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The Existence of Solutions for Local Dirichlet $(r(u), s(u))$-Problems

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Abstract: In this paper, we consider local Dirichlet problems driven by the $(r(u), s(u))$-Laplacian operator in the principal part. We prove the existence of nontrivial weak solutions in the case where the variable exponents r, s are real continuous functions and we have dependence on the solution u. The main contributions of this article are obtained in respect of: (i) Carathéodory nonlinearity satisfying standard regularity and polynomial growth assumptions, where in this case, we use geometrical and compactness conditions to establish the existence of the solution to a regularized problem via variational methods and the critical point theory; and (ii) Sobolev nonlinearity, somehow related to the space structure. In this case, we use a priori estimates and asymptotic analysis of regularized auxiliary problems to establish the existence and uniqueness theorems via a fixed-point argument.

Keywords: $(r(u), s(u))$-Laplacian operator; Palais-Smale condition; monotone operator; regularized problem; weak solution

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1. Introduction

Interest in general forms of differential problems, whose leading operator is of the (p, q)-Laplacian type, has greatly increased over the last few decades. The main reason is that this kind of nonlinear operator appears naturally in the study of nonlocal diffusion with special features (see Růžička [1]). Indeed, if the Laplacian operator (that is, $p = q = 2$) is recognized as a key mathematical prototype for the comprehensive study of linear elliptic equations in the context of physical phenomena, the (p, q)-Laplacian operator (in the case $q \neq 2$) extends the range of applications for nonlinear equations in the context of nonlinear physical phenomena as the viscosity analysis of materials with different hardening exponents in the growth rates, and the behavior of smart fluids with and without the influence of external fields (for example, an electromagnetic field). An interesting collection of monographies is available about the general theory of the Laplacian equation, p-Laplacian equation, (p, q)-Laplacian equation, $(p(z), q(z))$-Laplacian equation, and the suitable framework spaces (see, for example, [2–5]). Thus, the reader can find precise replies to any questions concerning the source of difficulties and extra complications in extending the regularity theory of the Laplacian equation up to $(p(z), q(z))$-Laplacian equations with variable exponent functions p and q (see also the book Rădulescu-Repovš [6]). Without being exhaustive, we briefly underline some facts which say that it is nontrivial to continue the theoretical study of $(p(z), q(z))$-Laplacian operators. Following the approach of the Calculus of Variations and Critical Point Theory, the natural starting point of the existing theory is qualitative works about the existence and regularity of solutions for variational integrals (total energy integrals) of the form

$$I(u) = \int_{\Omega} g(z, \nabla u(z)) dz,$$

where $u : \Omega \to \mathbb{R}^N$ (Ω is an open bounded domain) and ∇u is the $N \times N$ matrix of the deformation gradient. The study is carried out imposing a growth assumption of the form

$$c_0|u|^{c_1} \le |g(z,u)| \le c_2(1+|u|^{c_3}) \text{ for all } (z,u) \in \Omega \times \mathbb{R},$$

where c_0, c_1, c_2, c_3 are positive constants and $1 \le c_1 \le c_3$; we refer to [7–11] for a wide discussion on the topic. Moreover, we distinguish the case of constant exponents p, q (namely, isotropic equations) and the case of variable exponents $p(z), q(z)$ (namely, anisotropic equations). The existing results were developed in the abstract settings of Lebesgue and Sobolev spaces with and without variable exponents, namely, $L^p(\Omega)$, $W^{1,p}(\Omega)$, $L^{p(z)}(\Omega)$, $W^{1,p(z)}(\Omega)$. Now, it is well-known that $L^{p(z)}(\Omega)$ is not invariant with respect to translation (Kováčik-Rákosník [12]). This is a source of difficulties about convolutions and continuity of functions in the mean. Moreover, $W^{1,p(z)}(\Omega)$ presents difficulties about the density of smooth functions (Meyers-Serrin [13]), the Sobolev inequality, and embedding theorems (Edmunds-Rákosník [14], Kováčik-Rákosník [12]). This means that the passage from the constant exponent setting to the variable exponent setting needs attention to special cases, and thus, some challenging open problems remain (for further details, we refer to Barile-Figueiredo [15] and Cencelj-Rădulescu-Repovš [16], and the references therein).

In our paper, attention is focused on the following Dirichlet boundary value problem:

$$\begin{cases} -\Delta_{r(u)}u(z) - \Delta_{s(u)}u(z) = \mathcal{N} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_{m(u)}u := \operatorname{div}(|\nabla u|^{m(u)-2}\nabla u)$ is the $m(u)$-Laplacian, $\mathcal{N}$ is a given nonlinear reaction term, and $\Omega \subset \mathbb{R}^d$ is an open bounded domain with a smooth boundary. The main feature of problem (1) is that both r and s vary in respect of the solution u.

To the best of our knowledge, the study of qualitative behavior for anisotropic $(r(u), s(u))$-Laplacian equations with exponent functions depending on the solution u was not considered in previous works. Instead, there are two key papers dealing with the single $r(u)$-Laplacian equation, both with homogeneous Dirichlet boundary conditions (see Andreianov-Bendahmane-Ouaro [17] and Chipot-de Oliveira [18]). In these papers, the authors established existence and uniqueness results, assuming the nonlinearity $\mathcal{N}$ has an appropriate structure. However, due to the u-dependence, r and s continue to be strongly related to Ω. This leads us to keep in mind the natural position: $p(z) = r(u(z))$ and $q(z) = s(u(z))$. Consequently, in the sequel, we can work in the Lebesgue and Sobolev spaces with variable exponents according to Fan-Zhao [19] (see also Papageorgiou-Rădulescu-Repovš [20]).

The paper is organized as follows. In Section 2, we collect the preliminaries and give some auxiliary results. Section 3 is devoted to the proofs of existence theorems for regularized problems, in the case of a Carathéodory nonlinearity satisfying sign and polynomial growth conditions. Here, we use geometrical and compactness conditions to establish our results. Section 4 is devoted to the proofs of existence and uniqueness theorems in the case of a general nonlinearity with an appropriate structure of the Sobolev space. We work with regularized auxiliary problems and a priori estimates. Additionally, the Schauder fixed-point theorem is used in the proof of the second theorem.

2. Mathematical Background

The aim of this section is to recall some results about the variable exponent Lebesgue and Sobolev spaces, the known embedding theorems, and Hölder-type inequalities. We fix the notation as follows. By $\langle \cdot, \cdot \rangle$ we mean the duality brackets for the pair (X^*, X), where X and X^* are a Banach space and its topological dual, respectively.

To develop our results, we need some special features for the framework space. Thus, by $\mathcal{M}(\Omega)$ we mean the space of Lebesgue-measurable functions $m : \Omega \to [1, +\infty)$, where

$$m^- := \operatorname*{ess\,inf}_{z\in\Omega} m(z), \quad m^+ := \operatorname*{ess\,sup}_{z\in\Omega} m(z). \tag{2}$$

Fixed $m \in \mathcal{M}(\Omega) \cap L^\infty(\Omega)$, then the space $L^{m(z)}(\Omega)$ given as

$$L^{m(z)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \ u \text{ is measurable and } \rho_m(u) := \int_\Omega |u(z)|^{m(z)} dz < +\infty \right\},$$

contains all Lebesgue-measurable functions u over the variable space Ω, with the bounded integral value $\rho_m(u)$. This space is Banach whenever we consider the usual Luxemburg norm defined by

$$\|u\|_{L^{m(z)}(\Omega)} := \inf\left\{ \lambda > 0 : \rho_m\left(\frac{u}{\lambda}\right) \le 1 \right\}.$$

For further use, we recall the following properties of $L^{m(z)}(\Omega)$.

Theorem 1 ([19], Theorems 1.6 and 1.8)**.** *The space $(L^{m(z)}(\Omega), \|\cdot\|_{L^{m(z)}(\Omega)})$ is a separable Banach space. Moreover, $C_0^\infty(\Omega)$ is dense in the space $(L^{m(z)}(\Omega), \|\cdot\|_{L^{m(z)}(\Omega)})$.*

Theorem 2 ([19], Theorem 1.10)**.** *If $1 < m^- \le m^+ < +\infty$, then $(L^{m(z)}(\Omega), \|\cdot\|_{L^{m(z)}(\Omega)})$ is uniformly convex, thus reflexive too.*

To $m(\cdot)$, there corresponds $m'(\cdot)$ related by the relation $\frac{1}{m'(\cdot)} + \frac{1}{m(\cdot)} = 1$, and referred to as the Hölder conjugate exponent of $m(\cdot)$. Thus, we denote by $L^{m'(z)}(\Omega)$ the topological dual of $L^{m(z)}(\Omega)$. We recall the crucial inequality

$$1 < (m^+)' \le \operatorname*{ess\,inf}_{z\in\Omega} m'(z) \le \operatorname*{ess\,sup}_{z\in\Omega} m'(z) \le (m^-)' < +\infty.$$

Moreover, in the existing literature, some consolidated results link the norm with the integral in the definition of $L^{m(z)}(\Omega)$. Precisely, we need the following theorems.

Theorem 3 ([19], Theorem 1.2)**.** *Let $u \in L^{m(z)}(\Omega) \setminus \{0\}$, then $\|u\|_{L^{m(z)}(\Omega)} = a$ if, and only if $\rho_m(u/a) = 1$.*

Theorem 4 ([19], Theorem 1.3)**.** *Let $u \in L^{m(z)}(\Omega)$, then the following relations hold:*
(i) $\|u\|_{L^{m(z)}(\Omega)} < 1 \ (= 1, > 1) \Leftrightarrow \rho_m(u) < 1 \ (= 1, > 1)$;
(ii) *if* $\|u\|_{L^{m(z)}(\Omega)} > 1$, *then* $\|u\|^{m^-}_{L^{m(z)}(\Omega)} \le \rho_m(u) \le \|u\|^{m^+}_{L^{m(z)}(\Omega)}$;
(iii) *if* $\|u\|_{L^{m(z)}(\Omega)} < 1$, *then* $\|u\|^{m^+}_{L^{m(z)}(\Omega)} \le \rho_m(u) \le \|u\|^{m^-}_{L^{m(z)}(\Omega)}$.

Remark 1. *By Theorem 4, we can easily deduce that*

$$\|u\|^{m^-}_{L^{m(z)}(\Omega)} - 1 \le \rho_{m(z)}(u) \le \|u\|^{m^+}_{L^{m(z)}(\Omega)} + 1. \tag{3}$$

These inequalities will be used to obtain certain a priori estimates in the sequel.

Moreover, if $m^- > 1$, we recall the Hölder inequality in the form

$$\int_\Omega uv dz \le \left(\frac{1}{m^-} + \frac{1}{(m')^-}\right) \|u\|_{L^{m(z)}(\Omega)} \|v\|_{L^{m'(z)}(\Omega)} \le 2\|u\|_{L^{m(z)}(\Omega)} \|v\|_{L^{m'(z)}(\Omega)}, \tag{4}$$

for $u \in L^{m(z)}(\Omega)$, $v \in L^{m'(z)}(\Omega)$. This Hölder inequality leads to the proof of the following embedding theorem over a bounded domain Ω. Here, $|\Omega|$ denotes the Lebesgue measure of the set Ω.

Theorem 5 ([19], Theorem 1.11)**.** *Let $|\Omega| < +\infty$, $m_1, m_2 \in \mathcal{M}(\Omega) \cap L^\infty(\Omega)$. Then the necessary and sufficient condition for $L^{m_2(z)}(\Omega) \hookrightarrow L^{m_1(z)}(\Omega)$ is that for almost all $z \in \Omega$, we have $m_1(z) \leq m_2(z)$, and the embedding is continuous too.*

On this basis, we recall the precise definition of the Lebesgue−Sobolev space $W^{1,m(z)}(\Omega)$ given by

$$W^{1,m(z)}(\Omega) := \{u \in L^{m(z)}(\Omega) : |\nabla u| \in L^{m(z)}(\Omega)\},$$

endowed with the norm

$$\|u\|_{W^{1,m(z)}(\Omega)} = \|u\|_{L^{m(z)}(\Omega)} + \|\nabla u\|_{L^{m(z)}(\Omega)},$$

where $\|\nabla u\|_{L^{m(z)}(\Omega)} = \||\nabla u|\|_{L^{m(z)}(\Omega)}$.

Now, $W^{1,m(z)}(\Omega)$ is separable if $1 \leq m^- \leq m^+ < +\infty$ holds, and it is reflexive if $1 < m^- \leq m^+ < +\infty$ holds. Therefore, we get

$$W^{1,m_2(z)}(\Omega) \hookrightarrow W^{1,m_1(z)}(\Omega), \text{ whenever } m_1(z) \leq m_2(z) \text{ for a.e. } z \in \Omega. \tag{5}$$

A further step toward the correct definition of the framework space leads to an introduction of the set

$$W_0^{1,m(z)}(\Omega) := \{u \in W_0^{1,1}(\Omega) : |\nabla u| \in L^{m(z)}(\Omega)\},$$

normed by

$$\|u\|_{W_0^{1,m(z)}(\Omega)} = \|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^{m(z)}(\Omega)}. \tag{6}$$

Additionally, we have that $W_0^{1,m(z)}(\Omega)^* = W^{-1,m'(z)}(\Omega)$ is the topological dual of $W_0^{1,m(z)}(\Omega)$. It is well-known that, if $m \in C(\overline{\Omega}) \cap \mathcal{M}(\Omega)$ for the specific constant $c_4 = c_4(m, \Omega, d) > 0$, then we have

$$\|u\|_{L^{m(z)}(\Omega)} \leq c_4 \|\nabla u\|_{L^{m(z)}(\Omega)} \quad \text{for all } u \in W_0^{1,m(z)}(\Omega),$$

(see Theorem 8.2.18, p. 263, Diening-Harjulehto-Hästö-Rŭžicka [3]). Then, $\|u\|_{W^{1,m(z)}(\Omega)}$ and $\|\nabla u\|_{L^{m(z)}(\Omega)}$ are equivalent norms on $W_0^{1,m(z)}(\Omega)$. Thus, we will use $\|\nabla u\|_{L^{m(z)}(\Omega)}$ to replace $\|u\|_{W^{1,m(z)}(\Omega)}$ and put

$$\|u\| = \|\nabla u\|_{L^{m(z)}(\Omega)} \quad \text{in } W_0^{1,m(z)}(\Omega).$$

As pointed out in the Introduction, the density of smooth functions is a source of difficulties in $W_0^{1,m(z)}(\Omega)$. To solve this situation, by following the similar arguments in [18], we consider the set

$$\begin{aligned} &H_0^{1,m(z)}(\Omega) := \text{the closure of } C_0^\infty \text{ with respect to } \|\cdot\|_{W^{1,m(z)}(\Omega)}, \\ \Rightarrow \quad &H_0^{1,m(z)}(\Omega) \subsetneq W_0^{1,m(z)}(\Omega). \end{aligned}$$

Indeed, ([19], Theorem 2.6) gives us that $C_0^\infty(\Omega)$ is dense in $W_0^{1,m(z)}(\Omega)$, under the assumptions:

- Ω is a bounded domain with $\partial\Omega$ being Lipschitz-continuous;

- $m(\cdot)$ is log-Hölder continuous (that is, there exists $c_5 > 0$ such that

$$-|m(z) - m(x)| \ln |z - x| \leq c_5 \quad \text{for all } z, x \in \Omega, \text{ with } |z - x| < \tfrac{1}{2}). \tag{7}$$

Moreover, (7) implies that $H_0^{1,m(z)}(\Omega) = W_0^{1,m(z)}(\Omega)$. We point out that

$$\lambda \in (0,1),\, m \in C^{0,\lambda}(\Omega) \text{ imply that } m(\cdot) \text{ is log-Hölder continuous.} \tag{8}$$

Finally, we recall the critical Sobolev exponent corresponding to $m(\cdot)$ given as

$$m^*(z) = \begin{cases} \dfrac{d\, m(z)}{d - m(z)} & \text{if } m(z) < d, \\ +\infty & \text{if } m(z) \geq d. \end{cases}$$

Additionally, the classical Sobolev embedding theorem was generalized by Fan-Zhao ([19], Theorem 2.3) in the following way.

Proposition 1. *If $m \in C(\overline{\Omega})$ with $m^- > 1$, $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(z) < m^*(z)$ for all $z \in \Omega$, then there exists a continuous and compact embedding $W^{1,m(z)}(\Omega) \hookrightarrow L^{\alpha(z)}(\Omega)$.*

In the fashion of inequality (7) with $m^- > d$, we know that

$$\|u\|_\infty \leq c_6 \|\nabla u\|_{L^{m(z)}(\Omega)} \quad \text{for all } u \in W_0^{1,m(z)}(\Omega), \text{ some } c_6 = c_6(m^-, d, c_5) > 0.$$

We conclude this section with a result concerning the features of the functionals related to monotonicity (see Chipot [21]).

Lemma 1. *For all $\xi, \eta \in \mathbb{R}^d$, the following assertions hold true:*

$$2 \leq m < +\infty \Rightarrow 2^{1-m}|\xi - \eta|^m \leq (|\xi|^{m-2}\xi - |\eta|^{m-2}\eta) \cdot (\xi - \eta);$$
$$1 < m < 2 \Rightarrow (m-1)|\xi - \eta|^2 \leq (|\xi|^{m-2}\xi - |\eta|^{m-2}\eta) \cdot (\xi - \eta) \cdot (|\xi|^m + |\eta|^m)^{\frac{2-m}{m}}.$$

3. Carathéodory Nonlinearity

In this section, we assume $r, s : \mathbb{R} \to (1, +\infty)$ is continuous and $\mathcal{N} := g(z, u(z))$, where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies some standard sign and polynomial growth assumptions. Here, $g(z, \xi)$ is a Carathéodory function (i.e., for each $\xi \in \mathbb{R}$, $z \to g(z, \xi)$ is measurable and for a.e. $z \in \Omega$, $\xi \to g(z, \xi)$ is continuous). The results are consistent with the theoretical analysis of Fan-Zhang [22], but we work on a regularized problem with a u-dependent $(r(\cdot), s(\cdot))$-Laplacian operator (instead of a problem with a z-dependent $p(z)$-Laplacian operator).

We recall that for a weak solution of the problem (P_g) (that is, (1) with $\mathcal{N} = g$) we mean a function $u \in W_0^{1,r(u)}(\Omega)$ (resp. $u \in W_0^{1,r(u)}(\Omega) \cap W_0^{1,s(u)}(\Omega)$) satisfying

$$\int_\Omega |\nabla u|^{r(u)-2} \nabla u \nabla v dz + \int_\Omega |\nabla u|^{s(u)-2} \nabla u \nabla v dz = \int_\Omega g(z, u) v dz,$$

for each $v \in W_0^{1,r(u)}(\Omega)$ (resp. $v \in W_0^{1,r(u)}(\Omega) \cap W_0^{1,s(u)}(\Omega)$). As usual, this means that we do not require more differentiability than it belongs to the first-order variable exponent Sobolev space. In order to perform an asymptotic analysis of the solutions to problem (1), we implemented an approximation strategy based on the following parametric auxiliary problem:

$$\begin{cases} -\operatorname{div}\left((|\nabla u|^{r(u)-2} + |\nabla u|^{s(u)-2})\nabla u\right) - \varepsilon \operatorname{div}\left(|\nabla u|^{\beta-2}\nabla u\right) = g(z, u(z)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $\Omega \subset \mathbb{R}^d$ is an open bounded domain with a smooth boundary, g given as above has some regularities, and r and s are such that $1 < r^-, s^-$ and $r^+, s^+ < \beta$, and $\varepsilon > 0$ is a parameter.

We revisit the definition of a weak solution to (P_g) in the following form:

$$\int_\Omega \left[|\nabla u_\varepsilon|^{r(u_\varepsilon)-2} + |\nabla u_\varepsilon|^{s(u_\varepsilon)-2} \right] \nabla u_\varepsilon \nabla v dz + \varepsilon \int_\Omega |\nabla u_\varepsilon|^{\beta-2} \nabla u_\varepsilon \nabla v dz = \int_\Omega g(z,u) v dz, \tag{10}$$

that is, a function $u_\varepsilon \in W_0^{1,\beta}(\Omega)$ satisfying (10) for all $v \in W_0^{1,\beta}(\Omega)$ is a weak solution to the problem (9).

Now, we focus on the operator $V_\varepsilon : W_0^{1,\beta}(\Omega) \to (W_0^{1,\beta}(\Omega))^* := W^{-1,\beta'}(\Omega)$ defined by

$$\langle V_\varepsilon(u), v \rangle = \int_\Omega \left[|\nabla u|^{r(u)-2} + |\nabla u|^{s(u)-2} \right] \nabla u \nabla v dz + \varepsilon \int_\Omega |\nabla u|^{\beta-2} \nabla u \nabla v dz \tag{11}$$

for all $u, v \in W_0^{1,\beta}(\Omega)$. Using the norm $\|u\| = \|\nabla u\|_{L^\beta(\Omega)}$, we discuss some properties of V_ε (see also ([22], Theorem 3.1)).

Proposition 2. *The following statements hold:*

(i) $V_\varepsilon : W_0^{1,\beta}(\Omega) \to W^{-1,\beta'}(\Omega)$ *is continuous, bounded, and strictly monotone;*

(ii) V_ε *is an operator of type* (S_+), *that is, if* $u_n \xrightarrow{w} u$ *in* $W_0^{1,\beta}(\Omega)$ *and* $\limsup_{n\to+\infty} \langle V_\varepsilon(u_n) - V_\varepsilon(u), u_n - u \rangle = 0$, *then* $u_n \to u$ *in* $W_0^{1,\beta}(\Omega)$;

(iii) V_ε *is coercive;*

(iv) V_ε *is a homeomorphism.*

Proof. (i) Clearly, V_ε is continuous and bounded (by definition). On the other hand, strict monotonicity follows by Lemma 1.

(ii) With respect to the third term in Equation (11), we note that $B : W_0^{1,\beta}(\Omega) \to W^{-1,\beta'}(\Omega)$ defined by

$$\langle B(u), v \rangle := \int_\Omega |\nabla u|^{\beta-2} \nabla u \nabla v dz$$

is an operator of type (S_+), and hence, V_ε is an operator of type (S_+) too.

(iii) We know that

$$\lim_{\|u\|\to+\infty} \frac{\langle V_\varepsilon(u), u \rangle}{\|u\|} \geq \lim_{\|u\|\to+\infty} \frac{\int_\Omega |\nabla u|^\beta dz}{\|u\|} = +\infty,$$

and hence, V_ε is coercive.

(iv) According to (i) and (iii), the operator V_ε is continuous, strictly monotone (hence, maximal monotone too), and coercive. It follows that V_ε is surjective (see Corollary 2.8.7, p. 135, Papageorgiou-Rădulescu-Repovš [20]). Consequently, V_ε admits an inverse operator, namely, $V_\varepsilon^{-1} : W^{-1,\beta'}(\Omega) \to W_0^{1,\beta}(\Omega)$. Now, the continuity of V_ε^{-1} is sufficient enough to say that V_ε is a homeomorphism. Moreover, for $h_n, h \in W^{-1,\beta'}(\Omega)$ with h_n convergent to h as n goes to infinity, we can set $u_n = V_\varepsilon^{-1}(h_n)$ and $u = V_\varepsilon(h)$ so that $V_\varepsilon(u_n) = h_n$ and $V_\varepsilon(u) = h$. It follows that the sequence $\{u_n\}$ is bounded in $W_0^{1,\beta}(\Omega)$. With no loss of generality, suppose that

$$u_n \xrightarrow{w} u_0 \quad \text{as } n \to +\infty.$$

By the convergence of h_n to h as n goes to infinity, we deduce that

$$\begin{aligned} &\lim_{n\to+\infty} \langle V_\varepsilon(u_n) - V_\varepsilon(u_0), u_n - u_0 \rangle = \lim_{n\to+\infty} \langle h_n, u_n - u_0 \rangle = 0, \\ \Rightarrow \quad & u_n \to u_0 \quad \text{as } n \to +\infty \text{ (recall that } V_\varepsilon \text{ is of type } (S_+)), \\ \Rightarrow \quad & V_\varepsilon(u_0) = V_\varepsilon(u). \end{aligned}$$

This fact leads to $u_n \to u$ as $n \to +\infty$, and hence, V_ε^{-1} is a continuous operator. This completes the proof. $\square$

If we relax the u-dependence of g, then we can establish the following proposition (i.e., existence and uniqueness result).

Proposition 3. *Let $g : \Omega \to \mathbb{R}$ be such that $g \in L^{\gamma(z)}(\Omega)$, where $\gamma \in C(\overline{\Omega})$ with $\frac{1}{\gamma(z)} + \frac{1}{\beta} < 1$, then (9) admits a unique weak solution.*

Proof. With respect to the right-hand side of Equation (10), Proposition 1 permits us to say that

$$\langle g, v\rangle = \int_\Omega g(z) v dz \quad \text{for all } v \in W_0^{1,\beta}(\Omega)$$

is a continuous linear operator on $W_0^{1,\beta}(\Omega)$. By the proof of Proposition 2 (iv), V_ε is a strictly monotone surjective operator, and hence, (9) admits a unique solution. $\square$

Notice that the inequality in the statement of Proposition 3 implies that $\beta > \gamma'(\cdot)$. On the other hand, using an appropriate growth condition, we establish the following existence result (this time we restore the u-dependence of g).

Theorem 6. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition*

$$|g(z,\xi)| \le c_7 + c_8|\xi|^{\alpha(z)-1} \text{ for all } z \in \Omega, \text{ all } \xi \in \mathbb{R}, \text{ some } c_7, c_8 \ge 0 \text{ and } \alpha \in C(\overline{\Omega}). \tag{12}$$

If $1 \le \alpha(z) \le \alpha^+ < \beta$, then (9) admits a weak solution.

Proof. We introduce the functional $H(u) = \int_\Omega G(z,u)dz$, whose integrand is defined by $G(z,u) = \int_0^u g(z,\xi)d\xi$ for all $(z,u) \in \Omega \times \mathbb{R}$. Consequently, the derivative $H' : W_0^{1,\beta}(\Omega) \to W^{-1,\beta'}(\Omega)$ is completely continuous, that is,

$$\begin{aligned} & u_n \xrightarrow{w} u \text{ implies that } H'(u_n) \to H'(u), \\ \Rightarrow \quad & H \text{ is weakly continuous.} \end{aligned}$$

The growth condition (12) suggests that $|G(z,u)| \le c_9(1 + |u|^{\alpha(z)})$ for some $c_9 > 0$. In addition, looking to the critical points of the Euler-Lagrange functional associated to problem (9) and using the position: $p(z) = r(u(z))$ and $q(z) = s(u(z))$, for all $u \in W_0^{1,\beta}(\Omega)$ with $\|u\| \ge 1$, we have

$$\begin{aligned} I(u) :=& \int_\Omega \left(\frac{1}{p(z)} |\nabla u|^{p(z)} + \frac{1}{q(z)} |\nabla u|^{q(z)} \right) dz + \varepsilon \int_\Omega \frac{1}{\beta} |\nabla u|^\beta dz - \int_\Omega G(z,u) dz \\ \ge& \frac{\varepsilon}{\beta} \|u\|^\beta - c_9 C \|u\|^{\alpha^+} - c_9|\Omega| \to +\infty, \text{ as } \|u\| \to +\infty, \text{ (for some } C > 0). \end{aligned}$$

Since the functional I is weakly lower semicontinuous, then it attains a minimum in $W_0^{1,\beta}(\Omega)$. Clearly, this minimum point (i.e., a critical point of I) is an exact weak solution to (9). $\square$

As an illustrative example of the growth condition (12) (and hence, to apply Theorem 6), we construct a nonlinear reaction term starting from a locally defined Carathéodory function. This choice is motivated by the fact that boundary value problems of the form (P_g) (or more generally, of the form (1)) are considered in the context of the regularization theory in image processing for various image restoration problems (see also the comments in the Introduction of [17,21] and the references cited therein). In particular, we recall that regularization techniques often combine local smoothing effects and estimates by suitable cut-off functions. Thus, let $f : \Omega \times [0, b] \to \mathbb{R}$ $(b > 0)$ be a Carathéodory function satisfying $f(z, 0) = 0$ for a.e. $z \in \Omega$ and

(i) $0 \le f(z, \xi) \le a_b(z)$ for a.e. $z \in \Omega$, all $0 \le \xi \le b$ with $a_b \in L^\infty(\Omega)$;

(ii) for $\alpha \in C(\overline{\Omega})$ such that $1 \le \alpha(z) \le \alpha^+ < \beta$ for all $z \in \Omega$, there exists $\nu \in (0, b)$ sufficiently small enough to have $f(z, \xi) \le \xi^{\alpha^+ - 1}$, $\xi^{\beta^*} \le \xi^\beta$ for a.e. $z \in \Omega$, all $0 \le \xi \le \nu$.

Next, we introduce a cut-off function $\psi \in C_c(\mathbb{R})$ satisfying

$$\operatorname{supp} \psi \subseteq [0, \nu], \quad \psi\Big|_{[0,\nu/2]} \equiv 1 \text{ and } 0 < \psi \le 1 \text{ on } (0, \nu],$$

and define the function

$$g(z, \xi) = \begin{cases} 0 & \text{for all } z \in \Omega, \text{ all } \xi \le 0, \\ \psi(\xi)[f(z, \xi) + \xi^{\beta^* - 1}] + (1 - \psi(\xi))\xi^{\alpha(z) - 1} & \text{for all } (z, \xi) \in \Omega \times [0, +\infty). \end{cases}$$

We note that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, also involving a critical growth term $\xi \to \xi^{\beta^* - 1}$, satisfying the condition

$$0 \le g(z, \xi) \le c_\psi[1 + |\xi|^{\alpha(z) - 1}] \quad \text{for a.e. } z \in \Omega, \text{ all } \xi \in \mathbb{R}, \text{ some } c_\psi > 0.$$

As already mentioned in the Introduction, our strategy of proof for the existence of at least a weak nontrivial solution to problem (9) also uses geometrical and compactness conditions, and it applies to the functional I. In detail, to use the Critical Point Theory, we impose a Palais-Smale condition (i.e., a compactness-type condition) in the sense of the following definition.

Definition 1. *The functional I has the Palais-Smale property in $W_0^{1,\beta}(\Omega)$ if every sequence $\{u_n\} \subset W_0^{1,\beta}(\Omega)$, such that $\{I(u_n)\}$ is bounded and $\|I'(u_n)\| \to 0$ in $W^{-1,\beta'}(\Omega)$ as n goes to infinity, admits a convergent subsequence.*

We recall that the main source of difficulty in the construction of a Palais-Smale sequence (that is, the sequence given in Definition 1) is in establishing the boundedness of the sequence. To this aim, we impose the classical Ambrosetti-Rabinowitz condition to the nonlinearity g.

Lemma 2. *If g satisfies the Ambrosetti-Rabinowitz condition, that is,*

(AR) there exist $\theta > \beta$ and $M > 0$ satisfying "$ug(z, u) \ge \theta G(z, u) > 0$" for a.e. $z \in \Omega$ and all $|u| \ge M$,

then the functional I has the Palais-Smale property.

Proof. We start with a sequence $\{u_n\} \subset W_0^{1,\beta}(\Omega)$ such that $\{I(u_n)\}$ is bounded and $\|I'(u_n)\|$ converges to zero in $W^{-1,\beta'}(\Omega)$ as n goes to infinity. Thus, we can find a constant $c_{10} > 0$ such that

$$\begin{aligned} c_{10} &\ge I(u_n) \\ &\ge \int_\Omega \left(\frac{1}{p(z)} |\nabla u_n|^{p(z)} + \frac{1}{q(z)} |\nabla u_n|^{q(z)} \right) dz + \varepsilon \int_\Omega \frac{1}{\beta} |\nabla u_n|^\beta dz - \int_\Omega \frac{u_n}{\theta} g(z, u_n) dz - c_{11} \\ &\qquad \text{(for some } c_{11} > 0\text{; using the position } p(z) = r(u_n(z)) \text{ and } q(z) = s(u_n(z))) \\ &\ge \varepsilon \left(\frac{1}{\beta} - \frac{1}{\theta} \right) \|u_n\|^\beta - \frac{1}{\theta} \|I'(u_n)\| \|u_n\| - c_{12} \quad \text{(for some } c_{12} > 0). \end{aligned}$$

Thus, we deduce that $\{u_n\}$ is a bounded sequence. Now, there is no loss of generality in assuming that $u_n \xrightarrow{w} u$ as $n \to +\infty$. Consequently, we have that $H'(u_n)$ converges to $H'(u)$ (recall the definition of H at the beginning of the proof of Theorem 6). Next, we use Proposition 2. We have

$$\begin{aligned}&\langle V_\varepsilon(u_n) - V_\varepsilon(u), u_n - u\rangle = \langle I'(u_n) - I'(u), u_n - u\rangle + \langle H'(u_n) - H'(u), u_n - u\rangle \\ &\to 0 \quad \text{as } n \to +\infty,\end{aligned}$$

and hence, $u_n \to u$ as $n \to +\infty$, since V_ε is a mapping of type (S_+). We conclude that the functional I admits the Palais-Smale property. □

The Ambrosetti-Rabinowitz condition (AR) is a strong condition in the sense that it determines the supercritical behavior of the nonlinearity g, but it gives us the so-called mountain pass geometry of Ambrosetti and Rabinowitz [23] (that is, a key result to establish the existence of critical points of the Euler-Lagrange functional associated to the problem under investigation). Of course, the common (without z-dependence) superlinear nonlinearity $g(\xi) = \xi^{\theta-1}$ if $\xi \geq 0$ satisfies the condition (AR). On this basis, we consider the local infinitesimal condition with respect to u:

$$g(z,u) = o(|u|^{\beta-1}) \quad \text{as } u \text{ goes to 0 uniformly for } z \in \Omega. \tag{13}$$

Then, we give the following result about the nontriviality of solutions to (9).

Theorem 7. *If* (12) *with* $\beta^* > \alpha^+ \geq \alpha^- > \beta$, (13) *and* (AR) *hold, then* (9) *admits at least one nontrivial weak solution.*

Proof. Lemma 2 ensures that we can find a Palais-Smale sequence for the functional I on the framework space $W_0^{1,\beta}(\Omega)$. Since $\beta < \alpha^- \leq \alpha(z) < \beta^*$ and the continuity of the embedding $W_0^{1,\beta}(\Omega) \hookrightarrow L^\beta(\Omega)$, then we can find $c_{13} > 0$ satisfying the inequality

$$\|u\|_{L^\beta(\Omega)} \leq c_{13}\|u\|, \quad \text{for all } u \in W_0^{1,\beta}(\Omega).$$

Now, we choose $\sigma > 0$ that is sufficiently small enough to have $\varepsilon \geq 2\sigma\beta c_{13}^\beta$. The growth condition (12), together with (13), leads to

$$G(z,u) \leq \sigma|u|^\beta + c_{14}|u|^{\alpha(z)} \quad \text{for all } z \in \Omega, \text{ all } u \in \mathbb{R}, \text{ and some } c_{14} = c_{14}(\sigma) > 0.$$

By routine calculations, for all $u \in W_0^{1,\beta}(\Omega)$ with $\|u\| \leq 1$, we have

$$\begin{aligned}I(u) &\geq \varepsilon \int_\Omega \frac{1}{\beta}|\nabla u|^\beta dz - \sigma \int_\Omega |u|^\beta dz - c_{14}\int_\Omega |u|^{\alpha(z)} dz \\ &\geq \frac{\varepsilon}{\beta}\|u\|^\beta - \sigma c_{13}^\beta \|u\|^\beta - c_{15}\|u\|^{\alpha^-} \quad (\text{for some } c_{15} = c_{15}(\sigma) > 0) \\ &\geq \frac{\varepsilon}{2\beta}\|u\|^\beta - c_{16}\|u\|^{\alpha^-} \quad \text{for some } c_{16} = c_{16}(\sigma) > 0,\end{aligned}$$

then we can find a couple of positive constants (ρ, τ) such that $I(u) \geq \tau > 0$ for all $u \in W_0^{1,\beta}(\Omega)$ with $\|u\| = \rho$.

The condition (AR) says that

$$G(z,u) \geq c_{17}|u|^\theta \text{ for a.e. } z \in \Omega, \text{ all } |u| \geq M, \text{ some } c_{17} > 0.$$

Next, we can find $t > 1$ such that for $w \in W_0^{1,\beta}(\Omega) \setminus \{0\}$ we get

$$\begin{aligned} I(tw) &= \int_\Omega \left(\frac{1}{p(z)} |\nabla tw|^{p(z)} + \frac{1}{q(z)} |\nabla tw|^{q(z)} \right) dz + \varepsilon \int_\Omega \frac{1}{\beta} |\nabla tw|^\beta dz - \int_\Omega G(z,tw) dz \\ &\leq t^\beta I(w) - c_{17} t^\theta - c_{18} \text{ for some } c_{18} > 0 \text{ (we use } p(z) = r(tw(z)), q(z) = s(tw(z))), \\ \Rightarrow \quad & \lim_{t \to +\infty} I(tw) = -\infty. \end{aligned}$$

Notice that $I(0) = 0$, and hence, the functional I admits mountain pass geometry. Consequently, I admits at least one non-zero critical point that is an exact nontrivial solution to (9). □

A typical function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the hypotheses of Theorem 7 is

$$g(z, \xi) = a(z) |\xi|^{\alpha(z)-2} \xi \quad \text{for all } z \in \Omega, \text{ all } \xi \in \mathbb{R},$$

with $a, \alpha \in C(\overline{\Omega})$ and $\beta < \alpha(z) < \beta^*$ for all $z \in \overline{\Omega}$.

4. Sobolev-Type Nonlinearity

This section is based on the ideas and results of Chipot and de Oliveira [18]. We mention that in the Ref. [18], the authors appealed to the pioneering work of Zhikov [24] about the appropriate way to take the limit in a sequence of nonlinear elliptic equations. In more detail, the Ref. [18] pointed out that to get how $|\nabla u|^{r(u)}$ is in the L^1 space over Ω, one can make use of the following lemma (see [18], p. 289).

Lemma 3 ([18], Lemma 3.1). *Let $\alpha, \beta \in (1, +\infty)$ be such that the following conditions hold:*

$$\begin{aligned} &\alpha \leq p_n(z) \leq \beta \quad \text{for a.e. } z \in \Omega, \text{ all } n \in \mathbb{N}, \\ &p_n \to p \quad \text{a.e. in } \Omega, \\ &\nabla u_n \xrightarrow{w} \nabla u \quad \text{in } L^1(\Omega)^d, \\ &\| |\nabla u_n|^{p_n(z)} \|_{L^1(\Omega)} \leq c_{19}, \quad \text{for some } c_{19} > 0 \text{ without } n\text{-dependence.} \end{aligned}$$

Then we deduce that $\nabla u \in L^{p(z)}(\Omega)^d$ and $\liminf_{n \to +\infty} \int_\Omega |\nabla u_n|^{p_n(z)} dz \geq \int_\Omega |\nabla u|^{p(z)} dz$.

The proof of this lemma in the Ref. [18] does not require all the hypotheses considered by Zhikov in the Ref. [24] (see Lemmas 2.4 and 3.3). Since here, we work with a $(r(u), s(u))$-equation, we need a similar argument to establish that both $|\nabla u|^{r(u)}$ and $|\nabla u|^{s(u)}$ are in $L^1(\Omega)$, in the form of the following lemma.

Lemma 4. *Let $\{p_n\}, \{q_n\} \in \mathcal{M}(\Omega)$ and $\alpha, \beta \in (1, +\infty)$ be such that the following conditions hold:*

(i) $\alpha \leq p_n(z), q_n(z) \leq \beta$ *for a.e. $z \in \Omega$ and all $n \in \mathbb{N}$,*
(ii) $p_n \to p$, $q_n \to q$ *a.e. in Ω,*
(iii) $\nabla u_n \xrightarrow{w} \nabla u$ *in $L^1(\Omega)^d$,*
(iv) $\| |\nabla u_n|^{p_n(z)} \|_{L^1(\Omega)}, \| |\nabla u_n|^{q_n(z)} \|_{L^1(\Omega)} \leq c_{20}$, *for some $c_{20} > 0$ without n-dependence.*

Then we deduce that $|\nabla u|^{p(z)}, |\nabla u|^{q(z)} \in L^1(\Omega)$ and

$$\liminf_{n \to +\infty} \int_\Omega \left[|\nabla u_n|^{p_n(z)} + |\nabla u_n|^{q_n(z)} \right] dz \geq \int_\Omega \left[|\nabla u|^{p(z)} + |\nabla u|^{q(z)} \right] dz. \tag{14}$$

Proof. Hypothesis (i) and Lemma 3 (separately for (p, p_n) and (q, q_n)) give us that

$$|\nabla u|^{p(z)} \in L^1(\Omega), \quad \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^{p_n(z)} dz \geq \int_\Omega |\nabla u|^{p(z)} dz,$$
$$|\nabla u|^{q(z)} \in L^1(\Omega), \quad \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^{q_n(z)} dz \geq \int_\Omega |\nabla u|^{q(z)} dz.$$

Then, summing the two obtained inequalities, we trivially deduce that (14) holds. □

Now, we focus on the existence problem of solutions to (1) in the case $\mathcal{N} := g \in W^{-1,\alpha'}(\Omega)$. According to the existing theory for $(p(z), q(z))$-Laplacian equations, we work on the same solution set used in the Ref. [18], that is,

$$W_0^{1,r(u)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{r(u)} dz < +\infty \right\}.$$

With respect to the previously defined norm (see (6)), we know that $W_0^{1,r(u)}(\Omega)$ is a Banach space, whenever the exponent is finite and greater than 1 (i.e., $r(u) \in (1, +\infty)$ for all $u \in \mathbb{R}$). Moreover, classical results say that $\|u\|_{W_0^{1,r(u)}(\Omega)}$ and $\|\nabla u\|_{L^{r(u)}(\Omega)}$ are equivalent norms whenever the exponent is a continuous function over the closure of a variable space Ω. The embedding in (5) gives us that $W_0^{1,r(u)}(\Omega) \subseteq W_0^{1,\alpha}(\Omega)$ is closed, whenever the inequality $1 < \alpha \leq r(\cdot)$ holds for a r continuous function and some constant α. Consequently, $W_0^{1,r(u)}(\Omega)$ is a separable and reflexive Banach space. Moreover, we also work on the set $W_0^{1,s(u)}(\Omega)$, which of course has similar properties given above for $W_0^{1,r(u)}(\Omega)$.

Now, for each $\varepsilon > 0$, we consider the auxiliary problem (namely, the regularized problem) in the following form

$$\begin{cases} -\mathrm{div}\,((|\nabla u|^{r(u)-2} + |\nabla u|^{s(u)-2})\nabla u) - \varepsilon \mathrm{div}\,(|\nabla u|^{\beta-2}\nabla u) = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where the exponent functions $r, s : \mathbb{R} \to (1, +\infty)$ are continuous and satisfy the range condition separately $d < \alpha \leq s(u), r(u) \leq \beta < +\infty$ (that is, r and s are not related to each other). Different from problem (9), here we impose that the nonlinearity is $g \in W^{-1,\alpha'}(\Omega)$. Consequently, (15) admits a weak solution $u \in W_0^{1,\beta}(\Omega)$ whenever

$$\int_\Omega \left[|\nabla u|^{r(u)-2} + |\nabla u|^{s(u)-2} \right] \nabla u \nabla v dz + \varepsilon \int_\Omega |\nabla u|^{\beta-2} \nabla u \nabla v dz = \langle g, v \rangle,$$

for each $v \in W_0^{1,\beta}(\Omega)$, and $\langle \cdot, \cdot \rangle$ are the duality brackets for the pair $(W^{-1,\alpha'}(\Omega), W_0^{1,\alpha}(\Omega))$.

Now, we have all the ingredients to produce a weak solution of (15). Our strategy to obtain this solution is the following. Using technical hypotheses on r, s and g, we first create the setting for an application of Proposition 3, which gives us the existence of a particular solution. Then, this solution is used to define a self-map over a set with precise bounds in a norm. On account of the Schauder fixed-point theorem and Lebesgue theorem (of dominated convergence), we establish the continuity of the above map and hence, the existence of a fixed point, which gives us the required solution.

Theorem 8. *If $g \in W^{-1,\alpha'}(\Omega)$, $r, s : \mathbb{R} \to (1, +\infty)$ are continuous and $d < \alpha \leq s(u), r(u) \leq \beta < +\infty$ for all $u \in \mathbb{R}$, then for each $\varepsilon > 0$ problem (15) admits a weak solution u_ε.*

Proof. We fix $x \in L^2(\Omega)$. Then, the hypotheses on r, s imply that $r(x), s(x) \in \mathcal{M}(\Omega)$ and the following inequalities hold:

$$d < \alpha \leq s(x(z)), r(x(z)) \leq \beta < +\infty \quad \text{for a.e. } z \in \Omega. \tag{16}$$

From (16), $g \in W^{-1,\alpha'}(\Omega)$ and the notion of a conjugate exponent, we get

$$g \in W^{-1,\alpha'}(\Omega) \subseteq W^{-1,\beta'}(\Omega).$$

Now, Proposition 3 leads us to a $u = u_x \in W_0^{1,\beta}(\Omega)$ unique solution of the equation

$$\int_\Omega \left[|\nabla u|^{r(x)-2} + |\nabla u|^{s(x)-2} \right] \nabla u \nabla v dz + \varepsilon \int_\Omega |\nabla u|^{\beta-2} \nabla u \nabla v dz = \langle g, v \rangle, \tag{17}$$

for each $v \in W_0^{1,\beta}(\Omega)$. We choose $v = u$ in (17) and use the Hölder inequality to obtain

$$\int_\Omega \left[|\nabla u|^{r(x)} + |\nabla u|^{s(x)} \right] dz + \varepsilon \int_\Omega |\nabla u|^{\beta} dz \leq \|g\|_{W^{-1,\alpha'}(\Omega)} \|\nabla u\|_{L^\alpha(\Omega)} \leq c_{21} \|\nabla u\|_{L^\beta(\Omega)},$$

where $c_{21} = c_{21}(\alpha, \beta, \Omega, g) > 0$. Recall that $\|\cdot\|_{W^{-1,\alpha'}(\Omega)}$ means the operator norm related to $\|\nabla \cdot \|_{L^\alpha(\Omega)}$ and hence, we have the inequalities

$$\begin{aligned} & \varepsilon \|\nabla u\|^{\beta}_{L^\beta(\Omega)} \leq c_{22} \|\nabla u\|_{L^\beta(\Omega)}, \\ \Rightarrow \quad & \|\nabla u\|_{L^\beta(\Omega)} \leq c_{23}, \end{aligned} \tag{18}$$

for some $c_{22} = c_{22}(\alpha, \beta, \Omega, \varepsilon, g) > 0$, $c_{23} = c_{23}(\alpha, \beta, \Omega, \varepsilon, g) > 0$. We remark that these constants do not present any x-dependences. By hypothesis $2 \leq d < \beta$, we hence have the compact embedding $W_0^{1,\beta}(\Omega) \hookrightarrow L^2(\Omega)$ and the inequality

$$\|u\|_{L^2(\Omega)} = \|u_x\|_{L^2(\Omega)} \leq c_{24},$$

this time $c_{24} = c_{24}(\alpha, \beta, \Omega, \varepsilon, g, d) > 0$, but again we have no dependence from the choice of x.

Next, we introduce the self-map $h : B \to B$ defined by $h(x) = u_x$, over the set $B := \{v \in L^2(\Omega) : \|v\|_{L^2(\Omega)} \leq c_{24}\}$. The compact embedding $W_0^{1,\beta}(\Omega) \hookrightarrow L^2(\Omega)$ implies that $h(B)$ is relatively compact in B. Appealing to the Schauder fixed-point theorem, we know that the continuity of h is required in obtaining a fixed point of h.

With the assumption that we work on a sequence $\{x_n\}$ in $L^2(\Omega)$ satisfying

$$x_n \to x \text{ in } L^2(\Omega), \tag{19}$$

we denote by u_n, for all $n \in \mathbb{N}$, the solution of (17) related to $x := x_n$. Therefore, the inequality in (18) leads to

$$\|\nabla u_n\|_{L^\beta(\Omega)} \leq c_{25}, \quad \text{for some } c_{25} > 0 \text{ (without } n\text{-dependence).}$$

Passing to a subsequence if necessary (namely again $\{u_n\}$), for a certain $u \in W_0^{1,\beta}(\Omega)$ we get

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,\beta}(\Omega) \text{ and } u_n \to u \text{ in } L^2(\Omega). \tag{20}$$

We return to (17), so that considering (u_n, x_n) instead of (u, x), we have

$$\begin{aligned} & \int_\Omega \left[|\nabla u_n|^{r(x_n)-2} + |\nabla u_n|^{s(x_n)-2} \right] \nabla u_n \nabla v dz \\ & \quad + \varepsilon \int_\Omega |\nabla u_n|^{\beta-2} \nabla u_n \nabla v dz = \langle g, v \rangle \text{ for each } v \in W_0^{1,\beta}(\Omega). \end{aligned} \tag{21}$$

Since the operator on the left-hand side of (21) is monotone, then we deduce that

$$\int_{\Omega}\left(|\nabla u_n|^{r(x_n)-2}+|\nabla u_n|^{s(x_n)-2}+\varepsilon|\nabla u_n|^{\beta-2}\right)\nabla u_n\nabla(u_n-v)dz \\ -\int_{\Omega}\left(|\nabla v|^{r(x_n)-2}+|\nabla v|^{s(x_n)-2}+\varepsilon|\nabla v|^{\beta-2}\right)\nabla v\nabla(u_n-v)dz\geq 0 \tag{22}$$

for each $v\in W_0^{1,\beta}(\Omega)$.

Considering (21) with u_n-v instead of v, we use (22) to deduce that

$$\langle g,u_n-v\rangle-\int_{\Omega}\left(|\nabla v|^{r(x_n)-2}+|\nabla v|^{s(x_n)-2}+\varepsilon|\nabla v|^{\beta-2}\right)\nabla v\nabla(u_n-v)dz\geq 0 \tag{23}$$

for each $v\in W_0^{1,\beta}(\Omega)$. The convergence in (19), passing eventually to a subsequence, implies

$$x_n\to x \text{ a.e. in } \Omega.$$

Consequently, since r,s are continuous functions, we use the Lebesgue theorem (in $L^{\beta'}(\Omega)^d$) to get

$$\lim_{n\to+\infty}\left[|\nabla v|^{r(x_n)-2}+|\nabla v|^{s(x_n)-2}\right]\nabla v=\left[|\nabla v|^{r(x)-2}+|\nabla v|^{s(x)-2}\right]\nabla v, \tag{24}$$

for all $v\in W_0^{1,\beta}(\Omega)$. Finally, passing to the limit in (23), by the weak convergence in (20) and using (24), we conclude that

$$\langle g,u-v\rangle-\int_{\Omega}\left(|\nabla v|^{r(x)-2}+|\nabla v|^{s(x)-2}+\varepsilon|\nabla v|^{\beta-2}\right)\nabla v\nabla(u-v)dz\geq 0 \tag{25}$$

for all $v\in W_0^{1,\beta}(\Omega)$.

Next, we choose $v=u\mp\delta y$, where $y\in W_0^{1,\beta}(\Omega)$ and $\delta>0$, so that by (25) we get

$$\pm\Big[\langle g,y\rangle-\int_{\Omega}\big[|\nabla(u\mp\delta y)|^{r(x)-2}+|\nabla(u\mp\delta y)|^{s(x)-2} \\ +\varepsilon|\nabla(u\mp\delta y)|^{\beta-2}\big)\nabla(u\mp\delta y)\nabla y dz\Big]\geq 0. \tag{26}$$

We pass to the limit as δ goes to zero in (26), and deduce that

$$\int_{\Omega}\left[|\nabla u|^{r(x)-2}+|\nabla u|^{s(x)-2}\right]\nabla u\nabla y dz+\varepsilon\int_{\Omega}|\nabla u|^{\beta-2}\nabla u\nabla y dz=\langle g,y\rangle,$$

for all $y\in W_0^{1,\beta}(\Omega)$. Consequently, $u=u_x$, and hence, by the strong convergence in (20) we conclude that

$$u_{x_n}\to u_x \quad \text{in } L^2(\Omega).$$

It follows that h is continuous, and this establishes the existence of the fixed point which is the exact weak solution to (15). □

The next theorem needs the following revised definition of a weak solution.

Definition 2. *Given two continuous functions $r,s:\mathbb{R}\to(1,+\infty)$ such that*

$$d<\alpha\leq s(u)\leq r(u)\leq\beta<+\infty \quad \textit{for all } u\in\mathbb{R}, \textit{ some } \alpha,\beta>0, \tag{27}$$

we assume that

$$g\in W^{-1,\alpha'}(\Omega). \tag{28}$$

Then, we say that $u \in W_0^{1,r(u)}(\Omega)$ *is a weak solution to* (P_g) *if*

$$\int_\Omega \Big[|\nabla u|^{r(u)-2} + |\nabla u|^{s(u)-2}\Big] \nabla u \nabla v dz = \langle g, v\rangle, \text{ for each } v \in W_0^{1,r(u)}(\Omega),$$

and $\langle \cdot,\cdot\rangle$ *is the duality brackets for the pair* $(W^{-1,r(u)'}(\Omega), W_0^{1,r(u)}(\Omega))$.

Note that $r(u), s(u) \in \mathcal{M}(\Omega)$ and the essential infimum and the essential supremum of $r(u), s(u)$ satisfy the condition $\alpha \le s^-(u) \le r^+(u) \le \beta$ for all $u \in W_0^{1,r(u)}(\Omega)$.

For the strategy to work, we need to slightly strengthen hypotheses on the framework structure. Thus, the new conditions on Ω and (r,s) are the following:

$$\Omega \subset \mathbb{R}^d,\ d \ge 2, \text{is a bounded domain with } \partial\Omega \text{ Lipschitz-continuous;} \tag{29}$$

$$r, s : \mathbb{R} \to (1, +\infty) \text{ are Lipschitz-continuous functions.} \tag{30}$$

Theorem 9. *If (27)–(30) hold, then problem* (P_g) *admits at least one weak solution* $u \in W_0^{1,r(u)}(\Omega)$.

Proof. We already know that for each $\varepsilon > 0$, one can find $u_\varepsilon \in W_0^{1,\beta}(\Omega)$ satisfying

$$\int_\Omega \Big[|\nabla u_\varepsilon|^{r(u_\varepsilon)-2} + |\nabla u_\varepsilon|^{s(u_\varepsilon)-2}\Big] \nabla u_\varepsilon \nabla v dz + \varepsilon \int_\Omega |\nabla u_\varepsilon|^{\beta-2} \nabla u_\varepsilon \nabla v dz = \langle g, v\rangle, \tag{31}$$

for each $v \in W_0^{1,\beta}(\Omega)$. A crucial key of this result is the chain of inequalities

$$d < \alpha \le s(u_\varepsilon(z)) \le r(u_\varepsilon(z)) \le \beta < +\infty \text{ for all } \varepsilon > 0, \text{ for a.e. } z \in \Omega.$$

Now, we choose $v = u_\varepsilon$ in (31) and obtain

$$\int_\Omega \Big[|\nabla u_\varepsilon|^{r(u_\varepsilon)} + |\nabla u_\varepsilon|^{s(u_\varepsilon)}\Big] dz + \varepsilon \|\nabla u_\varepsilon\|^\beta_{L^\beta(\Omega)} = \langle g, u_\varepsilon\rangle. \tag{32}$$

By Remark 1 (i.e., we focus on the first part of (3)) we deduce that

$$\|u_\varepsilon\|_{L^{r(u_\varepsilon)}(\Omega)} \le (\rho_{r(u_\varepsilon)}(u_\varepsilon) + 1)^{1/r^-(u_\varepsilon)} = \Big(\int_\Omega |u_\varepsilon|^{r(u_\varepsilon)} dz + 1\Big)^{1/r^-(u_\varepsilon)}.$$

By (4), it follows that

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^\alpha dz &\le c_{26} \||\nabla u_\varepsilon|^\alpha\|_{L^{\alpha^{-1} r(u_\varepsilon)}(\Omega)} \quad (\text{for some } c_{26} > 0) \\ &\le c_{27} \Big(\int_\Omega |\nabla u_\varepsilon|^{r(u_\varepsilon)} dz + 1\Big)^{1/(\alpha^{-1} r(u_\varepsilon))^-} \quad (\text{for some } c_{27} > 0) \\ &\le c_{27} \Big(\int_\Omega |\nabla u_\varepsilon|^{r(u_\varepsilon)} dz + 1\Big). \end{aligned} \tag{33}$$

We point out that the above constants c_{26}, c_{27} do not depend on ε; instead, they depend on the triplet (α, β, Ω). Therefore, for some $c_{28} > 0$ (without ε-dependence), we obtain the following estimate of the right-hand side of (32)

$$\langle g, u_\varepsilon\rangle \le \|g\|_{W^{-1,\alpha'}(\Omega)} \|\nabla u_\varepsilon\|_{L^\alpha(\Omega)} \le c_{28} \|g\|_{W^{-1,\alpha'}(\Omega)} \Big(\int_\Omega |\nabla u_\varepsilon|^{r(u_\varepsilon)} dz + 1\Big)^{1/\alpha}. \tag{34}$$

Using (32) and (34), we get

$$\begin{aligned} &\int_\Omega |\nabla u_\varepsilon|^{r(u_\varepsilon)} dz \le c_{28} \|g\|_{W^{-1,\alpha'}(\Omega)} \Big(\int_\Omega |\nabla u_\varepsilon|^{r(u_\varepsilon)} dz + 1\Big)^{1/\alpha}, \\ \Rightarrow\ &\int_\Omega |\nabla u_\varepsilon|^{r(u_\varepsilon)} dz \le c_{29} \quad \text{for some } c_{29} > 0. \end{aligned}$$

Consequently, we have

$$\int_{\Omega}\left[|\nabla u_{\varepsilon}|^{r(u_{\varepsilon})}+|\nabla u_{\varepsilon}|^{s(u_{\varepsilon})}\right]dz+\varepsilon\|\nabla u_{\varepsilon}\|^{\beta}_{L^{\beta}(\Omega)}\leq c_{30}, \tag{35}$$

for some $c_{30}>0$ (without ε-dependence). From (33) and (35), we obtain the estimate

$$\|\nabla u_{\varepsilon}\|_{L^{\alpha}(\Omega)}\leq c_{31}\quad \text{for some } c_{31}>0 \text{ independent of } \varepsilon. \tag{36}$$

Now we consider a sequence $\{\varepsilon_n\}$ of positive real numbers such that $\varepsilon_n\downarrow 0$. For every $n\in\mathbb{N}$, let u_{ε_n} be the solution to the problem (31) associated to ε_n. Recall that $W_0^{1,\alpha}(\Omega)\hookrightarrow L^2(\Omega)$ compactly, then after passing to a subsequence if needed, for some $u\in W_0^{1,\alpha}(\Omega)$ we have

$$u_{\varepsilon_n}\xrightarrow{w} u \text{ in } W_0^{1,\alpha}(\Omega) \text{ and } \nabla u_{\varepsilon_n}\xrightarrow{w}\nabla u \text{ in } L^{\alpha}(\Omega)^d, \tag{37}$$

$$u_{\varepsilon_n}\to u \text{ in } L^{\alpha}(\Omega) \text{ and } u_{\varepsilon_n}\to u \text{ a.e. in } \Omega. \tag{38}$$

The constraints on the exponent range (see (27)) imply that u is Hölder-continuous. Consequently, by (30), the same conclusion holds for $r(u)$ and $s(u)$. By (38), we deduce that

$$\lim_{n\to+\infty} r(u_{\varepsilon_n})=r(u),\quad \lim_{n\to+\infty} s(u_{\varepsilon_n})=s(u)\quad \text{a.e. in } \Omega. \tag{39}$$

Clearly, the following chain of inequalities is satisfied

$$d<\alpha\leq s(u_{\varepsilon_n}(z))\leq r(u_{\varepsilon_n}(z))\leq\beta<+\infty\quad \text{for all } n\in\mathbb{N}, \text{ for a.e. } z\in\Omega. \tag{40}$$

On this basis, (35) written for $(u_{\varepsilon_n},\varepsilon_n)$, the second convergence in (37), (39) and (40) lead to the conclusion that u is in $W_0^{1,r(u)}(\Omega)$ (by Lemma 3) and hence, $u\in W_0^{1,s(u)}(\Omega)$.

Now, the theory of monotone operators implies that

$$\begin{aligned}&\int_{\Omega}\left(|\nabla u_{\varepsilon_n}|^{r(u_{\varepsilon_n})-2}+|\nabla u_{\varepsilon_n}|^{s(u_{\varepsilon_n})-2}+\varepsilon|\nabla u_{\varepsilon_n}|^{\beta-2}\right)\nabla u_{\varepsilon_n}\nabla(u_{\varepsilon_n}-v)dz\\&-\int_{\Omega}\left(|\nabla v|^{r(u_{\varepsilon_n})-2}+|\nabla v|^{s(u_{\varepsilon_n})-2}+\varepsilon_n|\nabla v|^{\beta-2}\right)\nabla v\nabla(u_{\varepsilon_n}-v)dz\geq 0,\end{aligned} \tag{41}$$

for all $v\in W_0^{1,\beta}(\Omega)$. Thus, (31) written for $(u_{\varepsilon_n},\varepsilon_n)$, and the choice of test function "$u_{\varepsilon_n}-v$", imply that (41) reduces to the form

$$\langle g,u_{\varepsilon_n}-v\rangle-\int_{\Omega}\left(|\nabla v|^{r(u_{\varepsilon_n})-2}+|\nabla v|^{s(u_{\varepsilon_n})-2}+\varepsilon_n|\nabla v|^{\beta-2}\right)\nabla v\nabla(u_{\varepsilon_n}-v)dz\geq 0, \tag{42}$$

for all $v\in C_0^{\infty}(\Omega)$. Now using the Lebesgue theorem and (39), in $L^{\alpha'}(\Omega)^d$ we have

$$\lim_{n\to+\infty}\left[|\nabla v|^{r(u_{\varepsilon_n})-2}+|\nabla v|^{s(u_{\varepsilon_n})-2}\right]\nabla v=\left[|\nabla v|^{r(u)-2}+|\nabla v|^{s(u)-2}\right]\nabla v. \tag{43}$$

We take the limit as n goes to infinity in (42), and use (36), (43) and the first convergence in (37), then we get

$$\langle g,u-v\rangle-\int_{\Omega}\left(|\nabla v|^{r(u)-2}+|\nabla v|^{s(u)-2}\right)\nabla v\nabla(u-v)dz\geq 0,\ \text{for all } v\in C_0^{\infty}(\Omega). \tag{44}$$

Now, (30) and (27), by (8), imply that $r(u)$ and $s(u)$ are Hölder-continuous. Hence, $C_0^{\infty}(\Omega)$ is dense in $W_0^{1,r(u)}(\Omega)$, which means that (44) remains true for all $v\in W_0^{1,r(u)}(\Omega)$. We choose $v=u\mp\delta y$, where $y\in W_0^{1,r(u)}(\Omega)$ and $\delta>0$, in (44) and we have

$$\pm\left(\langle g,y\rangle-\int_{\Omega}\left(|\nabla u|^{r(u)-2}+|\nabla u|^{s(u)-2}\right)\nabla u\nabla y dz\right)\geq 0.$$

This implies that

$$\int_\Omega (|\nabla u|^{r(u)-2} + |\nabla u|^{s(u)-2})\nabla u \nabla y dz = \langle g, y\rangle \text{ for all } y \in W_0^{1,r(u)}(\Omega).$$

Finally, it is sufficient to recall that $u \in W_0^{1,r(u)}(\Omega)$ to conclude that we arrived to a solution for our problem (P_g) (see Definition 2). $\square$

Below, we show how to change the setting of Theorem 9 in the case where we relax the inequality $s(u) \le r(u)$ for all $u \in \mathbb{R}$. Precisely, we change condition (27) by the following one:

$$d < \alpha \le s(u), r(u) \le \beta < +\infty \quad \text{for all } u \in \mathbb{R}, \text{ some } \alpha, \beta > 0, \tag{45}$$

that is exactly the range condition we assumed after the definition of problem (15). Thus, the reader has to restate Definition 2 with (45) instead of (27), and with $u, v \in W_0^{1,r(u)}(\Omega) \cap W_0^{1,s(u)}(\Omega)$ instead of $u, v \in W_0^{1,r(u)}(\Omega)$.

Theorem 10. *If* (28)–(30) *and* (45) *hold, then problem* (P_g) *admits at least one weak solution* $u \in W_0^{1,r(u)}(\Omega) \cap W_0^{1,s(u)}(\Omega)$.

The proof of Theorem 10 follows the similar arguments to the ones used in the proof of Theorem 9, so to avoid repetition, we omit the details. However, we point out the following technical differences:

(i) Using (39) and the second convergence in (37), the inequality (35) written for $(u_{\varepsilon_n}, \varepsilon_n)$, together with (45) written for u_{ε_n}, leads to the conclusion that $u \in W_0^{1,r(u)}(\Omega)$ and $u \in W_0^{1,s(u)}(\Omega)$ (directly by Lemma 4).

(ii) Here, $C_0^\infty(\Omega)$ is dense in $W_0^{1,r(u)}(\Omega) \cap W_0^{1,s(u)}(\Omega)$, which means that (44) remains true for all $v \in W_0^{1,r(u)}(\Omega) \cap W_0^{1,s(u)}(\Omega)$.

(iii) In (44), we use the test function $v = u \mp \delta y$, where $y \in W_0^{1,r(u)}(\Omega) \cap W_0^{1,s(u)}(\Omega)$ to obtain (recall that $u \in W_0^{1,r(u)}(\Omega) \cap W_0^{1,s(u)}(\Omega)$, by (i)):

$$\int_\Omega (|\nabla u|^{r(u)-2} + |\nabla u|^{s(u)-2})\nabla u \nabla y dz = \langle g, y\rangle \text{ for all } y \in W_0^{1,r(u)}(\Omega) \cap W_0^{1,s(u)}(\Omega),$$

that is, $u \in W_0^{1,r(u)}(\Omega) \cap W_0^{1,s(u)}(\Omega)$ is a weak solution to (P_g).

5. Conclusions

The qualitative analysis of solutions to special forms of anisotropic equations can be helpful to identify the features and adaptability of materials and diffusion phenomena in applications. In particular, understanding the characteristics of framework structures is crucial for scientists working to identify the intrinsic mechanisms of natural systems. Sufficient criteria of the existence of weak solutions to local Dirichlet $(r(u), s(u))$-problems with certain nonlinearities have been presented in this work. Here, using variational methods of the critical point theory and analysis of regularized auxiliary problems together with a priori estimates, we discuss the cases when the exponents r, s are related by the inequality $s(u) \le r(u)$ for all $u \in \mathbb{R}$, and when r and s are not related to each other through an inequality. In both cases, the novelty is that they depend on the solution u, but as usual, we assume $r(u(z)), s(u(z)) \in (1, +\infty)$ for a.e. $z \in \Omega$ (more precisely, in Section 4 we have $r(u(z)), s(u(z)) \in (2, +\infty)$ for a.e. $z \in \Omega$). The results could be helpful for modelling equations when, as an effect the behavior of the operator switches between two different elliptic situations. Further investigations will be devoted to discuss the impact of different nonlinearities on the solvability of the Dirichlet $(r(u), s(u))$-problems. For example, the use of resonant and parametric nonlinearities could lead to multiplicity results (see Gasiński-Papageorgiou [25]) and bifurcation-type results (see Papageorgiou-Winkert [26]), where the solutions depend on a real parameter.

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