

Microstructured induced band pattern in Love wave propagation for novel non destructive testing (NDT) procedures

Abstract

We propose a new approach for assessing microstructural properties of materials via nondestructive testing (NDT). This approach lies on the observation that, accounting for the microstructure within the materials, reveals a nonclassical band propagation pattern for Love waves. Precisely this propagation structure may be directly related to the internal microstructure. To illustrate this, propagation of Love waves is first investigated within the linear theory of couple stress materials with micro-inertia. Proving wave existence by the argument principle provides a closed-form condition for propagation to occur. This connection defines propagation bands, whose limits correspond to the situation when Love waves move with the same speed as bulk waves in the underlying half-space (internal resonance). This condition is closely related to the layer-to-substrate microstructure and it may be used to assess either of the two. Furthermore, we show that the frequency equation is a three-term combination of antiplane Rayleigh and Rayleigh-Lamb functions (in a free and in a free/clamped plate). Consequently, investigation of any extra observable, such as Rayleigh waves, reduces the risk of multiple solutions at the signal processing stage. We finally consider the limit as either the half-space or the layer becomes classical elastic. We show that this unseemly bonding of dissimilar models, sometimes adopted in the literature, usually leads to inconsistencies.

Keywords: Love waves, Non destructive testing, Pass-bands, Internal resonance

1. Introduction

1 Love waves are antiplane waves localized near the free surface of a layer in
2 perfect contact with a half-space. These celebrated waves were first considered
3 by Love (1911) in a timely attempt to explain the defying appearance of an-
4 tiplane Rayleigh waves in seismograms, something that could not be achieved
5 in a homogeneous solid by the classical theory (Maugin, 1988; Gourgiotis and
6 Georgiadis, 2015). The absence of a wave component normal to the surface
7 makes Love waves especially attractive for NEMS and ultrasonic transducers
8 (Jin et al., 2005). Although Love waves have been studied in great detail in
9 the context of classical elasticity (Achenbach, 1984; Graff, 1991), no investiga-
10 tion is available that performs this analysis in the context of complex materials,
11 namely material models which account for microstructure. Besides, in recent
12 years there has been an increased interest in Love waves as potential candidates
13 for developing new non destructive testing (NDT) procedures, see Destuynder
14 and Fabre (2016) and references therein. Indeed, this approach falls in the wake
15 of existing successful applications in the field of defect detection for piezoelectric
16 ceramics (Jin et al., 2005). Also, consideration of multiple observables, beyond
17 the traditional Rayleigh waves, is capable of overcoming the well-known issue of
18 solution non-uniqueness at the signal post-processing stage, cf. Dal Moro (2020).

19 Couple stress theory is perhaps the simplest strain-gradient theory and it
20 aims to account for the discrete nature of materials at the micro-scale (Mindlin,
21 1964). Indeed, incorporating microstructural features inside classical elasticity
22 is a complex and yet important feat, for it allows to remediate many drawbacks
23 of the original theory. Among these, we mention the non-dispersive feature of
24 bulk and Rayleigh waves, the absence of antiplane Rayleigh waves, the impossi-
25 bility to predict a size for shear bands and the unbounded nature of stress near
26 defects, as in crack problems. A number of papers have investigated the way
27 microstructure alters such outcomes, through consideration of different com-
28 plex materials in the form of strain-gradient, micropolar, surface and non-local
29 elasticity. In their pioneering work, Graff and Pao (1967) studied plane strain

30 reflection of waves impinging onto a free surface of a couple stress solid in the
31 absence of micro-inertia. They proved that waves propagate dispersively and
32 possibly faster than Rayleigh waves. Shortly later, Sengupta and Ghosh (1974)
33 investigated wave propagation in a couple stress layer. The first recognition
34 that support of antiplane localized waves may be granted by "perturbation" of
35 the classical boundary conditions is due to Maugin (1988). In particular, Var-
36 doulakis and Georgiadis (1997) showed that introducing microstructural char-
37 acteristics in the form of strain-gradient and surface-energy terms allows the
38 theory to support antiplane Rayleigh waves only when accounting for a cer-
39 tain type of gradient anisotropy. Ottosen et al. (2000) considered dispersion
40 of Rayleigh waves in microstructured media which are described by the couple
41 stress theory without micro-inertia. This work has been later extended by Geor-
42 giadis and Velgaki (2003), where a number of insightful remarks are given with
43 regard to the importance of considering rotational inertia for successfully re-
44 producing results from lattice theories. Gourgiotis and Georgiadis (2015) show
45 that antiplane Rayleigh and torsional waves are supported in a homogeneous
46 half-space when a complete strain gradient theory is considered. Recently, No-
47 bili et al. (2020) proved existence of a novel antiplane evanescent wave arising
48 by mode conversion in couple stress elasticity with micro-inertia. It is precisely
49 this wave that has been seen radiating energy away in the dynamic loading of
50 a crack (Nobili et al., 2019).

51 The analysis is further complicated when two or more bodies are set in con-
52 tact. Li et al. (2018) study reflection and transmission of plane strain thermo-
53 elastic coupled waves in couple stress materials without micro-inertia. Wang
54 et al. (2017) investigate plane strain reflection and transmission of elastic waves
55 impinging onto a layer glued in between two couple stress half-spaces disregard-
56 ing rotational inertia. Results are very involved and the limiting case of classical
57 elastic half-spaces is also addressed. Recently, Nobili et al. (2021) considered
58 propagation of Stoneley waves at the boundary between two couple stress half-
59 spaces and found that incorporation of the microstructure greatly relaxes the
60 classical restrictions for wave existence. In this paper, we show that this fea-

61 ture may be put to advantage to relate the band propagation structure to the
 62 underlying material microstructure.

63 We also address the not trivial situation where a couple stress body is bonded
 64 to a classical solid. Several contributions are available in the literature dealing
 65 with such layout. As an example, Fan and Xu (2018) considered Love waves
 66 arising on the surface of a couple stress layer bonded to a classical half-space,
 67 while Ray and Singh (2020) studied a couple stress stratum imperfectly bonded
 68 to a viscoelastic substrate. Since couple stress entails a kinematical description
 69 that is richer than classical elasticity's, the question of what boundary conditions
 70 are to be imposed is not trivial. The matter is settled in this paper by taking
 71 the proper limit of the general solution for Love waves localized in a couple
 72 stress layer perfectly bonded to a couple stress half-space in the presence of
 73 micro-inertia.

74 2. Variational derivation of couple stress elasticity

75 We begin by considering microstructural features within our description of
 76 elastic materials. For this, as it occurs in polar materials, we supplement the
 77 classical displacement field \mathbf{u} by the micro-rotation field $\boldsymbol{\varphi}$ as the kinematical
 78 fundamental variables. However, in contrast to Cosserat micro-polar theories,
 79 wherein displacements and micro-rotations are independent fields, couple stress
 80 (CS) theory relates one to the other, through the connection

$$\boldsymbol{\varphi} = \frac{1}{2} \text{curl } \mathbf{u}. \quad (1)$$

81 Component-wise, this reads $\varphi_i = \frac{1}{2} E_{ijk} u_{k,j}$, where $i, j, k \in \{1, 2, 3\}$, E_{ijk} is
 82 the alternator tensor and Einstein's summation convention on twice repeated
 83 subscripts is assumed. Hereinafter, a subscript comma denotes partial differ-
 84 entiation, e.g. $(\text{grad } u_k)_j = u_{k,j} = \partial u_k / \partial x_j$, while subscript round brackets
 85 produce symmetrization, i.e. $u_{(i,j)} = (u_{i,j} + u_{j,i})/2$. Alongside the linear strain
 86 tensor $\boldsymbol{\varepsilon}$ commonly defined in classical elasticity (CE)

$$\varepsilon_{ij} = u_{(i,j)}, \quad (2)$$

87 we introduce the *torsion-flexure (wryness or curvature) tensor* χ

$$\chi_{ij} = \varphi_{i,j}, \quad (3)$$

88 that, in light of the connection (1), is purely deviatoric, i.e. $\chi = \chi^D$, being
 89 $\chi^D = \chi - \frac{1}{3}(\text{tr } \chi)\mathbf{1}$, where $\mathbf{1}$ is the rank-2 identity tensor and $\text{tr } \chi = \chi \cdot \mathbf{1}$
 90 the trace operator. Here, a dot denotes the scalar product, i.e. $\chi \cdot \mathbf{1} = \chi_{ii}$; in
 91 particular, in the case of vectors, it induces the natural norm $\mathbf{u}^2 = \|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$.

92 We define the action integral for the deformable body \mathcal{B} in the time frame
 93 $[0, t]$

$$\mathcal{A} = - \int_0^t \mathcal{L} d\tau,$$

having introduced the Lagrangian function $\mathcal{L} = \mathcal{K} - \mathcal{V}$ as the difference between
 the kinetic and the potential energy. For the former, we take

$$\mathcal{K} = \int_{\mathcal{B}} \left(\frac{1}{2} \rho \dot{\mathbf{u}}^2 + \frac{1}{2} J \dot{\varphi}^2 \right) dV,$$

where ρ and J are the mass- and the micro-inertia densities, respectively. Here-
 inafter, a superposed dot denotes time differentiation, i.e. $\dot{\mathbf{u}} = \partial \mathbf{u} / \partial t$. In the
 absence of body forces, the potential energy reads

$$\begin{aligned} \mathcal{V} = \int_{\mathcal{B}} \left[\frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} + \frac{1}{2} \boldsymbol{\mu}^T \cdot \boldsymbol{\chi} + \boldsymbol{\pi} \cdot \left(\boldsymbol{\varphi} - \frac{1}{2} \text{curl } \mathbf{u} \right) \right] dV \\ - \int_{\partial \mathcal{B}} (\mathbf{p}_n \cdot \mathbf{u} + \mathbf{q}_n \cdot \boldsymbol{\varphi}) dA, \quad (4) \end{aligned}$$

94 and the superscript T , denoting transposition, is introduced for compatibility
 95 with Koiter (1964). The surface integral in (4) accounts for externally applied
 96 force and couple stress tractions, respectively \mathbf{p}_n and \mathbf{q}_n .

As in CE, conjugated to the strain tensor $\boldsymbol{\varepsilon}$ is the Cauchy (or force) stress
 tensor, \mathbf{t} , which is generally non-symmetric. Consequently, it may be decom-
 posed into its symmetric and skew-symmetric part, respectively

$$\sigma_{ij} = t_{(ij)}, \quad \tau_{ij} = t_{ij} - \sigma_{ij}.$$

97 Since $\boldsymbol{\varepsilon}$ is symmetric, only $\boldsymbol{\sigma}$ really performs work.

98 Conjugated to the curvature tensor χ is the couple stress tensor $\boldsymbol{\mu}$, which, in
 99 general, performs work through all its components. However, it should be noted
 100 that χ is deviatoric, whence only the deviatoric part of $\boldsymbol{\mu}^D$ really matters. As a
 101 result, to any effect, $\boldsymbol{\mu}$ may be replaced by its deviatoric part $\boldsymbol{\mu}^D$. Indeed, this
 102 theory is sometimes named indeterminate after the observation that the first
 103 invariant of the couple stress tensor rests indeterminate. Consequently, it may
 104 be set equal to zero without any loss of generality, e.g. $\text{tr } \boldsymbol{\mu} = 0$. For the sake
 105 of brevity, in the following we shall write $\boldsymbol{\mu}$, with the understanding that $\boldsymbol{\mu}^D$ is
 106 meant. For any surface of unit normal \boldsymbol{n} , the tensor $\boldsymbol{\mu}$ determines the internal
 107 reduced couple vector $\bar{\boldsymbol{q}} = \boldsymbol{\mu}^T \boldsymbol{n}$ acting across that surface.

Finally $\boldsymbol{\pi}$ is a Lagrange multiplier enforcing the kinematical constraint (1)
 and allowing \boldsymbol{u} and $\boldsymbol{\varphi}$ to be varied independently. For it, we let $\boldsymbol{\pi} = -2$ axial $\boldsymbol{\tau}$,
 with $(\text{axial } \boldsymbol{\tau})_i = \frac{1}{2} E_{ijk} \tau_{jk}$ denoting the axial vector attached to the skew-
 symmetric stress \boldsymbol{t} (in Nobili et al. (2020) we dispensed with the factor 2 in the
 definition of the axial vector). With this definition, we have

$$\text{axial } \boldsymbol{\tau} = [\tau_{23}, \tau_{31}, \tau_{12}],$$

108 and the inverse formula $E_{ijk}(\text{axial } \boldsymbol{\tau})_k = \tau_{ij}$.

Component-wise, the Lagrangian volumetric density reads

$$\mathcal{L} = \frac{1}{2} \rho \dot{u}_i \dot{u}_i + \frac{1}{2} J \dot{\varphi}_i \dot{\varphi}_i - \frac{1}{2} \sigma_{ij} u_{i,j} - \frac{1}{2} \mu_{ji} \varphi_{i,j} - \pi_i \left(\varphi_i - \frac{1}{2} e_{ijk} u_{k,j} \right).$$

By Hamilton's principle, the problem's governing equations are the Euler-Lagrange equations for the action integral. In the absence of body forces, they read

$$\text{div } \boldsymbol{t} = \rho \ddot{\boldsymbol{u}}, \quad (5a)$$

$$2 \text{ axial } \boldsymbol{\tau} + \text{div } \boldsymbol{\mu} = J \ddot{\boldsymbol{\varphi}}, \quad (5b)$$

109 it being $\text{div } \boldsymbol{t} = t_{ji,j}$. Application of the permutation tensor to Eq.(5b) yields

$$\boldsymbol{\tau} = -\frac{1}{2} \mathbf{E} (\text{div } \boldsymbol{\mu} - J \ddot{\boldsymbol{\varphi}}), \quad (6)$$

110 whence the skew-symmetric part of the force stress tensor \boldsymbol{t} is determined by
 111 rotational equilibrium, beside any constitutive consideration. It follows that

Eq.(6) is generally not objective (Ottosen et al., 2000; Gourgiotis et al., 2013), although, for time-harmonic motions, this is no longer an issue (Shodja et al., 2015).

2.1. Antiplane deformations

Under antiplane deformations, the displacement field $\mathbf{u} = [u_1, u_2, u_3]$ is completely defined by the out-of-plane component $u_3 = u_3(x_1, x_2, t)$. Then, the non-zero components of the micro-rotation vector, of the strain and of the curvature tensor become (see Nobili et al. (2019, 2020, 2021))

$$\varphi_1 = \frac{1}{2}u_{3,2}, \quad \varphi_2 = -\frac{1}{2}u_{3,1}, \quad (7a)$$

$$\varepsilon_{13} = \frac{1}{2}u_{3,1}, \quad \varepsilon_{23} = \frac{1}{2}u_{3,2}, \quad (7b)$$

$$\chi_{11} = -\chi_{22} = \frac{1}{2}u_{3,12}, \quad \chi_{21} = -\frac{1}{2}u_{3,11}, \quad \chi_{12} = \frac{1}{2}u_{3,22}. \quad (7c)$$

Consequently, Eqs.(5) simplifies to

$$\sigma_{13,1} + \sigma_{23,2} + \tau_{13,1} + \tau_{23,2} = \rho\ddot{u}_3, \quad (8a)$$

$$\mu_{11,1} + \mu_{21,2} + 2\tau_{23} = J\ddot{\varphi}_1, \quad (8b)$$

$$\mu_{12,1} + \mu_{22,2} - 2\tau_{13} = J\ddot{\varphi}_2. \quad (8c)$$

For hyperelastic materials, a stored energy potential $U = U(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$ exists which connects the strain $\boldsymbol{\varepsilon}$ and the curvature $\boldsymbol{\chi}$ to the Cauchy stress and to the couple stress. Indeed, we have

$$\boldsymbol{\sigma} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}}, \quad \boldsymbol{\mu}^T = \frac{\partial U}{\partial \boldsymbol{\chi}},$$

which, to leading order for small deformations of an isotropic material, yield

$$\boldsymbol{\sigma} = 2G\boldsymbol{\varepsilon} + \Lambda(\text{tr } \boldsymbol{\varepsilon})\mathbf{1}, \quad \boldsymbol{\mu} = 2Gl^2 (\boldsymbol{\chi}^T + \eta\boldsymbol{\chi}). \quad (9)$$

Here, Λ and $G > 0$ are the classical Lamé moduli, with $3\Lambda + 2G > 0$. Also, $l > 0$ is a characteristic length connected to the microstructure and $-1 < \eta < 1$ is a dimensionless number similar to Poisson's ratio. The material parameters l and η may be determined experimentally as in Lakes (1986). The situation $\eta = 0$ corresponds to the strain gradient effect considered in Zhang et al. (1998),

122 while the limiting value $\eta = 1$ corresponds to the modified couple stress theory of
 123 elasticity introduced in Yang et al. (2002) via the balance of torques of torques.

The constitutive equations (9), in light of the definitions (2,3) and with the help of the kinematic relations (7), give the force stress and the couple stress in terms of the u_3 displacement alone (Nobili et al., 2019)

$$\sigma_{13} = Gu_{3,1}, \quad \sigma_{23} = Gu_{3,2}, \quad (10a)$$

$$\mu_{11} = -\mu_{22} = Gl^2(1 + \eta)u_{3,12}, \quad \mu_{21} = Gl^2(u_{3,22} - \eta u_{3,11}), \quad (10b)$$

$$\mu_{12} = -Gl^2(u_{3,11} - \eta u_{3,22}). \quad (10c)$$

124 Introducing Eqs.(7a,10) into (6) yields

$$\tau_{13} = -\frac{1}{2}Gl^2\hat{\Delta}u_{3,1} + \frac{J}{4}\ddot{u}_{3,1}, \quad \tau_{23} = -\frac{1}{2}Gl^2\hat{\Delta}u_{3,2} + \frac{J}{4}\ddot{u}_{3,2}, \quad (11)$$

125 which correspond to Eqs.(9) of Mishuris et al. (2012). Here, $\hat{\Delta}$ denotes the 2-D
 126 Laplace operator in the dimensional co-ordinates (x_1, x_2) . Finally, Eqs.(10a)
 127 and (11) allow to rewrite translational equilibrium (8a) as the meta biharmonic
 128 equation

$$G\left(\frac{1}{2}l^2\hat{\Delta}\hat{\Delta}u_3 - \hat{\Delta}u_3\right) - \frac{J}{4}\hat{\Delta}\ddot{u}_3 + \rho\ddot{u}_3 = 0. \quad (12)$$

129 2.2. Boundary conditions

Accounting for surface terms in Hamilton's principle gives the boundary conditions

$$\mathbf{p} = \mathbf{p}_n, \quad \text{and} \quad \mathbf{q} = \mathbf{q}_n,$$

where we have let the reduced traction and reduce couple stress vectors, respectively

$$\mathbf{p} = \mathbf{t}^T \mathbf{n} + \frac{1}{2} \text{grad } \mu_{nn} \times \mathbf{n}, \quad (13a)$$

$$\mathbf{q} = \boldsymbol{\mu}^T \mathbf{n} - \mu_{nn} \mathbf{n}. \quad (13b)$$

130 Here, $\mu_{nn} = \mathbf{n} \cdot \boldsymbol{\mu} \mathbf{n}$ is the normal component of the couple stress traction vector.
 131 Eq.(13b) shows that only the tangential part of the couple stress vector may be
 132 prescribed. Indeed, it can be proved that the normal component of the couple
 133 stress tension is precisely annihilated by the second term in (13a), see also Koiter
 134 (1964) and Ottosen et al. (2000).

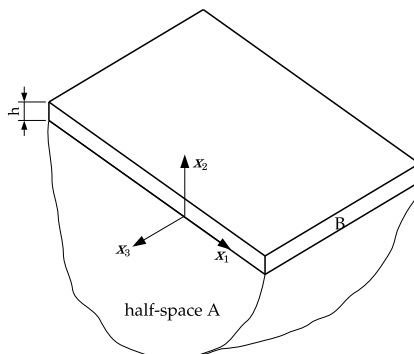


Figure 1: An infinite layer (B) perfectly bonded to an elastic half-space (A). The half-space is often named the substrate

135 *2.3. Nondimensionalization and time harmonic motions*

136 Let us consider a Cartesian co-ordinate system (O, x_1, x_2, x_3) and a layer,
 137 $\mathcal{B}_B = \{(x_1, x_2, x_3) : 0 < x_2 < h\}$, in perfect contact with the half-space
 138 $\mathcal{B}_A = \{(x_1, x_2, x_3) : x_2 < 0\}$, see Fig.1. The layer and the half-space are made
 139 of generally different isotropic elastic couple stress (CS) materials, for which
 140 antiplane deformations are considered.

141 At the layer top face, $x_2 = h$, it is $\mathbf{n} = [0, 1, 0]$ and, according to Eqs.(13),
 142 the out-of-plane component of the reduced force traction and the in-plane com-
 143 ponents of the couple stress traction read, respectively,

$$p_3 = (t_{23} + \frac{1}{2}\mu_{22,1}), \quad \text{and} \quad q_1 = \mu_{21}, \quad q_2 = 0. \quad (14)$$

144 We introduce the reference length Θl , by which we scale the spatial co-
 145 ordinates $(\xi_1, \xi_2, \xi_3) = (x_1, x_2, x_3)(\Theta l)^{-1}$ and the layer thickness $H = h/l$. In
 146 the same manner, we let the reference time $T = l/\tilde{c}_s$ and the dimensionless
 147 time $\tau = t/T$. Here, $\tilde{c}_s = \sqrt{G/\rho}$ is the bulk shear wave speed of CE. The
 148 parameter Θ is a convenient factor that is introduced so as to simplify notation.
 149 The equilibrium equation (12) becomes

$$\Delta\Delta u_3 - 2\Theta^2\Delta u_3 - 2\Theta^4\left(\frac{\ell^2}{\Theta^2}\Delta u_{3,\tau\tau} - u_{3,\tau\tau}\right) = 0, \quad (15)$$

150 where Δ is the 2-D Laplace operator in ξ_1 and ξ_2 and we have let the dynamic

151 characteristic length (Mishuris et al., 2012)

$$\ell = \frac{l_d}{l}, \quad \text{with} \quad l_d = \frac{1}{2} \sqrt{\frac{J}{\rho}}.$$

152 The latter is proportional to $2\sqrt{6}l_d$, that is introduced in Shodja et al. (2015).

153 Consideration of time-harmonic solutions brings

$$u_3 = W(\xi_1, \xi_2) \exp(-i\Omega\tau),$$

154 where i is the imaginary unit and $\Omega = \omega T > 0$ the dimensionless (time) fre-
 155 quency. Then, Eq.(15) yields the meta biharmonic PDE (Georgiadis and Vel-
 156 gaki, 2003, Eq.(19)) for the wave amplitude W :

$$[\Delta\Delta - 2(1 - \ell^2\Omega^2)\Theta^2\Delta - 2\Omega^2\Theta^4] W = 0. \quad (16)$$

157 This homogeneous equation is easily factored

$$(\Delta + \delta^2)(\Delta - 1)W = 0, \quad (17)$$

158 provided that Θ is let as in (Nobili et al., 2020, Eq.(3.4))

$$\Theta^2 = \frac{\sqrt{(1 - \ell^2\Omega^2)^2 + 2\Omega^2} - 1 + \ell^2\Omega^2}{2\Omega^2} \quad (18)$$

159 We then have (Nobili et al., 2021, Eq.(21))

$$\delta = 2\delta_{cr}\Theta^2, \quad \text{with} \quad \delta_{cr} = \ell_{cr}\Omega, \quad \ell_{cr} = 1/\sqrt{2}. \quad (19)$$

160 Eq.(17) is especially convenient for it shows that two (antiplane) bulk modes are
 161 supported: one, travelling, having wavenumber $\kappa = \pm\delta$, and another, evanes-
 162 cent, possessing wavenumber $\kappa = \pm i$ (Nobili et al., 2020).

In dimensionless form, the traction vectors (14) become

$$p_3 = -\frac{G}{2\Theta^3} [(\delta^2 - 1)W_{,2} + (\eta + 2)W_{,112} + W_{,222}], \quad (20a)$$

$$q_1 = \frac{Gl}{\Theta^2} (W_{,22} - \eta W_{,11}). \quad (20b)$$

163 *2.4. Extension to two materials*

164 We now consider the fact that A and B are constituted by different isotropic
 165 homogeneous materials, each having the relevant classical shear wave speed
 166 $\tilde{c}_{sA,B} = \sqrt{G_{A,B}/\rho_{A,B}}$. We can now choose to bring the problem in dimen-
 167 sionless form against either of the two materials: to fix ideas, we refer to the
 168 half-space A and let

$$l = l_A, \quad \text{and} \quad T = T_A = l_A/\tilde{c}_{sA},$$

whence $\Omega = \omega T_A$. Therefore, one should remember that, unless otherwise spec-
 ified, reference to the microstructure of A is made through the dimensionless
 variables. In order to move from A to B, we introduce the ratios

$$\beta = l_B/l_A, \quad v = T_A/T_B.$$

169 With these, we can introduce the dimensionless wavenumber $\kappa_B = \beta\kappa$ and
 170 frequency $\Omega_B = \omega T_B = \Omega/v$, both being normalized with respect to B. The
 171 limiting case when B is classical elastic, i.e. in the absence of microstructure for
 172 B, can be retrieved by taking

$$v \rightarrow +\infty, \quad \beta \rightarrow 0, \quad \text{such that} \quad \beta v = \frac{\tilde{c}_{sB}}{\tilde{c}_{sA}} < \infty. \quad (21)$$

173 The corresponding limit where A is deprived of microstructure may be obtained
 174 taking $\beta \rightarrow \infty$, with $\beta v < \infty$ and making use of variables normalised with
 175 respect to B.

We let the dynamic characteristic lengths of A and B

$$\ell_{A,B} = l_{dA,B}/l_{A,B},$$

176 being $l_{dA,B} = \frac{1}{2}\sqrt{J_{A,B}/\rho_{A,B}}$. Besides, we define the equivalent of δ for B,
 177 cf.(Nobili et al., 2021, Eq.(24))

$$\frac{\psi}{v} = \frac{\sqrt{(1 - \ell_B^2 \Omega_B^2)^2 + 2\Omega_B^2} - 1 + \ell_B^2 \Omega_B^2}{\sqrt{2}\Omega_B}. \quad (22)$$

Indeed, $\psi = \delta$ when $\ell_B = \ell_A$ and $v = 1$. Also, $\delta \sim \psi \sim \Omega/\sqrt{2}$ and $\Theta \sim 2^{-1/2}$ as $\Omega \rightarrow 0$. For ψ we have the asymptotics

$$\psi \rightarrow \sqrt{2}\Omega\ell_B + O(v^2), \quad \text{as } v \rightarrow 0,$$

178 and

$$\psi \rightarrow \delta_{cr} + O(v^{-2}), \quad \text{as } v \rightarrow +\infty. \quad (23)$$

179 The governing equation (16) specializes to

$$(\Delta + \delta^2)(\Delta - 1)W_A = 0, \quad (\xi_1, \xi_2) \in \mathcal{B}_A, \quad (24)$$

180 for the half-space A, and to

$$(\Delta + \delta_1^2)(\Delta - \delta_2^2)W_B = 0, \quad (\xi_1, \xi_2) \in \mathcal{B}_B, \quad (25)$$

for the layer B. Clearly, W_A and W_B are the solution amplitudes in medium A and B, respectively. Also, we have let the dimensionless wavenumbers

$$\delta_{1,2} = \frac{\kappa_{1,2}}{\beta}, \quad \text{with } \kappa_1 = \frac{\sqrt{\delta\psi}}{v}, \quad \kappa_2 = \sqrt{\delta/\psi},$$

181 normalized with respect to $\Theta\ell_A$ and $\Theta\ell_B$, respectively. They correspond to bulk
182 travelling and evanescent modes for B. With this notation, we can easily express
183 the corresponding bulk wave phase speed with respect to \tilde{c}_{sA} or \tilde{c}_{sB}

$$c_{SH} = V_{A,B}\Theta\tilde{c}_{sA,B},$$

where we have let the dimensionless phase speed

$$V_A = \frac{\Omega}{\delta}, \quad \text{and } V_B = \frac{\Omega_B}{\kappa} = \frac{\Omega}{v\kappa}.$$

184 In fact, we see that, in the limit (21), $V_B = \Theta^{-1}$ for $\kappa = \kappa_1$, which gives the
185 classical bulk wavespeed $c_{SH} = \tilde{c}_{sB}$. For $\kappa = \kappa_2$, $V_B \rightarrow 0$ and the evanescent
186 bulk mode becomes standing, as already pointed out in Nobili et al. (2019). We
187 have the Short-Wave High-Frequency (SWHF) approximation for bulk waves

$$\kappa^2 - 2\ell_A^4\Omega^2 = 0. \quad (26)$$

For material B, the boundary tractions (20) become (Nobili et al., 2021, Eqs.(27))

$$p_{3B} = -\frac{G_B}{2\Theta^3} \left\{ \beta^2 [(\eta_B + 2) W_{B,112} + W_{B,222}] + \frac{\delta}{\psi} \left(\frac{\psi^2}{v^2} - 1 \right) W_{B,2} \right\}, \quad (27a)$$

$$q_{1B} = \frac{G_B l_B}{\Theta^2} \beta^2 (W_{B,22} - \eta_B W_{B,11}), \quad (27b)$$

188 which, in the limiting case (21) and using of Eqs.(19,23), lends the classical
189 limit.

190 3. Dispersion relation for Love waves

191 For guided propagation along ξ_1 , we have (recall we chose $l = l_A$)

$$W_{A,B}(\xi_1, \xi_2) = l w_{A,B}(\xi_2) \exp(\iota \kappa \xi_1),$$

where $K = kl$ denotes the dimensionless (spatial) wavenumber in the propagation direction ξ_1 . Letting $V = \Omega/K$, we get the dimensional phase speed in the propagation direction

$$c = \omega/k = V \tilde{c}_{sA}.$$

192 The general decaying solution of Eq.(24), valid for the half-space A, reads

$$w_A(\xi_2) = s_1 \exp(A_1 \xi_2) + s_2 \exp(A_2 \xi_2), \quad (28)$$

193 where $s_{1,2}$ are undetermined amplitudes and

$$A_1 = \sqrt{\kappa^2 - \delta^2}, \quad A_2 = \sqrt{\kappa^2 + 1}, \quad (29)$$

194 are the decay indices in the thickness direction ξ_2 . For decay to occur, we need
195 to give proper and definite meaning to the square root multivalued function
196 (Noble, 1958). This is obtained introducing the cut complex plane and choosing
197 the particular branch of the square root such that $A_{1,2}(\kappa) \rightarrow |\kappa|$ as $\kappa \rightarrow \infty$ on
198 the real axis (Nobili et al., 2019). Cuts start at the branch points $\pm\delta$ and move
199 in opposing direction away from the real axis. As a result, $A_1(\kappa)$ is real positive
200 on the real domain $|\kappa| > \delta$, and $A_2(\kappa)$ is real positive on the whole real axis.

Similarly, we let

$$B_1 = \sqrt{\kappa^2 - \delta_1^2}, \quad B_2 = \sqrt{\kappa^2 + \delta_2^2},$$

and the general solution of Eq.(25) is given by Nobili et al. (2020)

$$w_B(\xi_2) = e_1 \cosh(B_1 \xi_2) + e_2 \cosh(B_2 \xi_2) \\ + o_1 B_1^{-1} \sinh(B_1 \xi_2) + o_2 B_2^{-1} \sinh(B_2 \xi_2), \quad (30)$$

201 where the dimensionless wavenumbers in the thickness direction ξ_2 are $P = \iota B_1$
 202 and $Q = \iota B_2$. Indeed, going back to the dimensional co-ordinate x_2 , we have
 203 $(p, q) = (P, Q)/(l\Theta)$ and

$$(p, q) = -k^2 \pm \frac{1}{l^2} \left(\sqrt{1 + 2l^2 \omega^2 c_{sA}^{-2}} - 1 \right),$$

204 which correspond to (3.5.1) and (3.5.2) of Fan and Xu (2018)¹.

205 3.1. Rayleigh function

206 Antiplane surface travelling waves (Rayleigh waves) occur in correspondence
 207 of the real zeros of the Rayleigh function (Nobili et al., 2021, §4.1)

$$R_0(\kappa, \lambda_1, \lambda_2, \eta) = (\eta \kappa^2 - \lambda_1 \lambda_2)^2 - \lambda_1 \lambda_2 (\lambda_1 + \lambda_2)^2, \quad (31)$$

that is analytic in the cut complex plane. This form of the Rayleigh function is especially simple, and it may be specialized for either A or B upon introducing the corresponding decay indices for $\lambda_{1,2}$ and microstructure ratio η . For example, it can be specialized for A

$$R_A(\kappa) = R_0(\kappa, A_1, A_2, \eta_A)$$

to give the form already adopted in Nobili et al. (2019, 2020)

$$(A_1 - A_2)R_A(\kappa) = \zeta_{11A}^2 A_1 - \zeta_{12A}^2 A_2,$$

208 having let $\zeta_{11A} = \eta_A \kappa^2 + A_2^2$ and $\zeta_{12A} = \eta_A \kappa^2 + A_1^2$.

¹Provided that we replace ℓ with $\sqrt{2}\ell$ and take the opposite of q^2

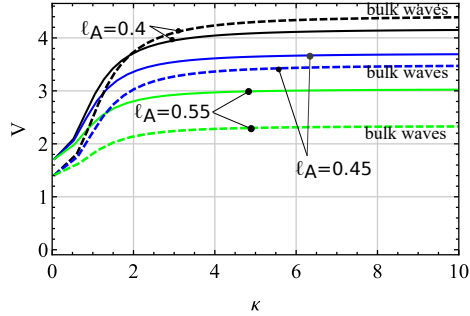


Figure 2: Dispersion curves for Rayleigh waves in medium B (solid) compared to bulk waves in medium A (dashed curves) with the parameters $\ell_A = 0.4$ (black), $\ell_A = 0.45$ (blue) and $\ell_A = 0.55$ (green). It appears that only in the case $\ell_A = 0.4$, bulk waves move faster than Rayleigh waves in the short-wave regime

209 Existence and uniqueness of antiplane Rayleigh waves is proved by the argu-
 210 ment principle in Nobili et al. (2021), where it is also shown that the Rayleigh
 211 root is the single real zero of a bi-quartic polynomial equation which is regularly
 212 perturbed in η_A . Indeed, for $\eta_A = 0$, we have $\kappa = \pm\delta_1$ whence, in general, we
 213 can write the solution as a power series in η_A

$$\kappa_R^2 = \delta_1^2 \left(1 + \frac{\delta_1^6}{(\delta_1^2 + \delta_2^2)^3} \eta_A^4 + \dots \right). \quad (32)$$

214 This expansion, once specialized for A and up to first order correction terms in
 215 η_A , corresponds to (Nobili et al., 2020, Eq.(3.37)). From (32) follows the SWHF
 216 approximation

$$\kappa^2 - 2\Omega^2 \frac{\ell_A^2 \ell_B^2}{\beta^2 v^2} (1 + \eta_A^4 + \dots) = 0, \quad (33)$$

217 which provides the asymptotic limit for antiplane Rayleigh-Lamb (RL) modes.

218 Hereinafter, numerical exploration is presented for the parameter set $\ell_B =$
 219 0.5 , $\beta = v = 1.1$, $H = 0.1$, $\eta_A = 0.8$ and $\eta_B = 0.5$. Fig.2 plots the dispersion
 220 curves for Rayleigh waves in the layer B, superposed onto bulk waves in medium
 221 A. It appears that, when micro-inertia in A is small enough, bulk waves in A
 222 move faster than Rayleigh waves in B.

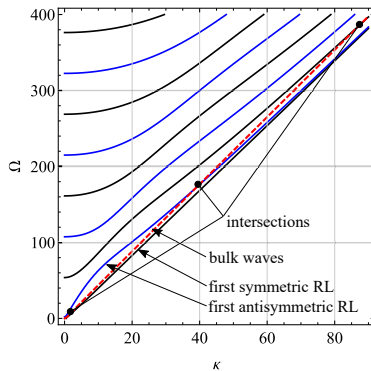


Figure 3: Frequency spectrum branches for symmetric (solid, black) and antisymmetric (solid, blue) Rayleigh-Lamb waves, with $\ell_A = 0.4$. The spectrum of bulk waves in medium A (dashed, red) intersects RL modes in an infinite succession of points. These are cut-on/cut-off points for Lamb waves.

223 3.2. Antiplane Rayleigh-Lamb waves in a free plate

224 Guided propagation in the free layer is described by the RL frequency equa-
 225 tion

$$D_{RL}(\kappa) = d_s(\kappa)d_o(\kappa), \quad (34)$$

where $d_s(\kappa)$ and $d_o(\kappa)$ are given in Nobili et al. (2020) respectively for symmetric and anti-symmetric modes. Looking at them, it is clear that $D_{RL}(\kappa)$ depends on B_1 and B_2 through even powers and therefore it is *analytic in the whole complex plane*, i.e. it is holomorphic. Physically, this means that there are no bulk waves associated to RL propagation. Eq.34 may be rewritten in terms of the Rayleigh function

$$D_{RL}(\kappa) = \frac{(B_1 - B_2)^2}{8B_1B_2} R_B(\kappa)^2 \left[\cosh\left(\frac{H(B_1 + B_2)}{\Theta}\right) - 1 \right] - \frac{(\zeta_{11B}^2 B_1 + \zeta_{12B}^2 B_2)^2}{8B_1B_2} \left[\cosh\left(\frac{H(B_1 - B_2)}{\Theta}\right) - 1 \right], \quad (35)$$

226 whence we retrieve the well-known result that, when it comes to very short
 227 waves, the layer behaves just like a half-space.

Fig.3 plots the frequency spectrum for antiplane RL symmetric and antisymmetric waves in medium B, cfr.Nobili et al. (2020). In this Figure, RL branches are superposed onto the spectrum for SH bulk waves in medium A, and it ap-

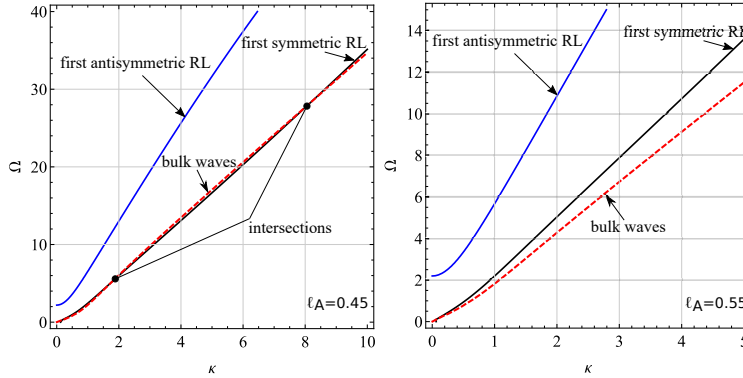


Figure 4: Frequency spectrum branches for symmetric (solid, black) and antisymmetric (solid, blue) Rayleigh-Lamb waves, with $\ell_A = 0.45$ (left) and $\ell_A = 0.55$ (right), superposed onto the spectrum of bulk waves in medium A (dashed, red). In the first case we have two intersections, in the second none.

pears that a infinite succession of intersection points occur, alternatively with symmetric and antisymmetric branches. These points are given by

$$D_{RL}(\delta) = 0,$$

228 and represent propagation states for which RL waves in medium B coexist with
 229 bulk waves in medium A (or, equally, they have the same phase speed).

230 In contrast, when micro-inertia in medium A increases, we move to two
 231 intersections and, eventually, to none, as it is illustrated in Figs.4. The two
 232 intersection condition is remarkable given that it occurs with the same first
 233 symmetric branch, i.e. there are two propagation frequencies that support lon-
 234 gitudinal waves in the layer and also in the half-space. The regime shift from
 235 an infinite to a finite number of intersections occurs when the phase speed of
 236 Rayleigh waves in medium B becomes greater than that of bulk waves in medium
 237 A, as demonstrated in Fig.2. Therefore, comparing expansion (26) with (33), an
 238 approximate criterion for existence of an infinite number of intersections may
 239 be obtained

$$\ell_{0A} < \frac{\ell_{0B}}{\beta v}. \quad (36)$$

240 For example, with the parameter set of Fig.2, we get $\ell_{0A} < 0.41$. This criterion
 241 may be improved for large values of η_A by taking into consideration more terms

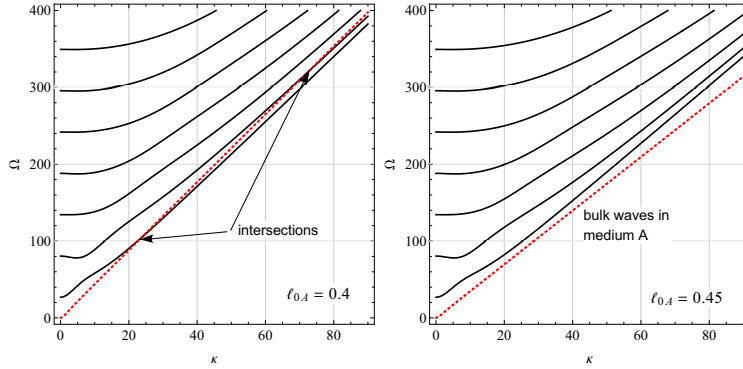


Figure 5: Frequency spectrum for Rayleigh-Lamb waves in a free/clamped plate, with $\ell_A = 0.4$ (left) and $\ell_A = 0.45$ (right). The frequency spectrum of bulk waves in medium A is also plotted (dotted, red) and it intersects RL modes only in the case $\ell_A = 0.4$

242 in the expansions.

243 3.3. Rayleigh-Lamb waves in a clamped-free plate

Proceeding as in Nobili et al. (2020), it can be proved that propagation in a free/clamped (FC) layer occurs according to the holomorphic function

$$\begin{aligned}
 D_{FC}(\kappa) = & \frac{1}{2} \frac{(B_1 - B_2)^2}{B_1 B_2} R_B(\kappa) \left[\cosh \left(\frac{H(B_1 + B_2)}{\Theta} \right) - 1 \right] - (\zeta_{11B} - \zeta_{12B})^2 \\
 & - \frac{1}{2} (B_2^{-1} + B_1^{-1}) (\zeta_{11B}^2 B_1 + \zeta_{12B}^2 B_2) \left[\cosh \left(\frac{H(B_1 - B_2)}{\Theta} \right) - 1 \right]. \quad (37)
 \end{aligned}$$

244 In the SWHF approximation, the first term of $D_{FC}(\kappa)$ grows exponentially and
 245 dominates, whence propagation occurs through Rayleigh waves, just as in a
 246 half-space.

247 Fig.5 compares RL spectra in a free/clamped plate with bulk modes for
 248 medium A. Again, we see that an infinite succession of intersection points be-
 249 tween the two is possible inasmuch as the phase speed of Rayleigh waves in
 250 medium B is lower than that of bulk waves in medium A, which requirement
 251 is approximated by Eq.(36). Indeed, increasing rotational inertia in medium A,
 252 intersections are no longer possible, while a finite number of intersections (as it
 253 occurs for RL waves) is not supported.

At the layer/half-space joining surface, we impose perfect adhesion

$$w_A(0) = w_B(0), \quad (38a)$$

$$w'_A(0) = w'_B(0), \quad (38b)$$

$$p_{3A}(0) = p_{3B}(0), \quad (38c)$$

$$q_{1A}(0) = q_{1B}(0), \quad (38d)$$

while the layer top surface $\xi_2 = H/\Theta$ is subjected to free conditions

$$p_{3B}(\Theta^{-1}H) = 0, \quad (39a)$$

$$q_{1B}(\Theta^{-1}H) = 0. \quad (39b)$$

Introducing the solutions (28,30) into the boundary conditions (39,38) yields a homogeneous system of linear equations in the unknown amplitudes $e_{1,2}, o_{1,2}, s_{1,2}$, which admits non-trivial solutions inasmuch as the *secular (or frequency)* equation

$$\Delta(\kappa) = 0,$$

255 is satisfied. Letting $\Gamma = G_B/G_A$, the ratio of the layer to the half-space shear
256 moduli, the general form of the secular equation can be written as

$$\Delta(\kappa) = \beta^6 \Gamma^2 (A_1 - A_2) D_0(\kappa),$$

257 having let the quadratic polynomial in Γ

$$D_0(\kappa) = d_0 + d_1 \Gamma + d_2 \Gamma^2, \quad (40)$$

with

$$d_0 = -\beta^{-2} R_A(\kappa) D_{FC}(\kappa),$$

$$d_1 = D_1(\kappa),$$

$$d_2 = -4\beta^2 D_{RL}(\kappa).$$

Here, $D_1(\kappa)$ expresses the coupling between the half-space and the layer

$$D_1(\kappa) = (B_1 - B_2)R_B(\kappa) \left\{ c_+ \left[\cosh \left(\frac{H(B_1 + B_2)}{\Theta} \right) - 1 \right] - \frac{s_+}{B_1 + B_2} \sinh \left(\frac{H(B_1 + B_2)}{\Theta} \right) \right\} - (\zeta_{11B}^2 B_1 + \zeta_{12B}^2 B_2) \times \left\{ c_- \left[\cosh \left(\frac{H(B_1 - B_2)}{\Theta} \right) - 1 \right] - \frac{s_-}{B_1 - B_2} \sinh \left(\frac{H(B_1 - B_2)}{\Theta} \right) \right\}, \quad (41)$$

where

$$c_{\pm} = (\eta_A \kappa^2 - A_1 A_2) \frac{\zeta_{11B} B_1 \mp \zeta_{12B} B_2}{B_1 B_2},$$

$$s_{\pm} = \pm \frac{1}{2} (B_2^2 - B_1^2) (A_1 + A_2) (B_1 \pm B_2) \left(1 \pm \frac{A_1 A_2}{B_1 B_2} \right).$$

258 Rewriting Eq.(41) as in Appendix Appendix A.1, it is seen that dependence on
 259 $B_{1,2}$ occurs only through even powers, whence only cuts associated with $A_{1,2}$
 260 remain.

When $A = B$, it is

$$(\Gamma, \beta, v, \psi, \eta_B) = (1, 1, 1, \delta, \eta)$$

261 and we retrieve the well expected result

$$D_0^{A=B}(\kappa) = (\zeta_{11A} - \zeta_{12A})^2 (A_1 - A_2) R_A(\kappa),$$

whereby only Rayleigh waves propagate. Similarly, for an exceedingly weak layer, that is for $\Gamma \rightarrow 0$, we find either Rayleigh waves confined to the half-space or RL waves trapped in the free/clamped layer, which bounce off the impenetrable rigid barrier posed by the half-space. In the special case of a vanishing layer, $H = 0$, one gets, to leading order,

$$D_0^{H=0}(\kappa) = (\zeta_{11B} - \zeta_{12B})^2 (A_1 - A_2) R_A(\kappa),$$

262 that gives Rayleigh waves again. Its classical limit (21) gives

$$D_0^{H=0}(\kappa) \rightarrow -\frac{2\delta^2}{\Omega^2} \left(\frac{c_{sB}}{c_{sA}} \right)^4 B_1,$$

263 whence Rayleigh waves collapse onto travelling bulk SH waves. On the opposite
 264 side of the spectrum, for an exceedingly strong layer, that is for $\Gamma \rightarrow +\infty$, we
 265 obtain, as expected, RL modes.

266 4. Wave pattern and microstructural features

267 We now consider the root landscape of $D_0(s)$ considered as a complex-valued
268 function of the argument $s = \Re(s) + i\Im(s)$. Eq.(40) is noteworthy for it shows
269 that the only branch cuts appearing in $D_0(s)$ are those brought by the Rayleigh
270 function for the half-space A. Accordingly, in this system, only three families of
271 waves may exist, namely:

- 272 1. Lamb waves, which are either travelling, when they correspond to the
273 real zeros of D_0 , or evanescent (Lamb-like waves), characterised by purely
274 imaginary zeros of D_0 ;
- 275 2. bulk travelling waves, which move like SH bulk waves for the half-space A,
276 possibly inhomogeneous: these are related to the branch cuts for $A_1(s)$;
- 277 3. bulk standing modes, related to the branch cuts for $A_2(s)$.

278 No bulk modes are possible that are related to SH bulk waves in B.

279 Existence and uniqueness of Love waves may be establish through the argu-
280 ment principle, following the procedure adopted in Cagniard (1962) and Nobili
281 et al. (2021) for Stoneley waves, respectively in CE and CS. This technical and
282 lengthy proof is sketched in Appendix A.2. One major result of this process is
283 the necessary condition for the existence of Love waves, namely

$$D_0(\delta) \geq 0, \tag{42}$$

284 equality setting the limits of the passbands, i.e. the cut-on and cut-off frequen-
285 cies. Physically, the passband ends correspond to the situation when Love waves
286 move with the same phase speed as bulk waves in medium A. In such states,
287 energy is no longer trapped in the layer, bouncing back and forth between its
288 boundaries, but leaks in the half-space in the form of bulk waves.

289 Eq.(42) is the fundamental result which enables to relates the band pattern to
290 the microstructural features in the material. Also, several qualitative outcomes
291 are possible in dependence of the ratio between the layer and the substrate
292 microstructure, as it is presently described.

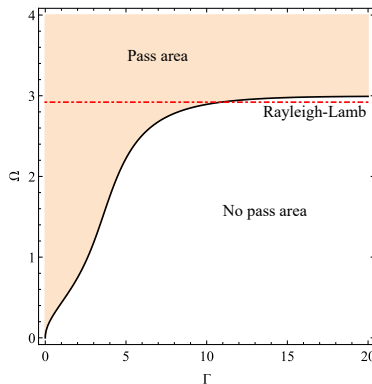


Figure 6: Cut-on frequency vs. shear stiffness ratio Γ for $\ell_A = 0.4 < \ell^B = 0.5$ (solid, black). Propagation occurs above the black curve, which asymptotes to the first symmetric Rayleigh-Lamb mode at the wavenumber $\kappa = \delta$ (red dash-dotted line).

293 4.1. Infinite passbands

294 Fig.6 plots the cut-on frequency as a function of Γ for $\ell_A = 0.4$, which,
 295 for large Γ , asymptotes to the first symmetric RL mode for a free plate at the
 296 wavenumber $\kappa = \delta$. Sweeping larger frequencies reveals a finite passband,
 297 that ranges from cut-on to cut-off, as it is shown in Fig.7. The corresponding
 298 frequency spectrum for Love waves is shown in Fig.8 for $\Gamma = 1$ and $\Gamma = 5$:
 299 the picture confirms that propagation only occurs within a finite frequency
 300 range. It also shows that the dispersive nature of propagation is restricted to
 301 low wavenumbers, a result which was already observed with respect to Rayleigh
 302 waves (Ottosen et al., 2000; Nobili et al., 2019).

303 From the discussion in Sec.3.2, we deduce that an infinite number of pass-
 304 bands is expected: for Γ large, these are framed in between symmetric and
 305 antisymmetric RL modes. Similarly, for $\Gamma = 0$, passbands are constrained by
 306 RL waves propagating in a clamped-free plate at $\kappa = \delta$, that is by the inter-
 307 sections indicated in Fig.5. In fact, the bounding curves for cut-on and cut-off
 308 start at $\Gamma = 0$, which is realized provided that $D_{FC}(\delta) = 0$.

309 4.2. Single passband

310 This propagation landscape, characterized by an infinite succession of pass-
 311 bands, holds only inasmuch as condition (36) stands. Indeed, when micro-inertia

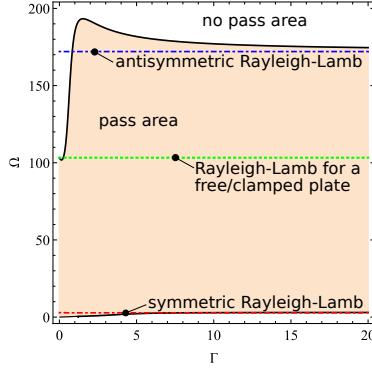


Figure 7: Passband frequency range vs. shear stiffness ratio Γ (solid, black), with $\ell_A = 0.4$. Propagation occurs in the (pink) region in between the black curves; these asymptote to the first symmetric/antisymmetric Rayleigh-Lamb frequencies, respectively for cut-on (red) and cut-off (blue dash-dotted lines). The cut-off curve sets off in correspondence to the occurrence of Rayleigh-Lamb waves in a free/clamped plate (green, dotted)

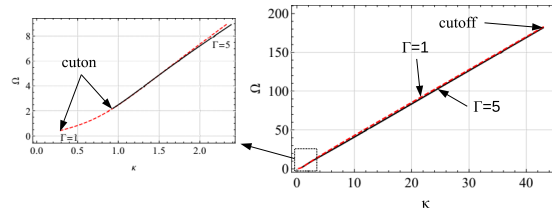


Figure 8: Frequency spectrum for $\Gamma = 1$ (dashed, red) and $\Gamma = 5$ (solid, black) with $\ell_A = 0.4 < \ell^B = 0.5$: Lamb frequency spectra asymptote to Rayleigh waves

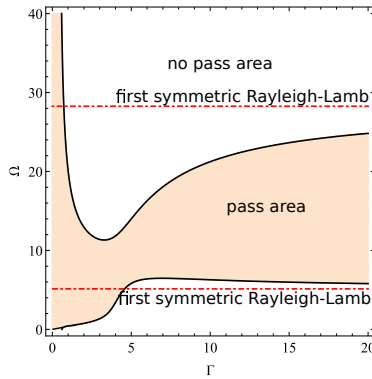


Figure 9: Cut-on frequency vs. shear stiffness ratio Γ for $\ell_A = 0.45$ (solid, black). Propagation occurs between the black curves, which asymptote to two frequencies in the first symmetric Rayleigh-Lamb branch at the wavenumber $\kappa = \delta$ (red dash-dotted lines). Since no intersection with RL waves in a free/clamped plate is possible, cut-off possesses a vertical asymptote

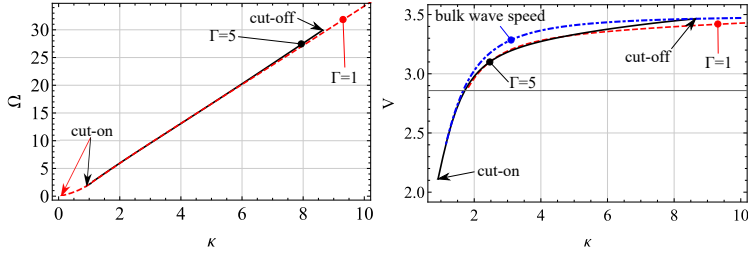


Figure 10: Frequency spectrum (left) for $\Gamma = 1$ (dashed, red) and $\Gamma = 5$ (solid, black), for $\ell_A = 0.45$: only for the latter cut-off is defined. Since spectra almost overlap, dispersion curves are also shown (right) and it is seen that cut-on and cut-off indeed occur at the intersections with the bulk wave speed (dash-dotted, blue)

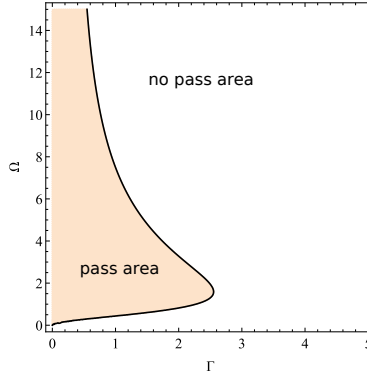


Figure 11: Cut-on frequency vs. shear stiffness ratio Γ for $\ell_A = 0.55 > \ell_B = 0.5$ (solid, black). Propagation occurs up to a limiting threshold for Γ .

312 of medium A is large enough (and therefore bulk waves move slow enough),
 313 propagation occurs through a single passband, which asymptotes to a pair of
 314 frequencies along the first symmetric RL mode, as in Fig.4. This situation is
 315 depicted in Fig.9, that plots cut-on and cut-off frequencies as a function of Γ for
 316 $\ell^A = 0.45$. Since no intersections of bulk waves with RL modes in a clamped/free
 317 plate are possible, cut-off displays a vertical asymptote and, as a consequence,
 318 for Γ small enough, the passband is semi-infinite. The frequency spectrum in
 319 Fig.10 shows that this is indeed the case for $\Gamma = 1$ and yet no longer for $\Gamma = 5$.

320 4.3. Love wave block-band

321 Increasing further micro-inertia in material A, an important change in the
 322 propagation features is encountered: namely the appearance of a bounding curve

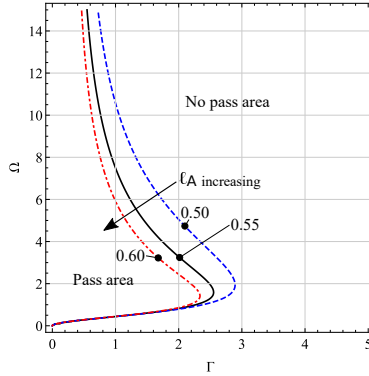


Figure 12: Cut-on frequency vs. shear stiffness ratio Γ for $\ell_A = 0.50$ (dashed, blue), 0.55 (solid, black) and 0.6 (dot-dashed, red): the pass region shrinks as rotational inertia increases in medium A

323 for Γ . Indeed, since no RL modes in the layer are possible which equally support
 324 bulk waves in the half-space, the passband region becomes bounded by $\Gamma <$
 325 $\Gamma_M(\Omega)$, as in Fig.11. This passband region shrinks for larger values of rotational
 326 inertia, as illustrated in Fig.12.

327 5. The classical limits

328 We now show that neglecting microstructure in only one of either the layer or
 329 the half-space generally leads to inconsistencies. Occasionally, however, a special
 330 set of boundary conditions is available which lends a well-posed problem.

331 5.1. Classical layer perfectly bonded to a couple stress half-space

332 We first consider the case when the layer B is classical elastic and the half-
 333 space A is made of couple stress material. This condition is obtained by taking
 334 the limit (21) of the general case. We note that, as $\beta \rightarrow 0$,

$$B_1 = \sqrt{\kappa^2 - \frac{\tilde{c}_{sA}}{\tilde{c}_{sB}} \Omega^2 \Theta^2}, \quad B_2 = \frac{\sqrt{2}\Theta}{\beta} + O(1).$$

335 Then, we can solve Eq.(39b) for o_2 to obtain

$$o_2 = \left[-\frac{\sqrt{2}\Theta}{\beta} \coth \frac{\sqrt{2}H}{\beta} + O(\beta) \right] e_2 + O\left(\frac{\beta}{\sinh(\sqrt{2}H/\beta)} \right), \quad (43)$$

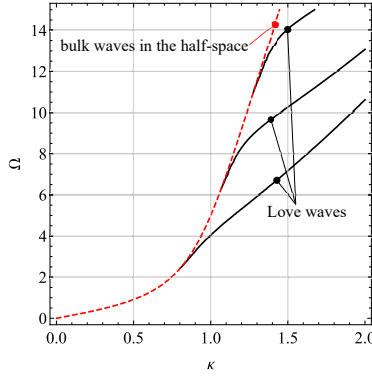


Figure 13: Love waves for a couple stress half-space in perfect contact with a classical layer (solid, black) for $H = 1, \eta_A = 0.5, \ell_A = 0.2, \beta v = 1.5$ and $\Gamma = 3$. For each branch, cut-on is located in correspondence of bulk waves in the half-space (dashed, red).

336 where the last term is exponentially small. Consequently, rotation continuity
 337 (38b) reads

$$o_1 + o_2 - A_1 s_1 - A_2 s_2 = 0,$$

338 which, assuming all quantities to be $O(1)$, demands $e_2 = O(\beta)$ to eliminate the
 339 $O(\beta^{-1})$ term given by o_2 . Solving the remaining equations in the system (38,39)
 340 with the assumption $e_2 = 0$, gives the frequency equation

$$-R_A(\kappa) \cosh\left(\frac{B_1 H}{\Theta}\right) + \Gamma \sqrt{2} \Omega^{-1} \delta(A_1 + A_2) B_1 \sinh\left(\frac{B_1 H}{\Theta}\right) = 0, \quad (44)$$

341 which clearly possesses the structure (40). As Fig.13 shows, this equation is
 342 remarkable in that it exhibits an infinite number of branches, which is un-
 343 expected for Love waves. Looking at the eigenforms, we find $o_2 = O(1)$,
 344 whence the corresponding displacement (30) blows to infinity as $\beta \rightarrow 0$. Also,
 345 although e_2 tends to zero, it combines with the exponentially large factor
 346 $B_2^{-1} \sinh(B_2 \xi_2) = \frac{\beta}{\sqrt{2}\Theta} \sinh(\sqrt{2}\Theta \beta^{-1} \xi_2)$ to produce an exponentially explod-
 347 ing contribution. As a consequence of these observations, the dispersion rela-
 348 tion (44) cannot be directly obtained assuming a classical solution for B and
 349 it is thereby defined as non-classical. We also observe that, to leading order,
 350 $q_{1A}(0) = 0$, which indeed corresponds to one of the boundary conditions used
 351 in Sharma and Kumar (2019); Sharma et al. (2020). However, rotation at the

352 interface is finite and continuous across

$$w'_A(0) = w'_B(0) = \left(A_1 - \frac{\zeta_{12A}}{\zeta_{11A}} A_2 \right) s_1,$$

353 and this condition is neglected.

354 *5.2. Couple stress layer perfectly bonded to a classical half-space*

Let's now consider the case when A is classical elastic and B is a couple stress layer. Then, $\kappa = O(\beta^{-1})$ and $\Omega = O(v)$, as $\ell_A \rightarrow 0$ and $\beta \rightarrow \infty$, while $v \rightarrow 0$ such that $\beta v < \infty$. Thus,

$$A_1 = \beta^{-1} \sqrt{\kappa_B^2 - \frac{1}{2} \beta^2 v^2 \Omega_B^2} + O(\beta^{-3}), \quad (45a)$$

$$A_2 = 1 + \frac{\kappa_B^2}{2\beta^2} + O(\beta^{-4}), \quad (45b)$$

$$B_1 = \beta^{-1} \sqrt{\kappa_B^2 - \frac{1}{2} \left(\sqrt{(1 - \ell_B^2 \Omega_B^2)^2 + 2\Omega_B^2} - (1 - \ell_B^2 \Omega_B^2) \right)} + O(\beta^{-3}), \quad (45c)$$

$$B_2 = \beta^{-1} \sqrt{\kappa_B^2 + \frac{\Omega_B^2}{\sqrt{(1 - \ell_B^2 \Omega_B^2)^2 + 2\Omega_B^2} - (1 - \ell_B^2 \Omega_B^2)}} + O(\beta^{-3}). \quad (45d)$$

Solving (38d) lends

$$s_2 = \Gamma \left\{ e_1 \left[(\eta_B + 1) \kappa_B^2 + \frac{1}{2} \left(1 - \Omega_B^2 \ell_B^2 - \sqrt{2\Omega_B^2 + (1 - \Omega_B^2 \ell_B^2)^2} \right) \right] \right. \\ \left. + e_2 \left[(\eta_B + 1) \kappa_B^2 + \frac{\Omega_B^2}{\Omega_B^2 \ell_B^2 - 1 + \sqrt{2\Omega_B^2 + (\Omega_B^2 \ell_B^2 - 1)^2}} \right] \right\} + O(\beta^{-2}), \quad (46)$$

355 which, substituted into the remaining set of boundary conditions, yields a system
 356 which is singular to leading order, owing to Eq.(39b) disappearing at leading
 357 order. Solving any regular subsystem gives

$$o_1 = o_2 = O(\beta^{-1}), \quad \text{and} \quad s_2 = O(\beta^{-2}), \quad (47)$$

358 whereupon, in the limit as $\beta \rightarrow \infty$, the half-space A becomes classical and the
 359 layer B admits the even solution only, i.e. $e_1 = e_2 = O(1)$. Consequently,
 360 we expect neither rotation nor couple stress in the layer B *to leading order* at
 361 the bonding interface $\xi_2 = 0$, for they arise from the odd part of the solution.

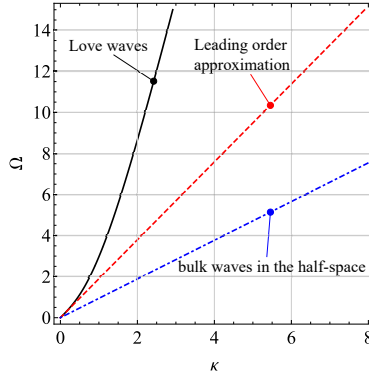


Figure 14: Love wave spectrum when A is classical elastic (solid, black), superposed onto the leading order approximation (49) (dashed, red) for $H_B = 10, \eta_B = 0.8$ and $\ell_B = 0.2$. Bulk waves in the classical half-space $\Omega_B - \sqrt{2}(\beta\nu)^{-1}\kappa_B = 0$ are also shown (blue, dot-dashed), with $\beta\nu = 1.5$

362 Equally, no rotation arises from the half-space A either, because, at $\xi_2 = 0$, it
 363 is simply given by

$$A_1 s_1 + A_2 s_2 = O(\beta^{-1}).$$

364 Thus, rotation continuity is trivially satisfied at leading order. Looking back at
 365 couple stress continuity (46), this appears compatible with the last of Eqs.(47)
 366 inasmuch as the leading term drops out. Since this is indeed the case, couple
 367 stress continuity is also trivially satisfied. Therefore, we are left with the classical
 368 system A in perfect contact with the even solution of the couple stress layer B
 369 subject to the conditions (38a,38c,39a). Letting $H_B = \beta^{-1}H$, we get the novel
 370 frequency equation

$$\zeta_{11B} B_1 \sinh\left(\sqrt{2}H_B \beta B_1\right) - \zeta_{12B} B_2 \sinh\left(\sqrt{2}H_B \beta B_2\right) = 0, \quad (48)$$

371 with the understanding that Eqs.(45c,45d) are used to eliminate β . It is observed
 372 that terms coming from the half-space A are not present in (48) because they
 373 factor out and never contribute a real wavenumber. In this condition, Love
 374 waves are always supported as in Fig.14, which also illustrates the leading order
 375 approximation

$$\Omega_B^2 = \sqrt{\alpha_1} \kappa_B^2, \quad \alpha_1 = \sqrt{2} \frac{1 - (\eta_B + 1) \frac{\sinh(\sqrt{2}H_B)}{\sqrt{2}H_B}}{1 - \frac{\sinh(\sqrt{2}H_B)}{\sqrt{2}H_B}}. \quad (49)$$

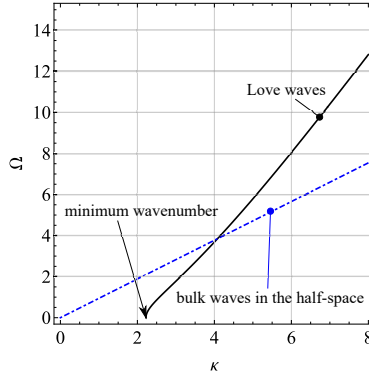


Figure 15: Love wave spectrum when A is classical elastic (solid, black), for $H_B = 2, \eta_B = -0.8$ and $\ell_B = 0.2$. Bulk waves in the classical half-space $\Omega_B - \sqrt{2}(\beta\nu)^{-1}\kappa_B = 0$ are also shown (blue, dot-dashed), with $\beta\nu = 1.5$

376 With the help of this approximation, it may be shown that, if we can find real
 377 solutions of

$$\frac{\sinh(\sqrt{2}H_B)}{\sqrt{2}H_B} = \frac{1}{1 + \eta_B},$$

378 then we get a minimum wavenumber for Love wave propagation, as in Fig.15.

379 It is worth pointing out that various subsets of boundary conditions have
 380 been adopted in the literature. As an example, in Fan and Xu (2018), a cou-
 381 ple stress layer is perfectly bonded to a classical half-space and the conditions
 382 (38a,38c,39) alongside $q_{1A} = 0$ are used instead. Since rotation continuity is
 383 missing, the resulting frequency equation does not match (48).

384 6. Conclusions

385 We show that incorporating microstructure into the material description
 386 leads to a nonclassical band structure for Love wave propagation, which may
 387 be conveniently back-processed for non destructive testing (NDT) evaluation of
 388 the material microstructure. In particular, an explicit expression is given for
 389 the propagation band limits, namely the cut-on and cut-off points. These are
 390 shown to correspond to the condition when the localized wave moves with the
 391 same speed as the SH bulk wave in the substrate. As a consequence, energy
 392 leaks to infinity and it is no longer confined (internal resonance). Precisely

393 this condition may be taken advantage of to assess the mechanical properties
394 of the two materials by non destructive testing. Indeed, it is simple matter
395 to experimentally locate cut-on/cut-off frequencies and therefrom compute the
396 relevant microstructural parameters. In general, the qualitative features of the
397 band pattern are related to the degree of similarity between the mechanical
398 properties of the layer and of the substrate.

399 Furthermore, to warrant uniqueness of the inverse problem, multiple ob-
400 servables needs to be collected, the easiest being the Rayleigh spectrum. In
401 this context, we show that the frequency equation for Love waves possesses an
402 elegant three term structure, where Rayleigh and Rayleigh-Lamb modes for a
403 free and free/clamped layer play an important role. Therefore, consideration
404 of Rayleigh waves besides Love waves is able to minimize the non-uniqueness
405 connected to signal back-processing on the surface.

406 Finally, we consider the special situation, well represented in the literature,
407 as either the substrate or the layer turns classical and therefore has no mi-
408 crostructure. We show that such glueing of dissimilar material models generally
409 leads to inconsistencies. Indeed, when the layer microstructure vanishes, the
410 resulting displacement field grows unbounded. In contrast, the case when the
411 substrate microstructure becomes exceedingly small still leads to a meaning-
412 ful solution, which may be obtained directly proceeding from the right set of
413 boundary conditions. A novel dispersion relation is obtained which lends a
414 single continuous branch with neither cut-on nor cut-off, as for classical Love
415 waves.

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419 **Appendix A. Appendix**

420 *Appendix A.1. Coupling coefficient*

The coupling term in the dispersion relation (40) may be rewritten as

$$D_1(\kappa) = c_c \left[\cosh\left(\frac{HB_1}{\Theta}\right) \cosh\left(\frac{HB_2}{\Theta}\right) - 1 \right] \\ + s_s B_1^{-1} B_2^{-1} \sinh\left(\frac{HB_1}{\Theta}\right) \sinh\left(\frac{HB_2}{\Theta}\right) + (A_1 + A_2) (\zeta_{11B} - \zeta_{12B}) \times \\ \left[s_c B_1^{-1} \sinh\left(\frac{HB_1}{\Theta}\right) \cosh\left(\frac{HB_2}{\Theta}\right) - c_s B_2^{-1} \cosh\left(\frac{HB_1}{\Theta}\right) \sinh\left(\frac{HB_2}{\Theta}\right) \right],$$

with the coefficients

$$c_c = 2\zeta_{11B}\zeta_{12B} (\zeta_{11B} + \zeta_{12B}) (A_1 A_2 - \kappa^2 \eta_A), \\ s_s = -2 (\zeta_{11B}^3 B_1^2 + \zeta_{12B}^3 B_2^2) (A_1 A_2 - \kappa^2 \eta_A), \\ s_c = A_1 A_2 \zeta_{12B}^2 - \zeta_{11B}^2 B_1^2, \\ c_s = A_1 A_2 \zeta_{11B}^2 - \zeta_{12B}^2 B_2^2.$$

421 *Appendix A.2. Existence and uniqueness*

We consider the complex-valued function

$$D(s) = D_0(s) \exp\left(-\frac{2H}{\Theta}|s|\right),$$

422 where the exponential factor is added to obtain algebraic growth as $|s| \rightarrow +\infty$.

423 Indeed, we find that $D(s) \sim |s|^4$ as $|s| \rightarrow \infty$. We recall that $D(s) = D(-s)$ is

424 central-symmetric. The path in the complex plane which is adopted to imple-

425 ment the argument principle is a very large circle of radius $R \rightarrow \infty$ with two

426 pairs of loops: one circles around the branch cuts at $\pm\delta$ and the other around

427 the branch cuts at $\pm i$, both being parallel to the imaginary axis, see Fig.A.16.

428 We thus see that the image of the path γ through the function $D(s)$ circles the

429 origin four times counter-clockwise and, possibly, two time clockwise if condi-

430 tion (42) is violated. In fact, this condition merely states that the image $D(\gamma_\delta)$

431 intersects the real axis to the right of the origin and contributes nothing. In this

432 case we have four roots, a real pair and a complex pair. In contrast, if condition

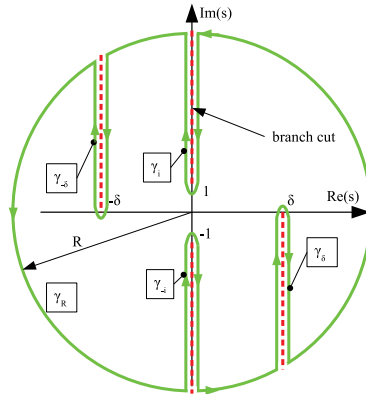


Figure A.16: Path $\gamma = \gamma_R \cup \gamma_{\pm\delta} \cup \gamma_{\pm i}$ for implementing the argument principle: the number of zeros of $D(s)$ equals the number of times the image $D(\gamma)$ winds around the origin

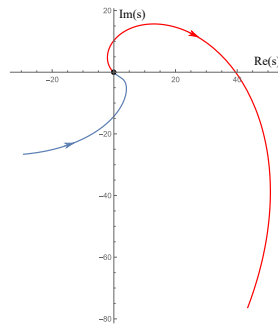


Figure A.17: The image $D(\gamma_\delta)$ winds around the origin in clockwise manner, because $D(\delta) < 0$. Hence, the number of zeros of $D(s)$ is diminished by two

433 (42) is violated, then the origin sits to the right of path, as in Fig.A.17, and
434 subtracts a pair of roots. Then, only the complex pair remains. It is finally
435 observed that the image $D(\gamma_i)$ never circles the origin.

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