# PRIMES REPRESENTED BY QUADRATIC POLYNOMIALS VIA EXCEPTIONAL CHARACTERS 

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#### Abstract

We estimate the number of primes represented by a general quadratic polynomial with discriminant $\Delta$, assuming that the corresponding real character is exceptional.


## 1. Introduction

Let $f(x)=a x^{2}+b x+c$ be a quadratic polynomial with integer coefficients such that $(a, b, c)=1, a+b$ or $c$ odd and discriminant $\Delta \neq \square$. Conjecture F in the classic work [4] claims that there are infinitely many prime numbers of the form $f(n)$ when $a>0$. Note that it is elementary that the imposed conditions on $f$ are necessary to represent infinitely many primes. If $a<0$, under the same conditions, we still expect to capture many primes if $\{n \in$ $\mathbb{Z}: f(n)>0\}$ is large and Conjecture H in [4] is an instance of it.

In [3] and in [2] special cases of these conjectures are addressed assuming the existence of exceptional characters. For instance, in the second paper it is proved that positive exceptional fundamental discriminants $D$ can be written as $D=m^{2}+p$ and that if $D$ is "exceptional enough" we have an asymptotic formula for the number of primes of the form $D-m^{2}$. In our setting it corresponds to $\Delta=4 D>0$ and $a=-1<0$. In [3] they are considered two families of polynomials with $\Delta<0$ and $a>0$. To interpret correctly this claims, it is important to keep in mind that the exceptional nature of a discriminant depends on our scale and in some sense an exceptional discriminant, zero or character is like a sequence. The existence of a real zero in $[1-c / \log q, 1]$ only ruins the generic de la Vallée Poussin zero free region if $c$ can be taken arbitrarily small when $q$ grows. In [3] and [2] a bona fide asymptotic formula is only achieved if $\Delta$ is allowed to grow.

Our goal in this paper is to adapt the techniques of 2 to get a result valid for every $f$ as above when $\Delta$ is exceptional. By the reasons explained before we prefer to present the result as a main term plus an error term

[^0]instead of as an asymptotic formula resembling the original statement of the conjectures.

For each $N \geq 1$ we denote

$$
\pi_{f}(N)=\#\{0 \leq f(n) \leq N: f(n) \text { prime }\}
$$

with $f$ as before and for each integer $d, \rho(d)=\#\{n: f(n) \equiv 0(\bmod d)\}$. Let $\chi_{\Delta}(n)$ be the Kronecker character modulo $\Delta$, $\chi_{\Delta}(n)=\left(\frac{\Delta}{n}\right)$. We will also consider the $L$ function associated to the character

$$
\begin{equation*}
L(s, \chi)=\sum_{n} \frac{\chi_{\Delta}(n)}{n^{s}}=\prod_{p \text { prime }}\left(1-\frac{\chi_{\Delta}(p)}{p^{s}}\right)^{-1} \tag{1.1}
\end{equation*}
$$

We will denote $\mathcal{A}=\{f(n) \in[0, N]: n \in \mathbb{Z}\}$ and $\mathcal{A}_{d}=\{k \in \mathcal{A}: d \mid k\}$ for $N, d \in \mathbb{Z}^{+}$and $A$ and $A_{d}$ stand for their cardinality. Also, we denote $V(x)=\prod_{p<x}\left(1-p^{-1} \rho(p)\right)$. The exceptionality of the character $\chi_{\Delta}$ will be measured by

$$
\beta=-\log \left(L\left(1, \chi_{\Delta}\right) \log |\Delta|\right)
$$

In the development of the proof it is convenient to introduce also

$$
L=-\log \left(L\left(1, \chi_{\Delta}\right) \log A\right) \quad \text { and } \quad B=\frac{3 \log |\Delta|}{\log A}
$$

To state our main result we introduce the function

$$
g(\Delta)= \begin{cases}\Delta e^{-\beta / 2} & \text { if } \Delta>0 \\ |\Delta|\left(4|a|-e^{-\beta / 2}\right)^{-1} & \text { if } \Delta<0\end{cases}
$$

Theorem 1.1. Let $1 \leq|a| \leq e^{\beta / 5}$ and $g(\Delta) \leq N \leq|a||\Delta|^{\beta / 2}$. Then

$$
\pi_{f}(N)=A V(A)\left(1+O\left(e^{-\sqrt{\beta} / 6}\right)\right)
$$

with an absolute $O$-constant.
Remark 1.2. An asymptotic formula is obtained only under $\beta \rightarrow+\infty$ or equivalently, under the usual definition of exceptionality $L\left(1, \chi_{\Delta}\right) \log |\Delta| \rightarrow$ 0 .

## 2. GUidelines.

Along the proof we follow the sieve techniques of [2]. As usual for $z \geq 2$ we write $P(z)=\prod_{p<z} p$ and $S(\mathcal{A}, z)=\#\{a \in \mathcal{A}:(a, P(z))=1\}$. We start with the trivial identity

$$
\pi_{f}(N)=S(\mathcal{A}, \sqrt{N})+O(1)
$$

and use the well known Buchstab identity

$$
\begin{equation*}
S(\mathcal{A}, z)-S(\mathcal{A}, \sqrt{N})=\sum_{z \leq p \leq \sqrt{N}} S\left(\mathcal{A}_{p}, p\right) \tag{2.1}
\end{equation*}
$$

The main term of the theorem will come from $S(\mathcal{A}, z)$, while the sum on the right will be part of the error term. In any event, in order to estimate
both terms we need to have a concrete knowledge of both $A$ and $A_{p}$. Then, the whole idea of the proof of Theorem 1.1 is to bound the right hand side of (2.1) using the exceptionality of the character $\chi_{\Delta}$. This comes by noting that for $d$ squarefree $\rho(d) \leq \lambda(d)$ where $\lambda(n)$ is given by the convolution $\lambda=1 * \chi_{\Delta}$, and in particular $S\left(\mathcal{A}_{p}, p\right)=0$ if $\chi_{\Delta}(p)=-1$. If the character is exceptional, this will happen often, giving many zero terms in the right hand side of (2.1).

In order to estimate the right hand side clearly we will need to have some control over $A_{p}$. Observe that

$$
A_{d}=\frac{\rho(d)}{d} A+r_{d}, \quad \text { with }\left|r_{d}\right| \leq \rho(d)
$$

Thanks to $\rho(d) \leq \lambda(d)$ finding good bounds for the sum in the right hand side of (2.1) will come from finding good estimates for the sum defined as

$$
\begin{equation*}
\delta(x)=\sum_{x \leq p \leq A} \frac{\lambda(p)}{p} . \tag{2.2}
\end{equation*}
$$

## 3. Proof of Theorem 1.1.

Along the proof we will assume $\beta$ large enough, since otherwise Theorem 1.1 is the classical upper bound from linear sieve theory. The size of $A$ grows with $\Delta$ (see Lemma 3.3 below) then we can assume that $A$ is bigger than a large constant. We will frequently use the inequality $\beta \leq \varepsilon \log |\Delta|$, which follows from Siegel's theorem.

Let us start by finding the asymptotics of $S(\mathcal{A}, z)$. For that we use the fundamental lemma of sieve theory (see e.g. [1, Cor. 6.10]), with level of distribution $A^{2 / 3}, z^{s}=A^{2 / 3}$ and any $1<s<\frac{2 \log A}{9 \log \log A}$ to get

$$
\begin{equation*}
S(A, z)=A V(z)\left(1+O\left(e^{-s}\right)\right)+O\left(A^{2 / 3} \log A\right) \tag{3.1}
\end{equation*}
$$

Observe that

$$
V(z)=V(A) e^{O(\delta(z))}=V(A)(1+O(\delta(z))),
$$

and hence we can replace $V(z)$ by $V(A)$ with an error term bounded by $\delta(z)$, which will be absorbed in the error term in (3.1).

The rest of the paper will be dedicated to bound the right hand side of (2.1), which we will split into three different sums, depending on the range of summation for the primes.

$$
\begin{align*}
\sum_{z \leq p \leq \sqrt{N}} S\left(\mathcal{A}_{p}, p\right) & =\sum_{z \leq p \leq A / z^{2}}+\sum_{A / z^{2}<p \leq A}+\sum_{A<p \leq \sqrt{N}} \\
& =S_{1}+S_{2}+S_{3} \tag{3.2}
\end{align*}
$$

We start with the sum $S_{1}$. The trivial bound $S\left(\mathcal{A}_{p}, p\right) \leq A_{p}$ is not good enough for the small primes in the sum $S_{1}$, and we need a better bound
gotten, as in [2], using an upper bound sieve of dimension 2 and level of distribution $A / p z$ (see e.g. [1, Cor. 6.10]). This gives us

$$
S\left(\mathcal{A}_{p}, p\right) \ll \frac{\rho(p)}{p} A V(z)+\sum_{d<A / p z} \rho(d),
$$

where the sum runs over squarefree integers $d$. The last term is trivially bounded by

$$
\sum_{d<A / p z} \rho(d) \leq \sum_{d<A / p z}\left(1 * \chi_{\Delta}\right)(d) \ll \sum_{d<A / p z} \tau(d) \ll \frac{A}{p z} \log A
$$

Noting that $z \geq(\log A)^{3}$, which follows by our assumption in $s$, and that $V(z) \geq V(A) \gg(\log A)^{-2}$, since $\rho(p) \leq 2$ for any prime $p$, we end up with $S\left(\mathcal{A}_{p}, p\right) \ll p^{-1} \lambda(p) A V(z)$ and hence

$$
\begin{equation*}
S_{1} \ll A V(z) \delta(z) . \tag{3.3}
\end{equation*}
$$

To bound $\delta(z)$ we use Lemma 3.4 of [2], which we include for reader's convenience.

Lemma 3.1. Let $2 \leq u \leq y \leq x$. Then,

$$
\sum_{\substack{y \leq n \leq x \\(n, P(u))=1}} \frac{\lambda(n)}{n} \ll W(u) L(1, \chi \Delta) \log \left(\frac{x}{y}\right)+|\Delta|^{1 / 8+\varepsilon} y^{-1 / 3} u^{1 / 3} \log u
$$

where $W(u)=\prod_{p<u}\left(1-p^{-1}\right)\left(1-p^{-1} \chi_{\Delta}(p)\right)$.
Remark 3.2. Observe that, since $W(u) \leq C \prod_{p<u}\left(1-p^{-1} \lambda(p)\right) \leq C V(u)$, for some absolute constant $C$, we can write either $W(u)$ or $V(u)$ indistinctly.

Further we will use the formula, also proved in [2, p.1106],

$$
\begin{equation*}
\delta(z)^{k} \ll k k!W(z) L\left(1, \chi_{\Delta}\right) \log A+k!|\Delta|^{1 / 8+\varepsilon} z^{(1-k) / 3} \tag{3.4}
\end{equation*}
$$

valid for any integer $k \geq 1$. It is worth to note that in order to establish the previous formula it is needed a bound of the type $\log z \ll \Delta^{\epsilon}$, which in our case follows assuming $L>0$. Indeed

$$
\log z<\log A<\frac{1}{L\left(1, \chi_{\Delta}\right)} \ll \Delta^{\epsilon}
$$

Dropping the contribution of $W(z)$ in (3.4) we get the more convenient form

$$
\delta(z) \ll k\left(k L\left(1, \chi_{\Delta}\right) \log A+|\Delta|^{1 / 8+\varepsilon} z^{(1-k) / 3}\right)^{1 / k}
$$

Our goal is to prove

$$
\begin{equation*}
\delta(z) \ll e^{-s} \tag{3.5}
\end{equation*}
$$

with a proper selection of $s$.

Taking any positive integer $k \geq\left(\frac{3}{16}+3 \varepsilon\right) B s+1$ and noting that $z=$ $|\Delta|^{2 / B s}$ we obtain

$$
|\Delta|^{1 / 8+\varepsilon} z^{(1-k) / 3} \ll|\Delta|^{-\epsilon}
$$

On the other hand

$$
k L\left(1, \chi_{\Delta}\right) \log A \geq|\Delta|^{-\epsilon}
$$

follows by Siegel's theorem, and then

$$
\delta(z) \ll k\left(L\left(1, \chi_{\Delta}\right) \log A\right)^{1 / k}
$$

Observe that, assuming again $L>0$, we have that the previous bound is increasing in $k$, and so we can relax the condition of $k$ being an integer, In particular, we can take $k=B s$, which is possible assuming $B s$ greater than a constant greater than $16 / 13$. Then, to prove (3.5) we need to select some

$$
s \leq-\log k+\frac{L}{k}
$$

which gives, replacing the value of $k$,

$$
\begin{equation*}
B s^{2}+B s \log (B s) \leq L \tag{3.6}
\end{equation*}
$$

The error term in Theorem 1.1 is in terms of $\beta$ instead of $L$. The comparison between both quantities comes from a proper control in $A$. We have the following lemma.

Lemma 3.3. Let $A \geq 1$ and assume the hypothesis in Theorem 1.1. Given $\varepsilon>0$, we have

$$
|\Delta|^{1 / 2-\varepsilon} \leq A<\frac{4 \sqrt{N}}{\sqrt{|a|}}
$$

for $\Delta$ large enough (depending on $\varepsilon$ ).
Remark 3.4. For the application of this in the proof of the main result we are going to choose $1 / 2-\varepsilon=7 / 16$. This is connected to the constant $16 / 13$ above.

Proof. The inequalities $0 \leq f(x) \leq N$ define one or two intervals for $x$, depending on the real zeros of $f$ and the sign of $a$ and $\Delta$, and it is straighforward to measure the length of those intervals to be

$$
X= \begin{cases}\frac{\sqrt{\Delta+4 a N}}{a} & \text { if } \Delta<0, a>0, N>\frac{|\Delta|}{4|a|}, \\ \frac{4 N}{\sqrt{\Delta+4 a N}+\sqrt{\Delta}} & \text { if } \Delta>0, a>0 \text { or if } \Delta>0, a<0, N \leq \frac{|\Delta|}{4|a|} \\ \frac{\sqrt{\Delta}}{|a|} & \text { if } \Delta>0, a<0, N>\frac{|\Delta|}{4|a|}\end{cases}
$$

The cases not listed above give empty intervals. From here the upper bound $X \leq 2 \sqrt{N /|a|}$ is trivial. Then, noting

$$
\begin{equation*}
X-2<A<X+2 \tag{3.7}
\end{equation*}
$$

we deduce $A \leq 4 \sqrt{N /|a|}$ for $|a|<N$, whenever $X \geq 2$, which follows from our assumptions $a<e^{\beta / 5}, N>g(\Delta)$ by Siegel's theorem because $\beta$ and $\Delta$ can be assumed sufficiently large.

We now prove $X>|\Delta|^{1 / 2-\varepsilon}$. In the last case in the definition of $X$, the result follows again from $|a|<e^{\beta / 5}$. If $\Delta<0$, then $X>a^{-1} e^{-\beta / 4} \sqrt{N}$, since $N \geq g(\Delta)$. Further if $\Delta>0$ and $N \leq \frac{\Delta}{4|a|}$, we have $X \gg N / \sqrt{\Delta}$, finally if $\Delta>0, a>0$ and $N>\frac{\Delta}{4|a|}$ we have the stronger bound $X \gg \sqrt{N /|a|}$. In any case $X>\Delta^{1 / 2-\varepsilon}$ is a consequence of $N / \Delta \gg|\Delta|^{-\epsilon}$ and $a \ll|\Delta|^{\epsilon}$.

Now, $A \leq \frac{4 \sqrt{N}}{\sqrt{|a|}}$ and the upper bound for $N$ give

$$
\beta=L+\log \left(\frac{\log A}{\log |\Delta|}\right)<2 L
$$

We select $s=\frac{1}{2} \sqrt{\beta / B}$. By Lemma $3.3 B$ is bounded, namely with the choice of $\varepsilon$ as in the Remark we have $B<7$. Then $s$ is arbitrarily large, in particular $s>1$. It is important to check that this selection of $s$ is compatible with the rest of our previous assumptions:

$$
\frac{16}{13 B}<s<\frac{2 \log A}{9 \log \log A}
$$

The first inequality is consequence of $A<4 \sqrt{N /|a|}$ and the upper bound in $N$. The second is equivalent to

$$
1<\frac{4 \sqrt{B} \log A}{9 \sqrt{\beta} \log \log A}
$$

which follows for $A$ large enough, by the definition of $B$ and Siegel's Theorem.

Let us prove with this selection of $s$ that

$$
\begin{equation*}
B s^{2}+B s \log (B s) \leq \frac{\beta}{2}<L \tag{3.8}
\end{equation*}
$$

As $B<7$ and $s$ is arbitrarily large, we can suppose $B s^{2} \geq B s \log (B s)$ and then (3.8) follows directly from our choice of $s$. This proves (3.8), and (3.5) with $s=\frac{1}{2} \sqrt{\beta / B}$ (and assures $L>0$ as assumed), which gives by (3.2)

$$
\begin{equation*}
S_{1} \ll A V(z) e^{-\sqrt{\beta / 4 B}} \ll A V(A) e^{-\sqrt{\beta} / 6} \tag{3.9}
\end{equation*}
$$

since $B<7$.
It remains to bound $S\left(\mathcal{A}_{p}, p\right)$ for medium and large $p$. If $A / z^{2}<p<A$ then

$$
S\left(\mathcal{A}_{p}, p\right) \leq A_{p} \leq \frac{\rho(p)}{p} A+\lambda(p) \ll \frac{\lambda(p)}{p} A
$$

so

$$
\begin{equation*}
S_{2} \ll A \delta\left(A / z^{2}\right) \tag{3.10}
\end{equation*}
$$

We apply Lemma $3.1 x=A$ and $y=A / z^{2}$ and $u=A z^{-2}|\Delta|^{-3 / 8-4 \varepsilon}$. Observe that $u=A^{\gamma}$ with $\gamma=1-\frac{4}{3 s}-\frac{B}{8}-\frac{4}{3} B \varepsilon$ and $\gamma>0$ for $\varepsilon$ sufficiently
small and $s$ sufficiently large, since $B<7$. With this selection the first term in the sum in Lemma 3.1 dominates the second and we deduce

$$
\delta\left(A / z^{2}\right) \ll V(u) L\left(1, \chi_{\Delta}\right) \log \left(z^{2}\right) \ll V(A) L\left(1, \chi_{\Delta}\right) \log \left(z^{2}\right)
$$

For the last inequality we have used that $u$ is a positive power of $A$. Now,

$$
\log \left(z^{2}\right) L\left(1, \chi_{\Delta}\right)=\frac{4}{3 s} e^{-L}<e^{-\beta / 2} \leq e^{-\sqrt{\beta} / 6},
$$

and putting everything together we get the desired result

$$
\begin{equation*}
S_{2} \ll A V(A) e^{-\sqrt{\beta} / 6} \tag{3.11}
\end{equation*}
$$

Finally, it remains to bound $S_{3}$ corresponding to the primes $A<p<\sqrt{N}$. We have

$$
S\left(\mathcal{A}_{p}, p\right) \leq A_{p} \leq \frac{\rho(p)}{p} A+\lambda(p) \ll \frac{\lambda(p)}{p} \sqrt{N},
$$

and again using Lemma 3.1 with the same parameter $u$ as before, $x=\sqrt{N}$ and $y=A$, we get

$$
\sum_{A \leq p \leq \sqrt{N}} S\left(\mathcal{A}_{p}, p\right) \ll \sqrt{N} V(A) L\left(1, \chi_{\Delta}\right) \log \sqrt{N}
$$

We separate the different cases giving $f(\mathbb{Z}) \cap \mathbb{Z}^{+} \neq \emptyset$ (see the definition of $X$ in the proof of Lemma 3.3).

If $\Delta<0, a>0$ and by hypothesis $N \geq g(\Delta)$, which gives, $\sqrt{N} \ll a e^{\beta / 4} A$ by (3.7) and, hence, since $f(x)=x \log x$ is increasing for $x>2$, we get

$$
\sqrt{N} \log \sqrt{N} \ll a e^{\beta / 4} A \log \left(a e^{\beta / 4} A\right) \ll a e^{\beta / 4} A \log A
$$

by our assumption in $a$ and Lemma 3.3. Also $a<e^{\beta / 5}$ implies

$$
a \ll e^{L-\beta / 4-\sqrt{\beta} / 6},
$$

since $\beta<2 L$, which gives

$$
\begin{equation*}
S_{3} \ll A V(A) e^{-\sqrt{\beta} / 6} \tag{3.12}
\end{equation*}
$$

in this case as desired.
If $\Delta>0, a<0$, we just need to consider the case $\Delta e^{-\beta / 2} \leq N \leq \frac{\Delta}{4|a|}$, since $f(n) \leq \frac{\Delta}{4|a|}$. Then $A \gg N / \sqrt{\Delta}>e^{-\beta / 4} \sqrt{N}$ and the proof of (3.12) follows in the same way as before.

Finally, if $\Delta>0, a>0$ we have $A \gg \sqrt{N / a} \gg e^{-\beta / 10} \sqrt{N}$ and the same proof applies getting again (3.12).

Substituting (3.9), (3.11) and (3.12) in (3.2) and recalling (2.1) and (3.1), the proof of Theorem 1.1 is complete.

## References

[1] J. Friedlander and H. Iwaniec. Opera de cribro, volume 57 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2010.
[2] J. B. Friedlander and H. Iwaniec. Exceptional discriminants are the sum of a square and a prime. Q. J. Math., 64(4):1099-1107, 2013.
[3] A. Granville and R. A. Mollin. Rabinowitsch revisited. Acta Arith., 96(2):139-153, 2000.
[4] G. H. Hardy and J. E. Littlewood. Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes. Acta Math., 44(1):1-70, 1923.

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