The giant component of the directed configuration model revisited

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Abstract

We prove a law of large numbers for the order and size of the largest strongly connected component in the directed configuration model. Our result extends previous work by Cooper and Frieze [5].

1 Introduction and notations

An SCC (strongly connected component) in a digraph (directed graph) is a maximal sub-digraph in which there exists a directed path from every node to every other node. In this short note, we analyse the size of the giant component, i.e., the largest SCC, in the directed configuration model. This is a continuation of our previous work [4], which studied the diameter of the model.

We briefly introduce the model and our assumptions. For further discussions and references, see [4]. Let $[n] \coloneqq \{1, \dots, n\}$ be a set of n nodes. Let $\vec{\mathbf{d}}_n = ((d_1^-, d_1^+), \dots, (d_n^-, d_n^+))$ be a bi-degree sequence with $m_n \coloneqq \sum_{i\in[n]} d_i^+ = \sum_{i\in[n]} d_i^-$. The directed configuration model, $\vec{\mathbb{G}}_n$, is the random directed multigraph on $[n]$ generated by giving d_i^- in half-edges (*heads*) and d_i^+ out half-edges (*tails*) to node i, and then pairing the heads and tails uniformly at random.

Let $D_n = (D_n^-, D_n^+)$ be the degrees (number of tails and heads) of a uniform random node. Let $n_{k,\ell}$ be the number of (k, ℓ) in $\vec{\mathbf{d}}_n$. Let $\Delta_n = \max_{i\in[n]}\{d_i^-, d_i^+\}$. Consider a sequence of bi-degree sequences $(\vec{\mathbf{d}}_n)_{n\geq 1}$. Throughout the paper, we will assume the following condition is satisfied,

Condition 1.1. There exists a discrete probability distribution $D = (D^-, D^+)$ on $\mathbb{Z}^2_{\geq 0}$ with $\lambda_{k,\ell} \coloneqq \mathbb{P}\{D = (k, \ell)\}$ such that

(i) D_n converges to D in distribution: $\lim_{n\to\infty} \frac{n_{k,\ell}}{n} = \lambda_{k,\ell}$ for every $k, \ell \in \mathbb{Z}_{\geq 0}$;

(ii) D_n converges to D in expectation and the expectation is finite:

$$\lim_{n\to\infty} \mathbb{E}[D_n^-] = \lim_{n\to\infty} \mathbb{E}[D_n^+] = \mathbb{E}[D^-] = \mathbb{E}[D^+] =: \lambda \in (0, \infty); \tag{1.1}$$

(iii) D_n converges to D in second moment and they are finite: for $i, j \in \mathbb{Z}_{\geq 0}$, $i + j = 2$,

$$\lim_{n\to\infty} \mathbb{E}[(D_n^-)^i(D_n^+)^j] = \mathbb{E}[(D^-)^i(D^+)^j] < \infty \tag{1.2}$$

To state the main result, some parameters of D are needed. Let

$$\nu \coloneqq \frac{\mathbb{E}[D^- D^+]}{\lambda} < \infty, \tag{1.3}$$

where the inequality follows from conditions (ii) and (iii). Let $f(z,w) \coloneqq \sum_{i,j\geq 0} \lambda_{i,j} z^i w^j$ be the bivariate generating function of D. Let s_- and s_+ be the survival probabilities of the branching processes with offspring distributions which have generating functions $\frac{1}{\lambda}\frac{\partial f}{\partial w}(z,1)$ and $\frac{1}{\lambda}\frac{\partial f}{\partial z}(1,w)$ respectively. In other words, $\rho_- \coloneqq 1 - s_-$ and $\rho_+ \coloneqq 1 - s_+$ are, respectively, the smallest positive solutions to the equations

$$z = \frac{1}{\lambda}\frac{\partial f}{\partial w}(z,1), \qquad w = \frac{1}{\lambda}\frac{\partial f}{\partial z}(1,w). \tag{1.4}$$

Let $\mathcal{G}_n$ be the largest SCC in $\vec{\mathbb{G}}_n$. (If there is more than one such SCC, we choose an arbitrary one among them as $\mathcal{G}_n$.) Let $v(\mathcal{G}_n)$ be the number of nodes in $\mathcal{G}_n$. Let $e(\mathcal{G}_n)$ be the number of edges in $\mathcal{G}_n$. Our main result is the following theorem on $\mathcal{G}_n$:

Theorem 1.2. *Suppose that $(\vec{\mathbf{d}}_n)_{n\geq 1}$ satisfies Condition 1.1. If $\nu > 1$, then*

$$\frac{v(\mathcal{G}_n)}{n} \to \eta < \infty, \tag{1.5}$$

$$\frac{e(\mathcal{G}_n)}{n} \to \lambda s_- s_+ < \infty, \tag{1.6}$$

in expectation, in second moment and in probability, where

$$\eta \coloneqq \sum_{i,j\geq 0} \lambda_{i,j}(1-\rho_-^i)(1-\rho_+^j) = 1 + f(\rho_-,\rho_+) - f(\rho_-,1) - f(1,\rho_+). \tag{1.7}$$

If $\nu < 1$, then for all a_n with $a_n \to \infty$

$$\frac{v(\mathcal{G}_n)}{a_n} \to 0, \tag{1.8}$$

in expectation and in probability.

Remark 1.3. Under Condition 1.1, the probability that $\vec{\mathbb{G}}_n$ is simple is bounded away from 0, see [2, 10]. Thus Theorem 1.2 holds for a uniform random *simple* digraph with degree sequence $\vec{\mathbf{d}}_n$.

The two cases $\nu < 1$ and $\nu > 1$ are often referred to as *subcritical* and *supercritical* regimes. As shown in [4], in the supercritical case, $s_\pm > 0$ and $\eta > 0$. In other words, whp (with high probability), the size of the largest SCC is *bounded* in the first case and *linear* in the second one.

Equation (1.5) in Theorem 1.2 was first proved by Cooper and Frieze [5] under stronger conditions including $\mathbb{E}[(D_n^+)^2 D_n^-] = o(\Delta_n)$, $\mathbb{E}[(D_n^-)^2 D_n^+] = o(\Delta_n)$ and $\Delta_n = o(n^{1/12})$. Graf [9, Theorem 4.1] extended the existence of a linear order SCC provided that $\mathbb{E}[D_n^+ D_n^-]$ converges uniformly and $\Delta_n = o(n^{1/4})$. Condition 1.1 only implies that $\Delta_n = o(\sqrt{n})$, see [4, Corollary 2.4]. In the subcritical case, the results in [5, 9] only show that whp the largest SCC has order $O(\Delta_n^2 \log n)$ instead of $O(1)$.

The paper is organized as follows: In Section 2, we study the probability of certain events for branching processes. In Section 3, we recall a graph exploration process defined in [4] and extend it. Section 4 studies the probability that a set of half-edges to reach a large number of other half-edges. Section 5 shows that the number of nodes which can reach and can be reached from many nodes is concentrated around its mean. Then in Section 6 we show that these nodes form the giant. Finally in Section 7 we give an application of Theorem 1.2 to binomial random digraphs.

2 Branching processes

Let ξ be a random variable on $\mathbb{Z}_{\geq 0}$ and let $(\xi_{i,t})_{i\geq 1, t\geq 0}$ be iid (independent and identically distributed) copies of ξ. Let h_ξ be the generating function of ξ and $\nu_\xi \coloneqq h'_\xi(1) = \mathbb{E}[\xi]$. Let $(X_t)_{t\geq 0}$ be a branching process with offspring distribution ξ. If $X_t > 0$ for all t, then the branching process is said to *survive*; otherwise, it is said to *become extinct*. The following are well-known in the branching process theory (see, e.g., [14, Theorem 3.1] and [1, Theorem I.10.3], respectively):

Lemma 2.1. *Let ρ_ξ be the smallest nonnegative solution of $z = h_\xi(z)$. The survival probability is*

$$s_\xi \coloneqq \mathbb{P}\{\cap_{t\geq 1}[X_t > 0]\} = 1 - \rho_\xi. \tag{2.1}$$

Moreover, $s_\xi > 0$ if and only if $\nu_\xi > 1$.

Lemma 2.2. *Assume that $\nu_\xi \in (1,\infty)$. Then there exists a sequence $(m_{\xi,t})_{t\geq 0}$ for which $m_{\xi,t}^{1/t} \to \nu$, such that $X_t/m_{\xi,t} \to W_\xi$, where W_ξ is a non-negative random variable for which $\mathbb{P}\{W_\xi = 0\} = 1 - s_\xi$ and which is continuously distributed on $(0,\infty)$.*

The main result of this section is the following:

Lemma 2.3. *Let $(X_t)_{t\geq 0}$ be a branching process with offspring distribution ξ with $\nu_\xi \in (1,\infty)$. Let*

$$T_\omega \coloneqq \inf\{t : X_t \geq \omega\}. \tag{2.2}$$

Then for all $\varepsilon > 0$ and as $\omega \to \infty$,

$$\mathbb{P}\{T_\omega \leq (1+\varepsilon)\log_{\nu_\xi} \omega\} \to s_\xi. \tag{2.3}$$

Proof. Let $t_1 = \lfloor (1+\varepsilon)\log_{\nu_\xi}\omega \rfloor + 1$. It suffices to show that $\mathbb{P}\{T_\omega > t_1\} \to q_\xi \coloneqq 1 - s_\xi$. We split this probability into

$$\mathbb{P}\{T_\omega > t_1\} = \mathbb{P}\{[T_\omega > t_1] \cap [X_{t_1} = 0]\} + \mathbb{P}\{[T_\omega > t_1] \cap [X_{t_1} \in (0,\omega)]\} =: I_1 + I_2. \tag{2.4}$$

By Theorem 3.4 of [4], there exist constants $C > 0$ and $\hat{\nu} \in (0,1)$ (both depending only on ξ) such that for all $\varepsilon > 0$,

$$I_2 = \mathbb{P}\{\cap_{i=0}^{t_1} X_i \in (0,\omega)\} \leq C\hat{\nu}^{(1+\varepsilon)\log_{\nu_\xi}\omega - (1+o(1))\log_{\nu_\xi}\omega - 1} \leq C\hat{\nu}^{(\varepsilon/2)\log_{\nu_\xi}\omega} = o(1). \tag{2.5}$$

Let $Y_t = \sum_{i=0}^t X_i$. Let E denote the event that $(X_t)_{t\geq 0}$ becomes extinct, i.e., $X_t = 0$ for some $t \in \mathbb{N}$. If $q_\xi = \mathbb{P}\{E\} = 0$, then $I_1 = 0$ and we are done. Thus we can assume that $q_\xi > 0$. Then

$$I_1 \leq \mathbb{P}\{[Y_{t_1} \leq (1+t_1)\omega] \cap [X_{t_1} = 0]\} \leq \mathbb{P}\{Y_{t_1} \leq (1+t_1)\omega \mid E\}\mathbb{P}\{E\} \to \mathbb{P}\{E\} = q_\xi, \tag{2.6}$$

since a branching process conditioned on becoming extinct has a finite total progeny.

For a lower bound of I_1, note that $Y_t < \omega$ implies $T_\omega > t$. Thus,

$$I_1 \geq \mathbb{P}\{[Y_{t_1} < \omega] \cap [X_{t_1} = 0]\} = \mathbb{P}\{Y_{t_1} < \omega\} - \mathbb{P}\{[Y_{t_1} < \omega] \cap [X_{t_1} > 0]\}. \tag{2.7}$$

Note that

$$\mathbb{P}\{Y_{t_1} < \omega\} \geq \mathbb{P}\{Y_{t_1} < \omega \mid E\}\mathbb{P}\{E\} \to \mathbb{P}\{E\} = q_\xi. \tag{2.8}$$

By Theorem 6 of [12], there exists a sequence $(r_t)_{t\geq 0}$ with $r_t^{1/t} \to \nu_\xi$ such that for all $x > 0$,

$$\mathbb{P}\left\{\frac{Y_{t_1}}{r_{t_1}} < x \;\middle|\; X_{t_1} > 0\right\} \to \mathbb{P}\left\{Z_\xi < x \mid Z_\xi > 0\right\}, \tag{2.9}$$

where Z_ξ is a non-negative random variable for which $\mathbb{P}\{Z_\xi = 0\} = q_\xi$ and which has continuous distribution on $(0, \infty)$. Therefore, for all $\delta > 0$,

$$\mathbb{P}\left\{Y_{t_1} < \omega \mid X_{t_1} > 0\right\} \leq \mathbb{P}\left\{\frac{Y_{t_1}}{r_{t_1}} < \delta \;\middle|\; X_{t_1} > 0\right\} \to \mathbb{P}\left\{Z_\xi < \delta \mid Z_\xi > 0\right\}, \tag{2.10}$$

as $\omega \to \infty$. Since δ is arbitrary, we have

$$\mathbb{P}\left\{Y_{t_1} < \omega \mid X_{t_1} > 0\right\} \to 0. \tag{2.11}$$

Putting (2.11) and (2.8) into (2.7) gives the desired lower bound. □

Lemma 2.3 can be generalized to multiple iid branching processes as follows:

Corollary 2.4. *Let $(X_{1,t})_{t\geq 0}, \dots, (X_{x,t})_{t\geq 0}$ be $x \in \mathbb{N}$ independent branching processes with offspring distribution ξ. Assume that $\nu_\xi \in (1, \infty)$. Let*

$$T_\omega^{(x)} := \inf\Big\{t : \sum_{i=1}^{x} X_{i,t} \geq \omega\Big\}. \tag{2.12}$$

Then for all $\varepsilon > 0$ and as $\omega \to \infty$,

$$\mathbb{P}\{T_\omega^{(x)} \leq (1+\varepsilon)\log_{\nu_\xi} \omega\} \to 1 - (1 - s_\xi)^x. \tag{2.13}$$

Proof. Let $t_1 = \lfloor (1+\varepsilon)\log_{\nu_\xi} \omega \rfloor + 1$. Let $T_{i,\omega} = \inf\{t \geq 1 : X_{i,t} \geq \omega\}$. By Lemma 2.3

$$\mathbb{P}\{T_\omega^{(x)} > t_1\} \leq \mathbb{P}\{\cap_{i=1}^x [T_{i,\omega} > t_1]\} = \prod_{i=1}^{x} \mathbb{P}\{T_{i,\omega} > t_1\} \to (1 - s_\xi)^x, \tag{2.14}$$

and

$$\mathbb{P}\{T_\omega^{(x)} > t_1\} \geq \mathbb{P}\{\cap_{i=1}^x [T_{i,\frac{\omega}{x}} > t_1]\} = \prod_{i=1}^{x} \mathbb{P}\{T_{i,\frac{\omega}{x}} > t_1\} \to (1 - s_\xi)^x. \tag{2.15}$$

□

3 Exploring the graph

We extend the Breadth First Search (BFS) graph exploration process of $\vec{\mathbb{G}}_n$ defined in [4].

For $\mathcal{I} \subseteq [n]$, let $\mathcal{E}^\pm(\mathcal{I})$ be the set of heads/tails incident to the nodes in $\mathcal{I}$. Let $\mathcal{E}^\pm := \mathcal{E}^\pm([n])$. For $\mathcal{X} \subseteq \mathcal{E}^\pm$, let $\mathcal{V}(\mathcal{X})$ be the set of nodes incident to $\mathcal{X}$. Let H be a partial pairing of half edges in $\mathcal{E}^\pm$. Let $\mathcal{P}^\pm(H) \subseteq \mathcal{E}^\pm$ be the set of heads/tails which are paired in H. Let $\mathcal{V}(H) = \mathcal{V}(\mathcal{P}^\pm(H))$. Let $\mathcal{F}^\pm(H) := \mathcal{E}^\pm(\mathcal{V}(H)) \setminus \mathcal{P}^\pm(H)$ be the unpaired heads/tails which are incident to $\mathcal{V}(H)$. Let E_H denote the event that H is part of $\vec{\mathbb{G}}_n$. We will explore the graph conditioning on E_H.

We start from an arbitrary set $\mathcal{X}^+$ of *unpaired* tails. In this process, we create random pairings of half-edges one by one and keep each half-edge in exactly one of the four states — *active*, *paired*, *fatal* or *undiscovered*. Let $\mathcal{A}_i^{\pm}$, $\mathcal{P}_i^{\pm}$, $\mathcal{F}_i^{\pm}$ and $\mathcal{U}_i^{\pm}$ denote the set of heads/tails in the four states respectively after the i-th pairing of half-edges. Initially, let

$$\mathcal{A}_0^+ = \mathcal{X}^+,\ \mathcal{A}_0^- = \mathcal{E}^-(\mathcal{V}(\mathcal{X}^+)),\ \mathcal{P}_0^{\pm} = \mathcal{P}^{\pm}(H),\ \mathcal{F}_0^{\pm} = \mathcal{F}^{\pm}(H),\ \mathcal{U}_0^{\pm} = \mathcal{E}^{\pm} \setminus (\mathcal{A}_0^{\pm} \cup \mathcal{P}_0^{\pm} \cup \mathcal{F}_0^{\pm}). \tag{3.1}$$

Then set $i = 1$ and proceed as follows:

(i) Let e_i^+ be one of the tails which became active earliest in $\mathcal{A}_{i-1}^+$.

(ii) Pair e_i^+ with a head e_i^- chosen uniformly at random from $\mathcal{E}^- \setminus \mathcal{P}_{i-1}^-$. Let $\mathcal{P}_i^{\pm} = \mathcal{P}_{i-1}^{\pm} \cup \{e_i^{\pm}\}$.

(iii) If $e_i^- \in \mathcal{F}_{i-1}^-$, then terminate; if $e_i^- \in \mathcal{A}_{i-1}^-$, then $\mathcal{A}_i^{\pm} = \mathcal{A}_{i-1}^{\pm} \setminus \{e_i^{\pm}\}$; and if $e_i^- \in \mathcal{U}_{i-1}^-$, then $\mathcal{A}_i^{\pm} = (\mathcal{A}_{i-1}^{\pm} \cup \mathcal{E}^{\pm}(v_i)) \setminus \{e_i^{\pm}\}$ where $v_i = \mathcal{V}(e_i^-)$.

(iv) If $\mathcal{A}_i^+ = \emptyset$ terminate; otherwise, $\mathcal{F}_i^{\pm} = \mathcal{F}_{i-1}^{\pm}$, $\mathcal{U}_i^{\pm} = \mathcal{E}^{\pm} \setminus (\mathcal{A}_i^{\pm} \cup \mathcal{P}_i^{\pm} \cup \mathcal{F}_i^{\pm})$, $i = i+1$ and go to (i).

Let $F_{\mathcal{X}^+}(0)$ be a forest with $|\mathcal{X}^+|$ isolated nodes corresponding to $\mathcal{X}^+$. Given $F_{\mathcal{X}^+}(i-1)$, $F_{\mathcal{X}^+}(i)$ is constructed as follows: if $e_i^- \in \mathcal{U}_{i-1}^-$, then construct $F_{\mathcal{X}^+}(i)$ from $F_{\mathcal{X}^+}(i-1)$ by adding $|\mathcal{E}^+(v_i)|$ child nodes to the node representing e_i^+, each of which representing a tail in $\mathcal{E}^+(v_i)$; otherwise, let $F_{\mathcal{X}^+}(i) = F_{\mathcal{X}^+}(i-1)$. While $F_{\mathcal{X}^+}(i)$ is an unlabelled forest, its nodes correspond to the tails in $(\mathcal{P}_i^+ \setminus \mathcal{P}_0^+) \cup \mathcal{A}_i^+$. So we can assign a label *paired* or *active* to each node of $F_{\mathcal{X}^+}(i)$.

Given half-edges e_1 and e_2, the distance $\mathrm{dist}(e_1, e_2)$ is the length of the shortest path from $\mathcal{V}(e_1)$ to $\mathcal{V}(e_2)$ which starts with the edge containing e_1 and ends with the edge containing e_2.

If i_t is the last step where a tail at distance t from $\mathcal{X}^+$ is paired, then $F_{\mathcal{X}^+}(i_t)$ satisfies: (i) the height is t; (ii) the set of actives nodes is the t-th level. We call a rooted forest F *incomplete* if it satisfies (i)-(ii). We let $p(F)$ be the number of *paired* nodes in F.

3.1 Size biased distributions

We recall some notation in [4]. The *in- and out-size biased* distributions of D_n and D are defined

$$\mathbb{P}\{(D_n)_{\text{in}} = (k-1, \ell)\} = \frac{k n_{k,\ell}}{m_n}, \qquad \mathbb{P}\{(D_n)_{\text{out}} = (k, \ell-1)\} = \frac{\ell n_{k,\ell}}{m_n}, \tag{3.2}$$

$$\mathbb{P}\{D_{\text{in}} = (k-1, \ell)\} = \frac{k \lambda_{k,\ell}}{\lambda}, \qquad \mathbb{P}\{D_{\text{out}} = (k, \ell-1)\} = \frac{\ell \lambda_{k,\ell}}{\lambda}. \tag{3.3}$$

Then, by (i) of Condition 1.1, $(D_n)_{\text{in}} \to D_{\text{in}}$ and $(D_n)_{\text{out}} \to D_{\text{out}}$, and by (iii) of Condition 1.1,

$$\lim_{n\to\infty} \mathbb{E}\left[(D_n)_{\text{in}}^+\right] = \lim_{n\to\infty} \mathbb{E}\left[(D_n)_{\text{out}}^-\right] = \mathbb{E}\left[D_{\text{in}}^+\right] = \mathbb{E}\left[D_{\text{out}}^-\right] = \frac{\mathbb{E}\left[D^+ D^-\right]}{\lambda} = \nu. \tag{3.4}$$

Let s_{n+}, s_{n-}, s_+ and s_- be the survival probabilities of the branching processes with distribution $(D_n)_{\text{in}}^+$, $(D_n)_{\text{out}}^-$, D_{in}^+ and D_{out}^- respectively. Then as we have shown in [4], $s_{n\pm} \to s_{\pm}$.

3.2 Coupling with branching processes

Consider the probability distribution $Q_n := (D_n)_{\text{in}}^+$ which satisfies for all $\ell \geq 0$,

$$\mathbb{P}\{Q_n = \ell\} = q_{n,\ell} := \frac{\sum_{k\geq 1} k n_{k,\ell}}{m_n}. \tag{3.5}$$

In [4, Section 3], it has been shown that $Q_n \to D_{\rm in}^+$ in distribution and in expectation. In particular, by (3.4) $\mathbb{E}[Q_n] \to \mathbb{E}[D_{\rm in}^+] = \nu$. Also in [4], we showed that the exploration process starting from one tail can be approximated by a branching process with offspring distribution Q_n. Similarly, the extended exploration process starting from $\mathcal{X}^+$ can be approximated by $|\mathcal{X}^+|$ independent branching processes with offspring distribution Q_n.

For $\beta \in (0, 1/10)$, consider the distributions $Q_n^\downarrow = Q_n^\downarrow(\beta)$ and $Q_n^\uparrow = Q_n^\uparrow(\beta)$ defined by

$$\mathbb{P}\left\{Q_n^\downarrow = \ell\right\} = q_{n,\ell}^\downarrow := \begin{cases} c^\downarrow q_{n,\ell} & \text{if } q_{n,\ell} \geq n^{-2\beta} \text{and } \ell \leq n^\beta \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

$$\mathbb{P}\left\{Q_n^\uparrow = \ell\right\} = q_{n,\ell}^\uparrow := \begin{cases} c^\uparrow q_{n,\ell} & \ell \geq 1 \\ c^\uparrow q_{n,0} + n^{-1/2+2\beta} & \ell = 0 \end{cases} \tag{3.7}$$

where $c^\downarrow$ and $c^\uparrow$ are normalising constants.

Let $\mathrm{GW}_\xi^{(x)} = (\mathrm{GW}_{1,\xi}, \dots, \mathrm{GW}_{x,\xi})$ be x independent Galton-Watson trees with offspring distribution ξ. Let $F = (T_1, \dots, T_x)$ be an incomplete forest. Let $\mathrm{GW}_\xi^{(x)} \cong F$ denote that for every $i \in [x]$, T_i is a root subtree of $\mathrm{GW}_{i,\xi}$ and all paired nodes of T_i have the same degree in $\mathrm{GW}_{i,\xi}$.

The following lemma is a straightforward extension of [4, Lemma 5.3] and we omit its proof:

Lemma 3.1. *Let $\beta \in (0, 1/10)$ and let H be a partial pairing with $|\mathcal{V}(H)| \leq n^{1-6\beta}$. Let $\mathcal{X}^+ \subset \mathcal{E}^+$ with $|\mathcal{X}^+| = x$. For every incomplete forest F with $p(F) \leq n^\beta$, we have*

$$(1 + o(1))\mathbb{P}\left\{\mathrm{GW}_{Q_n^\downarrow(\beta)}^{(x)} \cong F\right\} \leq \mathbb{P}\left\{F_{\mathcal{X}^+}(p(F)) = F \mid E_H\right\} \leq (1 + o(1))\mathbb{P}\left\{\mathrm{GW}_{Q_n^\uparrow(\beta)}^{(x)} \cong F\right\}. \tag{3.8}$$

4 Expansion probability

Let $\mathcal{N}_t^\pm(\mathcal{X}^\pm)$ and $\mathcal{N}_{\leq t}^\pm(\mathcal{X}^\pm)$ be the sets of heads/tails at distance t and at most t from $\mathcal{X}^\pm \subseteq \mathcal{E}^\pm$ respectively. From now on, let

$$\omega := \log^6 n, \qquad t_0 := \log_\nu \omega. \tag{4.1}$$

Let $t_\omega(\mathcal{X}^\pm)$ be the *expansion time* of $\mathcal{X}^\pm$ defined as

$$t_\omega(\mathcal{X}^\pm) := \inf\left\{t \geq 1 : \left|\mathcal{N}_t^\pm(\mathcal{X}^\pm)\right| \geq \omega\right\}. \tag{4.2}$$

For brevity, we write $\mathcal{N}^{\leq\omega}(\mathcal{X}^\pm) = \cup_{t=1}^{t_\omega}\mathcal{N}_t^\pm(\mathcal{X}^\pm)$.

Given H a partial pairing of $\mathcal{E}^\pm$ and $\mathcal{X}^\pm \subseteq \mathcal{E}^\pm$, we consider the following two events:

$$\begin{aligned} A_1(\mathcal{X}^\pm, \varepsilon) &:= [t_\omega(\mathcal{X}^\pm) \leq (1 + \varepsilon)t_0]. \\ A_2(\mathcal{X}^\pm, H) &:= \left[\mathcal{N}^{\leq\omega}(\mathcal{X}^\pm) \cap \mathcal{F}^\pm(H) = \emptyset\right]. \end{aligned} \tag{4.3}$$

The first lemma in this section shows that the probability that both these events happen is close to the survival probability of a branching process.

Lemma 4.1. *Assume that $\nu > 1$. Fix $x \in \mathbb{N}$, $\varepsilon \in (0, 1/2)$ and $\gamma \in (0, 1)$. Then uniformly for all choices of partial pairing H and $\mathcal{X}^\pm \subseteq \mathcal{E}^\pm$ with $|\mathcal{V}(H)| \leq n^{1-\gamma}$, $|\mathcal{X}^\pm| = x$, as $n \to \infty$,*

$$\mathbb{P}\left\{A_1(\mathcal{X}^\pm, \varepsilon) \cap A_2(\mathcal{X}^\pm, H) \;\middle|\; E_H\right\} = (1 + o(1))(1 - \rho_\pm^x). \tag{4.4}$$

Proof. Let $\mathcal{F}_{x,t,\omega}$ be the class of incomplete forests F with x trees, height t and such that only the last level has at least ω nodes. Let $t_1 = \lfloor (1+\varepsilon)t_0 \rfloor$. For $t \le t_1$ and $F \in \mathcal{F}_{x,t,\omega}$, we have $(t-1) \le p(F) \le x\omega t = O(\log^7 n)$. Let $\beta = \gamma/100$. Let $X^{\uparrow}_{1,t}, \dots, X^{\uparrow}_{x,t}$ be the sizes of the t-th generation of x iid branching processes with offspring distribution $Q^{\uparrow}_n(\beta)$ and let $s^{\uparrow}_{+n}$ be the survival probability of each one. Since $Q^{\uparrow}_n \to D^{+}_{\text{in}}$ in distribution, we have $s^{\uparrow}_{+n} \to s_+ = 1 - \rho_+ > 0$.

Let $T^{\uparrow}_{\omega} = \inf\{t \ge 1 : \sum_{i=1}^{x} X^{\uparrow}_{i,t} \ge \omega\}$. By Corollary 2.4 and Lemma 3.1, the LHS of (4.4) is

$$\begin{aligned}\sum_{t=1}^{t_1}\sum_{j=t-1}^{\lfloor x\omega t\rfloor}\sum_{\substack{F\in\mathcal{F}_{x,t,\omega}\\ p(F)=j}} \mathbb{P}\left\{F_{\mathcal{X}^+}(x) = F \mid E_H\right\} &\le (1+o(1))\sum_{t=1}^{t_1}\sum_{j=t-1}^{\lfloor x\omega t\rfloor}\sum_{\substack{F\in\mathcal{F}_{x,t,\omega}\\ p(F)=j}} \mathbb{P}\left\{\mathrm{GW}_{Q^{\uparrow}_n(\beta)} \cong F\right\} \\ &= (1+o(1))\mathbb{P}\left\{T^{\uparrow}_{\omega} \le t_1\right\} \\ &= (1+o(1))(1-(1-s^{\uparrow}_{+n})^{x}) \\ &= (1+o(1))(1-\rho_+^{x}),\end{aligned} \tag{4.5}$$

where we used that $\nu > 0$ implies $\rho_\pm < 1$. The lower bound follows from a similar argument. $\square$

Our next lemma shows that when $|\mathcal{X}^+||\mathcal{X}^-|$ is small, $\mathcal{X}^+$ and $\mathcal{X}^-$ are unlikely to be too close. We omit the proof since it follows from an easy adaptation of the proof in [4, Proposition 7.2].

Lemma 4.2. *Assume that $\nu > 1$. Fix $\varepsilon \in (0, 1/2)$ and $\gamma \in (0,1)$. Then uniformly for all choices of partial pairing H and $\mathcal{X}^\pm \subseteq \mathcal{E}^\pm$ with $|\mathcal{V}(H)| \le n^{1-\gamma}$ and $|\mathcal{X}^+||\mathcal{X}^-| \le \omega\sqrt{n}$, we have*

$$\mathbb{P}\left\{\mathrm{dist}(\mathcal{X}^+, \mathcal{X}^-) \le \left(\frac{1}{2} - \varepsilon\right)\log_\nu n \;\middle|\; E_H\right\} = o(n^{-\varepsilon/2}). \tag{4.6}$$

The previous lemma allows us to remove $A_2(\mathcal{X}^\pm, H)$ in Lemma 4.1.

Lemma 4.3. *Assume that $\nu > 1$. Fix $x^\pm \in \mathbb{N}$ and $\varepsilon \in (0, 1/2)$. Then uniformly for all choices of partial pairing H and $\mathcal{X}^\pm \subseteq \mathcal{E}^\pm$ with $|\mathcal{V}(H)| = o(\omega^2)$, $|\mathcal{X}^\pm| = x^\pm$, we have, as $n \to \infty$,*

$$\mathbb{P}\left\{A_1(\mathcal{X}^\pm, \varepsilon) \mid E_H\right\} = (1+o(1))(1-\rho_\pm^{x^\pm}), \tag{4.7}$$

$$\mathbb{P}\left\{A_1(\mathcal{X}^+, \varepsilon) \cap A_1(\mathcal{X}^-, \varepsilon) \mid E_H\right\} = (1+o(1))(1-\rho_-^{x^-})(1-\rho_+^{x^+}). \tag{4.8}$$

Proof. We will prove it for $\mathcal{X}^+$; a similar argument works for $\mathcal{X}^-$. Let

$$E_1 = A_1(\mathcal{X}^+, \varepsilon), \quad E_2 = A_2(\mathcal{X}^+, H), \quad E_3 = A_1(\mathcal{X}^-, \varepsilon). \tag{4.9}$$

Note that the event E_2 happens if and only if $\mathrm{dist}(\mathcal{X}^+, \mathcal{F}^+(H)) > t_\omega(\mathcal{X}^+)$.

By Lemma 4.1, the LHS of (4.7) equals

$$\begin{aligned}\mathbb{P}\left\{E_1 \mid E_H\right\} &= \mathbb{P}\left\{E_1 \cap E_2 \mid E_H\right\} + \mathbb{P}\left\{E_1 \cap E_2^c \mid E_H\right\} \\ &= (1+o(1))(1-\rho_+^{x^+}) + \mathbb{P}\left\{E_1 \cap E_2^c \mid E_H\right\}.\end{aligned} \tag{4.10}$$

Since $|\mathcal{V}(H)| = o(\omega^2)$, by [4, Lemma 2.2] we have $|\mathcal{E}^+(H)| = o(\omega\sqrt{n})$. By Lemma 4.2, for $\delta < 1/2$,

$$\begin{aligned}\mathbb{P}\left\{E_1 \cap E_2^c \mid E_H\right\} &\le \mathbb{P}\left\{\mathrm{dist}(\mathcal{X}^+, \mathcal{F}^+(H)) \le 4t_0 \mid E_H\right\} \\ &\le \mathbb{P}\left\{\mathrm{dist}(\mathcal{X}^+, \mathcal{F}^+(H)) \le \left(\frac{1}{2} - \delta\right)\log n \;\middle|\; E_H\right\} = o(1).\end{aligned} \tag{4.11}$$

Let $\mathcal{H}$ be the set of all possible partial pairings in $\mathcal{N}^{\le\omega}(\mathcal{X}^+)$ such that $E_1 \cap E_H$ happens. Then $H' \in \mathcal{H}$ implies that $|\mathcal{V}(H')|, |\mathcal{V}(H' \cup H)| = o(\omega^2)$. Using Lemma 4.1 again, we have

$$\begin{aligned}\mathbb{P}\{E_1 \cap E_3 \mid E_H\} &= \sum_{H' \in \mathcal{H}} \mathbb{P}\{E_3 \mid E_{H \cup H'}\} \mathbb{P}\{E_{H \cup H'} \mid E_H\} \\ &= \sum_{H' \in \mathcal{H}} (1+o(1))(1-\rho_-^{x^-})\mathbb{P}\{E_{H' \cup H} \mid E_H\} \\ &= (1+o(1))(1-\rho_-^{x^-})\mathbb{P}\{E_1 \mid E_H\} \\ &= (1+o(1))(1-\rho_-^{x^-})(1-\rho_+^{x^+}). \end{aligned} \tag{4.12}$$

□

Unsurprisingly, Lemma 4.3 can be extended to a fixed number of pairs of head-sets and tail-sets:

Lemma 4.4. *Assume that $\nu > 1$. Fix $i, x_1^\pm, \dots, x_i^\pm \in \mathbb{N}$ and $\varepsilon \in (0, 1/2)$. Then uniformly for all disjoint sets of tails $(\mathcal{X}_1^+, \dots, \mathcal{X}_i^-)$ and disjoint sets of heads $(\mathcal{X}_1^+, \dots, \mathcal{X}_i^-)$ with $\left|\mathcal{X}_j^\pm\right| = x_j^\pm$ for $j \in [i]$, we have, as $n \to \infty$,*

$$\mathbb{P}\left\{\cap_{j=1}^i [A_1(\mathcal{X}^+, \varepsilon) \cap A_1(\mathcal{X}^-, \varepsilon)]\right\} = (1+o(1)) \prod_{j=1}^i (1-\rho_-^{x_j^-})(1-\rho_+^{x_j^+}). \tag{4.13}$$

Proof. We prove it by induction. The case $i = 1$ follows by Lemma 4.3 with H an empty pairing.

Let E_j denote the event in the LHS of (4.13). Assume that the lemma holds for some $i \ge 1$. Let $\mathcal{H}$ be the sets of all possible partial pairings in $\cup_{j=1}^i [\mathcal{N}^{\le\omega}(\mathcal{X}_j^+) \cup \mathcal{N}^{\le\omega}(\mathcal{X}_j^-)]$ compatible with E_i. If $H \in \mathcal{H}$, then $|\mathcal{V}(H)| = o(\omega^2)$. Using Lemma 4.1 as in (4.13), we conclude

$$\mathbb{P}\{E_{i+1}\} = \sum_{H \in \mathcal{H}} \mathbb{P}\{E_{i+1} \mid E_H\} \mathbb{P}\{E_H\} = (1+o(1))(1-\rho_-^{x_{i+1}^-})(1-\rho_+^{x_{i+1}^+})\mathbb{P}\{E_i\}.$$

□

The last lemma shows that expansions are unlikely to happen very late.

Lemma 4.5. *Assume that $\nu > 1$. Fix $x^\pm \in \mathbb{N}$ and $\varepsilon \in (0, 1/2)$. Then uniformly for all choices of $\mathcal{X}^\pm \subseteq \mathcal{E}^\pm$ with $|\mathcal{X}^\pm| = x^\pm$, as $n \to \infty$,*

$$\mathbb{P}\left\{t_\omega(\mathcal{X}^\pm) \in ((1+\varepsilon)t_0, \infty)\right\} = o(1). \tag{4.14}$$

Proof. Let $t_1 = \lfloor (1+\varepsilon)t_0 \rfloor$. Note that

$$\mathbb{P}\left\{t_\omega(\mathcal{X}^\pm) \in (t_1, \infty)\right\} \le \sum_{e^\pm \in \mathcal{X}^\pm} \mathbb{P}\left\{t_\omega(e^\pm) \in (t_1, \infty)\right\}. \tag{4.15}$$

Thus we may assume that $\mathcal{X}^\pm = \{e^\pm\}$. Let $X_t^\uparrow$ be the size of the t-th generation of a branching process with offspring distribution $Q_n^\uparrow(\beta)$ for some $\beta \in (0, 1/10)$. Let $T_\omega = \inf\{t \ge 1 : X_t^\uparrow \ge \omega\}$. Then it follows from [4, Theorem 3.4] that there exist constants $C > 0$ and $\hat{\nu} \in (0, 1)$ such that

$$\mathbb{P}\{T_\omega \in (t_1, \infty)\} \le \mathbb{P}\left\{\cap_{t=0}^{t_1} [X_t \in (0, \omega)]\right\} \le C((1+o(1)\hat{\nu})^{(1+\varepsilon)t_0 - (1+o(1))t_0} = o(1). \tag{4.16}$$

By the same argument as in Lemma 4.1, this implies $\mathbb{P}\{t_\omega(e^\pm) \in (t_1, \infty)\} = o(1)$. □

5 Expectation and variance

Lemma 5.1. *Assume that* $\nu > 1$. *Let*

$$\mathcal{L} \coloneqq \left\{ v \in [n] : t_\omega(\mathcal{E}^+(v)) < \infty,\, t_\omega(\mathcal{E}^-(v)) < \infty \right\}. \tag{5.1}$$

Then

$$\frac{\mathbb{E}[|\mathcal{L}|]}{n} \to \eta, \qquad \frac{\mathbb{E}[|\mathcal{L}|^2]}{n^2} \to \eta^2, \tag{5.2}$$

where η *is defined as in* (1.7). *Thus,* $|\mathcal{L}|/n \to \eta$ *in probability.*

Proof. As $\rho_\pm < 1$ and $\sum_{i,j\ge 0} \lambda_{i,j} = 1$, we have $\eta \in (0,1)$. Fix $\varepsilon \in (0, 1/2)$. Define

$$\mathcal{L}(\varepsilon) \coloneqq \left\{ v \in [n] : t_\omega(\mathcal{E}^+(v)) < (1+\varepsilon)t_0,\, t_\omega(\mathcal{E}^-(v)) < (1+\varepsilon)t_0 \right\}. \tag{5.3}$$

and note that $\mathcal{L}(\varepsilon) \subseteq \mathcal{L}$. Given $v \in [n]$ with i heads and j tails, it follows from Lemma 4.4 that

$$p_{i,j} \coloneqq \mathbb{P}\left\{ v \in \mathcal{L}(\varepsilon) \right\} = (1+o(1))(1-\rho_-^i)(1-\rho_+^j). \tag{5.4}$$

Since there are $n_{i,j}$ such nodes, by (i) of Condition 1.1,

$$\frac{\mathbb{E}[|\mathcal{L}(\varepsilon)|]}{n} = \sum_{i,j\ge 0} \frac{n_{i,j}}{n} p_{i,j} = \sum_{i,j\ge 0} (1+o(1))\lambda_{i,j}(1-\rho_-^i)(1-\rho_+^j) \to \eta. \tag{5.5}$$

To see that the sum above converges to η, note that $\sum_{i,j\ge 0} \frac{n_{i,j}}{n} = 1$ and $p_{i,j} \le 1$. Thus we can apply the dominated convergence theorem by considering the double sum as an integral over $\mathbb{Z}_{\ge 0}^2$ with respect to the counting measure. Lemma 4.5 implies $\mathbb{P}\left\{ v \in \mathcal{L} \setminus \mathcal{L}(\varepsilon) \right\} = o(1)$. Thus $\mathbb{E}[|\mathcal{L} \setminus \mathcal{L}(\varepsilon)|] = o(n)$, which finishes the proof for the expectation.

Given distinct $v_1, v_2 \in [n]$ with degrees (i_1, j_1) and (i_2, j_2), again by Lemma 4.4

$$p_{i_1,j_1,i_2,j_2} \coloneqq \mathbb{P}\left\{ [v_1 \in \mathcal{L}(\varepsilon)] \cap [v_2 \in \mathcal{L}(\varepsilon)] \right\} = (1+o(1)) \prod_{r=1}^{2} (1-\rho_-^{i_r})(1-\rho_+^{j_r}). \tag{5.6}$$

By the same convergence argument used in (5.5), we have

$$\frac{\mathbb{E}[|\mathcal{L}(\varepsilon)|^2]}{n^2} = o(1) + \sum_{i_1,j_1,i_2,j_2\ge 0} \frac{n_{i_1,j_1} n_{i_2,j_2}}{n^2} p_{i_1,j_1,i_2,j_2} \to \eta^2. \tag{5.7}$$

As $\mathbb{E}[|\mathcal{L} \setminus \mathcal{L}(\varepsilon)|] = o(n)$, the following concludes the proof for the second moment:

$$\mathbb{E}[|\mathcal{L}|^2 - |\mathcal{L}(\varepsilon)|^2] \le 2n\mathbb{E}[|\mathcal{L} \setminus \mathcal{L}(\varepsilon)|] = o(n^2). \qquad \square$$

Lemma 5.2. *Assume that* $\nu > 1$. *Let* $\mathcal{L}_e$ *be the set of edges whose both endpoints are in* $\mathcal{L}$. *Then*

$$\frac{\mathbb{E}[|\mathcal{L}_e|]}{n} \to \lambda s_- s_+, \qquad \frac{\mathbb{E}[|\mathcal{L}_e|^2]}{n^2} \to (\lambda s_- s_+)^2. \tag{5.8}$$

Thus $|\mathcal{L}_e|/n \to \zeta$ *in probability.*

Proof. We only sketch the proof since the argument is very similar to that of Lemma 5.1.

Given $v_1, v_2 \in [n]$ with degrees (i_1, j_1) and (i_2, j_2) respectively, the number of edges X_{v_1,v_2} from v_1 to v_2 satisfies $\mathbb{E}[X_{v_1,v_2}] = j_1 i_2/m_n$. It follows from Lemma 4.4 that conditioning on X_{v_1,v_2}, the probability that both v_1 and v_2 are in $\mathcal{L}$ converges to $p_{v_1 v_2} := (1-\rho_-^{i_1})(1-\rho_+^{j_2})$. We have

$$
\begin{aligned}
\frac{\mathbb{E}\left[|\mathcal{L}_e|\right]}{n} &= \sum_{v_1,v_2\in[n]} \frac{\mathbb{E}[X_{v_1,v_2}]\cdot(1+o(1))p_{v_1,v_2}}{n} \\
&= \sum_{i_1,j_1,i_2,j_2\geq 0} (1+o(1))\frac{n_{i_1,j_1} n_{i_2,j_2}}{n}\frac{j_1 i_2}{m_n}(1-\rho_-^{i_1})(1-\rho_+^{j_2}) \\
&= \frac{1}{\lambda}\sum_{i_1,j_1\geq 0}\sum_{i_2,j_2\geq 0}(1+o(1))\lambda_{i_1,j_1} j_1 (1-\rho_-^{i_1})\cdot \lambda_{i_2,j_2} i_2 (1-\rho_+^{j_2}) \\
&\to \lambda\left(1-\frac{1}{\lambda}\frac{\partial f}{\partial w}(\rho_-,1)\right)\left(1-\frac{1}{\lambda}\frac{\partial f}{\partial z}(1,\rho_+)\right) = \lambda s_- s_+.
\end{aligned}
$$

The proof for the second moment is similar and we omit it. ☐

6 Proof of Theorem 1.2

If $\nu > 1$, it suffices to show that whp the set $\mathcal{L}$ defined in (5.1) exactly coincides with the largest SCC. Then (1.5) and (1.6) in Theorem 1.2 follow immediately from Lemma 5.1 and Lemma 5.2.

By [4, Proposition 7.2], uniformly for all $\mathcal{X}^{\pm} \subseteq \mathcal{E}^{\pm}$ with $|\mathcal{X}^{\pm}| \geq \omega$,

$$\mathbb{P}\left\{\mathrm{dist}(\mathcal{X}^+, \mathcal{X}^-) = \infty\right\} = o(n^{-100}), \tag{6.1}$$

and $\mathcal{L}$ is contained in a SCC whp. We will show that whp there is no other vertex in it. Let

$$A_3(e^{\pm}, t) = \left[\cap_{r=1}^{t}[0 < \mathcal{N}_r(e^{\pm}) < \omega]\right]. \tag{6.2}$$

By [4, Proposition 6.1], there exists a constant $\hat{\nu}_{\pm} \in (0,1)$ such that for $t = \Theta(\log n)$,

$$\mathbb{P}\left\{A_3(e^{\pm}, t)\right\} = \hat{\nu}_{\pm}^{(1+o(1))t}. \tag{6.3}$$

Thus, letting $t_2^{\pm} = \lceil 2\log_{1/\hat{\nu}_{\pm}}(n)\rceil$, we have

$$\mathbb{P}\left\{\cup_{e^+\in\mathcal{E}^+}\cup_{e^-\in\mathcal{E}^-}\left[A_3(e^+, t_2^+)\cup A_3(e^-, t_2^-)\right]\right\} \leq (m_n n^{-3/2})^2 = o(1). \tag{6.4}$$

Therefore, whp, each node $v \in [n]\setminus\mathcal{L}$ either can reach or can be reached from at most $\omega t_2^{\pm} = O(\log^7 n)$ other nodes. This implies that whp $\mathcal{L}$ is a SCC and that any other SCC has order $O(\log^7 n)$. This concludes the proof of the supercritical case.

For the subcritical case, we first show the following lemma.

Lemma 6.1. *Assume that $\nu < 1$. Let $C_{n,\geq \ell}$ be the number of directed simple cycles in $\vec{\mathbb{G}}_n$ of length at least ℓ. Then,*

$$\limsup_{n\to\infty} \mathbb{E}[C_{n,\geq 1}] \leq \log\left(\frac{1}{1-\nu}\right). \tag{6.5}$$

Moreover, for any $\ell_n \to \infty$,

$$\limsup_{n\to\infty} \mathbb{E}[C_{n,\geq \ell_n}] = 0. \tag{6.6}$$

Proof. Let $C_{n,k}$ be the number of directed cycles of length $k \geq 1$. (If $k = 1$, then $C_{n,1}$ is the number of loops.) Let $v \in [n]$ with degrees (i, j). By [4, Lemma 7.3] the expected number of simple paths of length k from $\mathcal{E}^+(v)$ to $\mathcal{E}^-(v)$ is at most $(1 + o(1))ij\nu^{k-1}/m_n$. As each cycle of length k is counted k times, we have

$$\mathbb{E}[C_{n,k}] \leq \frac{1}{k} \sum_{i,j \geq 0} (1 + o(1)) \frac{n_{i,j} ij \nu^{k-1}}{m_n} \to \frac{\nu^k}{k}. \tag{6.7}$$

We conclude that

$$\limsup_{n\to\infty} \mathbb{E}[C_{n,\geq 1}] = \limsup_{n\to\infty} \sum_{k\geq 1} \mathbb{E}[C_{n,k}] \leq \sum_{k\geq 1} \frac{\nu^k}{k} = \log\left(\frac{1}{1-\nu}\right), \tag{6.8}$$

$$\limsup_{n\to\infty} \mathbb{E}[C_{n,\geq \ell}] = \limsup_{n\to\infty} \sum_{k\geq \ell} \mathbb{E}[C_{n,k}] \leq \sum_{k\geq \ell} \frac{\nu^k}{k} \leq \frac{1}{\ell+1} \left(\frac{\nu}{1-\nu}\right)^{\ell}, \tag{6.9}$$

where the last inequality follows from the error bound on the Taylor approximation of $\log\left(\frac{1}{1-\nu}\right)$. □

The above lemma shows that, for any $\ell_n \to \infty$, whp (i) there are at most ℓ_n cycles in $\vec{\mathbb{G}}_n$, and (ii) all cycles have length at most ℓ_n. As any vertex in a SCC belongs to at least one cycle, it follows that any SCC has order at most ℓ_n^2. This finishes the proof of the subcritical case.

Remark 6.2. In [3], it was showed that the number of cycles outside the giant of a uniform random k-out digraph with $k \geq 2$ converges to a Poisson distribution. We believe that similar methods can be applied to derive that the law of $C_{n,\geq 1}$ converges to a Poisson distribution with mean $\log\left(\frac{1}{1-\nu}\right)$.

7 Binomial Random Digraphs

The binomial random digraph $\mathbb{D}_{n,p}$ is a simple digraph on $[n]$ in which each ordered pair of nodes is connected with an arc independently at random with probability p, see [7, Chapter 12].

Although the degrees of nodes in $\mathbb{D}_{n,p}$ are random, conditioning on its degree sequence, $\mathbb{D}_{n,p}$ has the same probability to be any simple digraph with such a degree sequence. Thus we can study its properties through the directed configuration model. Using this method, we were able to show that the diameter of $\mathbb{D}_{n,p}$ converges in probability in [4, Theorem 9.5].

The same argument can be applied to determine the largest SCC in $\mathbb{D}_{n,p}$. Assuming that $np \to \nu$, the degree of a uniform random node in $\mathbb{D}_{n,p}$ converges in distribution to two independent Poisson random variables with mean ν. Thus, by Theorem 1.2 we recover the following result by Karp [11]:

Theorem 7.1. *Assume that $np \to \nu$. Let ρ be the smallest solution of $\rho = e^{-\nu(1-\rho)}$ on $(0, 1]$. Let $\mathcal{G}_n$ be the largest* SCC *in $\mathbb{D}_{n,p}$. Then*

$$\frac{v(\mathcal{G}_n)}{n} \to (1-\rho)^2, \qquad \frac{e(\mathcal{G}_n)}{n} \to \nu (1-\rho)^2, \tag{7.1}$$

in expectation, in second moment and in probability.

The case $\nu = 1$ has attracted some attention recently. Coulson [6] determined the critical window of the model, and Goldschmidt and Stephenson [8] showed convergence of the sequence of rescaled largest SCC within the critical window.

Pittel and Poole [13] showed that in fact the joint distribution of $v(\mathcal{G}_n)$ and $e(\mathcal{G}_n)$ is asymptotically Gaussian in $\mathbb{D}_{n,p}$. It is interesting to see if this holds in the directed configuration model.

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