On Bipartite Sum Basic Equilibria

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Abstract: A connected and undirected graph G of size $n \geq 1$ is said to be a sum basic equilibrium iff for every edge uv from G and any node v' from G, when performing the swap of the edge uv for the edge uv' the sum of the distances from u to all the other nodes is not strictly reduced. This concept comes from the so called *Network Creation Games*, a wide subject inside Algorithmic Game Theory that tries to better understand how Internet-like networks behave. It has been shown that the diameter of sum basic equilibria is $2^{O(\sqrt{\log n})}$ in general and at most 2 for trees. In this paper we see that the upper bound of 2 not only holds for trees but for bipartite graphs, too. Specifically, we show that the only bipartite sum basic equilibrium networks are the complete bipartite graphs $K_{r,s}$ with $r, s \geq 1$.

1 Introduction

Definition of the model and context. In the *sum basic network creation game*, introduced by Alon et al. in 2010 [**3, 2**], it is assumed that $n \geq 1$ players are the nodes of an undirected graph of size n. If G is connected and for every edge uv and every node v', player u does not strictly reduce the sum of distances to all the other nodes by performing any single swap of the edge uv for the edge uv', then such network is said to be a *sum basic equilibrium graph.*

Given an undirected graph G and a node u from G, a *deviation* in u is any swap of an edge uv from G for any other edge uv' with $v' \neq u, v$ any other node from G. The *deviated graph* associated to any such deviation is the resulting graph obtained after applying the swap. Furthermore, the *cost difference* associated to any deviation in u is just the difference between the sum of distances from u to all the other nodes in the original graph minus the sum of the distances from u to all the other nodes in the deviated graph. Therefore, a connected and undirected graph G is a sum basic equilibrium iff for every node u in G the cost difference associated to every possible deviation in u is non-negative.

This network creation game is inspired by the *sum classical network creation game* introduced by Fabrikant et al. in 2003 [**5**] which is a relatively simple yet tractable model to better understand Internet-like networks. The main contributions of the several authors investigating the sum classical network creation game consist in showing improved bounds for the price of anarchy for this model, a measure that quantifies the loss of efficiency of the system due to the selfish behaviour of its agents. It turns to be that the price of anarchy in the sum classical network creation game is related to the diameter of equilibrium networks in the same model [**4**]. For this reason, one of the main interests in the sum basic network creation game is the study of the diameter of equilibrium networks, too.

One of the most important contributions from [**3**] is an upper bound on the diameter of any sum basic equilibrium of $2^{O(\sqrt{\log n})}$. However, this bound can be dramatically reduced if we restrict to the tree topology, in which case the diameter is shown to be at most 2. Furthermore, in [**3**] the authors establish a connection between sum basic equilibria of diameter larger than $2\log n$ and *distance-uniform graphs*. The authors then conjecture that distance-uniform graphs have logarithmic diameter which would imply, using this connection, that sum basic equilibria have poly-logarithmic diameter. Unfortunately the conjecture is later refuted in [**6**]. After some years, in [**7**], Nikoletseas et al. use the probability principle to establish structural properties of sum basic equilibria. As a consequence of some of these properties, in some extremal situations, like when the maximum degree of equilibrium network is at least $n/\log^l n$ with $l > 0$, it is shown that the diameter is polylogarithmic.

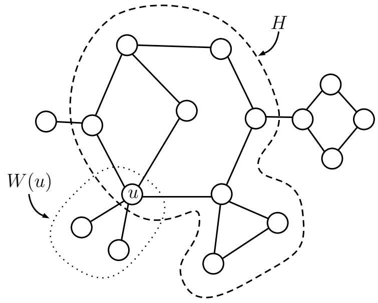
Our Contribution. In this paper we focus our attention to sum basic equilibria restricted to bipartite graphs topology. We show that such networks are the complete bipartite graphs $K_{r,s}$ with $r, s \geq 1$ thus dramatically reducing the diameter to 2 when restricting to this particular case. Our approach consists in considering any $2-$edge-connected component H of a non-tree sum basic equilibrium G. In section 2 we consider all the collection of individual swaps uv for uv' for each $u, v, v' \in V(H)$ and $uv, vv' \in E(H)$. We show that if $diam(H) > 2$, then the sum of the cost differences off all these swaps will be < 0, thus contradicting the fact that G is a sum basic equilibrium. In section 3, we study further properties of any $2-$edge-connected component of any non-tree sum basic equilibrium that work in general and which allow us to reach the main conclusion.

Notation. In this work we consider mainly undirected graphs G for which we denote by $V(G), E(G)$ its corresponding sets of vertices and edges, respectively.

Given an undirected graph G and any pair of nodes u, v from G we denote by $d_G(u, v)$ the distance between u, v. In this way, $D(u)$ is the sum of distances from u to all the other nodes, that is, $D(u) = \sum_{v \neq u} d_G(u, v)$ if G is connected or ∞ otherwise. Given a subgraph H from G, noted as $H \subseteq G$, the $i-$th distance layer in H with respect u is denoted as $\Gamma_{i,H}(u) = \{v \in V(H) \mid d_G(u, v) = i\}$. In particular, the neighbourhood of u in H, the set of nodes from $V(H)$ at distance one with respect u, is $\Gamma_{1,H}(u)$.

Given an undirected graph G and a property P, we say that H is a maximal subgraph of G satisfying P when for any other subgraph H' of G, if H' satisfies P then $H \not\subseteq H'$. An edge $e \in E(G)$ is said to be a *bridge* if its removal increases the number of connected components from G. In this way, a $2-$edge-connected component H from G is any maximal connected subgraph of G not containing any bridge. Moreover, for a given 2-edge-connected component H from G and a vertex $u \in V(H)$, $W(u)$ is the connected component containing u in the subgraph induced by the vertices $(V(G) \setminus V(H)) \cup \{u\}$.

Finally, remind that a bipartite graph is any graph for which all cycles, that is, closed paths, have even length.



2 Local swap deviations

Given a non-tree bipartite sum basic equilibrium graph G, let H be any of its 2−edge-connected components. In this section we show that $diam(H) = 2$.

Given $u \in V(H)$ and $w \in V(G)$, we define $\delta_w^-(u) = \{v \in \Gamma_{1,H}(u) \mid d_G(w,v) = d_G(w,u) - 1\}$ and $\delta_w^+(u) = \{v \in \Gamma_{1,H}(u) \mid d_G(w,v) = d_G(w,u) + 1\}$. Since G is bipartite, for any $u \in V(H)$ and $w \in V(G)$, $\delta_w^-(u) \cup \delta_w^+(u) = \Gamma_{1,H}(u)$.

Moreover, given $u \in V(H)$ and $w \in V(G)$ such that $|\delta_w^-(u)| = 1$, we define $u_w^- \in \delta_w^-(u)$ to be the neighbour of u in H closer from w than u. Recall that, for any $u \in V(H)$ and $w \in V(G)$, if $|\delta_w^-(u)| = 1$ then clearly $\delta_w^+(u) \neq \emptyset$ because H is 2−edge-connected.

Now, let u, v be nodes with $u \in V(H)$ and $v \in \Gamma_{1,H}(u)$. We define $S(u,v)$ to be the sum of the cost differences associated to the swaps of the edge uv by the edges uv' with $v' \in \Gamma_{1,H}(v) \setminus \{u\}$ divided over $deg_H(v) - 1$. Then we define $S = \sum_{u \in V(H)} \sum_{v \in \Gamma_{1,H}(u)} S(u,v)$.

Let u, v, w be nodes with $u \in V(H)$, $v \in \Gamma_{1,H}(u)$ and $w \in V(G)$ and define $\Delta_w(u,v)$ to be the sum of the distance changes from u to w due to the swaps of the edge uv by the edges uv' with $v' \in \Gamma_{1,H}(v) \setminus \{u\}$ divided over $deg_H(v) - 1$. Furthermore, let $\Delta_w = \sum_{u \in V(H)} \sum_{v \in \Gamma_{1,H}(u)} \Delta_w(u,v)$.

In this way we have that $S = \sum_{w \in V(G)} \Delta_w$.

Before going to the main result of this section we first find a formula to compute the value of $\Delta_w(u,v)$ allowing us to obtain an expression for Δ_w.

Lemma 1 *For any nodes $u, v \in V(H)$ and $w \in V(G)$ such that $v \in \Gamma_{1,H}(u)$, then:*

$$\Delta_w(u,v) = \begin{cases} \left(-|\delta_w^-(u_w^-)| + |\delta_w^+(u_w^-)| - 1\right)/(deg_H(v) - 1) & \text{If } |\delta_w^-(u)| = 1 \text{ and } v = u_w^- \\ -|\delta_w^-(v)|/(deg_H(v) - 1) & \text{If } |\delta_w^-(u)| > 1 \text{ and } v \in \delta_w^-(u) \\ 0 & \text{Otherwise} \end{cases}$$

Proof. If v is further from w than u, then clearly $\Delta_w(u,v) = 0$. Therefore, since G is bipartite the remaining case is that v is closer from w than u. We can see clearly that we need to distinguish the cases $|\delta_w^-(u)| = 1$ with $v = u_w^-$ and the case $|\delta_w^-(u)| > 1$ with $v \in \delta_w^-(u)$. In the first case the corresponding sum of distance changes from u to w could get positive when the set of nodes $\delta_w^+(u_w^-)$ has size at least two. In contrast, in the second case the sum of distance changes is always no greater than zero because having at least another node distinct than v in the subset $\delta_w^-(u)$ guarantees that when making the corresponding deviation the distance from u to w does not increase.

□

Now we are ready to show the main result of this section.

Theorem 2 *$diam(H) \leq 2$.*

Proof. First, we claim that for every $w \in V(G)$, $\Delta_w \leq 0$.

Applying Lemma 1 we get:

$$\Delta_w = \sum_{u \in V(H)} \sum_{v \in \Gamma_{1,H}(u)} \Delta_w(u,v) =$$

$$= \left(\sum_{\{u \in V(H) \wedge |\delta_w^-(u)|=1\}} \frac{-|\delta_w^-(u_w^-)| + |\delta_w^+(u_w^-)| - 1}{deg_H(u_w^-) - 1} + \sum_{\{u \in V(H) \wedge |\delta_w^-(u)|>1\}} \sum_{v \in \delta_w^-(u)} \frac{-|\delta_w^-(v)|}{deg_H(v) - 1} \right) =$$

$$= \left(\sum_{\{u \in V(H) \wedge |\delta_w^-(u)|=1\}} \frac{|\delta_w^+(u_w^-)| - 1}{deg_H(u_w^-) - 1} + \sum_{u \in V(H)} \sum_{v \in \delta_w^-(u)} \frac{-|\delta_w^-(v)|}{deg_H(v) - 1} \right)$$

On the one hand:

$$\sum_{u \in V(H)} \sum_{v \in \delta_w^-(u)} \frac{|\delta_w^-(v)|}{deg_H(v) - 1} = \sum_{v \in V(H)} \sum_{u \in \delta_w^+(v)} \frac{|\delta_w^-(v)|}{deg_H(v) - 1} = \sum_{v \in V(H)} \frac{|\delta_w^-(v)||\delta_w^+(v)|}{deg_H(v) - 1}$$

Now, let Z_w be the subset of nodes z from $V(H)$ such that $\delta_w^-(z) \neq \emptyset$ and $\delta_w^+(z) \neq \emptyset$. If $z \in Z_w$ then clearly $|\delta_w^-(z)||\delta_w^+(z)| \geq deg_H(z) - 1$. One possible way to see this is the following. Since H is bipartite, then $|\delta_w^-(z)|$ and $|\delta_w^+(z)|$ are positive integers that add up to $deg_H(z)$. Furthermore, any concave function defined on a closed interval attains its minimum in one of its extremes. Therefore, the conclusion follows when combining these two facts to the function $f(x) = x(deg_H(z) - x)$ defined in $[1, deg_H(z) - 1]$. In this way:

$$\sum_{u \in V(H)} \sum_{v \in \delta_w^-(u)} \frac{|\delta_w^-(v)|}{deg_H(v) - 1} = \sum_{v \in Z_w} \frac{|\delta_w^-(v)||\delta_w^+(v)|}{deg_H(v) - 1} \geq \sum_{v \in Z_w} 1 = |Z_w| \tag{1}$$

On the other hand for any u such that $|\delta_w^-(u)| = 1$:

$$\frac{|\delta_w^+(u_w^-)| - 1}{deg_H(u_w^-) - 1} \leq 1 \tag{2}$$

Notice that the equality in (2) holds exactly when $\delta_w^-(u_w^-) = \emptyset$. For any $w \in V(G)$ there exists exactly one node $t_w \in V(H)$ verifying $\delta_w^-(t_w) = \emptyset$ which is the unique node from $V(H)$ such that $w \in W(t_w)$. Therefore, the equality in (2) holds exactly for the nodes from $\Gamma_{1,H}(t_w)$.

In this way:

$$\sum_{\{u \in V(H) \wedge |\delta_w^-(u)|=1\}} \frac{|\delta_w^+(u_w^-)| - 1}{deg_H(u_w^-) - 1} \leq \left| \{u \in V(H) \wedge |\delta_w^-(u)| = 1\} \right| \tag{3}$$

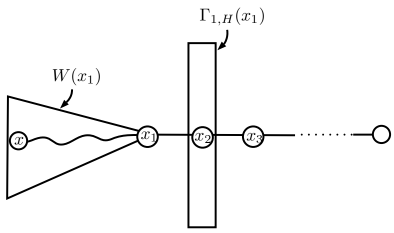
Notice that since H is bipartite, $\Gamma_{1,H}(t_w) \subseteq \{u \in V(H) \mid |\delta_w^-(u)| = 1\}$. Therefore, the equality in (3) holds only when $\Gamma_{1,H}(t_w) = \{u \in V(H) \mid |\delta_w^-(u)| = 1\}$, otherwise, the inequality in (3) is strict.

Now, recall that $\{u \in V(H) \wedge |\delta^-_w(u)| = 1\} \subseteq Z_w$ because H is $2-$edge-connected. Therefore, combining (1) with (3):

$$\Delta_w \leq -|Z_w| + | \ \left\{u \in V(H) \mid |\delta^-_w(u)| = 1\right\} | \leq 0$$

As we wanted to prove.

Now, suppose that $diam(H) > 2$ and take any path $\pi = x_1 - x_2 - x_3 - ...$ of length $diam(H)$ inside H. Then, pick $x \in W(x_1)$ any node inside $W(x_1)$. Setting $w = x$ we have that $x_1 = t_w$ and $x_3 \in Z_w$ but $x_3 \notin \Gamma_{1,H}(t_w)$. If $x_3 \notin \{u \in V(H) \mid |\delta^-_w(u)| = 1\}$ then the inclusion $\{u \in V(H) \mid |\delta^-_w(u)| = 1\} \subseteq Z_w$ is strict and then $\Delta_w < 0$. Otherwise, $x_3 \in \{u \in V(H) \mid |\delta^-_w(u)| = 1\}$ but $x_3 \notin \Gamma_{1,H}(t_w)$ so that the inclusion $\Gamma_{1,H}(t_w) \subseteq \{u \in V(H) \mid |\delta^-_w(u)| = 1\}$ is strict and then $\Delta_w < 0$, too. Therefore, $S = \sum_{w \in V(G)} \Delta_w < 0$ and this contradicts the fact that G is an equilibrium graph.



□

3 $2-$edge-connectivity in the sum basic equilibria

In this section, we investigate further topological properties of any $2-$edge-connected component H from any sum basic equilibrium G. These properties help us to derive the main result of this paper.

Lemma 3 *If $uv \in E(G)$ is a bridge, then $deg(u) = 1$ or $deg(v) = 1$.*

Proof. Let $u_1u_2 \in E(G)$ be a bridge between two connected components G_1, G_2 in such a way that $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$. Furthermore, assume wlog that $|V(G_1)| \leq |V(G_2)|$. If we suppose the contrary, then we can find a node $v \in V(G_1)$ such that $vu_1 \in E(G_1)$. Then, let ΔC be the cost difference associated to the deviation in v that consists in swapping the edge vu_1 for the edge vu_2. Clearly, we are getting one unit closer to every node from $V(G_2)$ and getting one unit distance further from at most all nodes in $V(G_1)$ except for the node v itself.

Therefore, using the assumption $|V(G_1)| \leq |V(G_2)|$:

$$\Delta C \leq |V(G_1)| - 1 - |V(G_2)| \leq -1 < 0$$

□

Lemma 4 *If H is any $2-$edge-connected component of G then there exists at most one node $u \in V(H)$ such that $W(u) \neq \{u\}$.*

Proof. Suppose the contrary and we reach a contradiction. Let u_1, u_2 be two distinct nodes such that $W(u_1) \neq \{u_1\}$ and $W(u_2) \neq \{u_2\}$. Let $v_1 \neq u_1$ and $v_2 \neq u_2$ be two nodes from $W(u_1)$ and $W(u_2)$ respectively. By Lemma 3, $W(u_1)$ and $W(u_2)$ are stars. Assume wlog that $D(u_1) \leq D(u_2)$. When swapping the link v_2u_2 for the link v_2u_1 we can reach the nodes from $V(G) \setminus \{v_1\}$ at the distances seen by v_1 and, also, we are reducing in at least one unit distance the distance from v_2 to v_1. Therefore, if ΔC is the cost difference associated to such swap, then:

$$\Delta C \leq D(u_1) - D(u_2) - 1 < 0$$

□

Therefore, combining these two lemmas with Theorem 2, we deduce that every non-tree bipartite sum basic equilibrium is the complete bipartite $K_{r,s}$ with some star S_k (the star with a central node and k edges) attached to exactly one of the nodes from $K_{r,s}$, let it be $x_0 \in V(K_{r,s})$. Then, if we consider any path $x_0 - x_1 - x_2$ in H of length 2, x_2 has an incentive to swap the link x_2x_1 for the link x_2x_0 unless $k = 0$, that is, unless $G = K_{r,s}$.

From here we reach the conclusion of this paper:

Corollary 5 *The set of bipartite sum basic equilibria is the set of complete bipartite graphs $K_{r,s}$, with $r, s \geq 1$ and therefore the diameter of every bipartite sum basic equilibrium graph is at most* 2.

4 Conclusion

Therefore, the diameter of sum basic equilibria is at most 2 not only when we consider trees, also when we consider bipartite graphs. However, it is known that there exist equilibrium networks of diameter 3 [**2**]. Therefore the open question is whether the general upper bound $2^{O(\sqrt{\log n})}$ can be substantially decreased in the non-bipartite case.

Furthermore, notice that the crucial results in this paper have been obtained considering a sum of the cost differences associated to a family of deviations. This is nothing more than a disguised application of the probability principle, a technique used also in [**7**] for the same model. These results show that maybe this technique can be further explored in order to reach new insights for this model or for related ones.

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