Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xxxxxxx

pp. **X–XX**

ON THE TIME DECAY FOR THE MGT-TYPE POROSITY PROBLEMS

Jacobo Baldonedo

CINTECX, Departamento de Ingeniería Mecánica
Universidade de Vigo, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain

José R. Fernández

Departamento de Matemática Aplicada I, Universidade de Vigo
Escola de Enxeñería de Telecomunicación
Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain

Ramón Quintanilla*

Departamento de Matemáticas, E.S.E.I.A.A.T.-U.P.C.
Colom 11, 08222 Terrassa, Barcelona, Spain

(Communicated by the associate editor name)

Abstract. In this work we study three different dissipation mechanisms arising in the so-called Moore-Gibson-Thompson porosity. The three cases correspond to the MGT-porous hyperviscosity (fourth-order term), the MGT-porous viscosity (second-order term) and the MGT-porous weak viscosity (zeroth-order term). For all the cases, we prove that there exists a unique solution to the problem and we analyze the resulting point spectrum. We also show that there is an exponential energy decay for the first case, meanwhile for the second and third case only a polynomial decay is found. Finally, we present some one-dimensional numerical simulations to illustrate the behaviour of the discrete energy for each case.

1. **Introduction.** In the last decade, a big interest has developed about the so-called Moore-Gibson-Thompson equation [10, 14, 15, 16, 21, 23, 24, 26, 28, 29]. This equation was proposed in the context of fluid dynamics [33], but recently it has also be used as a a heat conduction equation [31, 32]. Furthermore, it was obtained from the viscous theories of classical and non-classical elasticity [1, 4, 5, 6, 11, 12, 13, 17, 19, 20, 22, 27].

On the other hand, in a recent paper Ieşan [18] proposed the theory of porous-elasticity with strain gradient. The decay of solutions was studied when several dissipative effects were introduced [2] and, more recently, the equation corresponding to the porous part (without elastic effects) was considered including several

2020 *Mathematics Subject Classification.* Primary: 74F10, 35B40 ; Secondary: 65M60.

Key words and phrases. MGT-porosity, dissipation mechanisms, existence and uniqueness, energy decay, finite elements, numerical behaviour.

This paper is part of the projects PGC2018-096696-B-I00 and PID2019-105118GB-I00, funded by the Spanish Ministry of Science, Innovation and Universities and FEDER "A way to make Europe". The authors would also like to acknowledge the comments provided by the reviewer, which helped to improve the final quality of the paper.

*Corresponding author: Ramón Quintanilla.

dissipation mechanisms. A study of the point spectrum suggested that, when we introduce porous hyperviscosity, porous viscosity or porous weak viscosity we should obtain the exponential decay of solutions [3]. However, the position of the elements in the point spectrum suggests that the most efficient mechanism corresponds to porous viscosity.

In this paper, we consider a similar problem, but now the dissipation mechanisms are of MGT-type. The equations are obtained from the suitable choice of the relaxation functions. We will see that, in this case, the results are quite different. We will obtain the exponential decay when we introduce a MGT-porous hyperviscosity, but we will see that the decay is slow for the MGT-porous viscosity and MGT-porous weak viscosity. In fact, we will prove the polynomial decay in these two last cases. We will also see the point spectrum for each case and given constitutive parameters. Finally, we will show some numerical simulations in the one-dimensional setting to demonstrate the discrete energy decay.

2. **Basic equations.**

In this section we recall the evolution and constitutive equations which govern the problems we study in this paper by following the ideas proposed by Ieşan [18].

In this paper, we assume that B is a three-dimensional bounded domain with a boundary smooth enough to allow the use of the divergence theorem. We consider a rigid body where porosity is allowed to evolve. That is, we consider the equation proposed for the porosity, but without the elastic deformation. In this case, the evolution equation is

$$J\ddot{\varphi} = \Xi_{i,i} - \sigma_{ij,ij} + g.$$

Here, φ represents the volume fraction, Ξ is the equilibrated stress tensor, σ is the equilibrated hyperstress tensor and g is the equilibrated body force. As usual, $J > 0$ is the product of the mass density by the equilibrated inertia.

The constitutive equations in the case of viscoelastic isotropic bodies are:

$$\begin{aligned}
\Xi_i &= \int_{-\infty}^{t} \alpha^*(t-s)\dot{\varphi}_{,i}\, ds,\\
\sigma_{ij} &= \int_{-\infty}^{t} \kappa_1^*(t-s)\delta_{ij}\dot{\varphi}_{,rr}(s) + 2\kappa_2^*(t-s)\dot{\varphi}_{,ij}(s) + d^*(t-s)\delta_{ij}\dot{\varphi}(s)\, ds,\\
g &= -\int_{-\infty}^{t} \xi^*(t-s)\dot{\varphi}(s) + d^*(t-s)\dot{\varphi}_{,rr}(s)\, ds.
\end{aligned}$$

In this paper, we will consider three different cases:

Case (i) The constitutive functions are:

$$\begin{aligned}
&\kappa_i^*(s) = \kappa_i^* + (\tau^{-1}\kappa_i - \kappa_i^*)e^{-\tau^{-1}s} \quad i = 1, 2,\\
&\alpha^*(s) = \alpha^*, \quad d^*(s) = d^*, \quad \xi^*(s) = \xi^*.
\end{aligned}$$

Case (ii) The constitutive functions are:

$$\begin{aligned}
&\alpha^*(s) = \alpha^* + (\tau^{-1}\alpha - \alpha^*)e^{-\tau^{-1}s}, \quad d^*(s) = d^* + (\tau^{-1}d - d^*)e^{-\tau^{-1}s},\\
&\kappa_i^*(s) = \kappa_i^*, \quad \xi^*(s) = \xi^*.
\end{aligned}$$

Case (iii) The constitutive functions are:

$$\begin{aligned}
&\xi^*(s) = \xi^* + (\tau^{-1}\xi - \xi^*)e^{-\tau^{-1}s},\\
&\kappa_i^*(s) = \kappa_i^*, \quad d^*(s) = d^*, \quad \alpha^*(s) = \alpha^*.
\end{aligned}$$

If we assume that the volume fraction vanishes at time $t = -\infty$ our equations become:

Case (i) $J(\tau\dddot{\varphi} + \ddot{\varphi}) = -(\kappa_1^* + \kappa_2^*)\Delta^2\varphi - (\kappa_1 + \kappa_2)\Delta^2\dot{\varphi} + (\alpha^* - 2d^*)(\tau\Delta\dot{\varphi} + \Delta\varphi) - \xi^*(\tau\dot{\varphi} + \varphi)$.
Case (ii) $J(\tau\dddot{\varphi} + \ddot{\varphi}) = -(\kappa_1^* + \kappa_2^*)(\Delta^2\varphi + \tau\Delta^2\dot{\varphi}) + (\alpha^* - 2d^*)\Delta\varphi + (\alpha - 2d)\Delta\dot{\varphi} - \xi^*(\tau\dot{\varphi} + \varphi)$.
Case (iii) $J(\tau\dddot{\varphi} + \ddot{\varphi}) = -(\kappa_1^* + \kappa_2^*)(\Delta^2\varphi + \tau\Delta^2\dot{\varphi}) + (\alpha^* - 2d^*)(\Delta\varphi + \tau\Delta\dot{\varphi}) - \xi^*\varphi - \xi\dot{\varphi}$.

We can simplify the notation to write the equations as:

Case (i) $J(\tau\dddot{\varphi} + \ddot{\varphi}) = -\kappa^*\Delta^2\varphi - \kappa\Delta^2\dot{\varphi} + a^*(\tau\Delta\dot{\varphi} + \Delta\varphi) - \xi^*(\tau\dot{\varphi} + \varphi)$.
Case (ii) $J(\tau\dddot{\varphi} + \ddot{\varphi}) = -\kappa^*(\Delta^2\varphi + \tau\Delta^2\dot{\varphi}) + a^*\Delta\varphi + a\Delta\dot{\varphi} - \xi^*(\tau\dot{\varphi} + \varphi)$.
Case (iii) $J(\tau\dddot{\varphi} + \ddot{\varphi}) = -\kappa^*(\Delta^2\varphi + \tau\Delta^2\dot{\varphi}) + a^*(\Delta\varphi + \tau\Delta\dot{\varphi}) - \xi^*\varphi - \xi\dot{\varphi}$.

These equations will be studied in a three-dimensional bounded region B and we will assume the following initial and boundary conditions:

$$\varphi(\boldsymbol{x},0) = \varphi^0(\boldsymbol{x}), \quad \dot{\varphi}(\boldsymbol{x},0) = \varphi^1(\boldsymbol{x}), \quad \ddot{\varphi}(\boldsymbol{x},0) = \varphi^2(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in B, \tag{1}$$
$$\varphi(\boldsymbol{x},t) = \Delta\varphi(\boldsymbol{x},t) = 0 \quad \forall \boldsymbol{x} \in \partial B. \tag{2}$$

In this paper, we assume that

$$J > 0, \quad \kappa^* > 0, \quad a^* > 0, \quad \xi^* > 0.$$

We note that, when we consider Case (i), we also assume that $\kappa > \tau\kappa^*$. In the Case (ii) we assume that $a > \tau a^*$, and, for Case (iii), we suppose that $\xi > \tau\xi^*$.

3. **Cauchy problem for Case (i).** In this section we prove the existence of solution as well as the exponential decay to the problem determined by Case (i) with initial conditions (1) and boundary conditions (2). The remaining two cases will be shown later.

We first consider the Hilbert space

$$\mathcal{H} = H_0^1(B) \cap H^2(B) \times H_0^1(B) \cap H^2(B) \times L^2(B).$$

If we denote by $U = (\varphi, \psi, \zeta)$ the elements in this space, then we can write our problem as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = U^0, \tag{3}$$

where we have used the matrix operator $\mathcal{A}$ given by

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ \dfrac{-\kappa^*\Delta^2 + a^*\Delta - \xi^*}{\tau J} & \dfrac{-\kappa\Delta^2}{\tau J} + \dfrac{a^*\Delta - \xi^*}{J} & -\dfrac{1}{\tau} \end{pmatrix},$$

and $U^0 = (\varphi^0, \varphi^1, \varphi^2)$.

In our Hilbert space $\mathcal{H}$ we define the inner product

$$< (\varphi, \psi, \zeta), (\varphi^*, \psi^*, \zeta^*) >= \frac{1}{2}\int_B \Big[J(\tau\zeta + \psi)\overline{(\tau\zeta^* + \psi^*)} + \kappa^*(\Delta\varphi + \tau\Delta\psi)\overline{(\Delta\varphi^* + \tau\Delta\psi^*)}$$
$$+ \tau\overline{\kappa}\Delta\psi\Delta\overline{\psi^*} + a^*(\nabla\varphi + \tau\nabla\psi)\overline{(\nabla\varphi^* + \tau\nabla\psi^*)} + \xi^*(\varphi + \tau\psi)\overline{(\varphi^* + \tau\psi^*)} \Big] dV,$$

where, as usual, a bar over an element of the Hilbert space means the conjugated complex and $\overline{\kappa} = \kappa - \tau\kappa^*$. We note that we can define the norm:

$$\|(\varphi,\psi,\zeta)\|^2 = \frac{1}{2}\int_B \Big[J(\tau\zeta+\psi)\overline{(\tau\zeta+\psi)} + \kappa^*(\Delta\varphi+\tau\Delta\psi)\overline{(\Delta\varphi+\tau\Delta\psi)} \\ +\tau\overline{\kappa}\Delta\psi\Delta\overline{\psi} + a^*(\nabla\varphi+\tau\nabla\psi)\overline{(\nabla\varphi+\tau\nabla\psi)} + \xi^*(\varphi+\tau\psi)\overline{(\varphi+\tau\psi)}\Big]\, dV.$$

Moreover, we also point out that this is a scalar product which is the equivalent to the usual one in the Hilbert space.

It is easy to show that the operator $\mathcal{A}$ has a domain

$$\mathcal{D}(\mathcal{A}) = \{(\varphi,\psi,\zeta)\,;\, \zeta\in H^2(B)\cap H_0^1(B),\, \kappa^*\Delta^2\varphi+\kappa\Delta^2\psi\in L^2(B),\ \Delta\varphi=0 \text{ on } \partial B\},$$

and we can also obtain that

$$Re\langle \mathcal{A}U,U\rangle = -\frac{1}{2}\int_B \overline{\kappa}|\Delta\psi|^2\, dV.$$

Now, we need to prove that zero belongs to the resolvent of the operator. Let $(f_1,f_2,f_3)\in\mathcal{H}$ and so, we must solve the system:

$$\begin{aligned}
&\psi = f_1,\\
&\zeta = f_2,\\
&-\kappa^*\Delta^2\varphi - \kappa\Delta^2\psi + a^*\tau\Delta\psi + a^*\Delta\varphi - \xi^*\varphi - \xi^*\tau\psi - J\zeta = J\tau f_3.
\end{aligned}$$

The solution for ψ and ζ is clear. Therefore, we can introduce them in the last equation to find that

$$-\kappa^*\Delta^2\varphi + a^*\Delta\varphi - \xi^*\varphi = J\tau f_3 + \kappa\Delta^2 f_1 - a^*\tau\Delta f_1 + \xi^*\tau f_1 + J f_2.$$

It is clear that we can solve this equation for an $\varphi\in H^2(B)\cap H_0^1(B)$.

Therefore, an existence and uniqueness result follows.

Theorem 3.1. *Therefore, for each $U_0\in\mathcal{D}(\mathcal{A})$, there exists a unique solution $U(t)\in\mathcal{C}^1([0,\infty),\mathcal{H})\cap\mathcal{C}^0([0,\infty),\mathcal{D}(\mathcal{A}))$ to problem (3).*

In the remaining of this section we will prove the exponential decay of the solutions. To this end, we will use the arguments of Prüss [30]. This kind of argument has been used very often. Therefore, we will proceed in a fast way. First, we recall the following theorem shown in the book of Liu and Zheng [25].

Theorem 3.2. *Let $S(t)=\{e^{\mathcal{A}t}\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only the imaginary axis is contained in the resolvent of $\mathcal{A}$ and*

$$\overline{\lim_{|\lambda|\to\infty}}\|(i\lambda\mathcal{I}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{4}$$

We are going to use this result in our situation. Let us assume that there exists an element of the spectrum at the imaginary axis. Therefore, there will exist a sequence of elements of unit norm in the Hilbert space $\mathcal{H}$ and a sequence of real numbers λ_n such that $\lambda_n\to\lambda\neq 0$ and

$$\begin{aligned}
&i\lambda_n\varphi_n - \psi_n \to 0 \quad \text{in} \quad H^2(B),\\
&i\lambda_n\psi_n - \zeta_n \to 0 \quad \text{in} \quad H^2(B),\\
&i\lambda_n\zeta_n\tau J + \kappa^*\Delta^2\varphi_n - a^*\Delta\varphi_n + \xi^*\varphi_n + \kappa\Delta^2\psi_n\\
&\qquad -a^*\tau\Delta\psi_n - \xi^*\tau\psi_n - J\zeta_n \to 0 \quad \text{in} \quad L^2(B).
\end{aligned}$$

From the dissipation inequality we obtain that $\Delta^2\psi_n\to 0$ and so, $\lambda_n\Delta^2\psi_n\to 0$ and $\lambda_n^{-1}\Delta^2\zeta_n\to 0$.

It follows that

$$i\lambda_n\zeta_n\tau J - J\zeta_n \to 0 \quad \text{in} \quad L^2(B).$$

If we divide by λ_n we see that $\zeta_n \to 0$ in $L^2(B)$, which is a contradiction.

We note that the arguments used previously can be adapted to show that the asymptotic condition (4) also holds and therefore, we have proved the exponential decay of the solutions.

Theorem 3.3. *The operator $\mathcal{A}$ generates a semigroup exponentially stable. That is, there exist two positive constants M, ω such that*

$$||U(t)|| \leq Me^{-\omega t}||U^0||$$

for every $U^0 \in \mathcal{D}(\mathcal{A})$.

Finally, we consider the point spectrum in the one-dimensional case. Furthermore, we assume that $B = [0, \pi]$. So, if we consider solutions of the form:

$$\varphi(x,t) = e^{\omega t}\sin nx, \quad n \in \mathbb{N},$$

then it follows that ω must satisfy

$$J\tau\omega^3 + J\omega^2 + (\kappa n^4 + a^*\tau n^2 + \xi^*\tau)\omega + \kappa^* n^4 + a^* n^2 + \xi^* = 0.$$

After a tedious calculation, we can obtain an explicit formula in this case. In Figure 1 we can see the roots for the following values of the constitutive parameters:

$$J = 1, \quad \tau = 1, \quad \kappa^* = 0.1, \quad a^* = 0.1, \quad \xi^* = 0.1, \quad \kappa = 1.$$

We plot a range of 40 different roots beginning with $n = 1$ and taking a step of $n = 1$ (up to $n = 40$). We can see how almost all the roots lie on the line $x = -0.46$ but far away of the imaginary axis.

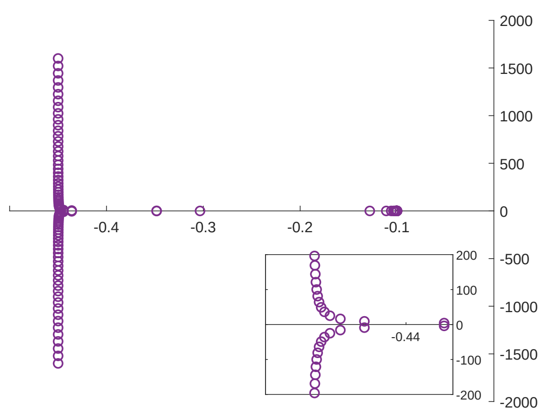

FIGURE 1. Roots behaviour for the fourth-order dissipation mechanism.

4. **Cauchy problem for Case (ii).** This section is devoted to the study of the corresponding equation to Case (ii) with suitable initial and boundary conditions. We first note that if we denote by $\varphi_1 = \varphi + \tau\dot{\varphi}$ and $\overline{a} = a - \tau a^*$ we can write our equation in the following form:

$$J\ddot{\varphi}_1 = -\kappa^*\Delta^2\varphi_1 + a^*\Delta\varphi_1 + \overline{a}\Delta\psi - \xi^*\varphi_1, \tag{5}$$

where $\psi = \dot{\varphi}$.

We impose the initial conditions:

$$\begin{aligned} &\varphi_1(\boldsymbol{x},0) = \varphi^0(\boldsymbol{x}) + \tau\varphi^1(\boldsymbol{x}),\\ &\psi(\boldsymbol{x},0) = \varphi^1(\boldsymbol{x}),\\ &\varphi_3(\boldsymbol{x},0) = \varphi^1(\boldsymbol{x}) + \tau\varphi^2(\boldsymbol{x}), \end{aligned} \tag{6}$$

where $\varphi_3 = \dot{\varphi} + \tau\ddot{\varphi}$, and we consider the boundary conditions:

$$\varphi_1 = \Delta\varphi_1 = \psi = 0 \quad \text{on} \quad \partial B. \tag{7}$$

It is clear that, given φ_1, ψ and φ_3, we can recover the volume fraction φ and its derivatives.

We will study now the problem defined by equation (5), initial conditions (6) and boundary conditions (7) in the Hilbert space

$$\mathcal{H} = H^2(B) \cap H_0^1(B) \times H_0^1(B) \times L^2(B),$$

where we consider the element $(\varphi_1, \psi, \varphi_3)$.

In this space $\mathcal{H}$ we define the inner product:

$$\begin{aligned} <(\varphi_1,\psi,\varphi_3),(\varphi_1^*,\psi^*,\varphi_3^*)>= \frac{1}{2}\int_B \Big[&J\varphi_3\overline{\varphi_3^*} + \kappa^*\Delta\varphi_1\overline{\Delta\varphi_1^*} + a^*\nabla\varphi_1\overline{\nabla\varphi_1^*} \\ &+\tau\overline{a}\nabla\psi\nabla\overline{\psi^*} + \xi\varphi_1\overline{\varphi_1^*}\Big]\, dV. \end{aligned}$$

It is clear that it defines an inner product which is equivalent to the usual one in the space.

Our problem can be written in the following matrix form:

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = U^0 = (\varphi_1(\boldsymbol{x},0), \psi(\boldsymbol{x},0), \varphi_3(\boldsymbol{x},0)), \tag{8}$$

where we have used now the matrix operator $\mathcal{A}$ given by

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & I \\ 0 & -\tau^{-1}I & \tau^{-1}I \\ \dfrac{-\kappa^*\Delta^2 + a^*\Delta - \xi^*}{J} & \dfrac{\overline{a}\Delta}{J} & 0 \end{pmatrix}.$$

We note that the domain of this operator $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = \{(\varphi_1,\psi,\varphi_3)\,;\, \varphi_3 \in H^2(B)\cap H_0^1(B),\ \kappa^*\Delta^2\varphi_1 - \overline{a}\Delta\psi \in L^2(B),\ \Delta\varphi_1 = 0 \text{ on } \partial B\},$$

which is dense.

On the other hand, it follows that

$$Re\langle \mathcal{A}U, U\rangle = -\frac{\overline{a}}{2}\int_B |\nabla\psi|^2\, dV \leq 0.$$

In a similar way to the previous section, we can also prove that zero belongs to the resolvent of operator $\mathcal{A}$.

However, in this case we cannot obtain the exponential decay of the solutions. In fact, we can show that we can find elements of the point spectrum as near as we want to the imaginary axis.

Let us consider solutions of the form:

$$\varphi(\boldsymbol{x},t) = e^{\omega t}\Phi_n(\boldsymbol{x}), \tag{9}$$

where $\Phi_n(\boldsymbol{x})$ is a solution of the problem:

$$\begin{aligned} &\Delta\Phi_n + \lambda\Phi_n = 0 \quad \text{in} \quad B, \\ &\Phi_n = 0 \quad \text{on} \quad \partial B. \end{aligned}$$

We can see that ω must satisfy the equation

$$J\tau\omega^3 + J\omega^2 + (\kappa^*\tau\lambda_n^2 + a\lambda_n + \xi^*\tau)\omega + \kappa^*\lambda_n^2 + a^*\lambda_n + \xi^* = 0.$$

Our aim is to show that there always exist elements at the point spectrum at the right-hand of the line $Im(z) = -\varepsilon$, for every $\varepsilon > 0$ sufficiently small. To this end, we change ω by $\omega - \varepsilon$ to obtain the equation

$$\begin{aligned} &J\tau\omega^3 + (J - 3J\tau\varepsilon)\omega^2 + (\kappa^*\tau\lambda_n^2 + a\lambda_n + \xi^*\tau + 3J\tau\varepsilon^2 - 2J\varepsilon)\omega \\ &\quad +\kappa^*\lambda_n^2 + a^*\lambda_n + \xi^* - J\tau\varepsilon^3 + J\varepsilon^2 - (\kappa^*\tau\lambda_n^2 + a\lambda_n + \xi^*\tau)\varepsilon = 0. \end{aligned}$$

We know that the necessary and sufficient condition to guarantee that the solutions to the polynomial

$$A_0\omega^3 + A_1\omega^2 + A_2\omega + A_3 = 0 \tag{10}$$

are on the left-hand of the imaginary axis is that all the A_i are positive and

$$A_1A_2 - A_0A_3 > 0.$$

We note that in our case we obtain

$$A_1A_2 - A_0A_3 = -2J\tau^2\kappa^*\varepsilon\lambda_n^2 + A_1^*\lambda_n + A_2^*, \tag{11}$$

where A_1^* and A_2^* depend on the coefficients of the problem and ε.

As the sequence of λ_n becomes unbounded we can always find λ_n large enough to guarantee that the expression (11) is negative. As the point spectrum of the equation is as near as we want to the imaginary axis, we cannot obtain an uniform exponential decay of the solutions.

Although we cannot expect the exponential decay of the solutions to our problem, we will prove that the solution decays in a polynomial way. In fact, we will show that the decay is controlled by a term of the form $t^{-1/2}$.

Theorem 4.1. *The semigroup $S(t)$ generated by the operator $\mathcal{A}$ is polynomially stable of order $1/2$. That is, for every $U \in \mathcal{D}(\mathcal{A})$ there exists a positive constant C, which is independent of the initial data, such that*

$$\|S(t)U\|_{\mathcal{H}} \leq C\|U\|_{\mathcal{D}(\mathcal{A})}t^{-1/2}. \tag{12}$$

Proof. In order to show the decay it will be sufficient (see Borichev and Tomilov [7]) to prove that the imaginary axis is contained in the resolvent of the operator $\mathcal{A}$ and that the asymptotic condition:

$$\overline{\lim_{|\lambda|\to\infty}} \lambda^{-2}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{13}$$

holds.

We first suppose that the imaginary axis is not included in the resolvent. Therefore, there will exist a sequence of real numbers $\lambda_n \to \lambda \neq 0$ and a sequence $U_n = (\varphi_{1n}, \psi_n, \varphi_{3n})$, with unit norm, in the domain of the operator $\mathcal{A}$ such that

$$\|(i\lambda_n I - \mathcal{A})^{-1}U_n\|_{\mathcal{H}} \to 0. \tag{14}$$

Hence, we have the following convergences:

$$i\lambda_n\varphi_{1n} - \varphi_{3n} \to 0 \quad \text{in} \quad H^2(B), \tag{15}$$

$$i\lambda_n\psi_n + \tau^{-1}\psi_n - \tau^{-1}\varphi_{3n} \to 0 \quad \text{in} \quad H^1(B), \tag{16}$$

$$i\lambda_n\varphi_{3n} + \kappa^*\Delta^2\varphi_{1n} - a^*\Delta\varphi_{1n} + \xi^*\varphi_{1n} + a\Delta\psi_n \to 0 \quad \text{in} \quad L^2(B). \tag{17}$$

In view of the dissipation inequality, we have $\psi_n \to 0$ in $H^1(B)$ and then, using (16) we find that $\varphi_{3n} \to 0$ in $H^1(B)$. As λ_n is bounded, we obtain from (15) that $\varphi_{1n} \to 0$ in $H^1(B)$. Now, if we multiply convergence (17) by φ_{1n} we obtain that $\varphi_{1n} \to 0$ in $H^2(B)$ and we arrive to a contradiction. It follows that the imaginary axis should be contained in the resolvent.

We now prove the asymptotic condition. Assume that it is not true. We see that there exist a sequence of real numbers λ_n such that $|\lambda_n| \to \infty$ and a unit norm sequence of vectors at the domain of $\mathcal{A}$, $U_n = (\varphi_{1n}, \psi_n, \varphi_{3n})$, such that

$$\lambda_n^2(i\lambda_n\varphi_{1n} - \varphi_{3n}) \to 0 \quad \text{in} \quad H^2(B), \tag{18}$$

$$\lambda_n^2(i\lambda_n\psi_n + \tau^{-1}\psi_n - \tau^{-1}\varphi_{3n}) \to 0 \quad \text{in} \quad H^1(B), \tag{19}$$

$$\lambda_n^2(iJ\lambda_n\varphi_{3n} + \kappa^*\Delta^2\varphi_{1n} - a^*\Delta\varphi_{1n} + \xi^*\varphi_{1n} + a\Delta\psi_n) \to 0 \quad \text{in} \quad L^2(B). \tag{20}$$

The dissipation inequality implies that $\lambda_n\psi_n$ tends to zero in $H^1(B)$. So, from (19) we see that φ_{3n} also tends to zero in $H^1(B)$ and, from (18), $\lambda_n\varphi_{1n}$ also tends to zero in $H^1(B)$. If we multiply convergence (20) by φ_{1n} we conclude that $\varphi_{1n} \to 0$ in $H^2(B)$ and we arrive again to a contradiction. Therefore, the asymptotic condition also holds and the theorem is proved. $\square$

Finally, we consider again the point spectrum in the one-dimensional case when $B = [0, \pi]$. Taking solutions of the form:

$$\varphi(x,t) = e^{\omega t}\sin nx, \quad n \in \mathbb{N},$$

then ω must satisfy now the equation:

$$J\tau\omega^3 + J\omega^2 + (\kappa^*\tau n^4 + an^2 + \xi^*\tau)\omega + \kappa^* n^4 + a^* n^2 + \xi^* = 0.$$

Proceeding as in the previous case, in Figure 2 we can see the roots for the following values of the constitutive parameters:

$$J = 1, \quad \tau = 1, \quad \kappa^* = 0.1, \quad a^* = 0.1, \quad \xi^* = 0.1, \quad a = 1.$$

We plot a range of 30 different roots beginning with $n = 1$ and taking a step of $n = 1$ (up to $n = 30$). In this case, the roots tend to the imaginary axis when n is large enough.

5. **Cauchy problem for Case (iii).** Last question we study in this paper corresponds to the equation labelled as case (iii). If we consider the notation proposed in the last section, we can write our equation as

$$J\ddot{\varphi}_1 = -\kappa^*\Delta^2\varphi_1 + a^*\Delta\varphi_1 - \xi^*\varphi_1 - \overline{\xi}\psi.$$

Again, we should consider the initial conditions (6) but we only impose the first two boundary conditions in (7).

We can study this problem in the Hilbert space

$$\mathcal{H} = H^2(B) \cap H_0^1(B) \times L^2(B) \times L^2(B),$$

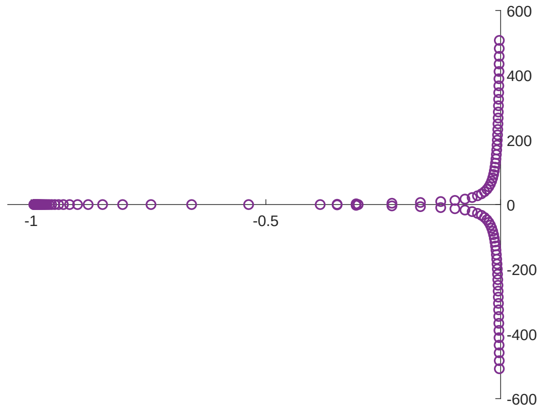

FIGURE 2. Roots behaviour for the second-order dissipation mechanism.

and to consider the inner product:

$$< (\varphi_1, \psi, \varphi_3), (\varphi_1^*, \psi^*, \varphi_3^*) >= \frac{1}{2} \int_B \Big[J\varphi_3 \overline{\varphi_3^*} + \kappa^* \Delta\varphi_1 \overline{\Delta\varphi_1^*} + a^* \nabla\varphi_1 \overline{\nabla\varphi_1^*} + \xi^* \varphi_1 \overline{\varphi_1^*} + \tau\overline{\xi}\psi\overline{\psi^*} \Big] \, dV.$$

Again, our problem can be written in the matrix form (8), where we have used now the matrix operator $\mathcal{A}$ given by

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & I \\ 0 & -\tau^{-1}I & \tau^{-1}I \\ \dfrac{-\kappa^*\Delta^2 + a^*\Delta - \xi^*}{J} & -\dfrac{\overline{\xi}}{J} & 0 \end{pmatrix}.$$

In this case, we can see that the domain of this operator $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = \{(\varphi_1, \psi, \varphi_3) \, ; \, \varphi_3 \in H^2(B) \cap H_0^1(B), \, \Delta^2\varphi_1 \in L^2(B), \; \Delta\varphi_1 = 0 \text{ on } \partial B\}.$$

It is clear that the domain is dense in $\mathcal{H}$ and we also have

$$Re\langle \mathcal{A}U, U\rangle = -\frac{\overline{\xi}}{2} \int_B |\psi|^2 \, dV \le 0.$$

It is not difficult to show that zero belongs to the resolvent of the operator and, therefore, we can conclude the existence and uniqueness of the solutions.

Again, we can obtain that the decay is slow. In fact, if we consider functions of the form (9), we find that ω must satisfy the equation:

$$J\tau\omega^3 + J\omega^2 + (\kappa^*\tau\lambda_n^2 + a^*\tau\lambda_n + \xi)\omega + \kappa^*\lambda_n^2 + a^*\lambda_n + \xi^* = 0.$$

After the change of ω by $\omega - \varepsilon$ we can obtain an equation of the form (10), where $A_0 = J$, $A_1 = J - 3J\tau\varepsilon$, $A_2 = \kappa^*\tau\lambda_n^2 + a^*\tau\lambda_n + \xi + 3J\tau\varepsilon^2 - 3J\varepsilon$ and $A_3 = \kappa^*\lambda_n^2 + a^*\lambda_n + \xi^* - J\tau\varepsilon^3 + J\varepsilon^2 - (\kappa^*\tau\lambda_n^2 + a^*\tau\lambda_n + \xi)\varepsilon$.

A direct calculation shows that

$$A_1A_2 - A_0A_3 = -2J\tau^2\kappa^*\varepsilon\lambda_n^2 + \text{ other terms of order less than } \lambda_n^2.$$

Again, we see that the decay cannot be of exponential type. However, it is also possible to prove that the solutions decay as $t^{-1/2}$. To this end, we can follow the same argument proposed in the previous section. That is, to show that the imaginary axis is contained in the resolvent of the operator and that the asymptotic condition (13) is fulfilled.

To prove that the imaginary axis is contained in the resolvent we follow the same argument proposed in the analysis of Case (ii), and (14) leads to the convergences:

$$
\begin{array}{l}
i\lambda_n\varphi_{1n}-\varphi_{3n}\to 0 \quad \text{in} \quad H^2(B),\\
i\lambda_n\psi_n+\tau^{-1}\psi_n-\tau^{-1}\varphi_{3n}\to 0 \quad \text{in} \quad L^2(B),\\
i\lambda_n\varphi_{3n}+\kappa^*\Delta^2\varphi_{1n}-a^*\Delta\varphi_{1n}+\xi^*\varphi_{1n}+\overline{\xi}\psi_n\to 0 \quad \text{in} \quad L^2(B).
\end{array}
$$

Dissipation inequality implies that $\psi_n\to 0$ in $L^2(B)$. Therefore, $\varphi_{3n}\to 0$ in $L^2(B)$ and $\lambda_n\varphi_{1n}\to 0$ in $L^2(B)$. If we multiply the last convergence by φ_{1n} we see that $\Delta\varphi_{1n}\to 0$ in $L^2(B)$ and so, we arrive to a contradiction.

We can also prove the condition (13) by a similar argument. In this case, the convergences will be:

$$
\begin{array}{l}
\lambda_n^3(i\lambda_n\varphi_{1n}-\varphi_{3n})\to 0 \quad \text{in} \quad H^2(B),\\
\lambda_n^2(i\lambda_n\psi_n+\tau^{-1}\psi_n-\tau^{-1}\varphi_{3n})\to 0 \quad \text{in} \quad L^2(B),\\
\lambda_n^2(i\lambda_n\varphi_{3n}+\kappa^*\Delta^2\varphi_{1n}-a^*\Delta\varphi_{1n}+\xi^*\varphi_{1n}+\overline{\xi}\psi_n)\to 0 \quad \text{in} \quad L^2(B).
\end{array}
$$

Again, we obtain that $\lambda_n\psi_n$ tends to zero in $L^2(B)$ and then, φ_{3n} and $\lambda_n\varphi_{1n}$ also converge to zero in $L^2(B)$. From the above last convergence, we also obtain that $\varphi_{1n}\to 0$ in $H^2(B)$, and we arrive to another contradiction. Therefore, we have seen that, in this case (iii), the estimate (12) also holds.

Remark 1. *To finish the analysis of this section we consider the case when $\kappa^*=0$. It corresponds to the porous deformations, but without the strain gradient effects. We have the same initial conditions, but we do not assume that $\Delta\varphi_1$ vanishes at the boundary. In this case, the Hilbert space is*

$$\mathcal{H}=H_0^1(B)\times L^2(B)\times L^2(B).$$

The matrix operator can be defined as previously (assuming $\kappa^*=0$), but the domain is:

$$\mathcal{D}(\mathcal{A})=\{(\varphi_1,\psi,\varphi_3)\ ;\ \varphi_3\in H_0^1(B),\,\Delta\varphi_1\in L^2(B)\}.$$

Existence and uniqueness can be obtained following similar arguments but, in this case, we have

$$A_1A_2-A_2A_3=-2J\tau a^*\lambda_n\varepsilon+\text{ terms of order zero in }\lambda_n.$$

Therefore, the decay is also shown (not exponential), but we can also prove the polynomial decay.

Finally, we also show the point spectrum in the one-dimensional case when $B=[0,\pi]$. Thus, ω must satisfy now the equation:

$$J\tau\omega^3+J\omega^2+(\kappa^*\tau n^4+a^*\tau n^2+\xi)\omega+\kappa^*n^4+a^*n^2+\xi^*=0.$$

Proceeding as in the previous cases, in Figure 3 we can see the roots for the following values of the constitutive parameters:

$$J=1,\quad \tau=1,\quad \kappa^*=0.1,\quad a^*=0.1,\quad \xi^*=0.1,\quad \xi=1.$$

We plot a range of 40 different roots beginning with $n=1$ and taking a step of $n=1$ (up to $n=40$). In this case, the roots tend to the imaginary axis when n is large enough.

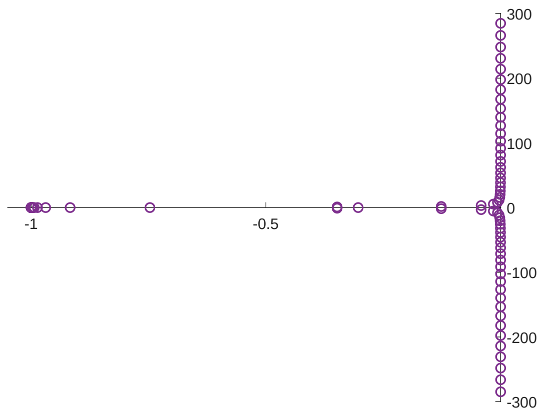

FIGURE 3. Roots behaviour for the zero-order dissipation mechanism.

6. **Numerical simulations for the three cases.** In this final section, we describe a finite element approximation of the problems studied in the previous sections, corresponding to cases (i), (ii) and (iii). Then, we derive their variational formulation. For the sake of simplicity we restrict ourselves to the one-dimensional case and so, let us consider the domain $B = (0, 1)$ and denote by $Y = L^2(0, 1)$, and by $(\cdot, \cdot)$ the scalar product in this space, with corresponding norm $\|\cdot\|$.

We note that we will only focus on the first case for the sake of clarity. It is straightforward to obtain the fully discrete approximations in the remaining cases.

We must change the boundary conditions (2) as

$$\varphi(0, t) = \varphi(\ell, t) = \varphi_x(0, t) = \varphi_x(\ell, t) = 0. \tag{21}$$

Therefore, after integrating by parts we obtain the weak form of problem given by case (i), the initial conditions (1) and the modified boundary conditions (21).

Find the porous acceleration $\xi : [0, T] \to H_0^2(0, \ell)$ such that $\xi(0) = \varphi^2$, and, for a.e. $t \in (0, T)$ and $r \in H_0^2(0, \ell)$,

$$\begin{aligned} &J(\tau\dot{\xi}(t) + \xi(t), r) + (\kappa^*\varphi_{xx}(t) + \kappa\psi_{xx}(t), r_{xx}) + a^*(\tau\psi_x(t) + \varphi_x(t), r_x) \\ &\quad + \xi^*(\tau\psi(t) + \varphi(t), r) = 0, \end{aligned}$$

where the porous velocity ψ and the porosity φ are given by

$$\psi(t) = \int_0^t \xi(s)\, ds + \varphi^1, \quad \varphi(t) = \int_0^t \psi(s)\, ds + \varphi^0.$$

Now, we show the fully discrete approximation of this variational problem in two steps. In order to obtain the spatial approximation, let us assume that the interval $[0, 1]$ is divided into M subintervals $a_0 = 0 < a_1 < \ldots < a_M = 1$ of length $h = a_{i+1} - a_i = 1/M$ and so, we consider the finite dimensional space $V^h \subset H_0^2(0, 1)$ given by

$$\begin{aligned} V^h = \{r^h \in C^1([0, 1])\ ;\ r^h_{|_{[a_i, a_{i+1}]}} \in P_3([a_i, a_{i+1}]) \quad i = 0, \ldots, M-1, \\ r^h(0) = r^h(\ell) = r^h_x(0) = r^h_x(1) = 0\}, \end{aligned}$$

where $P_3([a_i, a_{i+1}])$ denotes the space of polynomials of degree less or equal to three in the subinterval $[a_i, a_{i+1}]$; that is, the space has C^1 and piecewise cubic functions.

Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we define the discrete initial conditions φ_0^h, φ_1^h and φ_2^h be defined as

$$\varphi_0^h = \mathcal{P}^h \varphi^0, \quad \varphi_1^h = \mathcal{P}^h \varphi^1, \quad \varphi_2^h = \mathcal{P}^h \varphi^2,$$

where $\mathcal{P}^h$ is the classical finite element interpolation operator over V^h (see [9]).

Secondly, in order to obtain the discretization of the time derivatives, we consider a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, with step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \ldots, N$.

Therefore, using the classical implicit Euler scheme, the fully discrete approximations of the above variational problem are the following.

Find the discrete porous acceleration $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset V^h$ *such that* $\xi_0^{hk} = \varphi_2^h$ *and, for all* $r^h \in V^h$ *and* $n = 1, \ldots, N$,

$$\begin{aligned} J(\tau(\xi_n^{hk} - \xi_{n-1}^{hk})/k + \xi_n^{hk}, r^h) + (\kappa^*(\varphi_n^{hk})_{xx} + \kappa(\psi_n^{hk})_{xx}, r_{xx}^h) \\ + a^*(\tau(\psi_n^{hk})_x + (\varphi_n^{hk})_x, r_x^h) + \xi^*(\tau\psi_n^{hk} + \varphi_n^{hk}, r^h) = 0, \end{aligned}$$

where the discrete porous velocity and the discrete porosity ψ_n^{hk} and φ_n^{hk} are now recovered from the relations:

$$\psi_n^{hk} = k\xi_n^{hk} + \psi_{n-1}^{hk}, \quad \varphi_n^{hk} = k\psi_n^{hk} + \varphi_{n-1}^{hk}.$$

It is straightforward to obtain that this fully discrete problem has a unique solution applying the well-known Lax Milgram lemma and the required assumptions on the constitutive parameters.

In all the numerical simulations described below, we have chosen the discretization parameters $h = 0.025$ and $k = 10^{-5}$ and we have used the following data:

$$T = 100, \quad J = 1, \quad \kappa^* = 0.1, \quad a^* = 0.1, \quad \xi^* = 0.1, \quad \tau = 1,$$

and the initial conditions, for all $x \in (0, 1)$:

$$\varphi^0(x) = 10x^3(x-1)^3, \quad \varphi^1(x) = \varphi^2(x) = 0.$$

We have assumed to vary the respective parameters κ, a and ξ for each case.

In the first example, we numerically study the dependence of the solution with respect to parameter κ in the case (i) (i.e. it corresponds to the MGT-porous hyperviscosity). Therefore, the evolution in time of the discrete energy given by

$$\begin{aligned} E_n^{hk} = \frac{1}{2}\int_0^1 \Big[& J(\tau\xi_n^{hk} + \psi_n^{hk})^2 + \kappa^*((\varphi_n^{hk})_{xx} + \tau(\psi_n^{hk})_{xx})^2 \\ & + \tau(\kappa - \tau\kappa^*)(\psi_n^{hk})_{xx}^2 + a^*((\varphi_n^{hk})_x + \tau(\psi_n^{hk})_x)^2 + \xi^*(\varphi_n^{nk} + \tau\psi_n^{hk})^2 \Big]\, dx \end{aligned}$$

is shown in Figure 4 for some values of parameter κ ($\kappa = 0.2$, 1, 10, 100) in both natural and semi-log scales.

As can be clearly seen, an asymptotic exponential behavior is observed for the discrete energy when κ is less than 5, which agrees with the theoretical result. Moreover, we can also appreciate that, when higher values of parameter κ are chosen, the asymptotic behavior degenerates. Even, for the largest value $\kappa = 100$ the decay is not observed because a much larger final time is needed.

In a second example, we consider now the dependence of the solution with respect to parameter a in the case (ii) (i.e. it corresponds to the MGT-porous viscosity).

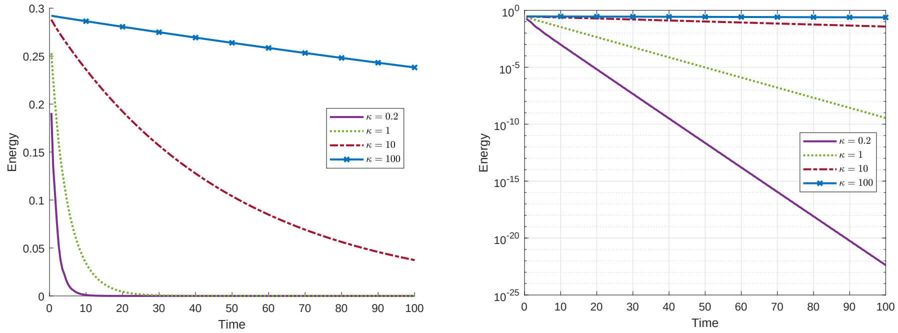

FIGURE 4. Example 1: Dependence of the solution with respect to parameter κ (fourth-order dissipation mechanism- Case (i)).

Therefore, the evolution in time of the discrete energy given by

$$E_n^{hk} = \frac{1}{2}\int_0^1 \Big[J(\varphi_{3n}^{hk})^2 + \kappa^*((\varphi_{1n}^{hk})_{xx})^2 \\ + a^*((\varphi_{1n}^{hk})_x)^2 + \tau(a - \tau a^*)((\psi_n^{hk})_x)^2 + \xi(\varphi_{1n}^{nk})^2 \Big]\, dx$$

is shown in Figure 5 for some values of parameter a ($a = 0.2$, 1, 10, 100) in both natural and semi-log scales.

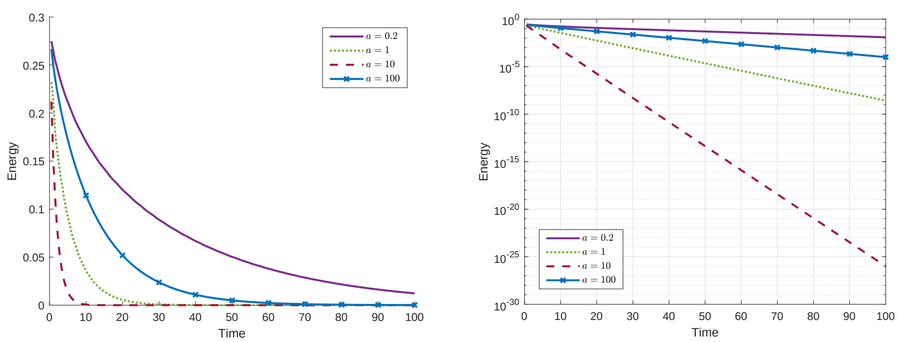

FIGURE 5. Example 2: Dependence of the solution with respect to parameter a (second-order dissipation mechanism- Case (ii)).

Again, we can see clearly the exponential decay of the discrete energy for all the values of the parameter. However, it seems that there is a critical optimal value for its decreasing between 10 and 100 since its ratio changes.

In the final example, we study the dependence of the solution with respect to parameter ξ in the case (iii) (i.e. it corresponds to the MGT-porous weak viscosity). Therefore, the evolution in time of the discrete energy given by

$$E_n^{hk} = \frac{1}{2}\int_0^1 \Big[J(\varphi_{3n}^{hk})^2 + \kappa^*((\varphi_{1n}^{hk})_{xx})^2 + a^*((\varphi_{1n}^{hk})_x)^2 \\ + \xi^*(\varphi_{1n}^{nk})^2 + \tau(\xi - \tau\xi^*)(\psi_n^{nk})^2 \Big]\, dx$$

is shown in Figure 6 for some values of parameter ξ ($\xi = 0.2$, 1, 10, 100) in both natural and semi-log scales.

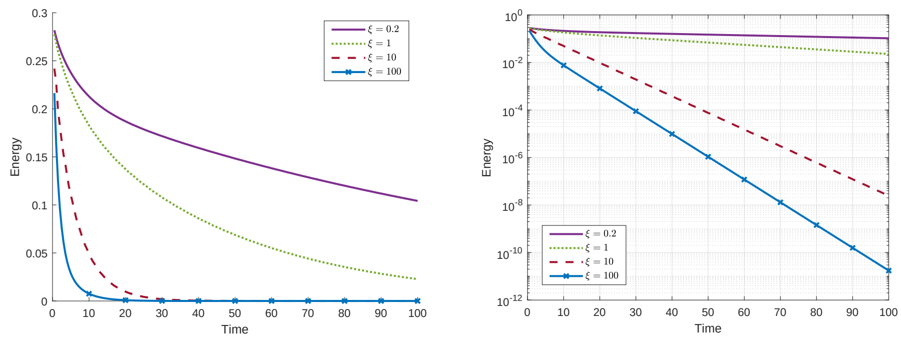

Figure 6. Example 3: Dependence of the solution with respect to parameter ξ (zeroth-order dissipation mechanism- Case (iii)).

As in the previous cases, the exponential energy decay is found for all the values although, contrary to the results of case (i), when parameter ξ increases the convergence ratio becomes higher.

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E-mail address: jacobo.gonzalez.baldonedo@uvigo.es
E-mail address: jose.fernandez@uvigo.es
E-mail address: ramon.quintanilla@upc.edu