) \leq g_\perp(X)$ and $J_\perp(X) \leq J(X)$ always. Sometimes we shall simply write J, $J_\perp$, g, $g_\perp$ when it is clear which is the underlying space.
Note that $J(X) = \sup\{J(Y) : Y \subset X,\ dim(Y) = 2\}$; and a similar remark applies for $J_\perp$, g, $g_\perp$.
This indicates that studying these parameters for 2-dimensional spaces (where some specific pathologies can also arise) is essentially studying them in general.

3 Studying $g_\perp(X)$

An equivalent formulation for (UNS) is the following: there exists $\epsilon > 0$ such that for all $x, y \in S_X$ either $||x-y|| > 1-\epsilon$ or $||x+y|| > 1-\epsilon$.
Also: reading the proof of Theorem 3.2 in [2], we see that the following fact (based on orthogonal vectors) is true:

X is not (UNS) if and only if there exist $x, y \in S_X$, $x \perp y$, such that $||x \pm \lambda y|| \approx 1$ with $\lambda \approx 2$.

Next result gives a sharper result concerning orthogonal pairs.

Theorem 3.1 *Let X be a real Banach space; assume that S_X contain two points x, y such that*

$$||x \pm y|| \leq 1+\epsilon, \ \epsilon \in [0,1). \tag{3.1}$$

Then S_X contains y' such that: $x \perp y'$ and $||x \pm y'|| \leq \dfrac{1-\epsilon^2+2\epsilon}{1-\epsilon}$.

Proof Let x, y as indicated; assume that x is not orthogonal to y, thus $\epsilon > 0$ (otherwise there is nothing to prove). Take a norm-one functional f_x such that $f_x(x) = 1$ and $f_x(y) \neq 0$. Eventually exchanging y and $-y$ we can assume that $f_x(y) > 0$. Let $y' = \alpha x + \beta y \in S_X$ be such that $f_x(y') = 0$ (so $x \perp y'$). We have $\beta \neq 0$ (otherwise also $\alpha = 0$ against $y' \in S_X$).
Again we can assume $\beta > 0$ (eventually exchanging y' and $-y'$). Then: $f_x(y') = \alpha + \beta f_x(y) = 0$, so $\alpha = -\beta f_x(y) < 0$. Also: $1 + f_x(y) = f_x(x+y) \leq ||x+y|| \leq 1+\epsilon$, so $0 \leq f_x(y) \leq \epsilon$.
This implies $|\alpha| \leq \beta\epsilon$. We have:
$1 = ||y'|| \geq \beta - |\alpha| \geq \beta(1-\epsilon)$; thus $\beta \leq \dfrac{1}{1-\epsilon}$;
$1 = ||y'|| \leq |\alpha| + \beta \leq \beta(1+\epsilon)$; so we have: $\dfrac{1}{1+\epsilon} \leq \beta \leq \dfrac{1}{1-\epsilon}$. Therefore $||y-y'|| = ||y - \alpha x - \beta y|| = ||\alpha x + (\beta-1)y|| \leq |\alpha| + |\beta - 1|$.
If $\beta \geq 1$, then $||y-y'|| \leq |\alpha| + \beta - 1 \leq \beta(\epsilon+1) - 1 \leq \dfrac{2\epsilon}{1-\epsilon}$;
if $\beta \leq 1$, then $||y-y'|| \leq |\alpha| + 1 - \beta \leq \beta(\epsilon-1) + 1 \leq \dfrac{\epsilon-1}{1+\epsilon} + 1 = \dfrac{2\epsilon}{1+\epsilon} < \dfrac{2\epsilon}{1-\epsilon}$.
So we obtain: $||x-y'|| = ||x-y-y'+y|| \leq ||x-y|| + ||y-y'|| \leq 1+\epsilon+\dfrac{2\epsilon}{1-\epsilon}$;
$||x+y'|| = ||x+y+y'-y|| \leq ||x+y|| + ||y-y'|| \leq 1+\epsilon+\dfrac{2\epsilon}{1-\epsilon}$ and so: $||x \pm y'|| \leq \dfrac{1-\epsilon^2+2\epsilon}{1-\epsilon}$.
Note that the last function of $\epsilon \in [0,1)$ is increasing. ∎

By using Theorem 3.1 we can prove the following result:

Theorem 3.2 *Fon any space X we have:*

$$g_\perp(X) \leq \frac{-g^2(X) + 4g(X) - 2}{2 - g(X)}. \tag{3.2}$$

In particular, $g_\perp(X) = 1$ characterizes non (UNS) spaces (in fact $g_\perp(X) = 1$ if and only if $g(X) = 1$ since $1 \leq g(X) \leq g_\perp(X)$ always).

Proof Take $\epsilon > g(X) - 1$: thus S_X contains pair x, y satisfying (3.1). According to Theorem 3.1 we have:

$$g_\perp(X) \leq \frac{1-\epsilon^2+2\epsilon}{1-\epsilon}.$$

Since this is true for all $\epsilon > g(X) - 1$ we obtain:

$$g_\perp(X) \le \frac{1-(g(X)-1)^2+2(g(X)-1)}{1-(g(X)-1)} = \frac{-g^2(X)+4g(X)-2}{2-g(X)}$$

. ■

In the last theorem, the majorizing function (of g(X)) is increasing.
We note that Theorem 3.2 gives an estimate that is not very sharp in general; for example if X is a Hilbert space then $g_\perp(X) = g(X) = \sqrt{2}$ but that estimate gives $g_\perp(X) \le 2\sqrt{2}$; on the contrary that estimate is "fine" if $g(X) \approx 1$.

4 Studying $J_\perp(X)$

We start with a remark concerning $g(X)$ and $J(X)$.

Remark 4.1 *It is not difficult to see that:*

$$J(X) = \sup\{\min\{||x-y||, ||x+y||\} : x, y \in S_X;\ ||x-y|| = ||x+y||\};$$

$$g(X) = \inf\{\max\{||x-y||, ||x+y||\} : x, y \in S_X;\ ||x-y|| = ||x+y||\}.$$

Proof: see for example [5] for this and a general discussion of this. ■

We prove now a result relating $J(X)$ and $J_\perp(X)$.

Theorem 4.1 *In any space X we have:*

$$J_\perp(X) \ge 2J(X) - 2\,. \tag{4.1}$$

In particular $J(X) = 2$ implies (so it is equivalent to) $J_\perp(X) = 2$, and this condition is equivalent to X being not (UNS).

Proof According to Remark 4.1, given $\epsilon > 0$ there exist $x, y \in S_X$ such that $||x-y|| = ||x+y|| = \beta$ for some $\beta \in (J(X)-\epsilon, J(X))$. Set $f(t) = ||x+ty||$: this is a convex, 1-Lipschitz function and $f(0) = 1 < \beta = f(1) = f(-1)$. Let $t_0 \in (-1,1)$ a point where the function f attains its minimum $\alpha \in (0,1]$. This means that $x + t_0 y \perp y$. We can suppose $t_0 > 0$ (eventually exchanging y and $-y$); if $t_0 = 0$ then there is nothing to prove. Also, by considering the slope of f in $[0,1]$ and the fact that f is 1-Lipschitz, we have: $\beta - \alpha + 1 - \alpha = f(1) - f(t_o) + f(0) - f(t_0) \le 1$, so $\alpha \ge \beta/2$.
Set $z = (x+t_0y)/\alpha$ (so $z \in S_X$; $||z-(x+t_oy)|| = 1-\alpha$; $z \perp y$). We have:
$||z+y|| = ||z-(x+t_0y)+(x+t_0y)+y|| \ge ||x+(t_0+1)y|| - (1-\alpha) = f(t_0+1) - 1 + \alpha \ge \frac{\beta-\alpha t_0}{1-t_0} - 1 + \alpha$.
Hence: $||z+y|| \ge \dfrac{\beta-1+\alpha+t_0(1-2\alpha)}{1-t_0}$. Since $J(X) \ge \sqrt{2}$ we can assume $\beta > 4/3$; $\alpha \ge \beta/2$ implies $\beta - 1 + \alpha \ge \frac{3}{2}\beta - 1$; $1 - 2\alpha \ge -1$ implies:

$$||z+y|| \ge \frac{(3/2)\beta - 1 - t_0}{1-t_0} > \frac{3}{2}\beta - 1\,.$$

Considering the average slope of f in $[t_0, 1]$ we have $\frac{\beta-\alpha}{1-t_0} \le 1$, so $t_0 \le 1 + \alpha - \beta$. Then we have: $f(t_0 - 1) = ||x + t_0y - y|| \ge \beta - t_0 \ge 2\beta - \alpha - 1$. Therefore:

$||z-y|| = ||z-(x+t_0y)+(x+t_0y)-y|| \geq f(t_0-1)-1+\alpha \geq 2(\beta-1)$. Since $2(\beta-1) \leq \frac{3}{2}\beta - 1$ (in fact $\beta \leq 2$) we obtain:

$$J_\perp(X) \geq \min\{2(\beta-1);\ \frac{3}{2}\beta-1\} = 2(\beta-1).$$

But we can choose $\epsilon > 0$ arbitrarily small so β can be arbitrarily near to $J(X)$, then we obtain the result. ■

We note that the inequality (4.1) is "fine" if $J(X) \approx 2$, but it is not sharp in general: for example in Hilbert spaces it only gives $J_\perp(X) \geq 2(\sqrt{2}-1)$; it gives $J_\perp(X) \geq \sqrt{2}$ if $J(X) \geq 1+1/\sqrt{2}$. Of course $g(X) = g_\perp(X)$ and/or $J(X) = J_\perp(X)$ when $g(X)$ and/or $J(X)$ is realized by orthogonal pairs $x, y \in S_X$.

5 Examples

In this section we collect several examples of 2-dimensional spaces, where we compute the values of our parameters. We recall (see [3]) that the value of $J(X)$ depend on the modulus of convexity, defined for $\epsilon \in [0,2]$ in this way:

$$\delta_X(\epsilon) = \inf\{1 - \frac{||x+y||}{2} : x, y \in S_X; ||x-y|| \geq \epsilon\}.$$

Namely, we have

$$J(X) = \sup\{\epsilon > 0 : \epsilon \leq 2 - \delta_X(\epsilon)\}.$$

Thus if $J(X) < 2$ we have $J(X) = 2 - 2\delta_X(J(X)$. Computing the values of $g_\perp(X)$ and $J_\perp(X)$ is often not very simple: in many cases the calculation is not difficult but rather tedious; due also to this we shall not give all details here. We shall use these examples later, to clearify the behaviour of our parameters and see which properties of $g(X)$ and $J(X)$ remain true for $g_\perp(X)$ and $J_\perp(X)$.

Example 5.1 Consider $X = \mathbb{R}^2$ with the norm determined by a regular hexagon whose vertices are $(\pm1, 0)$; $(\pm1, \pm1)$; $(0, \pm1)$. In other words the norm in X is so given:

$$||(x,y)|| = \begin{cases} \max\{|x|,|y|\} & xy \geq 0 \\ |x|+|y| & xy < 0 \end{cases}$$

As known, for this space we have: $\delta(\epsilon) = \max\{0, (\epsilon-1)/2\}$. So $J(X) = 3/2$; $g(X) = 4/3$. We can see that $g_\perp(X) = 3/2$ (achieved when $x = (0,1)$; $y = (1, 1/2)$). Therefore $J_\perp(X) \leq J(X) = 3/2$; for $x = (1/2, -1/2)$ and $y = (1,1)$, $x, y \in S_X$, $x \perp y$ we obtain $J_\perp(X) = 3/2$.

Example 5.2 Let $X = \mathbb{R}^2$ endowed with the norm determined by a different hexagon, whose vertices are: $(\pm1, 0)$; $(\mp1, \pm1)$; $(\pm1/2, \pm1)$. Concerning the modulus of convexity in this space we have:

$$\delta_X(\epsilon) = \begin{cases} 0 & \epsilon \leq 3/2 \\ (1/2)\epsilon - (3/4) & 3/2 < \epsilon \leq 2 \end{cases}$$

This implies: $J(X) = 7/4$ so $g(X) = 8/7$. Concerning our parameters we have:
$J_\perp(X) = 5/3$ (achieved, for example, for $x = (-1/4, 1), y = (1,0)$); $g_\perp(X) = 5/4$ (achieved, for example, for $x = (-1,1)$, $y = (2/3, 2/3)$)

Example 5.3 Let $X = \mathbb{R}^2$ endowed with the norm determined by a regular octagon, whose vertices are: $(\pm(\sqrt{2}-1), \pm 1)$, $(\pm(1-\sqrt{2}), \pm 1)$, $(\pm 1, \pm(\sqrt{2}-1))$, $(\pm 1, \pm(1-\sqrt{2}))$ so:

$$||(x,y)|| = \min\{\max\{|x|,|y|\}, \frac{|x|+|y|}{\sqrt{2}}\}.$$

As known, in this case we have $g(X) = J(X) = \sqrt{2}$ (consider for example $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$; but we observe that $(1/\sqrt{2}, 1/\sqrt{2}) \perp (-1/\sqrt{2}, 1/\sqrt{2})$ and so we have also: $g_\perp(X) = J_\perp(X) = \sqrt{2}$.

Example 5.4 Let $X = \mathbb{R}^2$ endowed with the norm so defined:

$$||(x,y)|| = \begin{cases} \sqrt{x^2+y^2} & xy \geq 0 \\ |x|+|y| & xy < 0 \end{cases}$$

We have (see for example [4, page 280]): $J(X) = \sqrt{8/3}$. Therefore $g(X) = \sqrt{3/2}$. Concerning $J_\perp(X)$ (in this case the calculation is non trivial), it is slightly smaller than $J(X)$:
in fact $J_\perp(X) \approx 1,626$ achieved by $(1,0)$ and $(a, \sqrt{1-a^2})$ with $a \approx 0,321$; or by $(0,1)$ and $(a, \sqrt{1-a^2})$ with $a \approx 0.948$.

Example 5.5 Let $X = \mathbb{R}^2$ endowed with the norm so defined:

$$||(x,y)|| = \begin{cases} \sqrt{x^2+y^2} & xy \geq 0 \\ \max\{|x|,|y|\} & xy < 0 \end{cases}$$

As known, $J(X) = 1 + \sqrt{2}/2$; $g(X) = 4 - 2\sqrt{2}$. $J(X)$ is achieved by $(-1,1)$, $(1/\sqrt{2}, 1/\sqrt{2})$, and $(-1,1) \perp (1/\sqrt{2}, 1/\sqrt{2})$. So $J_\perp(X) = J(X)$. $g(X)$ is not achieved by an orthogonal pair: we should take $(1,-\alpha)$, $(\alpha, 1)$ with $\alpha = 3 - \sqrt{8}$ and so $||x \pm y|| = 1 + \alpha$. We obtain: $g_\perp(X) = 5/4$, with the orthogonal pair $(-1/2, 1)$ and $(1,0)$.
We note that $g_\perp(X)\, J_\perp(X) > 2$.

Example 5.6 Let $X = \mathbb{R}^2$ endowed with the norm l^p :

$$||(x,y)|| = \begin{cases} (|x|^p + |y|^p)^{1/p} & p \geq 1 \\ \\ max\{|x|,|y|\} & p = +\infty \end{cases}$$

For $p \in \{1, +\infty\}$ X is not (UNS) so $J(X) = J_\perp(X) = 2$, $g(X) = g_\perp(X) = 1$. Otherwise (see [3]) by using for example the modulus of convexity, we obtain: $J(X) = \max\{2^{1/p}, 2^{1-1/p}\}$ and $g(X) = \min\{2^{1/p}, 2^{1-1/p}\}$. Directly these values can be obtained bu using Clarkson's inequality and (respectively) the following pairs of orthogonal vectors: $(0,1)$, $(1,0)$ and $(1/2^{1/p}, 1/2^{1/p})$, $(-1/2^{1/p}, 1/2^{1/p})$. So we have $J(X) = J_\perp(X)$ and $g(X) = g_\perp(X)$.

Example 5.7 Let $X = \mathbb{R}^2$ endowed with the norm so defined:

$$||(x,y)|| = \begin{cases} (|x|^3 + |y|^3)^{1/3} & xy \geq 0 \\ |x|+|y| & xy < 0 \end{cases}$$

According to [7] we have $J(X) \approx 1.5573$ and $g(X) = 2/J(X) \approx 1.2843$. For our parameters we have: $J_\perp(X) = J(X)$; $g_\perp(X) \approx 1.2987 > g(X)$. The calculations are not simple. We only indicate here how the modulus of continuity behaves. Clearly $\delta_X(\epsilon) = 0$ if $\epsilon \leq 2^{1/3}$. For $\epsilon > 2^{1/3}$ the graph of δ_X is formed by two segments intersecting (approximately) at $(1.55,\ 0.23)$ (the other extremes being $(1.26, 0)$ and $(2,\ 0.23)$).

6 Comparing our parameters with the old ones

We collect a few properties concerning the parameters $g(X)$ and $J(X)$. We always have:

$$1 \leq g(X) \leq J(X) \leq 2 \tag{6.1}$$

$$g(X)\, J(X) = 2 \tag{6.2}$$

$$g(X) = \sqrt{2} \iff J(X) = \sqrt{2} \iff g(X) = J(X) \iff g(X) = J(X) = \sqrt{2} \tag{6.3}$$

$$X\ is\ (UNS) \iff J(X) < 2 \iff g(X) > 1 \tag{6.4}$$

$$g(X) = J(X) = \sqrt{2} \text{ if X is Hilbert;} \tag{6.5}$$

the converse of the last statement is true if dim(X) >2, but not in general: see Example 5.3.

$$\text{It may happen that } J(X) \neq J(X^*), \;\; g(X) \neq g(X^*). \tag{6.6}$$

Concerning (6.6), we can consider Example 5.4 and Example 5.5, namely $X = \mathbb{R}^2$ endowed with the norm so defined:

$$||(x,y)|| = \begin{cases} \sqrt{x^2+y^2} & xy \geq 0 \\ |x| + |y| & xy < 0 \end{cases}$$

and then $X^* = \mathbb{R}^2$ endowed with the norm so defined:

$$||(x,y)|| = \begin{cases} \sqrt{x^2+y^2} & xy \geq 0 \\ \max\{|x|,|y|\} & xy < 0 \end{cases}$$

Since X is reflexive we see that passing to the dual the value of these parameters can both increase or decrease.

The examples we have described in the previous section show that some of these properties fail for $g_\perp(X)$ and $J_\perp(X)$. Now we list the situation with some details.

$$1 \leq g(X) \leq g_\perp(X) \leq J_\perp(X) \leq J(X) \leq 2 \tag{6.7}$$

This chain of inequalities strengthen (6.1); the only non trivial is the central one.
The proof is based on the following result (see [1], Theorem 6.6):

Theorem 6.1 *In every 2-dimensional normed plane there exist* $x, y \in S_X$ *such that* $x \perp y$ *and* $||x-y|| = ||x+y||$.

Theorem 6.2 *In any space* X *we have:* $g_\perp(X) \leq J_\perp(X)$.

Proof It is enough to prove this for X 2-dimensional. Set for $x \in S_X$

$$\alpha_\perp(x) = \inf\left\{ \max\{||x \pm y||\} : x \perp y;\, y \in S_X \right\}$$

and

$$\beta_\perp(x) = \sup\{\min\{||x \pm y||\} : x \perp y;\, y \in S_X\}$$

Of course $g_\perp(X) = \inf\{\alpha_\perp(x) : x \in S_x\}$ and $J_\perp(X) = \sup\{\beta_\perp(x) : x \in S_x\}$.
According to the Theorem 6.1, if dim(X) = 2, then there is a pair $x_0, y_0 \in S_X$ such that $x_0 \perp y_0$ and

$||x_0 - y_0|| = ||x_0 + y_0|| = k$ so we have: $\alpha_\perp(x_0) \le k \le \beta_\perp(x_0)$; thus
$g_\perp(X) \le \alpha_\perp(x_0) \le k \le \beta_\perp(x_0) \le J_\perp(X)$. ■

We note that given two different spaces X, Y we always have $g(X) \le J(Y)$ but our examples show that instead we can have $g_\perp(X) > J_\perp(Y)$.
(6.2): The analogue of (6.2) is not true for our parameters: for example, we have $g_\perp(X)J_\perp(X) > 2$ in Example 5.1.
(6.3): Example 5.2 shows that both $g(X) \neq g_\perp(X)$ and $J(X) \neq J_\perp(X)$ are possible.
(6.4) According to Theorem 3.2 and Theorem 4.1 we see that this result extends to $g_\perp(X)$ and $J_\perp(X)$ giving new characterizations of (UNS) spaces.
(6.5) $g(X) = J(X) = \sqrt{2}$ implies $g_\perp(X) = J_\perp(X) = \sqrt{2}$ so this does not imply that X is Hilbertian (see Example 5.3).
(6.6) We already noticed that the same results hold for our parameters.

7 Bounds concerning the new parameters

We know (see Example 5.2) that we can have $g_\perp(X) = 3/2 > \sqrt{2}$. We can ask how large $g_\perp(X)$ can be in general. Note that: $g(X) \le g_\perp(X) \le J_\perp(X) \le J(X) = 2/g(X)$, thus we have:

$$g(X)\, g_\perp(X) \le 2\,, \quad J(X)\, J_\perp(X) \ge 2. \tag{7.1}$$

Consider the first inequality; in general it only gives $g_(X) \le \sqrt{2}$; it says that $g(X) = \sqrt{2}$ implies $g_\perp(X) = \sqrt{2}$, so the equality (we already noticed this). We know (Theorem 3.2) that:

$$g_\perp(X) \le \frac{-g^2(X) + 4g(X) - 2}{2 - g(X)}\,.$$

(The function on the right increases with $g \in [1, \sqrt{2}]$). Since also $g_\perp(X) \le 2/g(X)$ (the majorizing function is decreasing with g) we compute when we have:

$$\frac{-g^2(X) + 4g(X) - 2}{2 - g(X)} = 2/g(X)\,.$$

This happens for $g(X) \approx 1.194$ and from this we obtain $g_\perp(X) \le a \approx 1.675$. We can also estimate

$$g_\perp(X) - g(X) \le \min\left\{\frac{2g(X) - 2}{2 - g(X)};\ \frac{2}{g(X)} - g(X)\right\};$$

again we have

$$\frac{2g(X) - 2}{2 - g(X)} = \frac{2}{g(X)} - g(X)$$

if and only if $g(X) \approx 1.194$ so $g_\perp(X) - g(X) \le b \approx 0.481$.
Consider now $J_\perp(X)$. According to Theorem 4.1 we have:

$$J_\perp(X) \ge 2J(X) - 2\,;$$

but also

$$J_\perp(X) \ge \frac{2}{J(X)}\,.$$

The first minorizing function is increasing and the second is decreasing: moreover

$$2J(X) - 2 = \frac{2}{J(X)}$$

exactly for

$$J(X) = \frac{1+\sqrt{5}}{2};$$

so we have

$$J_\perp(X) \geq \sqrt{5} - 1.$$

Also:

$$J(X) - J_\perp(X) \leq \min\left\{J(X) - (2J(X) - 2)\,;\, J(X) - \frac{2}{J(X)}\right\}$$

and since the two majorizing functions coincide for $J(X) = \frac{1+\sqrt{5}}{2}$ we obtain:

$$J(X) - J_\perp(X) \leq \frac{3-\sqrt{5}}{2} \approx 0.382.$$

Also the estimates given in this section seem to be not so sharp; in fact for example $J_\perp(X) \geq 2J(X) - 2$ implies $J_\perp(X) \geq \sqrt{2}$ if $J(X) \geq 1 + 1/\sqrt{2}$: but in our examples we have $J_\perp(X) \geq \sqrt{2}$ always.

8 References

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