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Article

Controllability of Brain Neural Networks in Learning Disorders—A Geometric Approach †

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Abstract: The human brain can be interpreted mathematically as a linear dynamical system that shifts through various cognitive regions promoting more or less complicated behaviors. The dynamics of brain neural network play a considerable role in cognitive function and therefore of interest in the bid to understand the learning processes and the evolution of possible disorders. The mathematical theory of systems and control makes available procedures, concepts, and criteria that can be applied to ease the perception of the dynamic processes that administer the evolution of the brain with learning and its control with treatment in case of disorder. In this work, a geometric study through the conception of exact controllability is comprehended to detect the minimum set and the location of the driving nodes of learning. We will describe the different roles of the nodes in the control of the paths of brain networks and show the transition of some driving nodes and the preservation of the rest in the course of learning in patients with some learning disability.

Keywords: neural network; controllability; exact controllability; eigenvalues; eigenvectors; linear systems

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1. Introduction

The brain structure is a complex recurrent neuronal network that can be easily described by a graph (see Figure 1, where the nodes represent brain areas and the edges the strength of connections between these areas that emerge when certain tasks are performed.

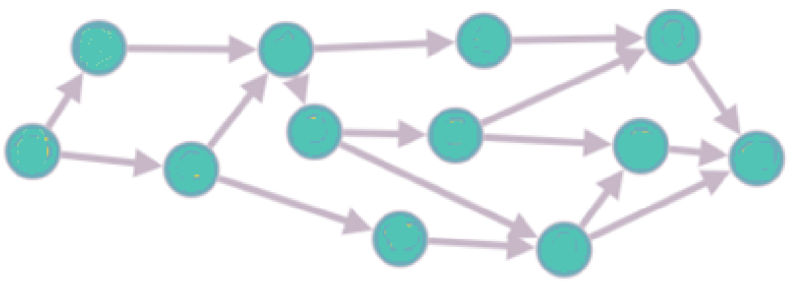

Figure 1. Recurrent Neuronal Network.

The locution neuronal network makes reference to a particular model for comprehending brain function, in which neurons are the basic computational units and computation is interpreted in terms of network interactions.

It has been shown [1] that cognitive control and the ability to control brain dynamics holds great suggestive of improvement of cognitive functions and reversing the possible disorder in learning processes. The human brain seems to be able to travel between diverse cognitive states. Its most imposing role is in connecting multiple sources of information in large-scale networks that are required to solve complex cognitive problems and strengthen memory.

As Kriegeskorte asserts in [2], neuronal network models indicate a starting point of a new period of computational neuroscience, in which participants bear a part in real-world labours that require wide knowledge and elaborate calculations.

In the interest of controlling their functions, neural networks have been treated by means of dynamic linear control systems. In this work, neural networks are treated as multi-agent systems, that is, systems of linear dynamic systems related to each other through a previously established topology.

Multi-agent systems are used in different areas of engineering, to solve synchronization problems and address consensus problems of the systems (see for example, [3,4]). On the other hand, it should be said that neural networks are also being studied as non-linear dynamic systems (see, for example, [5]).

García-Planas in [6] showed that a noise-free multisystem of linear discrete-time and time-invariant model

$$\left.\begin{array}{rl} \dot{x}^1(t) & = A_1x^1(t) + B_1u^1(t) \\ \vdots & \\ \dot{x}^k(t) & = A_kx^k(t) + B_ku^k(t) \end{array}\right\} \quad (1)$$

where $A_i \in M_n(\mathbb{R})$, $B_i \in M_{n\times m}(\mathbb{R})$, $x^i(t) \in \mathbb{R}^n$, $u^i(t) \in \mathbb{R}^m$, $1 \leq i \leq k$ could be employed to describe the neuronal dynamics.

A block diagram is plotted in Figure 2.

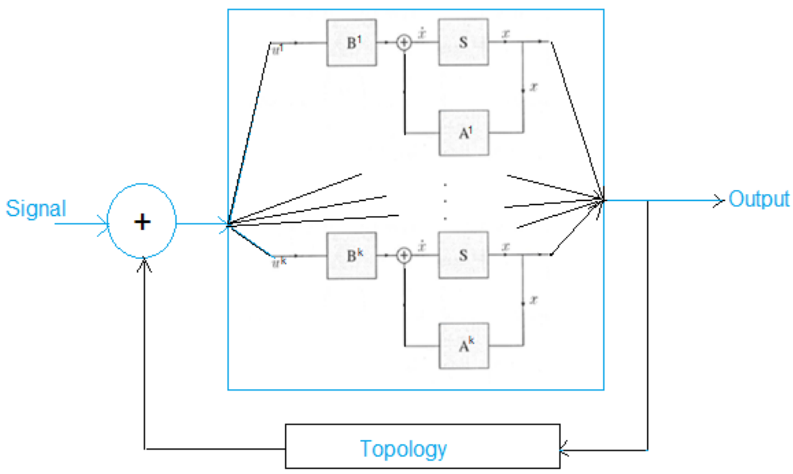

Figure 2. Diagram of a multi-agent system.

Systems and control theory can help answer the question about the theoretical control of the human brain. Some results on brain interfaces and neuromodulation suggest that modifications in regional activity (measured by evoked potentials or other means) can cause alterations in the dynamics of brain function [7].

Notwithstanding the complete comprehension of the relationship between mathematical control measures and the slight knowledge of cognitive control of neuroscience are difficult to reach, small advances in the study can stimulate the study and action against learning difficulties such as dyscalculia or other disturbances such as the phenomena of forgetting [7]).

Structural controllability theory could be a good tool to control structured linear systems, in this way Garcia-Planas in ([6], showed that structural controllability in a mathematical instrument which could be inferred to multi-agent systems in which each of the agents has a previously determined structure.

2. Preliminaries

To study the control problems proposed, the complexity of the brain structure requires that, the global model be divided into several local submodels, each one with its own complex and interrelated network structure. In this way, it could be possible to structure the brain as a neuronal multi-network with a common objective.

Let us consider a group of k agents as (1).

In our particular setup, the agents are communicating by the topology defined by the graph $\mathcal{G}$, with

(i) Set of Vertices: $V = \{1, \ldots, k\}$
(ii) Set of Edges: $\mathcal{E} = \{(i, j) \mid i, j \in V\} \subset V \times V$

Figure 3 shows the graph that defines the topology on the participating agents in the system.

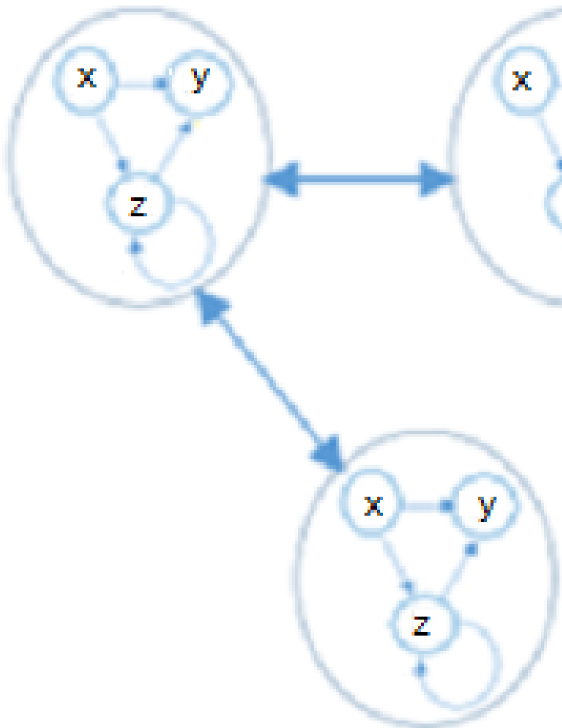

Figure 3. Multiagent graph.

It is well known that each graph has an associated matrix called Laplacian, this matrix is defined as

$$\mathcal{L} = (l_{ij}) = \begin{cases} |\mathcal{N}_i| & \text{if } i = j \\ -1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$$

Writing

$$\mathcal{X}(t) = \begin{pmatrix} x^1(t) \\ \vdots \\ x^k(t) \end{pmatrix}, \quad \dot{\mathcal{X}}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \vdots \\ \dot{x}^k(t) \end{pmatrix}, \quad \mathcal{U}(t) = \begin{pmatrix} u^1(t) \\ \vdots \\ u^k(t) \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}, \mathcal{B} = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix},$$

With these notations it is possible to describe the multisystem can be described as a system:

$$\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{B}\mathcal{U}(t).$$

The description of the local interrelation between systems defined by the considered topology is given by the control:

$$u^i(t) = F_i \sum_{j \in \mathcal{N}_i} (x^i(t) - x^j(t)),\ 1 \leq i \leq k \tag{2}$$

That in a matrix description is

$$\mathcal{F}\mathcal{U}(t) = \mathcal{F}(\mathcal{L} \otimes I_n)\mathcal{X}(t)$$

where $\mathcal{F} = \begin{pmatrix} F_1 & & \\ & \ddots & \\ & & F_k \end{pmatrix}$.

Then, the multisystem with interrelation control is described as:

$$\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{B}\mathcal{F}\mathcal{U}(t) = (\mathcal{A} + \mathcal{B}\mathcal{F}(\mathcal{L} \otimes I_n))\mathcal{X}(t). \tag{3}$$

3. Controllability and Exact Controllability

Controllability is one of the most important properties of dynamical systems, and that is why a great portion of the literature refers to this concept ([6,8,9], among others).

First of all, and for a good understanding of the work, the notion of controllability for linear dynamic systems in the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4}$$

is remembered.

Definition 1 ([6])**.** *The linear dynamical system* (4) *is called controllable if, for any* $t_1 > 0$, $x(0) \in \mathbb{C}^n$ *and* $w \in \mathbb{C}^n$, *there exists a control input* $u(t)$ *sufficiently smooth such that* $x(t_1) = w$.

The controllability character can be measured using the well-known Kalman's rank condition.

Proposition 1 ([9])**.** *The dynamical system* (4) *is controllable if and only if:*

$$\operatorname{rank} \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = n \tag{5}$$

or via the Hautus Test for controllability of linear time-invariant dynamical systems.

Proposition 2 ([8])**.** *The dynamical system* (4) *is controllable if and only if:*

$$\operatorname{rank} \begin{pmatrix} sI - A & B \end{pmatrix} = n,\ \forall s \in \mathbb{C}\ . \tag{6}$$

We give evidence of the work, applying it to simple example of an undirected graph represented in Figure 4.

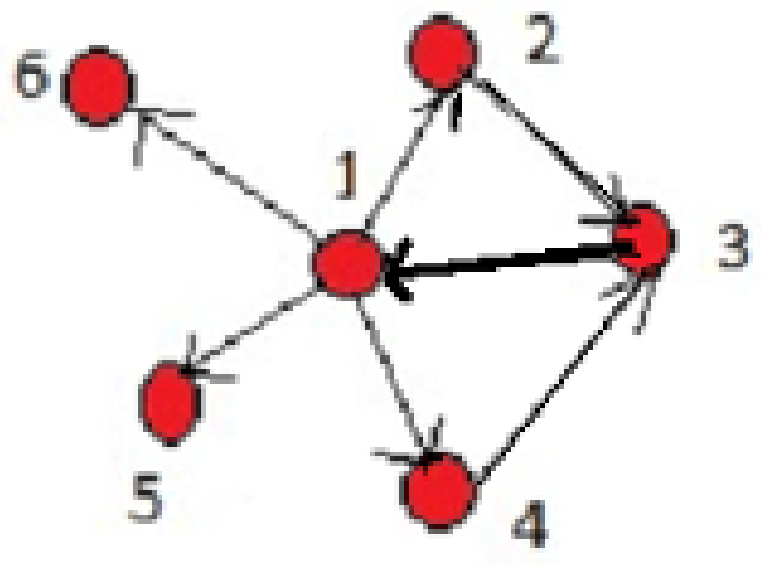

Figure 4. Example of a graph.

The system A

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

is controllable:

$$\operatorname{rank}\begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix} = 6$$

See image Figure 5 for a better understanding in which the matrix B is represented by the arrows that go from b_i to the nodes.

Therefore, taking controls u_1, u_2, and u_3, it is possible to reach a desired state from a fixed initial state in a finite time.

For example, taking $u_1 = (0,0,0)$, $u_2 = (0,-1,0)$ and $(0,0,1)$, it is possible to reach node 5 from node 6:

$$A^3x_6 + (A^2B)u_1 + ABu_2 + Bu_3 = x_5.$$

It is a challenge to find out which B matrices are valid for the system to be controllable and even more so if these matrices have the minimum number of inputs. Liu et al. [10] suggest "the maximum coincidence algorithm" based on the network representation of the matrix, to select the control nodes to ensure that systems are controllable; Yuan et al. in [11] exhibit a general framework based on the maximum multiplicity theory to investigate the exact controllability of multiplex interrelated networks, focusing the study on the controllability amount defined by the minimum set of drivers that are needed to control steering the whole system toward any desired state but the authors do not construct the possible drivers. García-Planas in [12] builds the matrices (drivers) based on the eigenvalues of the matrix A and of its geometric multiplicity.

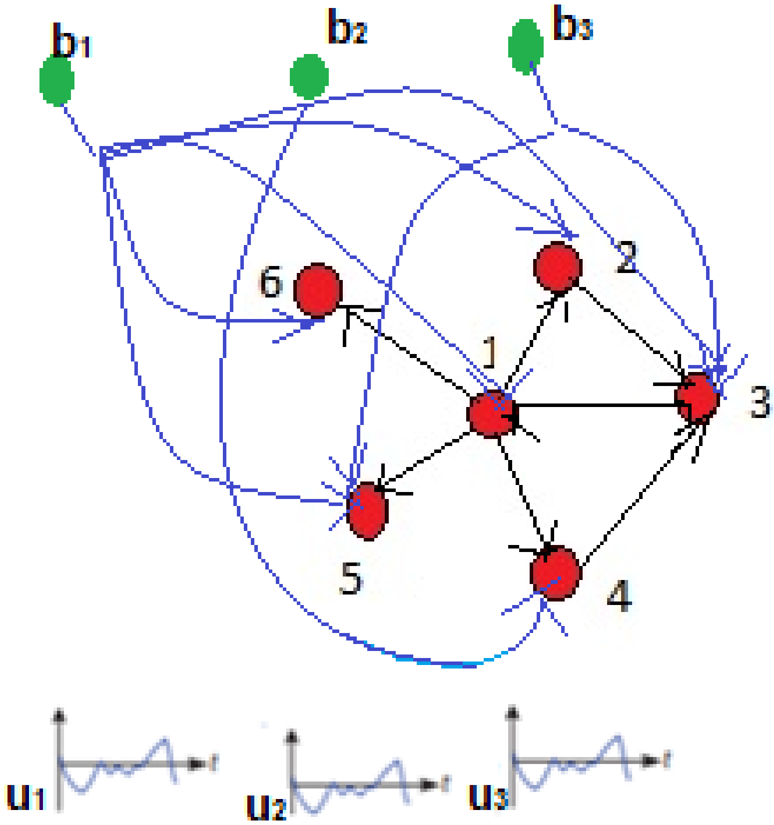

Figure 5. Example of a feedback graph.

Given a linear dynamical system such as (4) for plainness, from now on, we will write the pair of matrices as (A, B). It is well known that there are many possible control matrices B in the system that can assure the controllability condition, for it suffices to consider invertible matrices $B \in Gl(n;\mathbb{R})$.

The goal is to find the collection of all possible matrices B, having the minimum number of columns corresponding to the minimum number $n_D(A)$ of independent controllers that are required to control the entirety network.

Controllability with a minimal number of inputs is known as *exact controllability*.

Definition 2 ([11])**.** *Let $A \in M_n(\mathbb{R})$ be a matrix. The exact controllability number $n_D(A)$ is the minimum of the ranks of all possible matrices B making the system $\dot{x} = Ax + Bu$ controllable.*

$$n_D(A) = \min\{\text{rank}\, B, \forall B \in M_{n\times i}\, 1 \leq i \leq n\ (A, B) \text{ being controllable}\}.$$

For simplicity, we will write it as n_D.

It is easy to prove that n_D is invariant under similarity equivalence relation, that is to say: for any invertible matrix $S \in Gl(n;\mathbb{R}$ we have $n_D(A) = n_D(S^{-1}AS)$. Therefore, and if necessary, we can consider A in a reduced form, for example in its Jordan canonical form.

Proposition 3 ([11])**.** *Let $\mu(\lambda_i) = \dim \text{Ker}\,(A - \lambda_i I)$ be the geometric multiplicity of the eigenvalue λ_i. Then,*

$$n_D = \max_i \{\mu(\lambda_i)\}$$

In [12], a manner to obtain a set of minimal number of controls is presented.

4. Controllability of Multiagent Neural Networks

We are concerned about bringing the output of the system (1) to a reference value and keeping it there; we can ensure that it is possible when the system is controllable. Unquestionably, the system (1) is controllable if and only if each subsystem is controllable, and, in this case, there is feedback in which we obtain the requested solution.

We can be interested with the control (2) and ask for the stability of the system (3).

If, having considered this control, the resulting system (3) has not the desired eigenvalues, we can try to consider different feedback F_i so that, with the new control (with feedback $= K_i$),

$$u^i(t) = K_i \sum_{j\in\mathcal{N}_i} (x^i(t) - x^j(t)),\ 1 \le i \le k \tag{7}$$

the system has appointed eigenvalues to take a requested output of the system.

In some cases, this could be attentive in a solution such that

$$\lim_{t\to\infty} \|x^i - x^j\| = 0,\ 1 \le i, j \le k,$$

namely, finding solutions for each subsystem, all reaching the same point.

Proposition 4. *Considering the control $u^i(t) = K_i \sum_{j\in\mathcal{N}_i}(x^i(t) - x^j(t)),\ 1 \le i \le k$ the closed-loop system can be detailed as*

$$\dot{\mathcal{X}}(t) = (\mathcal{A} + \mathcal{B}\mathcal{K}(\mathcal{L} \otimes I_n))\mathcal{X}(t).$$

where $\mathcal{K}$ is the diagonal matrix $\begin{pmatrix} K_1 & & \\ & \ddots & \\ & & K_k \end{pmatrix}$.

Computing the matrix $\mathcal{A} + \mathcal{B}\mathcal{K}(\mathcal{L} \otimes I_n)$, we obtain

$$\begin{pmatrix} A_1 + l_{11}B_1K_1 & l_{12}B_1K_1 & \dots & l_{1k}B_1K_1 \\ l_{21}B_2K_2 & A_2 + l_{22}B_2K_2 & \dots & l_{2k}B_2K_2 \\ \vdots & \vdots & \ddots & \vdots \\ l_{k1}B_kK_k & l_{k2}B_kK_k & \dots & A_k + l_{kk}B_kK_k \end{pmatrix}$$

In this specific case, proposition 4 can be rewritten in the following manner (see [13]).

Proposition 5. *Considering the control $u^i(t) = K \sum_{j\in\mathcal{N}_i}(x^i(t) - x^j(t)),\ 1 \le i \le k$ the closed-loop system for a multiagent with identical linear dynamical mode is detailed as*

$$\dot{\mathcal{X}} = ((I_k \otimes A) + (I_k \otimes BK)(\mathcal{L} \otimes I_n))\mathcal{X}.$$

It is also interesting to study the case that we can consider external controls that allow us to obtain the desired eigenvalues.

5. Selection of Control Nodes

It is of interest to recognize the minimum set of driver nodes needed to achieve full control of networks having arbitrary structures and link-weight distributions.

In our particular setup, the objective is to find the collection of all possible matrices E, having the minimum number of columns corresponding to the minimum number $n_D((\mathcal{A} + \mathcal{B}\mathcal{K}(\mathcal{L} \otimes I_n)))$ of independent drivers that are necessary to control the whole network.

Given the protocol as (7) with K the feedback gain matrix, and defining

$$\mathcal{U}_{\text{ext}}(t) = \begin{pmatrix} u^1_{\text{ext}}(t) \\ \vdots \\ u^k_{\text{ext}}(t) \end{pmatrix}, \ \mathcal{D} = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_k \end{pmatrix}$$

Proposition 6. *With these notations the system can be described as*

$$\dot{\mathcal{X}}(t) = (\mathcal{A} + \mathcal{B}\mathcal{K}(\mathcal{L} \otimes I_n))\mathcal{X}(t) + \mathcal{D}\mathcal{U}_{ext}(t). \tag{8}$$

Considering the case where the dynamics depend only of the topology interrelating the agents, the system can be described as

$$\dot{\mathcal{X}}(t) = (\mathcal{L} \otimes I_n)\mathcal{X}(t) + \mathcal{D}\mathcal{U}_{\text{ext}}(t)$$

and, if it is considered in such a way, that each agent just follows every one in front of it, on a higher level. That is to say, the Laplacian has a triangular form.

Example 1. *Consider the graph of Figure 6*

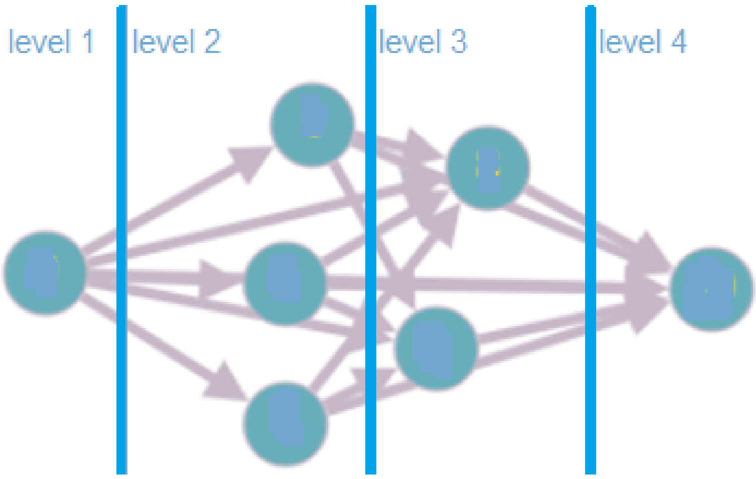

Figure 6. Neural Network.

The Laplacian matrix is:

$$\begin{pmatrix} 6 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 3 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 3 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then, in this case, if the agents have n variables, and following proposition 3, the minimum number of controls to make the system controllable are $3n$*.*

In the case where the nodes of graph are all in a different level and each agent follows only the one on the following level or every one in front of it, on a higher level, (see Figure 7), the minimum number of controls to make the system controllable are n in the first case and $3n$ in the second one.

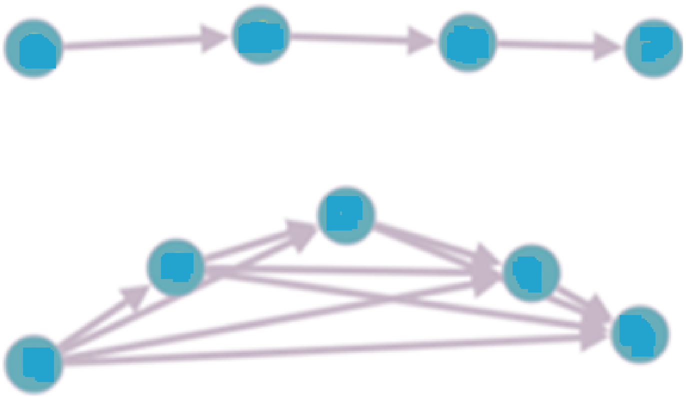

Figure 7. Neural Networks.

The Laplacian matrices in these cases are

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{respectively.}$$

In a more general case we have:

Proposition 7. *Let us consider a directed graph where the nodes are classified by levels and each one follows only some nodes of a high level. Then, the minimum number of controls making the system controllable is*

$$n_D(\mathcal{L} \otimes I_n) = n \cdot n_D(\mathcal{L}).$$

Proof. In this particular setup, the Laplacian matrix has a triangular form. □

In the case, where the graph is undirected (see Figure 8), the Laplacian matrix is symmetric, therefore it is diagonalizable. Therefore, let $P \in Gl(n;\mathbb{C})$ be an invertible matrix such that $\mathcal{L} = P^{-1}DP$, with $D = \text{diag}\,(\lambda_1, \ldots, \lambda_k)$ a diagonal matrix.

$$\begin{array}{l} \dot{\mathcal{X}}(t) = (\mathcal{L} \otimes I_n)\mathcal{X}(t) + \mathcal{D}\mathrm{U}_{\text{ext}}(t) = \\ (P^{-1}DP \otimes I_n)\mathcal{X}(t) + \mathcal{D}\mathrm{U}_{\text{ext}}(t) = \\ (P^{-1} \otimes I_n)(D \otimes I_n)(P \otimes I_n)\mathcal{X}(t) + \mathcal{D}\mathrm{U}_{\text{ext}}(t) \end{array}$$

$$\begin{array}{l} \dot{\mathcal{Y}}(t) = (P \otimes I_n)\mathcal{X}(t) = (D \otimes I_n)(P \otimes I_n)\mathcal{X}(t) + (P \otimes I_n)\mathcal{D}\mathrm{U}_{\text{ext}}(t) \\ \dot{\mathcal{Y}}(t) = (D \otimes I_n)\mathcal{Y}(t) + \mathcal{D}_1\mathcal{U}_{\text{ext}}(t) \end{array}$$

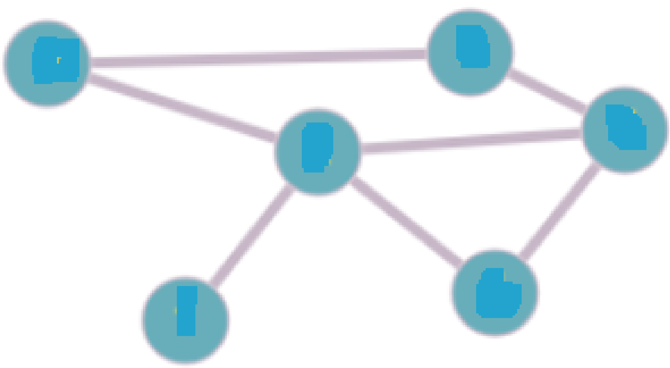

Figure 8. Undirected graph.

Then, the eigenvalues, as well its geometrical multiplicity of

$$(\mathcal{L} \otimes I_n)$$

are the same as the diagonal matrix

$$(D \otimes I_n)$$

Proposition 8.

$$n_D(\mathcal{L} \otimes I_n) = n \cdot n_D(D)$$

We want to emphasize that, although the minimum number of controls is uniquely determined, the set of controls is not unique and each one of them can be chosen within a subspace.

Now we present a set of a minimal number of controls for each of these cases.

In example 1, and for $n = 1$, a basis of eigenvectors is

$$\begin{aligned} u_1 &= (1,0,0,0,0,0,0) \\ u_2 &= (1,2,0,0,0,0,0) \\ u_3 &= (1,0,3,0,0,0,0) \\ u_4 &= (1,0,0,3,0,0,0) \\ u_5 &= (1,1,1,1,2,0,0) \\ u_6 &= (1,1,1,1,0,2,0) \\ u_7 &= (1,1,1,1,1,1,1) \end{aligned}$$

To construct the matrix B, we consider the following vectors

$$\begin{aligned} u &= u_1 + u_2 + u_5 + u_7 \\ v &= u_3 + u_6 \\ w &= u_4 \end{aligned}$$

Then,

$$B = \begin{pmatrix} 4 & 2 & 1 \\ 4 & 1 & 0 \\ 2 & 4 & 0 \\ 2 & 1 & 3 \\ 3 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and,

$$\operatorname{rank}\begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix} = 7$$

Therefore, the system (A, B) is controllable

In this case, where each agent just follows every one in front of it, on a higher level, and for k agents of dimension n, the matrix $\mathcal{L} \otimes I_n$ is a triangular block matrix whose blocks on the diagonal are scalar matrices. Therefore, this matrix is diagonalizable.

The eigenvalues of the matrix $\mathcal{L} \otimes I_n$ are n copies of the eigenvalues of the matrix $\mathcal{L}$.

Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of $\mathcal{L} \otimes I_n$ and $n_1, \ldots, n_r$ the respective multiplicities. The greatest multiplicity of the eigenvalues corresponds to the level of nodes that contains the greatest number of them.

Proposition 9. *Let $v_{1_1}, \ldots, v_{1_{n_1}}, \ldots, v_{r_1}, \ldots, v_{r_{n_r}}$ a basis of eigenvectors corresponding to the eigenvalues $\lambda_1, \overset{n_1}{\ldots}, \lambda_r, \ldots, \lambda_r, \overset{n_r}{\ldots}, \lambda_r$. Then the matrix B making $(\mathcal{L} \otimes I_n, B)$ controllable is the matrix whose columns are:*

$\alpha_{11}v_{1_1} + \alpha_{12}v_{2_1} + \ldots + \alpha_{1r}v_{r_1}, \alpha_{21}v_{1_2} + \alpha_{22}v_{2_2} + \ldots + \alpha_{2r}v_{r_2}, \ldots, \alpha_{\ell 1}v_{1_\ell} + \alpha_{\ell 2}v_{2_\ell} + \ldots + \alpha_{\ell r}v_{r_\ell}$ *where* $\ell = n_D(\mathcal{L} \otimes I_n)$. *with* $\alpha_{ij} \neq 0$.

(If $n_i < \ell$ then $v_{i_{\ell+j}} = 0$).

Proof. The maximal minors of the controllability matrix are generalized Vandermonde determinants. □

The nodes of graph are all in a different level and each agent follows only the one on the following level. Considering the example, the matrix $\mathcal{L} \otimes I_n$ for $n = 3$ is

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Jordan canonical form is a diagonal by blocks matrix all identical to the Jordan reduction form of the matrix $\mathcal{L}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and a corresponding basis change matrix can be

$$\begin{aligned} V &= [v_{11}, v_{12}, v_{13}, v_{14}; v_{21}, v_{22}, v_{23}, v_{24}; v_{31}, v_{32}, v_{33}, v_{34}] \\ &= \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and the matrix B making the system $(\mathcal{L} \otimes I_n, B)$ controllable is

$$B = [v_{11} + v_{12} + v_{13} + v_{14}; v_{21} + v_{22} + v_{23} + v_{24}; v_{31} + v_{32} + v_{33} + v_{34}]$$

6. Discussion

The word control implies action and reflects the human effort to intervene in the environment that surrounds it to guarantee its survival and a permanent improvement in the quality of life. Many of the control problems can be analyzed through a mathematical model that describes the physical system under consideration through equations that show the state of the system.

Being a central problem in many network systems, there are few studies to date regarding how to explore this issue quantitatively, or how we can control a directed network, which is the configuration that is usually found more frequently in real systems.

The fundamental problem is the size. Liu et al. [10], have developed the tools to undertake the study of controllability for arbitrary network sizes and topologies using the controllability matrix considering a few driver nodes on the network.

In [7], Gu et al. define controllability (global, regional, average, modal, and boundary) from different points of view to use on the neural systems which can be treated. In this work, the authors suggest that the differences between the different points of view of controllability can help to analyze different roles in the control of the dynamic trajectories of the function of the brain network.

In this paper, we consider the brain network as a multisystem of linear discrete-time and time-invariant mode that permits us to consider a larger number of nodes. In 2018, Abiodun et al. [14], carried out a survey on the state of the art on the applications of artificial neural networks.

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Notations

In this paper, we will use the following notations:

- $\mathbb{R}$: Set of real numbers;
- $\mathbb{R}^n$: n-dimensional vector space over the real numbers;
- $M_{n\times m}$: $(\mathbb{R})$: Set of matrices with n rows and m columns;
- $M_n(\mathbb{R})$: the set square matrices of n-order;
- $Gl(n;\mathbb{R})$; the set of n-square real matrices invertible;
- $x^i = (x^i_1, \dots, x^i_n)$ a vector state in $\mathbb{R}^n$;
- $u^i = (u^i_1, \dots, u^i_m)$ and input vector in $\mathbb{R}^m$;
- $x^i(t)$ a time variant state vector in $\mathbb{R}^n$ for each t;
- $\dot{x}^i(t)$ the derivative of the time variant state vector;
- $u^i(t)$ a time variant input vector in $\mathbb{R}^m$ for each t.

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