

# Asymptotically consistent size-dependent plate models based on the couple-stress theory with micro-inertia

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## Abstract

Several beam and plate models have been recently developed in the literature to accommodate for size-dependence. These are usually obtained starting from a generalized continuum theory (such as the couple-stress, strain-gradient or non-local theory or their modifications) and then deducing the governing equations through Hamilton's principle and ingenuous kinematical assumptions. This approach, originated by Kirchhoff, usually fails to reproduce the dispersion features of the equivalent 3D theory. Besides, it produces a variety of models, in dependence of the different assumptions, such as Kirchhoff's or Mindlin's. In contrast, in this paper we adopt asymptotic reduction: moving from the couple-stress linear theory of elasticity with micro-inertia, we deduce new models for elongation and flexural deformation of microstructured plates. The resulting models are consistent, in the sense that they reproduce the dispersion features of the corresponding 3D body. Also, models are unique, for they may only differ by the order of the approximation. We find that microstructure especially affects inertia terms, which can be hardly captured by a-priori kinematical assumptions. For static flexural deformations, our results match those already obtained assuming plane cross-sections within the modified couple-stress theory. In fact, we show that couple-stress, reduced couple-stress and strain gradient theories all lead to equivalent results. Higher order models are also given, that describe the near first-cut-off behaviour and account for thickness deformations in the

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spirit of Timoshenko.

*Keywords:* Asymptotic model, Microstructured plate, Couple stress,  
Rayleigh-Lamb waves

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## 1. Introduction

Modern technology heavily relies on micro- and nano-components for sensing, actuating, filtering and testing. It is an established experimental fact that mechanical response of these structures depends on their scale (Chong et al., 2001; Radi et al., 2020). However, the classical theory of elasticity lacks an internal length-scale and it is therefore incapable of incorporating this dependence. Precisely to remedy this deficiency, which leads to several limitations (Nobili et al., 2020), many generalized continuous theories (GCT) have been proposed, such as surface elasticity (Gurtin and Murdoch, 1975), non-local elasticity (Eringen, 1984) and strain gradient theories (Yang et al., 2002). Couple stress theory is perhaps the simplest micropolar theory (Toupin, 1962; Koiter, 1969), where an extra rotational displacement is considered (the micro-motion), that is related to the skew symmetric part of the displacement (the macro-motion) gradient. In this respect, it is also a strain gradient theory. Recently, adaptations of the original theory have been proposed, in an attempt to reduce the number of material parameters, such as the modified or reduced couple stress theory (Yang et al., 2002), which postulates a new equilibrium equation for the moment of momenta, the symmetric theory (Hadjesfandiari and Dargush, 2011), which restricts the curvature tensor to being symmetric, and the strain gradient effect, originating from plasticity and relating to geometrically necessary dislocations (Fleck and Hutchinson, 1993). Also, the majority of studies concerning couple stress (CS) theory neglects the role of micro-rotational inertia, owing to the increased mathematical complications.

In this context, a large number of contributions has recently appeared in the literature introducing new dimensional-reduced models, such as beams, plates, shells, incorporating microstructural features. The generality of these is ob-

27 tained moving from an enhanced continuum theory and then deducing the new  
28 equilibrium equations through an application of Hamilton's principle, in con-  
29 junction with ingenious kinematical constraints. This is indeed the method  
30 originally adopted by Kirchhoff (1859) in developing the classical plate theory  
31 that bears his name, and later extended to shells by Love, see (Love, 1888) and  
32 the excellent historical introduction therein. Along these lines, Ma et al. (2008)  
33 developed a Timoshenko-like beam model moving from the differential non-  
34 local model of Eringen, which was later found to lead to paradoxes Mikhasev  
35 and Nobili (2020). In Park and Gao (2006), a static Bernoulli-Euler theory of  
36 size-dependent beams is introduced from the reduced couple stress (RCS) theory  
37 by the minimum energy principle. Tsiatas (2009) developed a static Kirchhoff  
38 plate model based on the RCS theory, which was later extended to dynamics  
39 by Yin et al. (2010) simply by addition of translational inertia. On a similar  
40 basis, Jomehzadeh et al. (2011) introduced size-dependence in micro-plates, ac-  
41 counting for plate extension and adding rotational inertia. Mindlin-like models,  
42 where cross-section rotation is an extra degree of freedom, have been presented  
43 in Ma et al. (2011); Zhou and Gao (2014), again moving from the modified cou-  
44 ple stress theory. Similar contributions, but departing from the strain gradient  
45 (SG) theory, are presented in Lazopoulos (2004) and, successively, in Lazopou-  
46 los (2009) with the addition of surface elasticity, both being restricted to statics  
47 and arriving at a higher-order plate theory.

48 In this paper, we develop a novel size-dependent plate model by asymp-  
49 totic reduction of the elastodynamics of a thin strip of couple stress elastic  
50 isotropic material with micro-inertia, encompassing for extension and flexure.  
51 As described by Kaplunov et al. (1998) in their excellent monograph, asymp-  
52 totic reduction is a powerful technique by which dimensional-reduced models are  
53 built which preserve the fundamental dispersion feature of the original 3D body.  
54 Precisely in this sense, models are defined *consistent*. Models are also *unique*,  
55 for a given range of approximation (say, long-wave low-frequency). Higher or-  
56 der (or, better, long-wave high-frequency) models are also constructed, which  
57 reproduce dispersion near the first cut-on frequency (the first overtone) and add

58 thickness-stretch and thickness-shear effects. In the absence of microstructure,  
 59 models correctly reduce to those developed within classical elasticity. Interest-  
 60 ingly, restriction to statics lends results already obtained by assuming plane  
 61 cross-sections in the context of the RCS theory. In fact, we show that CS, RCS  
 62 and SG theories are all equivalent in terms of asymptotic reduction.

## 63 2. Fundamentals of couple stress theory

In a linear elastic couple stress solid, the traditional displacement field  $\mathbf{u}$  is supplemented by the rotation vector  $\boldsymbol{\varphi}$ , which represents the motion at the micro-level. In contrast to micropolar theories, the latter is related to the former through

$$\boldsymbol{\varphi} = \frac{1}{2} \operatorname{curl} \mathbf{u}.$$

We introduce the classical measure of deformation (strain tensor)

$$\boldsymbol{\varepsilon} = \operatorname{Sym} \operatorname{grad} \mathbf{u},$$

alongside the torsion-flexure or curvature tensor, which is typical of higher gradient theories (the wryness tensor)

$$\boldsymbol{\chi} = \operatorname{grad} \boldsymbol{\varphi}.$$

For an isotropic hyperelastic material, we introduce the stored elastic potential  $U = U(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$  and obtain the constitutive equations

$$\boldsymbol{\sigma} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}} \quad \Rightarrow \quad \boldsymbol{\sigma} = 2G\boldsymbol{\varepsilon} + \Lambda(\operatorname{tr} \boldsymbol{\varepsilon})\mathbf{1}, \quad (1a)$$

$$\boldsymbol{\mu} = \frac{\partial U}{\partial \boldsymbol{\chi}} \quad \Rightarrow \quad \boldsymbol{\mu} = 2G\ell^2 (\boldsymbol{\chi}^T + \eta\boldsymbol{\chi}), \quad (1b)$$

64 where  $\mathbf{1}$  is the identity tensor and  $\ell > 0$ ,  $-1 < \eta < 1$  is a pair of material  
 65 constants connected to the microstructure, while  $G$  and  $\Lambda$  are the traditional  
 66 Lamé parameter. Eqs.(1) specify the symmetric part of the stress tensor,  $\boldsymbol{\sigma} =$   
 67  $\operatorname{Sym} \boldsymbol{t}$ , and the couple stress tensor  $\boldsymbol{\mu}$ , respectively. The latter is a deviatoric  
 68 tensor, which becomes symmetric simply by taking  $\eta = 1$ , that is indeed the

69 assumption in the RCS theory. Similarly, the case  $\eta = \frac{1}{2}$  yields the strain  
70 gradient effect.

The equations of motion express equilibrium of linear and angular momentum

$$\operatorname{div} \mathbf{t} = \rho \ddot{\mathbf{u}}, \quad (2a)$$

$$2 \operatorname{axial} \mathbf{t} + \operatorname{div} \boldsymbol{\mu} = J \ddot{\varphi}, \quad (2b)$$

71 where  $\rho$  is the mass density and  $J \geq 0$  is the micro-inertia, with physical di-  
72 mensions of mass over length. Here,  $\operatorname{div}$  operates on the first tensor component  
73 and  $(\operatorname{axial} \mathbf{t})_i = e_{ijk} t_{kj}$  denotes the axial vector attached to any skew symmetric  
74 tensor (here  $e_{ijk}$  is the permutation tensor).

75 Hereinafter, we restrict our analysis to plane-strain conditions

$$u_1(x_1, x_2, t), \quad u_2(x_1, x_2, t), \quad u_3(x_1, x_2, t) = 0, \quad (3)$$

whence, the non-zero components of  $\boldsymbol{\varepsilon}$ ,  $\boldsymbol{\varphi}$  and  $\boldsymbol{\chi}$  read

$$\varepsilon_{11} = u_{1,1}, \quad \varepsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) = \varepsilon_{21}, \quad \varepsilon_{22} = u_{2,2}, \quad (4a)$$

$$\varphi_3 = \frac{1}{2}(u_{2,1} - u_{1,2}), \quad (4b)$$

$$\chi_{31} = \frac{1}{2}(u_{2,11} - u_{1,21}), \quad \chi_{32} = \frac{1}{2}(u_{2,12} - u_{1,22}). \quad (4c)$$

Together, Eqs.(4) with the constitutive equations (1) give

$$\sigma_{11} = 2Gu_{1,1} + \Lambda(u_{1,1} + u_{2,2}), \quad \sigma_{22} = 2Gu_{2,2} + \Lambda(u_{1,1} + u_{2,2}), \quad (5a)$$

$$\sigma_{33} = \Lambda(u_{1,1} + u_{2,2}), \quad (5b)$$

$$\sigma_{12} = \sigma_{21} = G(u_{1,2} + u_{2,1}), \quad (5c)$$

$$\mu_{13} = G\ell^2(u_{2,11} - u_{1,21}), \quad \mu_{23} = G\ell^2(u_{2,12} - u_{1,22}), \quad (5d)$$

$$\mu_{31} = G\ell^2\eta(u_{2,11} - u_{1,21}), \quad \mu_{32} = G\ell^2\eta(u_{2,12} - u_{1,22}), \quad (5e)$$

In plane strain, Eqs.(2) become

$$\sigma_{11,1} + \sigma_{21,2} + \tau_{21,2} = \rho \ddot{u}_1, \quad (6a)$$

$$\sigma_{12,1} + \tau_{12,1} + \sigma_{22,2} = \rho \ddot{u}_2, \quad (6b)$$

$$2\tau_{12} + \mu_{13,1} + \mu_{23,2} = J \ddot{\varphi}_3, \quad (6c)$$

76 and the last, with the help of (5e), directly lends the skew-symmetric part of  
 77 the stress tensor  $\boldsymbol{\tau} = \text{Skw } \mathbf{t}$

$$\tau_{12} = -\tau_{21} = -\frac{1}{2}G\ell^2(\hat{\Delta}u_{2,1} - \hat{\Delta}u_{1,2}) + \frac{J}{4}(\ddot{u}_{2,1} - \ddot{u}_{1,2}). \quad (7)$$

Here, the notation  $\hat{\Delta}$  indicates the 2-D Laplace operator in the  $x_1, x_2$  coordinates. Substituting (5) and (7) into (6a) and (6b) leads to the following expression for the governing equations in terms of displacement

$$\begin{aligned} (\Lambda + G)(u_{1,11} + u_{2,12}) + G(u_{1,11} + u_{1,22}) + \frac{J}{4}(\ddot{u}_{1,22} - \ddot{u}_{2,12}) \\ + \frac{1}{2}G\ell^2(u_{2,1222} + u_{2,1112} - u_{1,1122} - u_{1,2222}) = \rho\ddot{u}_1, \\ (\Lambda + G)(u_{1,12} + u_{2,22}) + G(u_{2,11} + u_{2,22}) + \frac{J}{4}(\ddot{u}_{2,11} - \ddot{u}_{1,12}) \\ + \frac{1}{2}G\ell^2(u_{1,1222} - u_{2,1122} + u_{1,1112} - u_{2,1111}) = \rho\ddot{u}_2. \end{aligned}$$

78 In vector form, the governing equation of plane strain elastodynamics of couple  
 79 stress bodies with micro-inertia reads

$$(\Lambda + G) \text{grad}_2(\text{div}_2 \mathbf{u}) + G\hat{\Delta}\mathbf{u} + \frac{J}{4} \text{curl}_2(\text{curl}_2 \ddot{\mathbf{u}}) - \frac{1}{2}G\ell^2 \hat{\Delta} [\text{curl}_2(\text{curl}_2 \mathbf{u})] = \rho\ddot{\mathbf{u}}, \quad (8)$$

80 where  $\text{grad}_2$  and  $\text{div}_2$  indicate the gradient and the divergence in the  $x_1, x_2$   
 81 coordinates, respectively, whereas  $\text{curl}_2$  operates differently on a vector and on  
 82 a scalar, i.e.  $2\varphi_3 = \text{curl}_2 \mathbf{u} = u_{2,1} - u_{1,2}$  and  $\text{curl}_2 f = (-f_{,2}, f_{,1})$ . To the best  
 83 of the author's knowledge, Eq.(8) is original. It correctly particularises to the  
 84 expressions reported in Graff and Pao (1967) in the absence of micro-inertia.  
 85 Naturally, upon letting  $\ell = J = 0$ , we retrieve the classical Navier's equations.  
 86 Eq.(8) may be rewritten in the equivalent form

$$(\Lambda + 2G) \text{grad}_2(\text{div}_2 \mathbf{u}) + \frac{J}{4} \text{curl}_2(\text{curl}_2 \ddot{\mathbf{u}}) + G \left[ 1 - \frac{1}{2}\ell^2 \hat{\Delta} \right] \text{curl}_2(\text{curl}_2 \mathbf{u}) = \rho\ddot{\mathbf{u}}, \quad (9)$$

87 which, again in the absence of micro-inertia, reduces to Eq.(141) of Hadjesfan-  
 88 diari and Dargush (2011) (provided that our  $\ell$  is replaced by  $\sqrt{2}\ell$  and that we  
 89 understand  $\nabla \times \nabla \times \mathbf{u}$  as  $-\epsilon_{\alpha\beta\epsilon\gamma\delta}u_{\delta,\gamma\beta}$ ).

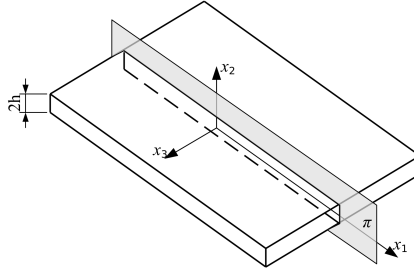


Figure 1: Plate in plane strain setting: the co-ordinate plane  $(x_1, x_3)$  is the plate mid-plane

90 *2.1. Boundary conditions*

91 We follow Koiter (1969) and introduce the *reduced force traction* vector,  $\mathbf{p}$ ,  
 92 and the *CS traction* vector,  $\mathbf{q}$ . For the latter, only the tangential part may be  
 93 prescribed at any point of a smooth surface with unit normal  $\mathbf{n}$ , the longitudinal  
 94 component being absorbed into the traction vector. Accordingly, we have

$$\mathbf{p} = \mathbf{t}^T \mathbf{n} + \frac{1}{2} \nabla \mu_{nn} \times \mathbf{n}, \quad (10a)$$

95 with  $\mu_{nn} = \mathbf{n} \cdot \boldsymbol{\mu} \mathbf{n}$ , and

$$\mathbf{q} = \boldsymbol{\mu}^T \mathbf{n} - \mu_{nn} \mathbf{n}. \quad (10b)$$

We now consider a flat plate in plane strain, see Fig.1. At the top/bottom face of the plate,  $x_2 = \pm h$ , the unit normal vector is  $\mathbf{n} = (0, \pm 1, 0)$  such that, by referring to Eqs.(10), we have ( $\mu_{nn} = 0$ )

$$p_1 = \pm t_{21} = \pm(\sigma_{21} + \tau_{21}), \quad (11a)$$

$$p_2 = \pm t_{22} = \pm \sigma_{22}, \quad (11b)$$

$$q_3 = \pm \mu_{23}. \quad (11c)$$

96 *2.2. Potentials*

97 We may express the displacement field by introducing the scalar potential  
 98  $\phi(x_1, x_2, x_3, t)$  and the vector potential  $\mathbf{H}(x_1, x_2, x_3, t)$ , such that (Graff and  
 99 Pao, 1967, Eq.(13))

$$\mathbf{u} = \text{grad } \phi + \text{curl } \mathbf{H}, \quad \text{with} \quad \text{div } \mathbf{H} = 0. \quad (12)$$

100 In particular, under plane strain conditions (3), we have

$$u_1 = \frac{\partial\phi}{\partial x_1} + \frac{\partial H}{\partial x_2}, \quad u_2 = \frac{\partial\phi}{\partial x_2} - \frac{\partial H}{\partial x_1}, \quad (13)$$

where we have let the shorthand  $H = H_3(x_1, x_2, t)$ , given that the other pair of components of  $\mathbf{H}$  vanish. We observe that the gauge condition  $\text{div } \mathbf{H} = 0$  is trivially satisfied. By substituting Eqs. (13) in (8) we obtain

$$G \left(1 - \frac{1}{2}\ell^2 \hat{\Delta}\right) \hat{\Delta}H = \rho \ddot{H} - \frac{J}{4} \hat{\Delta} \ddot{H}, \quad (14a)$$

$$(2G + \Lambda) \hat{\Delta}\phi = \rho \ddot{\phi}, \quad (14b)$$

101 which, in the absence of micro-inertia, correspond to Eqs.(3) of Sengupta and  
102 Ghosh (1974) and Eqs.(8) of Wang et al. (2017).

Boundary conditions (11) may now be rewritten as

$$p_1 = \pm \left[ G(H_{,22} + 2\phi_{,12} - H_{,11}) + \frac{J}{4} \hat{\Delta} \ddot{H} - \frac{1}{2} G \ell^2 \hat{\Delta}^2 H \right], \quad (15a)$$

$$p_2 = \pm \left[ \Lambda \hat{\Delta}\phi + 2G(\phi_{,22} - H_{,12}) \right], \quad (15b)$$

$$q_3 = \mp G \ell^2 \hat{\Delta} H_{,2}, \quad (15c)$$

103 where  $\hat{\Delta}^2 = \nabla^4$  indicates the bi-harmonic operator.

### 104 3. Travelling wave solutions

In order to develop dimensional-reduced models, we need a good understanding of wave propagation in a thin plate. We begin by looking at solutions in the form of travelling waves

$$\phi(x_1, x_2) = \Phi(x_2) \exp[i(kx_1 - \omega t)],$$

$$H(x_1, x_2) = \mathcal{H}(x_2) \exp[i(kx_1 - \omega t)].$$

105 These, plugged into Eqs.(14), lends a pair of ODEs for the amplitude functions  
106  $\Phi(x_2)$  and  $\mathcal{H}(x_2) = \mathcal{H}_1(x_2) + \mathcal{H}_2(x_2)$

$$\Phi'' - \frac{1}{\ell^2} \lambda_L^2 \Phi = 0, \quad \mathcal{H}_1'' - \frac{1}{\ell^2} \lambda_{z1}^2 \mathcal{H}_1 = 0, \quad \mathcal{H}_2'' - \frac{1}{\ell^2} \lambda_{z2}^2 \mathcal{H}_2 = 0, \quad (16)$$

107 whereupon  $\ell^{-1} \lambda_L$  and  $\ell^{-1} \lambda_{z1,2}$  are wavenumbers in the  $x_2$  direction.



108 We let the dimensionless frequency  $\Omega = \omega T$ , where  $T = \ell/\tilde{c}_S$  is a refer-  
 109 ence time and  $\tilde{c}_S = \sqrt{G/\rho}$  is the classical shear wavespeed. Also, we let the  
 110 dimensionless wavenumber  $\kappa = k\ell$ , the rotational inertia characteristic length  
 111  $\tilde{h} = \frac{1}{2}\sqrt{J/\rho}$ , as well as the dimensionless counterpart

$$\mathfrak{J} = \tilde{h}/\ell. \quad (17)$$

112 With this, it is

$$\lambda_L = \sqrt{\kappa^2 - \theta_L^2}, \quad \theta_L^2 = \Omega^2/\psi^2, \quad (18)$$

113 and

$$\lambda_{z1} = \sqrt{\kappa^2 - \theta_1^2}, \quad \lambda_{z2} = \sqrt{\kappa^2 + \theta_2^2}, \quad (19)$$

114 where

$$\theta_{1,2}^2 = \sqrt{(\mathfrak{J}^2\Omega^2 - 1)^2 + 2\Omega^2} \pm (\mathfrak{J}^2\Omega^2 - 1) \geq 0.$$

115  $\pm(\theta_L, \theta_1, \imath\theta_2)$  define the branch points of the square roots. In the absence of  
 116 rotational inertia,  $\theta_{1,2}$  reduce to  $\sqrt{2}\ell(p, q)$  of Sengupta and Ghosh (1974) and  
 117 to  $\sqrt{2}\ell\beta_{1,2}$  of Graff and Pao (1967), respectively (just substitute our  $\ell$  with  $\sqrt{2}\ell$ ).  
 118 For  $\kappa = 0$ , we get the so-called *thickness-stretch or thickness-shear resonance*  
 119 *frequencies* (see Fig.5), with real wavenumber  $\theta_L$  and  $\theta_1$ . We also have the  
 120 non-classical evanescent thickness mode with wavenumber  $\imath\theta_2$  (for a definition  
 121 of evanescent modes see (Graff, 2012, §1.5.4)).

We observe that the classical limit is obtained taking  $\ell \rightarrow 0$  and, if necessary,  
 $J = 0$ . This limit, in terms of  $\Omega$  and  $\kappa$ , amounts to taking  $\Omega \sim \kappa \ll 1$ ,  
 i.e. it is the long-wave low-frequency (LWLF) approximation. Within this  
 approximation, we get

$$\begin{aligned} \theta_1 &= \Omega - \frac{1}{2}(\frac{1}{2} - \mathfrak{J}^2)\Omega^3 + \dots \\ \theta_2 &= \sqrt{2} + \frac{1}{\sqrt{2}}(\frac{1}{2} - \mathfrak{J}^2)\Omega^2 + \dots, \end{aligned}$$

122 whence it is clear that, comparing with the classical results, for instance in the  
 123 notation of (Graff, 2012, §8), we have

$$\lambda_L^2 \rightarrow -\ell^2\alpha^2, \quad \lambda_1^2 \rightarrow -\ell^2\beta^2, \quad \lambda_2^2 \rightarrow 2. \quad (20)$$

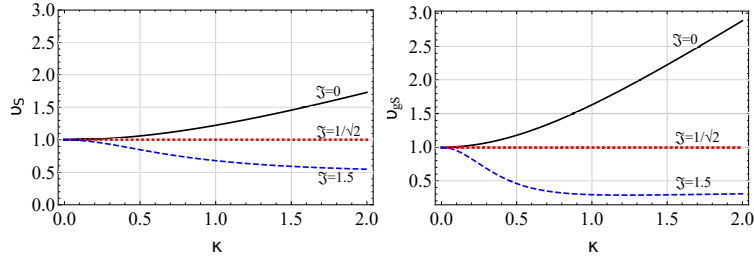


Figure 2: Phase (left) and group (right) speed of shear bulk waves, respectively Eq.(25) and (26), normalized over the classical wave speed,  $\tilde{c}_S$ . Propagation is regular inasmuch as  $\tilde{\mathfrak{J}} > \tilde{\mathfrak{J}}_{\text{crit}} = 1/\sqrt{2}$

124 This limit corresponds to Case I of Sengupta and Ghosh (1974), although there  
 125 it is claimed that the wavelength  $L = 2\pi/k$  is large compared to  $h$ , while it  
 126 really should be  $L \gg \ell$ .

127 We let the ratio between the longitudinal and the transversal wavespeed in  
 128 classical media  $\psi = \tilde{c}_L/\tilde{c}_S$ , with  $\tilde{c}_L = \sqrt{(2G + \Lambda)/\rho}$ . It is important to observe  
 129 that  $\theta_1 > \theta_L$  demands

$$0 < \Omega < \psi \sqrt{\frac{\psi^2 - 1}{\frac{1}{2} - \tilde{\mathfrak{J}}^2 \psi^2}}, \quad (21)$$

130 provided

$$\tilde{\mathfrak{J}} < \frac{1}{\sqrt{2}\psi}, \quad (22)$$

131 otherwise  $\theta_1 > \theta_L$  always.

132 The general solutions of Eqs.(16) are

$$\Phi(\xi_2) = \frac{A_1}{\lambda_L} \sinh(\lambda_L \xi_2) + A_2 \cosh(\lambda_L \xi_2), \quad (23a)$$

$$\mathcal{H}(\xi_2) = \frac{B_1}{\lambda_{z1}} \sinh(\lambda_{z1} \xi_2) + B_2 \cosh(\lambda_{z1} \xi_2) + \frac{C_1}{\lambda_{z2}} \sinh(\lambda_{z2} \xi_2) + C_2 \cosh(\lambda_{z2} \xi_2), \quad (23b)$$

134 where we have let the dimensionless transversal coordinate  $\xi_2 = x_2/\ell$ .

### 135 3.1. Bulk waves

136 Bulk waves are travelling homogeneous waves, whence we get the dimension-  
 137 less speed of longitudinal waves setting  $\lambda_L = 0$

$$v_L = \psi, \quad (24)$$

138 which is obviously non-dispersive. Similarly, we get the wavespeed of shear  
 139 waves

$$v_S = \sqrt{1 + \frac{\frac{1}{2} - \mathfrak{J}^2}{1 + \mathfrak{J}^2 \kappa^2} \kappa^2}, \quad (25)$$

140 that generalizes the result (Ottosen et al., 2000, Eq.(33)), obtained in the ab-  
 141 sence of micro-inertia. Depending on the latter, two regimes are possible: when  
 142  $\mathfrak{J} \leq \mathfrak{J}_{\text{crit}} \equiv 2^{-1/2}$ , the shear bulk speed is greater/lesser than the classical bulk  
 143 wave speed  $\tilde{c}_S$ , while the particular case  $\mathfrak{J} = \mathfrak{J}_{\text{crit}}$  corresponds to non-dispersive  
 144 waves. This behaviour has already been observed for antiplane shear waves  
 145 Nobili et al. (2019). The bulk shear wave speed asymptotes to

$$v_{S\text{lim}} = \frac{\mathfrak{J}_{\text{crit}}}{\mathfrak{J}},$$

146 which may be greater than the longitudinal wave speed  $v_L = \psi$  if (22) holds. We  
 147 conclude that shear waves are generally slower than bulk waves, unless condition  
 148 (22) holds, in which case they still are but only in the frequency range expressed  
 149 by (21). Therefore, the case where micro-inertia disappears is very special, in  
 150 that shear waves are always dispersive and they eventually become faster than  
 151 longitudinal waves, see Fig.2.

152 Looking at group velocity

$$V_{gS} = v_S + \frac{\mathfrak{J}_{\text{crit}}^2 - \mathfrak{J}^2}{v_S(1 + \mathfrak{J}^2 \kappa^2)^2} \kappa^2, \quad (26)$$

we see that it is always positive. Besides, *propagation is regular* (i.e. group  
 velocity is less than phase speed) inasmuch as  $\mathfrak{J} > \mathfrak{J}_{\text{crit}}$ , see Fig.2. The corre-  
 sponding dimensional speeds easily follow through

$$c = \frac{\omega}{k} = \frac{\Omega}{\kappa} \tilde{c}_S = v \tilde{c}_S,$$

153 whereupon we see that the phase speed of longitudinal waves  $c_L$  corresponds  
 154 to the classical result  $\tilde{c}_L$ . In the absence of micro-inertia, longitudinal (24)  
 155 and transversal (25) wave speeds match the corresponding results (Sharma and  
 156 Kumar, 2014, Eq.(7)). Dimensionalizing factors have been chosen such that in  
 157 the classical limit, that is for  $J$  and  $\ell$  tending to zero, we rightly get  $v_S \rightarrow 1$ .

158 We conclude that, in CS materials, two bulk wave speeds are admitted, one  
 159 longitudinal and one transversal, just like in classical media, the difference being  
 160 that the latter is now dispersive and supports two families of thickness-stretch  
 161 modes. In fact, when it comes to partial waves (and equally localized waves,  
 162 Rayleigh-Lamb waves and all forms of non-uniform waves), shear waves comes  
 163 in family pairs.

#### 164 4. Rayleigh waves

Assuming an exponentially decaying solution and imposing free boundary conditions

$$p_1(x_1, 0, t) = p_2(x_1, 0, t) = q_3(x_1, 0, t) \equiv 0,$$

165 we obtain the Rayleigh equation in CS elasticity with micro-inertia

$$R(\kappa, \Omega) = 0.$$

Here, we let

$$\begin{aligned} d_0 &= -4\kappa^2 (\theta_1^2 + \theta_2^2), \\ d_1 &= \theta_2^2 (\Omega^2 - 2\kappa^2)^2, \\ d_2 &= -\theta_1^2 (\Omega^2 - 2\kappa^2)^2, \end{aligned}$$

166 together with the Rayleigh function

$$R(\kappa, \Omega) = \frac{d_0 \lambda_L \lambda_{z1} \lambda_{z2} + d_1 \lambda_{z2} - d_2 \lambda_{z1}}{\lambda_{z1} - \lambda_{z2}}. \quad (27)$$

This function should be compared with Eq.(75) Graff and Pao (1967) of which it is a generalization in that (i) it encompasses for micro-inertia and (ii) it extracts the factor  $\lambda_{z1} - \lambda_{z2}$  which cancels out spurious branch-cuts. The Rayleigh function may be rewritten as

$$R_0(\kappa, \Omega) = (\Omega^2 - 2\kappa^2)^2 \left[ 1 - \frac{\theta_1^2}{\lambda_{z2} (\lambda_{z1} + \lambda_{z2})} \right] - 4\kappa^2 \lambda_L \lambda_{z1}.$$

167 which, in the limit as  $\ell$  goes to zero (and letting  $J = 0$ ), corresponds, at leading  
 168 order, to the classical Rayleigh function. Indeed, we have  $\kappa \sim \Omega \rightarrow 0$  and

$$R_0(\kappa, \Omega) = (\lambda_{z1}^2 + \kappa^2)^2 - 4\kappa^2 \lambda_L \lambda_{z1} + O(\Omega^6).$$

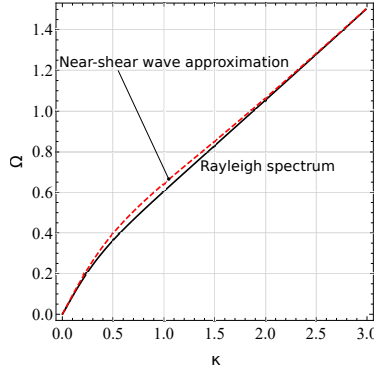


Figure 3: Frequency spectrum of Rayleigh waves (solid, black) superposed onto the approximation (28) (dashed, red) for the parameter set  $\psi = 1.5$ ,  $\mathfrak{J} = 1.5$

169 Rayleigh wave are perturbations of bulk shear waves and, as such, they may  
 170 be sought in their close neighbourhood: for  $\theta_1 > \theta_L$  we have

$$\kappa_R^2 = \theta_1^2 + \frac{(\theta_1^2 + \theta_2^2) \theta_2^4 (\Omega^2 - 2\theta_1^2)^4}{\theta_1^4 (4\theta_1^4 - 4\theta_1^2 (S + \Omega^2) - 4\theta_2^2 S + \Omega^4)^2}, \quad (28)$$

171 where we let the shorthand  $S = \sqrt{(\theta_1^2 + \theta_2^2)(\theta_1^2 - \theta_L^2)}$ . The quality of this  
 172 approximation is excellent at both ends of the wavelength spectrum (i.e. for  
 173 both short and long waves), as illustrated in Fig.3.

## 174 5. Rayleigh-Lamb waves

For Rayleigh-Lamb (R-L) waves, we now impose free boundary conditions

$$p_1(x_1, \pm h, t) = p_2(x_1, \pm h, t) = q_3(x_1, \pm h, t) \equiv 0.$$

175 where  $h$  is the plate half-thickness, and consider symmetric and anti-symmetric  
 176 waves separately.

### 177 5.1. Symmetric waves

178 For symmetric waves we obtain the frequency equation

$$D_s = d_0 \lambda_L \tanh(H \lambda_L) + \frac{d_1}{\lambda_{z1}} \tanh(H \lambda_{z1}) - \frac{d_2}{\lambda_{z2}} \tanh(H \lambda_{z2}), \quad (29)$$

179 being  $H = h/\ell$ . It is straightforward matter to recognize that, in either the  
 180 short-wave (SW) or high-frequency (HF) limit, respectively  $\kappa \gg 1$  or  $\Omega \gg 1$ ,

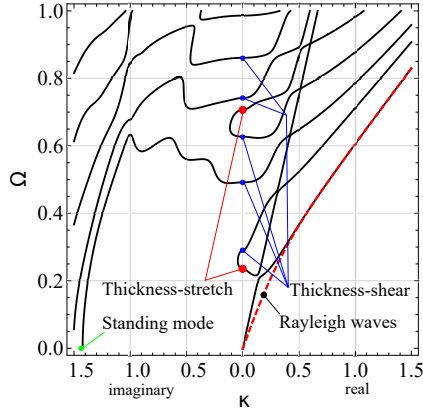


Figure 4: Frequency spectrum of symmetric Rayleigh-Lamb waves (black, solid) and Rayleigh waves (red, dashed) with the parameters  $\psi = 1.5$ ,  $\mathfrak{J} = 1.5$  and  $H = 10$ . Clearly, the fundamental mode quickly asymptotes to Rayleigh waves. Thickness-stretch (red, large dots), thickness-shear (blue, small dots) cut-offs and standing modes (green dots) are also shown.

181  $D_s$  reduces to the numerator of the Rayleigh function (27), whence the first  
 182 branch in the spectrum asymptotes to the Rayleigh branch. In fact, in such  
 183 limit, the phase speed of all branches tends to that of Rayleigh waves, and this  
 184 is in contrast to CE, where branches other than the first tend to the bulk shear  
 185 wave speed. Similarly, in the classical limit (20), we have, at leading order,

$$d_0 = -8\ell^2 k^2, \quad d_1 = 2\ell^4 \left( 2k^2 - \frac{\omega^2}{c_S^2} \right)^2 = 2\ell^4 (k^2 - \beta^2)^2, \quad d_2 = O(\ell^6),$$

186 whence Eq.(29) reduces to the classical result (Graff, 2012, Eq.(8.1.54))

$$8\ell^3 k^2 \alpha \tan(h\alpha) + \frac{2\ell^3 (k^2 - \beta^2)^2}{\beta} \tan(h\beta) + O(\ell^6) = 0.$$

187 Eq.(29) does not correspond to (16) of Sengupta and Ghosh (1974) because  
 188 the latter is obtained taking a linear approximation in  $\ell$ . However, it is observed  
 189 that, owing to the scaling,  $\ell$  never appears explicitly in Eq.(29). This means  
 190 that, within the natural scaling and in plane-strain conditions, the couple-stress  
 191 problem becomes self-similar. The frequency spectrum for symmetric waves is  
 192 drawn in Fig.4.

193 Cut-on frequencies (below which the wave is evanescent and no longer prop-

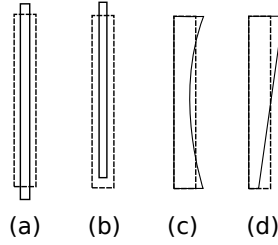


Figure 5: Thickness modes: (a) symmetric thickness-stretch, (b) antisymmetric thickness-stretch, (c) symmetric thickness-shear and (d) antisymmetric thickness-shear

agate Bhaskar (2009)) are determined setting  $\kappa = 0$  in (29), which lends

$$\cos(H\theta_L) \left[ \frac{\theta_2^3 \tan(H\theta_1)}{\theta_1^3 \tanh(H\theta_2)} + 1 \right] = 0. \quad (30)$$

The presence of two factors is a consequence of the block structure of the linear system. Consequently, there are two families of modes (Graff, 2012, §8.7.5):

1. *thickness-stretch modes*, related to the first factor in (30),

$$\Omega_{sn} = \left(\frac{1}{2} + n\right) \pi \frac{\psi}{H}, \quad n = 0, 1, 2, \dots, \quad (31)$$

which are purely classical, cf.(Kaplunov et al., 1998, Eq.(1.2.22)) and (Graff, 2012, Eq.(8.1.99)). For these modes,  $A_2$  is arbitrary and  $B_1 = C_1 = 0$ , i.e. the displacement is *irrotational*. In this case,  $u_1 \equiv 0$  for  $\kappa = 0$  and indeed this is thickness-stretch, as in Fig.5(a).

2. *thickness-shear modes*, which are non-classical. For such modes,  $A_2 = 0$ , whence  $\phi \equiv 0$  and the displacement field is *solenoidal* according to the decomposition (12), with rotational potential

$$\mathcal{H}(\xi_2) = \left( \frac{\lambda_{z2}^2 \sinh(\lambda_{z1}\xi_2)}{\lambda_{z1} \cosh(\lambda_{z1}H)} - \frac{\lambda_{z1}^2 \sinh(\lambda_{z2}\xi_2)}{\lambda_{z2} \cosh(\lambda_{z2}H)} \right) R,$$

where  $R$  is a free parameter. Displacement easily follows from Eqs.(13), which give  $u_2 \equiv 0$  at cut-off (because  $\kappa = 0$ ) and confirm that this is indeed thickness-shear, as in Fig.5(c)

We observe that, for both thickness mode families, cut-on frequencies are  $O(H^{-1})$ .

Standing modes, i.e. wave solutions for which  $\Omega = 0$ , are given by either  $\kappa = 0$ , corresponding to rigid body motions, or by the solutions of the transcendental

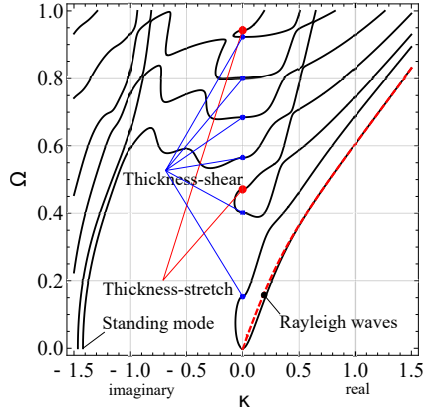


Figure 6: Frequency spectrum of antisymmetric (flexural) Rayleigh-Lamb waves (black, solid) superposed onto Rayleigh waves (red, dashed). The fundamental mode quickly asymptotes to Rayleigh waves. Antisymmetric thickness-shear (blue, small dots) and thickness-stretch (red, big dots) are shown, for the parameter set  $\psi = 1.5$ ,  $\mathfrak{J} = 1.5$  and  $H = 10$ .

equation in  $k_2 = \mathfrak{S}(k)$

$$2Hk_2(\psi^2 - 1) - ((k_2^2 - 1)\psi^2 + 1)\sin(2Hk_2) + \frac{2k_2^3\psi^2}{\sqrt{2 - k_2^2}}\cos^2(Hk_2)\tanh\left(H\sqrt{2 - k_2^2}\right) = 0.$$

Assuming  $H^{-1} \ll 1$ , these may be found as perturbations of classical standing modes

$$\kappa = \pm i\sqrt{2 + \left(\frac{1}{2} + j\right)^2 \frac{\pi^2}{H^2}} + O(H^{-2}), \quad j = 0, 1, 2, \dots$$

206 Standing modes are associated with a displacement field where  $C_1 = 0$  and  $A_2$   
 207 is linearly proportional to  $B_1$ . Unlike cut-ons, standing modes are  $O(1)$  even  
 208 when  $H$  is large and cannot be captured in a long-wave approximation.

### 209 5.2. Antisymmetric waves

210 Similarly, for anti-symmetric waves, we get

$$D_o = d_0\lambda_L \coth(H\lambda_L) + \frac{d_1}{\lambda_{z1}} \coth(H\lambda_{z1}) - \frac{d_2}{\lambda_{z2}} \coth(H\lambda_{z2}), \quad (32)$$

211 whose first branch, once again, collapses into Rayleigh waves (27) in the short-  
 212 wave regime, see Fig.6. Proceeding as for symmetric waves, it is easily shown



213 that, in the classical limit, the frequency equation (32) reduces to the well-known  
 214 result

$$-8\ell^3 k^2 \alpha \tan(h\alpha) + \frac{2\ell^3 (k^2 - \beta^2)^2}{\beta} \tan(h\beta) + O(\ell^6) = 0.$$

215 Cut-on frequencies are given by

$$\sin(H\theta_L) \left[ \frac{\theta_2^3 \cot(H\theta_1)}{\theta_1^3 \coth(H\theta_2)} - 1 \right] = 0, \quad (33)$$

216 whence, again, two families of modes arise

- 217 1. *antisymmetric thickness-stretch modes*, purely classical, with frequency  
 218 (Kaplunov et al., 1998, Eq.(1.2.25))

$$\Omega_{on} = n\pi \frac{\psi}{H}, \quad n = 0, 1, 2, \dots$$

219 for which  $A_1$  is arbitrary and  $B_2 = C_2 = 0$ , i.e. the displacement is  
 220 *irrotational*. The last feature accounts for the fact that  $u_1 \equiv 0$  at cut-off  
 221 and this is thickness-stretch, as in Fig.5(b).

2. *antisymmetric thickness-shear modes*, non-classical, for which  $A_1 = 0$  and  
 the displacement field is *solenoidal*, with potential

$$\mathcal{H}(\xi_2) = \left( \frac{\lambda_{z2}^3 \cosh(\lambda_{z1}\xi_2)}{\lambda_{z1} \sinh(\lambda_{z1}H)} - \frac{\lambda_{z1}^3 \cosh(\lambda_{z2}\xi_2)}{\lambda_{z2} \sinh(\lambda_{z2}H)} \right) R,$$

222 where  $R$  is a free parameter. For a solenoidal field, only  $u_1$  can be non-  
 223 zero at cut-off, and again we have thickness-shear, although this time in  
 224 antisymmetric fashion, as in Fig.5(d).

225 Standing modes are given by

$$\kappa = \pm i \sqrt{2 + n^2 \frac{\pi^2}{H^2}}, \quad n = 1, 2, \dots, \quad (34)$$

226 and are non-classical. They are associated with eigenforms where  $B_2, C_2$  are  
 227 linear functions of  $A_1$ .

## 228 6. Construction of the asymptotic models

229 Asymptotic models are constructed on the assumption that the wavelength  
 230 of interest is much greater than the microstructural parameter  $\ell$ , i.e.  $\kappa \ll 1$ , and

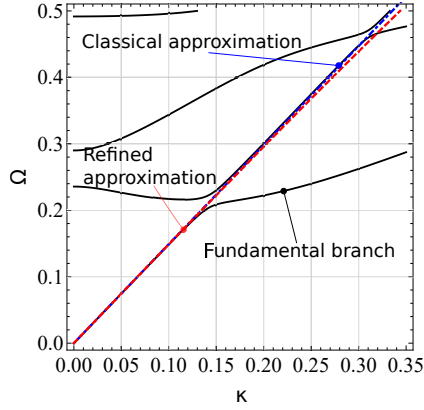


Figure 7: Frequency spectrum of symmetric Rayleigh-Lamb waves superposed onto the leading order (blue, dot-dashed) and first correction approximation (red, dashed) for the parameter set  $\psi = 1.5$ ,  $\mathfrak{J} = 1.5$  and  $H = 10$ .

231 that the time scale of wave propagation is much greater than  $T$ , i.e.  $\Omega \ll 1$ . We  
 232 emphasize that this not necessarily corresponds to the classical limit, whereby  
 233  $\ell \rightarrow 0$ . In the following, unless otherwise stated, we assume  $H \sim 1$ .

#### 234 6.1. Extensional plate model

Looking at the frequency spectrum of the symmetric fundamental mode, we see that  $\Omega \sim \kappa$  and we write the LWLF approximation

$$D_s = \Omega^2 - 4(1 - \psi^{-2})\kappa^2 + O(\kappa^4)$$

235 corresponding to classical longitudinal waves in a plate

$$\square_{P1} u = c_{P1}^{-2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} = 0, \quad (35)$$

236 being  $\square_{P1}$  the wave (or D’Alambert’s) operator with wave speed  $c_{P1}$  (sometimes  
 237 denoted  $c_3$ )

$$c_{P1}^2 = 4\tilde{c}_S^2(1 - \psi^{-2}) = \frac{E}{\rho(1 - \nu^2)} = \frac{2}{1 + \nu}\tilde{c}_S^2,$$

where  $E = G(2G + 3\Lambda)/(G + \Lambda)$  is Young’s modulus and  $\nu = \frac{1}{2}\Lambda/(\Lambda + G)$  Poisson’s ratio. The wave phase speed  $c_{P1} > \tilde{c}_S$  corresponds to that of classical plate theory, see (Graff, 2012, Eq.(8.3.57)), and indeed Eq.(35) is the plane-strain counterpart of the classical equation of extensional vibrations of thin plates,

originally developed by Poisson (1829), Cauchy (1828) and Filon (Mindlin and Yang, 2006, Eq.(4.043))

$$\square_{P1} u = \frac{1 - \nu^2}{2hE} b_1,$$

where  $b_1$  is the longitudinal force acting on the plate. This linear approximation (of the dispersion relation) is very robust (see Fig.7) and classical, for it is independent of  $\ell$ ,  $H$  and  $\mathfrak{J}$ . As it might be expected, rotational inertia appears when accounting for the first order correction terms in the asymptotic expansion of the frequency equation. Indeed, we have

$$\begin{aligned} D_s &= \Omega^2 - 4(1 - \psi^{-2})\kappa^2 \\ &+ \frac{(1 - 2\psi^{-2})^2}{4(1 - \psi^{-2})} \left[ \frac{4}{3}(1 - \psi^{-2})H^2 + \frac{\sqrt{2}}{H} \tanh(\sqrt{2}H) - 2 \right] \Omega^2 \kappa^2 + \dots \end{aligned}$$

238 whence we obtain the asymptotic model

$$\frac{EA}{1 - \nu^2} \frac{\partial^2 u}{\partial x_1^2} - \mathfrak{J}_{\text{tg}} u = 0, \quad (36)$$

where we have defined the operator of *modified tangential inertia* in analogy with (Kaplunov et al., 1998, §7)

$$\mathfrak{J}_{\text{tg}} = \rho A \left\{ 1 - \frac{\nu^2}{2(1 - \nu)} \left[ \frac{2}{3(1 - \nu)} h^2 - \ell^2 \left( 2 - \sqrt{2} \frac{\ell}{h} \tanh \frac{\sqrt{2}h}{\ell} \right) \right] \frac{\partial^2}{\partial x_1^2} \right\} \frac{\partial^2}{\partial t^2}.$$

239 Remarkably, the contribution of the microstructure affects plate elongation by  
240 *reducing rotational inertia* (that is the term with the mixed derivative), inde-  
241 pendently of  $J$ . Indeed, the term in round bracket is always positive and ranges  
242 from 0 to 2, respectively at  $H = 0$  and as  $H \rightarrow \infty$ . As a result, the fundamental  
243 spectrum *decays less* than in the classical situation.

In the classical limit  $\ell \rightarrow 0$ , we get

$$\frac{EA}{1 - \nu^2} \frac{\partial^2 u}{\partial x_1^2} - I_{\text{tg}} u = 0,$$

244 where  $A = 2h$  is the transversal thickness of the plate and

$$I_{\text{tg}} = \rho A \left( 1 - h^2 \frac{\nu^2}{3(1 - \nu)^2} \frac{\partial^2}{\partial x_1^2} \right) \frac{\partial^2}{\partial t^2},$$

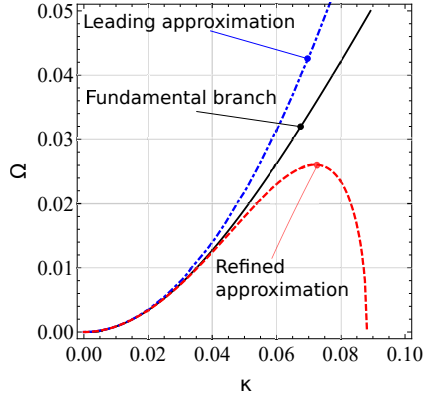


Figure 8: Frequency spectrum of antisymmetric (flexural) Rayleigh-Lamb waves superposed onto the leading order (green, dashed) and first correction approximation (red, dotted) for the parameter set  $\psi = 1.5$ ,  $\mathfrak{J} = 1.5$  and  $H = 10$ .

245 corresponds to the result given in (Kaplunov et al., 1998, Eq.(7.4.3)) in the  
 246 context of classical elasticity.

247 The asymptotically consistent result (36) differs from any engineering theory  
 248 of plate extensional vibrations. For example, by analogy with (Kaplunov et al.,  
 249 1998, Eq.(7.4.3)), we can easily generalize our result outside plane strain

$$\frac{EA}{1-\nu^2} \left( \frac{1}{1+\nu} \hat{\Delta} + \frac{1}{1-\nu} \text{grad}_2 \text{div}_2 \right) \mathbf{u} - \mathfrak{J}_{\text{tg}} \mathbf{u} = \mathbf{o}, \quad (37)$$

250 and this equation is in disagreement with Eqs.(16a,b) of Jomehzadeh et al.  
 251 (2011). In fact, they differ by the very way microstructure operates: by affect-  
 252 ing elastic stiffness and disregarding inertia there, the opposite occurring here.  
 253 Interestingly, models agree to leading order under plane strain.

254 We now consider the limit  $H \rightarrow 0$  of a plate whose thickness is much smaller  
 255 than the microstructural length  $\ell$ . Then, we have

$$\frac{EA}{1-\nu^2} \frac{\partial^2 u}{\partial x_1^2} - \rho A \left\{ 1 + \frac{\nu^2(1-2\nu)}{3(1-\nu)^2} h^2 \frac{\partial^2}{\partial x_1^2} \right\} \frac{\partial^2 u}{\partial t^2} = 0, \quad (38)$$

256 and rotational inertia operates in the same direction as translational inertia, i.e.  
 257 the spectrum grows. Remarkably, in this limit,  $\ell$  drops out.

258 *6.2. Flexural plate model*

As in the classical situation, we seek for a quadratic approximation for the fundamental flexural branch  $\Omega \sim \kappa^2 \ll 1$

$$D_o = \Omega^2 - 2 \left[ 1 + \frac{2}{3} H^2 (1 - \psi^{-2}) \right] \kappa^4 + O(\kappa^6),$$

corresponding to dispersive waves in a flexural plate

$$\mathfrak{L}w = \left[ \rho A \frac{\partial^2}{\partial t^2} + (D^\ell + D) \frac{\partial^4}{\partial x_1^4} \right] w = 0,$$

259 where  $\mathfrak{L}$  is the biharmonic wave operator,  $D = \frac{2}{3} \frac{Eh^3}{1-\nu^2}$  is the classical flexural  
 260 rigidity of the plate and  $D^\ell = 2AG\ell^2$  is the contribution of rotation gradients  
 261 to bending rigidity. It is worth mentioning that the lack of the factor 2 in the  
 262 expression for  $D^\ell$  given in (Tsiatas, 2009, Eq.(23b)) is a consequence of the fact  
 263 that the modified couple stress theory is adopted, whose curvature tensor off-  
 264 diagonal components are half those given by the couple stress theory. Also, we  
 265 note that  $D^\ell = 3(1-\nu)DH^{-2}$ , whereupon this term is significant when  $H \sim 1$ .  
 266 In contrast, for  $H^{-1} \ll 1$ , the classical term dominates.

Micro-inertia appears when first order correction terms are incorporated

$$D_o = \Omega^2 - 2 \left[ 1 + \frac{2}{3} H^2 (1 - \psi^{-2}) \right] \kappa^4 + \left[ 2 + 4\mathfrak{J}^2 + \frac{3}{2H^2(1-\psi^{-2})+3} \times \right. \\ \left. \times \left( \frac{2}{45} (1 - \psi^{-2}) (27 - 20\psi^{-2}) H^4 - 2 + \frac{H}{\sqrt{2}} \coth(\sqrt{2}H) \right) \right] \kappa^2 \Omega^2 + \dots,$$

267 which corresponds to the asymptotic model

$$(D^\ell + D) \frac{\partial^4 w}{\partial x_1^4} + \mathfrak{I}_{\text{tr}} w = 0. \quad (39)$$

where we have defined the operator of *modified transverse inertia*

$$\mathfrak{I}_{\text{tr}} = \rho A \left\{ 1 - \left[ 2\ell^2 + \frac{J}{\rho} + \frac{D^\ell}{D + D^\ell} \times \right. \right. \\ \left. \left. \left( \frac{17 - 7\nu}{45(1-\nu)^2} \frac{h^4}{\ell^2} - \ell^2 \left( 2 - \frac{h}{\sqrt{2}\ell} \coth \frac{\sqrt{2}h}{\ell} \right) \right) \right] \frac{\partial^2}{\partial x_1^2} \right\} \frac{\partial^2}{\partial t^2}.$$

Here, microstructure adds to rotational inertia directly through  $2\ell^2 + J/\rho$ , but it also contribute to/subtract from the classical term inasmuch as  $H \gtrsim 2.8265$ . Eqs.(36) and (40) extend to microstructured plates the *classical Kirchhoff-Love theory with modified inertia* presented in Kaplunov et al. (1998). Indeed, in the limit as  $\ell \rightarrow 0$  and setting  $J = 0$ , we get

$$D \frac{\partial^4 w}{\partial x_1^4} + I_{\text{tr}} w = 0,$$

268 where  $I_{\text{tr}}$  is the operator defined in (Kaplunov et al., 1998, Eq.(7.4.2)) for clas-  
269 sical elasticity and a general situation of strain (in contrast to plane-strain)

$$I_{\text{tr}} = \rho A \left( 1 - h^2 \frac{17-7\nu}{15(1-\nu)} \frac{\partial^2}{\partial x_1^2} \right) \frac{\partial^2}{\partial t^2}.$$

When we consider the first dispersion branch of the Timoshenko theory (Bhaskar, 2009, Eq.(2.7))

$$D_T = \Omega^2 - \frac{4}{3} H^2 (1 - \psi^{-2}) \kappa^4 \left( 1 - (1 + \varsigma) \frac{I_2}{A} \kappa^2 \right), \quad I_2 = \frac{2}{3} h^3 = \frac{1}{3} A h^2,$$

270 we immediately see that the contribution of rotational inertia differs from that  
271 of the Kirchhoff-Rayleigh theory by the factor  $1 + \varsigma$ , where  $\varsigma = E/(\varkappa_T G) =$   
272  $2(1 + \nu)\varkappa_T^{-1}$  and  $\varkappa_T$  is the shear correction factor. Assuming, with Bhaskar  
273 (2009), that  $\varsigma \approx 4$ , the final contribution is close to the asymptotic result for  
274  $\nu = \frac{1}{2}$ .

275 Extension outside plane strain is straightforward

$$(D^\ell + D) \nabla^4 w + \mathfrak{I}_{\text{tr}} w = 0, \quad (40)$$

276 where  $\nabla^4$  is the bi-harmonic operator and  $\mathfrak{I}_{\text{tr}}$  is generalized with the rotational  
277 inertia term  $\hat{\Delta} \frac{\partial^2}{\partial t^2}$ . Again, this result is at variance with the engineering theo-  
278 ries. In particular, in the classical limit, it does not correspond to the classical  
279 Kirchhoff-Rayleigh theory of plates

$$\left[ D \nabla^4 + \rho A \left( 1 - \frac{I_2}{A} \hat{\Delta} \right) \frac{\partial^2}{\partial t^2} \right] w = 0,$$

280 by the factor  $\frac{12}{5} \leq \frac{17-7\nu}{5(1-\nu)} \leq \frac{27}{5}$  in the rotational inertia term. Even more  
281 it disagrees with the engineering theory with size-dependence. For example,

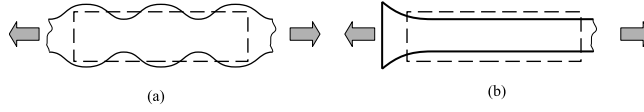


Figure 9: Thickness-stretch modes in a plate: real wavespeed (a) and evanescent mode for a semi-infinite domain (b)

282 according to (Jomehzadeh et al., 2011, Eq.(16c)), microstructure only affects  
 283 the plate flexural rigidity and contributes nothing to rotational inertia, that is  
 284 purely geometrical, just as in the classical case of Kirchhoff-Rayleigh plates. The  
 285 same occurs, among many, in Ma et al. (2008) and in the Mindlin-like model  
 286 Ma et al. (2011).

287 Finally, we consider the limit  $H \rightarrow 0$ , corresponding to the microstructure-  
 288 dominated regime. Then, we find

$$D^\ell \frac{\partial^4 w}{\partial x_1^4} + \rho A \left\{ 1 - \left( \frac{1}{2} \ell^2 + \frac{J}{\rho} \right) \frac{\partial^2}{\partial x_1^2} \right\} \frac{\partial^2 w}{\partial t^2} = 0, \quad (41)$$

289 which is independent of  $h$ . It so appears that, in this limit, flexural and elonga-  
 290 tional waves depend in disjoint fashion on  $\ell$  and  $h$ , respectively.

## 291 7. High order modes

292 We now consider the second branch of the spectrum (the first overtone) and  
 293 its leading order approximation for  $\Omega, \kappa \ll 1$ . This limit, which is named *long-*  
 294 *wave high-frequency* (Kaplunov et al., 1998, §1.2), is capable of capturing the  
 295 first cut-on frequency inasmuch as  $\epsilon \sim \Omega \ll 1$  and, as it will be presently seen,  
 296  $\epsilon = H^{-1}$  for elongation or  $\epsilon = \sqrt{H^2 + \frac{\pi^2}{4} \mathfrak{J}^2}$  for flexural vibrations.

### 297 7.1. Symmetric waves

298 For symmetric waves, we assume  $\Omega \sim \kappa \sim \epsilon$  and find the expansion

$$D_{2s} = \Omega^2 - \psi^2 \varkappa_s \kappa^2 - \Omega_{s0}^2, \quad (42)$$

299 where  $\Omega_{s0}$  is the first cut-on frequency for symmetric waves (31) and refers to a  
 300 thickness-stretch mode. In terms of the reduced model, this mode corresponds  
 301 to squeezing and thickening of the cross-section, see Fig.9. The coefficient  $\varkappa_s$

302 may be given in closed form, but its expression is cumbersome and it is not  
 303 reported here. Instead, we write it in the form of an asymptotic expansion in  $\epsilon$ ,  
 304 namely  $\varkappa_s = \varkappa_s^{(0)} + \epsilon^2 \varkappa_s^{(2)} + \dots$ , and find

$$\varkappa_s = 1 + \frac{16}{\pi\psi^3} \cot\left(\frac{1}{2}\pi\psi\right) + \dots, \quad (43)$$

305 which corresponds to (Kaplunov et al., 1998, Eq.(1.2.23)). As expected, mi-  
 306 crostructural effects only appear as first order correction terms. It is worth  
 307 emphasizing that two regimes are possible: for  $\varkappa_s^{(0)} > 0$ ,  $\Omega_{s0}^2$  is really a cut-on  
 308 and we have the wave equation

$$\left[ \square_{2s} + \frac{\pi^2}{4h^2 \varkappa_s^{(0)}} \right] u = 0, \quad (44)$$

where the wave operator  $\square_{2s}$  is connected to the real wave speed  $c_{2s} = \tilde{c}_L \sqrt{\varkappa_s^{(0)}}$ .  
 In this regime, the first branch describes a travelling thickness-stretch deforma-  
 tion mechanism as a taut string with the addition of pre-stress. In contrast,  
 for  $\varkappa_s^{(0)} < 0$  (that first occurs in the range  $1.21612 < \psi < 2$ ),  $\Omega_{s0}^2$  is locally  
 a cut-off and we obtain a decaying (evanescent) mode, which is important for  
 semi-infinite or finite domains, Fig.9(b) Nobili et al. (2020). Thus, an asymp-  
 totically consistent theory accounting for the first two branches of symmetric  
 waves is purely classical

$$\left( \square_{2s} + \frac{\pi^2}{4h^2 \varkappa_s^{(0)}} \right) \square_{P1} u = 0,$$

309 to leading order in  $\epsilon$ .

310 When moving to first correction terms, it may be observed that the coefficient  
 311 of  $\kappa^4$  in the expansion of the frequency equation is  $O(\epsilon^2)$  and therefore simply  
 312 the correction term

$$\varkappa_s^{(2)} = \frac{\pi}{\psi} \left[ \pi \left( \frac{1}{2} - \mathfrak{J}^2 \right) \psi + \left( \mathfrak{J}^2 + \frac{1}{2} \right) \sin(\pi\psi) \right] \csc^2 \frac{\pi\psi}{2} \quad (45)$$

313 needs to be considered in (43). The quality of this refined approximation is  
 314 very good and, in Fig.10, it is compared with the leading order approximation  
 315  $\varkappa_s^{(0)}$  and the full coefficient  $\varkappa_s$ . It is most interesting to observe that there are  
 316 special values for  $\psi$ , the first of which being  $\psi = 1.21612$ , such that  $\varkappa_s^{(0)} = 0$   
 317 and the correction term (45) becomes leading order.



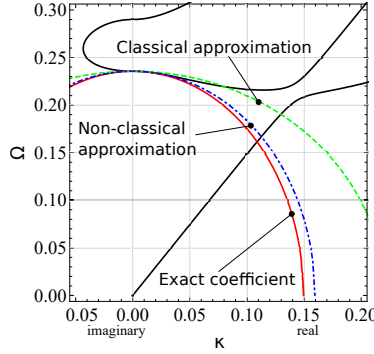


Figure 10: Frequency spectrum for symmetric R-L waves (solid, black) superposed onto the approximation (42) taking the leading order term (43) (green, dashed), the first correction approximation  $\varkappa_s^{(0)} + \epsilon^2 \varkappa_s^{(2)}$  (blue, dot-dashed) and the full coefficient  $\varkappa_s$  (red, solid). It is clear that the first order approximation is very efficient. Here,  $1.21612 < \psi = 1.5 < 2$  and we see spectrum decay, which means that this is an evanescent mode for the plate ( $H = 10, \mathfrak{J} = 1.5$ )

### 318 7.2. Antisymmetric waves

319 We now consider the second branch for antisymmetric waves, which is an  
 320 antisymmetric thickness-shear mode. In terms of the cross-section of a plate  
 321 model, this resembles shear-warping in the fashion considered by Timoshenko  
 322 theories. Although now an explicit solution for the cut-on frequency equation  
 323 (33) is not available, we can observe that, for  $H = 0$ , we have the single solution  
 324  $\Omega_{o0} = \mathfrak{J}^{-1}$ , while for  $H^{-1} \rightarrow 0$ , we have the expansion

$$\Omega_{o0} = \frac{\pi}{2H} \left( 1 + \frac{\frac{1}{2} - \mathfrak{J}^2}{8H^2} + \dots \right).$$

325 Consequently, to leading order, we adopt the composite expansion

$$\Omega_{o0} = \frac{\pi}{2\sqrt{H^2 + \frac{\pi^2}{4}\mathfrak{J}^2}}, \quad (46)$$

326 which proves extremely accurate for  $\epsilon = (H^2 + \frac{\pi^2}{4}\mathfrak{J}^2)^{-1/2}$  small, see Fig.11.  
 327 Clearly, this expression for the first cut-on is non-classical. Thus, we assume  
 328  $\Omega \sim \epsilon$  and the dispersion relation takes the form

$$D_{2o} = \Omega^2 - \frac{\pi^2}{4H^2 + \pi^2\mathfrak{J}^2} - \varkappa_o \kappa^2. \quad (47)$$

329 For the coefficient  $\varkappa_o$ , an explicit expression is available but it is rather  
 330 involved and we do not present it here. Instead, we observe that, in the

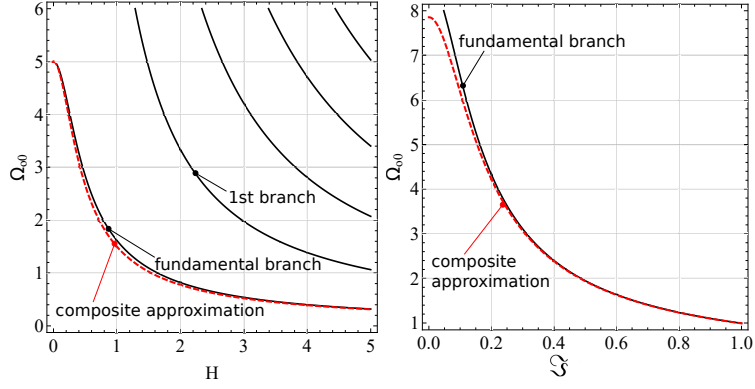


Figure 11: Cut-on frequencies for antisymmetric waves as a function of  $H$  for  $\mathfrak{J} = 0.2$  (left) and as a function of  $h$  for  $H = 0.2$  (right), superposed onto the composite expansion (46) for the first branch (red, dashed)

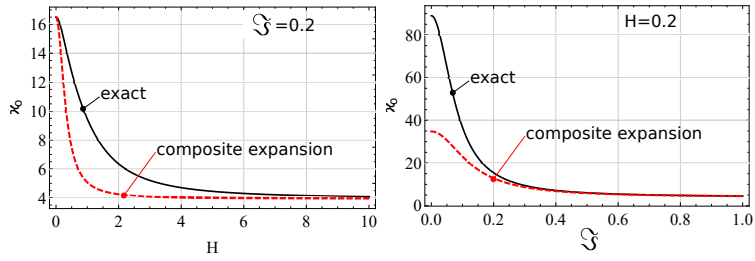


Figure 12: Coefficient  $\varkappa_0$  as a function of  $H$  for  $\mathfrak{J} = 0.2$  (left) and as a function of  $\mathfrak{J}$  for  $H = 0.2$  (right), superposed onto the composite expansion (49) (red, dashed)

331 microstructure-dominated regime,

$$\varkappa_o = 4 + \frac{1}{2\mathfrak{J}^2} + \frac{8\mathfrak{J}^2 \left( -\frac{2}{\psi^2} + \frac{60}{\pi^2} - 5 \right) + \frac{60}{\pi^2} - 7}{12\mathfrak{J}^4} H^2 + \dots, \quad H \ll 1, \quad (48)$$

332 while, in the classical limit, we retrieve (Kaplunov et al., 1998, Eq.(1.2.27))

$$\varkappa_o = 1 + \frac{16}{\pi\psi} \cot \frac{\pi}{2\psi} + \dots, \quad H \rightarrow \infty.$$

333 Hence, we introduce the composite expansion

$$\varkappa_o = \frac{\left( 1 + \frac{16}{\pi\psi} \cot \frac{\pi}{2\psi} \right) H^2 + \frac{\pi^2}{8} (1 + 8\mathfrak{J}^2)}{H^2 + \frac{\pi^2}{4} \mathfrak{J}^2} \geq 0, \quad (49)$$

which is  $O(1)$  and achieves excellent accuracy for  $\epsilon$  small, see Fig.12. Thus, we have  $\kappa = O(\epsilon)$  and we disregard the closest neighbourhood of the cut-on frequency, that is to say that  $\Omega - \Omega_{o0} \sim \epsilon$ , see Erbaş et al. (2018) for more details in the context of a Kirchhoff plates resting on a Winkler foundation. Therefore, we retrieve the wave equation

$$\square_o w + \frac{\pi^2}{(4h^2 + \frac{\pi^2}{4} J/\rho) \varkappa_o} w = 0,$$

334 with real speed  $c_o = \tilde{c}_S \sqrt{\varkappa_o} > \tilde{c}_S$ .

In this equation, microstructure appears mostly (but not only) through micro-inertia. In the classical limit, this approximation becomes purely classical and it can be compared with the second branch of the Timoshenko theory. As observed in Bhaskar (2009), this second branch is able to capture the cut-on frequency upon letting  $\varkappa_T = \pi^2/12$ , whence the requirement  $\sigma \approx 4$  demands  $\nu = 0.644$ , which is unacceptable. Besides, with this choice, Timoshenko second branch has the form (Bhaskar, 2009, Eq.(2.7))

$$D_{2o} = \Omega^2 - \Omega_{o0}^2 \left( 1 + \frac{1 + \sigma}{6} h^2 \kappa^2 \right),$$

335 where  $\Omega_{o0} = \frac{\pi}{2H}$ . This expression is clearly missing the  $O(\epsilon^2)$  term, in favour  
336 of the higher order contribution  $\Omega_{o0}^2 \kappa^2$ .

We conclude that an higher order theory accounting for the first two branches of flexural waves is given by

$$\left( \square_o + \frac{\pi^2}{(4h^2 + \frac{\pi^2}{4} J/\rho) \varkappa_o} \right) \left[ (D^\ell + D) \nabla^4 + \rho A \frac{\partial^2}{\partial t^2} \right] w = 0,$$

337 to leading order in  $\epsilon$ .

## 338 8. Conclusions

339 We develop a new model for plate elongation and flexure, incorporating size-  
340 dependence, by asymptotic reduction of the elastodynamics of a thin plate made  
341 of elastic isotropic couple-stress material with micro-inertia. This model is con-  
342 sistent, in that it reproduces the dispersion features of the fundamental mode  
343 of the 3D body. It is also unique, it being only possible to refine the degree  
344 of approximation or the number of modes it captures. This is in contrast with  
345 the numerous models recently presented in the literature, which are obtained  
346 enforcing specific kinematical assumptions onto Hamilton's principle (engineer-  
347 ing models). Consequently, these models may well differ in the elastic part,  
348 depending on the assumed kinematics, be it either Kirchhoff's or Mindlin's or  
349 many other's. Conversely, models are very similar in the inertia terms, in which  
350 microstructure usually does not appear. This is because it is hardly possible to  
351 anticipate the actual distribution of inertia forces. Asymptotic reduction shows  
352 that the opposite holds true, in that the classical solution is the leading order  
353 approximation, while microstructure appears, as the first order correction, in  
354 the form of rotational inertia terms. As a noticeable exception, couple stresses  
355 contribute at leading order to the flexural rigidity of the plate, and this is indeed  
356 a result that we share with the engineering models.

357 It is also noteworthy that asymptotic results are independent of the specific  
358 modification of the couple-stress theory that we may start from, namely reduced  
359 couple-stress or strain gradient effect. This shows that all couple-stress origi-  
360 nated models are equivalent, when it comes to dimensional-reduced theories.  
361 Conversely, micro-inertia plays an important role in some fundamental features  
362 of the Rayleigh-Lamb spectrum, and it cannot be neglected.

363 When we turn to higher order models, we see that symmetric modes are fun-  
364 damentally classical, and therefore microstructure appears, again, as a correc-  
365 tion term. This is perhaps expected, since couple-stress theory provides results  
366 very similar to classical elasticity in terms of longitudinal waves. Conversely,

367 microstructure already determines the leading term approximation for antisym-  
368 metric thickness-shear modes. The resulting size-dependent approximation can  
369 be intended to incorporate shear deformations, in the spirit of Timoshenko.

370 Finally, it is perhaps useful to compare the asymptotic approach with Mindlin's  
371 (Mindlin and Yang, 2006; Graff, 2012), which adopts Taylor expansions in the  
372 thickness direction (an idea originally proposed by Poisson, cf. Love (1888)). De-  
373 spite the two methods being vastly different, the latter still exploits consistency  
374 with the 3D body dispersion features to set tunable parameters (Graff, 2012,  
375 §8.3.1). We conclude that spectrum consistency is a transversal concept, which  
376 is not tied to the realm of asymptotic reduction.

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## 380 10. Declaration of interest

381 The author has no competing interest to declare.

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