

A summary on symmetries and conserved quantities of autonomous Hamiltonian systems

NARCISO ROMÁN-ROY*

*Departament of Mathematics. Universitat Politècnica de Catalunya.
Edificio C-3, Campus Norte UPC. C/ Jordi Girona 1. 08034 Barcelona. Spain*

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Abstract

A complete geometric classification of symmetries of autonomous Hamiltonian systems is established; explaining how to obtain their associated conserved quantities in all cases. In particular, first we review well-known results and properties about the symmetries of the Hamiltonian and of the symplectic form and then some new kinds of non-symplectic symmetries and their conserved quantities are introduced and studied.

Key words: *Symmetries, Conserved quantities, Hamiltonian systems, Noether theorem, Symplectic manifolds.*

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1 Introduction

The existence of symmetries of Hamiltonian and Lagrangian systems is related with the existence of *conserved quantities* (or *constants of motion*). All of them allow us to simplify the integration of dynamical equations, applying suitable reduction methods [1, 3, 19, 28]. The use of geometrical methods is a powerful tool in the study of these topics. In particular, we are interested in the case of *regular* (i.e., symplectic) Hamiltonian systems. The most complete way to deal with these problems is using the *theory of actions of Lie groups on (symplectic) manifolds*, and the subsequent theory of reduction [1, 19, 26, 29] (see also [14, 27] for a extensive list of references that cover many aspects of the problem of reduction by symmetries in a lot of different situations). Nevertheless, the problem of reduction will not be addressed in this dissertation.

As it is well known, the standard procedure to obtain conserved quantities consists in introducing the so-called *Noether symmetries*, and then use the *Noether theorem* which is stated both for the Lagrangian and the Hamiltonian formalism in mechanics (and field theories). Noether's theorem gives a procedure to associate conservation laws to Noether symmetries [1, 3, 19, 25, 28]. However, these kinds of symmetries do not exhaust the set of symmetries. As is known, there are symmetries which are not of Noether type, but they also generate conserved quantities (see, for instance, [7, 8, 9, 10, 12, 20, 22, 21, 31, 33]), and they are sometimes called *hidden symmetries*. Different attempts have been made to extend Noether's results or state new theorems in order to include and obtain the conserved quantities corresponding to these symmetries, for dynamical systems (for instance, see [4, 10, 21, 23, 36, 37]) and also for field theories [15, 17, 34].

*e-mail: narciso.roman@upc.edu / ORCID: 0000-0003-3663-9861.

The aim of this paper is to make a broad summary about the geometric study of symmetries of dynamical Hamiltonian systems (autonomous and regular) in the environment of symplectic mechanics. In particular, we establish a complete scheme of classification of all the different kinds of symmetries of Hamiltonian systems, explaining how to obtain the associated conserved quantities in each case. We follow the same lines of argument as in the analysis made in [36] for nonautonomous Lagrangian systems, where the authors obtain conserved quantities for different kinds of symmetries that do not leave the *Poincaré-Cartan form* invariant.

In particular, in Section 2, after stating the main concepts about the geometric (symplectic) description of (autonomous) Hamiltonian systems, we introduce the concept and characterization of symmetries and conserved quantities and we classify the symmetries in two groups: those leaving invariant the geometric structure (the symplectic form), which are called *geometric symmetries*, and those leaving invariant the dynamics (the Hamiltonian function), which are called *Hamiltonian symmetries*. Then, we review *Noether symmetries*; that is, those which are both geometrical and Hamiltonian, and their conserved quantities; stating the *Noether theorem* and its inverse [1, 19, 28]. Section 3 is devoted to study *non-Noether symmetries*. First, non-Hamiltonian symmetries are also reviewed, explaining how to obtain their associated conserved quantities, depending on whether the symmetry is or not geometric too. The most original part of the paper is in Section 3.2, where different kinds of non-geometric symmetries are defined, depending on how the symplectic form transforms under the symmetry. All of them are studied in detail, showing how to obtain conserved quantities depending on whether the symmetry is or not Hamiltonian. Finally, in Section 4 we present some typical examples of dynamical systems that illustrate some of the cases presented.

All manifolds are real, paracompact, connected and C^∞ . All maps are C^∞ . Sum over crossed repeated indices is understood.

2 Symplectic mechanics

2.1 Hamiltonian systems. Symmetries. Conserved quantities

(See, for instance, [1, 3, 11, 18, 19, 28, 29] for more information on the topics in this section).

Definition 1 A (regular) Hamiltonian system is a triad (M, ω, h) , such that (M, ω) is a symplectic manifold; where M represents the phase space of a dynamical system, and $h \in C^\infty(M)$ is the Hamiltonian function, which gives the dynamical information of the system (and can be locally or globally defined). If ω is a degenerate form (i.e.; a presymplectic form), then (M, ω, h) is said to be a non-regular (or singular) Hamiltonian system

Usually $M = T^*Q$, where Q is the *configuration space* of the system.

In this paper, only regular Hamiltonian systems are considered. In these cases there exists a unique vector field $X_h \in \mathfrak{X}(M)$, which is the *Hamiltonian vector field* associated with h :

$$i(X_h)\omega = dh, \quad (1)$$

and the dynamical trajectories are the integral curves $\sigma: \mathbb{R} \rightarrow M$ of this Hamiltonian vector field $X_h \in \mathfrak{X}(M)$. In a chart of symplectic (Darboux) coordinates $(U; q^i, p_i)$ in M we have that $X_h|_U = \frac{\partial h}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i} \frac{\partial}{\partial p_i}$, and the integral curves $\sigma(t)$ of X_h are the solution to the *Hamilton equations*. If $\dot{\sigma}: \mathbb{R} \rightarrow TM$ denotes the canonical lifting of σ to the tangent bundle TM , as $\dot{\sigma} = X_h \circ \sigma$, Hamilton's equations can be written in an intrinsic way (equivalent to (1)) as

$$i(\dot{\sigma})(\omega \circ \sigma) = dh \circ \sigma.$$

Definition 2 A symmetry (or a dynamical symmetry) of a Hamiltonian system is a diffeomorphism $\Phi: M \rightarrow M$ such that, if σ is a solution to the Hamilton equations, then $\Phi \circ \sigma$ is also a solution; which is equivalent to $\Phi_* X_h = X_h$. If $M = T^*Q$ and $\Phi = T^*\varphi$ (Φ is the canonical lifting of a diffeomorphism $\varphi: Q \rightarrow Q$ to the cotangent bundle), then Φ is a natural symmetry.

An infinitesimal symmetry (or a infinitesimal dynamical symmetry) is a vector field $Y \in \mathfrak{X}(M)$ whose local flows are local symmetries. which is equivalent to $L(Y)X_h = [Y, X_h] = 0$. If $M = T^*Q$ and $Y = Z^{C^*}$ (Y is the canonical lifting of $Z \in \mathfrak{X}(Q)$ to the cotangent bundle), then Y is a natural infinitesimal symmetry.

Definition 3 $f \in C^\infty(M)$ is a conserved quantity (or a constant of motion) if $L(X_h)f = 0$.

In particular, the Hamiltonian function h is a conserved quantity since $L(X_h)h = i(X_h)dh = i^2(X_h)\omega = 0$ (conservation of energy). Furthermore, it is immediate to prove that:

Proposition 1 if $\Phi: M \rightarrow M$ is a symmetry and $f \in C^\infty(M)$ is a conserved quantity, then Φ^*f is a conserved quantity. As a consequence, if $Y \in \mathfrak{X}(M)$ is an infinitesimal symmetry and $f \in C^\infty(M)$ is a conserved quantity, then $L(Y)f$ is a conserved quantity.

A symmetry of a Hamiltonian system leaves the Hamiltonian vector field, X_h , invariant. But, as X_h is determined by the geometrical structure (the symplectic form) and the dynamics (the Hamiltonian function) through the equation (1), some relationship is to be expected between the invariance of X_h and the invariance of these two elements. This leads to define:

Definition 4 A diffeomorphism $\Phi: M \rightarrow M$ is a geometric symmetry of the Hamiltonian system if $\Phi^*\omega = \omega$ (that is, Φ is a symplectomorphism).

A vector field $Y \in \mathfrak{X}(M)$ is an infinitesimal geometric symmetry if $L(Y)\omega = 0$ (that is, it is a local Hamiltonian vector field, $Y \in \mathfrak{X}_{lh}(M)$).

A diffeomorphism $\Phi: M \rightarrow M$ is a Hamiltonian symmetry if $\Phi^*h = h$.

A vector field $Y \in \mathfrak{X}(M)$ is an infinitesimal Hamiltonian symmetry if $L(Y)h = 0$.

Proposition 2 Every (infinitesimal) geometrical and Hamiltonian symmetry is a (infinitesimal) symmetry.

(Proof) $0 = \Phi^*(i(X_h)\omega - dh) = i(\Phi_*^{-1}X_h)\Phi^*\omega - \Phi^*dh = i(\Phi_*^{-1}X_h)\omega - dh = i(X_h)\omega - dh$
 $\implies \Phi_*^{-1}X_h = X_h$.

$$i([Y, X_h])\omega = L(Y)i(X_h)\omega - i(X_h)L(Y)\omega = L(Y)dh = dL(Y)h = 0 \implies [Y, X_h] = 0. \quad \blacksquare$$

2.2 Noether symmetries. Noether's theorem

(See, for instance, [1, 3, 18, 19, 25, 28] for more details on these topics).

Definition 5 Let (M, ω, h) be a Hamiltonian system.

A Noether symmetry is a diffeomorphism $\Phi: M \rightarrow M$ such that:

$$(i) \Phi^*\omega = \omega \quad ; \quad (ii) \Phi^*h = h.$$

If $M = T^*Q$ and $\Phi = T^*\varphi$, for a diffeomorphism $\varphi: Q \rightarrow Q$, then Φ is a natural Noether symmetry.

An infinitesimal Noether symmetry is a vector field $Y \in \mathfrak{X}(M)$ such that:

$$(i) L(Y)\omega = 0 \quad ; \quad (ii) L(Y)h = 0.$$

If $M = T^*Q$ and $Y = Z^{C^*}$, for $Z \in \mathfrak{X}(Q)$, then Y is a natural infinitesimal Noether symmetry.

Thus, a (infinitesimal) Noether symmetry is a (infinitesimal) geometric and Hamiltonian symmetry and hence it is a symmetry. From now on we consider only infinitesimal symmetries.

Theorem 1 (Noether): *Let $Y \in \mathfrak{X}(M)$ be an infinitesimal Noether symmetry.*

1. *The form $\omega_{(0)} \equiv i(Y)\omega \in \Omega^1(M)$ is closed. Then, for every $p \in M$, there is $U_p \ni p$, there exists $f_Y \in C^\infty(U_p)$, unique up to a constant function, such that $i(Y)\omega = df_Y$ (on U_p).*
2. *f_Y is a conserved quantity on U_p ; that is, $L(X_h)f_Y = 0$.*

(Proof) 1. $d i(Y)\omega = L(Y)\omega - i(Y)d\omega = 0$.

$$2. L(X_h)f_Y = i(X_h)df_Y = i(X_h)i(Y)\omega = -i(Y)i(X_h)\omega = -i(Y)dh = -L(Y)h = 0. \blacksquare$$

Corollary 1 *The function f_Y is invariant by Y . (No new conserved quantities are generated by the action of Y on f_Y).*

(Proof) As f_Y is a Hamiltonian function of Y , then $L(Y)f_Y = i(Y)df_Y = i^2(Y)\omega = 0$. \blacksquare

Corollary 2 *If $\omega = d\theta$, for $\theta \in C^\infty(U_p)$, then there exists $\xi_Y \in C^\infty(U_p)$ verifying that $L(Y)\theta = d\xi_Y$, on U_p ; and then $f_Y = \xi_Y - i(Y)\theta$ (up to a constant function).*

(Proof) In U_p we have that $0 = L(Y)\omega = L(Y)d\theta = dL(Y)\theta$. Then there exists $\xi_Y \in C^\infty(U_p)$ such that $L(Y)\theta = d\xi_Y$, on U_p , and the result follows from

$$df_Y = i(Y)\omega = i(Y)d\theta = L(Y)\theta - d i(Y)\theta = d\xi_Y - d i(Y)\theta .$$

\blacksquare

Theorem 2 (Inverse Noether): *For every conserved quantity $f \in C^\infty(M)$, its Hamiltonian vector field $Y_f \in \mathfrak{X}_{lh}(M)$ is an infinitesimal Noether symmetry.*

(Proof) As $Y_f \in \mathfrak{X}_{lh}(M)$, then $L(Y_f)\omega = 0$. In addition,

$$L(Y_f)h = i(Y_f)dh = i(Y_f)i(X_h)\omega = -i(X_h)i(Y_f)\omega = -i(X_h)df = -L(X_h)f = 0 .$$

\blacksquare

3 Non-Noether symmetries

3.1 Non-Hamiltonian symmetries

Now we study all the symmetries which are not of Noether's type (that is, symmetries which are not Hamiltonian and/or geometrical), and how they generate conserved quantities. First we analyze the (infinitesimal) *non-Hamiltonian symmetries*; that is, such that

$$[Y, X_h] = 0 \quad , \quad L(Y)h \neq 0 .$$

Theorem 3 *Let $Y \in \mathfrak{X}(M)$ be an infinitesimal non-Hamiltonian symmetry. Then $f = L(Y)h$ is a conserved quantity (which may be trivial).*

(Proof) $L(X_h)L(Y)h = L([X_h, Y])h + L(Y)L(X_h)h = 0$. \blacksquare

Remark 1 Bearing in mind that $0 = i([Y, X_h])\omega = L(Y) i(X_h)\omega - i(X_h) L(Y)\omega$, we get

$$i(X_h) L(Y)\omega = L(Y) i(X_h)\omega = L(Y)dh = dL(Y)h = df ; \quad (2)$$

(then X_h is said to be a *bi-Hamiltonian vector field* for $\tilde{\omega} = L(Y)\omega$ and $f = L(Y)h$, and the dynamical system is called a *bi-Hamiltonian system*. For a deeper analysis of bi-Hamiltonian systems and their symmetries and conserved quantities see, for instance, [5, 6, 7, 8, 9, 13, 33]). Taking this into account, if in addition $L(Y)\omega = 0$, from (2) we obtain that $df = 0$, and then f is locally constant. Furthermore, in the particular case that $L(Y)\omega = c\omega$, $c \in \mathbb{R}$, we have

$$i(X_h) L(Y)\omega = c i(X_h)\omega = c dh ,$$

and (2) lead to the general result that $f = ch$ (up to a constant) is the conserved quantity.

As a straightforward consequence of Proposition 1 we have:

Theorem 4 *If $Y \in \mathfrak{X}(M)$ is an infinitesimal symmetry and $L^N(Y)h = \overbrace{L(Y) \dots L(Y)}^N h \neq 0$, for $N > 1$, then they are conserved quantities (which may be trivial).*

3.2 Non-geometric symmetries

Next we analyze the (infinitesimal) *non-geometric symmetries*; that is, such that

$$[Y, X_h] = 0 \quad , \quad L(Y)\omega \neq 0 .$$

Although Theorem 3 also applies to a particular case of this situation (see Remark 1), there are other possibilities which we study in the next sections. Our analysis is based on the methods introduced in [36, 37] for non-Noether symmetries in the nonautonomous Lagrangian context.

3.2.1 Higher-order Noether symmetries

Definition 6 *$Y \in \mathfrak{X}(M)$ is an infinitesimal Noether symmetry of order N if:*

1. *Y is an infinitesimal symmetry.*
2. *There exists $N > 1$ such that $L^N(Y)\omega = 0$.*
3. *$L(Y)h = 0$ (that is, Y is a Hamiltonian symmetry).*

Remark 2 If condition (3) does not hold (Y is not a Hamiltonian symmetry) then, as stated in Prop. 1 and Theor. 4, $L(Y)h$ and, eventually, $L^m(Y)h$, for $m > 1$, are conserved quantities.

Theorem 5 (Noether generalized): *Let $Y \in \mathfrak{X}(M)$ be an infinitesimal Noether symmetry of order N . Then:*

1. *The form $L^{N-1}(Y) i(Y)\omega \in \Omega^1(M)$ is closed.
Then, for every $p \in M$, there is a neighborhood $U_p \ni p$ such that there exists $f \in C^\infty(U_p)$, which is unique up to a constant function, satisfying that $L^{N-1}(Y) i(Y)\omega = df$.*
2. *The function $f \in C^\infty(U_p)$ is a conserved quantity; that is, $L(X_h)f = 0$ (on U_p).*

(Proof) As $L^N(Y)\omega = 0$, we have

$$dL^{N-1}(Y)i(Y)\omega = L^{N-1}(Y)d i(Y)\omega = L^N(Y)\omega - L^{N-1}(Y)i(Y)d\omega = 0.$$

Furthermore, as $Y \in \mathfrak{X}(M)$ it is a symmetry, then $[Y, X_h] = 0$ and therefore

$$\begin{aligned} L(X_h)f &= i(X_h)df = i(X_h)L^{N-1}(Y)i(Y)\omega = i(X_h)L(Y)L^{N-2}(Y)i(Y)\omega \\ &= [L(Y)i(X_h) - i([Y, X_h])]L^{N-2}(Y)i(Y)\omega = L(Y)i(X_h)L^{N-2}(Y)i(Y)\omega, \end{aligned}$$

and repeating the reasoning $N - 2$ times we arrive at the result

$$\begin{aligned} L(X_h)f &= L^{N-1}(Y)i(X_h)i(Y)\omega = -L^{N-1}(Y)i(Y)i(X_h)\omega = \\ &= -L^{N-1}(Y)i(Y)dh = -L^N(Y)h = 0. \end{aligned}$$

■

Corollary 3 *The function f given in the above theorem is invariant by Y . (No new conserved quantities are generated by the action of Y on f).*

(Proof) $L(Y)f = i(Y)df = i(Y)L^{N-1}(Y)i(Y)\omega = L^{N-1}(Y)i^2(Y)\omega = 0$.

■

Corollary 4 *If $\omega = d\theta$, for $\theta \in \Omega^1(U_p)$, then there exists $\xi \in C^\infty(U_p)$ verifying that $L^N(Y)\theta = d\xi$, on U_p ; and then $f = \xi - L^{N-1}(Y)\theta$ (up to a constant function).*

(Proof) In fact, we have that, in U_p

$$0 = L^N(Y)\omega = L^N(Y)d\theta = dL^N(Y)\theta,$$

then there exists $\xi \in C^\infty(U_p)$ such that $L^N(Y)\theta = d\xi$, on U_p . Furthermore,

$$\begin{aligned} df &= L^{N-1}(Y)i(Y)\omega = L^{N-1}(Y)i(Y)d\theta = L^{N-1}(Y)L(Y)\theta - L^{N-1}(Y)d i(Y)\theta \\ &= L^N(Y)\theta - dL^{N-1}(Y)i(Y)\theta = d\xi - dL^{N-1}(Y)i(Y)\theta \end{aligned}$$

and the result follows.

■

3.2.2 Other non-geometric symmetries

If $Y \in \mathfrak{X}(M)$ is not an infinitesimal geometric symmetry and it is not a higher-order Noether symmetry, then we have that $L^m(Y)\omega \neq 0$, $\forall m \in \mathbb{N}$. Then, as the module of 2-forms in a finite-dimensional manifold is locally finite generated, after a finite number of Lie derivations we have that the following condition holds (maybe only locally):

$$L^N(Y)\omega = f_0\omega + f_1L(Y)\omega + \dots + f_{N-1}L^{N-1}(Y)\omega, \quad (3)$$

being $\omega, L(Y)\omega, \dots, L^{N-1}(Y)\omega$ independent forms and $\{f_0, \dots, f_{N-1}\} \subset C^\infty(M)$. Therefore:

Theorem 6 *Let $Y \in \mathfrak{X}(M)$ be an infinitesimal symmetry such that condition (3) holds.*

1. *If $\{f_0, \dots, f_{N-1}\} \subset C^\infty(M)$ are not all constant functions, then these non-constant functions f_j are (non-trivial) local conserved quantities.*
2. *If $\{f_0, \dots, f_{N-1}\}$ are constant functions such that $f_0 = 0$ (and some of the other are non-vanishing; that is, $L^N(Y)\omega = C_1L(Y)\omega + \dots + C_{N-1}L^{N-1}(Y)\omega$), and $L(Y)h = 0$, (Y is an infinitesimal Hamiltonian symmetry), then:*

- (a) The form $\gamma \equiv L^{N-1}(Y) i(Y)\omega - C_{N-1} L^{N-2}(Y) i(Y)\omega - \dots - C_1 i(Y)\omega$ is closed. Then, for every $p \in M$, there exist an open neighbourhood $U_p \ni p$ and a function $f \in C^\infty(U_p)$ (unique up to a constant), such that $\gamma = df$.
- (b) f is a local conserved quantity.

(Proof) 1. Remember that $[X_h, Y] = 0$, because Y is an infinitesimal symmetry. Then, if $\{f_0, \dots, f_{N-1}\} \subset C^\infty(M)$ are not all constant functions, taking Lie derivatives with respect to X_h in both sides of the equation (3); for the left-hand side first we observe that

$$L(X_h) L(Y)\omega = L([X_h, Y])\omega + L(Y) L(X_h)\omega = 0 ;$$

then, assuming that $L(X_h) L^{N-1}(Y)\omega = 0$, we obtain

$$L(X_h) L^N(Y)\omega = L([X_h, Y]) L^{N-1}(Y)\omega + L(Y) L(X_h) L^{N-1}(Y)\omega = 0 . \quad (4)$$

For the right-hand side, bearing in mind (4), we have that

$$\begin{aligned} L(X_h)(f_{N-1} L^{N-1}(Y)\omega) &= (L(X_h)f_{N-1}) L^{N-1}(Y)\omega + f_{N-1} L(X_h) L^{N-1}(Y)\omega \\ &= (L(X_h)f_{N-1}) L^{N-1}(Y)\omega . \end{aligned}$$

Then, as $\omega, L(Y)\omega, \dots, L^{N-1}(Y)\omega$ are independent forms, from (3) and (4) we conclude

$$L(X_h)f_0 = \dots = L(X_h)f_{N-1} = 0 .$$

2. Bearing in mind the hypothesis, a direct calculation leads to

$$\begin{aligned} d\gamma &= L^{N-1}(Y)d i(Y)\omega - C_{N-1} L^{N-2}(Y)d i(Y)\omega - \dots - C_1 d i(Y)\omega \\ &= L^N(Y)\omega - C_{N-1} L^{N-1}(Y)\omega - \dots - C_1 L(Y)\omega = 0 . \end{aligned}$$

As $\gamma = df$ (locally) and $[Y, X_h] = 0$, we obtain

$$\begin{aligned} L(X_h)f &= i(X_h)df = i(X_h)\gamma = \\ &= i(X_h)[L^{N-1}(Y) i(Y)\omega - C_{N-1} L^{N-2}(Y) i(Y)\omega - \dots - C_1 i(Y)\omega] \\ &= [-i([Y, X_h]) + L(Y) i(X_h)] L^{N-2}(Y) i(Y)\omega \\ &\quad - C_{N-1} [-i([Y, X_h]) + L(Y) i(X_h)] L^{N-3}(Y) i(Y)\omega \\ &\quad - \dots - C_2 [-i([Y, X_h]) + L(Y) i(X_h)] L(Y) i(Y)\omega + C_1 i(Y) i(X_h)\omega \\ &= L(Y) i(X_h)[L^{N-2}(Y) - C_{N-1} L^{N-3}(Y) - \dots - C_2 L(Y)] i(Y)\omega + C_1 L(Y)h \\ &= L(Y) i(X_h)[L^{N-2}(Y) - C_{N-1} L^{N-3}(Y) - \dots - C_2 L(Y)] i(Y)\omega , \end{aligned}$$

and repeating the procedure $N - 2$ times we arrive to the result

$$L(X_h)f = L^{N-1}(Y) i(X_h) i(Y)\omega = -L^{N-1}(Y) i(Y) i(X_h)\omega = -L^{N-1}(Y) i(Y)dh = -L^N(Y)h = 0 .$$

■

Remark 3 In the remaining cases; that is, item 2 with $L(Y)h \neq 0$, or when

$$L^N(Y)\omega = C_0\omega + C_1 L(Y)\omega + \dots + C_{N-1} L^{N-1}(Y)\omega , \text{ with } C_0 \neq 0 ,$$

we are, in general, in the situation of Theorems 3 and 4, and hence h and, eventually, $L^N(Y)h$ (with $N \geq 1$) are conserved quantities.

In the case of item 1, the conserved quantities f_0, \dots, f_{N-1} are not invariant by Y necessarily and their Lie derivatives could generate new conserved quantities. In the case 2, no new conserved quantities are generated by the action of Y on f , since

$$\begin{aligned} L(Y)f &= i(Y)df = i(Y)[L^{N-1}(Y) i(Y)\omega - C_{N-1} L^{N-2}(Y) i(Y)\omega - \dots - C_1 i(Y)\omega] \\ &= [L^{N-1}(Y) - C_{N-1} L^{N-2}(Y) - \dots - C_1] i^2(Y)\omega = 0 . \end{aligned}$$

Remark 4 Theorems 5 and 6 give new ways to obtain conserved quantities generated by non-Noether symmetries; nevertheless, we are not aware of any examples of their application.

4 Some examples

We illustrate some applications of Theorems 3 and 4 that yield trivial and nontrivial first integrals respectively in Sections 4.1 and 4.2. In addition to these ones, other interesting examples of non-Noether symmetries and their associated conserved quantities can be found, for instance, in [10, 16, 24, 32, 38] (see also [2, 30], and the references quoted therein, for another collection of (quantum-mechanical) systems having nontrivial integrals of motion).

4.1 Example 1: 2-dimensional harmonic oscillator

(See also [8, 20, 36] for this and other similar models). In this case, $Q = \mathbb{R}^2$ and $M = T^*Q \simeq \mathbb{R}^2 \times \mathbb{R}^2$, with canonical coordinates (q^1, q^2, p_1, p_2) in T^*Q , and the symplectic form reads

$$\omega = dq^1 \wedge dp_1 + dq^2 \wedge dp_2 . \quad (5)$$

Now, the Hamiltonian function is

$$h = \frac{1}{2}((p_1)^2 + (p_2)^2 + (\Omega_1)^2(q^1)^2 + (\Omega_2)^2(q^2)^2) , \quad (6)$$

where Ω_1, Ω_2 are constants. The Hamiltonian vector field is

$$X_h = p_1 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial q^2} - (\Omega_1)^2 q^1 \frac{\partial}{\partial p_1} - (\Omega_2)^2 q^2 \frac{\partial}{\partial p_2} . \quad (7)$$

This system has two (geometric but non-Hamiltonian) infinitesimal non-Noether symmetries

$$Y_1 = \frac{\Omega_1}{(\Omega_1)^2(q^1)^2 + (p_1)^2} \left(q^1 \frac{\partial}{\partial q^1} + p_1 \frac{\partial}{\partial p_1} \right) , \quad Y_2 = \frac{\Omega_2}{(\Omega_2)^2(q^2)^2 + (p_2)^2} \left(q^2 \frac{\partial}{\partial q^2} + p_2 \frac{\partial}{\partial p_2} \right) ;$$

in fact, we have that

$$L(Y_i)h = -\Omega_i \quad , \quad L(Y_i)\omega = 0 \quad , \quad [Y_i, X_h] = 0 \quad ; \quad (i = 1, 2) ;$$

and the corresponding constants of motion are $f_i = L(Y_i)h = \Omega_i$; which, in this case, are constant functions; that is, trivial conserved quantities. (See also [35] for an analysis of the algebra of symmetries of this model in the case of commensurable frequencies).

4.2 Example 2: 2-dimensional isotropic harmonic oscillator

This is a particular case of the above example, with $\Omega_1 = \Omega_2 = \Omega$. Then $Q = \mathbb{R}^2$ and $M = T^*Q \simeq \mathbb{R}^2 \times \mathbb{R}^2$, as above, and the symplectic form is again (5). The Hamiltonian function is (6) and the Hamiltonian vector field is (7) with $\Omega_1 = \Omega_2 = \Omega$. For this system, the vector field

$$Y = q^2 \frac{\partial}{\partial q^1} + q^1 \frac{\partial}{\partial q^2} + p_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_2}$$

verifies that

$$\begin{aligned} [Y, X_h] &= 0 & , & & L(Y)h &= 2(p_1 p_2 + \Omega^2 q^1 q^2) & , \\ L(Y)\omega &= 2(dq^1 \wedge dp_2 + dq^2 \wedge dp_1) & , & & L^2(Y)\omega &= 4(dq^1 \wedge dp_1 + dq^2 \wedge dp_2) = 4\omega & , \end{aligned}$$

so it is an infinitesimal non-Noether symmetry which is a non-Hamiltonian and non-geometric symmetry. Then, according to Theorem 3, a conserved quantity is $f = p_1 p_2 + \Omega^2 q^1 q^2$. Now we have that $L(Y)f = L^2(Y)h = 4h$, and no new conserved quantities arise from f .

Nevertheless, it is well known that this dynamical system is an example of a *superintegrable system* [8, 20]. In fact, the Hamiltonian function can be split as $h = h_1 + h_2$, where $h_i =$

$\frac{1}{2}((p_i)^2 + \Omega^2(q^i)^2)$ ($i = 1, 2$), and h_1 and h_2 are also constants of motion, in addition to h , since $L(X_h)h_i = 0$, for $i = 1, 2$. Thus, we have $3 = 2n - 1$ independent conserved quantities (notice that h_1 , h_2 and h are not independent, but h_1 , h_2 and f are).

As stated in Theorem 2, there are infinitesimal Noether symmetries which originate these new conserved quantities: their Hamiltonian vector fields, which are

$$X_{h_i} = p_i \frac{\partial}{\partial q^i} - \Omega^2 q^i \frac{\partial}{\partial p_i} \quad ; \quad (i = 1, 2) \quad ;$$

and $X_h = X_{h_1} + X_{h_2}$. Nevertheless, they can be also associated with other kinds of infinitesimal symmetries. In fact, the infinitesimal symmetry Y can be split into $Y = Y_1 + Y_2$, where

$$Y_1 = q^2 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial p_1} \quad , \quad Y_2 = q^1 \frac{\partial}{\partial q^2} + p_1 \frac{\partial}{\partial p_2} \quad ,$$

and these vector fields are non-Hamiltonian and non-geometric infinitesimal symmetries. In fact,

$$\begin{array}{llll} [Y_1, X_h] = 0 & , & [Y_2, X_h] = 0 & , \\ L(Y_1)\omega = dq^2 \wedge dp_1 + dq^1 \wedge dp_2 & , & L(Y_2)\omega = dq^2 \wedge dp_1 + dq^1 \wedge dp_2 & , \\ L^2(Y_1)\omega = 2 dq^2 \wedge dp_2 & , & L^2(Y_2)\omega = 2 dq^1 \wedge dp_1 & , \\ L^3(Y_1)\omega = 0 & , & L^3(Y_2)\omega = 0 & , \\ L(Y_1)h = p_1 p_2 + \Omega^2 q^1 q^2 = f & , & L(Y_2)h = p_1 p_2 + \Omega^2 q^1 q^2 = f & , \\ L^2(Y_1)h = (p_1)^2 + \Omega^2 (q^1)^2 = 2h_1 & , & L^2(Y_2)h = (p_2)^2 + \Omega^2 (q^2)^2 = 2h_2 & , \\ L^3(Y_1)h = 0 & , & L^3(Y_2)h = 0 & . \end{array}$$

Therefore, as it is stated in Theorem 4, h_1 , h_2 and f are three independent conserved quantities.

Finally, it is interesting to notice that there are other independent non-Noether infinitesimal symmetries having h_1 , h_2 and f as their associated conserved quantities; in particular (see [4]):

$$\begin{aligned} Z_1 &= [(p_2)^2 + \Omega^2 (q^2)^2] \left(q^2 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial p_1} \right) \quad , \quad Z_2 = [(p_1)^2 + \Omega^2 (q^1)^2] \left(q^1 \frac{\partial}{\partial q^2} + p_1 \frac{\partial}{\partial p_2} \right) \quad , \\ Z_3 &= [q^1 p_2 - q^2 p_1] \left(p_1 \frac{\partial}{\partial q^1} - p_2 \frac{\partial}{\partial q^2} - q^1 \frac{\partial}{\partial p_1} + q^2 \frac{\partial}{\partial p_2} \right) \quad . \end{aligned}$$

5 Conclusions and outlook

A classification of the symmetries for (*autonomous and regular*) *Hamiltonian systems* has been done, obtaining the associated conserved quantities in each case. In this way, we have reviewed and completed previous results on this topic (for instance, in [4, 7, 10, 20, 21, 23, 33, 36, 37]).

We have reviewed the Noether symmetries (which are both geometrical and dynamical) and the Hamiltonian version of Noether's theorem (and its converse). Next, we have considered the non-Noether symmetries. First, we have analyzed the non-Hamiltonian symmetries and their conserved quantities; but the main contribution of the paper is the analysis of the non-geometric symmetries (although we are not aware of any instances of application of Theorems 5 and 6). We have seen that there are several types of them, according to the behaviour of the symplectic structure under the action of the symmetry. The procedure for obtaining the conserved quantities depends on whether the symmetry is also Hamiltonian or not. In particular, in some cases, it consist in applying a suitable generalization of the Noether theorem.

A similar study to what we have done here could be done for *autonomous Lagrangian systems*, although in this case the *symmetries of the Lagrangian* must be also considered. Finally all these results could also be extended to classical field theories in order to do a classification of their symmetries and the corresponding conservation laws; completing, in this way, the partial results already obtained in [15, 17, 34] for non-Noether symmetries.

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