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Combinatorial and analytic techniques for lattice path enumeration

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Supervised by Juanjo Rué Perna

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I would like to thank all the people without whom this thesis would not have been possible.
First of all, I am so grateful to Juanjo, for being my tutor and guiding me throughout all the process of this thesis. He has helped me a lot explaining totally new concepts for me and offering his help in any moment.
I would also like to thank my friends and specially my family, who have all been supporting and encouraging me during these years in the faculty.

Abstract

This bachelor's degree thesis studies two type of combinatorial objects. The first ones are exact models of which we can get exact counting formulas, whereas the second type of models are the ones of which we don't have enumerative exact results. The first part of the work focuses on the exact models and two methods useful to find counting formulas are described: the *Symbolic method* and the *Kernel method*. In the second part of this thesis, an specific type of non exact model is addressed: *self-avoiding walks*. Although there are no exact counting formulas, we are going to study its asymptotic behaviour and we will proof a theorem which states that the *connective constant* of self-avoiding walks in the hexagonal lattice $\mathbb{H}$ equals $\sqrt{2+\sqrt{2}}$.

Keywords

Symbolic method, Catalan numbers, Dyck paths, Motzkin numbers, Kernel method, generating function, bivariate generating function, algebraic functions, resultant, algebraic elimination, self-avoiding walks, Fekete's Lemma, bridges, polygons, holomorfic observable, hexagonal lattice, connective constant.

Contents

Introduction

Enumerative combinatorics is an area of combinatorics concerned about counting the number of certain combinatorial objects. It is one of the most basic and important aspects of combinatorics, since in many branches of mathematics and its applications it is necessary to know the number of different ways of doing something and many of the problems that appear in applications can be simply described in a combinatorial way.

Given an infinite collection of finite sets S_n indexed by the natural numbers, enumerative combinatorics tries to determine the number of elements of all the sets S_n, describing a method which gives a counting function that for every n gives the number of elements of S_n. The simplest such functions are closed formulas, expressed as a composition of elementary functions such as factorial, powers, and so on. There are many techinques used to find this counting formulas: decomposition, refinement, recursion, bijections methods, $\dots$. In this thesis we are going to learn two methods to find counting formulas for combinatorial objects: the *Symbolic method* and the *Kernel method*. We are going to develop them in the first part of this work and we are going to show some specific examples to see how they work.

However, often there is no closed formula or there appear complicated formulas in such a way that we have no idea of the behaviour of this counting formula as the number of counted objects grows. In these cases we are happy obtaining asymptotic estimates for such numbers. We will address it in the second part of the thesis, where we are going to study the asymptotic behaviour of *self-avoiding walks*, concluding the work with the first mathematical proof that the exponential growth (also known as *connective constant*) in the hexagonal lattice is equal to $\sqrt{2+\sqrt{2}}$. It is an important result first conjectured by B. Nienhuis in 1982, who saw it using Coulomb gas approach from theoretical physics. This proof was published in the distinguished journal Annals of Mathematics, by Hugo Duminil-Copin and Stanislav Smirnov. Such a result contributed in the merits of Smirnov to achieve the Fields Medal in 2010.

Part I

Exact models

In combinatorics there are lots of combinatorial families from which we can get exact enumerative results. In this first part of the thesis, we are going to focus on these families, and we will see a very useful method to find their terms: the *Kernel method.* In particular, we are going to focus on the study of *lattice paths.*

Definition (Definition I)**.** *Let* $\mathcal{S} = \{(a_1, b_1), \dots, (a_r, b_r)\}$ *be a finite set of vectors of* $\mathbb{Z} \times \mathbb{Z}$*, whose elements are called* steps*. A* lattice path *or* walk *relative to* $\mathcal{S}$ *is a sequence* $v = (v_1, \dots, v_n)$ *such that each* v_j *is in* $\mathcal{S}$*. The geometric realization of a lattice path* $v = (v_1, \dots, v_n)$ *is the sequence of points* $(p_0, p_1, \dots, p_n)$ *such that* $p_0 = (0, 0)$ *and* $\overrightarrow{p_{j-1}p_j} = v_j$*.*
The quantity n *is called the* size *of the path.*

We will identify a lattice path with the polygonal line with $p_0, \dots, p_n$ as vertices. We will also impose some restrictions to them. We are going to work with *directed paths*, which means that if $(a, b) \in \mathcal{S}$, then $a > 0$. Therefore, the path will live in the half plane $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$.
There is an extensive bibliography to study these models, but in this work we are going to focus in [1], which talks about directed lattice paths with $a = 1$.

This part consists of 3 chapters. The first one is as an introduction, talking about Catalan numbers and introducing an useful method to find generating functions for some combinatorial objects: the *Symbolic method.* As an example of how it works, Catalan and Motzkin numbers are found by using this method. However, this method does not allow us to find all the generating functions. In Chapter 2, an example is presented and to solve it a new method is introduced: the *Kernel method.* But before, some algebraic concepts have to be reviewed or defined. Finally, in Chapter 3, the Kernel method as well as the Symbolic method are used to find the generating functions of all the possible lattice paths.

Chapter 1

Introduction. Symbolic method

1.1 Catalan numbers

Catalan numbers is a well known sequence of numbers of which we have a formula for every term of it: it is an example of an exact model. Catalan numbers are named after the French-Belgian mathematician Eugène Charles Catalan and a lot of enumeration problems are counted by these numbers (see [10] for many examples of combinatorial objects counted by Catalan numbers). In this section we are going to define this sequence, which is a particular type of lattice path, and we are going to find the expression of its terms with two different methods.

Definition 1.1.1. *A* Dyck path *of length* $2n$ *is a random walk on* $\mathbb{Z}^2$ *which starts at* $(0,0)$ *and ends at* $(2n,0)$*, with only two possible steps:* $(1,1)$ *or* $(1,-1)$*, and never dipping below the height it began on, i.e. never crossing the* x*-axis. See Figure 1.1 for an example.*
Catalan numbers *is the sequence* $(c_n)_{n\geq 0}$*, where* c_n *is the number of Dick paths of length* $2n$*.*

Note that the length of a Dyck path must be even, because as it has to start and end at the same height, for every step upwards another step downwards is needed to reach the x-axis.

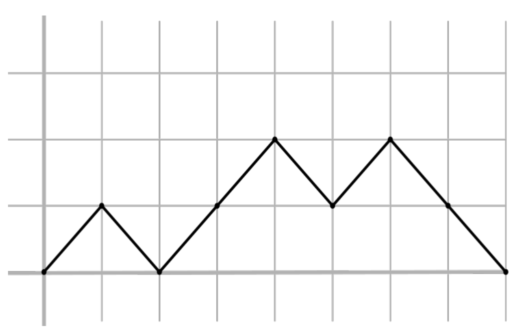

Figure 1.1: Example of a Dyck path of length 8.

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Proposition 1.1.1. *For all $n \geq 0$*

$$c_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1.1.1}$$

It is easy to find the exact formula for the coefficients c_n by simple arguments of recurrences. However, we are going to use the reflection principle to prove Proposition 1.1.1.

Proof. We start counting all the walks from $(0,0)$ to $(2n,0)$ with steps $\nearrow := (1,1)$ and $\searrow := (1,-1)$, without any restriction about crossing the x-axis. As the walks have to start and end at the same height, there must be n steps $\nearrow$ and n steps $\searrow$. All the possibilities to choose how these steps are distributed is the number of walks without restrictions we are searching for and it is $\binom{2n}{n}$.
Among all these walks, there are the walks that do not cross the x-axis. To know the number of this set of walks (c_n) we need to subtract the number of walks from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ and crossing the x-axis, i.e. with at least one point with the second coordinate negative. If we denote this last set of paths by $\mathcal{S}$, we will have

$$c_n = \binom{2n}{n} - \#\{\mathcal{S}\}$$

It only remains to find the size of $\mathcal{S}$. Now, we are going to apply the reflection principle, which is based on the following idea: we take one walk $\omega \in \mathcal{S}$. Note that there will always be a step from $(i,0)$ to $(j,-1)$ for some $0 \leq i < j \leq 2n-1$. Let us define $m = \min\{j \text{ such as } \omega(j) = (j,-1)\}$. We keep the walk ω until m as it is originally and we reflect the rest of the walk with respect the line $y=-1$; see Figure 1.2. The resulting walk will end at $(2n,-2)$. Thus, we have shown that there exists a bijection between $\mathcal{S}$ and $\mathcal{R}$, where $\mathcal{R} = \{\text{walks from } (0,0) \text{ to } (2n,-2) \text{ with steps } (1,1) \text{ and } (1,-1)\}$.

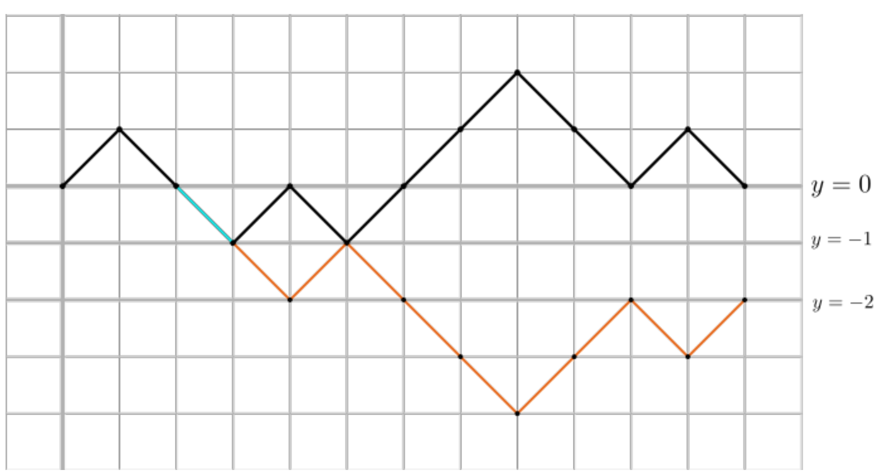


Figure 1.2: In black the original Dyck path $\omega \in \mathcal{S}$, in blue the first step from $y=0$ to $y=-1$ and in orange the reflection of the path with respect the line $y=-1$.

To compute the size of $\mathcal{R}$ it suffices to notice that walks on $\mathcal{R}$ must have $n-1$ steps $\nearrow$ and $n+1$ $\searrow$ steps. Then, choosing where to place the $n-1$ steps $\nearrow$, we obtain

$$\#\mathcal{R} = \#\mathcal{S} = \binom{2n}{n-1}.$$

Therefore,

$$\begin{aligned} c_n &= \binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} = \frac{(2n)!}{n!(n+1)!}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{(2n)!}{n!(n+1)!}\frac{1}{n(n+1)} = \frac{1}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}, \end{aligned}$$

as we wanted to see. $\square$

1.2 Symbolic method

We have found the coefficients of the Catalan numbers' sequence by the reflection principle. In this section we are going to introduce the *Symbolic method*, a technique used in combinatorics to count some combinatorial objects, developed in [5] by Flajolet and Sedgewick. It is based on the translation of some internal structures of the object into formulas for their generating functions. As an example of application, we will use this method to find Catalan and the so-called Motzkin numbers.

First of all, let us remember some concepts and definitions:

Definition 1.2.1. *A* combinatorial class*, or simply a* class *is a countable set of mathematical objects on which there is defined a* size *function which satisfies:*

i. the size of any element is a non-negative integer

ii. the number of elements of any size is finite

If $\mathcal{A}$ is a class and α an element in $\mathcal{A}$, we denote by $|\alpha|$ its size.

Definition 1.2.2. *The* counting sequence *of a combinatorial class $\mathcal{A}$ is the sequence of integers $(a_n)_{n\geq 0}$ where $a_n = card(\mathcal{A}_n)$ is the number of objects of class $\mathcal{A}$ that have size n.*

Definition 1.2.3. *The* ordinary generating function (OGF) *or simply the* generating function (GF) *of a sequence $(a_n)_{n\geq 0}$ is the formal power series*

$$A(z) = \sum_{n\geq 0} a_n z^n.$$

Therefore, the ordinary generating function of a class $\mathcal{A}$ is the OGF of its counting sequence. But *it can also be written as*

$$A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|}.$$

From this last form, we can see that generating functions are a simple representation of a class, without considering internal structures and where elements contributing to size are represented with the variable z.

1.2.1 Basic constructions

Let us now define the principal constructions or operations that are used to describe combinatorial classes and we will see how these operations can be translated in terms of OGF.

First of all, we need to define the *neutral class* $\mathcal{E}$, which consists of a single object of size 0. Any object of size 0 is called a *neutral object* and it denoted by ε or 1. Secondly, we also define an *atomic class* $\mathcal{Z}$ made by a single element of size 1. Any object of size one is called an *atom*. Examples of atoms could be a generic node in a tree or a graph (represented by a circle) or a generic letter in a word (`a`,`b`,`c`,...), among others.
The generating function of a neutral class $\mathcal{E}$ is $E(z) = 1$ and the corresponding one to an atomic class $\mathcal{Z}$ is $Z(z) = z$.

Now, let us defined the 3 principal operations or constructions used in combinatorial classes:

1. **COMBINATORIAL SUM/DISJOINT UNION**
   This operation reflects the idea of a disjoint union but without the need of having the disjointness restriction. To achieve that, we use 'markers' such as $\Box$ and $\Diamond$ in order to differentiate both classes and we define the disjoint union $\mathcal{B}+\mathcal{C}$ of $\mathcal{B}$ and $\mathcal{C}$ as:

   $$\mathcal{A} = \mathcal{B} + \mathcal{C} := (\Box \times \mathcal{B}) \cup (\Diamond \times \mathcal{C}),$$

   with size inherited from its size in its class of origin.

2. **CARTESIAN PRODUCT**
   The cartesian product construction applied to two classes $\mathcal{B}$ and $\mathcal{C}$ is the set of ordered pairs:

   $$\mathcal{A} = \mathcal{B} \times \mathcal{C} := \{\alpha = (\beta, \gamma) \,|\, \beta \in \mathcal{B}, \gamma \in \mathcal{C}\},$$

   with the size of a pair $\alpha = (\beta, \gamma)$ defined by:

   $$|\alpha|_A = |\beta|_B + |\gamma|_C.$$

3. **SEQUENCE CONSTRUCTION**

   The sequence of a class $\mathcal{B}$ is defined as the infinite sum:

$$\mathcal{A} = \texttt{Seq}(\mathcal{B}) := \epsilon + \mathcal{B} + (\mathcal{B} \times \mathcal{B}) + (\mathcal{B} \times \mathcal{B} \times \mathcal{B}) + \dots$$

   or, alternatively:

$$\mathcal{A} := \{\alpha = (\beta_1, ..., \beta_k) \,|\, \beta_i \in \mathcal{B}, k \geq 0\}.$$

   From the definitions of sizes in sums and products, we deduce that for any $\alpha \in \mathcal{A}$:

$$|\alpha|_A = |\beta_1|_B + ... + |\beta_l|_B.$$

   We admit restrictions on the number of component in a sequence. Therefore, we will use the notation

$$\texttt{Seq}(\mathcal{B})_{=k} \text{ (or simply } \texttt{Seq}(\mathcal{B})_k) \qquad \texttt{Seq}(\mathcal{B})_{\geq k} \qquad \texttt{Seq}(\mathcal{B})_{1\dots k}$$

   to denote sequences with exactly k numbers of components, larger or equal than k, or in the interval $1 \dots k$, respectively.

   Finally, we are going to introduce an other construction that will we useful in later moments. It is named *pointing*.

4. **POINTING**

   We know that combinatorial structures are formed of atoms (letters, nodes, etc), which determine their sizes. *Pointing* means pointing at a distinguished atom and if $\mathcal{A}$ is the pointing class of $\mathcal{B}$, we will denote $\mathcal{A} = \mathcal{B}^{\circ}$.

   We are going now to see how this operation is translated on terms of the generating functions. By definition of pointing, we have $a_n = n \cdot b_n$, because for every element of size n there are n possible atoms to mark and there are b_n elements of size n. Then,

$$A(z) = B^{\circ}(z) = \sum_{n\geq 0} a_n z^n = \sum_{n\geq 0} n \cdot b_n z^n = z \sum_{n\geq 1} n \cdot b_n z^{n-1} = z \frac{d}{dz} B(z).$$

In Table 1.1, you can see a summary of these constructions and which are the formulas for the OGF of the resulting classes, which can be proved easily (see [5]).

Let us see some basic examples of how to use this method:

Example 1.2.1. *Let $\mathcal{A} = \{\bullet\}$, with $|\bullet|_{\mathcal{A}} = 1$ ($\bullet$ is an atom). Then $B = \mathit{Seq}(\mathcal{A}) = \{\emptyset, \bullet, \bullet\bullet, \bullet\bullet\bullet, \dots\}$ and the corresponding OGF is $B(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$, because $A(z) = z$.*

OPERATION	NOTATION	SEMANTICS	OGF
Disjoint union/ Combinatorial sum	$\mathcal{A} = \mathcal{B} + \mathcal{C}$	disjoint copies of objects from $\mathcal{B}$ and $\mathcal{C}$	$A(z) = B(z) + C(z)$
Cartesian product	$\mathcal{A} = \mathcal{B} \times \mathcal{C}$	ordered pairs of copies of objects, one from $\mathcal{B}$ and the other from $\mathcal{C}$	$A(z) = B(z)C(z)$
Sequence	$\mathcal{A} = \mathtt{Seq}(\mathcal{B})$	sequence of objects from $\mathcal{B}$	$A(z) = \frac{1}{1-B(z)}$
Pointing	$\mathcal{A} = \mathcal{B}^{\circ}$	pointing a distinguished atom from $\mathcal{B}$	$A(z) = z\frac{d}{dz}B(z)$

Table 1.1: Constructions or operations with their translation into generating functions.

Example 1.2.2. *Let* $\mathcal{A} = \{0,1\}$, *with* $|0|_{\mathcal{A}} = |1|_{\mathcal{A}} = 1$. *Then* $B = \mathit{Seq}(\mathcal{A}) = \{\emptyset, 0, 1, 00, 01, 10, 11, 000, \ldots\}$ *and the corresponding OGF is* $B(z) = \frac{1}{1-2z} = \sum_{n\geq 0} 2^n z^n$, *because* $A(z) = 2z$.

Example 1.2.3. *We want to count the number of possible words with the alphabet* $\{0,1\}$ *without two consecutive* 1. *We denote* $A(z)$ *its generating function.*
To use the symbolic method we need to imagine the words that we want to count as constructions as the ones in Table 1.1. A type of word as the one in this example can be seen as it follows:

$$\mathit{Seq}_{\geq 0}(0) \quad 1 \quad \mathit{Seq}_{\geq 1}(0) \quad 1 \quad \mathit{Seq}_{\geq 1} \quad \cdots \quad 1 \quad \mathit{Seq}_{\geq 1}(0) \quad \emptyset/1. \tag{1.2.1}$$

Note that the sequences of 0, *except the first one, must have at least one element. If not, then two consecutive* 1 *would be in the word. Compacting (1.2.1) and translating it into more operations, we get*

$$\mathit{Seq}_{\geq 0}(0) \times \mathit{Seq}_{\geq 0}\Big(\{1\} \times \mathit{Seq}_{\geq 1}(0)\Big) \times \{\emptyset, 1\}.$$

Translating these operations in terms of the generating function (using Table 1.1):

$$A(z) = \frac{1}{1-z} \cdot \frac{1}{1 - \frac{z^2}{1-z}} \cdot (1+z) = \frac{1+z}{1-z-z^2}.$$

1.2.2 Catalan numbers with the symbolic method

Our objective is to find the coefficients of the sequence of Catalan numbers, that we already know that they are (1.1.1), by using the symbolic method that has been exposed. Therefore, we want to find an expression for the generating function of Catalan numbers $C(z) = \sum_{n=0}^{\infty} c_n z^n$, which encodes the expression for the coefficients c_n.

We denote $\mathcal{D}$ the combinatorial class of Dyck paths. We will find the expression of the generating function of Dyck paths, $D(z) = \sum_{n\geq 0} d_n z^n$ and we will use it to find $C(z)$ or c_n, because $d_{2n} = c_n$. Hence, as there only exists Dyck path of even length we have

$$D(z) = \sum_{n\geq 0} d_n z^n = \sum_{n\geq 0} d_{2n} z^{2n} = \sum_{k\geq 0} c_k z^k = C(z).$$

First of all note that any not empty Dyck path must start with a step $\nearrow = (1,1)$ and end with one step $\searrow = (1,-1)$. We pay attention to the first time that the path reaches the x-axis, and we define the walk until this point as an *arch*. Then, we can also define $\mathcal{A} = \{\text{arches}\}$, where $\mathcal{A} = \nearrow \times \mathcal{D} \times \searrow$; see Figure 1.3. It is also easy to see that $\mathcal{D} = \texttt{Seq}_{\geq 0}(\mathcal{A})$; see Figure 1.4.

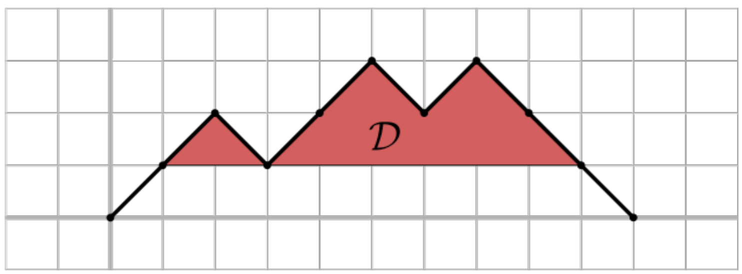


Figure 1.3: In this picture you can see that an arch is the combination of an up step, a dick path and a down step, i.e. $\mathcal{A} = \nearrow \times \mathcal{D} \times \searrow$.

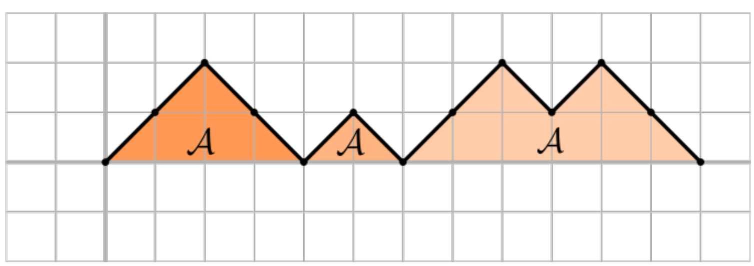


Figure 1.4: This picture visually shows that Dick paths can be seen as a sequence of arches, i.e. $\mathcal{D} = \texttt{Seq}_{\geq 0}(\mathcal{A})$.

Now, we have to translate these operations into operations for the generating functions, $D(z)$ for the Dyck paths and $A(z)$ for arches. In order to do it, we will use Table 1.1. Thus,

$$\mathcal{D} = \texttt{Seq}_{\geq 0}(\mathcal{A}) \Longrightarrow D(z) = \frac{1}{1 - A(z)},$$
$$\mathcal{A} = \nearrow \times \mathcal{D} \times \searrow \Longrightarrow A(z) = z^2 D(z).$$

For $D(z)$ we obtain the equation $D(z) = \frac{1}{1-z^2D(z)} \iff D(z) = 1 + z^2D(z)^2$.
Now, remember that $D(z) = \sum_{n\geq 0} d_{2n}z^{2n}$. Then if we define $x = z^2$, we have the new equation for $D(x)$: $D(x) = 1 + xD(x)^2$.

$$D(x) = \frac{1}{1-xD(x)} \iff xD(x)^2 - D(x) + 1 = 0 \iff D(x) = \frac{1 \pm \sqrt{1-4x}}{2x}. \tag{1.2.2}$$

We need now to choose the correct sign of $D(x)$ from (1.2.2). We know that $d_0 = 1$ (the empty path), so $D(0) = 1$. Evaluating $D(0)$, we get a division by zero with both the two signs. Let us now study the limit $\lim_{x\to 0^+} D(x)$. With the positive sign choice:

$$\lim_{x\to 0^+} D(x) = \lim_{x\to 0^+} \frac{1+\sqrt{1-4x}}{2x} = \infty.$$

Whereas, if we choose the negative choice, applying the Hôpital's rule, we obtain the desired value:

$$\lim_{x\to 0^+} D(x) = \lim_{x\to 0^+} \frac{1-\sqrt{1-4x}}{2x} \overset{\text{(Hôpital)}}{=} \lim_{x\to 0^+} \frac{4}{4\sqrt{1-4x}} = \lim_{x\to 0^+} \frac{1}{\sqrt{1-4x}} = 1.$$

Hence, the choice is clear: $D(x) = \frac{1-\sqrt{1-4x}}{2x}$. By Newton's generalized binomial Theorem

$$D(x) = \frac{1}{2x}(1-\sqrt{1-4x}) = \frac{1}{2x}\left(1 - \sum_{n=0}^{\infty}\binom{1/2}{n}(-4x)^n\right) \Leftrightarrow xD(x) = -\frac{1}{2}\sum_{n=1}^{\infty}\binom{1/2}{n}(-4x)^n,$$

where $\frac{1}{2}$ cancels with the first term of the summation.
If we set equal coefficients:

$$\begin{aligned}
[x^n]D(x) &= -\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1} = -\frac{1}{2}\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n\right)}{(n+1)!}(-4)^{n+1} \\
&= \cancel{-\frac{1}{4}}\frac{\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n\right)}{(n+1)!}(-4)^n\cancel{(-4)} \\
&= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(n+1)!}2^n, \qquad \text{distributing } (-2)^n \text{ among the } n \text{ factors} \\
&= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(n+1)!}\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 2\cdot 3\cdots n}, \qquad \text{substituting } 2^n \text{ for } \frac{2}{1},\frac{4}{2},\frac{6}{3},\ldots \\
&= \frac{(2n)!}{(n+1)!\,n!} = \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

Hence, $c_n(x) = d_n(x) = \frac{1}{n+1}\binom{2n}{n}$, as we expected.

1.2.3 Another example: Motzkin numbers

To conclude this section about the symbolic method, we are going to see another example of how to use it. We are going to find the *Motzkin numbers*, that are closely related with Catalan numbers and they are also associated to many counting problems.

Definition 1.2.4. *A* Motzkin path *of size n is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(n,0)$ with steps $u := (1,1)$, $d := (1,-1)$ and $h := (1,0)$ and which never crosses the x-axis.*
We denote by $\mathcal{M}_n$ the set of all Motzkin paths of length n and $\mathcal{M}_0$ denote the empty path.
Motzkin numbers *M_n count the number of Motzkin paths of size n, i.e $M_n = card(\mathcal{M}_n)$.*

Note that denoting by u, d and h the up, down and horizontal steps respectively, a Motzkin path of size n can be encode as a word of length n with the alphabet $\{u, d, h\}$ in such a way that, counting from the left, the u count is always greater or equal to the d count (to ensure that the path never passes below the x-axis) and such that the total count of u and d is equal (because the path has to start and end on the x-axis).
The first numbers of the Motzkin numbers' sequence are $1, 1, 2, 4, 9, 21, 52, 127, \ldots$. In Figure 1.5 the first cases are depicted with its representation in words with the alphabet $\{u, d, h\}$.

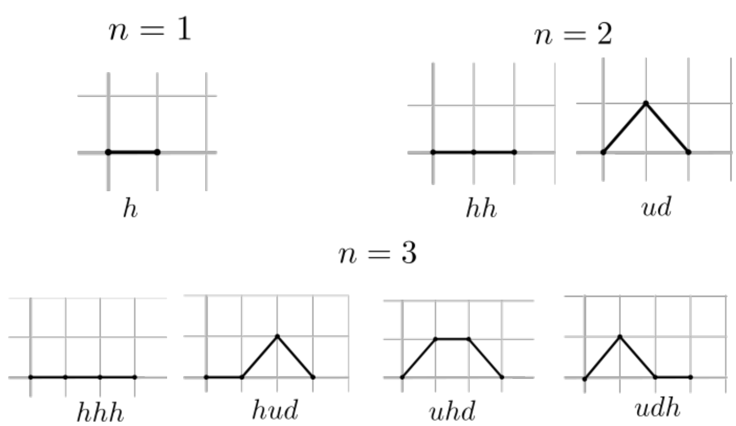


Figure 1.5: $M_0 = 1$, the empty path $M_1 = 1$ represented at the top left side, $M_2 = 2$ represented at top right side of the picture and $M_3 = 4$ represented at the bottom of the picture.

The exact formula for Motzkin numbers M_n is

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k, \tag{1.2.3}$$

where C_k are Catalan numbers.
For n even, $n = 2r$, the paths in M_n that contain only up and down steps but no horizontal steps are precisely the Dyck paths of size r, which are counted by the Catalan numbers.

Let us define $M(z)=\sum_{n\geq 0} M_n z^n$ as the generating function of Motzkin paths. With the Symbolic method we are going to find its expression. From it, doing some long and complex calculus we could get its coefficients (1.2.3).

First of all, notice that the set of all Motzkin paths $\mathcal{M}$ can be divided into the empty path ϵ, the set $\mathcal{M}_h$ that denotes the paths which start with an horizontal step h and the set $\mathcal{M}_u$ which is the set of walks that start with an up step u. That is

$$\mathcal{M}=\{\epsilon\}\sqcup\mathcal{M}_h\sqcup\mathcal{M}_u.$$

If we want to write it all in function of $\mathcal{M}$ it would be

$$\mathcal{M}=\{\epsilon\}\sqcup h\mathcal{M}\sqcup u\mathcal{M}d\mathcal{M}.$$

Now, by the symbolic method (Table 1.1) we can translate all this constructions into operations for the generating function $M(z)$ and we get

$$M(z)=1+zM(z)+z^2M(z)^2 \iff z^2M(z)^2+(z-1)M(z)+1=0 \tag{1.2.4}$$

Isolating $M(z)$ from (1.2.4):

$$M(z)=\frac{(1-z)\pm\sqrt{(z-1)^2-4z^2}}{2z^2}=\frac{(1-z)\pm\sqrt{-3z^2-2z+1}}{2z^2}.$$

Now we need to choose the right sign. Again, we are going to use the fact that we know $M_0=M(0)=1$. If we evaluate $M(z)$ on $z=0$, we get a division by 0 in both cases (with the positive and the negative sign). Therefore, we are also going to study the limit $\lim_{z\to 0^+} M(z)$.
With the positive sign choice:

$$\lim_{z\to 0^+} M(z)=\lim_{z\to 0^+}\frac{(1-z)+\sqrt{-3z^2-2z+1}}{2z^2}=\infty.$$

The positive sign does not give the correct value for $z=0$. Let us see what happens with the negative sign. For the negative sign choice, using the Hôpital's rule:

$$\begin{aligned}\lim_{z\to 0^+} M(z)&=\lim_{z\to 0^+}\frac{(1-z)+\sqrt{-3z^2-2z+1}}{2z^2}\overset{\text{(Hôpital)}}{=}\lim_{z\to 0^+}\frac{-1-\frac{-6z-2}{2\sqrt{-3z^2-2z+1}}}{4z}\\&=\lim_{z\to 0^+}\frac{-2\sqrt{-3z^2-2z+1}+6z+2}{8z\sqrt{-3z^2-2z+1}}\overset{\text{(Hôpital)}}{=}\lim_{z\to 0^+}\frac{\frac{6z+2}{\sqrt{-3z^2-2z+1}}+6}{8\sqrt{-3z^2-2z+1}+\frac{4z(-6z-2)}{\sqrt{-3z^2-2z+1}}}\\&=\lim_{z\to 0^+}\frac{6z+2+6\sqrt{-3z^2-2z+1}}{-48z^2-24z+8}=\frac{2+6}{8}=1.\end{aligned}$$

So, the correct sign for $M(z)$ is the negative one because we have obtained the desired value of the limit. Thus,

$$M(z) = \frac{(1-z) + \sqrt{-3z^2 - 2z + 1}}{2z^2}.$$

Chapter 2

Algebraic techniques for generating functions

Since now we have seen two examples of combinatorial objects, Dyck and Motzkin paths. Via the Symbolic method we have obtained equations for their generating functions and we have been able to solve them. But we are going to see that it is not always that easy to solve the equations for the generating functions and we will have to introduce another method: the *Kernel method*.

2.1 Motivation

We have seen two types of paths starting at $(0, 0)$ and ending at $(n, 0)$ which never cross the x-axis:

- Dyck paths, where $n = 2k$ and where there are only two allowed steps: $(1, 1)$ and $(1, -1)$.
- Motzkin paths, where the allowed steps are $(1, 1)$, $(1, -1)$ and also $(1, 0)$.

Let us now define a new set of paths.

Definition 2.1.1. *A family of paths is called* simple *if each allowed step in $\mathcal{S}$ (Definition 1) is of the form $(1, b)$ with $b \in \mathbb{Z}$. Then we abbreviate $\mathcal{S}$ as $\mathcal{S} = \{b_1, \ldots, b_r\}$.*

With simple paths the size n of the path coincides with its *length*, its span along the horizontal direction.

Definition 2.1.2. *We are going to consider the set $\mathcal{G}$ consisting of all the simple walks from $(0, 0)$ to $(n, 0)$ that never cross the x-axis with the set of possible steps $S = \{b_1, b_2, \ldots, b_r\}$.*

We want to find its generating function $G(z) = \sum_{n \geq 0} g_n z^n$, where g_n counts the number of these paths of length n. In order to do it we are going to use bivariate generating functions.

2.1.1 Bivariate generating functions

Many times, not only will be interested in the size of elements of a combinatorial class but also in auxiliary parameters. Multivariate generating functions allows us to encode different parameters defined over a combinatorial object all in one structure.
In this section, we are going to define a particular case of multivariate generating functions: bivariate generating functions, which will be so helpful in later moments to find $G(z)$.

Definition 2.1.3. *The* bivariate generating function (BGFs) *of a sequence of numbers* $(f_{n,k})$ *is the formal power series in two variables defined as*

$$F(z,u) = \sum_{n,k} f_{n,k} z^n u^k.$$

Usually, in our context, $(f_{n,k})$ will denote the number of elements ω in some class $\mathcal{F}$, such that $|\omega| = n$ and some parameter $\chi(\omega)$ is equal to k.

Remark. Observe that with $u = 1$, we get the univariate generating function.

We can refer to the *horitzontal* and *vertical* generating functions definend by:

Definition 2.1.4. *Horizontal GF:* $f_n(u) := \sum_k f_{n,k} u^k$ *(number of elements of $\mathcal{F}$ of size n).*
Vertical GF: $f^{\langle k \rangle}(z) := \sum_n f_{n,k} z^n$ *(number of elements $\omega \in \mathcal{F}$ with $\chi(\omega)$ equal to k).*

We have,

$$F(z,u) = \sum_k f^{\langle k \rangle}(z)\ u^k = \sum_n f_n(u)\ z^n.$$

The concept of bivariate generating functions can be also extended with more variables and we obtain multivariate generating functions, but we will not address them in this document (see [5] to know more about multivariate generating functions).

2.1.2 Equation for the bivariate generating function

To find the generating function $G(z)$ we define another set of paths.

Definition 2.1.5. *We denote by $\mathcal{F}$ the set of all simple paths starting at* $(0,0)$ *without crossing the x-axis and with possible steps $S = \{b_1, b_2, \dots, b_r\}$.*

We are also going to use the bivariate generating function $F(z,u) \equiv F(u)$ of $\mathcal{F}$ with z counting the length (number of steps) of the path and u the height of the endpoint. We are going to call u as a *catalytic variable* because we are not interested in the information that this variable encodes, but it is going to help us to find our generating function $G(z)$. Note that $G(z) = F(0) \equiv F(z,0)$.

By construction or by a recurrence relation, we can find an equation for this bivariate generating function $F(u)$.

In instance, if the possible steps are $S = \{+1, -1\}$, in this case $F(0)$ is the generating function of Dyck paths, and this is going to be the resulting equation for $F(u)$:

$$F(u) = 1 + zuF(u) + \frac{z}{u}(F(u) - F(0)) = 1 + zuF(u) + \frac{z}{u}(F(u) - F_0), \tag{2.1.1}$$

where on the right-hand side 1 counts the empty path, the second term counts the walks ending with a $+1$ step and the third term counts the walks ending with a -1 step. Note that in this last case, we need to discount the walks that in the penultimate step are at height $y = 0$, because if not with the final step -1 they would cross the x-axis. F_0 is a priori an unknown function and it is $F_0(z) = F(0) = \sum_{n \geq 0} f_{n,0} z^n$ (in this case it is known because it is the generating function of Dyck paths). Equation (2.1.1) can be rewritten as a polynomial equation $P(F(u), F_0, z, u) = 0$, where

$$P(x_0, x_1, z, v) = (v - z(1 + v^2))x_0 - v + zx_1.$$

Let us now study an other problem with more unknown functions. We are considering the set of paths $\mathcal{F}$ with possible steps $S = \{-2, +3\}$. Here, the equation for $F(u)$ is the following one:

$$F(u) = 1 + zu^3F(u) + \frac{z}{u^2}(F(u) - F_0 - uF_1). \tag{2.1.2}$$

In this case, in the last term of the equation we need to discount the paths that in the penultimate step are at height $y = 0$ or $y = 1$. F_0 and F_1 are unknown functions, where $F_k = F_k(z) = f^{\langle k \rangle}(z)$, from Definition 2.1.4.

Equation (2.1.2) can also be written as a polynomial equation $P(F(u), F_0, F_1, z, u) = 0$ with

$$P(x_0, x_1, x_2, z, v) = (v^2 - z(1 + v^5))x_0 - v^2 + zx_1 + zvx_2.$$

At this point, we have obtained equations (2.1.1) and (2.1.2) for $F(u)$ but now we need to know if they can be solved and how to do it. If we consider paths of $\mathcal{F}$ with a general set of steps $S = \{s_1, s_2, \ldots, s_r\}$, we will obtain equations for $F(u)$ similar to the ones of these examples and they will also can we written as an equation of the form

$$P(F(u), F_0, F_1, \ldots, F_{k-1}, z, u) = 0, \tag{2.1.3}$$

where $P(x_0, x_1, ..., x_k, t, v)$ is a non-trivial polynomial in $k+3$ variables, with coefficients in $K = \mathbb{C}$. To answer the question of the existence of solution of these equations of the form (2.1.3), we need to resort to some algebraic results and next we are going to describe a method to solve this type of equations: the *Kernel method*.

2.2 Algebraic preliminaries

In this section we are going to see that the equations of the form (2.1.3) have indeed solution in a particular space. To see it we are going to expose the main concepts to understand it but we are not going to enter into any details because the aim of this work is not an algebraic study. For more information on the topic we are going to develop in this section you can find it in [5] and [10] .
We are going to work with the field $K = \mathbb{C}$, but the results that we are going to see in this section can be extended to other fields K.

2.2.1 Algebraic generating functions

In this section we are going to talk about *algebraic functions*, which are a natural generalization of rational functions. Algebraic functions also have a natural generalization and they are the *D-finite functions*. Thus we have the hierarchy

$$\begin{array}{c} D\text{-finite} \\ | \\ \text{algebraic} \\ | \\ \text{rational} \end{array}$$

Definition 2.2.1. *A formal power series $\eta \in \mathbb{C}[[x]]$ is said to be* algebraic *if there exist polynomials $P_0(x), \ldots, P_d(x) \in \mathbb{C}[x]$, not all 0, such that*

$$P_0(x) + P_1(x)\eta + \ldots + P_d(x)\eta^d = 0. \tag{2.2.1}$$

The smallest positive integer d for which (2.2.1) holds it is called the degree *of η.*
The set of all the algebraic power series over $\mathbb{C}$ is denoted $\mathbb{C}_{alg}[[x]]$.

Remark. Note that an algebraic series η has degree one if and only if η is rational.

We are not going to address D-finite power series in this work, so we just define them and if wanted, you can search for more information about them in Stanley's book [10].

Definition 2.2.2. *Let $u \in \mathbb{C}[[x]]$. We say that u is a D*-finite *(short for* differentiably finite*) power series if there exist polynomials $P_0(x), \ldots, P_d(x) \in \mathbb{C}[x]$, with $P_d(x) \neq 0$, such that*

$$P_d(x)u^{(d)} + P_{d-1}(x)u^{(d-1)} + \cdots + P_1(x)u' + P_0(x)u = 0,$$

where $u^{(j)} = d^j u/dx^j$.

Let us now focus on algebraic series, which are the power series we are interested in. When dealing with them, it is convenient to work over fields rather than over the rings $\mathbb{C}[x]$ and $\mathbb{C}[[x]]$. For this reason we are going to consider their quotient fields. The quotient field of $\mathbb{C}[x]$ is just $\mathbb{C}(x) = \left\{\frac{P(x)}{Q(x)} | Q(x) \neq 0, P(x), Q(x) \in \mathbb{C}[x]\right\}$ while the quotient field of $\mathbb{C}[[x]]$ is given by $\mathbb{C}((x)) = \mathbb{C}[[x]][1/x]$. The elements of $\mathbb{C}((x))$ can be regarded as Laurent series. If $\eta \in \mathbb{C}((x))$ then $\eta = \sum_{n \geq n_0} a_n x^n$ for some $n_0 \in \mathbb{Z}$ (depending on η) and it is easy to see that such Laurent series indeed form a field.

Definition 2.2.3. *Let D be an integral domain containing the field $\mathbb{C}(x)$. Then, $\eta \in D$ is* algebraic *over $\mathbb{C}(x)$ if there exist elements $F_0(x), \ldots, F_d(x) \in \mathbb{C}(x)$, not all 0, such that*

$$F_0(x) + F_1(x)\eta + \ldots + F_d(x)\eta^d = 0. \tag{2.2.2}$$

The smallest positive integer d for which (2.2.2) holds is the degree *of η over the field $\mathbb{C}(x)$ and we denote it by $\deg_{\mathbb{C}(x)}(\eta)$.*

The degree $d = \deg_{\mathbb{C}(x)}(\eta)$ is also the dimension of the field $\mathbb{C}(x, \eta)$ as a vector space over $\mathbb{C}(x)$. Equivalently, η is algebraic over $\mathbb{C}(x)$ if only if the $\mathbb{C}(x)$-vector space spanned by $\{1, \eta, \eta^2, \ldots\}$ is finite-dimensional and with dimension $\deg_{\mathbb{C}(x)}(\eta)$. In addition, the set of $\eta \in D$ that are algebraic over $\mathbb{C}(x)$ form a subring of D containing $\mathbb{C}(x)$.

We denote $P(y) = F_0(x) + F_1(x)y + \ldots F_d(x)y^d \in \mathbb{C}(x)[y]$ ($P(y)$ is a polynomial in the indeterminate variable y with coefficients in $\mathbb{C}(x)$). Suppose that (2.2.2) holds. Then,

$$d = \deg_{\mathbb{C}(x)}(\eta) \Longleftrightarrow P(y) \text{ is } \textit{irreducible}.$$

If we divide (2.2.2) by $F_d(x)$ so that $P(y)$ is monic and $d = \deg_{\mathbb{C}(x)}(\eta)$, then the equation (2.2.2) is unique (otherwise we could subtract two such equations and obtain one of smaller degree).
We can also multiply the equation (2.2.2) by a common denominator of the F_i's so we can assume that the F_i's are polynomials, i.e. $F_i \in \mathbb{C}[x]$. By Definition 2.2.1 we see that $\eta \in \mathbb{C}[[x]]$ is algebraic if and only if it is algebraic over $\mathbb{C}(x)$. The same is true for $\eta \in \mathbb{C}((x))$ and the set of all algebraic Laurent series over $\mathbb{C}(x)$ is denoted $\mathbb{C}_{\text{alg}}((x))$. Therefore we have seen

$$\mathbb{C}_{\text{alg}}[[x]] = \mathbb{C}_{\text{alg}}((x)) \cap \mathbb{C}[[x]].$$

For most enummerative and combinatorial purposes which involve algebraic series it suffices to work with Laurent series, that is to work with $\mathbb{C}((x))$. However, there exist elements η in some extension field of $\mathbb{C}(x)$ that are algebraic over $\mathbb{C}(x)$ but cannot be represented as elements of $\mathbb{C}((x))$. A simple example could be the elements μ defined by $\mu^N = x$ for $N \geq 2$. This suggest to introduce an other type of formal series.

Definition 2.2.4. *A* fractional (Laurent) series *or* Puiseux series *with coefficients in* $\mathbb{C}$ *is the formal series*

$$\eta = \sum_{n \geq n_0} a_n x^{n/N},$$

where N *is a positive integer and* $a_n \in \mathbb{C}$ *for each* n.
If $n_0 = 0$ *then we have a* fractional power series.
We denote by $\mathbb{C}^{fra}((x))$ *the ring of all Puiseux series over* $\mathbb{C}$ *and by* $\mathbb{C}^{fra}[[x]]$ *the set of fractional power series over* $\mathbb{C}$.

$\mathbb{C}^{\text{fra}}((x)) = \mathbb{C}((x))[x^{1/2}, x^{1/3}, x^{1/4}, \ldots]$, or what is the same, every $\eta \in \mathbb{C}^{\text{fra}}((x))$ can be written as a polynomial in $x^{1/2}, x^{1/3}, \ldots$ (and hence involving only a finite number of them) with coefficients in $\mathbb{C}((x))$. Conversely, every polynomial of this form is a fractional series. For example, $\sum_{N \geq 1} x^{1/N}$ is not a Puiseux series.
It is easy to verify that $\mathbb{C}^{\text{fra}}[[x]]$ is a ring and that $\mathbb{C}^{\text{fra}}((x))$ is the quotient field of $\mathbb{C}^{\text{fra}}[[x]]$.

The next theorem, known as *Puiseuix's theorem* is an important result what we are not going to prove it because it is a complex proof. (see [8] for the details of the proof).

Theorem 2.2.1 (Puiseux's Theorem)**.** *The field* $\mathbb{C}^{fra}((x))$ *is algebraically closed and it is the algebraic closure of the field of Laurent series over* $\mathbb{C}$*,* $\mathbb{C}((x))$.

This theorem will allow us to work in $\mathbb{C}^{\text{fra}}((x)$ without problems when studying Puiseux's series, in the same way as we work in $\mathbb{C}$ instead of $\mathbb{R}$ when we study roots of polynomials.

2.2.2 Algebraic elimination

For linear algebra we are capable of solving a given lineal system of equations. Moreover, given a polynomial system

$$\{P_j(z, y_1, y_2, \ldots, y_m) = 0\}, \qquad j = 1, \ldots m, \tag{2.2.3}$$

with *resultants* we are also capable of extracting a single equation satisfied by one of the indeterminates y_j.

Definition 2.2.5. *Consider a field* K*. A polynomial of degree* d *in* $K[x]$ *has at most* d *roots in* K *and exactly* d *in the algebraic closure* $\overline{K}$ *of* K*. Given two polynomials,* $P(x) = \sum_{i=0}^{n} a_i x^{n-i}$ *and* $Q(x) = \sum_{j=0}^{m} b_j x^{m-j}$*, their* resultant *(with respect to the* x *variable) is the determinant of order* $(n + m)$*,*

$$\mathbf{R}(P,Q,x) = \det \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots a_{n-1} & a_n & \\ b_0 & b_1 & b_2 & \cdots & 0 & 0 \\ 0 & b_0 & b_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots b_{m-1} & b_m, & \end{vmatrix} \tag{2.2.4}$$

also called the Sylvester determinant.

By definition, the resultant is a polynomial form in the coefficients of P and Q. Let us now see the next proposition that gathers the main properties of the resultants.

Proposition 2.2.1. *Let K be a field.*

i) If $P(x), Q(x) \in K[x]$ have a common root in $\overline{K}$, then $\mathbf{R}(P(x), Q(x), x) = 0$.

ii) Conversely, if $\mathbf{R}(P(x), Q(x), x) = 0$, then either $a_0 = b_0 = 0$ or else $P(x), Q(x)$ have a common root in $\overline{K}$.

Therefore, the condition $\mathbf{R}(P, Q, x) = 0$ will capture all the situations in which P and Q have a common root, but it may also include situations where there is a reduction in degree, although the polynomial have no common roots.

Now, to eliminate all but one indeterminate variables y_j from the system (2.2.3), we take resultants with P_m and we eliminate all occurrences of the variable function y_m from the first $m-1$ equations, achieving a new system with $m-1$ equations in $m-1$ unknown functions $(y_1, y_2, \ldots, y_{m-1})$. Repeating the process we can eliminate $y_{m-1}, \ldots, y_2$ and obtain the polynomial $Q(y_1, z)$ that we wanted.

2.2.3 Singularities of algebraic functions

Let $P(z, y)$ be an irreducible polynomial of $\mathbb{C}[z, y]$,

$$P(z, y) = p_0(z)y^d + p_1(z)y^{d-1} + \cdots + p_d(z).$$

The solutions of the polynomial equation $P(z, y) = 0$ define a set of points (z, y) in $\mathbb{C} \times \mathbb{C}$ known as a *complex algebraic curve*. If d is the y-degree of P, for each z there are at most d possible values of y. In fact, there are always d different values for y unless:

- z_0 is such that $p_0(z_0) = 0$ and there is a reduction in the degree in y. The points that disappear can be seen as "points to infinity".

- z_0 is such that $P(z_0, y)$ has a multiple root. In this case, some values of y coalesce.

Let us define a new set that will be useful to locate these exceptions, providing the set of possible candidates for the singularities of an algebraic function.

Definition 2.2.6. *We define* the exceptional set of P *as the set*

$$\Xi[P] := \{z | R(z) = 0\}, \text{ where } R(z) := \mathbf{R}(P(z, y), \partial_y P(z, y), y).$$

$R(z)$ *is called the* discriminant *of* $P(z, y)$ *and* $\mathbf{R}$ *is the resultant of Definition 2.2.5.*

- If $z \notin \Xi[P]$ we can assure that there exist d different solutions to $P(z, y) = 0$. This is because from Proposition 2.2.1,

$$R(z) = 0 \Longleftrightarrow \begin{cases} p_0(z) = dp_0(z) = 0 \\ \text{or} \\ P(z, y) \text{ and } \partial_y P(z, y) \text{ have common roots.} \end{cases}$$

  As $R(z) \neq 0$, neither $p_0(z) = dp_0(z) = 0$ nor $P(z, y)$ and $\partial_y P(z, y)$ having common roots can happen. This guarantees that there is no reduction in degree and that $P(z, y)$ has no multiple roots.

  Knowing it, by the Implicit Function Theorem, for each solution y_j we have a locally analytic function $y_j(z)$. A *branch* of the algebraic curve $P(z, y) = 0$ is the choice of such a $y_j(z)$ together with a simply connected region of the complex plane where this particular $y_j(z)$ is analytic.

- If $z \in \Xi[P]$ then we may have singularities. If we are in a point z_0 such that $p_0(z_0) = 0$, some of the branches escape to infinity, stopping being analytic, whereas if we are at a point z_0 such that $R(z) = 0$ but $p_0(z_0) \neq 0$, then two or more branches collide. These collision point can be either *multiple points*, if two or more branches reach the same value but each one exists as an analytic function around z_0, or *branch points*, if some of the branches cease to be analytic. See Figure 2.1 to see an example of multiple and branch points.

The following lemma allow us to extend an analytic function at the origin in a analytical way. We are only going to announce it; see [5] for the proof.

Lemma 2.2.1 (Location of algebraic singularities)**.** *Let $y(z)$ be a function analytic at the origin such that satisfies a polynomial equation $P(z, y) = 0$. Then, $y(z)$ can be analytically continued along any simple path emanating from the origin that does not cross any point of the exceptional set $\Xi[P]$.*

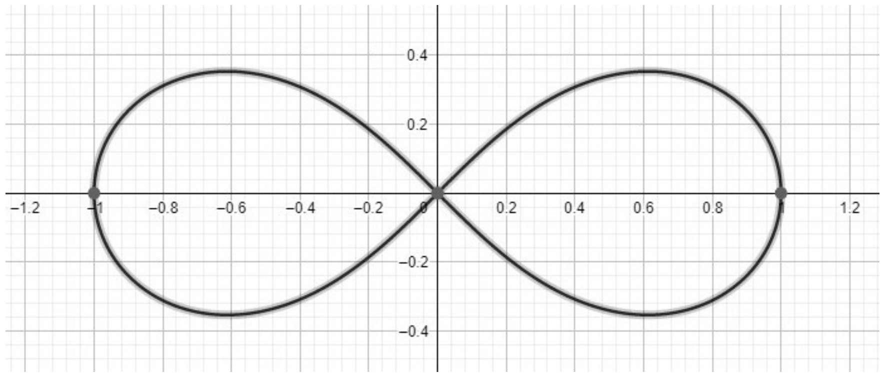


Figure 2.1: Real section of the lemniscate of Bernoulli defined by $P(z, y) = (z^2 + y^2)^2 - (z^2 - y^2) = 0$. At the origin there is a multiple point (a double point) where two analytic branches collapse, whereas at $z = \pm 1$ there are two branch points.

Another important result is the following, that gives us the certainty that an algebraic function have a Puiseux expansion near a singularity.

Theorem 2.2.2 (Newton-Puiseux expansions at a singularity)**.** *Let $f(z)$ be a branch of an algebraic function $P(z, f(z)) = 0$. In a circular neighbourhood of a singularity ξ, $f(z)$ admits a fractional series expansion (Puiseux expansion) that is locally convergent of the form*

$$f(z) = \sum_{k \geq k_0} c_k (z - \xi)^{k/N},$$

for a fixed determination of $(z - \xi)^{1/N}$, where $k_0 \in \mathbb{Z}$ and N in an integer ≥ 1, called the branching type*.*

The next step is how to find this Puiseux expansion mentioned in Theorem 2.2.2. There exists a method to do it and it is the *Newton's polygon method.*

NEWTON'S POLYGON METHOD

By Lemma 2.2.1, to determine a Puiseux expansion near a point (z_0, y_0) we can do it near $(0, 0)$. If we have $P(z, y) = 0$ with a singularity at (z_0, y_0), via a translation of the variables z, y we can write an other polynomial equation $Q(Z, Y) = 0$, where now $(0, 0)$ is the singularity. If the new polynomial $Q(Z, Y)$ is

$$Q(Z, Y) = \sum_{j \in J} c_j Z^{a_j} Y^{b_j}$$

we define the *Newton diagram* as the set formed by the finite set of point (a_j, b_j) in $\mathbb{N} \times \mathbb{N}$.
We search for asymptotic solutions for Y of the form $Y \sim cZ^{\alpha}$, with $c \neq 0$, such that satisfy asymptotically $Q(Z, Y) = 0$, it means that the main asymptotic order of $Q(Z, Y)$ must be 0. This can only happen if two or more exponents of the polynomial coincide and the coefficients of their

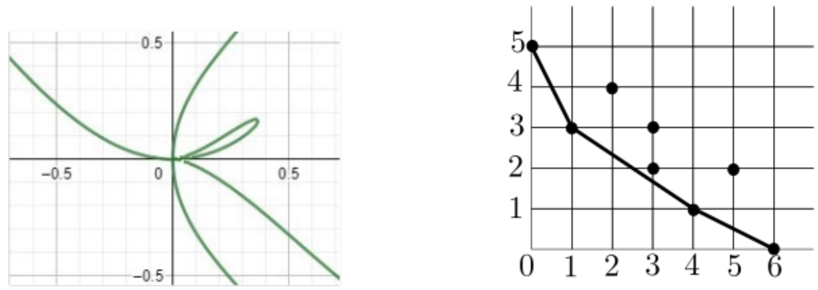


Figure 2.2: The real algebraic curve defined by the equation $P(z,y) = (y - z^2)(y^2 - z)(y^2 - z^3) - z^3y^3$ near $(0,0)$ (*left*) and the corresponding Newton diagram (*right*).

monomials cancel, which is an algebraic constraint on the constant c.
Therefore, the only solutions of the form $Y \sim cZ^{\alpha}$ correspond to the values of α that are inverse slopes (i.e. $\Delta x/\Delta y$) of lines connecting two or more points of the Newton diagram (this expresses the cancellation condition between two monomials of Q) and such that all other points of the diagram are on the line or to the right of it (this assures that the other monomials are of smaller order). For each viable α, a polynomial equation constrains the possible values of the corresponding coefficient c. The complete expansion for Y is obtained by repeating the process, by way of the substitution $Y \mapsto Y - cZ^{\alpha}$.

Example 2.2.1. *Let us consider the curve $P = 0$ where*

$$\begin{aligned} P(z,y) &= (y - z^2)(y^2 - z)(y^2 - z^3) - z^3y^3 \\ &= y^5 - y^3z - y^4z^2 + y^2z^3 - 2z^3y^3 + z^4y + z^4y^2 - z^6. \end{aligned} \tag{2.2.5}$$

In Figure 2.2 we can see an illustration of the curve near the origin as well as its Newton diagram. From it, we find the possible exponents $y \sim cz^{\alpha}$ at the origin:

$$\alpha = 2, \quad \alpha = \frac{1}{2}, \quad \alpha = \frac{3}{2},$$

which are the inverse slopes of the segments composing the envelope of the Newton diagram. To determine the constant c, we have an equation for each value of α.
For $\alpha = 2$,

$$P(z, cz^2) = c^5z^{10} - c^3z^7 - c^4z^{10} + c^2z^7 - 2c^3z^9 + cz^6 + c^2z^7 - z^6.$$

We want $cz^6 - z^6 = 0 \Longrightarrow c = 1 \Longrightarrow \boxed{y \sim z^2}$.

For $\alpha = \frac{1}{2}$,

$$P(z, cz^{1/2}) = c^5 z^{5/2} - c^3 z^{5/2} - c^4 z^4 + c^2 z^4 - 2c^3 z^{9/2} + cz^{9/2} + c^2 z^6 - z^6.$$

We need $c^5 z^{5/2} - c^3 z^{5/2} = 0 \Longrightarrow c^3(c^2-1) = 0 \overset{c\neq 0}{\Longrightarrow} c = \pm 1 \Longrightarrow \boxed{y \sim \pm\sqrt{z}}$.
Finally, for $\alpha = \frac{3}{2}$,

$$P(z, cz^{3/2}) = c^5 z^{15/2} - c^3 z^{11/2} - c^4 z^8 + c^2 z^6 - 2c^3 z^{15/2} + cz^{11/2} + c^2 z^8 - z^6.$$

In this case, we want $-c^3 z^{11/2} + cz^{11/2} = 0 \Longrightarrow c(1-c^2) = 0 \overset{c\neq 0}{\Longrightarrow} c = \pm 1 \Longrightarrow \boxed{y \sim \pm z^{3/2}}$.

We have obtained, as the factored part (2.2.5) *suggests, that the curve locally at* $(0,0)$ *is the union of two orthogonal parabolas (*$y = z^2$ *and* $y = \pm\sqrt{z}$*) and of a curve* $y = \pm z^{3/2}$ *having a cusp. The full expansion of* y *can be recovered bu deflating the function from its first terms and repeating the Newton diagram construction as we have just done.*

2.3 The method

In this section, we are finally going to describe the Kernel method, developed in [1].

Definition 2.3.1. *Let* $\mathcal{S} = \{b_1, \ldots, b_r\}$ *be a simple set of steps. the* characteristic polynomial *of* $\mathcal{S}$ *is defined as the Laurent polynomial*

$$C(u) := \sum_{j=1}^{r} u^{b_r}.$$

We denote by $c = -\min_j b_j$ *and* $d = \max_j b_j$ *the two extreme vertical amplitudes of any step. We assume* $c, d > 0$*. The* characteristic curve *of the lattice path determined by* $\mathcal{S}$ *is the plane algebraic curve defined by the equation*

$$1 - zC(u) = 0, \text{ or equivalently } u^c - z(u^c C(u)) = 0. \tag{2.3.1}$$

The quantity $N(z,u) := u^c - zu^c C(u)$ *is called* kernel *and the equation* (*2.3.1*) *is called* kernel equation*.*

The kernel equation is satisfied if and only if

$$1 = zC(u) = z\left(u^{-c} + \ldots + u^d\right).$$

The asymptotic study of the equation near $z = 0$, tells us that it can only be satisfied if

$$zu^{-c} \sim 1 \quad \text{or} \quad zu^{d} \sim 1.$$

From these conditions we obtain c *small branches* that we will denote as $u_1, \dots, u_c$ and d *large branches* $v_1 \equiv u_{c+1}, \dots, v_d \equiv u_{c+d}$. They will be asymptotically the cth and the dth roots of z and $1/z$, respectively.

Let us remember that we had an equation for the generating function $F(z, u)$ of the form (2.1.3) and we had two examples, (2.1.1) and (2.1.2). They can all be rewritten as the *fundamental functional equation*

$$F(z, u) = 1 + zC(u)F(z, u) - z\{u^{<0}\}(C(u)F(z, u)), \tag{2.3.2}$$

where $C(u)$ is the characteristic polynomial of the set of possible steps $\mathcal{S}$ (Definition 2.3.1) and $\{u^{<l}\}Q(u)$ means the sum of all the monomials of Q that have exponent less than l. For instance, in (2.1.1) we had $\mathcal{S} = \{-1, +1\}$, so $C(u) = u^{-1} + u$. We verify that (2.1.1) is equal to (2.3.2):

$$\begin{aligned} F(z, u) &= 1 + zC(u)F(z, u) - z\{u^{<0}\}(C(u)F(z, u)) \\ &= 1 + zuF(z, u) + \frac{z}{u}F(z, u) - z\{u^{<0}\}((u^{-1} + u)F(z, u)). \end{aligned}$$

Now,

$$(u^{-1} + u)F(z, u) = \sum_{n,k\geq 0} f_{n,k} z^n u^{k+1} + \sum_{n,k\geq 0} f_{n,k} z^n u^{k-1}.$$

Then we only have a term with negative exponent in u (when $k = 0$ in the second summation) and we have that

$$F(z, u) = 1 + zuF(z, u) + \frac{z}{u}F(z, u) - zF_0(z)u^{-1} = 1 + zuF(z, u) + \frac{z}{u}(F(z, u) - F_0(z)),$$

with $F_0(z) = \sum_{n\geq 0} f_{n,0} z^n$. we have obtained exactly the same equation that we got in (2.1.1). We could verify exactly the same with (2.1.2) and any other example. Therefore, it is proved that all the equations for the generating functions $F(z, u)$ that encode the information of the path class $\mathcal{F}$ are fundamental equations (2.3.2).
Rewriting (2.3.2) we obtain

$$F(z, u)(1 - zC(u)) = 1 - z\sum_{k=0}^{c-1} r_k(u)F_k(z), \tag{2.3.3}$$

with $F(z,u) = \sum_{k\geq 0} F_k(z)u^k$ ($F_k = f^{\langle k\rangle}(z)$ in Definition 2.1.4) and $r_k(u)$ the Laurent polynomials

$$r_k(u) := \{u^{<0}\}(C(u)u^k) \equiv \sum_{j=-c}^{-k-1} u^{j+k}.$$

Looking into the fundamental equation in its form (2.3.3) is easy to see that it involves $(c+1)$ unknown functions: the bivariate $F(z,u)$ and the univariate $\{F_k(z)\}_{k=0}^{c-1}$. Multiplying it all by u^c we obtain

$$F(z,u)(u^c - zu^cC(u)) = u^c - z\sum_{k=0}^{c-1} u^c r_k(u)F_k(z),$$

appearing the kernel term in the left-hand side. The Kernel method consists in imposing the kernel equation. From it, we have seen that we obtain c small branches and we can restrict z to a small neighbourhood on the origin in such a way that all the small branches are distinct and satisfy $\left|u_j(z)\right| < 1$. In this way we obtain a system of c equations in the unknown functions $F_0,\ldots,F_{c-1}$

$$\begin{cases} u_1^c - z\displaystyle\sum_{k=0}^{c-1} u_1^c r_k(u_1)F_k(z) = 0 \\ \qquad\qquad\vdots \\ u_c^c - z\displaystyle\sum_{k=0}^{c-1} u_c^c r_k(u_c)F_k(z) = 0. \end{cases} \tag{2.3.4}$$

The system (2.3.4) has a solution because it have a non-zero determinant. To see it, working a little bit with the expression of the determinant we arrive to the Vandermonde determinant, which we know that has determinant $\displaystyle\prod_{1\leq i<j\leq c}(u_i - u_j)$, different of zero because the branches u_i are chosen all distinct.

Let us see how we can transform the determinant of the system into a Vandermonde determinant:

$$\begin{vmatrix} zu_1^c r_0(u_1) & zu_1 r_1(u_1) & \cdots & zu_1^c r_{c-1}(u_1) \\ zu_2^c r_0(u_2) & zu_2 r_1(u_2) & \cdots & zu_2^c r_{c-1}(u_2) \\ \vdots & \vdots & \cdots & \vdots \\ zu_c^c r_0(u_c) & zu_c r_1(u_c) & \cdots & zu_c^c r_{c-1}(u_c) \end{vmatrix} = z^c \begin{vmatrix} u_1^c r_0(u_1) & u_1^c r_1(u_1) & \cdots & u_1^c r_{c-1}(u_1) \\ u_2^c r_0(u_2) & u_2^c r_1(u_2) & \cdots & u_c^2 r_{c-1}(u_2) \\ \vdots & \vdots & \cdots & \vdots \\ u_c^c r_0(u_c) & u_c^c r_1(u_c) & \cdots & u_c^c r_{c-1}(u_c) \end{vmatrix}$$

But now,

$$u_i^c r_k(u_i) = \sum_{j=0}^{c-k-1} u_i^{j+k}.$$

If we add to a column a linear combination of the other columns, the determinant of the matrix does not change. Let us denote by C_m the mth column of the matrix.

Doing the change $C_1 = C_1 + (-1)C_2$ the coefficient of the first column and the ith line is going to be:

$$u_i^c r_0(u_i) - u_i^c r_1(u_i) = \sum_{j=0}^{c-1} u_i^j - \sum_{j=0}^{c-2} u_i^{j+1} = \sum_{j=0}^{c-1} u_i^j - \sum_{j=1}^{c-1} u_i^j = u_i^0 = 1.$$

The operation $C_2 = C_2 + (-1)C_3$ leads to the following value for the coefficient of the second column and the ith line:

$$u_i^c r_1(u_i) - u_i^c r_2(u_i) = \sum_{j=0}^{c-2} u_i^{j+1} - \sum_{j=0}^{c-3} u_i^{j+2} = \sum_{j=0}^{c-2} u_i^{j+1} - \sum_{j=1}^{c-2} u_i^{j+1} = u_i.$$

In general, we can do the change of columns $C_m = C_m + (-1)C_{m+1}$ for $1 \leq m \leq c-1$ and the coefficient in the mth column and ith line will be u_i^{m-1}. Note that the coefficients of the last column, by definition of the r_{c-1} polynomial will be also u_i^{c-1}. Therefore, the determinant of the system (2.3.4) is equal to

$$z^c \begin{vmatrix} 1 & u_1 & \cdots & u_1^{c-1} \\ 1 & u_2 & \cdots & u_{c-1}^2 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & u_c & \cdots & u_c^{c-1} \end{vmatrix},$$

which is exactly the determinant of a Vandermonde matrix.

Hence, the system (2.3.4) can be solved and we obtain an algebraic expression for the functions F_k in terms of the u_i's, which we can rewrite to the form $P_0(F_k, u_1, \ldots, u_c, z) = 0$. Then, for each function F_k, with $k = 0, \ldots c-1$, a following system needs to be solved:

$$\begin{cases} P_0(F_k, u_1, \ldots, u_c, z) = 0 \\ P_1(F_k, u_1, \ldots, u_c, z) = 0 \\ \qquad\qquad \vdots \\ P_c(F_k, u_1, \ldots, u_c, z) = 0, \end{cases} \tag{2.3.5}$$

where for $1 \leq i \leq c$, $P_i(F_k, u_1, \ldots, u_c, z) = u^c - z(u^c C(u_i))$, the Kernel equation.
It is a system with $c+1$ equations and $c+1$ unknown functions $(F_k, u_1, \ldots, u_c)$, which can be solved with the algebraic elimination explained before and obtain a single polynomial equation $Q(F_k, z) = 0$, where u_i for Theorem 2.2.2 can be written as a Puiseux expansion found by the Newton's polygon method. Therefore, we can obtain an expression in terms of z for every function F_k, with $k = 0, \ldots, c-1$. Finally, from (2.3.3) we obtain the solution of $F(z, u)$ that we were searching for.

Instead of following the direction of determinantal calculations, we can make use of an observation of Bousquet-Mélou: the quantity

$$N(z,u) := u_1^c - z\sum_{k=0}^{c-1} u_1^c r_k(u_1)F_k(z). \tag{2.3.6}$$

is by (2.3.4) a polynomial in u whose roots are all the u_i. Hence, the polynomial can be written as the factorization

$$N(z,u) = \prod_{i=1}^{c}\left(u - u_i(z)\right). \tag{2.3.7}$$

From (2.3.3), we know $F(z,u)(1 - zC(u)) = u^{-c}N(z,u)$. Now, the result for the BFG $F(z,u)$ is obtained using the factorization (2.3.7):

$$F(z,u) = \frac{N(z,u)}{u^c\left(1 - zC(u)\right)} = \frac{\prod_{i=1}^{c}\left(u - u_i(z)\right)}{u^c\left(1 - zC(u)\right)}. \tag{2.3.8}$$

Remember that we were looking for the generating function $G(z)$ of the set of walks $\mathcal{G}$ from Definition 2.1.2. We know that $G(z) = F(z,0) = F_0$, where $F(z,u)$ is the bivariating function of the set $\mathcal{F}$ defined at Definition 2.1.5. Note that if we evaluate $u = 0$ in (2.3.8) the denominator cancels. But we have another method to find $G(z)$. We just need to match the constant terms from (2.3.6) and (2.3.7). The constant term from the factorization is $(-1)^c u_1 \cdots u_c$ whereas the constant term from (2.3.6) is $-zF_0$. Then,

$$G(z) = F_0 = \frac{(-1)^{c-1}}{z}\prod_{i=1}^{c} u_i(z). \tag{2.3.9}$$

Remember that u_i are Puiseux series but surprisingly when we multiply them in formulas (2.3.8) and (2.3.9) we simply obtain a power series, since when the exponents of the u_i for each $1 \le i \le c$ are multiplied they become natural numbers.

To conclude this part, let us see an specific example from [1], to see how the Kernel method works.

Lattice paths with steps $\mathcal{S}$= $\{-2, -1, 0, +1, +2\}$

The characteristic polynomial of this set of paths is $C(u) = u^{-2} + u^{-1} + 1 + u + u^2$ and in this case the Kernel equation is $u^2 - z(1 + u + u^2 + u^3 + u^4)$. We have obtained the following formula for the generating function $G(z)$ of $\mathcal{G}$, the class of the walks finishing at the horizontal axis and never crossing it:

$$G(z) = \frac{(-1)^{c-1}}{z}\prod_{i=1}^{c} u_i(z).$$

In this example, $c = 2$ so we have two small branches which are conjugated and are given by:

$$u_1(z) = +z^{1/2} + \frac{1}{2}z + \frac{5}{8}z^{3/2} + z^2 + \frac{231}{128}z^{5/2} + 3z^6 + \cdots,$$
$$u_2(z) = -z^{1/2} + \frac{1}{2}z - \frac{5}{8}z^{3/2} + z^2 - \frac{231}{128}z^{5/2} + 3z^6 + \cdots.$$

The first terms of $G(z)$ are the following ones:

$$G(z) = -\frac{u_1(z)u_2(z)}{z} = 1 + z + 3z^2 + 9z^3 + 32z^4 + 120z^5 + 473z^6 + 1925z^7 + \cdots.$$

We have used the formula and we have not solved the system (2.3.4) from the Kernel method, that in this case is

$$\begin{cases} u_1^2 - z\left(u_1^2 r_0(u_1)F_0(z) + u_1^2 r_1(u_1)F_1(z)\right) = 0 \\ u_2^2 - z\left(u_2^2 r_0(u_2)F_0(z) + u_2^2 r_1(u_2)F_1(z)\right) = 0, \end{cases}$$

with $r_0(u_i) = \sum_{j=0}^{1} u_i^j$ and $r_1(u_i) = u_i^{j+1}$, for $i = 1, 2$.

But solving it, we would have obtained an algebraic expression for F_0 and F_1 in terms of u_1 and u_2 ($P_0(F_k, u_1, u_2, z) = 0$). For instance, for $F_0 = G(z)$ the algebraic expression is $P_0 = zG + u_1u_2$, the same that we have obtained via the formula. Moreover, we can also obtain the equation satisfied directly for $G(z)$, solving a system like (2.3.5) with the kernel equations for u_1 and u_2:

$$\begin{cases} zG + u_1u_2 = 0 \\ u_1^2 - z(1 + u_1 + u_1^2 + u_1^3 + u_1^4) = 0 \\ u_2^2 - z(1 + u_2 + u_2^2 + u_2^3 + u_2^4) = 0, \end{cases} \tag{2.3.10}$$

Solving it, it is found that $G(z)$ satisfies a polynomial equation of degree 4:

$$z^4y^4 - z^2(1+z)y^3 + z(2+z)y^2 - (1+z)y + 1 = 0.$$

Chapter 3

Lattice paths: walks, bridges, meanders and excursions

In the first chapter we have seen Dyck and Motzkin paths, of which we have been able to find their generating functions by the Symbolic method, whereas in Chapter 2 we have developed a new method that allows us to find lots of more generating functions, even for complicated paths. In this chapter we are going to make use of both Symbolic and Kernel methods to determine the generating function of four type of paths or walks in the quadrangular lattice $\mathbb{Z}^2$.

Definition 3.0.1. *A* bridge *is a path whose end-point P_n lies on the x-axis. A* meander *is a path that lies in the quarter plane $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. An* excursion *is a path that is at the same time a meander and a bridge, it means that it finishes on the x-axis and has no point with negative y-coordinate.*

The fourth type of path is the one that has no restrictions. In Table 3.1 you can find the four types of lattice paths with their corresponding generating functions. In this chapter we want to proof the expression of the generating function of these 4 types of lattice paths.
Let us start for meanders and excursions, where we are going to use the Kernel method just seen in the previous chapter.

Theorem 3.0.1. *For a simple set of steps $\mathcal{S}$, the generating functions of all meanders is given in terms of the small branches of the characteristic curve of $\mathcal{S}$ and it is*

$$M(z) = \frac{1}{1 - zC(1)} \prod_{i=1}^{c} \left(1 - u_i(z)\right),$$

where $C(u)$ is the characteristic polynomial of the set $\mathcal{S}$ and u_i are the small branches.

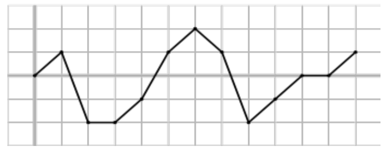
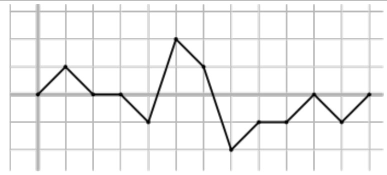
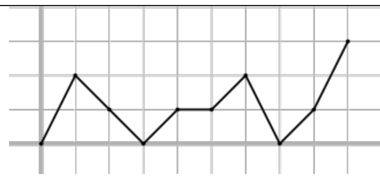
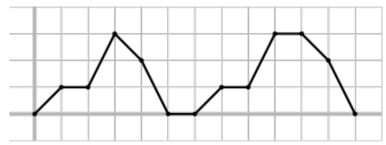
	ENDING ANYWHERE	ENDING AT 0
UNCONSTRAINED (ON $\mathbb{Z}$)	 walk/path ($\mathcal{W}$) $W(z) = \frac{1}{1-zC(1)}$	 bridge ($\mathcal{B}$) $B(z) = z \sum_{i=1}^{c} \frac{u_i'(z)}{u_i(z)}$
CONSTRAINED (ON $\mathbb{Z}_{\geq 0}$)	 meander($\mathcal{M}$) $M(z) = \frac{1}{1-zC(1)} \prod_{i=1}^{c}(1-u_i(z))$	 excursion ($\mathcal{E}$) $E(z) = \frac{(-1)^{c-1}}{z} \prod_{i=1}^{c} u_i(z)$

Table 3.1: The four types of lattice paths and their generating functions.

In particular, the GF of excursions, $E(z)$, satisfies

$$E(z) = \frac{(-1)^{c-1}}{z} \prod_{i=1}^{c} u_i(z).$$

Proof. In the previous chapter we found the expression (2.3.8) for the BGF $F(z,u)$ for the set $\mathcal{F}$ (Definition 2.1.5). We just need to note that $\mathcal{F}$ is in fact exactly the set of all possible meanders. We also know that to recover the univariate generating function we only need to evaluate $F(z,u)$ at $u = 1$. It is

$$M(z) \equiv F(z,1) \overset{(2.3.8)}{=} \frac{1}{1-zC(1)} \cdot \prod_{i=1}^{c} \left(1 - u_i(z)\right).$$

For the generating function of excursions $E(z)$ we have also done all the work before, since $\mathcal{E} \equiv \mathcal{G}$ (from Definition 2.1.2). Therefore,

$$E(z) = G(z) \overset{(2.3.9)}{=} \frac{(-1)^{c-1}}{z} \prod_{i=1}^{c} u_i(z).$$

$\square$

The generating functions for walks or paths can be easily obtained using the Symbolic method, explained in Chapter 2.

Theorem 3.0.2. *The generating function of all the simple walks with a set of possible steps $\mathcal{S}$ is*

$$W(z) = \frac{1}{1 - zC(1)},$$

with $C(u)$ the characteristic polynomial of $\mathcal{S}$.

Proof. Let be $\mathcal{W}$ the set of all walks with simple steps $\mathcal{S} = \{b_1, \dots, b_r\}$. $\mathcal{W}$ can be seen as a sequence of $\mathcal{S}$, i.e. $\mathcal{W} = \mathtt{Seq}(\mathcal{S})$. Therefore, by Table 1.1 of the Symbolic method, the generating function of $\mathcal{W}$ is

$$W(z) = \frac{1}{1 - S(z)},$$

where $S(z)$ is the GF of the set of possible steps $\mathcal{S}$. Each element of the combinatorial class $\mathcal{S}$ has size 1, therefore $S(z) = rz$. It can be also expressed in terms of the characteristic polynomial $C(u)$, since $C(1) = r$. Hence, we obtain

$$W(z) = \frac{1}{1 - zC(1)}.$$

□

Finally, let us find the generating function for all the simple bridges $\mathcal{B}$. We are going to see that it is related to the generating function of excursions.

Theorem 3.0.3. *The generating function of all the possible simple bridges $\mathcal{B}$ with the set of possible steps $\mathcal{S}$ is*

$$B(z) = z \sum_{i=1}^{c} \frac{u_i'(z)}{u_i(z)},$$

with u_i the small branches of the characteristic curve of $\mathcal{S}$.

Proof. We first define a particular type of excursions: *arches.*

Definition 3.0.2. *An* arch *is an excursion of size > 0 whose only contact with the horizontal axis is its end-point. We denote by $\mathcal{A}$ the set of arches and $A(z)$ its generating function.*

We have that $\mathcal{E} = \mathtt{Seq}(\mathcal{A})$. Therefore, by the Symbolic method (Table 1.1),

$$E(z) = \frac{1}{1 - A(z)} \Longleftrightarrow A(z) = 1 - \frac{1}{E(z)}. \tag{3.0.1}$$

Now we want to relate bridges and arches. Consider any bridge and name m (with $m \leq 0$) the minimal altitude of any vertex. Any not empty walk β decomposes uniquely into a walk ω_1 of size ≥ 1 that goes from height 0 to m reaching level m only at its endpoint, followed by an excursion ϵ from level m to m and followed by a path ω_2 of size ≥ 1 from level m to 0 touching only the height

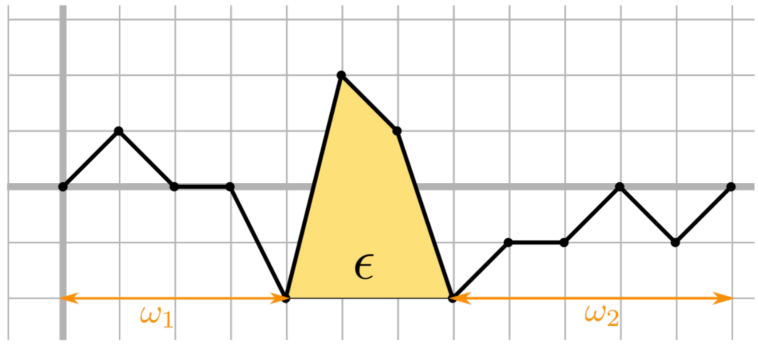


Figure 3.1: A bridge is decomposed into a walk ω_1, followed by an excursion ϵ and followed by another walk ω_2.

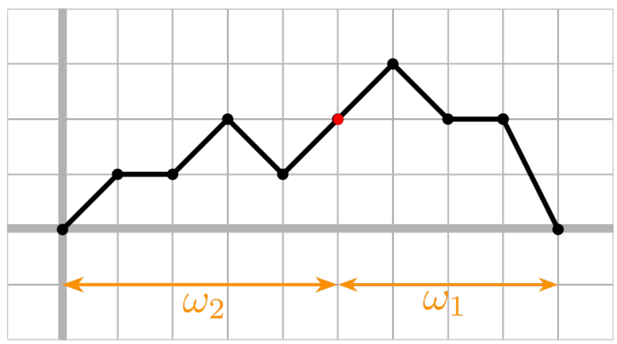


Figure 3.2: Walks ω_2 and ω_1 from the previous decomposition can be concatenated to form an arch with a marked point (in red), which is the union point of both walks.

m at its beginning; see Figure 3.1. Paths ω_2 and ω_1 can be concatenated to form an arch α with a marked point, the union point of both walks; see Figure 3.2. Therefore, from a bridge β we have obtained an excursion ϵ with an arch α with a marked point. Conversely, given an excursion $\bar{\epsilon}$ and an arch $\bar{\alpha}$ with a distinguished point we can obtain a unique bridge $\bar{\beta}$. It means that there is a bijection between the set of non-empty bridges and the product of the set of excursions by the set of arches with a distinguished point, that is $\mathcal{A}^{\circ}$ (the pointing class of arches, defined in Chapter 1). By the Symbolic method (see Table 1.1) the set of marked arches has generating function $z\frac{d}{dz}A(z)$. Hence, by the bijection

$$B(z) - 1 = E(z) \cdot \left(z\frac{d}{dz}A(z) \right).$$

Applying (3.0.1) into the previous expression, we get

$$B(z) = 1 + E(z) \cdot z\frac{d}{dz}\left(1 - \frac{1}{E(z)}\right) = 1 + E(z) \cdot z\frac{E'(z)}{E(z)^2} = 1 + z\frac{E'(z)}{E(z)}.$$

Now we have a relation between the generating functions of bridges and excursions and by Theorem

3.0.1 we have an expression for $E(z)$:

$$E(z) = \frac{(-1)^{c-1}}{z}\prod_{i=1}^{c} u_i(z). \tag{3.0.2}$$

Using the rule of derivation of a product

$$E'(z) = (-1)^{c-1}\left(-\frac{1}{z^2}\prod_{i=1}^{c} u_i(z) + \frac{1}{z}\sum_{i=1}^{c}\left(u_i'(z)\prod_{\substack{k=1\\k\neq i}}^{c} u_k(z)\right)\right) \tag{3.0.3}$$

Therefore, it is easy now to obtain the expression for $B(z)$.

$$B(z) = 1 + z\frac{E'(z)}{E(z)} \overset{(3.0.2),(3.0.3)}{=} 1 + \frac{(-1)^{c-1}}{(-1)^{c-1}}\left(-\frac{z^2}{z^2}\frac{\prod_{i=1}^{c} u_i(z)}{\prod_{i=1}^{c} u_i(z)} + \frac{z^2}{z}\sum_{i=1}^{c}\left(u_i'(z)\underbrace{\frac{\prod_{\substack{k=1\\k\neq i}}^{c} u_k(z)}{\prod_{k=0}^{c} u_k(z)}}_{=\frac{1}{u_i(z)}}\right)\right)$$

$$= 1 + \left(-1 + z\sum_{i=1}^{c}\frac{u_i'(z)}{u_i(z)}\right) = z\sum_{i=1}^{c}\frac{u_i'(z)}{u_i(z)}.$$

□

We have obtained the generating function from the 4 types of lattice paths, which are gathered together into Table 3.1. With this information we can know the generating function of any path in the quadrangular lattice $\mathbb{Z}^2$. To conclude this part, we are going to see an example to clarify how to apply these results. We are going to use the example in page 31 and we are going to complete it to find the generating functions for the 4 types of lattice paths seen in this chapter.

Lattice paths with steps $\mathcal{S}= \{-2, -1, 0, +1, +2\}$.

We are now going to use the formulas in Table 3.1 to find the generating functions of walks, bridges, meanders and excursions with simple steps $\mathcal{S}$.

Let us start with walks. We know

$$W(z) = \frac{1}{1 - zC(1)}.$$

As we have seen in this same example in page 31, the characteristic polynomial is $C(u) = u^{-2} + u^{-1} + 1 + u + u^2$. Therefore, just evaluating $C(u)$ at $u = 1$ we have the GF for walks.

$$C(1) = 5 \Longrightarrow W(z) = \frac{1}{1-5z}.$$

For excursions $\mathcal{E}$, we have obtained the following formula:

$$E(z) = \frac{(-1)^{c-1}}{z} \prod_{i=1}^{c} u_i(z),$$

which, as we know, is the same generating function of the combinatorial class $\mathcal{G}$ (from Definition 2.1.2) and we have calculated its generating function before in page 31 of Chapter 2. Therefore, we know that the two small branches are:

$$u_1(z) = +z^{1/2} + \frac{1}{2}z + \frac{5}{8}z^{3/2} + z^2 + \frac{231}{128}z^{5/2} + 3z^6 + \cdots,$$
$$u_2(z) = -z^{1/2} + \frac{1}{2}z - \frac{5}{8}z^{3/2} + z^2 - \frac{231}{128}z^{5/2} + 3z^6 + \cdots.$$

And the first terms of $E(z)$ are the following ones:

$$E(z) = -\frac{u_1(z)u_2(z)}{z} = 1 + z + 3z^2 + 9z^3 + 32z^4 + 120z^5 + 473z^6 + 1925z^7 + \cdots.$$

For meanders, with the formula $M(z) = \frac{1}{1-zC(1)} \prod_{i=1}^{c}(1 - u_i(z))$, we obtain

$$M(z) = \frac{(1 - u_1(z))(1 - u_2(z))}{1 - 5z} = 1 + 3z + 12z^2 + 51z^3 + 226z^4 + 1025z^5 + \cdots.$$

It only remains the generating function formula for bridges $\mathcal{B}$, $B(z) = z \sum_{i=1}^{c} \frac{u_i'(z)}{u_i(z)}$.

$$u_1'(z) = \frac{1}{2}z^{-1/2} + \frac{1}{2} + \frac{15}{16}z^{1/2} + 2z + \frac{1155}{256}z^{3/2} + 18z^5 + \cdots,$$
$$u_2'(z) = -\frac{1}{2}z^{-1/2} + \frac{1}{2} - \frac{15}{16}z^{1/2} + 2z - \frac{1155}{256}z^{3/2} + 18z^5 + \cdots.$$

Via the formula we can compute the first terms of $B(z)$:

$$B(z) = z\left(\frac{u_1'(z)}{u_1(z)} + \frac{u_2'(z)}{u_2(z)}\right) = 1 + z + 5z^2 + 19z^3 + 85z^4 + 381z^5 + 1751z^6 + \cdots.$$

Part II

Non exact models

During the first part of this thesis we have studied exact models, i.e. combinatorial objects from whom we can extract exact counting formulas and know their entire generating function. However, we do not have so much information for all the combinatorial classes and we can not find exact enumerative results for them. We call these models *non exact models.* To find an example it suffices to pay attention to another type of walks in the quadrangular lattice $\mathbb{Z}^2$ that we have not addressed in Part I: they are the *self-avoiding walks.* In this second part, we are going to focus on these walks and even though we are not going to be able to find exact counting formulas we will try to present some asymptotic results.

In Chapter 4 we are going to compare the asymptotic behaviour of self-avoiding walks in the d-dimensional lattice $\mathbb{Z}^d$ with the ones of other related objects such as *bridges* and *polygons*, which are particular types of self-avoiding walks. However, we are not going to get any asymptotic exact result for them. Lastly, in Chapter 5 we are going to consider self-avoiding walks in the hexagonal lattice $\mathbb{H}$. Due to some interesting properties of this lattice we are going to be able to find an exact result for the asymptotic behaviour of these type of walks and we are going to proof an important theorem published in the journal Annals of Mathematics, which states that the *connective constant* on the hexagonal lattice equals $\sqrt{2+\sqrt{2}}$.

Chapter 4

Self-avoiding walks (SAWs)

Self-avoiding walks (also called SAWs) are difficult to study and many of the important problems remain unsolved. They are a discrete model of which we are not able to obtain exact formulas of it. We are going to focus on one of the natural questions about self-avoiding walks developed in [2] that concern the asymptotic behaviour as the length of the path tend to infinity.

4.1 Fekete's subadditive lemma

Our objective in this section is to prove Fekete's Lemma, which gives us a result about the convergence of a sequence that presents *subadditivity*, and an analogue result for *sub-multiplicative* sequences. This last result, that is mentioned in a corollary, will be so useful with self-avoiding walks. We are going to follow the proofs from [9].
First of all, we define the *subadditive* and *sub-multiplicative* sequences:

Definition 4.1.1. *A sequence* $\{a_n\}_{n\geq 1}$ *is called* subadditive *if it satisfies the inequality*

$$a_{n+m} \leq a_n + a_m \quad \forall n, m \geq 1.$$

A sequence $\{a_n\}_{n\geq 1}$ *is called* sub-multiplicative *if it satisfies the inequality*

$$a_{n+m} \leq a_n a_m \quad \forall n, m \geq 1.$$

Note that the positivity of the sequence, i.e. $a_n > 0\ \forall n \geq 1$*, is a necessary condition for a sub-multiplicative sequence.*

To prove Fekete's Lemma (Lemma 4.1.2) we first need an auxiliary lemma:

Lemma 4.1.1. *Let $\{a_n\}_{n\geq 1}$ be a subadditive sequence. Then for any arbitrary fixed natural number n, $a_{kn} \leq ka_n$ $\forall k \geq 1$.*

Proof. Fix a natural number $n \geq 1$. We will prove it by induction on k. We first see some base cases:

- $\underline{k=1}$: $a_n \leq 1 \cdot a_n$
- $\underline{k=2}$: $a_{2n} \leq a_n + a_n = 2a_n$, where the first inequality is due to the subadditivity of the sequence.

We assume the claimed statement to be true for a particular value of k, i.e. $a_{kn} \leq ka_n$. It remains to see if the inequality is satisfied with $k+1$:

$$a_{(k+1)n} = a_{kn+n} \leq a_{kn} + a_n \leq ka_n + a_n = (k+1)n,$$

where in the first inequality we have used the fact that $\{a_n\}$ is a subadditive sequence and in the second one we have applied the induction hypotesis. □

Now, we can announce and prove Fekete's Lemma:

Lemma 4.1.2 (Fekete's lemma)**.** *Let $\{a_n\}_{n\geq 1}$ be a subadditive sequence. Then, the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to the infimum $\inf_{n\geq 1} \frac{a_n}{n}$.*

Proof. To prove this lemma we will use Lemma 4.1.1. We choose arbitrary $n, m \geq 1$ such that $n > m$. By Euclid's Division Lemma, we can write $n = km + r$, with $0 \leq r \leq m-1$. Using the property of subbaditivity of the sequence, it is clear that

$$\frac{a_n}{n} = \frac{a_{km+r}}{km+r} \leq \frac{a_{km}+a_r}{km+r} = \frac{a_{km}}{km+r} + \frac{a_r}{\underbrace{km+r}_{=n}} \leq \frac{a_{km}}{km} + \frac{a_r}{n} \leq \frac{\cancel{k}a_m}{\cancel{k}m} + \frac{a_r}{n},$$

where the last inequality is a consequence of Lemma 4.1.1. So, we have

$$\frac{a_n}{n} \leq \frac{a_m}{m} + \frac{a_r}{n}. \tag{4.1.1}$$

Now, we fix m and define $M := \max\{a_s \mid 0 \leq s \leq m-1\}$. Therefore $a_r \leq M$ $\forall n \geq 1$. If we apply this result to (4.1.1), we obtain

$$\frac{a_n}{n} = \frac{a_m}{m} + \frac{M}{n}.$$

Taking limits,

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \limsup_{n\to\infty} \left(\frac{a_m}{m} + \frac{M}{n} \right) \overset{\text{m fixed}}{\Longrightarrow} \limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m} + \limsup_{n\to\infty} \frac{M}{n} = \frac{a_m}{m}.$$

As the previous result holds for all values of m

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \inf_{m\geq 1} \frac{a_m}{m}. \tag{4.1.2}$$

Now we consider $\inf\limits_{n\geq 1} \frac{a_n}{n}$. We are going to prove the following claim:
Claim:

$$\inf_{n\geq 1} \frac{a_n}{n} \leq \liminf_{n\to\infty} \frac{a_n}{n}. \tag{4.1.3}$$

Proof of the claim. Let us assume the contrary and we will get a contradiction.
We assume

$$\inf_{n\geq 1} \frac{a_n}{n} > \liminf_{n\to\infty} \frac{a_n}{n} \Longrightarrow \inf_{n\geq 1} \frac{a_n}{n} - \liminf_{n\to\infty} \frac{a_n}{n} := h > 0.$$

By definition of limit inferior of a sequence $\{x_n\}_{n\geq 1}$, if α is greater than the limit inferior there are infinitely many x_n less than α. Hence, in our case it implies that there are infinitely many terms of the sequence $\{\frac{a_n}{n}\}_{n\geq 1}$ less than $\liminf_{n\to\infty} \frac{a_n}{n} + h$. Hence there are infinite terms less than $\inf_{n\geq 1} \frac{a_n}{n}$, which is a contradiction. ■

Changing the notation from n to m in the expression $\inf_{m\geq 1} \frac{a_m}{m}$ and combining (4.1.2) and (4.1.3),

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \inf_{m\geq 1} \frac{a_m}{m} \leq \liminf_{n\to\infty} \frac{a_n}{n}.$$

But, $\liminf\limits_{n\to\infty} \frac{a_n}{n} \leq \limsup\limits_{n\to\infty} \frac{a_n}{n}$ so:

$$\liminf_{n\to\infty} \frac{a_n}{n} = \limsup_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}.$$

This means that the limit $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists (since the limit superior and inferior agree) and it is equal to $\inf\limits_{n\geq 1} \frac{a_n}{n}$, as we wanted to see. □

As an application of Fekete's Lemma, we have a similar result for sub-multiplicative sequences, which is announced in the next corollary:

Corollary 4.1.1. *Let $\{a_n\}_{n\geq 1}$ be a postive sub-multiplicative sequence. Then $\lim\limits_{n\to\infty} a_n^{\frac{1}{n}}$ exists.*

Proof. We define a new sequence $\{b_n\}_{n\geq 1}$ such that $b_n := \log(a_n)$ $\forall n \geq 1$. Then, using the sub-multiplicative property of $\{a_n\}_{n\geq 1}$,

$$b_{n+m} = \log(a_{n+m}) \leq \log(a_n a_m) = \log(a_n) + \log(a_m) = b_n + b_m.$$

Clearly, $\{b_n\}_{n\geq 1}$ is a subadditive sequence and we can use the Fekete's Lemma to conclude that $\lim_{n\to\infty} \frac{b_n}{n}$ exists. Then, $\lim_{n\to\infty} \frac{\log(a_n)}{n} = \lim_{n\to\infty} \log a_n^{1/n}$ exists as well.
Due to the continuity of the exponential function $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) := e^x$: if $\lim_{n\to\infty} \log a_n^{1/n}$ exists, then it also exists $\lim_{n\to\infty} e^{\log a_n^{1/n}}$, which is equal to $\lim_{n\to\infty} a_n^{1/n}$. Hence, we have proved the corollary. □

4.2 Bridges and polygons

This section focus on several results for self-avoiding walks on the d-dimensional lattice $\mathbb{Z}^d$. In this section we are going to relate its asymptotic behaviour with the one of *bridges* and *polygons*.
First of all we need to give a rigorous definition for self-avoiding walks and we also need to define other concepts.

Let $\Omega \subset \mathbb{Z}^d$ be the set of possible steps. In this notes we are going to consider the *nearest-neighbour model*: $\Omega := \{x \in \mathbb{Z}^d : \|x\|_1 = 1\}$.

Definition 4.2.1. *An* n-step walk *is a sequence* $\omega = (\omega(0), \omega(1), \ldots, \omega(n))$ *where* $\omega(i)$ *is a point of* $\mathbb{Z}^d$ *for* $i = 0, \ldots, n$ *and* $\omega(j) - \omega(j-1) \in \Omega$ *for* $j = 1, \ldots, n$. *We can define the sets*

$$\mathcal{W}_n(0, x) = \{\omega : \omega \textit{ is an n-step walk with } \omega(0) = 0 \textit{ and } \omega(n) = x\}, \;\; \mathcal{W}_n = \bigcup_{x\in\mathbb{Z}^d} \mathcal{W}_n(0, x).$$

The *strictly self-avoiding walks* (or also called simply *self-avoiding walks*) are the main subject of this section. They are random paths on $\mathbb{Z}^d$ defined as follows:

Definition 4.2.2. *Given an* n*-step walk* $\omega \in \mathcal{W}_n$, *and integers* i, j *with* $0 \leq i < j \leq n$, *let*

$$U_{ij} = U_{ij}(\omega) = -\mathbb{1}_{\{\omega(i)=\omega(j)\}} = \begin{cases} -1 & if\ \omega(i) = \omega(j), \\ 0 & if\ \omega(i) \neq \omega(j). \end{cases}$$

Fix $\lambda \in [0, 1]$. *We assign to each path* $\omega \in \mathcal{W}_n$ *the weighting factor*

$$\prod_{0\leq i<j\leq n} (1 + \lambda U_{ij}(\omega)). \tag{4.2.1}$$

The choice $\lambda = 0$ is the case of simple random walks, where all walks in $\mathcal{W}_n$ have the same weight. For $\lambda \in (0, 1)$, self-intersections are penalised but not forbidden, and this model is called *weakly* *self-avoiding walk*. Finally, with $\lambda = 1$ returning to a previously visited site is forbidden, and this model is the *self-avoiding walk (SAW)*. In Figure 4.1 you can see the first numbers of the sequence c_n of the number of n-step self-avoiding walks on a square lattice and its corresponding walks.

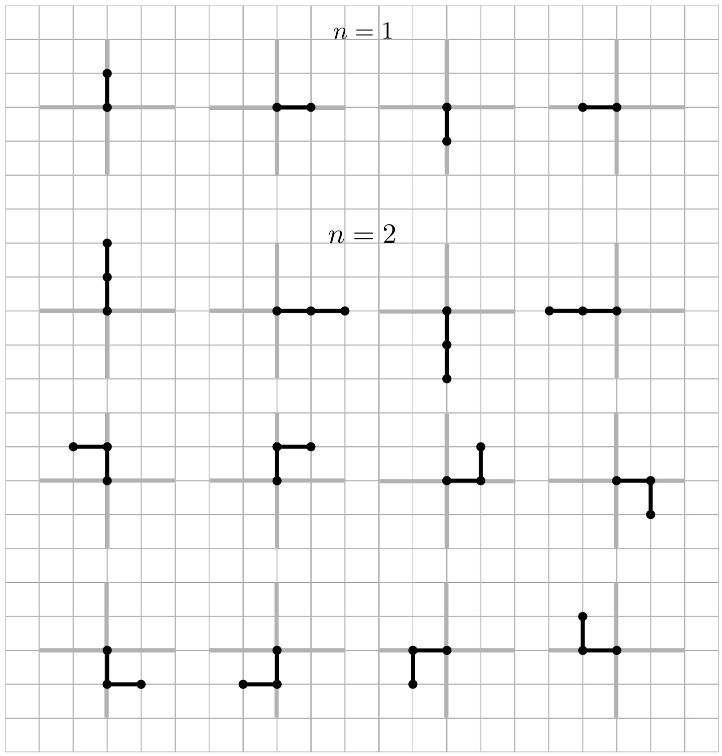


Figure 4.1: $c_0 = 1$ (the empty walk), $c_1 = 4,\ c_2 = 12,\ c_3 = 36,\ c_4 = 100, \ldots$ and the sequence grows rapidly with n.

Observe that an n-step walk ω is a self-avoiding walk if only if (4.2.1) is non 0 for $\lambda = 1$, which means that ω visits each site at most once, and for these walks their weight is equal to 1.
We denote

$$c_n^{(\lambda)}(x) = \sum_{\omega \in \mathcal{W}_n(0,x)} \prod_{0 \le i < j \le n} (1 + \lambda U_{ij}(\omega)), \quad c_n^{(\lambda)} = \sum_{x \in \mathbb{Z}^d} c_n^{(\lambda)}(x).$$

In the case $\lambda = 1$, $c_n^{(1)}(x)$ counts the number of self-avoiding walks of length n ending at x, and $c_n^{(1)}$ counts the number of all n-step self-avoiding walks. In this case we can omit the superscript (1) and we simply write $c_n(x)$ and c_n.

Now, observe that for any non-negative integers n, m,

$$c_{n+m} \le c_n c_m. \tag{4.2.2}$$

This is true as a SAW of $n+m$ steps can be divided into two self-avoiding walks, one of length n and the other with m steps. Conversely, if we concatenate together two different self-avoiding walks, the union will not necessary be another self-avoiding walk ($c_n c_m$ denote the number of possible unions of SAWs of lengths n and m respectively, and they are not all going to be a self-avoiding walk of length $n+m$). See Figure 4.2.

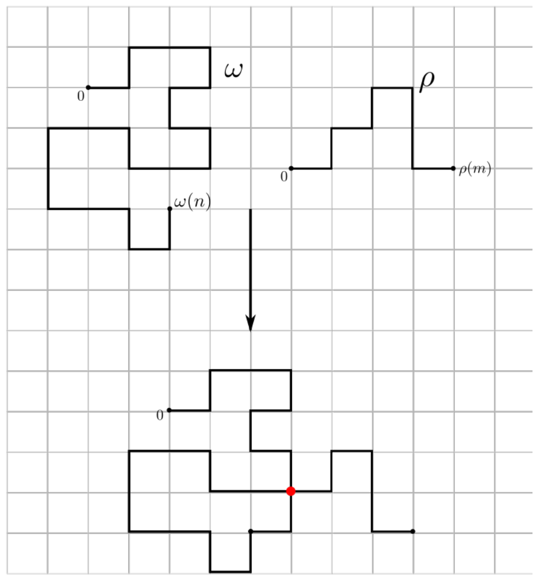


Figure 4.2: The concatenation of two self-avoiding walks ω and ρ is not necessary another self-avoiding walk.

According to Definition 4.1.1, the number of self-avoiding walks of length n, c_n, is a sub-multiplicative sequence. Therefore, Corollary 4.1.1 applies on this sequence and we can assure that $\mu = \lim_{n\to\infty} c_n^{1/n}$ exists. This μ, which depends on d, is called *connective constant.*

We can find some lower and upper bounds for μ depending on d thinking some bounds for the number of self-avoiding walks c_n.

Proposition 4.2.1. *For the nearest-neighbour model, we obtain*

$$d^n \leq c_n \leq 2d(2d-1)^{n-1} \quad \textit{which implies, by definition of } \mu, \quad d \leq \mu \leq 2d-1.$$

Proof. The lower bound for c_n is found counting the number of walks that only make steps moving into positive coordinate directions (for each step there are d options) and the upper bound is obtained by counting the walks that are only restricted to prevent returning to the immediate site that the walk have just visited (the first step is completely free, $2d$ options, and the other $n-1$ remaining ones have one direction forbidden). $\square$

Note, that if we denote by $C(z) = \sum_{n=0}^{\infty} c_n z^n$ the generating function of the number of self-avoiding walks, its radius of convergence is exactly μ^{-1} (by definition).
In particular, as the series has positive Taylor coefficients, it means that

$$C\left(\mu^{-1}\right) = \infty. \tag{4.2.3}$$

4.2.1 Bridges

Throughout this section, we only consider the nearest-neighbour (strictly) self-avoiding walks on the d-dimensional lattice $\mathbb{Z}^d$. We will introduce *bridges*, a particular class of SAW, and we will show that its number grow with the same exponential rate as the number of self-avoiding walks, namely as μ^n. To show this result we will need some other theorems and new concepts.

For a self-avoiding walk ω, we denote by $\omega_1(i)$ the first spatial coordinate of $\omega(i)$.
First of all we define the main object of this section:

Definition 4.2.3. *An n-step* bridge *is an n-step SAW ω such that*

$$\omega_1(0) < \omega_1(i) \leq \omega_1(n) \qquad \textit{for } i = 1, \dots, n.$$

We denote by b_n the number of bridges of length n with $\omega(0) = 0$ for $n \geq 1$ and $b_0 = 1$ (the empty walk). See Figure 4.3 for an example.

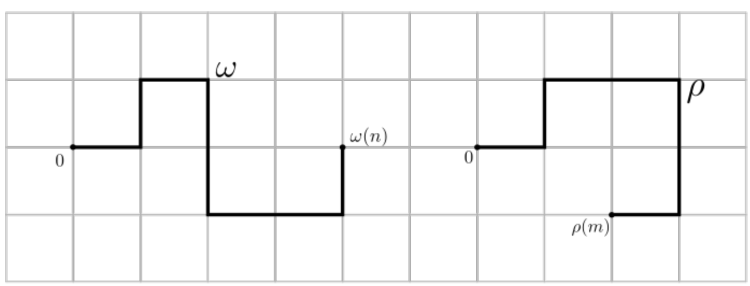


Figure 4.3: ω is a bridge of length $n = 8$ whereas ρ is not a bridge.

While the number of self-avoiding walks is a sub-multiplicative sequence (remember (4.2.2)), the number of bridges is a super-multiplicative sequence:

$$b_{n+m} \geq b_n b_m \qquad \forall\, n, m \in \mathbb{N} \tag{4.2.4}$$

This property is due to the fact that any bridges of length n and m can be concatenated to form another bridge of length $n+m$, whereas not all the bridges of $n+m$ steps can be divided into two different bridges of length n and m respectively, for any integers $n, m \geq 0$.

We can redo the proof of Corollary 4.1.1 with the sequence $-\log b_n$ and we obtain the existence of the bridge growth constant μ_{Bridge} defined by

$$\mu_{\text{Bridge}} = \lim_{n\to\infty} b_n^{1/n} = \sup_{n\geq 1} b_n^{1/n}.$$

As a bridge is a particular SAW we also have $\mu_{\text{Bridge}} \leq \mu$ and we can conclude

$$b_n \leq \mu_{\text{Bridge}}^n \leq \mu^n. \tag{4.2.5}$$

The main objective of this section is to prove the equality

$$\mu_{\text{Bridge}} = \mu \tag{4.2.6}$$

In order to achieve it we need some more new concepts to be defined.

Definition 4.2.4. *An n-step* half-space walk *is an n-step SAW ω with*

$$\omega_1(0) < \omega_1(i) \qquad \text{for } i = 1, \ldots, n.$$

For $n \geq 1, h_n$ denote the number of n-step half-space walks with $\omega(0) = 0$ and we define $h_0 = 1$.

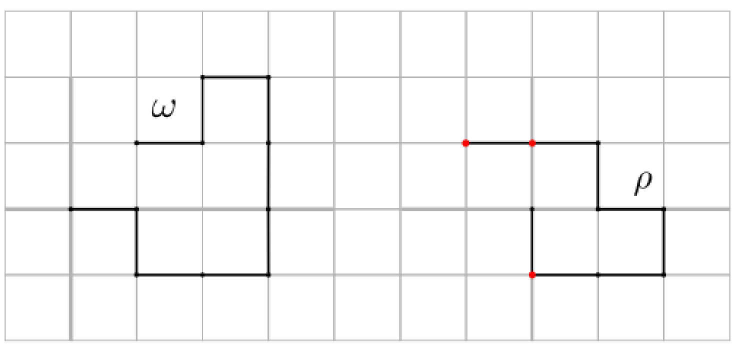


Figure 4.4: ω is a 10-step half-space walk whereas ρ is not a half space walk because there are three points (in red) with $\rho_1(i) \leq \rho_1(0)$.

Definition 4.2.5. *The* span *of an n-step SAW ω is*

$$\max_{0\leq i\leq n} \omega_1(i) - \min_{0\leq i\leq n} \omega_1(i).$$

See Figure 4.5 for an example. We denote by $b_{n,A}$ the number of n-step bridges with span A.

We will also use the following result on integer partitions from Hardy and Ramanujan in [6], which

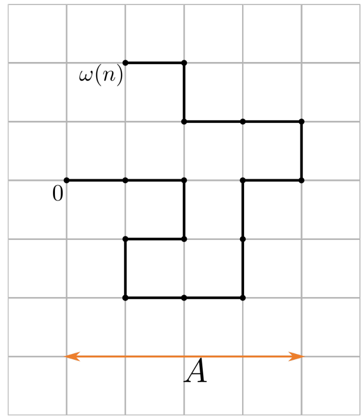


Figure 4.5: A half-space walk with span $A = 4$.

we just state it.

Theorem 4.2.1. *For an integer $A > 1$, let $P_D(A)$ denote the number of ways of writing $A = A_1 + \cdots + A_k$ with $A_1 > \cdots > A_k > 1$, for any $k \geq 1$. Then, as $A \to \infty$*

$$\log P_D(A) \sim \pi \left(\frac{A}{3}\right)^{1/2}.$$

Now, we are able to announce and prove several results that will lead us to the equality (4.2.6) that we want to see. The next proposition gives us a relation between the number of n-step bridges b_n and the number of n-step half space walks h_n.

Proposition 4.2.2. *For all $n \geq 1$, $h_n \leq P_D(n) b_n$.*

Proof. Let ω be a half-space walk with n steps. We set $n_0 = 0$ and inductively we define

$$A_{i+1} = \max_{j>n_i}(-1)^i \left(\omega_1(j) - \omega_1(n_i)\right), \quad n_{i+1} = \max\{j > n_i : (-1)^i \left(\omega_1(j) - \omega_1(n_i)\right) = A_{i+1}\}.$$

In words, $j = n_1$ maximises $\omega_1(j)$, $j = n_2$ minimises $\omega_1(j)$ for $j > n_1$, $j = n_3$ maximises $\omega_1(j)$ for $j > n_2$, and so on in an alternating pattern. In addition, the n_i are chosen to be the last time these extremes are reached. Then, $A_1 = \omega_1(n_1) - \omega_1(n_0)$, $A_2 = \omega_1(n_1) - \omega_1(n_2)$, and so on (see Figure 4.6). Since the n_i are chosen maximal, we have $A_{i+1} < A_i$.

This procedure stops at some step $K \geq 1$, when $n_K = n$. Note that $K = 1$ if and only if ω is a bridge, and in that case $A_1 = A$, the span of ω. $(*)$

Let $H_n[a_1, \ldots, a_k]$ denote the subset of n-step half-space walks with $K = k, A_i = a_i$ for $i = 1, \ldots, k$. and $h_n[a_1, \ldots, a_k]$ the number of elements of this subset .We observe that

$$h_n[a_1, a_2, a_3, \ldots, a_k] \leq h_n[a_1 + a_2, a_3, \ldots, a_k]. \tag{4.2.7}$$

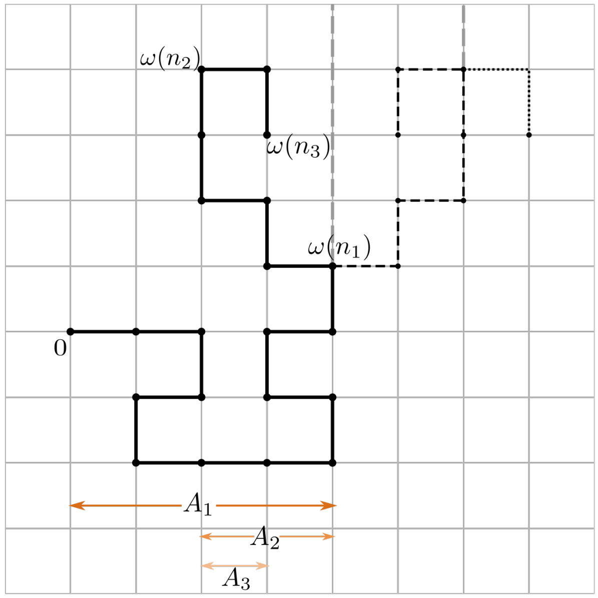


Figure 4.6: A half-space walk ω is decomposed into bridges, which are reflected to form a single bridge.

To see this inequality we take a half-space walk $\omega \in H_n[a_1, a_2, a_3, \ldots, a_k]$ and we reflect the part of the walk $(\omega(j))_{j \geq n_1}$ across the line $\omega_1 = A_1$; see Figure 4.6. We obtain a new half-space walk $\tilde{\omega} \in H_n[a_1+a_2, a_3, \ldots, a_k]$. Therefore, it is clear that given a half-space walk of $H_n[a_1, a_2, a_3, \ldots, a_k]$ we also have a half-space walk belonging to $H_n[a_1 + a_2, a_3, \ldots, a_k]$. Hence, the inequality is true. By induction, repeating the inequality (4.2.7) we obtain:

$$h_n[a_1, \ldots, a_k] \leq h_n[a_1 + \ldots + a_k] = b_{n, a_1 + \ldots + a_k},$$

where the last equality is due to the previous observation $(*)$. Using this, we have

$$h_n = \sum_{k \geq 1} \sum_{a_1 > \cdots > a_k > 0} h_n[a_1, \ldots, a_k] \leq \sum_{k \geq 1} \sum_{a_1 > \cdots > a_k > 0} b_{n, a_1 + \cdots + a_k} = \sum_{A=1}^{n} P_D(A) b_{n,A},$$

where we obtain the last equality realising that the double summation of the term $A = a_1 + \cdots + a_k$ is the number of ways of writing A like a partition as the described in Theorem 4.2.1, and here A refers to the span of the bridge and its maximum value is the length n.

Finally, bounding $P_D(A)$ by $P_D(n)$ we obtain $h_n \leq P_D(n) \sum_{A=1}^{n} b_{n,A} = P_D(n) b_n$, as we wanted to prove. $\square$

We are ready now to announce the theorem that will allow us to see the equality (4.2.6) as a consequence of it.

Theorem 4.2.2. *Fix $B > \pi\left(\frac{2}{3}\right)^{\frac{1}{2}}$. Then there exists $n_0 = n_0(B)$ independent of the dimension $d \geq 2$ such that*

$$c_n \leq b_{n+1}e^{B\sqrt{n}} \leq \mu^{n+1}e^{B\sqrt{n}}, \qquad \text{for } n \geq n_0.$$

Proof. We will first prove the next claim:

Claim:

$$c_n \leq \sum_{m=0}^{n} h_{n-m}h_{m+1}. \tag{4.2.8}$$

Proof of the claim. Given an n-step SAW ω, we define

$$x_1 = \min_{0\leq i\leq n} \omega_1(i), \qquad m = \max\{i : \omega_1(i) = x_1\}, \tag{4.2.9}$$

and we denote by e_1 the unit vector in the first coordinate direction of $\mathbb{Z}^d$.

Then, the walk $(\omega(m), \omega(m+1), \ldots, \omega(n))$ is an $(n-m)$-step half-space walk. We can assure that it will be a half-space walk due to the choice of m: if $\omega_1(j) \leq \omega_1(m)$ for any $j > m$, then m can not be the $\max\{i : \omega_1(i) = x_1\}$. Similarly, we see that the walk $(\omega(m)-e_1, \omega(m), \omega(m-1), \ldots, \omega(1), \omega(0))$ is an $(m+1)$-step half-space walk; see Figure 4.7.

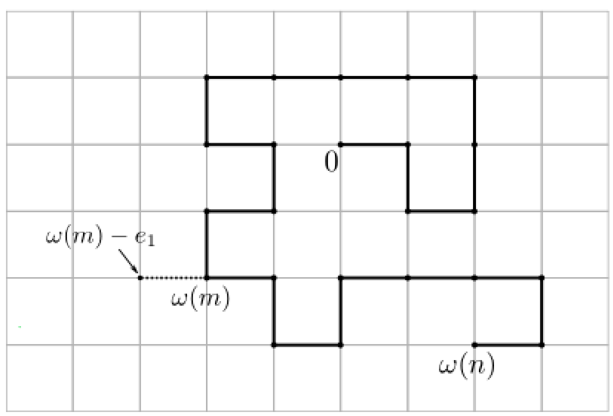


Figure 4.7: A self-avoiding walk decomposed into two half-space walks.

We have just seen that each self-avoiding walk has a unique decomposition into two half-space walks, one of length $n-m$ and the other one of length $m+1$, for a specific m, the one that satisfies (4.2.9). Hence the inequality of the claim is proved. ∎

Now, we apply Proposition 4.2.2 to the Claim (4.2.8) to get

$$c_n \leq \sum_{m=0}^{n} P_D(n-m)P_D(m+1)b_{n-m}b_{m+1} \leq b_{n+1} \sum_{m=0}^{n} P_D(n-m)P_D(m+1), \tag{4.2.10}$$

where in the last inequality we have used the super-multiplicative property (4.2.4) of the sequence $(b_n)_n$: $b_{n-m}b_{m+1} \leq b_{(n-m)+(m+1)} = b_{n+1}$.
To finalise the proof, we fix B' such that $B > B' > \pi\left(\frac{2}{3}\right)^{\frac{1}{2}}$. By Theorem 4.2.1 there is a constant $K > 0$ such that $P_D(A) \leq Ke^{\pi\left(\frac{A}{3}\right)^{\frac{1}{2}}} = Ke^{\pi\left(\frac{2}{3}\right)^{\frac{1}{2}}\left(\frac{A}{2}\right)^{\frac{1}{2}}} \leq Ke^{\pi B'\left(\frac{A}{2}\right)^{\frac{1}{2}}}$. Consequently,

$$P_D(n-m)P_D(m+1) \leq K^2 e^{B'\left[\left(\frac{n-m}{2}\right)^{\frac{1}{2}}+\left(\frac{m+1}{2}\right)^{\frac{1}{2}}\right]}. \tag{4.2.11}$$

If we use the obvious inequality $x^{\frac{1}{2}} + y^{\frac{1}{2}} \leq (2x+2y)^{\frac{1}{2}}$ with $x = \frac{n-m}{2}$ and $y = \frac{m+1}{2}$ we obtain $\left(\frac{n-m}{2}\right)^{\frac{1}{2}} + \left(\frac{m+1}{2}\right)^{\frac{1}{2}} \leq (n+1)^{\frac{1}{2}}$. Putting it into (4.2.11) and combining it with (4.2.10), gives us:

$$c_n \leq b_{n+1}(n+1)K^2 e^{B'\sqrt{n+1}} \leq b_{n+1}e^{B\sqrt{n}}, \qquad \text{if } n \geq n_0(B).$$

We have obtained the first inequality of the theorem. The second one is immediate by (4.2.5) $(b_{n+1} \leq \mu^{n+1})$. $\square$

As a consequence of this theorem, we obtain the equality that we wanted to see:

Corollary 4.2.1. *For $n \geq n_0(B)$,*

$$b_n \geq c_{n-1}e^{-B\sqrt{n-1}} \geq \mu^{n-1}e^{-B\sqrt{n-1}}.$$

In particular, $b_n^{1/n} \to \mu$ and so $\mu_{Bridge} = \mu$.

Proof. The chain of inequalities is obtained by changing n for $n-1$ in Theorem (4.2.2) and isolating b_n. Then, raising all the expression to the power of $\frac{1}{n}$ and using (4.2.5) we obtain

$$\mu \geq b_n^{1/n} \geq \mu^{\frac{n-1}{n}} e^{-B\frac{\sqrt{n-1}}{n}}.$$

Taking the limit when n tends to infinity in this expression we have

$$\mu \geq \lim_{n\to\infty} b_n^{1/n} \geq \mu.$$

So, $\mu_{\text{Bridge}} = \lim_{n\to\infty} b_n^{1/n} = \mu$. $\square$

Finally, to conclude this section about bridges, we are going to see another property regarding their generating functions as a consequence of Theorem 4.2.2.

Corollary 4.2.2. *If* $B(z)=\sum_{n=0}^{\infty} b_n z^n$ *is the generating function of bridges, then*

$$C(z)\leq \frac{1}{z}e^{2(B(z)-1)}.$$

In particular $B\left(\mu^{-1}\right)=\infty$.

Proof. In the proof of Proposition 4.2.2, we decomposed a half-space walk into subwalks on $[n_{i-1},n_i]$ for $i=1,\dots,K$. Note that in fact these subwalks were bridges of span A_i. With this information, we conclude that

$$h_n\leq \sum_{k=1}^{\infty}\sum_{A_1>\dots>A_k}\sum_{0=n_0<n_1<\dots<n_k}\prod_{i=1}^{k} b_{n_i-n_i-1,A_i}$$

The first sum fixes K, the second one fixes the spans A_i of every bridge, the third one fixes the points n_{i-1} and n_i where every bridge starts and the product means the concatenation of all these bridges.

Multiplying the last expression by z^n and summing for all values of n from 0 to infinite,

$$\sum_{n=0}^{\infty} h_n z^n \leq \prod_{A=1}^{\infty}\left(1+\sum_{m=1}^{\infty} b_{m,A}z^m\right).$$

If now we use the well known inequality $1+z\leq e^z$, we obtain for all real z

$$\sum_{n=0}^{\infty} h_n z^n \leq \prod_{A=1}^{\infty}\exp\left(\sum_{m=1}^{\infty} b_{m,A}z^m\right)\leq \exp\left(\sum_{A=1}^{\infty}\sum_{m=1}^{\infty} b_{m,A}z^m\right)=e^{B(z)-1},$$

where the term -1 is because the summation does not start with $m=0$.

Now, we use (4.2.8) and we obtain

$$\begin{aligned}C(z)=\sum_{n=0}^{\infty} c_n z^n &\leq \frac{1}{z}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n-m}z^{n-m}h_{m+1}z^{m+1}\\ &=\frac{1}{z}\left(\sum_{n=0}^{\infty} h_n z^n\right)\left(\sum_{m=0}^{\infty} n_m z^m\right)\\ &\leq \frac{1}{z}e^{2(B(z)-1)},\end{aligned}$$

as we wanted to see.

Once we have proved this inequality the fact that $B\left(\mu^{-1}\right)=\infty$ is obvious, because from (4.2.3) at $z=\mu^{-1}$, $C(z)$ diverges. □

4.2.2 Polygons

In this last part about self-avoiding walks in $\mathbb{Z}^d$, we are going to see the analogous result of the asymptotic behaviour that we have seen for bridges for another particular type of walks: *self-avoiding polygons*. Although all of the results about polygons that we are going to see are applicable for any $d \geq 2$, we are going to study most of the results on the bidimensional lattice.

First of all, let us define polygons.

Definition 4.2.6. *A 2n-step* self-avoiding return *is a walk* $\omega \in \mathcal{W}_{2n}$ *with* $\omega(2n) = \omega(0) = 0$ *and* $\omega(i) \neq \omega(j)$ *for every pair* $i \neq j$ *different from* $0, 2n$.

Observation. Note that an even number of steps is necessary for the condition $\omega(2n) = \omega(0) = 0$: for every step that the walk moves away from the origin, another one is needed to return to the origin. Also we must have $n \geq 2$ so that the walk can be closed and verify $\omega(2n) = \omega(0)$.

Definition 4.2.7. *A* self-avoiding polygon *is a self-avoiding return with both the orientation and the location of the origin forgotten.*

By definition, counting the number of self-avoiding polygons is equivalent to count the number of self-avoiding returns up to orientation and translation invariance.
If we denote by q_{2n} the number of self-avoiding polygons we have

$$q_{2n} = \frac{2dc_{2n-1}(e_1)}{2 \cdot 2n} \qquad \text{for } n \geq 2, \tag{4.2.12}$$

where $e_1 = (1, 0, \ldots, 0)$ is the first standard basis vector and $c_{2n-1}(x)$ denote the number of self-avoiding walks of length $2n - 1$ finishing at position x.
Notice that the 2 of the denominator cancels the choice of orientation, whereas the $2n$ cancels the choice of origin in the polygon.

Next, let us see some simple properties of these polygons that will be helpful later. For general dimensions $d \geq 2$ similar arguments can be done but form this point we are only going to consider the bidimensional lattice ($d = 2$).

First, notice that any two self-avoiding polygons can be concatenated to form a larger self-avoiding polygon: for the first polygon ω of length $2n$ we choose one edge $\omega(i) - \omega(j)$ with $\omega_1(i) = \omega_1(j) = \max_{0 \leq k \leq 2n} \omega_1(k)$, and for the second one $\bar{\omega}$ of length $2m$ we pick an edge $\bar{\omega}(s) - \bar{\omega}(t)$, with $\bar{\omega}_1(s) = \bar{\omega}_1(t) = \min_{0 \leq k \leq 2n} \bar{\omega}_1(k)$. As there is translation invariance, we can put $i = 0$ and $j = 2n - 1$ and in the same way $s = 0$ and $t = 2m - 1$. Then, we join these two edges and create the new self-avoiding polygon $\tilde{\omega}$ of length $2(n + m)$ defined as it follows: with $\tilde{\omega}(k) = \omega(k)$ for $k = 0, \ldots, j = 2n - 1$, $\tilde{\omega}(2n + k) = \bar{\omega}(s + k) = \bar{\omega}(k)$ for $k = 0, \ldots, t = 2m - 1$ and $\tilde{\omega}(2n + 2m) = \omega(i) = \omega(0)$; see Figure 4.8.

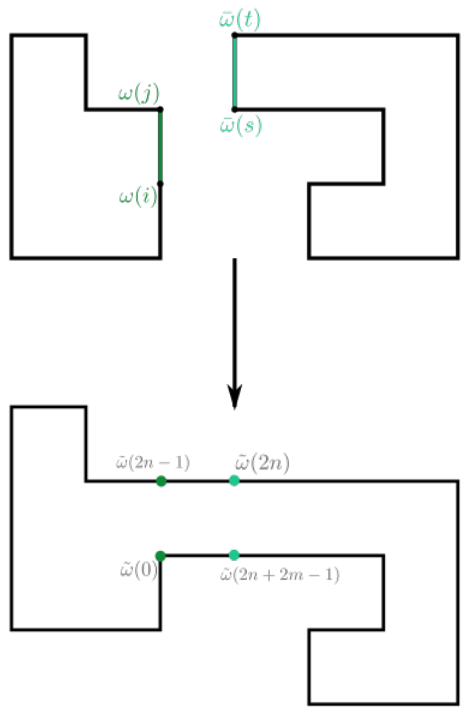


Figure 4.8: Concatenation of a 10-step polygon and a 14-step polygon to create a polygon of length 24.

So, any two self-avoiding polygons of length $2n$ and $2m$ can be concatenated to form another self-avoiding polygon of $2(n+m)$ steps. Therefore, the next inequality holds for $d = 2$ and any $n, m \geq 2$:

$$q_{2n}q_{2m} \leq q_{2(n+m)}. \tag{4.2.13}$$

If we set $q_2 = 1$ (the void walk) and verify that $q_{2n} \leq q_{2n+2}$ then (4.2.13) holds for every $n, m \geq 1$. Note that the inequality $q_{2n} \leq q_{2n+2}$ is true because for any given polygon belonging to q_{2n} we can create a polygon of $2n+2$ steps just modifying one of its corners in the way depicted in Figure 4.9.

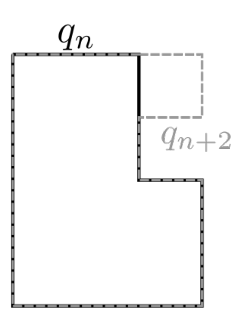


Figure 4.9: Construction of a polygon belonging to q_{2n+2} (in grey) from a given $2n$-step polygon (in black).

We define $\mu_{\text{Polygon}} = \lim_{n\to\infty} q_{2n}^{1/2n}$. If we take out one edge from a self-avoiding polygon we obtain a SAW, so $q_{2n} \leq c_{2n-1}$. Taking limits:

$$\mu_{\text{Polygon}} \leq \mu, \qquad q_{2n} \leq \mu_{\text{Polygon}}^{2n} \leq \mu^{2n} \quad \text{for all } n \geq 2. \tag{4.2.14}$$

Remember that our goal is to see $\mu_{\text{Polygon}} = \mu$. The following theorem will bring us closer to our objective.

Theorem 4.2.3. *In the bidimensional lattice $\mathbb{Z}^2$ there is a constant K such that, for all $n \geq 1$,*

$$c_{2n+1}(e_1) \geq \frac{K}{n^4} b_n^2. \tag{4.2.15}$$

Proof. First of all we show the next inequality.
Claim:

$$\sum_{x\in\mathbb{Z}^2} b_n(x)^2 \leq 4(n+1)^2 c_{2n+1}(e_1),$$

where $b_n(x)$ denotes the number of n-step bridges ending at x.
Proof of the claim. We are going to prove it geometrically. We are going to see that given two different n-step bridges ω and v starting at 0 and ending at x, we can create a self-avoiding walk ρ of length $2n+1$ finishing at e_1.

We start tracing a straight line r that connects 0 with x and we denote by $\mathbf{v}$ a non-zero vector with origin at 0 and perpendicular direction to the line r that we have drawn in such a way that $\vec{0x}$ and $\mathbf{r}$ make a positive base of $\mathbb{R}^2$. Now, we translate the line r with the direction and sense of $\mathbf{v}$ and we stop when the line does not cross any more points of the bridge ω. We denote by M the point $\omega(i)$ with the smallest index i which is last crossed by the line. We do the same with the other bridge v but now we translate the line with the direction of $\mathbf{v}$ but with its contrary sense. We denote by m the point $v(j)$ with the smallest index j which is last crossed by the line. See Figure 4.10.

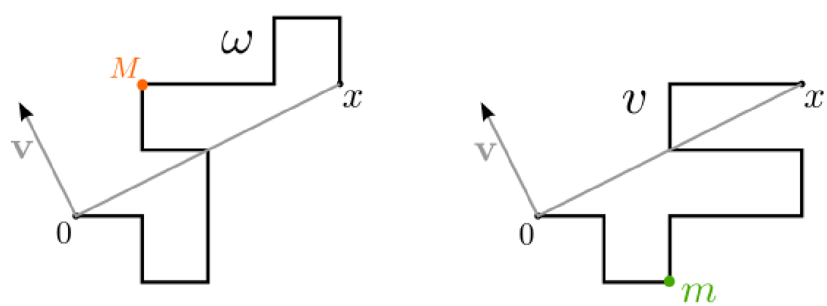


Figure 4.10: Two n-step bridges ω and v with vector the $\mathbf{v}$ and points M and m marked.

Next step consists in creating two new walks $\bar{\omega}$ and $\bar{v}$. To create $\bar{\omega}$ we split ω into two pieces before and after M and interchange them. We do the same for v, splitting the bridge before and after m. Note that in these new walks if we trace a line connecting the origin and the end of each walk respectively, each walk is contained in one side of the line; see Figure 4.11.

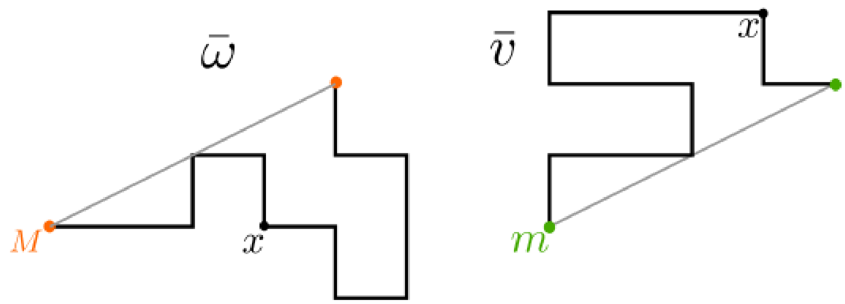


Figure 4.11: The new walks $\bar{\omega}$ and $\bar{v}$, both contained in one side of the grey line that connects the origin and end point of each walk.

Finally, we create the new self-avoiding walk ρ joining $\bar{\omega}$ and $\bar{v}$. To join them we put an extra edge or step $(0,1)$ between the end of both walks. The result is a self-avoiding walk with $2n+1$ steps starting at the origin and ending at $e_1 = (1,0)$ or any rotation of this vector and with two marked points: the initials $\omega(n)$ and $v(n)$. See Figure 4.12.

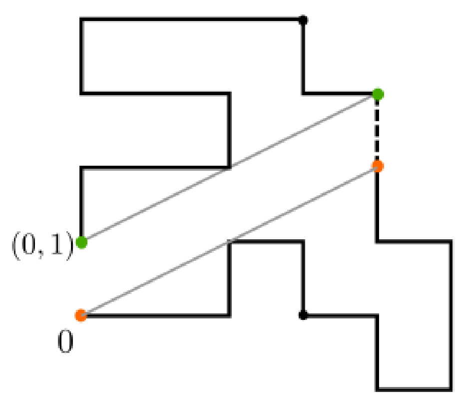


Figure 4.12: The resulting $(2n+1)$-step self-avoiding walk finishing at $(0,1)$, a rotation of the vector e_1, with two marked points in black.

The inequality of the claim is already proved, because the left side of the inequality represents the concatenation of two n-step bridges ending at x and the right side counts the number of $(2n+1)$-step self-avoiding walks ending at e_1 with the 4 possible rotations of this vector and with the choice of two different points of the walks (what is counted in the term $(n+1)^2$). ■

To prove the theorem it only remains some inequalities. Using the claim and Cauchy-Schwarz inequality we obtain:

$$b_n^2 = \left(\sum_{x\in\mathbb{Z}^2} b_n(x)\mathbb{1}_{\{b_n(x)\neq 0\}}\right)^2 \overset{(C-S)}{\leq} \sum_{x\in\mathbb{Z}^2} b_n(x)^2 \sum_{x\in\mathbb{Z}^2} \mathbb{1}_{\{b_n(x)\neq 0\}}$$
$$\leq n(2n+1)\sum_{x\in\mathbb{Z}^2} b_n(x)^2 \overset{(claim)}{\leq} 4n(2n+1)(n+1)^2 c_{2n+1}(e_1),$$

where we say that an upper boundary for $\sum_{x\in\mathbb{Z}^2} \mathbb{1}_{\{b_n(x)\neq 0\}}$ (the amount of possible points $x \in \mathbb{Z}^2$ that can be reached with an n-step bridge) is $n(2n+1) = \#\{x \in \mathbb{Z}^2, | \, \|x\|_1 \leq n\}$.
Now, we isolate $c_{2n+1}(e_1)$ and we have that for a constant K

$$c_{2n+1}(e_1) \geq b_n^2 \frac{1}{4n(2n+1)(n+1)^2} = b_n^2 \frac{1}{8n(n+\frac{1}{2})(n+1)^2} \overset{(n\geq 1)}{\geq} b_n^2 \frac{1}{8n2n(2n)^2} = b_n^2 \frac{K}{n^4}.$$

□

Finally, as a consequence of this theorem, we obtain our expected result.

Corollary 4.2.3. *There is a constant $C > 0$ such that*

$$\mu^{2n} e^{-C\sqrt{n}} \leq c_{2n+1}(e_1) \leq (n+1)\mu^{2n+2} \tag{4.2.16}$$

In particular, $\mu_{Polygon} = \mu$.

Proof. We are going to prove both inequalities independently:

$$\mu^{2n} e^{-C\sqrt{n}} \overset{(1)}{\leq} c_{2n+1}(e_1) \overset{(2)}{\leq} (n+1)\mu^{2n+2}$$

Inequality (1): Using Theorem 4.2.2 and Corollary 4.2.1 we have:

$$c_{2n+1}(e_1) \geq \frac{K}{n^4} b_n^2 \geq \frac{K}{n^4}\mu^{2n-2} e^{-2B\sqrt{n-1}} \geq \mu^{2n} e^{-2C\sqrt{n}}, \quad \text{for some constant } C.$$

Inequality (2): From (4.2.12) with $d = 2$ and (4.2.14) we obtain:

$$c_{2n+1}(e_1) \overset{(4.2.12)}{=} q_{2n+2}(n+1) \overset{(4.2.14)}{\leq} (n+1)\mu^{2n+2}.$$

To see $\mu_{Polygon} = \mu$ we first elevate all the expression (4.2.16) to the power of $(2n)^{-1}$:

$$\mu^{\frac{2n}{2n}} e^{-C\frac{\sqrt{n}}{2n}} \leq c_{2n+1}^{1/2n}(e_1) \leq (n+1)^{\frac{1}{2n}} \mu^{\frac{2n+2}{2n}}.$$

Then, $c_{2n+1}^{1/2n}(e_1) = q_{2n+2}^{1/2n}(n+1)^{1/2n}$ and taking limits when n tend to ∞:

$$\mu \leq \mu_{\text{Polygon}} \leq \mu \Longrightarrow \mu_{\text{Polygon}} = \mu.$$

$\square$

Chapter 5

The connective constant on the hexagonal lattice

Until now, we have worked on the lattice $\mathbb{Z}^d$. Whereas we do not have exact results for the asymptotic behaviour of self-avoiding walks on $\mathbb{Z}^d$, we have an important result about it on the 2-dimensional hexagonal lattice $\mathbb{H}$ and we are going to prove it in this section. We are going to prove the following theorem from [4] , an article published in the important mathematical journal Annals of Mathematics. This theorem was a conjecture of B. Nienhuis in 1982 and one of the authors of this paper, Stanislav Smirnov, received the Fields Medal in 2010 partly because of this publication. We will see that we are able to proof such a result thanks to the special "discrete analyticity" property that the hexagonal lattice has, captured in Lemma 5.1.1.

Theorem 5.0.1. *For the hexagonal lattice $\mathbb{H}$,*

$$\mu = \sqrt{2 + \sqrt{2}}.$$

Fist of all we need to define some technical issues about the hexagonal lattice $\mathbb{H}$.
For convenience we are going to consider extended walks starting and ending at *mid-edges*, i.e., centres of edges of $\mathbb{H}$. We denote the set of mid-edges by H. We dispose the hexagonal lattice of mesh size 1 in $\mathbb{C}$ so that there exists an horizontal edge e with mid-edge a being 0.
We now denote by c_n the number of n-step SAWs on the hexagonal lattice $\mathbb{H}$ starting at 0 and $C(z) = \sum_{n=0}^{\infty} c_n z^n$ is the generating function.
We also need to redefine *bridges*.

Definition 5.0.1. *A* bridge *on $\mathbb{H}$ is a SAW which never revisits the vertical line through its starting point, never visits a vertical line to the right of the vertical line through its endpoint, and moreover*

starts and ends at the midpoint of an horizontal edge.
In this section, we use b_n *to denote the number of* n*-step bridges on* $\mathbb{H}$ *which start at* 0 *and* $B(z) = \sum_{n=0}^{\infty} b_n z^n$ *for* $z > 0$.

The first step needed to prove Theorem 5.0.1 is to notice that adapting the arguments that we have used in the previous section to prove Corollary 4.2.1 in $\mathbb{Z}^d$ we can conclude that $\mu_{\text{Bridge}} = \mu$ also on $\mathbb{H}$. Thus it suffices to show that

$$\mu_{\text{Bridge}} = \sqrt{2+\sqrt{2}}. \tag{5.0.1}$$

Throughout this section we will use the following notation: $z_c = \frac{1}{\sqrt{2+\sqrt{2}}}$.
Therefore, proving (5.0.1) is equivalent to see that $B(z_c) = \infty$ and that $B(z) < \infty$ for any $z < z_c$. And this is what we are going to prove.

5.1 The holomorfic observable

The proof is based on what we call the *holomorphic observable*. In this section we are going to define it and prove its discrete analyticity. We are not going into detail on it, but we will see that the discrete analyticity has an analogous result to a classic complex analysis theorem. First, some other definitions are required.

Definition 5.1.1. *A* domain $\Omega \in H$ *is a union of all mid-edges emanating form a given connected collection of vertices* $V(\Omega)$ *(see Figure 5.1). In other words, a mid-edge* x *belongs to* Ω *if at least one end-point of its associate edge is in* $V(\Omega)$*. The* boundary $\partial\Omega$ *consists of mid-edges whose associate edge has exactly one endpoint in* Ω*. We assume* Ω *to be simply connected (i.e. having a connected complement).*

Definition 5.1.2. *The* winding $W_\omega(a,b)$ *of a SAW* ω *between mid-edges* a *and* b *(not necessary the start and end of* ω*) is the total rotation in radians when* ω *is traversed from* a *to* b*. See Figure 5.2 for an example.*

We note that this definition is equivalent to say that

$$W_\omega(a,b) = \#\{\text{turns to the left}\} \times \frac{\pi}{3} - \#\{\text{turns to the right}\} \times \frac{\pi}{3}. \tag{5.1.1}$$

Here is an example:
In Figure 5.3 there is a walk ω with winding $W_\omega(a,b) = \pi$, according to Definition 5.1.2. Let us

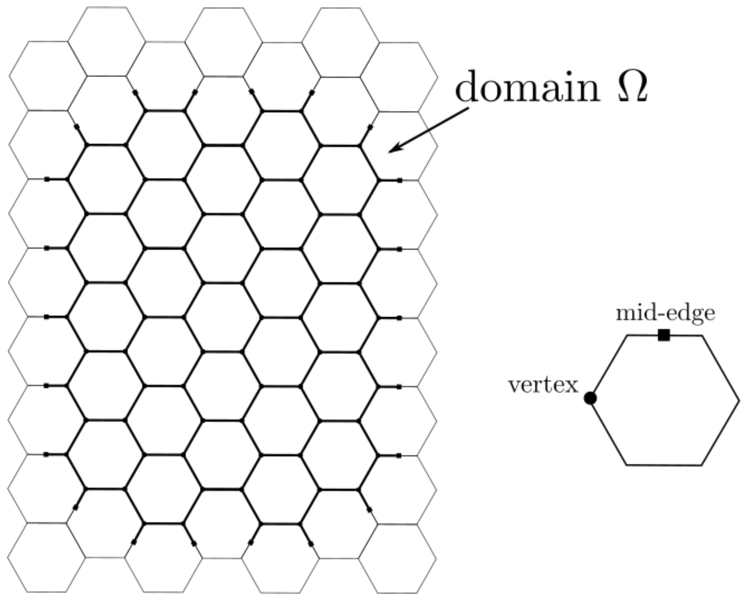


Figure 5.1: A domain Ω, where vertices are represented with small black points and mid-edges are represented with small black squares.

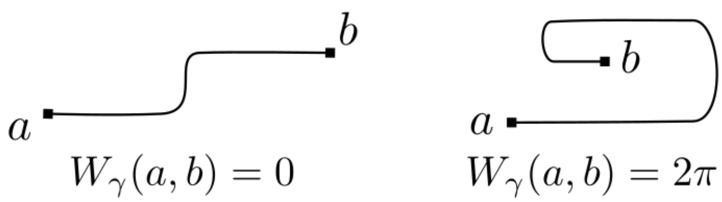


Figure 5.2: Winding of two different SAWs γ.

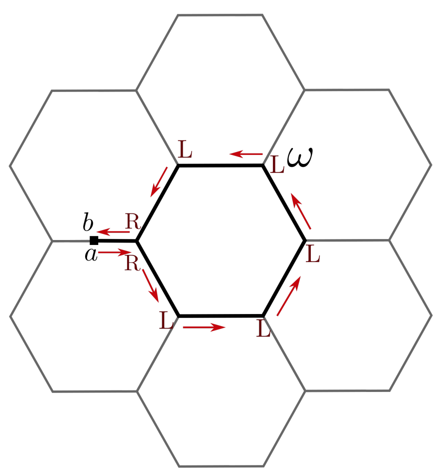


Figure 5.3: A walk ω from a to b with left turns denoted by L and right ones by R.

now check that with (5.1.1) we also obtain the winding number equal to π.

$$\begin{aligned} W_\omega(a,b) &= \#\{\text{turns to the left}\} \times \frac{\pi}{3} - \#\{\text{turns to the right}\} \times \frac{\pi}{3} \\ &= 5 \times \frac{\pi}{3} - 2 \times \frac{\pi}{3} = \frac{3\pi}{3} = \pi. \end{aligned}$$

Therefore, we have also obtained $W_\omega(a,b) = \pi$ and definitions are equivalent.

Even if we not mention it, throughout this section ω will represent a self-avoiding walk on $\mathbb{H}$. We write $\omega : a \longrightarrow E$ if a walk ω starts at mid-edge a and ends at some mid-edge of $E \subset H$. In the case where $E = b$ we simply write $\omega : a \longrightarrow b$.

Proposition 5.1.1. *For any different SAWs ω_1 and ω_2 and from any mid-edge a to any mid-edge b with at least one of them in the boundary of a domain Ω, $W_{\omega_1}(a,b) = W_{\omega_2}(a,b)$.*

Proof. First of all, we notice that any SAW with either the start point a or the end point b on the boundaries of Ω can not go round any mid-edge. If it happened, then the walk would have a self-intersection, which is forbidden in SAWs.
Consequently, locally we have:

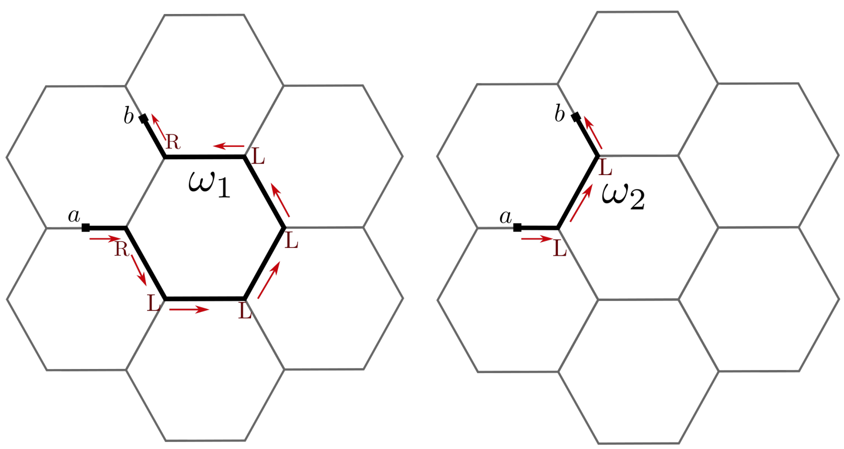


Figure 5.4: Two different walks ω_1 and ω_2 from a to b round an edge of an hexagon, with the same winding number.

For two different walks from a to b round an edge of an hexagon, like in Figure 5.4, the winding number of both walks is exactly the same.

$$W_{\omega_1}(a,b) = 4 \times \frac{\pi}{3} - 2 \times \frac{\pi}{3} = \frac{2\pi}{3}, \qquad W_{\omega_2}(a,b) = 2 \times \frac{\pi}{3} = \frac{2\pi}{3}.$$

And it happens the same with all the edges of an hexagon. Doing an induction argument on the steps number, we see that the winding number of a walk on Ω that starts and ends in mid-edges of the boundary does not depend on the path followed. □

Definition 5.1.3. *The* length $\ell(\omega)$ *of the walk* ω *is the number of vertices belonging to it.*

We can finally define the *holomorphic observable*.

Definition 5.1.4. *Let* a *be a fixed mid-edge on the boundary* $\partial\Omega$ *and* $\sigma \in \mathbb{R}$ *a fixed constant. For* $x \in \Omega$ *and* $z \geq 0$ *the* holomorphic observable *is defined as the following function.*

$$F_z(x) = \sum_{\omega \in \Omega:\ a \to x} e^{-i\sigma W_\omega(a,x)} z^{\ell(\omega)}.$$

The following lemma shows that for the special case $z = z_c$ and $\sigma = \frac{5}{8}$, F_{z_c} satisfies an important relation that can be regarded as a weak form of discrete analyticity, since it bears a resemblance to the mean value principle for harmonic functions, which states that the value of a harmonic function at a point is equal to its average value over spheres or balls centered at that point.

Lemma 5.1.1. *If* $z = z_c$ *and* $\sigma = \frac{5}{8}$, *then, for every vertex* $v \in V(\Omega)$,

$$(p - v)F_{z_c}(p) + (q - v)F_{z_c}(q) + (r - v)F_{z_c}(r) = 0, \tag{5.1.2}$$

where p, q, r *are the mid-edges of the three edges adjacent to* v.

Proof. For now we consider $z \geq 0$ and $\sigma \in \mathbb{R}$ (we are going to specialise later to $z = z_c$ and $\sigma = \frac{5}{8}$). We assume without loss of generality that p, q and r are orientated counter-clockwise around v in this order. By definition of the holomorphic observable, $(p-v)F_{z_c}(p)+(q-v)F_{z_c}(q)+(r-v)F_{z_c}(r)$ is a sum of contributions $c(\omega)$ over all the SAWs ω finishing at p, q or r. For example, if ω is a self-avoiding walk ending at p, its contribution will be

$$c(\omega) = (p - v)e^{-i\sigma W_\omega(a,p)} z^{\ell(\omega)}.$$

The set of walks ending at p, q and r can be partitioned into pairs and triplets of walks in the following way (represented on Figure 5.5):

- If a walk ω_1 visits all the three mid-edges p, q and r, then it can be seen like a SAW until one of this mid-edges plus (up to a half-edge) a self-avoiding return from v to v. We can associate to ω_1 the walk ω_2 that goes through the same edges, but traverse the return from v to v in the opposite direction. Hence, walks visiting the three mid-edges can be grouped in pairs.

- If a walk ω_1 visits only one mid-edge, it can be associated to two other walks ω_2 and ω_3 which visit two mid-edges, just adding one step (there are two possibilities). The reverse is also true:

a walk visiting two mid-edges can be associated with a walk just visiting one mid-edge by erasing the last step. Hence, walks visiting one or two mid-edges can be grouped in triplets.

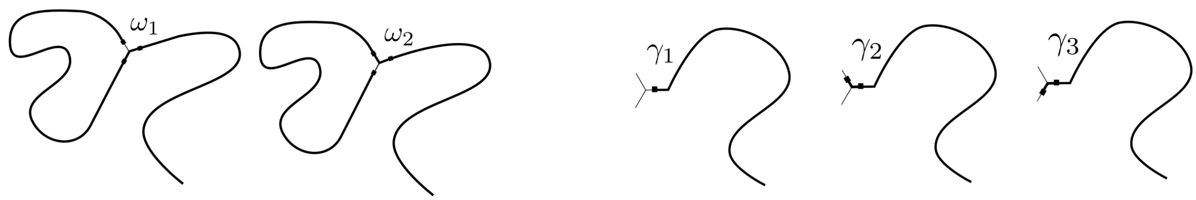


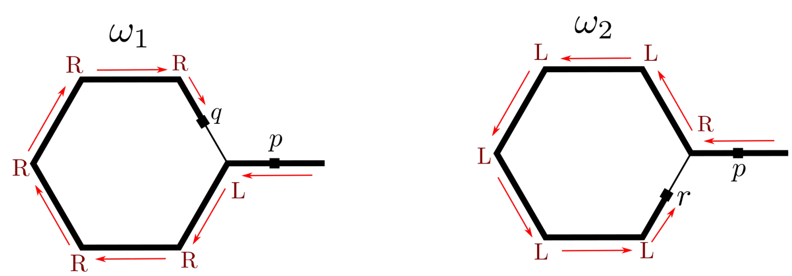
Figure 5.5: On the left a pair of walks ω_1 and ω_2 visiting the three mid-egdes, matched together. On the right a triplet of walks γ_1, γ_2 and γ_3, one visiting one mid-edge and the other ones visiting two mid-edges, also matched together.

Now, we are going to verify the equation (5.1.2) for the two cases separately:

- <u>First case:</u> Let ω_1 and ω_2 be two-self avoiding walks that are grouped as in the first case. Without loss of generality, we suppose that ω_1 ends at q and ω_2 ends at r. Therefore, ω_1 and ω_2 coincide up to the mid-edge p, since they are associated. Then

$$\ell(\omega_1) = \ell(\omega_2) \quad \text{and} \quad \begin{cases} W_{\omega_1}(a,q) = W_{\omega_1}(a,p) + W_{\omega_1}(p,q) = W_{\omega_1}(a,p) - 4\frac{\pi}{3} \\ W_{\omega_2}(a,r) = W_{\omega_2}(a,p) + W_{\omega_2}(p,r) = W_{\omega_1}(a,p) + 4\frac{\pi}{3}, \end{cases} \tag{5.1.3}$$

where the winding numbers $W_{\omega_1}(p,q)$ and $W_{\omega_2}(p,r)$ have been calculated according to (5.1.1), using Proposition 5.1.1 and having in mind that Ω is simply connected and $a \in \partial\Omega$ (so that we can not go round any mid-edge):



$$W_{\omega_1}(p,q) = \frac{\pi}{3} - \frac{5\pi}{3} = -\frac{4\pi}{3}. \qquad W_{\omega_2}(p,r) = \frac{5\pi}{3} - \frac{\pi}{3} = \frac{4\pi}{3}.$$

Writing $\lambda = \exp\left(-i\sigma\frac{\pi}{3}\right)$ and $j = \exp\left(-i\frac{2\pi}{3}\right)$ we obtain

$$\begin{aligned}
c(\omega_1) + c(\omega_2) &= (q-v)e^{-i\sigma W_{\omega_1}(a,q)} z^{\ell(\omega_1)} + (r-v)e^{-i\sigma W_{\omega_2}(a,r)} z^{\ell(\omega_2)} \\
&\overset{(5.1.3)}{=} (q-v)e^{-i\sigma\left(W_{\omega_1}(a,p)-4\frac{\pi}{3}\right)} z^{\ell(\omega_1)} + (r-v)e^{-i\sigma\left(W_{\omega_1}(a,p)+4\frac{\pi}{3}\right)} z^{\ell(\omega_1)} \\
&= e^{-i\sigma W_{\omega_1}(a,p)} z^{\ell(\omega_1)} \left((q-v)\bar{\lambda}^4 + (r-v)\lambda^4\right). \qquad (5.1.4)
\end{aligned}$$

Now, we want to relate $(q-v)$ and $(r-v)$ with $(p-v)$. Thinking it as vectors in the complex plain (see figure 5.6), we have:

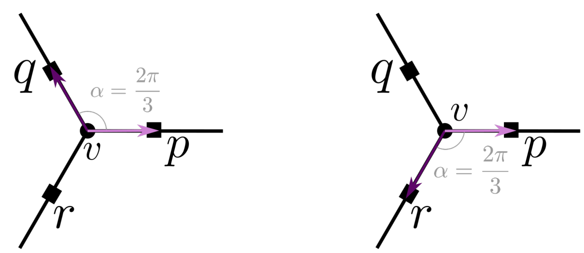


Figure 5.6: Representation of $(q-v)$ and $(p-v)$ as vectors and its relation on the left and the same with $(r-v)$ and $(p-v)$ on the right.

$$(q-v) = (p-v)e^{i\frac{2\pi}{3}} = (p-v)j \qquad (5.1.5)$$
$$(r-v) = (p-v)e^{-i\frac{2\pi}{3}} = (p-v)\bar{j} \qquad (5.1.6)$$

Thus, (5.1.4) turns into

$$c(\omega_1) + c(\omega_2) = (p-v)e^{-i\sigma W_{\omega_1}(a,p)} z^{\ell(\omega_1)} \left(j\bar{\lambda}^4 + \bar{j}\lambda^4\right).$$

If now we set $\sigma = \frac{5}{8}$,

$$j\bar{\lambda}^4 + \bar{j}\lambda^4 = e^{i\left(\frac{2\pi}{3}+\frac{5}{8}\frac{4\pi}{3}\right)} + e^{-i\left(\frac{2\pi}{3}+\frac{5}{8}\frac{4\pi}{3}\right)} = 2\cos\left(\frac{3\pi}{2}\right) = 0.$$

Therefore $c(\omega_1) + c(\omega_2) = 0$.

- Second case: Let $\omega_1, \omega_2, \omega_3$ be three walks grouped as in the second case. Without loss of generality, we assume that ω_1 finishes at p and that ω_2 and ω_3 prolong ω_1 to q and r respectively. We proceed as before and in this case we obtain

$$\ell(\omega_2) = \ell(\omega_3) = \ell(\omega_1) + 1 \quad \text{and} \quad \begin{cases} W_{\omega_2}(a,q) = W_{\omega_2}(a,p) + W_{\omega_2}(p,q) = W_{\omega_1}(a,p) - \frac{\pi}{3} \\ W_{\omega_3}(a,r) = W_{\omega_3}(a,p) + W_{\omega_3}(p,r) = W_{\omega_1}(a,p) + \frac{\pi}{3}. \end{cases}$$

Now,

$$\begin{aligned}c(\omega_1)+c(\omega_2)+c(\omega_3) &= (p-v)e^{-i\sigma W_{\omega_1}(a,p)}z^{\ell(\omega_1)}+(q-v)e^{-i\sigma W_{\omega_2}(a,q)}z^{\ell(\omega_2)} \\ &\quad +(r-v)e^{-i\sigma W_{\omega_3}(a,r)}z^{\ell(\omega_3)} \\ &= (p-v)e^{-i\sigma W_{\omega_1}(a,p)}z^{\ell(\omega_1)}+(q-v)e^{-i\sigma\left(W_{\omega_1}(a,p)-\frac{\pi}{3}\right)}z^{\ell(\omega_1)+1} \\ &\quad +(r-v)e^{-i\sigma\left(W_{\omega_1}(a,p)-\frac{\pi}{3}\right)}z^{\ell(\omega_1)+1} \\ &\overset{(5.1.5)}{\underset{(5.1.6)}{=}} (p-v)e^{-i\sigma W_{\omega_1}(a,p)}z^{\ell(\omega_1)}\left(1+zj\bar{\lambda}+z\bar{j}\lambda\right).\end{aligned}$$

At this point, we want $1+zj\bar{\lambda}+z\bar{j}\lambda=0$. So, let us find the value of z that verifies this equality. Remember that we have already chosen $\sigma=\frac{5}{8}$. Then,

$$0=1+zj\bar{\lambda}+z\bar{j}\lambda=1+z\left(e^{i\left(\frac{2\pi}{3}+\frac{5\pi}{24}\right)}+e^{-i\left(\frac{2\pi}{3}+\frac{5\pi}{24}\right)}\right)=1+2z\cos\left(\frac{7\pi}{8}\right).$$

Now, using the trigonometric property $\cos(\pi-\theta)=-\cos\theta$, the value of z that we are looking for is

$$z_c=\left(2\cos\frac{\pi}{8}\right)^{-1}=\left(\sqrt{2+\sqrt{2}}\right)^{-1}.$$

Finally, the identity (5.1.2) that we wanted to prove follows by summing over all the pairs and triplets of walks. □

5.1.1 Proof of Theorem 5.0.1

By applying Lemma 5.1.2 we will be able to prove Theorem 5.0.1. But first, we need to define some concepts that will be used in the proof.

We consider a vertical strip domain S_T composed of the vertices of T strips of hexagons, and $S_{T,L}$ is its finite version, cutting at height L at an angle of $\frac{\pi}{3}$; see Figure 5.7. We denote by α and β the left and right boundaries of S_T, respectively, and the top and bottom boundaries of $S_{T,L}$ by ϵ and $\bar{\epsilon}$ respectively.

We consider a fixed point $a\in\alpha$ and we define the following functions:

$$A_{T,L}(z)=\sum_{\omega\subset S_{T,L}:\ a\to\alpha\setminus\{a\}} z^{\ell(\omega)}, \tag{5.1.7}$$

$$B_{T,L}(z)=\sum_{\omega\subset S_{T,L}:\ a\to\beta} z^{\ell(\omega)}, \tag{5.1.8}$$

$$E_{T,L}(z)=\sum_{\omega\subset S_{T,L}:\ a\to\epsilon\cup\bar{\epsilon}} z^{\ell(\omega)}. \tag{5.1.9}$$

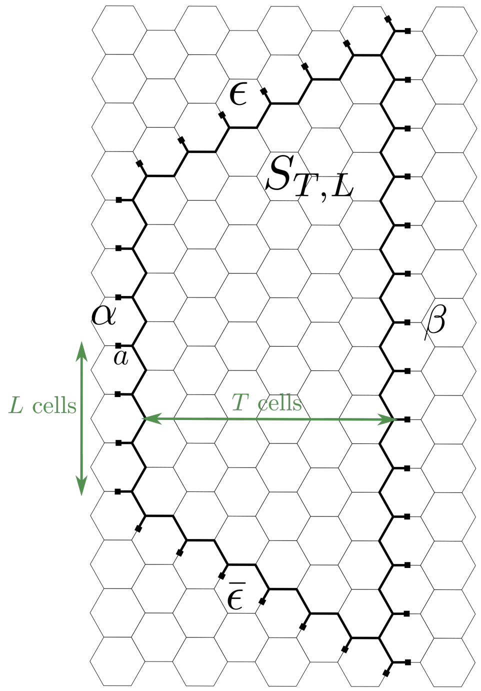


Figure 5.7: Domain $S_{T,L}$ with boundary parts α, β, ϵ and $\bar{\epsilon}$.

Let us now present a lemma that shows a relation between the last three functions in the case $z = z_c$.

Lemma 5.1.2. *For $z = z_c$ and $\sigma = \frac{5}{8}$,*

$$1 = c_\alpha A_{T,L}(z_c) + B_{T,L}(z_c) + c_\epsilon E_{T,L}(z_c), \tag{5.1.10}$$

with $c_\alpha = \cos\left(\frac{3\pi}{8}\right)$ and $c_\epsilon = \cos\left(\frac{\pi}{4}\right)$.

Proof. We fix $z = z_c$, $\sigma = \frac{5}{8}$ and we continue denoting $j = e^{i\frac{2\pi}{3}}$. We sum over all vertices in $V(S_{T,L})$ the relation (5.1.2). The contributions of interior mid-edges cancel and it only remains the contributions of the mid-edges on the boundary.

Then, if we consider the mid-edge $x \in \alpha$ and v its end-point vertex belonging to $V(S_{T,L})$, $(x - v) = -1$ (remember that the hexagonal lattice is a mesh of size 1). In the same way, if $x \in \beta$, $(x - v) = 1$; if $x \in \epsilon$, $(x - v) = j$ and if $x \in \bar{\epsilon}$, $(x - v) = \bar{j}$. Then

$$-\sum_{x\in\alpha} F(x) + \sum_{x\in\beta} F(x) + j\sum_{x\in\epsilon} F(x) + \bar{j}\sum_{x\in\bar{\epsilon}} F(x) = 0. \tag{5.1.11}$$

We are going now, to develop all this terms. We need to calculate the winding number of any SAW from a to any mid-edge of every part of the boundary. As in Proposition 5.1.1 we have proved that

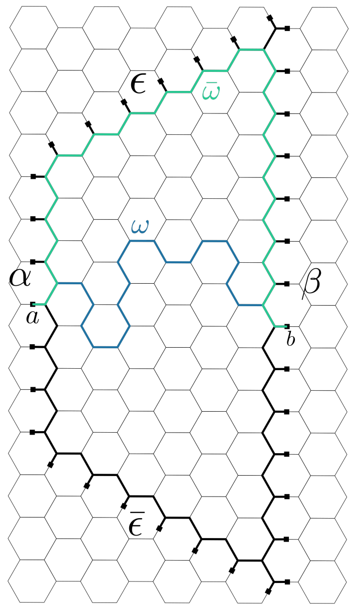


Figure 5.8: The winding number of the walk ω from a point a to another b both in $\partial S_{T,L}$ is the same as the winding number of the walk $\bar{\omega}$ from a to b through the edges nearer to $\partial S_{T,L}$.

the winding number of any SAW from a to b both in $\partial S_{T,L}$ does not depend on the path followed, for any walk $\omega \subset S_{T,L}$we are going to calculate $W_\omega(a,b)$ by calculating the winding number of the walk that goes from a to b along the edges nearer to the boundary of $S_{T,L}$; see Figure 5.8.

To calculate the winding number of any SAW on $S_{T,L}$ we also need to take into account the next claim:

Claim: *If a and b are mid-edges on the boundary of $S_{T,L}$, then for any SAWs ω_1 and ω_2, any mid-edge $\tilde{a}$ from the same boundary part of $S_{T,L}$ as a and any mid-edge $\tilde{b}$ from the same boundary part of $S_{T,L}$ as b, $W_{\omega_1}(a,b) = W_{\omega_2}(\tilde{a},\tilde{b})$.*

Proof of the claim. We have seen that for calculating the winding number of a walk we can restrict to consider the walks passing through the edges nearer to $\partial S_{T,L}$. Then, we only need to verify that the winding number of a walk through these edges starting at one part of $\partial S_{T,L}$ to a different part of it does not depend on the start and end mid-edges. We can prove it by considering all the possible cases and checking all of them, which is easy to do by using (5.1.1). Note that it is only necessary to prove the cases of adjacent parts of the boundary, i.e. from α to ϵ, from ϵ to β, from β to $\bar{\epsilon}$ and from $\bar{\epsilon}$ to α. The reverse cases will have the same winding number but with the opposite sign and any other walk can be considered as a concatenation of walks between contiguous parts of the boundary and the winding number will be the sum of the winding numbers of these concatenated walks. ■

Now, we are ready to calculate the winding number of walks from the fixed mid-edge $a \in \alpha$ to any other mid-edge on the boundaries and we are going to use (5.1.1) and consider the claims.
The winding of any SAW from a to the bottom part of α is $-\pi$, while the winding number to the top part is π. Then, we can develop the first term of (5.1.11):

$$\sum_{x\in\alpha} F(x) = F(a) + \sum_{x\in\alpha\setminus\{a\}} F(x) = 1 + \frac{e^{-i\sigma\pi} + e^{i\sigma\pi}}{2} A_{T,L}, \tag{5.1.12}$$

where we have used that $F(a) = 0$ because the only SAW from a to a is the empty walk with length 0 and the 2 on the denominator is because (for symmetry) in $A_{T,L}$ we are counting all the walks to α and we have that a half is to the bottom part and the other to the top part.
On the other hand, using trigonometric properties

$$\frac{e^{-i\sigma\pi} + e^{i\sigma\pi}}{2} = \cos(\sigma\pi) = \cos\left(\frac{5\pi}{8}\right) = -\cos\left(\frac{3\pi}{8}\right) := -c_\alpha.$$

Gathering it with (5.1.12) we obtain

$$\sum_{x\in\alpha} F(x) = 1 - c_\alpha A_{T,L}, \qquad \text{with } c_\alpha = \cos\left(\frac{3\pi}{8}\right). \tag{5.1.13}$$

The winding number from a to a mid-edge in β is 0. Therefore

$$\sum_{x\in\beta} F(x) = B_{T,L} \tag{5.1.14}$$

Finally, the winding from a to any mid-edge in ϵ is $\frac{2\pi}{3}$, whereas to a mid-edge in $\bar{\epsilon}$ is $-\frac{2\pi}{3}$. In this case

$$\begin{aligned} j\sum_{x\in\epsilon} F(x) + \bar{j}\sum_{x\in\bar{\epsilon}} F(x) &= e^{i\frac{2\pi}{3}} \sum_{x\in\epsilon}\ \sum_{\omega\subset S_{T,L}:\ a\to x} e^{-i\sigma W_\omega(a,x)} z^{\ell(\omega)} \\ &\quad + e^{-i\frac{2\pi}{3}} \sum_{x\in\bar{\epsilon}}\ \sum_{\omega\subset S_{T,L}:\ a\to x} e^{-i\sigma W_\omega(a,x)} z^{\ell(\omega)} \\ &= e^{i\frac{2\pi}{3}(1-\sigma)} \sum_{x\in\epsilon}\ \sum_{\omega\subset S_{T,L}:\ a\to x} z^{\ell(\omega)} + e^{-i\frac{2\pi}{3}(1-\sigma)} \sum_{x\in\epsilon}\ \sum_{\omega\subset S_{T,L}:\ a\to x} z^{\ell(\omega)} \\ &\overset{\text{(symmetry)}}{=} \frac{1}{2} E_{T,L}\left(e^{i\frac{2\pi}{3}(1-\sigma)} + e^{-i\frac{2\pi}{3}(1-\sigma)}\right) \overset{(\sigma=\frac{5}{8})}{=} \frac{1}{2} E_{T,L}\left(e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}\right) \\ &= \cos\left(\frac{\pi}{4}\right) E_{T,L} := c_\epsilon E_{T,L}. \end{aligned} \tag{5.1.15}$$

The proof is completed by inserting (5.1.13), (5.1.14) and (5.1.15) into (5.1.11). □

By definition, the sequences $\left(A_{T,L}(z)\right)_{L>0}$ and $\left(B_{T,L}(z)\right)_{L>0}$ are increasing in L and bounded for

$z < z_c$, thanks to (5.1.10) and the monotonicity of z. Thus, the following limits exist

$$A_T(z) := \lim_{L\to\infty} A_{T,L}(z) = \sum_{\omega\subset S_T:\ a\to\alpha\setminus\{a\}} z^{\ell(\omega)},$$
$$B_T(z) := \lim_{L\to\infty} B_{T,L}(z) = \sum_{\omega\subset S_T:\ a\to\beta} z^{\ell(\omega)}.$$

Again using (5.1.10), for $z = z_c$ we conclude that $\big(E_{T,L}(z_c)\big)_{L>0}$ is decreasing and converges to $E_T(z_c) = \lim_{L>0} E_{T,L}(z_c)$. Hence, taking limits into (5.1.10)

$$1 = c_\alpha A_T(z_c) + B_T(z_c) + c_\epsilon E_T(z_c). \tag{5.1.16}$$

We conclude this section by providing a complete proof of Theorem 5.0.1.

Proof of Theorem 5.0.1. Remember that to prove it, it suffices to show that $B(z_c) = \infty$ and that $B(z) < \infty$ for any $z < z_c$, where $B(z)$ is the generating function of bridges that we can rewrite as $B(z) = \sum_{T=0}^{\infty}$.

We first see the case with $z < z_c$. Since $B_T(z)$ involves only bridges of length at least T we have

$$\left(\frac{z}{z_c}\right)^T B_T(z_c) = b_{T,T} z^T + b_{T+1,T} z_c z^T + \ldots \overset{(z_c>z)}{\geq} B_T(z) = b_{T,T} z^T + b_{T+1,T} z^{T+1} + \ldots.$$

Then, using (5.1.16)

$$B_T(z) \leq \left(\frac{z}{z_c}\right)^T B_T(z_c) \leq \left(\frac{z}{z_c}\right)^T$$

Therefore, for $z < z_c$, $B(z)$ is finite since the right-hand side of the inequality is summable (it is a geometric series with ratio $\frac{z}{z_c} < 1$).

It only remains to prove that $B(z_c) = \infty$. To do it, we are going to consider two different cases:

<u>$E_T(z_c) > 0$ for some T</u>: As said before, $E_{T,L}(z_c)$ is decreasing in L. Then, if $C(z) = \sum_{n=0}^{\infty} c_n z^n$ is the generating function of the number of self-avoiding walks

$$C(z_c) \geq \sum_{L=1}^{\infty} E_{T,L}(z_c),$$

since the right-hand side only takes into account the SAWs ending at one mid-edge in ϵ or $\bar{\epsilon}$, while the left-hand side takes into account all the possible self-avoiding walks.

Now, using that $E_{T,L} \searrow E_T$ and $E_T(z_c) > 0$

$$C(z_c) \geq \sum_{L=1}^{\infty} E_{T,L}(z_c) \geq \sum_{L=1}^{\infty} E_T(z_c) = \infty.$$

Finally, by Corollary 4.2.2 if $C(z_c)$ diverges, then we also have $B(z_c) = \infty$ as we wanted to see.

$E_T(z_c) = 0$ for all T: In this case, the equality (5.1.16) simplifies to

$$1 = c_\alpha A_T(z_c) + B_T(z_c). \tag{5.1.17}$$

Note that if ω is contributing to $A_{T+1}(z_c)$ but not to $A_T(z_c)$, then ω must visit some vertex adjacent to β. Cutting such a walk at the first such point (and adding mid-edges to the two halves), we obtain two bridges of span $T+1$ in S_{T+1}. Thus,

$$A_{T+1}(z_c) - A_T(z_c) \leq z_c(B_{T+1}(z_c))^2, \tag{5.1.18}$$

where the term z_c in the right side is because adding the two mid-edges we have added an step. Combining (5.1.17) for T and $T+1$ with (5.1.18), we obtain

$$\begin{aligned} 0 &= \big(c_\alpha A_{T+1}(z_c) + B_{T+1}(z_c)\big) - \big(c_\alpha A_T(z_c) + B_T(z_c)\big) \\ &= \big(c_\alpha A_{T+1}(z_c) - c_\alpha A_T(z_c)\big) + B_{T+1}(z_c) - B_T(z_c) \\ &\leq c_\alpha z_c (B_{T+1}(z_c))^2 + B_{T+1}(z_c) - B_T(z_c). \end{aligned}$$

Thus,

$$c_\alpha z_c (B_{T+1}(z_c))^2 + B_{T+1}(z_c) \geq B_T(z_c) \tag{5.1.19}$$

Then, for $B_T(z_c)$ we have:

Claim:

$$B_T(z_c) \geq \frac{1}{T} \min\left\{B_1(z_c), \frac{1}{c_\alpha z_c}\right\} \qquad \textit{for every } T \geq 1.$$

Proof of the claim. We are going to prove it by induction on T. To simplify notation we denote $c = c_\alpha z_c$ and we are going to write just B_T instead of $B_T(z_c)$. We are also going to denote $m = \min\left\{B_1, \frac{1}{c}\right\}$. We want to see $B_T \geq \frac{m}{T}$ for all values of $T \geq 1$.

Base case: $T = 1$: We trivially have $B_1 \geq \frac{1}{1} m$.

Induction: $T \to T+1$: From (5.1.19),

$$cB_{T+1}^2 + B_{T+1} \geq B_T \overset{\text{(induction)}}{\geq} \frac{m}{T} \Longrightarrow cB_{T+1}^2 + B_{T+1} - \frac{m}{T} \geq 0.$$

Solving the inequality and only taking the positive solution (because B_T are positive terms), we obtain

$$B_{T+1} \geq \frac{-1+\sqrt{1+4c\frac{m}{T}}}{2c}.$$

Now,

$$\begin{aligned}
\frac{-1+\sqrt{1+4c\frac{m}{T}}}{2c} \geq \frac{m}{T+1} &\iff 1+4\frac{cm}{T} \geq \left(\frac{2cm}{T+1}+1\right)^2 = 1+\frac{4cm}{T+1}+\frac{4c^2m^2}{(T+1)^2} \\
&\iff \frac{1}{T} \geq \frac{1}{T+1}+\frac{cm}{(T+1)^2} \iff \frac{1}{T(T+1)} = \frac{1}{T}-\frac{1}{T+1} \geq \frac{cm}{(T+1)^2} \\
&\iff \frac{1}{T} \geq \frac{cm}{T+1}.
\end{aligned}$$

Since $cm = \min\{cB_1, 1\} \leq 1$ and $\frac{1}{T} \geq \frac{1}{T+1}$ the last inequality holds and due to all the implications we finally obtain

$$B_{T+1} \geq \frac{m}{T+1}$$

Hence, the claim is proved. ■

Finally, using the claim

$$B(z_c) = \sum_{T=0}^{\infty} B_T(z_c) \geq \sum_{T=1}^{\infty} B_T(z_c) \geq \sum_{T=1}^{\infty} \frac{cm}{T} = \infty.$$

Therefore, we have proved that $B(z_c) = \infty$. □

We have just proved Theorem 5.0.1, which gives us an exact number for the connective constant μ on the hexagonal lattice $\mathbb{H}$. However, this proof can not be adapted to the d-dimensional lattice $\mathbb{Z}^d$, due to the special property of discrete analyticity, among other characteristics, that only take place in $\mathbb{H}$. Therefore, the connective constant in the hexagonal lattice is the only one exactly known. Nevertheless, there are numerical approximations known for some connective constants in planar lattices, including $\mathbb{Z}^2$, which you can find in [7].

Conclusions

As the title of this work suggests, the goal of this bachelor's thesis was to study different techniques to find enumerative results for a combinatorial class. In this work, we have learned two useful techniques: the Symbolic method and the Kernel method. These methods can be used for any combinatorial objects but we have focused on the study of lattice paths. Using them, we have arrived to a formula for the generating function of any walk, bridge, meander or excursion with any simple set of steps $\mathcal{S}$ in the quadrangular lattice $\mathbb{Z}^2$. These formulas only involve small branches of the characteristic polynomial of $\mathcal{S}$, that we have also learned how to find. Therefore, given any simple set of steps $\mathcal{S}$, as complicated as you want, this thesis provides a way to find the generating function of any of these four types of lattice paths.

We have also learned that it is not possible to find exact counting formulas for all the combinatorial objects, but in theses cases we can still study the asymptotic behaviour of these counting numbers. To show it, we have focused on another type of lattice paths that we can generalize into a d-dimensional lattice, with $d \geq 2$ arbitrary. They are the self-avoiding walks, of which we have seen that have the same connective constant (or exponential growth) as the one of bridges and polygons, two new models that we have studied.

To conclude the work we have considered SAWs into the hexagonal lattice $\mathbb{H}$ and, due to the special properties of this lattice, we have proved and understood the celebrated solution by Smirnov and Duminil-Copin of the asymptotic counting of self-avoiding walks on the honeycomb. It is a result that, as we have seen, appeals to discrete complex analysis results, an area which has recently become very powerful to attack both combinatoric and probability problems. As further steps it would be interesting to investigate more about discrete complex analysis and see more applications of it, as it has been discovered that it has an spectacular range of applications, from analysis and mathematical physics to image processing and architecture.

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