Solution method for the time-fractional hyperbolic heat equation

Ioannis K. Dassios[1], Francesc Font[2]

[1]AMPSAS, University College Dublin, Ireland

[2]Centre de Recerca Matemàtica, Campus de Bellaterra Edifici C, 08193 Bellaterra, Barcelona, Spain

Abstract. In this article we propose a method to solve the time-fractional hyperbolic heat equation. We first formulate a boundary value problem for the standard hyperbolic heat equation in a finite domain and provide an analytical solution by means of separation of variables and Fourier series. Then, we consider the same boundary value problem for the fractional hyperbolic heat equation. The fractional problem is solved using three different definitions of the fractional derivative: the Caputo fractional derivative and two recently defined alternative versions of this derivative, the Caputo–Fabrizio and the Atangana–Baleanu. A closed form of the solution is provided for each case. Finally, we compare the solutions of the fractional and the standard problem and show numerically that the solution of the standard hyperbolic heat equation can be retrieved from the solution of the fractional equation in the limit $\gamma \to 2$, where γ represents the exponent of the fractional derivative.

Keywords: fractional calculus, caputo derivative, initial conditions, boundary value problem, heat equation.

1 Introduction

In the last decade many authors have studied problems of fractional partial differential equations (FPDEs), fractional differential equations and fractional difference equations, and have derived interesting results on different type of problems for given initial or boundary conditions, see [4], [5], [6], [11], [12], [13], [14], [15], [20], [21], [22], [26], [28].

Focus has also been given in the mathematical modelling of many phenomena by using fractional operators. The theory of FPDEs is a promising tool for applications in physics [18], nanotechnology [4], biology [5], gravitation theory [6], and applications where memory effects appear [24]. Other examples of the use of FPDEs involve self-similar protein dynamics [17], applications to control processing [25], fractional PID controllers for industry application [29] and the study of kinetic phenomena emerging from the self-similar structure of the medium [30].

Fractional-order operators are not just a generalization of the classical integer-order operators and because of the way they are defined more elegant techniques are required for qualitative studies. Of course in many practical cases these techniques are not enough. In addition, the definition of the fractional derivative is not unique and hence when modelling phenomena with FPDEs one can not rely on a single definition and has to study the FPDE also with the alternative definitions. In this article we will use three different definitions of the fractional derivative to investigate a FPDE relevant in the context of heat transport phenomena.

It is well known that the Fourier law for the heat flux, $q = -k\nabla T$, breaks down at very small length and time scales [33, 34]. The classical heat conduction equation, $T_t = aT_{xx}$, derived from the Fourier law and the energy conservation principle, fails to reproduce thermal transport at these scales. In addition, the classical heat equation assumes an infinite speed for the heat propagation, which is incompatible with the principles of relativity. In the 1950s, Cattaneo introduced a hyperbolic heat conduction equation (HHE), $T_t + \tau T_{tt} = aT_{xx}$, that fixes the problem of infinite speed of propagation [31]. However, recent studies, showed that the HHE does not provide a satisfactory description of thermal transport at very small length and time scales either [32]. Other variations of the classical heat conduction equation to capture the anomalous phenomena at the nanoscale involve the introduction of an effective thermal conductivity that depends on the size of the material [35] or a modified expression of the thermal flux to account for non-local effects [36]. In the last few years, with the advance on the theory of fractional partial differential equations, the question of whether a fractional heat equation could provide a consistent explanation of the heat transport phenomena observed at small length and time scales has become more relevant. In this article we will look for solutions to the fractional hyperbolic heat equation (FHHE).

We will use three different definitions of the fractional derivative to solve the FHHE. It is then convenient to provide their definitions:

Definition 1.1. (see [5], [7], [20]) Let $Y : [0, +\infty) \to \mathbb{R}$, $t \to Y$, denote a continuous and differentiable function. Then, the Caputo (C) fractional derivative of order a, $0 < a < 1$, is defined by

$$Y_C^{(a)}(t) := Y^{(a)}(t) = \frac{1}{\Gamma(1-a)} \int_0^t \left[(t-x)^{-a} Y'(x)\right] dx. \tag{1}$$

Recently, a new fractional derivative was defined by Caputo and Fabrizio (see [8]) and it was followed by some related theoretical and applied results (see [1], [2], [23] and the references therein). The aim of this fractional derivative was to introduce of a new derivative with exponential kernel. Its anti-derivative was reported in [1] and it was found to be the average of a given function. We believe that the main idea presented in [8] was to find a way to describe even better the dynamics of systems with memory effect than other existing definitions of fractional derivatives in the literature.

Definition 1.2. (see [8], [9], [23]) Let $Y : [0, +\infty) \to \mathbb{R}$, $t \to Y$, denote a continuous and differentiable function. Then, the Caputo–Fabrizio (CF) fractional derivative of order a, $0 \le a \le 1$, is defined by

$$Y_{CF}^{(a)}(t) := Y^{(a)}(t) = \frac{1}{1-a} \int_0^t \left[e^{-\frac{a}{1-a}(t-x)} Y'(x)\right] dx. \tag{2}$$

Following the question "what is the most accurate kernel which better describes it?", Atangana and Baleanu, see [3], suggested a possible answer to this by introducing a new fractional derivative which has a non-local kernel.

Definition 1.3. (see [3]) Let $Y : [0, +\infty) \to \mathbb{R}$, $t \to Y$, denote a differentiable function. Then, the modified Caputo (AB) fractional derivative of order $0 \le a \le 1$, is defined by

$$Y_{AB}^{(a)}(t) := Y^{(a)}(t) = \frac{B(a)}{1-a} \int_0^t E_a \left[-a\frac{(t-x)^a}{1-a}\right] Y'(x) dx. \tag{3}$$

Where $E_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+ak)}$, $a, z \in \mathbb{C}$, $Re(a) > 0$ (see [5], [7], [20]). $B(a)$ denotes a normalization function obeying $B(0) = B(1) = 1$.

The article is organised as follows: in Section 2 we use the (C), (CF), (AB) fractional derivatives as defined in (1), (2), (3) respectively and study the solutions of an initial & boundary value proble of a FPDE which we propose as a suitable physical model to describe sub-nanometric thermal transport. In Section 3 we provide several numerical examples to justify our theory and compare the effect of the different type of derivatives.

2 Main Results

In this section, we first introduce a boundary value problem for the hyperbolic heat equation and provide a solution using separation of variables and Fourier series. Then, we formulate the same boundary value problem for the fractional hyperbolic heat equation. The problem is solved using the three different definitions of the fractional derivative. The method that we use to solve the fractional problem involves separation of variables and Laplace transforms for the resulting fractional equations in time.

Hyperbolic heat equation. We will first discuss the hyperbolic heat equation:

$$\beta \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad \text{on} \quad 0 < x < 1, \tag{4}$$

where $0 \leq \beta \leq 1$, subject to the initial and boundary conditions

$$T(x, 0) = \phi(x), \quad \left.\frac{\partial T}{\partial t}\right|_{t=0} = \psi(x), \quad T(0, t) = T(1, t) = 0. \tag{5}$$

Equation (4) is separable and the general solution can be expressed as

$$T(x, t) = \sum_{n=1}^{\infty} X_n(x) Y_n(t).$$

Where

$$X_n(x) = \sin(\sqrt{\lambda_n} x), \quad \lambda_n = n^2 \pi^2, \quad n = 1, 2, \dots,$$

and

$$Y_n(t) = C_{n,1} \exp\left(\frac{-1 + \sqrt{1 - 4\lambda_n \beta}}{2\beta} t\right) + C_{n,2} \exp\left(\frac{-1 - \sqrt{1 - 4\lambda_n \beta}}{2\beta} t\right). \tag{6}$$

Note that from the initial conditions (5) we have

$$T(x, 0) = \sum_{n=1}^{\infty} X_n(x) Y_n(0), \quad T_t(x, 0) = \sum_{n=1}^{\infty} X_n(x) Y_n'(0),$$

or, equivalently,

$$\phi(x) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n} x) Y_n(0), \quad \psi(x) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n} x) Y_n'(0),$$

or, equivalently, by expanding into their corresponding Fourier series

$$Y_n(0) = \phi_n = 2 \int_0^1 \phi(x) \sin(\sqrt{\lambda_n} x) dx, \quad Y_n'(0) = \psi_n = 2 \int_0^1 \psi(x) \sin(\sqrt{\lambda_n} x) dx. \tag{7}$$

Using (7) into $Y_n(t)$ we obtain

$$C_{n,1} = \phi_n - C_{n,2}\,, \qquad C_{n,2} = -\frac{\left[2\beta\psi_n - \phi_n(-1+\sqrt{1-4\lambda_n\beta})\right]}{2\sqrt{1-4\lambda_n\beta}}$$

which completely determines the solution $T(x,t)$. Finally, we note that $\sqrt{1-4\lambda_n\beta}$ will become complex for $n > 1/2\pi\sqrt{\beta}$ and (6) will admit a form in terms of sine and cosine. In particular, (6) can be expressed as

$$Y_n(t) = \begin{cases} C_{n,1}\exp\left(\frac{-1+\sqrt{1-4\lambda_n\beta}}{2\beta}t\right) + C_{n,2}\exp\left(\frac{-1-\sqrt{1-4\lambda_n\beta}}{2\beta}t\right) & n < 1/2\pi\sqrt{\beta} \\ e^{-t/2\beta}\left[C_{n,3}\cos\left(\frac{\sqrt{4\lambda_n\beta-1}}{\beta}\right) + C_{n,4}\sin\left(\frac{\sqrt{4\lambda_n\beta-1}}{2\beta}\right)\right] & n > 1/2\pi\sqrt{\beta} \end{cases} \tag{8}$$

where

$$C_{n,3} = \phi_n\,, \qquad C_{n,4} = \frac{1}{\sqrt{4\beta\lambda_n - 1}}\left(2\beta\psi_n + \phi_n\right)\,.$$

Time-fractional hyperbolic heat equation. We now propose the following fractional hyperbolic heat equation:

$$\beta\frac{\partial^\gamma T}{\partial t^\gamma} + \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad \text{on} \qquad 0 < x < 1, \tag{9}$$

with initial and boundary conditions given by (5), and state the following Theorem:

Theorem 2.1. We consider the FPDE (9) with initial and boundary conditions given by (5). Then the solution of the initial & boundary value problem is given by:

$$T(x,t) = \sum_{n=1}^{\infty} Y_n(t) X_n(x),$$

with

$$X_n(x) = \sin(\sqrt{\lambda_n}x)\,, \qquad \lambda_n = n^2\pi^2\,, \qquad n = 1, 2, \ldots,$$

and:

(a) If we use the (C) fractional derivative:

$$Y_n(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{4\beta^{r+1}d}\Big[(-2\beta(1-d)\mu_1^r + 2\beta\mu_2^r(1+d))\frac{t^r\phi_n}{\Gamma(r+1)} + (-4\beta^2\mu_1^r + 4\beta^2\mu_2^r)\frac{t^{ar}\psi_n}{\Gamma(ar+1)}\Big];$$

(b) If we use the (CF) fractional derivative:

$$Y_n(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{4\beta^{r+1}d}\Big[(-2\beta(1-d)\mu_1^r + 2\beta\mu_2^r(1+d))\frac{t^r\phi_n}{\Gamma(r+1)} + (-4\beta^2\mu_1^r + 4\beta^2\mu_2^r)\sum_{j=0}^{r}\binom{r}{j}(1-a)^j a^{r-j}\frac{t^{r-j}\psi_n}{\Gamma(r-j+1)}\Big];$$

(c) If we use the (AB) fractional derivative:

$$Y_n(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{4\beta^{r+1}d}\Big[(-2\beta(1-d)\mu_1^r + 2\beta\mu_2^r(1+d))\frac{t^r\phi_n}{\Gamma(r+1)} + (-4\beta^2\mu_1^r + 4\beta^2\mu_2^r)[\frac{B(a)}{1-a}]^{-r}\sum_{j=0}^{r}\binom{r}{j}(\frac{a}{1-a})^{r-j}\frac{t^{-aj+ar}\psi_n}{\Gamma(-aj+ar+1)}\Big].$$

In all cases where

$$\mu_1 = \frac{1}{2} + \frac{d}{2}, \quad \mu_2 = \frac{1}{2} - \frac{d}{2}, \quad d = \sqrt{1 - 4\beta n^2 \pi^2}$$

and ϕ_n, ψ_n are given by (7).

Proof. Equation (9) is separable and the solution can be expressed as

$$T(x,t) = X(x)Y(t),$$

and by replacing the above expression into (9) we get

$$\frac{\beta Y^{(\gamma)}(t) + Y'(t)}{Y(t)} = \frac{X''(x)}{X(x)},$$

and hence

$$\frac{\beta Y^{(\gamma)}(t) + Y'(t)}{Y(t)} = \frac{X''(x)}{X(x)} = -\lambda_n, \quad n = 1, 2, ...$$

with

$$X(x) = X_n(x) = \sin(\sqrt{\lambda_n} x), \quad \lambda_n = n^2 \pi^2, \qquad n = 1, 2, \ldots$$

Thus, we have to solve the following fractional differential equation for $Y(t) = Y_n(t)$:

$$\beta Y^{(\gamma)}(t) + Y'(t) + n^2 \pi^2 = 0.$$

Let $a + 1 = \gamma$. If we set

$$\begin{aligned} y_1 &= Y, \\ y_2 &= Y', \end{aligned}$$

then we have

$$\begin{aligned} y_1' &= y_2, \\ y_2^{(a)} &= Y^{(\gamma)} = -\tfrac{1}{\beta} y_2 - \tfrac{n^2\pi^2}{\beta} y_1, \end{aligned}$$

or, in matrix form

$$\begin{bmatrix} y_1' \\ y_2^{(a)} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Let $\mathcal{L}\{y_1(t)\} = Y_1(s)$, $\mathcal{L}\{y_2(t)\} = Y_2(s)$ be the Laplace transforms of $y_1(t)$, $y_2(t)$ respectively with $Y_1 = Y_1(s)$, $Y_2 = Y_2(s) \in \mathbb{C}$, being inverse functions. Using the fractional derivative as defined in (1), (2), (3), and by applying the Laplace transform $\mathcal{L}$ into the above matrix fractional differential equation we get

$$\begin{bmatrix} \mathcal{L}\{y_1'\} \\ \mathcal{L}\{y_2^{(a)}\} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \mathcal{L}\{y_1\} \\ \mathcal{L}\{y_2\} \end{bmatrix}.$$

or, equivalently,

$$\begin{bmatrix} sY_1(s) - c_1 \\ zY_2(s) - wc_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}. \tag{10}$$

Where $c_1, c_2 \in \mathbb{R}$ are unknown constants and z, w are defined, see [3], [5], [7], [8], [9], [20], as follows:

(i) $z = s^a$, $w = s^{a-1}$, if we use the (C) fractional derivative;

(ii) $z=\frac{s}{s+a(1-s)}$, $w=\frac{1}{s+a(1-s)}$, if we use the (CF) fractional derivative;

(iii) $z=\frac{B(a)}{1-a}\frac{s^a}{s^a+\frac{a}{1-a}}$, $w=\frac{B(a)}{1-a}\frac{s^{a-1}}{s^a+\frac{a}{1-a}}$, if we use the (AB) fractional derivative.

In the case of (a), we use (i) and (10) takes the form

$$\begin{bmatrix} sY_1(s)-c_1 \\ s^aY_2(s)-s^{a-1}c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} sY_1(s)-Y_2(s) \\ \frac{n^2\pi^2}{\beta}Y_1(s)+(s^a+\frac{1}{\beta})Y_2(s) \end{bmatrix} = \begin{bmatrix} c_1 \\ s^{a-1}c_2 \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} s & -1 \\ \frac{n^2\pi^2}{\beta} & s^a+\frac{1}{\beta} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} c_1 \\ s^{a-1}c_2 \end{bmatrix}. \tag{11}$$

Where

$$\left(\begin{bmatrix} s & -1 \\ \frac{n^2\pi^2}{\beta} & s^a+\frac{1}{\beta} \end{bmatrix}\right)^{-1} = \left(\begin{bmatrix} s & 0 \\ 0 & s^a \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}\right)^{-1}.$$

Let

$$S=\begin{bmatrix} s & 0 \\ 0 & s^a \end{bmatrix}, \quad A=\begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}.$$

Then,

$$(S-A)^{-1}=\sum_{r=0}^{\infty} A^r S^{-r-1}. \tag{12}$$

Hence

$$\left(\begin{bmatrix} s & -1 \\ \frac{n^2\pi^2}{\beta} & s^a+\frac{1}{\beta} \end{bmatrix}\right)^{-1} = \sum_{r=0}^{\infty} \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s & 0 \\ 0 & s^a \end{bmatrix}^{-r-1},$$

or, equivalently,

$$\left(\begin{bmatrix} s & -1 \\ \frac{n^2\pi^2}{\beta} & s^a+\frac{1}{\beta} \end{bmatrix}\right)^{-1} = \sum_{r=0}^{\infty} \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s^{-r-1} & 0 \\ 0 & s^{-ar-a} \end{bmatrix}.$$

and by replacing the above expression in (11) we get

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \sum_{r=0}^{\infty} \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s^{-r-1} & 0 \\ 0 & s^{-ar-a} \end{bmatrix} \begin{bmatrix} c_1 \\ s^{a-1}c_2 \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \sum_{r=0}^{\infty} \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s^{-r-1} & 0 \\ 0 & s^{-ar-a} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s^{a-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \sum_{r=0}^{\infty} \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s^{-r-1} & 0 \\ 0 & s^{-ar-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

and by using the inverse of the Laplace transform we have

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \sum_{r=0}^{\infty} \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} \mathcal{L}^{-1}\{s^{-r-1}\} & 0 \\ 0 & \mathcal{L}^{-1}\{s^{-ar-1}\} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

or, equivalently and by taking into account that $\mathcal{L}^{-1}\{s^{-m-1}\} = \frac{t^m}{\Gamma(m+1)}$,

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \sum_{r=0}^{\infty} \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} \frac{t^r}{\Gamma(r+1)} & 0 \\ 0 & \frac{t^{ar}}{\Gamma(ar+1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} Y(t) \\ Y'(t) \end{bmatrix} = \sum_{r=0}^{\infty} \frac{(-1)^r}{\beta^r} \begin{bmatrix} 0 & -\beta \\ n^2\pi^2 & 1 \end{bmatrix}^r \begin{bmatrix} \frac{t^r}{\Gamma(r+1)} & 0 \\ 0 & \frac{t^{ar}}{\Gamma(ar+1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Note that

$$\begin{bmatrix} Y(0) \\ Y'(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

where $Y(0) = Y_n(0) = \phi_n$ and $Y'(0) = Y_n'(0) = \psi_n$ are given by (7), i.e.

$$\begin{bmatrix} Y(t) \\ Y'(t) \end{bmatrix} = \sum_{r=0}^{\infty} \frac{(-1)^r}{\beta^r} \begin{bmatrix} 0 & -\beta \\ n^2\pi^2 & 1 \end{bmatrix}^r \begin{bmatrix} \frac{t^r}{\Gamma(r+1)} & 0 \\ 0 & \frac{t^{ar}}{\Gamma(ar+1)} \end{bmatrix} \begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix}.$$

The matrix $\begin{bmatrix} 0 & -\beta \\ n^2\pi^2 & 1 \end{bmatrix}$ has eigenvalues

$$\mu_1 = \frac{1}{2} + \frac{d}{2}, \quad \mu_2 = \frac{1}{2} - \frac{d}{2}.$$

with corresponding eigenvectors

$$\begin{bmatrix} -2\beta \\ 1+d \end{bmatrix}, \quad \begin{bmatrix} -2\beta \\ 1-d \end{bmatrix}.$$

Where $d = \sqrt{1 - 4\beta n^2\pi^2}$

$$\begin{bmatrix} 0 & -\beta \\ n^2\pi^2 & 1 \end{bmatrix}^r = \frac{1}{4\beta d} \begin{bmatrix} -2\beta & -2\beta \\ 1+d & 1-d \end{bmatrix} \begin{bmatrix} \mu_1^r & 0 \\ 0 & \mu_2^r \end{bmatrix} \begin{bmatrix} 1-d & 2\beta \\ -1-d & -2\beta \end{bmatrix}$$

and hence

$$Y_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{4\beta^{r+1}d} \Big[(-2\beta(1-d)\mu_1^r + 2\beta\mu_2^r(1+d)) \frac{t^r\phi_n}{\Gamma(r+1)} + (-4\beta^2\mu_1^r + 4\beta^2\mu_2^r) \frac{t^{ar}\psi_n}{\Gamma(ar+1)}\Big].$$

In the case of (b) we use (ii) and (10) takes the form

$$\begin{bmatrix} sY_1(s) - c_1 \\ \frac{s}{s+a(1-s)}Y_2(s) - \frac{1}{s+a(1-s)}c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} sY_1(s) - Y_2(s) \\ \frac{n^2\pi^2}{\beta}Y_1(s) + (\frac{s}{s+a(1-s)} + \frac{1}{\beta})Y_2(s) \end{bmatrix} = \begin{bmatrix} c_1 \\ \frac{1}{s+a(1-s)}c_2 \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} s & -1 \\ \frac{n^2\pi^2}{\beta} & \frac{s}{s+a(1-s)}+\frac{1}{\beta} \end{bmatrix}\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} c_1 \\ \frac{1}{s+a(1-s)}c_2 \end{bmatrix}. \tag{13}$$

Where

$$\left(\begin{bmatrix} s & -1 \\ \frac{n^2\pi^2}{\beta} & \frac{s}{s+a(1-s)}+\frac{1}{\beta} \end{bmatrix}\right)^{-1} = \left(\begin{bmatrix} s & 0 \\ 0 & \frac{s}{s+a(1-s)} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}\right)^{-1}.$$

Let

$$S = \begin{bmatrix} s & 0 \\ 0 & \frac{s}{s+a(1-s)} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}.$$

Then from (12) we have

$$(S-A)^{-1} = \sum_{r=0}^{\infty} A^r S^{-r-1},$$

or, equivalently,

$$\left(\begin{bmatrix} s & -1 \\ \frac{n^2\pi^2}{\beta} & \frac{s}{s+a(1-s)}+\frac{1}{\beta} \end{bmatrix}\right)^{-1} = \sum_{r=0}^{\infty}\begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s & 0 \\ 0 & \frac{s}{s+a(1-s)} \end{bmatrix}^{-r-1},$$

or, equivalently,

$$\left(\begin{bmatrix} s & -1 \\ \frac{n^2\pi^2}{\beta} & \frac{s}{s+a(1-s)}+\frac{1}{\beta} \end{bmatrix}\right)^{-1} = \sum_{r=0}^{\infty}\begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s^{-r-1} & 0 \\ 0 & \frac{s^{-r-1}}{[s+a(1-s)]^{-r-1}} \end{bmatrix}.$$

and by replacing the above expression in (13) we get

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \sum_{r=0}^{\infty}\begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s^{-r-1} & 0 \\ 0 & \frac{s^{-r-1}}{[s+a(1-s)]^{-r-1}} \end{bmatrix}\begin{bmatrix} c_1 \\ \frac{1}{s+a(1-s)}c_2 \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \sum_{r=0}^{\infty}\begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s^{-r-1} & 0 \\ 0 & \frac{s^{-r-1}}{[s+a(1-s)]^{-r-1}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{s+a(1-s)} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \sum_{r=0}^{\infty}\begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s^{-r-1} & 0 \\ 0 & \frac{s^{-r-1}}{[s+a(1-s)]^{-r}} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

and by using the inverse of the Laplace transform we have

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \sum_{r=0}^{\infty}\begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} \mathcal{L}^{-1}\{s^{-r-1}\} & 0 \\ 0 & \mathcal{L}^{-1}\{\frac{s^{-r-1}}{[s+a(1-s)]^{-r}}\} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Where

$$\frac{s^{-r-1}}{[s+a(1-s)]^{-r}} = \frac{[s+a(1-s)]^r}{s^{r+1}} = s^{-r-1}[s+a(1-s)]^r = s^{-r-1}[(1-a)s+a]^r$$

or, equivalently,

$$\frac{s^{-r-1}}{[s+a(1-s)]^{-r}} = s^{-r-1}\sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right) s^j (1-a)^j a^{r-j} = \sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right)(1-a)^j a^{r-j} s^{-(r-j+1)}.$$

By using the above expression and taking into account that $\mathcal{L}^{-1}\{s^{-m-1}\} = \frac{t^m}{\Gamma(m+1)}$, we have

$$\left[\begin{array}{c} y_1(t) \\ y_2(t) \end{array}\right] = \sum_{r=0}^{\infty}\left[\begin{array}{cc} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{array}\right]^r \left[\begin{array}{cc} \frac{t^r}{\Gamma(r+1)} & 0 \\ 0 & \sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right)(1-a)^j a^{r-j}\frac{t^{r-j}}{\Gamma(r-j+1)} \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right],$$

or, equivalently,

$$\left[\begin{array}{c} T(t) \\ T'(t) \end{array}\right] = \sum_{r=0}^{\infty}\left[\begin{array}{cc} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{array}\right]^r \left[\begin{array}{cc} \frac{t^r}{\Gamma(r+1)} & 0 \\ 0 & \sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right)(1-a)^j a^{r-j}\frac{t^{r-j}}{\Gamma(r-j+1)} \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right],$$

Note that

$$\left[\begin{array}{c} T(0) \\ T'(0) \end{array}\right] = \left[\begin{array}{c} c_1 \\ c_2 \end{array}\right],$$

where $T(0) = T_n(0) = \phi_n$ and $T'(0) = T_n'(0) = \psi_n$ are given by (7), i.e.

$$\left[\begin{array}{c} T(t) \\ T'(t) \end{array}\right] = \sum_{r=0}^{\infty}\frac{(-1)^r}{\beta^r}\left[\begin{array}{cc} 0 & -\beta \\ n^2\pi^2 & 1 \end{array}\right]^r \left[\begin{array}{cc} \frac{t^r}{\Gamma(r+1)} & 0 \\ 0 & \sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right)(1-a)^j a^{r-j}\frac{t^{r-j}}{\Gamma(r-j+1)} \end{array}\right]\left[\begin{array}{c} \phi_n \\ \psi_n \end{array}\right].$$

The matrix $\left[\begin{array}{cc} 0 & -\beta \\ n^2\pi^2 & 1 \end{array}\right]$ has eigenvalues

$$\mu_1 = \frac{1}{2} + \frac{d}{2}, \quad \mu_2 = \frac{1}{2} - \frac{d}{2}.$$

with corresponding eigenvectors

$$\left[\begin{array}{c} -2\beta \\ 1+d \end{array}\right], \quad \left[\begin{array}{c} -2\beta \\ 1-d \end{array}\right].$$

Where $d = \sqrt{1-4\beta n^2\pi^2}$. Then

$$\left[\begin{array}{cc} 0 & -\beta \\ n^2\pi^2 & 1 \end{array}\right]^r = \frac{1}{4\beta d}\left[\begin{array}{cc} -2\beta & -2\beta \\ 1+d & 1-d \end{array}\right]\left[\begin{array}{cc} \mu_1^r & 0 \\ 0 & \mu_2^r \end{array}\right]\left[\begin{array}{cc} 1-d & 2\beta \\ -1-d & -2\beta \end{array}\right]$$

and hence

$$Y_n(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{4\beta^{r+1}d}\Big[(-2\beta(1-d)\mu_1^r + 2\beta\mu_2^r(1+d))\frac{t^r\phi_n}{\Gamma(r+1)} +$$

$$(-4\beta^2\mu_1^r + 4\beta^2\mu_2^r)\sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right)(1-a)^j a^{r-j}\frac{t^{r-j}\psi_n}{\Gamma(r-j+1)}\Big].$$

In the case of (c), we use (iii) and (10) takes the form

$$\left[\begin{array}{c} sY_1(s) - c_1 \\ \frac{B(a)}{1-a}\frac{s^a}{s^a+\frac{a}{1-a}}Y_2(s) - \frac{B(a)}{1-a}\frac{s^{a-1}}{s^a+\frac{a}{1-a}}c_2 \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{array}\right]\left[\begin{array}{c} Y_1(s) \\ Y_2(s) \end{array}\right].$$

By working similarly to (a) and (b) we arrive at

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \sum_{r=0}^{\infty} \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s^{-r-1} & 0 \\ 0 & [\frac{B(a)}{1-a}\frac{s^a}{s^a+\frac{a}{1-a}}]^{-r-1} \end{bmatrix} \begin{bmatrix} c_1 \\ \frac{B(a)}{1-a}\frac{s^{a-1}}{s^a+\frac{a}{1-a}}c_2 \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \sum_{r=0}^{\infty} \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} s^{-r-1} & 0 \\ 0 & [\frac{B(a)}{1-a}]^{-r}\frac{s^{-ar-1}}{(s^a+\frac{a}{1-a})^{-r}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

Where

$$\frac{s^{-ar-1}}{(s^a+\frac{a}{1-a})^{-r}} = s^{-ar-1}(s^a+\frac{a}{1-a})^r = s^{-ar-1}\sum_{j=0}^{r}\begin{pmatrix} r \\ j \end{pmatrix} s^{aj}(\frac{a}{1-a})^{r-j},$$

or, equivalently,

$$\frac{s^{-ar-1}}{(s^a+\frac{a}{1-a})^{-r}} = \sum_{j=0}^{r}\begin{pmatrix} r \\ j \end{pmatrix}(\frac{a}{1-a})^{r-j}s^{-(-aj+ar+1)}.$$

By using the above expression and taking into account that $\mathcal{L}^{-1}\{s^{-m-1}\} = \frac{t^m}{\Gamma(m+1)}$, we have

$$\begin{bmatrix} T(t) \\ T'(t) \end{bmatrix} = \sum_{r=0}^{\infty} \begin{bmatrix} 0 & 1 \\ -\frac{n^2\pi^2}{\beta} & -\frac{1}{\beta} \end{bmatrix}^r \begin{bmatrix} \frac{t^r}{\Gamma(r+1)} & 0 \\ 0 & [\frac{B(a)}{1-a}]^{-r}\sum_{j=0}^{r}\begin{pmatrix} r \\ j \end{pmatrix}(\frac{a}{1-a})^{r-j}\frac{t^{-aj+ar}}{\Gamma(-aj+ar+1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Note that similarly to (a), (b), where $c_1 = T(0) = T_n(0) = \phi_n$ and $c_2 = T'(0) = T'_n(0) = \psi_n$ are given by (7), i.e.

$$\begin{bmatrix} T(t) \\ T'(t) \end{bmatrix} = \sum_{r=0}^{\infty} \frac{(-1)^r}{\beta^r} \begin{bmatrix} 0 & -\beta \\ n^2\pi^2 & 1 \end{bmatrix}^r \begin{bmatrix} \frac{t^r}{\Gamma(r+1)} & 0 \\ 0 & [\frac{B(a)}{1-a}]^{-r}\sum_{j=0}^{r}\begin{pmatrix} r \\ j \end{pmatrix}(\frac{a}{1-a})^{r-j}\frac{t^{-aj+ar}}{\Gamma(-aj+ar+1)} \end{bmatrix} \begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix}.$$

The matrix $\begin{bmatrix} 0 & -\beta \\ n^2\pi^2 & 1 \end{bmatrix}$ has eigenvalues

$$\mu_1 = \frac{1}{2} + \frac{d}{2}, \quad \mu_2 = \frac{1}{2} - \frac{d}{2}.$$

with corresponding eigenvectors

$$\begin{bmatrix} -2\beta \\ 1+d \end{bmatrix}, \quad \begin{bmatrix} -2\beta \\ 1-d \end{bmatrix}.$$

Where $d = \sqrt{1-4\beta n^2\pi^2}$. Then

$$\begin{bmatrix} 0 & -\beta \\ n^2\pi^2 & 1 \end{bmatrix}^r = \frac{1}{4\beta d}\begin{bmatrix} -2\beta & -2\beta \\ 1+d & 1-d \end{bmatrix}\begin{bmatrix} \mu_1^r & 0 \\ 0 & \mu_2^r \end{bmatrix}\begin{bmatrix} 1-d & 2\beta \\ -1-d & -2\beta \end{bmatrix}$$

and hence

$$Y_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{4\beta^{r+1}d}\Big[(-2\beta(1-d)\mu_1^r + 2\beta\mu_2^r(1+d))\frac{t^r\phi_n}{\Gamma(r+1)} +$$

$$(-4\beta^2\mu_1^r + 4\beta^2\mu_2^r)[\frac{B(a)}{1-a}]^{-r}\sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right)(\frac{a}{1-a})^{r-j}\frac{t^{-aj+ar}\psi_n}{\Gamma(-aj+ar+1)}\Big].$$

The proof is completed.

3 Numerical Examples

In this section we present and discuss particular solutions of the HHE and the FHHE found in the previous sections and compare them when possible. For doing so we assume a particular set of initial conditions given by $\phi(x) = \psi(x) = 4x(1-x)$. The parameter β in our problem represents the relative importance of wave-driven heat transfer to diffusive heat transfer. In practical situations, wave-driven heat transfer is much less important than diffusive heat transfer and, therefore, the parameter beta is typically very small. Hence, we only use values $\beta \ll 1$ in our examples.

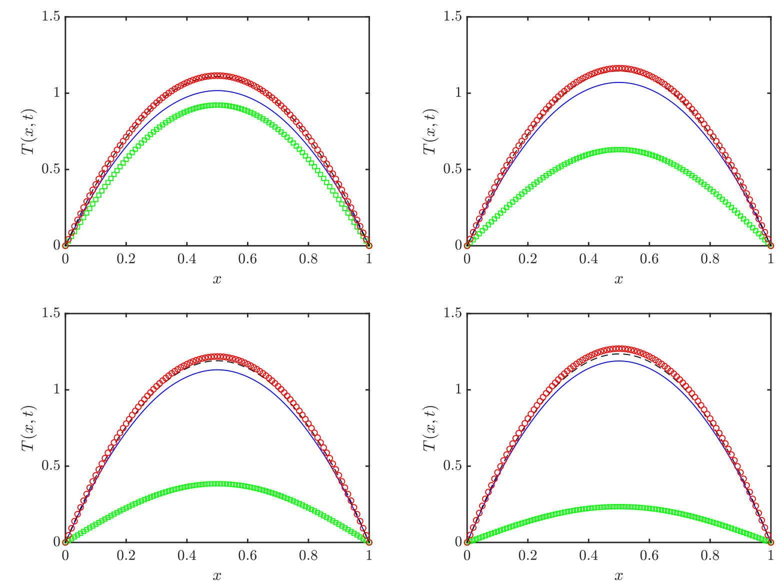

Figure 1: Solution of the HHE and the FHHE with $\beta = 2.5 \cdot 10^{-4}$, $\gamma = 1.9$ and $r = 2$ for increasing time.

In Figure 1 we present the solutions of the FHHE for each fractional derivative used along with the solution of the standard HHE at four different times. The three solutions for the fractional problem evolve in a similar fashion, with the solution for the CF and AB cases being very close for all times and the C case becoming closer to them for longer times. While the three solutions of the FHHE increase with time the solution of the HHE decreases with time. The fact that the solutions of the FHHE systematically increase could be related to the convergence of the series in (a)-(c).

In Figure 2, we show the variation in the solution of the FHHE (using C, CF and AB) for values of γ increasingly close to 2, along with the solution of the HHE. For $\gamma = 1.985$ the solutions from the CF and AB derivatives are far from the solution of the HHE. For $\gamma = 1.99$ the solutions CF and AB are close to the solution of the HHE and for $\gamma = 1.995$ all four profiles collapse into a single curve. This behaviour indicates that the solution of the HHE can be retrieved by taking the limit $\gamma \to 2$ in the solutions of the HHHE. The solution obtained by the C derivative is almost indistinguishable from the solution of the HHE for the values of γ in Figure 2. In fact, the solution by the C derivative tends to the solution of the HHE much faster than the CF and AB cases and for $\gamma \approx 1.8$ (not shown in the figure) is already virtually identical.

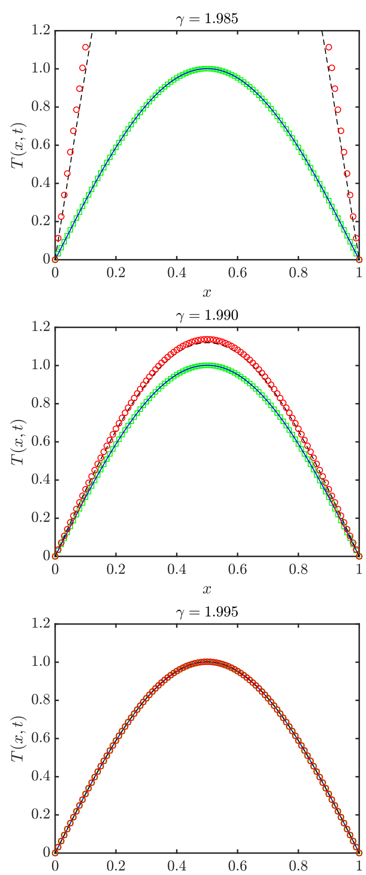

Figure 2: Solution of the HHE (square symbols) and solutions to the FHHE using the C (solid line), CF (dashed line) and AB (circles) derivatives, for $\beta = 0.01$ at time $t = 0.01$, for γ values increasingly close to 2.

4 Conclusions

In this work we have presented a solution method for the fractional hyperbolic heat equation. We have used the method to solve the hyperbolic heat equation using three different definitions of the fractional derivative (Caputo, Caput-Fabrizio and Atangana- Baleanu). A closed form of the solution was found for each case. A numerical comparison of the solutions shows that the three solutions behave in a similar fashion. Finally, the solutions of the fractional equation have been compared with the solutions of the standard hyperbolic heat equation. The comparison indicates that the solution of the standard equation can be retrieved from the solution of the fractional problem by taking the limit $\gamma \to 2$, where γ is the exponent of the frational derivative.

Acknowledgement

This work is supported by the Science Foundation Ireland (SFI), by funding Ioannis Dassios under Investigator Programme Grant No. SFI/15 /IA/3074. Francesc Font acknowledges financial support from the *Juan de la Cierva* programme (grant IJC2018-038463-I) from the Spanish MICINN, from the *Obra Social "la Caixa"* through the programme *Recerca en Matemàtica Col·laborativa* and from the CERCA Programme of the *Generalitat de Catalunya*.

References

[1] Atangana, A., On the new fractional derivative and application to nonlinear Fisher's reaction- diffusion equation, Applied Mathematics and Computation, 273(2016), pp. 948–956

[2] Atangana, A., Nieto, J. J., Numerical Solution For The Model Of RLC Circuit Via the Fractional Derivative Without Singular Kernel, Advances in Mechanical Engineering, 7(2015) 10, pp. 1–7.

[3] Atangana, A., Baleanu, D., New Fractional Derivatives with Nonlocal and Non-Singular Kernel: Theory and Application to Heat Transfer Model,THERMAL SCIENCE International Scientific Journal (2016).

[4] D. Baleanu, G. Ziya Burhanettin and JA Tenreiro Machado, eds. *New trends in nanotechnology and fractional calculus applications.* New York, NY, USA: Springer, 2010.

[5] D. Baleanu, JA Tenreiro Machado, and ACJ Luo, eds. *Fractional dynamics and control.* Springer Science & Business Media, 2011.

[6] D. Baleanu, K. Diethelm, E. Scalas, *Fractional Calculus: Models and Numerical Methods,* World Scientific (2012).

[7] Bonilla, B., Margarita Rivero, and Juan J. Trujillo. *On systems of linear fractional differential equations with constant coefficients.* Applied Mathematics and Computation 187.1 (2007): 68-78.

[8] Caputo, M., Fabrizio M., A New Definition of Fractional Derivative Without Singular Kernel, Progress in Fractional Differentiation and Applications, 1(2015) 2.

[9] Caputo, M., Fabrizio M., Applications of new time and spatial fractional derivatives with exponential kernels, 2(2016) 2.

[10] L. Dai, *Singular Control Systems*, Lecture Notes in Control and information Sciences Edited by M.Thoma and A.Wyner (1988).

[11] Dassios I.K., and Baleanu D., *Caputo and related fractional derivatives in singular systems*, Applied Mathematics and Computation, Elsevier, Volume 337, pp. 591-606 (2018).

[12] I. Dassios, D. Baleanu, G. Kalogeropoulos, *On non-homogeneous singular systems of fractional nabla difference equations,* Applied Mathematics and Computation, Volume 227, 112–131 (2014).

[13] A. Dzielinski, W. Malesza, *Point to point control of fractional differential linear control systems.* Adv. Difference Equ. 2011, 2011:13, 17 pp. (2011).

[14] A. Dzielinski, G. Sarwas, D. Sierociuk, *Comparison and validation of integer and fractional order ultracapacitor models.* Adv. Difference Equ. 2011, 2011:11, 15 pp. (2011).

[15] A. Dzielinski, D. Sierociuk, *Fractional order model of beam heating process and its experimental verification.* New trends in nanotechnology and fractional calculus applications, 287â294, Springer, New York, (2010).

[16] R.F. Gantmacher, *The theory of matrices I, II*, Chelsea, New York (1959).

[17] W.G. Glockle, T.F. Nonnenmacher, *A fractional calculus approach to self-similar protein dynamics*, Biophysical Journal. 68(1), 46–53 (1995).

[18] R. Hilfe (ed.), Applications of Fractional Calculus in Physics, p. 463. World Scientific, River Edge, NJ, USA (2000).

[19] G. I. Kalogeropoulos, *Matrix pencils and linear systems*, Ph.D Thesis, City University, London (1985).

[20] Kaczorek, Tadeusz. *Fractional Continuous-Time Linear Systems.* Selected Problems of Fractional Systems Theory. Springer Berlin Heidelberg, 2011. 27-52.

[21] Kilbas, A. Anatolii Aleksandrovich, Hari Mohan Srivastava, and Juan J. Trujillo. *Theory and applications of fractional differential equations.* Vol. 204. Elsevier Science Limited, 2006.

[22] J. Klamka, *Controllability of dynamical systems. A survey.* Bulletin of the Polish Academy of Sciences: Technical Sciences 61.2 (2013): 335-342.

[23] Losada, J., Nieto, J.J., Properties of a new fractional derivative without singular Kernel, Progress in Fractional Differentiation and Applications 1(2015)2, pp. 87-92

[24] J. A. Machado, M. E. Mata, and A. M. Lopes. *Fractional State Space Analysis of Economic Systems.* Entropy 17, Number 8 (2015): 5402–5421.

[25] Matignon, Denis. *Stability results for fractional differential equations with applications to control processing.* Computational engineering in systems applications. Vol. 2. Lille France, 1996.

[26] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*,p. xxiv+340. Academic Press, San Diego, Calif, USA (1999).

[27] W.J. Rugh; *Linear system theory*, Prentice Hall International (Uk), London (1996).

[28] Trujillo, J. J., M. Rivero, and B. Bonilla. *On a Riemann -Liouville generalized Taylor's formula.* Journal of Mathematical Analysis and Applications 231.1 (1999): 255-265.

[29] Vinagre BM, Monje CA, Calderón AJ, Suárez JI. Fractional PID controllers for industry application. A brief introduction. Journal of Vibration and Control. 2007 Sep 1;13(9-10):1419-29.

[30] G. M. Zaslavsky, *Chaos, fractional kinetics, and anomalous transport.* Physics Reports 371.6 (2002): 461-580.

[31] Cattaneo, C. R. *Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanée*. Comptes Rendus, 247 (4): 431 (1958).

[32] Chen, G. *Ballistic-Diffusive Heat-Conduction Equations*. Physical Review Letters, 86:11, 2297–2300 (2001)

[33] C. W. Chang, D. Okawa, H. Garcia, A. Majumdar, and A. Zettl. *Breakdown of Fourier's law in nanotube thermal conductors*, Physical Review Letters, 101(7), 075903 (2008)

[34] F. Font, F. Bresme. *Transient melting at the nanoscale: a continuum heat transfer and nonequilibrium molecular dynamics approach*, Journal of Physical Chemistry C, 122 (30), 17481–17489 (2018)

[35] F. Font. *A one-phase Stefan problem with size-dependent thermal conductivity*, Applied Mathematical Modelling, 63, 172–178 (2018)

[36] R. A. Guyer, J. A. Krumhansl, *Thermal conductivity, second sound, and phonon hydrodynamic phenomena in nonmetallic crystals*, Physical Review, 148(2), 778–788 (1966)