Bachelor Degree in Informatics Engineering, Specialization in Computing

Bachelor Thesis

Network Formation Games under adversary attack with immunization: an introduction to the scientific research

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Abstract

In this bachelor thesis we study the Network Formation Games model with attack and immunization introduced by Goyal et al. The model consists of agents who want to maximize their benefit by connecting with other agents, which involves the cost of creating connections. Furthermore, an adversary will attack the network generated by agents and the agents can decide to immunize against the attack paying an additional cost. Different types of adversaries were proposed by Goyal et al. The objective is to gain an understanding of the model focusing on the *random attack adversary* and obtain results of connectivity, social welfare, diversity of equilibrium networks and best response dynamics based on previous works. With respect to the *random attack adversary* we prove that when cost of creating an edge $C_E > 1$ the resulting non-trivial equilibrium network (with at least one edge and at least one immunized vertex) G is a connected graph, and when C_E and cost of immunization C_I are constants and $C_E > 1$ then the welfare of G is $n^2 - O(n^{5/3})$. We also study the diversity of equilibrium networks and we show that among the types of equilibrium networks for the *maximum carnage adversary* presented by Goyal et al., *empty graph*, *trees*, *cycles*, *flowers* and *complete bipartite graph* are also equilibria for the *random attack adversary* with slight difference in parameters, the *forest* networks have a particular case of equilibrium for the *random attack adversary* but in general they are only equilibrium with respect to the *maximum carnage adversary.* Finally, we study the convergence of the best response dynamics with respect to the *random attack adversary* and we prove that it can cycle. We conclude our research with a simulation of the best response dynamics, we observe that it converges rapidly to an efficient equilibrium for the *random attack adversary.*

Key words: Network Formation Games, Social Welfare, Equilibrium Networks, Best Response Dynamics

Resumen

En esta tesis de grado estudiamos el modelo de Juegos de Formación de Red con ataque e inmunización introducido por Goyal et al. El modelo consta de agentes que quieren maximizar su beneficio por conectarse con otros agentes, lo que implica el coste de crear conexiones. Además, un adversario atacará la red generada por los agentes y los agentes pueden decidir inmunizarse contra el ataque pagando un coste adicional. Goyal et al. propusieron diferentes tipos de adversarios. El objetivo es comprender el modelo enfocándose en el *adversario de ataque aleatorio* y obtener resultados de conectividad, bienestar social, diversidad de redes de equilibrio y dinámicas de mejor respuesta basados en trabajos previos. Con respecto al *adversario de ataque aleatorio*, demostramos que cuando el coste de crear una arista $C_E > 1$ la red de equilibrio no trivial (con al menos una arista y al menos un vértice inmunizado) resultante G es un grafo conexo, y cuando C_E y el coste de la inmunización C_I son constantes y $C_E > 1$, entonces el bienestar de G es $n^2 - O(n^{5/3})$. También estudiamos la diversidad de redes de equilibrio y demostramos que entre los tipos de redes de equilibrio para el *adversario de máxima matanza* presentado por Goyal et al., *grafo nulo*, *árboles*, *ciclos*, *flores* y *grafo bipartito completo* también son equilibrios para el *adversario de ataque aleatorio* con ligera diferencia en los parámetros, las redes *bosque* tienen un caso particular de equilibrio para el *adversario de ataque aleatorio* pero en general son solo equilibrio con respecto al *adversario de máxima matanza*. Finalmente, estudiamos la convergencia de la dinámica de mejor respuesta con respecto al *adversario de ataque aleatorio* y demostramos que puede ciclar. Concluimos nuestra investigación con una simulación de la dinámica de mejor respuesta, observamos que converge rápidamente a un equilibrio eficiente para el *adversario de ataque aleatorio*.

Palabras clave: Juegos de Formación de Red, Bienestar Social, Redes de Equilibrio, Dinámica de Mejor Respuesta

Resum

En aquesta tesi de grau estudiem el model de Jocs de Formació de Xarxa amb atac i immunització introduït per Goyal et al. El model consta d'agents que volen maximitzar el benefici per connectar-se amb altres agents, fet que implica el cost de crear connexions. A més, un adversari atacarà la xarxa generada pels agents i els agents poden decidir immunitzar-se contra l'atac pagant un cost addicional. Goyal et al. van proposar diferents tipus d'adversaris. L'objectiu és comprendre el model enfocant-se a l'*adversari d'atac aleatori* i obtenir resultats de connectivitat, benestar social, diversitat de xarxes d'equilibri i dinàmiques de millor resposta basats en treballs previs. Pel que fa a l'*adversari d'atac aleatori*, demostrem que quan el cost de crear una aresta $C_E > 1$ la xarxa d'equilibri no trivial (amb almenys una aresta i almenys un vèrtex immunitzat) resultant G és un graf connex, i quan C_E i el cost de la immunització C_I són constants i $C_E > 1$, llavors el benestar de G és $n^2 - O(n^{5/3})$. També estudiem la diversitat de xarxes d'equilibri i demostrem que entre els tipus de xarxes d'equilibri per l'*adversari de màxima matança* presentat per Goyal et al., *graf nul*, *arbres*, *cicles*, *flors* i *graf bipartit complet* també són equilibris per l'*adversari d'atac aleatori* amb lleugera diferència als paràmetres, les xarxes *bosc* tenen un cas particular d'equilibri per l'*adversari d'atac aleatori* però en general són només equilibri respecte a l'*adversari de màxima matança.* Finalment, estudiem la convergència de la dinàmica de millor resposta pel que fa a l'*adversari d'atac aleatori* i demostrem que pot ciclar. Concloem la nostra recerca amb una simulació de la dinàmica de millor resposta, observem que convergeix ràpidament a un equilibri eficient per l'*adversari d'atac aleatori.*

Paraules clau: Jocs de Formació de Xarxa, Benestar Social, Xarxes d'Equilibri, Dinàmica de Millor Resposta

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1 Introduction

1.1 Context

In Computer Science, a problem can involve many different selfish entities, like players who want to maximize their own benefit. A method of modeling this type of problem is to use the concept of strategic games in Game Theory. Game Theory is the area that studies decisions made by players in those environments and the behavior of players following the objective of optimizing their benefit. Game Theory was first applied in economics and is one of the standard economic analysis tools. Along with its continuous development, the application of Game Theory has expanded to Computer Science, Biology and many other fields. The combination of Game Theory and Computer Science is called Algorithmic Game Theory, it focuses on issues related to the Internet and human behavior, and investigates many problems in Computer Science from a point of view of Game Theory.

Models in Game Theory are referred to as games. A classical model is the so-called strategic game in which there are a set of players (agents) and players interact between them by choosing an available action according to the rules of the model, that define their costs and payoffs which depend upon the actions selected by all players, and their utility as payoffs minus costs. A strategy of a player is the method the player uses to determine which action is chosen throughout the game.

Game theory looks for states of equilibrium when no player can do better by choosing a different action and properties in equilibrium states. We have different definitions for states of equilibrium, with Nash Equilibrium being the most important among them. To introduce the notion of Nash Equilibrium, first we define the best response. The best response of a player is a strategy which maximizes the utility of the player, assuming that all other players do not change their strategies. Then, we define Nash Equilibrium as the state of a strategic game in which all players' strategy is their best response, this means that no player can reach a higher utility by changing her strategy. Generally, computing the best response and Nash Equilibrium can be computationally hard, as it has been shown that the problem is NP-hard. To find a Nash Equilibrium in the case that we can compute the best response efficiently, we often "play" a strategic game by rounds and update strategies of players according to their best response in each round. This method is called best response dynamics, there exist models that their best response dynamics always converge to a Nash Equilibrium, for example potential games[12], and models that their best response dynamics do not always converge.

One particular subfamily of strategic games is Strategic Network Formation Model[1][13]. It consists of games that model creation and maintenance of a network. The network is represented by a graph, in which each player is a node of the graph and can buy edges to other players, paying a price per edge. The players can benefit by forming connections with other players. This type of games can be used to analyze networks in real life such as the internet and are also used in physics and biology.

Traditionally, in Strategic Network Formation Games we want to build a low-cost network determined by the behavior of various selfish agents that satisfies certain properties. However, when networks like the internet evolve and increase their size, they can be vulnerable to adversarial attack. For example, on the internet there are connections between different computers that are vulnerable under attacks like viruses or DDoS. All computers have the option of taking measures such as antivirus to defend against the attack with an additional cost for it. In biology, networks can represent social relation and contacts between humans, an attack may occur in the form of infection of a virus that spreads to other individuals through the network. Again, individuals can choose to immunize against the attack.[10]

To model the above situation, different variations of Network Formation Games have been considered, in which concepts like adversary attack and immunization were introduced. There is an enemy who can examine the network and choose a player to attack. The attack will spread through edges of the network and destroy vertices of players. Players have the option to pay an additional cost to be immunized against the attack.

Bala and Goyal in [9] studied the *original reachability network formation game* without attack or immunization in which players buy edges to each other and benefit from the size of their connected component. Networks with contagion risk have been studied previously by Cerdeiro et al. [8] and Goyal et al. [1] introduced Network Formation Games model with attack and immunization. In the model introduced by Goyal et al., an adversary can choose a player to start the spread of the attack, and the players can buy edges and decide to immunize against the attack. Players benefit from the expected size of their connected component after the adversary attack. Different types of adversaries were proposed, such as *maximum carnage adversary*, *random attack adversary* and *maximum disruption adversary*. Goyal et al. studied the equilibria of such games focusing on the *maximum carnage adversary*. The *maximum carnage adversary* chooses a player to attack with the objective of maximizing the spread of the attack and the number of destroyed players. The main results of such work can be summarized as follows:

- Diversity of equilibria: Graphs formed in equilibrium states can have different structures: empty graphs, trees, forest, cycles, flowers, complete bipartite graph...
- Edge density of equilibrium networks: It is shown that an equilibrium network of n players can have at most $2n - 4$ edges. This result also holds for *random attack adversary* and *maximum disruption adversary*.
- Connectivity and social welfare of equilibrium networks: It is shown that an equilibrium network of n players is a connected graph when the cost of buying an edge is greater than 1 and the total utility is $n^2 - O(n^{5/3})$.
- Best response dynamics: Goyal et al. conjectured a fast convergence of the best response dynamics in most cases for *maximum carnage adversary*. However,

depending on tie breaking rules for multiple best responses, there are examples in which the best response dynamics does not converge.

Our main objective in this project is to gain an understanding of the model focusing on the *random attack adversary* and obtain results based on previous works. The *random attack adversary* chooses randomly an unimmunized player to attack. In contrast with the *maximum carnage adversary*, all unimmunized players have the possibility of being attacked by the *random attack adversary* and it does not maximize the number of destroyed players. We might think the *random attack adversary* is weaker by its definition, our main questions are how it affects the structures of equilibrium networks, social welfare and best response dynamics. To answer these questions we will study diversity of equilibrium networks, connectivity and social welfare, compare our results with those presented by Goyal et al. [1] to see similarities and differences between *random attack adversary* and *maximum carnage adversary*. With respect to the *random attack adversary* we prove that when the cost of creating an edge $C_E > 1$ the resulting non-trivial equilibrium network G is a connected graph, and when C_E and cost of immunization C_I are constants and $C_E > 1$ then the welfare of G is $n^2 - O(n^{5/3})$. We show that among the types of equilibrium networks for the *maximum carnage adversary* presented by Goyal et al. [1], *empty graph*, *trees*, *cycles*, *flowers* and *complete bipartite graph* are also equilibria for the *random attack adversary* with slight difference in parameters, the *forest* networks have a particular case of equilibrium for the *random attack adversary* but in general they are only equilibrium with respect to the *maximum carnage adversary*. Besides of that, we will also study the convergence of the best response dynamics when all players choose their best response until an equilibrium is reached, we will experiment it with an implemented algorithm that calculates the best response for one player, different from the approach of the article [1] that uses Swapstable dynamics to update the action of a player by considering only deviations of: buying an edge, deleting an edge, buying and deleting an edge or any of the above modifying immunization. We show that the best response dynamics for the *random attack adversary* can cycle under a certain tie breaking rule as in the case of *maximum carnage adversary* and we observe that it converges rapidly to an efficient equilibrium in experiments.

Finally, this project is an introduction to scientific research for me. Variations of Network Formation Games and their properties is an interesting problem and it still has a lot of unanswered questions for researchers to investigate.

1.2 Scope

1.2.1 Objectives

We have the following general objectives of our project:

1. Study the model of *Network Formation Games under adversary attack with immunization* in depth.

2. Study the results shown in the article *Strategic Network Formation with Attack and Immunization* to have an understanding.

3. Study the following properties focusing on the *random attack adversary* based on previous studies: Diversity of equilibrium networks, Connectivity and social welfare and Convergence of best response dynamics.

4. Compare different models of adversaries.

5. Simulate the network for the *random attack adversary*.

6. Obtain conclusions from results of our research.

1.2.2 Requirements

Some functional and non-functional requirements we need to meet for the quality of our project are:

- Have good knowledge of concepts and definitions of Game Theory, Network Formation Games, and the extended model.
- Understand the results of the article [1] and their proofs.
- Write clearly in demonstrations of properties.
- Design a correct and efficient algorithm to simulate the network.
- Use plots and graphics to represent the obtained results.

1.2.3 Potential Obstacles and Risks

There are several potential obstacles and risks that we may encounter during the development of this project:

- The first obstacle is the **timing** of the project. We have three months for the development of the project before the final delivery. This is enough to complete the project but is not expected to leave time in case of incidents, so we have to organize well regarding the planning.
- The **lack of theoretical knowledge** in the field is also an obstacle. I have not studied subjects related to Game Theory before and I had to read books and other materials to learn about this field.
- A risk of our project is **the computation time of the simulation** because the computational cost and time complexity of the implemented algorithm can be high.
- **Mistakes in theoretical proofs and algorithm implementation** is another risk of our project. To prevent them, we must check our demonstrations carefully and spend more time on code debugging.
- **Mistakes in writing style** used in documentations is also a risk. We have to show our ideas to readers clearly, using a formal language free of grammar and spelling mistakes.

- Finally, in the time of pandemic, the **meeting** with the director of this project will be online. This makes the explanation of the results harder because we do not have a physical blackboard as in real life.

1.3 Methodology and rigour

1.3.1 Work Methodology

In order to develop the project with success, it is important to choose the right work methodology in relation to methodologies studied in the degree and adapted to the project.

The work methodology used in this project is the Kanban methodology. This methodology consists of a board with tasks represented as cards. The board is divided into columns, each column represents a status of tasks. We have in total three columns:

- **To do**: contains all tasks that have not been started.
- **In progress**: contains all started but unfinished tasks.
- **Completed**: contains all finished tasks.

The Kanban methodology has the advantage of being very visual, which allows us to quickly see the status of all tasks of the project.

To track the progress, we will use Trello, a web-based application of Kanban style in which we can organize our tasks by putting them under different tags.

1.3.2 Monitoring Tools

An online meeting with the director of this project will be arranged every week using Google Meet with the objective of discussing the progress of the project and the next tasks to do. We will also use Google Jamboard for illustration in case of questions.

Due to the theoretic nature of the project, all the documents that contain demonstrations, questions and newer results will be sent via email. For version control of the coding part, we will use a GitHub repository because it is simple and stores previous versions that can be easy to recover. We will use a master branch for the functioning code and a develop branch for all the code that is in development.

2 Preliminars

2.1 Strategic Game

A strategic game Γ is defined by:

- A finite set N of n players, $\{1, 2, ...n\}$.
- Each player i has her own nonempty *set of possible strategies*, S_i.
- Each player chooses her action $s_i \in S_i$ once. Players choose actions simultaneously.
- When a player i chooses her action $s_i \in S_i$ she is not informed of the actions chosen by others.
- $S = S_1 \times ... \times S_n$ the set of all possible ways in which players can pick strategies and $s = (s_1, ..., s_n) \in S$ is a strategy profile or vector of strategies selected by the players.
- For each player i, a *preference relation* (a complete, transitive, reflexive binary relation) over the set S. Given two elements of S, the relation for player i says which of these two outcomes i weakly prefers. We say that i *weakly prefers* S_1 to S_2 if i either prefers S_1 to S_2 or considers them as equally good outcomes.
- *Pay-off functions* $u_i(s_1, ..., s_n)$ to specify preferences by assigning for each player a value to each outcome. We can also consider *cost functions* $c_i(s_1, ..., s_n) = -u_i(s_1, ..., s_n)$ to specify costs and each player prefers to minimize her individual cost.[5][6]

An example of the strategic game is the Prisoner's Dilemma. Two prisoners are held in separate cells on trial for a crime and each one faces a choice of confessing to the crime or remaining silent. If they both remain silent, they will be convicted for a minor offense (2 years). If only one of them confesses, his term will be reduced to 1 year and the other will be convicted for a major offense (5 years). If both confess, each one will be convicted for a major offense with a reward for cooperation (4 years each).

In this example, we have two players, each player has silent, confess as the set of possible strategies. The set of all possible actions is (silent, silent), (silent, confess), (confess, silent), (confess, confess). The utilities of players are represented in the Table 1 below.

2.2 Nash Equilibrium

In a strategic game, given a player i and the strategies of the rest of players $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, the set of *best responses* for player i respect to s_{-i} are the actions that give maximum pay-off provided the other players do not change

player/actions	(silent,silent)	(silent,confess)	(confess,silent)	(confess,confess)
player1	2	0	3	1
player2	2	3	0	1

Table 1: Utilities of players

their strategies. Formally it is the set of strategies $s_i \in S_i$ that maximizes the players' pay-off function u_i taking current strategies of other players s_{-i} into account.

The best response dynamics is a process of game-playing that in each iteration a player updates her strategy by making a best response move. The best response dynamics converges if the process stops at an equilibrium in which no player can do better by changing her strategy.

A pure Nash equilibrium is a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ such that no player i can do better choosing an action different from s_i^*, given that every other player j adheres to s_j^*. Formally, for every player i and for every action s_i $u_i(s_1^*, \ldots, s_i^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_i, \ldots, s_n^*)$. From now on, we use the term equilibrium or Nash equilibrium to refer to pure Nash equilibrium.[5][6]

2.3 Network Formation Games

Network Formation Games are games that:

- Model creation and maintenance of a network.
- Have n players as vertices $V = \{1, ..., n\}$ in an undirected graph.
- Each player can buy/create edges to other players paying a price $\alpha > 0$ per edge.
- A strategy s_u of player u is a subset $s_u \in V - \{u\}$ that represents the set of nodes for which u pays for a link.
- As a result of a strategy profile $s = (s_1, ..., s_n)$ an undirected graph $G = (V, E)$ is created with $E = \{(u, v) | u \in s_v \vee v \in s_u\}$.
- The goal of player u is to minimize the cost function $c_u(s) = \textit{creation cost} + \textit{usage cost}$, with $\textit{creation cost} = \alpha|s_u|$. The *usage cost* depends on our definition, for example in SUM game the *usage cost* = sum of all distances between u and other vertices $\sum_{v \in V} d_G(u, v)$ and in MAX game the *usage cost* = maximum of all their distances $\max_{v \in V} d_G(u, v)$.[5][6][7]

Here we have an example of Network Formation Game:

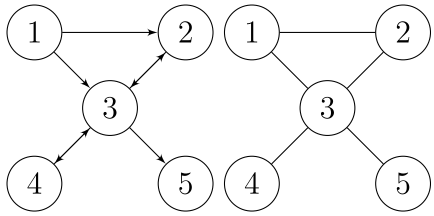

Figure 1: Example of Network Formation Game. Left: Strategy profile. Right: Resulted network.

In the strategy profile graph, each arrow represents who bought the edge. We have 5 players and strategy profile $s = (\{2,3\},\{3\},\{2,4,5\},\{3\},\{\})$. The creation cost is $\{2,1,3,1,0\}$ for $\alpha = 1$, in the resulted network the usage cost using SUM definition is $\{6,6,4,7,7\}$ and the usage cost using MAX definition is $\{2,2,1,2,2\}$.

2.4 Graph Theory

Given a graph $G = (V, E)$ and a subset of its vertices $U \subseteq V$, its *induced subgraph* is defined as $G[U] = (U, \{(u,v)|(u,v) \in E \wedge u,v \in U\})$.

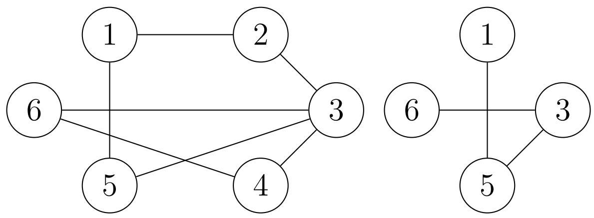

Figure 2: Example of a graph G and its induced subgraph $G[\{1,3,5,6\}]$

3 Network Formation Game with Attack and Immunization

In this section, we define the model introduced by Goyal et al. in [1] and review their main results:

3.1 Definition

This Network Formation Game with Attack and Immunization has n players as vertices of a graph. Each player can buy/create edges to other players paying a price $C_E > 0$ per edge and each player can spend a cost $C_I > 0$ to immunize against the adversary attack.

A strategy s_i for player i is defined by a pair $s_i = (x_i, y_i)$ where $x_i \subseteq \{1, ...n\}$ represents the set of edges i has bought to a subset of players and $y_i \in \{0, 1\}$ her decision of immunization, $y_i = 1$ iff i immunizes.

Given a strategy profile s, the set of edges bought by all players induce an undirected graph $G = (V, E)$ with $V = \{1, ..., n\}$ and $E = \{(u, v)|u \in x_v \vee v \in x_u\}$.

The immunization choices induce a partition of V in the set of immunized players $\mathcal{I} \subseteq V$ and vulnerable players $\mathcal{U} = V - \mathcal{I}$. The maximally connected components of the induced graph $G[\mathcal{U}]$ are called vulnerable regions.[8]

Given a game state $(G, \mathcal{I})$, the adversary can examine the whole graph G and choose a vulnerable vertex $v \in \mathcal{U}$ to attack. The attack will spread to all vertices reachable from v without immunization (the vulnerable region that contains v) and killing them. The adversary is specified a function Pr that defines the probability of attack for each vulnerable region. $\mathcal{T} = \{\mathcal{T}_1, ...\mathcal{T}_k\}$ is the set of vulnerable regions with non-zero probability of attack, which are called as *targeted regions*. Formally, let $Pr[\mathcal{T}']$ the probability of attack of a targeted region $\mathcal{T}'$ and $CC_i(\mathcal{T}')$ the size of connected component of player i after an attack to $\mathcal{T}'$. Then the expected utility $u_i(s)$ of player i in strategy profile s is defined by the expected size of her connected component after attack minus her costs of edges and immunization:[9]

$$u_i(s) = \sum_{\mathcal{T}' \in \mathcal{T}} (Pr[\mathcal{T}']CC_i(\mathcal{T}')) - |x_i|C_E - y_i C_I.$$

The social welfare of a strategy profile s is defined by the sum of expected utilities of all the players:

$$welfare(s) = \sum_{i \in V} u_i(s).$$

The adversary can use various strategies to select the vertex to attack, we considered three examples of adversary strategy:

- The *maximum carnage* adversary attacks of the vulnerable region of maximum size to attack with the goal of maximizing the number of agents killed. If there are more than one of such regions, the adversary selects one of them uniformly at random.
- The *random attack* adversary attacks a vulnerable vertex uniformly at random.
- The *maximum disruption* adversary selects randomly one of the vulnerable regions that minimizes the total utility or social welfare after attack.

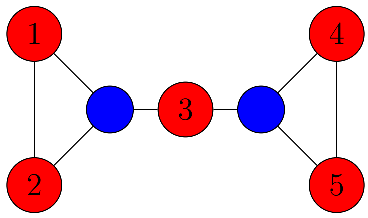

Figure 3: We use blue nodes for $\mathcal{I}$ and red nodes for $\mathcal{U}$. We have vulnerable regions $\mathcal{V}_1 = \{1,2\}$, $\mathcal{V}_2 = \{3\}$ and $\mathcal{V}_3 = \{4,5\}$. The probability of attack to $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ for each adversary are: maximum carnage: 0.5,0,0.5; random attack: 0.4,0.2,0.4; maximum disruption: 0,1,0.

We have defined previously the concept of Nash Equilibrium for the strategic game in general. In addition to this, we also define two more types of equilibria to analyze for our model.

Given a player i in G with strategy s_i, we define four possible *swap deviations* for it:

1. Dropping one purchased edge.
2. Buying one unpurchased edge.
3. Dropping one purchased edge and buying one unpurchased edge.
4. Making any one of the deviations above and changing the immunization status.

A *swapstable equilibrium* is a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ such that no player i can do better making any of the swap deviations above, given that every other player j adheres to s_j^*. A *linkstable equilibrium* is a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ such that no player i can do better making deviations:

1. Dropping one purchased edge.
2. Buying one unpurchased edge.
3. Making deviation 1 or 2 above and changing the immunization status, given that every other player j adheres to s_j^*.

As Goyal et al. stated in [1] it is easy to see that linkstable equilibria and swapstable equilibria are both generalizations of Nash equilibria and linkstable equilibria is also a generalization of swapstable equilibria. This implies that any Nash equilibrium is also a swapstable and linkstable equilibrium and any swapstable equilibrium is also a linkstable equilibrium. Furthermore, in the same article it is shown that there exists a swapstable equilibrium which is not a Nash equilibrium and there exists a linkstable equilibrium which is not a Nash or swapstable equilibrium, focusing on the maximum carnage adversary.

3.2 Previous results

Goyal et al. show in [1] some results of the model focusing on the maximum carnage adversary. Some of these results jointly with the techniques used to show them have inspired our work.

In [1] the authors show that under a mild restriction on the adversary, the number of edges of any Nash, swapstable or linkstable equilibrium network is at most $2n-4$ for $n \geq 4$, thus proving the sparsity of the equilibrium.

With respect to maximum carnage adversary, assuming $C_E > 1$, the authors prove first that any Nash, swapstable or linkstable equilibrium network of our model with at least one edge and at least one immunized vertex is a connected graph. Then, assuming $C_E > 1$, the authors show that if $C_E > 1$ and C_E and C_I are constants (independent of the size n of G), the social welfare of any non-trivial Nash or swapstable equilibrium network G is $n^2 - O(n^{5/3})$. They show that the resulting equilibrium network has a high probability of preserving a connected component of large size after the adversary attack, thus giving a high upper bound of social welfare.

Focusing on the maximum carnage adversary, the authors also study the diversity of equilibrium networks. For each type of equilibrium networks like empty graph, trees, forest, cycles, flowers and complete bipartite graph, they prove that there exists a range of values for C_E and C_I for which such strategies are equilibrium.

Finally, they do simulations of swapstable best response dynamics to prove its general and fast convergence to an equilibrium, although they show that an infinite loop can happen in the best response dynamics depending on tie-breaking rules.

One year later, Friedrich et al. show in [2] that the best response computation of our game under maximum carnage adversary and random attack adversary can be done in polynomial time by providing an efficient algorithm. In 2021, Zhang et al. in [3] implement this algorithm efficiently with improvements and optimizations. This result is useful for the simulation of the best response dynamics because we can now decide efficiently if our game state has reached Nash equilibrium. Nevertheless, this question is still open for the maximum disruption adversary.

We use the article [1] as our main source of reference and we focus our study on the random attack adversary. By remarks of the article, their connectivity and welfare results of the maximum carnage adversary are based on specific properties of maximum carnage adversary. We extend these results to random attack adversary by showing that it holds the same properties. We also study the diversity of equilibrium networks to make comparison between maximum carnage adversary and random attack adversary and see their difference in structures of equilibrium networks and parameters. Afterwards, we show that the best response dynamics may cycle for random attack adversary, as it also happens to the best response dynamics for maximum carnage adversary according to [1]. Finally, we simulate the best response dynamics with an implemented version [3] of the algorithm given recently in [2], we observe that the best response dynamics for the random attack adversary converges rapidly to an efficient equilibrium in experiments.

4 Connectivity and Social Welfare for Random Attack Adversary

One of our interests in this project is to study the connectivity of the equilibrium networks and their social welfare considering the random adversary attack. First, in the Section 5.1 we show that the empty graph is an equilibrium network with respect to the random attack adversary when $C_E \geq 1$ and its social welfare is $O(n)$. Afterwards, we consider any equilibrium network that contains at least one edge and at least one immunized vertex as a *non-trivial* equilibrium network. We show that the sparsity results shown in [1] also apply to the random attack adversary because the random attack adversary is a well-behaved adversary.

Let us review the main concepts and sparsity results of [1]:

Let $G_1 = (V, E_1)$, and $G_2 = (V, E_2)$ be two networks, G_1 and G_2 are *equivalent* if for all vertices $v \in V$ the connected component of v is the same in both G_1 and G_2 for every possible choice of initial attack vertex in V.

An adversary is *well-behaved* if on any pair of equivalent networks G_1 and G_2 the probability that a vertex $v \in V$ is chosen for attack is the same.

As it is pointed in [1], the maximum carnage, random attack and maximum disruption adversaries are all well-behaved.

Lemma 4.1 *Let $G = (V, E)$ be a Nash, swapstable or linkstable equilibrium network. Then all the vulnerable regions in G are trees if the adversary is well-behaved.*

Theorem 4.1 *Let $G = (V, E)$ be a Nash, swapstable or linkstable equilibrium network on $n \geq 4$ vertices. Then $|E| \leq 2n - 4$ for any well-behaved adversary.*

As it is stated in [1], all the proofs of connectivity and welfare results with respect to the maximum carnage adversary are based on the following properties:

Property 4.1 *Adding an edge between any 2 vertices (at least 1 of which is immunized) does not change the distribution of the attack.*

Property 4.2 *Breaking a link inside a targeted region does not increase the probability of attack to the targeted region while at the same time does not decrease the probability of attack to any other vulnerable regions.*

In the following paragraphs, we show that these properties also hold for the random attack adversary.

Remind that the random attack adversary attacks a vulnerable vertex uniformly at random (so all vulnerable regions are targeted) and we use $\mathcal{U} = V \backslash \mathcal{I}$ to denote the vulnerable vertices.

Lemma 4.2 *Let* $(G,\mathcal{I})$ *be a game state with* $\mathcal{I} \subsetneq V$. *Then the game states defined by* $(G[\mathcal{U}],\emptyset)$ *and* $(G,\mathcal{I})$ *have the same distribution of the attack with respect to the random attack adversary, respectively.*

Proof. If $|\mathcal{U}| > 0$, the probability of attack to a vulnerable (also targeted) region $\mathcal{T}$ in G by the random attack adversary is $Pr[\mathcal{T}] = |\mathcal{T}|/|\mathcal{U}|$. Note that G with immunized vertices $\mathcal{I}$ and its induced subgraph $G[\mathcal{U}]$ without immunized vertices will have the same distribution of the attack under the random attack adversary because each connected component of $G[\mathcal{U}]$ is a vulnerable/targeted region in G. Hence, the set of vulnerable/targeted regions is unchanged and every targeted region will have the same size. □

Using this lemma we can show that Property 4.1 and Property 4.2 also hold for the random attack adversary as it was stated in [1]:

Proof of Property 4.1. Let $G' = (V, E \cup (u,v))$ be the graph after adding an edge, $(u,v) \notin E, u \neq v$ with $u \in \mathcal{I}$ and $v \in V$ to the graph $G = (V,E)$. Hence, $G'[\mathcal{U}] = G[\mathcal{U}]$ by definition. By Lemma 4.2 $(G',\mathcal{I}), (G'[\mathcal{U}],\emptyset), (G[\mathcal{U}],\emptyset)$ and $(G,\mathcal{I})$ have the same distribution of the attack. □

Proof of Property 4.2. Let $G' = (V, E - (u,v))$ be the graph after breaking a link $(u,v) \in E$ with $u, v \in \mathcal{T}$ inside a targeted region $\mathcal{T}$ of $G = (V,E)$. We have $G'[\mathcal{U}] = (\mathcal{U}, \{(u',v')|(u',v') \in E \wedge u',v' \notin \mathcal{I}\} - (u,v))$ graph of vulnerable/targeted regions in G' that by Lemma 4.2 $(G',\mathcal{I})$ and $(G'[\mathcal{U}],\emptyset)$ have the same distribution of the attack. After deleting (u,v), the targeted region $\mathcal{T}$ can be still connected or be cut into connected components $\mathcal{T}'$ and $\mathcal{T}''$ of smaller size. In both cases the total probability of attack does not increase with respect to $(G,\mathcal{I})$ because it is directly proportional to the size of the targeted region. The other connected components in $G'[\mathcal{U}]$ will remain with the same size and probability of attack with respect to $(G,\mathcal{I})$. □

As Goyal et al. remark in [1], they prove the sparsity result with a rather mild restriction on the adversary. Moreover, their proofs of the following lemma and subsequent theorem of welfare results for the maximum carnage adversary essentially relay on the properties 4.1 and 4.2 stated above. Hence, we can extend the connectivity and welfare results of maximum carnage adversary to random attack adversary as it is pointed in [1]:

Lemma 4.3 *Let* G *be a non-trivial Nash or swapstable equilibrium network with respect to the random attack adversary. Then in any component of* G *with at least one immunized vertex and at least one edge, the targeted regions (if they exist) are singletons when* $C_E > 1$.

Theorem 4.2 *Let* G *be a non-trivial Nash, swapstable or linkstable equilibrium* *network with respect to the random attack adversary. Then, G is a connected graph when $C_E > 1$.*

Theorem 4.3 *Let G be a non-trivial Nash or swapstable equilibrium network on n vertices with respect to the random attack adversary. If C_E and C_I are constants (independent of n) and $C_E > 1$ then the welfare of G is $n^2 - O(n^{5/3})$.*

5 Diversity of Equilibrium Networks for Random Attack Adversary

In the article [1] it was shown that there exists a range of values for parameters C_E and C_I for which the empty graph/tree/forest/cycles/flowers/complete bipartite graph are (Nash, swapstable and linkstable) equilibrium network with respect to the maximum carnage adversary.

In this section we study the equilibria for the random attack adversary. We examine all the different examples of equilibria presented in [1] for the maximum carnage adversary and we show that not all are equilibria for the random attack adversary. In our figures, we use blue nodes for immunized players and red nodes for targeted players. In our demonstrations we give the utility function of a player only if it has changed with respect to proofs from the appendix D of [1], otherwise we will cite its results. Some of our proofs are identical to the proofs of the mentioned appendix D and some are inspired from them.

Remind that the expected utility $u_i(s)$ of player i in strategy profile s is defined by the expected size of her connected component after attack minus her costs of edges and immunization:

$$u_i(s) = \sum_{\mathcal{T}' \in \mathcal{T}} (Pr[\mathcal{T}']CC_i(\mathcal{T}')) - |x_i|C_E - y_i C_I$$

To show that a strategy profile s is an equilibrium, we compare for every player i her current expected utility $u_i(s)$ with her expected utility after each deviation $u_i(s_{-i}, s'_i) = u_i(s')$ and show that she can not achieve a better expected utility by making any of the deviations.

5.1 Empty Graphs

We show that an empty graph with all immunized or all targeted vertices can form in equilibria (Nash, swapstable or linkstable) with respect to the random attack adversary and these are the only empty equilibrium networks of our game. The ranges of the parameters C_E and C_I are the same as those of the maximum carnage adversary.

Lemma 5.1 *When $C_E \geq 1$ and $C_I > 0$ the empty graph is a (Nash, swapstable or linkstable) equilibrium network with respect to the random attack adversary.*

Proof. When $C_E \geq 1$ and $C_I \geq 1/n$ the empty network with all targeted vertices is an equilibrium because no player would strictly prefer to purchase more edges. Notice that in the empty network with all targeted vertices, the expected utility of any player i is:

$$u_i(s) = (1 - \frac{1}{n}).$$

If we assume that player i buys $k > 0$ edges, then her expected utility after the deviation is:

$$u_i(s') = \left(1 - \frac{(k+1)}{n}\right) \cdot (k+1) - (k \cdot C_E).$$

And

$$C_E \geq 1 \Longrightarrow u_i(s') = \left(1 - \frac{(k+1)}{n}\right) \cdot (k+1) - (k \cdot C_E) \leq 1 - \frac{(k+1)^2}{n} \leq u_i(s) = (1 - \frac{1}{n}).$$

Hence, player i has no incentive to buy $k > 0$ edges. If the player i immunizes, then her expected utility after the deviation is:

$$u_i(s') = 1 - C_I.$$

And we have:

$$C_I \geq \frac{1}{n} \Longrightarrow u_i(s') = 1 - C_I \leq u_i(s) = (1 - \frac{1}{n}).$$

Hence, player i has no incentive to immunize. If the player i buys $k > 0$ edges and immunizes, then her expected utility after the deviation is:

$$u_i(s') = \left(1 - \frac{k}{(n-1)}\right) \cdot (k+1) + \left(\frac{k}{(n-1)}\right) \cdot k - (k \cdot C_E) - C_I \leq (k+1) - (k \cdot C_E) - C_I.$$

And

$$C_E \geq 1 \wedge C_I \geq \frac{1}{n} \Longrightarrow u_i(s') \leq (k+1) - (k \cdot C_E) - C_I \leq u_i(s) = (1 - \frac{1}{n}).$$

Hence, player i has no incentive to buy $k > 0$ edges and immunize.
Furthermore, when $C_E \geq 1$ and $C_I \leq 1$ the empty network with all immunized vertices is an equilibrium. Notice that in the empty network with all immunized vertices, the expected utility of any player i is :

$$u_i(s) = 1 - C_I.$$

If we assume that player i buys $k > 0$ edges, then her expected utility after the deviation is:

$$u_i(s') = (k+1) - (k \cdot C_E) - C_I.$$

We have:

$$C_E \geq 1 \Longrightarrow u_i(s') = (k+1) - (k \cdot C_E) - C_I \leq u_i(s) = 1 - C_I.$$

Hence, player i has no incentive to buy $k > 0$ edges. Every player prefers to remain immunized, otherwise she will be the unique targeted vertex and be killed by the adversary and her expected utility after the deviation will be 0:

$$C_I \leq 1 \Longrightarrow u_i(s') = 0 \leq u_i(s) = 1 - C_I.$$

Similarly, no player wants to buy $k > 0$ edges and unimmunize. This shows that regardless of value of C_I when $C_E \geq 1$ the empty network is an equilibrium with respect to the random attack adversary. □

Lemma 5.2 *Let G be a (Nash, swapstable or linkstable) equilibrium network with respect to the random attack adversary. If G is the empty network, then the vertices in G are either all immunized or all targeted.*

Proof. see proof of Lemma 7 in the section D.1 of appendix D of the article [1], page 31 □.

5.2 Trees

We show that trees can be equilibria (Nash, swapstable or linkstable) with respect to the random attack adversary.

We consider two types of trees, one in which all vertices are immunized and another one in which all the vertices are immunized except for the leaves.

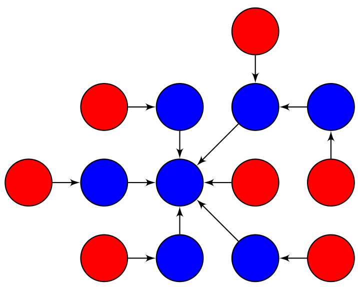

Figure 4: Example of tree equilibrium with respect to random attack adversary when $n = 14$, $k = 7$, $C_E = 2$ and $C_I = 2$

For the tree equilibrium with all immunized vertices, the ranges of the parameters C_E and C_I are the same as those of the maximum carnage adversary.

Lemma 5.3 *Consider any tree on n vertices. Suppose $C_E \in (0, n/2)$ and $C_I \in (0, n/2)$. Then, there exists an edge purchasing pattern which makes that tree an equilibrium with respect to the random attack adversary when all the vertices are immunized.*

Proof. see proof of Lemma 8 in the section D.2 of appendix D of the article [1], page 32. □

Lemma 5.4 (Jordan[4]). *Consider a graph $G = (V, E)$ where $|V| = n$. If G is a tree, then there exists a vertex $v \in V$ such that rooting the tree on v, no sub-tree has size more than $n/2$.*

For the tree equilibrium with immunized non-leaves and vulnerable leaves, the ranges of the parameters C_E and C_I are slightly different from those of the maximum carnage adversary.

Lemma 5.5 *Consider any tree on n vertices so that $k \leq n/2$ of them are immunized non-leaves and the remaining $n-k$ are vulnerable leaves. Then, for $C_E \in (0, k/2), C_I \in [(n-1)/(n-k), min(k/2-1, 2(n-1)/(n-k+1)))$, there exists an edge purchasing pattern that makes this network an equilibrium with respect to the random attack adversary.*

Proof. Root the tree of immunized vertices as described in Proposition 2 of appendix C of the article [1], page 30. Consider the edge purchasing pattern in which every immunized vertex buys an edge towards its parent in the tree of immunized vertices and every unimmunized vertex buys an edge towards its immunized parent. Notice that in this network the expected utility of any immunized player i is:

$$u_i(s) = (n-1) - (m \cdot C_E) - C_I.$$

With $m = 0$ for the root vertex and $m = 1$ for the rest of the vertices. And the expected utility of any unimmunized player j is:

$$u_j(s) = \left(1 - \frac{1}{(n-k)}\right) \cdot (n-1) - C_E.$$

If any immunized player i decides to unimmunize, then her expected utility after the deviation is:

$$u_i(s') = \left(1 - \frac{l}{(n-k+1)}\right) \cdot (n-1) - (m \cdot C_E).$$

With $2 \leq l \leq n-k$ being the size of the targeted region she belongs. We have:

$$C_I \leq \frac{2(n-1)}{(n-k+1)} \leq \frac{l \cdot (n-1)}{(n-k+1)}.$$

$$\Longrightarrow u_i(s') = \left(1 - \frac{l}{(n-k+1)}\right) \cdot (n-1) - (m \cdot C_E) \leq u_i(s) = (n-1) - (m \cdot C_E) - C_I.$$

Hence, player i has no incentive to unimmunize. If the player i buys more edges it would be redundant and only decreases her expected utility, hence the player i has no incentive to buy more edges or do both deviations above. Furthermore, player i would not drop her only purchased edge nor unimmunize and drop her edge because with $C_E < k/2$ and $C_I < k/2 - 1$, the current expected utility of the player i, $u_i(s)$, is greater than $n-k$, thus greater than the expected utility after the deviation as the size of her connected component after dropping her edge does not surpass $n-k$ by Lemma 5.4. It is easy to see that her current purchased edge is the best one and the player i would not drop her purchased edge and buy one unpurchased edge.
If any unimmunized player j decides to immunize, then her expected utility after the deviation is:

$$u_j(s') = (n-1) - C_I - C_E.$$

We have:

$$C_I \geq \frac{(n-1)}{(n-k)} \Longrightarrow u_j(s') = (n-1) - C_I - C_E \leq u_j(s) = \left(1 - \frac{1}{(n-k)}\right) \cdot (n-1) - C_E.$$

Hence, player j has no incentive to immunize. If the player j buys more edges it would be redundant and only decreases her expected utility, hence the player j has no incentive to buy more edges or do both deviations above. It is easy to see that her current purchased edge is the best one and the player j would not drop her purchased edge and buy one unpurchased edge. This shows that all players have the best current strategy and the described network is an equilibrium with respect to the random attack adversary. ☐

We also show that Hub-Spoke, a particular case of tree, is an equilibrium with respect to the random attack adversary for a wider range of parameters in comparison to Lemma 5.5. It consists of one immunized vertex (hub) and $n-1$ unimmunized vertices (spokes) and the spokes buy the edges to the hub. The ranges of the parameters C_E and C_I are the same as those of the maximum carnage adversary.

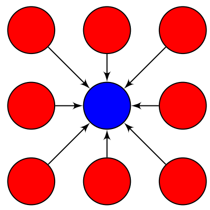

Figure 5: Example of Hub-Spoke equilibrium with respect to random attack adversary when $C_E = 2$ and $C_I = 2$

Lemma 5.6 *If $C_E \in (0, n-3]$ and $C_I \in [1, n-1]$ then a hub-and-spoke network is an equilibrium with respect to the random attack adversary when the hub immunizes and the spokes buy the edges to the hub.*

Proof. Notice that in this network the current expected utility of the hub vertex i is $u_i(s) = (n-1) - C_I$ and the current expected utility of any spoke vertex j is $u_j(s) = (1 - 1/(n-1)) \cdot (n-1) - C_E = (n-2) - C_E$. We consider all deviations the players can make and show that they can not increase their expected utility comparing the utilities before and after the deviation.
The hub can deviate by:

1. Changing her immunization. Then she will be the part of the unique targeted region, with the expected utility $u_i(s') = 0 \leq u_i(s) = (n-1) - C_I$ because $C_I \leq n-1$.

2. Buying more edges. It is redundant and only decreases her expected utility because she is already connected to all the rest of the vertices.

3. Changing her immunization and buying more edges. Her expected utility is negative and worse than other deviations.

Any spoke vertex can deviate by:

1. Immunizing. Her expected utility after the deviation is $u_j(s') = (n-1) - C_E - C_I$.

Because the expected utilities before and after the deviation coincide for both maximum carnage adversary and random attack adversary, by Lemma 10 in the section D.2 of appendix D of the article [1], page 33, we have $u_j(s') \leq u_j(s)$.

2. Dropping her purchased edge. Her expected utility after the deviation is $u_j(s') = 1 - 1/(n-1)$. Because the expected utilities before and after the deviation coincide for both maximum carnage adversary and random attack adversary, by Lemma 10 in the section D.2 of appendix D of the article [1], page 33, we have $u_j(s') \leq u_j(s)$.

3. Dropping her purchased edge and immunizing. Her expected utility after the deviation is $u_j(s') = 1 - C_I$. Because the expected utilities before and after the deviation coincide for both maximum carnage adversary and random attack adversary, by Lemma 10 in the section D.2 of appendix D of the article [1], page 33, we have $u_j(s') \leq u_j(s)$.

4. Dropping her purchased edge and buying new edge(s). After dropping her bought edge and adding $k \geq 1$ new edges, the size of her targeted region is $k+1$ and her expected utility is:

$$u_j(s') = \left(1 - \frac{(k+1)}{(n-1)}\right) \cdot (n-1) - (k \cdot C_E) = (n-2-k) - (k \cdot C_E) < u_j(s) = (n-2) - C_E.$$

5. Dropping her purchased edge, buying new edge(s) and immunizing. Her expected utility after dropping her bought edge, adding $k \geq 1$ new edges and immunizing is $u_j(s') \leq (n-1) - (k \cdot C_E) - C_I$ as $n-1$ is the maximum expected size of her connected component. Because the expected utilities before and after the deviation coincide for both maximum carnage adversary and random attack adversary, by Lemma 10 in the section D.2 of appendix D of the article [1], page 33, we have $u_j(s') \leq u_j(s)$.

6. Buying more edges. After adding $k \geq 1$ new edges the size of her targeted region is $k+1$ and her expected utility is:

$$u_j(s') = \left(1 - \frac{(k+1)}{(n-1)}\right) \cdot (n-1) - (k+1) \cdot C_E = (n-2-k) - (k+1) \cdot C_E < u_j(s) = (n-2) - C_E.$$

7. Buying more edges and immunizing. Her expected utility after adding $k \geq 1$ new edges and immunizing is $u_j(s') = (n-1) - (k+1) \cdot C_E - C_I$. This is worse than the case 1 because the new edges are redundant. Thus $u_j(s') \leq u_j(s)$ and the player does not prefer to deviate in this case.
Hence, no player has incentive to change her current strategy and the described network is an equilibrium with respect to the random attack adversary. □

5.3 Forest

In [1] it is shown that a forest consisting of targeted trees of equal size can form in equilibria with respect to maximum carnage adversary. We show for random adversary that in general this is not true, but a particular case of forest, 2 disjoint targeted trees of the same size, can form an equilibrium.

Lemma 5.7 *Let $n = kF + n'$ with $(k-2)F + n' > 1$, $k \geq 2, F \geq 2, n' \geq 0$. If $C_E > 0$ and $C_I > 0$, then k disjoint targeted trees of size F with n' vulnerable singleton vertices can not form an equilibrium with respect to the random attack adversary.*

Proof. For any edge purchasing pattern, in all trees of size F there exist at least one vertex i that has bought all its incident edges, otherwise it will have at least F edges and is not a tree. This vertex i prefers more to drop all its $m \geq 1$ purchased edges and connect to another tree. The expected utility of the player i after the deviation is

$$u_i(s') = \left(1 - \frac{(F+1)}{(kF+n')}\right) \cdot (F+1) - C_E = (F+1) - \frac{(F+1)^2}{(kF+n')} - C_E$$

$$= F + \frac{((k-2) \cdot F + n' - 1)}{(kF+n')} - \frac{F^2}{(kF+n')} - C_E > F - \frac{F^2}{(kF+n')} - C_E.$$

Since

$$F - \frac{F^2}{(kF+n')} - C_E \geq \left(1 - \frac{F}{(kF+n')}\right) \cdot F - (m \cdot C_E) = u_i(s)$$

her expected utility before the deviation, thus there is at least one deviation for vertex i with better payoff and the forest can not form an equilibrium with respect to the random attack adversary. ☐

Contrasting with this, we show that 2 disjoint targeted trees of the same size can form an equilibrium with respect to the random attack adversary. The ranges of the parameters C_E and C_I are different from those of the maximum carnage adversary.

Lemma 5.8 *Let $n = 2F, F \geq 4, C_E \in (0, F/2 - 1], C_I \geq 3F/2$, then 2 disjoint targeted trees of size F can form an equilibrium with respect to the random attack adversary.*

Proof. We can fix a root for each tree and consider the edge purchasing pattern in which every non-root vertex buys an edge towards the root of its tree. Let us analyze all the possible deviations for any player i who bought an edge with her current expected utility $u_i(s) = F/2 - C_E$:

1. Dropping her purchased edge: After dropping her bought edge we have that

$$u_i(s') = 1 - \frac{1}{2F} < u_i(s) = \frac{F}{2} - C_E$$

since $C_E \leq F/2 - 1$.

2. Dropping her purchased edge and immunizing: Then her expected utility

$$u_i(s') = 1 - C_I < u_i(s) = \frac{F}{2} - C_E$$

since $C_E \leq F/2 - 1$.

3. Dropping her purchased edge and buying other edges: buying other edges to the same connected component will have at most the same expected utility as her current strategy. Buying $m \geq 1$ edge(s) to another connected component will have the expected utility

$$u_i(s') = \left(1 - \frac{(F+1)}{2F}\right) \cdot (F+1) - m \cdot C_E = \frac{(F^2-1)}{2F} - m \cdot C_E < u_i(s) = \frac{F}{2} - C_E.$$

Buying more edges to both connected components will form a unique targeted region and be killed by the adversary, her expected utility after the deviation will be negative.

4. Dropping her purchased edge, buying other edges and immunizing: Buying other edges to the same connected component will have at most the same expected utility as the case 5. Buying $m \geq 1$ edges to another connected component will have the expected utility

$$u_i(s') = \frac{F}{(2F-1)} + \frac{(F-1)(F+1)}{(2F-1)} - (m \cdot C_E) - C_I < 2F - 1 - C_E - C_I < u_i(s) = \frac{F}{2} - C_E$$

since $C_I \geq 3F/2$. Buying other edges to the both connected components will have at most the same expected utility as the case 7.

5. Immunizing: her expected utility will be

$$u_i(s') = \frac{F^2}{(2F-1)} + \frac{(F-1)}{(2F-1)} - C_E - C_I < 2F - 1 - C_E - C_I < u_i(s) = \frac{F}{2} - C_E$$

since $C_I \geq 3F/2$.

6. Buying more edges: buying more edges to another component will form a unique targeted region and be killed by the adversary, her expected utility after the deviation will be negative. Buying more edges to the same component is redundant.

7. Buying more edges and immunizing: She would benefit the most by buying an edge to another component. Her expected utility will be

$$u_i(s') = \frac{F^2}{(2F-1)} + \frac{(F-1)(F+1)}{(2F-1)} - 2C_E - C_I < 2F - 1 - C_E - C_I < u_i(s) = \frac{F}{2} - C_E$$

since $C_I \geq 3F/2$.

Possible deviations for the root j who did not purchase any edge with her current expected utility $u_j(s) = F/2$:
1. Immunizing. Her expected utility will be

$$u_j(s') \leq \frac{F^2}{(2F-1)} + \frac{(F-1)^2}{(2F-1)} - C_I < 2F - 1 - C_I < u_j(s) = \frac{F}{2}$$

since $C_I \geq 3F/2$.

2. Buying more edges: buying more edges to another component will form a unique targeted region and be killed by the adversary, her expected utility after the deviation will be negative. Buying more edges to the same component is redundant.

3. Buying more edges and immunizing: She would benefit the most by buying an edge to another component. Her expected utility will be

$$u_j(s') \leq \frac{F^2}{(2F-1)} + \frac{(F-1)(2F-1)}{(2F-1)} - C_E - C_I < 2F - 1 - C_I < u_j(s) = \frac{F}{2}$$

since $C_I \geq 3F/2$.

Hence, no player has incentive to change her current strategy and the described network is an equilibrium with respect to the random attack adversary. □

5.4 Cycles

We show that an alternating cycle of immunized and targeted vertices can form in equilibria (Nash, swapstable or linkstable) with respect to the random attack adversary. The ranges of the parameters C_E and C_I are slightly different from those of the maximum carnage adversary.

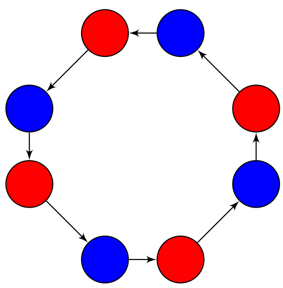

Figure 6: Example of cycle equilibrium with respect to random attack adversary when $k = 4$, $C_E = 1.5$ and $C_I = 2.5$

Lemma 5.9 *A cycle of* $n = 2k$ *alternating immunized and targeted vertices can form in equilibria with respect to the random attack adversary when every vertex buys an edge to the vertex in her clockwise direction in the cycle and* $C_E \in (1, n/2 - 2), C_I \in (2, 2(n-1)/(n/2+1)), C_E + C_I \leq k$ *and* $k \geq 4$.

Proof. For any immunized player i her expected utility in this network is $u_i(s) = (n-1) - C_E - C_I$. She can deviate by:
1. Dropping her purchased edge and buying other edges. After adding $m \geq 1$ new edges the expected connected component size she can achieve is at most $n-1$ so her expected utility after deviation $u_i(s')$ can not surpass her current expected utility $u_i(s)$:

$$u_i(s') \leq (n-1) - (m \cdot C_E) - C_I \leq u_i(s) = (n-1) - C_E - C_I.$$

2. Dropping her purchased edge. Her expected utility after the deviation is:

$$u_i(s') = \frac{1}{k}\,(1 + 3 + \ldots + (2k-1)) - C_I.$$

Because the expected utilities before and after the deviation coincide for both maximum carnage adversary and random attack adversary, by Lemma 13 in the section D.4 of appendix D of the article [1], page 36, we have $u_i(s') \leq u_i(s)$.

3. Buying more edges. The $m \geq 1$ new edges she purchased are redundant and only decreases her expected utility because the expected connected component size she can achieve is at most $n-1$:

$$u_i(s') = (n-1) - ((m+1) \cdot C_E) - C_I \leq u_i(s) = (n-1) - C_E - C_I.$$

4. Changing her immunization. She would not change her immunization because her expected utility after the deviation is:

$$u_i(s') = \left(1 - \frac{3}{(n/2+1)}\right) \cdot (n-1) - C_E < u_i(s) = (n-1) - C_E - C_I$$

since $C_I < 2(n-1)/(n/2+1) < 3(n-1)/(n/2+1)$.

5. Changing her immunization, dropping her purchased edge and buying other edges. After adding $m \geq 1$ new edges the expected connected component size she can achieve is at most $(1 - 2/(n/2+1))(n-1)$ so her expected utility after deviation $u_i(s')$ can not surpass her current expected utility $u_i(s)$:

$$u_i(s') \leq \left(1 - \frac{2}{(n/2+1)}\right) \cdot (n-1) - (m \cdot C_E) < u_i(s) = (n-1) - C_E - C_I$$

since $C_I < 2(n-1)/(n/2+1)$.

6. Changing her immunization and dropping her purchased edge. Her expected utility after the deviation is:

$$u_i(s') = \frac{1}{(k+1)}\,(3 + \ldots + (2k-1)) = k - 1 \leq u_i(s) = (n-1) - C_E - C_I$$

since $C_E + C_I \leq k$.

7. Changing her immunization and adding more edges. The $m \geq 1$ new edges she purchased are redundant and only decreases her expected utility. Thus her expected utility in this case is worse than the case 4.

For any unimmunized player j her expected utility in this network is $u_j(s) = (1 - 1/k)(n-1) - C_E$. She can deviate by:
1. Dropping her purchased edge and buying other edges. After adding $m \geq 1$ new edges the expected connected component size she can achieve is at most $(1-1/k)(n-1)$

so her expected utility after deviation $u_j(s')$ can not surpass her current expected utility $u_j(s)$:

$$u_j(s') \leq (1 - \frac{1}{k})(n-1) - (m \cdot C_E) - C_I \leq u_j(s) = (1 - \frac{1}{k})(n-1) - C_E - C_I.$$

2. Dropping her purchased edge. Her expected utility after the deviation is:

$$u_j(s') = \frac{1}{k}\left(2 + \ldots + (2k-2)\right).$$

3. Buying more edges.The $m \geq 1$ new edges she purchased are redundant and only decreases her expected utility because the expected connected component size she can achieve is at most $(1 - 1/k)(n-1)$:

$$u_j(s') \leq (1 - \frac{1}{k})(n-1) - ((m+1) \cdot C_E) - C_I \leq u_j(s) = (1 - \frac{1}{k})(n-1) - C_E - C_I.$$

4. Immunizing. Her expected utility after the deviation is:

$$u_j(s') = (n-1) - C_E - C_I.$$

5. Dropping her purchased edge, buying other edges and immunizing. With $m \geq 1$ new edges the expected connected component size she can achieve is at most $(n-1)$. Her maximum expected utility after the deviation is:

$$u_j(s') = (n-1) - (m \cdot C_E) - C_I.$$

6. Dropping her purchased edge and immunizing. Her expected utility after the deviation is:

$$u_j(s') = \frac{1}{k-1}\left(2 + \ldots + (2k-2)\right) - C_I.$$

7. Adding more edges and immunizing.With $m \geq 1$ new edges the expected connected component size she can achieve is at most $(n-1)$. Her maximum expected utility after the deviation is:

$$u_j(s') = (n-1) - ((m+1) \cdot C_E) - C_I.$$

In cases 2,4,5,6 and 7 above, because the expected utilities before and after the deviation coincide for both maximum carnage adversary and random attack adversary, by Lemma 13 in the section D.4 of appendix D of the article [1], page 36, we have $u_j(s') \leq u_j(s)$ for all of them.
Hence, no player has incentive to change her current strategy and the described network is an equilibrium with respect to the random attack adversary. □

5.5 Flowers

We show that multiple cycles of immunized and targeted vertices can form in equilibria (Nash, swapstable or linkstable) with respect to the random attack adversary. The ranges of the parameters C_E and C_I are slightly different from those of the maximum carnage adversary.

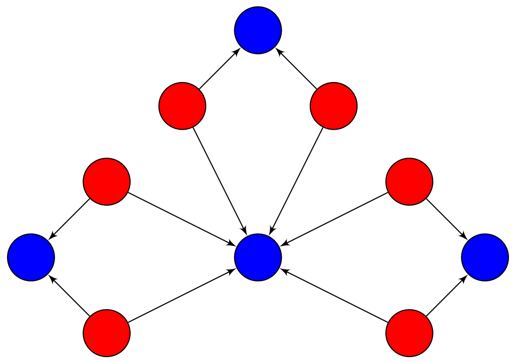

Figure 7: Example of flower equilibrium with respect to random attack adversary when $F = 3$, $k = 2$, $C_E = 0.4$ and $C_I = 2.5$

Lemma 5.10 *Let $n = F(2k-1)+1$. Consider a flower network containing F petals (cycles) of size $2k$ where all the cycles share exactly one vertex. Assume each petal is composed of alternating immunized and targeted vertices, and the shared vertex is immunized. Then the flower network can form in the equilibrium with respect to the random attack adversary when in each petal, the targeted vertices buy both of the edges to their immunized neighbors and $C_I \in (2, 3(F(2k-1))/(kF+1)), C_E \in (0, min\{(k-1)F-2, ((k-1)^2+5)/(2kF)\}), k \geq 2$ and $F \geq 3$.*

Proof. For any immunized player i her expected utility in this network is $u_i(s) = (n-1) - C_I$. She would not buy any edge because she already has the maximum expected size of connected component $n-1$ and it would be redundant. She would not change her immunization decision because her expected utility after the deviation is:

$$u_i(s') \leq \left(1 - \frac{3}{(kF+1)}\right) \cdot (F(2k-1)) < u_i(s) = (n-1) - C_I$$

since $C_I < 3(F(2k-1))/(kF+1)$.

For any unimmunized player j her expected utility in this network is $u_j(s) = (1 - 1/kF)(2k-1)F - 2C_E$. She would not buy more than two edges because she already achieves the maximum expected size of connected component $(1-1/kF)(2k-1)F$ buying only two edges. Her remaining deviations are:

1. Buying two edges and immunizing. The maximum expected size of connected component she can achieve buying two edges and immunizing is $(2k-1)F$. Her maximum expected utility after the deviation is:

$$u_j(s') = (2k-1)F - 2CE - C_I.$$

2. Buying one edge. To maximize the expected size of connected component she has to be the vertex with distance $k-1$ with respect to the shared immunized vertex and buy edge towards the shared immunized vertex. Her maximum expected utility after the deviation is:

$$u_j(s') = (1 - \frac{1}{kF})(2k-1)F - \frac{1}{kF}(1+3+\ldots+(k-3)+1+3+\ldots+(k-1)) - C_E.$$

3. Buying one edge and immunizing. Similar to case 2, her maximum expected utility after the deviation is:

$$u_j(s') = (2k-1)F - \frac{1}{kF}(1+3+\ldots+(k-3)+1+3+\ldots+(k-1)) - C_E - C_I.$$

4. Buying no edges. Her expected utility after the deviation is:

$$u_j(s') = (1 - \frac{1}{kF}).$$

5. Buying no edges and immunizing. Her expected utility after the deviation is:

$$u_j(s') = (1 - C_I).$$

In cases 1,2,3,4 and 5 above, because the expected utilities before and after the deviation coincide for both maximum carnage adversary and random attack adversary, by Lemma 14 in the section D.5 of appendix D of the article [1], page 38, we have $u_j(s') \leq u_j(s)$ for all of them.
Hence, no player has incentive to change her current strategy and the described network is an equilibrium with respect to the random attack adversary. □

5.6 Complete Bipartite Graph

We show that a complete bipartite graph can form in equilibria (Nash, swapstable or linkstable) with respect to the random attack adversary. The ranges of the parameters C_E and C_I are slightly different from those of the maximum carnage adversary.

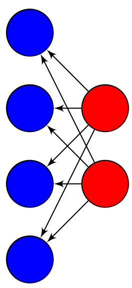

Figure 8: Example of complete bipartite graph equilibrium with respect to random attack adversary when $C_E = 0.3$ and $C_I = 3$

Lemma 5.11 *Consider a complete bipartite graph $G = (\mathcal{U} \cup V, E)$ with $|\mathcal{U}| = 2$ and $|V| \geq 1$. G can form in the equilibrium with respect to the random attack adversary if all the vertices in $\mathcal{U}$ are targeted, all the vertices in V are immunized, the vertices in U purchase all the edges in $E, C_E \in (0, 1/2]$ and $C_I \in ((n-1)/2, 2(n-1)/3)$.*

Proof. For any immunized player i her expected utility in this network is $u_i(s) = (n-1) - C_I$. She would not buy any edge because she already has the maximum expected size of connected component $n-1$ and it would be redundant. She would not change her immunization decision because her expected utility after deviation is

$$u_i(s') = \frac{(n-1)}{3} < u_i(s) = (n-1) - C_I$$

since $C_I < 2(n-1)/3$.

For any unimmunized player j her expected utility in this network is $u_j(s) = (n-1)/2 - (n-2)C_E$. Notice that she would not buy an edge to the other unimmunized player. If she does not immunize and purchases an edge to the other unimmunized player then she will be the part of the unique targeted region, with negative expected utility. If she immunizes, the other unimmunized player will be the unique targeted vertex and be killed by the adversary, so she prefers to not purchase the edge to reduce her cost. Her remaining deviations are:

1. Buying $k \in \{0, \ldots, n-3\}$ edges to immunized vertices. Her expected utility after the deviation is:

$$u_j(s') = \frac{k+1}{2} - kC_E.$$

2. Buying $k \in \{0, \ldots, n-3\}$ edges to immunized vertices and immunizing. Her expected utility after the deviation is:

$$u_j(s') = (k+1) - kC_E - C_I.$$

In above cases, because the expected utilities before and after the deviation coincide for both maximum carnage adversary and random attack adversary, by Lemma 15 in the section D.6 of appendix D of the article [1], page 39, we have $u_j(s') \leq u_j(s)$ for all of them.

Hence, no player has incentive to change her current strategy and the described network is an equilibrium with respect to the random attack adversary. $\square$

6 Convergence of Best Response Dynamics for Random Attack Adversary

A problem we face in the computation of equilibrium of our game is the convergence of the best response dynamics. In the best response dynamics we "play" a strategic game by rounds, in each round we compute current best response for each player and update their strategies based on their best response by a fixed order. We stop when no player can do better by changing her strategy, by definition the game reaches to an equilibrium and the best response dynamics converges at this point, but this does not always happen. We suppose that this process will converge in most cases of our game. However, under some tie breaking rules, the best response dynamics can cycle and the process will not converge. The appendix G of the article [1], page 59 gave us an example of cycle in the best response dynamics by tie breaking rule for maximum carnage adversary.

We now prove that there exists a tie breaking rule which causes the best response dynamics to cycle with respect to random attack adversary. Our example is a subgraph of the example given by Goyal et al. [1] with $n = 8$.

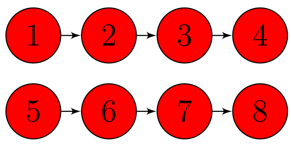

Figure 9: Example 6.1

Example 6.1 *Consider the network with $n = 8$ unimmunized vertices, $C_I = 8, C_E = 1/4$ and all vertex $i, 1 \leq i \leq n$ purchases one edge to vertex $i + 1$ except vertices 4 and 8, to be the initial configuration in running the Nash best response dynamics. If the vertices Nash best response in the increasing order of their labels, then there exists a tie breaking rule which causes the best response dynamics to cycle with respect to a random attack adversary.*

Proof. Since the components are symmetric, we only analyze one of the components. Vertices 1 and 2 are best responding with utility $(1 - 1/2) * 4 - 1/4 = 7/4$, vertex 3's best response is to drop her edge with utility $(1 - 3/8) * 3 = 15/8 > 7/4$, vertex 4's best response is to connect back to the same component she was a part of before vertex 3's best response, we break ties by forcing vertex 4 to purchase an edge to vertex 1.
After the first round, we are in the same pattern as before, but the labels of the vertices are different. So in the next round vertex 2 would drop her edge and vertex 3 would buy an edge to vertex 4. In the third round, vertex 1 would drop its edge. In the fourth round, and vertex 2 would buy an edge to vertex 3. In the fourth round, vertex 4 would drop its edge. In the fifth round, vertex 1 would buy an edge to 2, vertex 3 would drop its edge and vertex 4 would buy an edge to vertex 1. So we are back in the same configuration that we were at the beginning. If we break ties by

forcing vertex 4 to purchase an edge to vertex 3 then all vertices are in their best response and there is no cycle. □

As in the case of the maximum carnage adversary, we suspect that this phenomenon is the result of adversary tie-breaking and the ordering of the vertices.

7 Experimentation

Goyal et al. study in [1] about swapstable best response dynamics in simulations with different values of C_E and C_I. They observe a rapid convergence (a sublinear growth of simulation rounds when the number of players increases) to the equilibrium and thus conjecture the general and fast convergence of swapstable best response dynamics, contrasting with the fact that they showed that best response dynamics may cycle. Moreover, they leave whether there exists an efficient algorithm of computing the (Nash) best response of the model as an open question.

In 2017, Friedrich et al. in [2] have answered this question by giving an algorithm of computing the (Nash) best response of the model in polynomial time with respect to n. We decide to use this algorithm for our experiments as recently Zhang et al. in [3] have implemented this algorithm efficiently. In our experiments, we will simulate the best response dynamics with respect to random attack adversary for different values of parameters. While we know that best response dynamics can also cycle with random attack adversary, we also experiment in order to find out if by changing tie-breaking policy or changing the order of players, the best response dynamics converges rapidly empirically.

Our method of simulation is described as the follow: The parameters are the number of players n, the cost of buying an edge C_E and the cost of immunization C_I. We first create a graph of n unimmunized vertices with edges generated randomly according to an initial edge density p with the Erdős–Rényi model [11]. For each iteration, for each player in the increasing order of their labels we compute their best response with respect to random attack adversary in current game state fixing the strategies of other players using the implemented algorithm in [3], then for players with better utility after the best response we replace their strategies by their best response modifying the graph and for players with equal utility after the best response we do not modify their strategies. We will repeat those steps until we obtain a Nash equilibrium or the process exceeds the limit of rounds and does not converge.

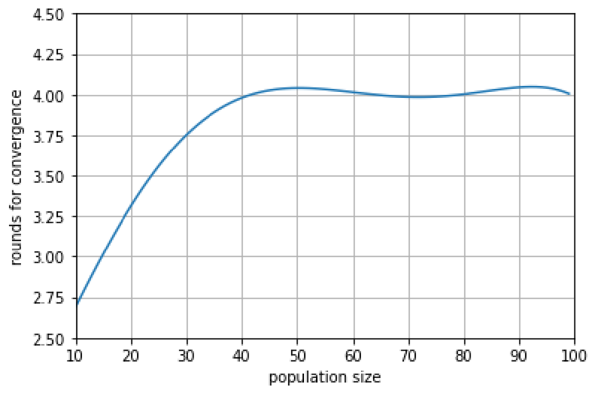

Figure 10: Average number of rounds until convergence for $C_E = C_I = 2$

The figure above shows results of our first experiment. For $n = 10, 20...100$, we generate 20 graphs for each n randomly with the Erdős–Rényi model such that their initial average degree $= 1$ and the number of immunized vertices is $n/10 + 1$. Then we set $C_E = C_I = 2$ and compute the average number of iterations of simulation until convergence for each n using the generated graphs as the input of our simulation algorithm. We have observed a fast convergence (average number of rounds until convergence < 5) with respect to random attack adversary for all population sizes and the convergence does not become harder (the average number of rounds approximates to 4) when we increase n.

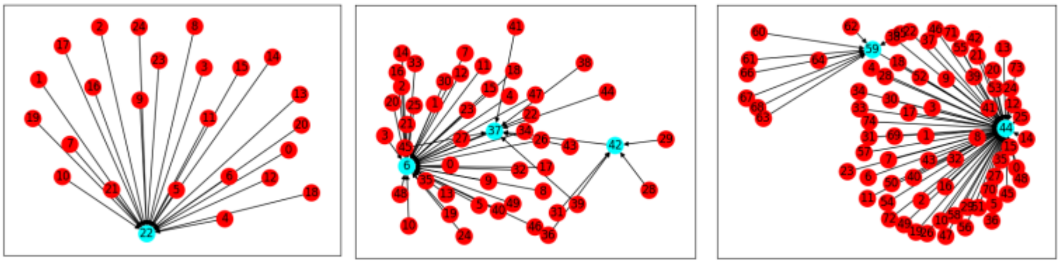

Figure 11: Results of Hub-Spoke and tree equilibrium. Left: $n = 25, C_E = C_I = 1.5$. Middle: $n = 50, C_E = C_I = 2$. Right: $n = 75, C_E = C_I = 2.5$.

We have also observed that in most cases the simulation will converge to a Hub-Spoke equilibrium which is an efficient equilibrium with one immunized vertex (hub) and $n - 1$ unimmunized vertices (spokes) buying an edge towards the hub vertex, the utility of the hub vertex is $(n - 1) - C_I$ and the utility of any spoke vertex is $(n - 2) - C_E$. Sometimes it will converge to a tree equilibrium with few immunized vertices ($k \leq n/2$ immunized non-leaves), the utility of any immunized vertex is $(n - 1) - C_I$ or $(n - 1) - C_I - C_E$ and the utility of any unimmunized vertex is $(1 - 1/(n - k))(n - 1) - C_E$. The social welfare is better than examples of resulted equilibrium networks for maximum carnage adversary presented in [1] which have cycles and/or multiple immunized vertices. This phenomenon is explained by the behavior of the best response algorithm. When there exist multiple disconnected immunized vertices (potential hubs), the players will prefer to connect to the hub with the largest connected component and the non-hub immunized players will prefer to connect to the hub if they have enough neighbors or connect to the hub and unimmunize if most of their neighbors have switched to the unique hub. The result is either a Hub-Spoke equilibrium or a tree equilibrium in which the connected immunized vertices can be seen as a single hub.

In the rest of the cases the obtained equilibrium network is an empty graph with all unimmunized vertices, the utility of any player in this equilibrium is $1 - 1/n$ so it has a low social welfare. The figure above shows results of our experiment to estimate the probability that the resulting equilibrium network is empty for different values of C_E and C_I. For $C_E = C_I = 2, 4 \ldots 22$, we generate 20 graphs for $n = 50$, initial average degree $= 1$ and the number of immunized vertices $n/10 + 1$. Then we compute the percentage of empty equilibrium in convergence. The estimated probability of empty equilibrium is higher for higher values of C_E and C_I because

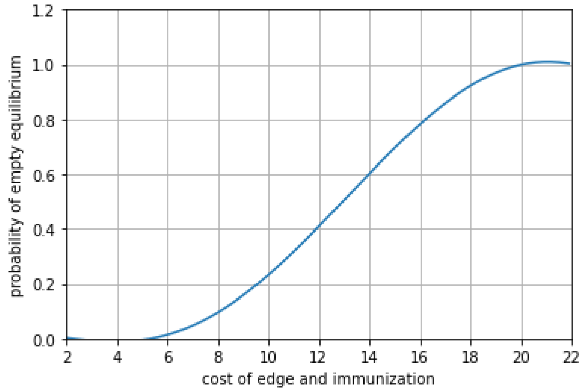

Figure 12: Estimated probability of empty equilibrium for different values of C_E and C_I, $n = 50$.

the players prefer more to drop edges and unimmunize to reduce the cost.

8 Conclusion and Future Works

The initial objective of our research was to gain an understanding of the model focusing on the random attack adversary. Based on previous works, we studied connectivity, social welfare, diversity of equilibrium networks and convergence of the best response dynamics with respect to the random attack adversary. We then concluded our research with simulations of the best response dynamics using the algorithm that computes the (Nash) best response efficiently implemented by Zhang et al. [3]

Goyal et al. in [1] have studied connectivity and social welfare for the maximum carnage adversary and their results are based on properties of maximum carnage adversary. By proving that these properties also hold for the random attack adversary, we can extend their results of connectivity and social welfare to random attack adversary: when $C_E > 1$ the resulting non-trivial equilibrium network G is a connected graph, when C_E and C_I are constants and $C_E > 1$ then the social welfare of non-trivial Nash or swapstable equilibrium network G is $n^2 - O(n^{5/3})$.

In comparison with maximum carnage adversary, random attack adversary has similar structures of equilibrium networks, although not all equilibrium networks with respect to the maximum carnage adversary are equilibria with respect to the random attack adversary and vice versa. While the majority of the types of equilibrium networks for the maximum carnage adversary studied by Goyal et al. in [1] (empty graph, trees, cycles, flowers, complete bipartite graph) are also equilibria for the random attack adversary with slight difference in parameters, the *forest* networks have a particular case of equilibrium (2 disjoint targeted trees of size F) with respect to the random attack adversary but the most of the *forest* network can only form in equilibria with respect to the maximum carnage adversary.

Regarding the best response dynamics for the random attack adversary, we proved that it can cycle under a certain tie breaking rule as in the case of maximum carnage adversary. In our experiments, we have observed a fast convergence of the best response dynamics for the random attack adversary and in most cases it will converge to either a Hub-Spoke equilibrium or a tree equilibrium with few immunized vertices, which is more efficient than examples of resulted equilibria for maximum carnage adversary in [1]. In the rest of the cases the best response dynamics will converge to an empty equilibrium with all unimmunized vertices, the probability of empty equilibrium increases for higher values of C_E and C_I.

Personally, during the development of this project I have learned to read research articles and study their works, then combine them with knowledge obtained from subjects of the degree to inspire my research. Moreover, I have learned to organize my research results into a formal paper. I was able to achieve the initial objective of the research and was satisfied with the results. Finally, this project has introduced me to scientific research, especially in the field of Game Theory and Network Formation Games.

For possible future works, we enumerate some topics which can be interesting to researchers:

- The welfare of non-trivial Nash or swapstable equilibrium network G for maximum carnage adversary and random attack adversary is $n^2 - O(n^{5/3})$ when C_E and C_I are constants. How it can be changed when C_E and C_I are functions depending on n ($C_E = f(n)$ and $C_I = g(n)$)?
- Can the upper bound of achievable welfare of non-trivial Nash or swapstable equilibrium network G for maximum carnage adversary and random attack adversary, $n^2 - O(n^{5/3})$, be improved?
- The adversary of the model can only choose 1 player to spread the attack. What will be the behavior of game dynamics if the adversary can attack $k \geq 1$ players?
- We suppose that the immunization is always perfect and protects against the adversary attack with 100% probability. What will be the effects of an imperfect immunization which can fail to survive the attack with probability p?
- The Section 7 is short due to the timing and planning of the project. We could do experiments with more variations in order to be able to reach better-founded conclusions.

9 References

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A Appendix - Project Management

A.1 Temporal Planning

A.1.1 Task Definition

In this section we will identify and describe all the tasks that will be done throughout the project by order. For each task we will give its definition, duration and dependencies. This will be summarized in the Section A.1.3.

Project management
Project management is the first group of tasks to be done because of its importance on the planning of the final work. It defines context, scope and temporal planning of the project and studies its budget and sustainability. The tasks of the project management are shown below:

- **Context and scope**: Contextualize the thesis and describe its scope. Identify the problem to be resolved, previous works, requirements, risks and the work methodology.
- **Time planning**: Define all the tasks of the project and its dependencies, resources and estimated hours. Propose solutions to potential obstacles and risks.
- **Budget and sustainability**: Analyze the budget of the project and make a self-assessment of the sustainability report. Describe elements and costs of the budget. Propose mechanisms for controlling potential budget deviations.
- **Final document**: Integration of all the tasks above in a final document, correcting mistakes.
- **Meetings**: Online meeting with the director of the project arranged every week with the objective of discussing the progress of the project and the next tasks to do.

Previous studies
As we have said in the title, this project is an introduction to the scientific research. Before starting the project, we have to do research of previous studies of the past. This will help us to familiarize with the area of the project. Some of our previous studies are:

- **Basics of Game Theory**: Study of basic concepts and definitions of Game Theory.
- **Study of the model** of Network Formation Games with adversary attack and immunization.
- **Study of previous works**: results of *maximum carnage adversary*, efficient algorithm for computing the best response, etc.

Theoretical part
The theoretical studies are important in this project, since we need them for the interpretation of experimental results. In the theoretical part we will focus our study mainly on the *random attack adversary* and prove the following properties of it using previous results:

- Proof of **diversity of equilibrium networks** regarding the *random attack adversary*.
- Proof of **connectivity and social welfare** regarding the *random attack adversary*.
- Proof of **convergence of best response dynamics** regarding the *random attack adversary*.

Programming part
In the programming part we will implement the algorithm of simulation to prepare the experiment of network formation for the *random attack adversary*. It can be divided into two tasks:

- Design and implement the **algorithm of simulation** for the *random attack adversary*.
- **Testing**: test the correct functioning of the code implemented.

Experimentation, analysis and conclusion
After the implementation of the algorithm we can start our simulation and extract conclusions from its results. We will compare experimental results with our theoretical studies and make comparisons between different models of adversaries. This part has the following tasks:

- **Simulation** of the network formation for the *random attack adversary* using the implemented code with changes in various parameters.
- **Comparison of models**: compare both theoretical and experimental results between different models, try to find other properties of *random attack adversary*.
- **Conclusion**: analyze our theoretical and experimental results and obtain conclusions of our research.

Finally, we will write the **documentation** of the project which contains all results obtained in previous tasks and **prepare for the oral defense**, considering possible questions regarding our work.

A.1.2 Resources

In the development of this project, various human and material resources are required:

Human resources
The human resources needed in this project are:

1. **The researcher**, who works on the planning of the project, develops the project and presents at the oral defense.

2. **The director of the thesis**, Carme Alvarez Faura, who will mentor the researcher in the technical part of the project.

3. **The GEP tutor** who will guide the researcher to manage the project correctly.

Material resources
We will use some software and hardware resources in this project to perform experiments and write documentations. We also consider material resources used for communication.

1. **Books and papers** are needed for the research of the theoretical part and previous works.

2. **Google Meet** is needed for online meetings with the director.

3. **Google Jamboard** is needed for illustration in online meetings in case of questions.

4. **Gmail** will be used to communicate with the director, send documents that contain demonstrations, questions and newer results.

5. **Google Drive** will be used to write documents and make backups of them.

6. **Atenea and Racó** where we will deliver documentations of the project.

7. **Overleaf and *Latex*** We will use Overleaf for the text formatting using *Latex* in the final document because we already have some experiences with it.

8. A **programming language** to implement the algorithm for the experimentation part.

9. **Github** as the version control tool of the code since it is simple and stores previous versions for recovery.

10. **Computer**. All documents and codes in this project are created with an ASUS Zenbook computer, with 4GB of RAM and Intel(R) Core (TM) M-5Y10c CPU, the same used to execute all experiments.

A.1.3 Summary of the Tasks

In the following table we have summarized all the tasks, with their dependencies, required time and resources (Hx indicates human resource x, Mx indicates material resource x).

ID	Name	Time(h)	Dependencies	Resources
T1	**Project management**	94		
T1.1	Context and scope	28	-	H1, H3, M5, M6, M10

T1.2	Time Planning	12	-	H1, H3, M5, M6, M10
T1.3	Budget and sustainability	12	-	H1, H3, M5, M6, M10
T1.4	Final Document	23	T1.1, T1.2, T1.3	H1, H3, M5, M6, M10
T1.5	Meetings	19	-	H1, H2, M2, M3, M4, M10
T2	**Previous studies**	21		
T2.1	Basics of Game Theory	5	-	H1, M1, M10
T2.2	Study of the model	5	-	H1, M1, M10
T2.3	Study of previous works	11	-	H1, M1, M10
T3	**Theoretical part**	135		
T3.1	Diversity of equilibrium networks	45	T2	H1, M1, M10
T3.2	Connectivity and social welfare	45	T2	H1, M1, M10
T3.3	Convergence of best response dynamics	45	T2	H1, M1, M10
T4	**Programming part**	90		
T4.1	Algorithm of simulation	45	T2	H1, M8, M9, M10
T4.2	Testing	45	T2, T4.1	H1, M8, M9, M10
T5	**Experimentation, analysis and conclusion**	70		
T5.1	Simulation	45	T2, T4	H1, M8, M9, M10
T5.2	Comparison of models	10	T2, T3, T4, T5.1	H1, M10
T5.3	Conclusion	15	T5.2	H1, M10
T6	**Documentation**	70		
T6.1	Theoretical part documentation	15	T2, T3	H1, M5, M10
T6.2	Programming part documentation	15	T2, T4	H1, M5, M10
T6.3	Final documentation	40	T5, T6.1, T6.2	H1, M5, M6, M7, M10
T7	Prepare for the oral defense	20	T6.3	H1, M10

Total	500

Table 2: Summary of the tasks

A.1.4 Gantt Chart

The Gantt chart of the project is shown below. We have considered the number of hours of each task and dependencies between tasks. We can see that T2 (previous studies) can be done in the first week before T1 (project management), T1.5 (online meetings) must be arranged every week and we have to start T3 (theoretical part) concurrently with the project management to meet the deadline. The project lasts approximately 500 hours and it is planned to work 3.75 hours approximately every day for a total of 133 days (19 weeks).

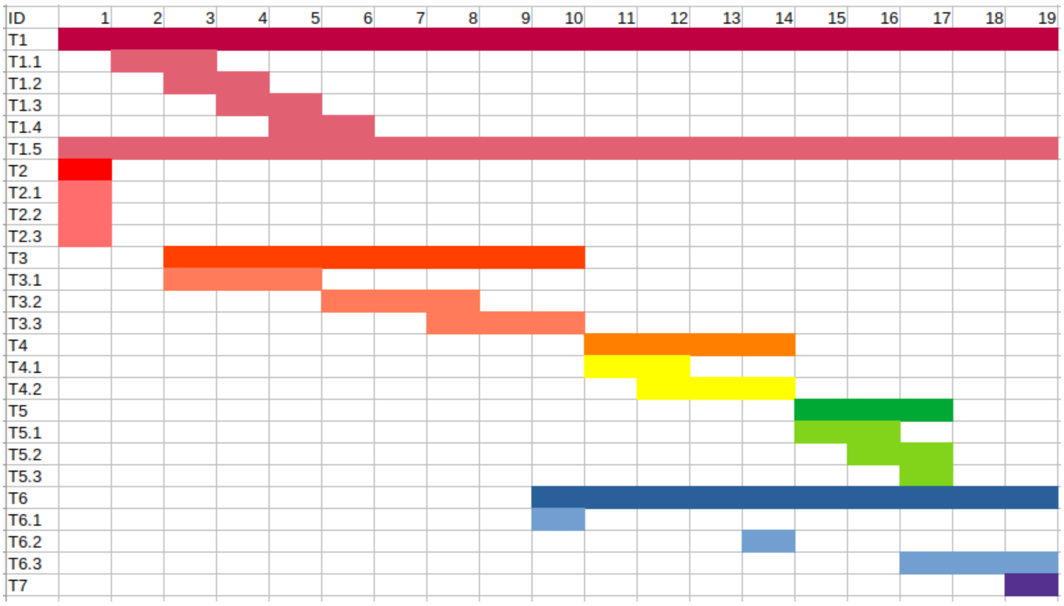

Figure 13: Gantt chart of the project

A.1.5 Risk Management

We have presented in the Section 1.2.3 several potential obstacles and risks that we may encounter during the development of this project. Here we will analyze their effects, our alternatives and we will propose solutions to them.

- **Timing of the project** [**High risk**]. We have three months for the development of the project before the final delivery. This is enough to complete the project but is not expected to leave time for optional tasks. A risk of this is the underestimation of the duration of some task. In this case as plan B we have to redo the planning and re-estimate hours needed for each task. We can also increase working hours per day if it is necessary.

- **Lack of theoretical knowledge** [**Medium risk**]. The lack of theoretical knowledge in the field leads to the risk of getting stuck in the theoretical part and increases the duration of the project unexpectedly. Our solution to this obstacle as plan B is to redo the planning, dedicate more time in theoretical studies and increase working hours per day if it is necessary.

- **Computation time of the simulation** [**Medium risk**]. Our method of simulation is to consider each iteration a round in which players can make moves to improve their welfare and stop when no player can do better by making any move. A risk of our project is that the computational cost and time complexity of the implemented algorithm can be high. This can increase the duration of the simulation task, our alternative for it is to stop the simulation when it reaches a maximum number of iterations and will result in an approximate solution of the optimum. This alternative will replace the task T5.1 Simulation with the same time, dependencies and resources.
- **Mistakes in theoretical proofs and algorithm implementation** [**Medium risk**]. As the proposed solution, we will work with our project director to check our demonstrations in the theoretical part by organizing extra meetings in case of doubts. We will spend more time on debugging if the code does not work.
- **Mistakes in writing style used in documentations** [**High risk**]. We have to present our ideas to readers clearly and there is a high risk of making grammar and spelling mistakes. To learn how to use formal language, we will study examples of research works. Software resources like auto-checking tools are also useful for avoiding grammar and spelling mistakes.
- **Online meeting** [**Low risk**]. In the time of pandemic, the meeting with the director of this project will be online. This makes the explanation of the results harder because we do not have a physical blackboard as in real life. The solution of this problem is to use tools like Google Jamboard to help our explanations and send documents to be discussed by email before the meeting so we have time to understand it.

A.2 Budget

In this section we will discuss the budget of the project. We will include staff costs, generic costs and costs of contingencies and incidentals. Moreover, we will propose mechanisms for controlling potential budget deviations.

A.2.1 Staff Costs

To compute all personnel costs properly, we must define first which roles are needed in this project and their cost per hour. The total cost for one task will be the sum of costs of the personnel involved in the task (multiplying their cost per hour by the number of hours of their working time). The gross salary per hour is obtained by dividing average salary per month (average annual salary/12) by the average number of working hours per month (160 approximately). The cost of social security is included by multiplying the gross salary by 1.35.

In this project 5 types of personnel are defined and their roles are going to be performed by the researcher (myself), the director and the GEP tutor.

- **Project Manager**: responsible for the planning and development of the project.
- **Researcher**: studies theoretical aspects of the project as the model, makes experiments with them.
- **Developer**: implements the algorithm of simulation of the model for the *random attack adversary*.
- **Tester**: verifies the correct functioning of the code implemented by the programmer.
- **Analyst**: analyzes theoretical and experimental results of the project and obtains conclusions.

We will show cost per hour for each role[14][15][16], time per task, cost per task and total personnel costs in the following tables:

Role	**Gross salary (€)/h**	**Cost (€)/h including SS**	**Role played by**
Project Manager	16	21.6	Researcher, director, GEP tutor
Researcher	14	18.9	Researcher
Developer	15	20.25	Researcher
Tester	11	14.85	Researcher
Analyst	10	13.5	Researcher

Table 3: Cost per hour of the different roles

Task	**Time (h)**	**Project Manager**	**Researcher**	**Developer**	**Tester**	**Analyst**	**Cost (€)**
Project management	**94**	**94**	**19**	**19**	**19**	**19**	**3312.9**
Context and scope	28	28	0	0	0	0	604.8
Time Planning	12	12	0	0	0	0	259.2
Budget and sustainability	12	12	0	0	0	0	259.2
Final document	23	23	0	0	0	0	496.8
Meetings	19	19	19	19	19	19	1692.9
Previous studies	**21**	**0**	**21**	**0**	**0**	**0**	**396.9**

Basics of Game Theory	5	0	5	0	0	0	94.5
Study of the model	5	0	5	0	0	0	94.5
Study of previous works	11	0	11	0	0	0	207.9
Theoretical part	**135**	**0**	**135**	**0**	**0**	**0**	**2551.5**
Diversity of equilibrium networks	45	0	45	0	0	0	850.5
Connectivity and social welfare	45	0	45	0	0	0	850.5
Convergence of best response dynamics	45	0	45	0	0	0	850.5
Programming part	**90**	**0**	**0**	**45**	**45**	**0**	**1579.5**
Algorithm of simulation	45	0	0	45	0	0	911.25
Testing	45	0	0	0	45	0	668.25
Experimentation, analysis, conclusion	**70**	**0**	**12.5**	**15**	**15**	**27.5**	**1134**
Simulation	45	0	0	15	15	15	729
Comparison of models	10	0	5	0	0	5	162
Conclusion	15	0	7.5	0	0	7.5	243
Documentation	**70**	**0**	**27.5**	**5**	**5**	**32.5**	**1134**
Theoretical part documentation	15	0	7.5	0	0	7.5	243
Programming part documentation	15	0	0	5	5	5	243
Final documentation	40	0	20	0	0	20	648
Prepare for the oral defense	**20**	**20**	**0**	**0**	**0**	**0**	**432**
Total (CPA)	**500**	**114**	**215**	**84**	**84**	**79**	**10540.8**

Table 4: Time and cost per task

Role	**Hours**	**Cost (€)**

Project Manager	114	2462.4
Researcher	215	4063.5
Developer	84	1701
Tester	84	1247.4
Analyst	79	1066.5

Table 5: Total personnel costs

A.2.2 Generic Costs

Amortization costs
Due to the fact that all the software resources used for this project are free, we will only take into account the amortization of hardware resources.

The project lasts approximately 500 hours and we planned to work 3.75 hours per day, for a total of 133 days. We will use an ASUS Zenbook laptop to develop 100% of the project. The formula to compute the amortization is the following:

$$Amortization = \frac{Resource\,price \times Hours\,used}{Years\,of\,use \times Days\,of\,work \times Hours\,per\,day}$$

Hardware	**Price (€)**	**Years of use**	**Hours used (h)**	**Amortization (€)**
ASUS Zenbook Laptop	750	5	500	150.4
Total	**750**	-	-	**150.4**

Table 6: Amortization costs of hardware resources

Indirect costs
We do not have to consider travel costs in this project because during the time of pandemic we are working from home and will only arrange online meetings. We must take into account the following indirect costs:

- **Internet cost**: The internet service costs approximately 40€ per month. Thus the total cost is 40(€/month)*133(days)*3.75(hours per day)/30(days per month)/24(hours per day) = 27.7€.
- **Electricity cost**: The electricity price of the provider is 0.127003 €/kWh.[17]
  The power of our ASUS laptop is 45W, we will consume in total 45(W)*133(days)*3.75(hours per day) = 22.44kWh and the total electricity cost is 22.44(kWh)*0.127003(€/kWh) = 2.85€.
- **Rent cost**: During the development of this project I will be working from my house all the time. The rent cost of this house is 700€/month and the duration of this project is 5 months. Hence the total rent cost is 3500€.

Generic cost of the project
A summary of all generic costs of the project is shown in the table below:

Description	**Cost (€)**
Amortization	150.4
Internet	27.7
Electricity	2.85
Rent	3500
Total (CG)	**3680.95**

Table 7: Generic cost of the project

A.2.3 Contingencies and Incidentals

Contingencies
We have to define a contingency margin for our project as a percentage of the total value of the budget, this can be useful to cover unexpected obstacles due to incomplete information or oversights. The contingency margin to add for CPA is 15%, because an unexpected event can delay the working time and it is necessary to increase the worker salary. We will use this contingency margin to cover possible cost deviations (variance in price of labour, price of resources and consumption of time) after finishing a task. However the CG is less variant due to the deadline of this project so its contingency margin will be 5%.

Incidentals
We have to include the incidental cost as well, which are related to the obstacles that are identified in previous sections. The cost for each incident is calculated by multiplying the estimated cost with the likelihood of occurrence. We have also considered all tasks in which they may cause deviations.

Incident	**Estimated cost per hour (€)**	**Estimated cost (€)**	**Risk (%)**	**Cost (€)**	**Tasks affected**
Timing of the project (40h)	17.82	712.8	50	356.4	All tasks
Lack of theoretical knowledge (30h)	18.9	567	25	141.75	Previous studies, Theoretical part
Computation time of the simulation (10h)	17.55	175.5	25	43.88	Simulation

Mistakes in proofs and codes (40h)	16.875	675	25	168.75	Theoretical part, Programming part
Mistakes in writing style (20h)	18.9	378	50	189	Documentation
Online meeting (19h)	0	0	10	0	All tasks
Total	-	-	-	**899.78**	-

Table 8: Incidental cost of the project

A.2.4 Total Cost of the Project

We have computed the total cost of the project based on calculations we did in previous sections:

Activity	**Cost (€)**
CPA	**10540.8**
Project management	3312.9
Previous studies	396.9
Theoretical part	2551.5
Programming part	1579.5
Experimentation, analysis, conclusion	1134
Documentation	1134
Prepare for the oral defense	432
CG	**3680.95**
Amortization	150.4
Internet	27.7
Electricity	2.85
Rent	3500
Contingencies	**1765.17**
Incidentals	**899.78**
Timing of the project	356.4
Lack of theoretical knowledge	141.75
Computation time of the simulation	43.88
Mistakes in proofs and codes	168.75
Mistakes in writing style	189

Online meeting	0
Total	**16886.7**

Table 9: Total cost of the project

A.2.5 Management Control

To control cost deviations during project execution, we have defined our cost control model to compare and assess variances between the budget and the actual costs incurred at the end of a stage of the project. This can help us to keep our budget flexible.

We will take into account the following parameters to evaluate where, why and how much deviations can happen during the process of a task: its actual consumption in hours, its estimated consumption in hours and estimated/actual costs for human resources, internet and electricity.

Every time we finish a task on the Gantt chart we will calculate all its cost deviations (CPA, CG) using the following formulas:

- **Variance in cost by rate**: this can be caused by the variance in price of labour (personnel salary) or price of provider (internet, electricity). For each element we can compute its deviation with: *Price deviation = (estimated cost - actual cost) * actual consumption in hours.* This determines if we have underestimated the cost of a resource.
- **Efficiency variance**: this can be caused by the variance in consumption of time (task time, hardware/internet/electricity usage hours). For each element we can compute its deviation with: *Consumption deviation = (estimated consumption in hours - actual consumption in hours) * estimated cost.* This determines if we have underestimated the consumption time of a resource.
- **Total deviation** is the sum of the price deviation and the consumption deviation. *Total deviation = Price deviation + Consumption deviation.*

In the following table we will show all elements of the cost deviation:

Name	**Variance in cost by rate**	**Efficiency variance**
Human resources	(estimated cost - actual cost) * actual consumption	(estimated consumption - actual consumption) * estimated cost
Amortization	No deviation	(estimated consumption - actual consumption) * estimated cost
Internet	(estimated cost - actual cost) * actual consumption	(estimated consumption - actual consumption) * estimated cost

Electricity	(estimated cost - actual cost) * actual consumption	(estimated consumption - actual consumption) * estimated cost
Rent	No deviation	No deviation
Incidental	(estimated cost per hour - actual cost per hour) * actual consumption	(estimated consumption - actual consumption) * estimated cost per hour
Total	Human resources + Amortization + Internet + Electricity + Incidental	

Table 10: Types of cost deviations

We will register all deviations in a table, in case the total deviation is positive it means that we have overestimated the money needed for the task and can reserve this money for future incidents. Otherwise we have to use our budget of contingencies to cover the negative deviation. We will know where the variance occurred by looking at each type of cost deviations.

A.3 Sustainability

In this section we will study the sustainability of the project by answering questions of the sustainability matrix. For this project, a great part of its economic dimension is the budget. We have to look not only at the costs of the project, but also its future economic impacts. The application of its results in practice affects its economic and social dimensions. In the environmental dimension the primary topic is the resources used and reusability and we have to analyze the ecological footprint of the project.

A.3.1 Environmental Dimension

We have quantified the environmental impact of the project in the Section A.2. The environmental impact of the project is little due to its theoretical nature. We have avoided waste of printed paper by using online resources. We consume in total 22.44kWh of electricity and the mix of the Spanish electricity grid published by the CNMC on April 16, 2021 [18] is 0.25 kg CO_2/kWh, thus the total CO_2 emission of our project is 22.44kWh*0.25kg/kWh = 5.61kg. To minimize the environmental impact of our project, we have reused resources like online tools and other research results. For the experimental part we have reused implemented codes of Zhang et al. in [3]. We could have reduced the use of paper to zero if we had only used informatic resources. Furthermore, we could have reduced the consumption of electricity by using a less powerful computer.

During the lifespan of this project its environmental impact will be low because the article contains its theoretical and experimental results and does not use other resources. Other researchers can make references of it in their works and reuse our results. Therefore the energy consumption will be decreased as they do not have to recalculate theoretical results and redo experiments we have done.

There are environmental risks that can increase the ecological footprint of the project. If we use the computer longer, we will consume more electricity and emit more CO_2. If the computer stopped working we would have used computers of FIB labs to continue the project which increases the consume of electricity.

A.3.2 Economic Dimension

We have quantified the cost (human and material resources) of the project in the Section A.2. We have calculated personnel costs by activity, generic costs like amortization, internet, electricity and rent, contingencies and incidentals. The staff cost of the project is reasonable (10540.8€) and the generic cost is small due to the theoretical nature of the project (3680.95€). We have used free software resources to reduce the cost of the project.

During the lifespan of this project it will not produce maintenance costs because the article contains its theoretical and experimental results and does not use other resources.

The economic risk of this project is that its results may not be enough for the use of companies and society because the model it was based is simple and approximates less to the reality.

A.3.3 Social Dimension

This project is a good introduction to scientific research for me, it makes me interested to learn more about an area in which I do not have experience. I have analyzed the planning and the budget in the project management part and have started to study the concept of sustainability because of the questions proposed in the sustainability report.

Researchers, students and organizations like companies and governments can benefit from our solution. It can be used in research works, education or practice in real life, such as financial decisions, computer systems and epidemic control. It does not have any social risks since its theoretical and experimental results cannot harm any segment of the population.

B Appendix - Code for Simulation

For simulation of best response dynamics in the Section 7, we implemented our algorithm in the file *main.py* below, using packages implemented by Zhang et al.[3]:

```python
from bestResponse import bestResponse
import networkx as nx
import random as rd

from utils.graph_utils import drawNetwork, utility_s, paintTarget
from PossibleStrategy.PossibleStrategy import getTargetRegion

# Generate a random graph using the Erdos Renyi model with parameters n and p
def randomGraph(n, p):
    G = nx.generators.random_graphs.erdos_renyi_graph(n, p, directed=True)
    immunized = rd.sample(range(0, n), int(n / 10.0 + 1))
    dict_immunization = {node: (False if node not in immunized else True) for node in G.nodes()}
    dict_size = {node: 1 for node in G.nodes}
    nx.set_node_attributes(G, dict_immunization, 'immunization')
    nx.set_node_attributes(G, dict_size, 'size')
    return G

# Get the current utility of player v in G with strategy r, edge cost = alpha and immunization cost = beta
def getutility(G,v,r,alpha,beta):
    G1 = G.copy()
    G_undirected = G1.to_undirected()
    G2_undirected = paintTarget(G_undirected)
    G2_undirected.nodes[v]['immunization'] = r[1]
    G2_undirected.nodes[v]['target'] = not r[1]
    TR = getTargetRegion(G2_undirected, [node for node in G2_undirected])
    return utility_s(G2_undirected, TR, v) - len(r[0]) * alpha - r[1] * beta

# Simulation of best response dynamics for G, edge cost = alpha and immunization cost = beta
def sim(G, alpha, beta):
    G1 = G.copy()
    r = [] # current strategy
    for i in range(G1.number_of_nodes()):
        G2 = G1.copy()
        resp, utility = bestResponse(G2, i, alpha, beta)
        current = []
```

```
        current.append(G1.edges(i))
        current.append(G1.nodes[i]['immunization'])
        u = getutility(G1,i,current,alpha,beta)
        if utility > u: # compare the utility of current strategy with best response and update graph
            G1.nodes[i]['immunization'] = resp[1]
            current_strategy = [(i, k) for k in G1.adj[i]]
            G1.remove_edges_from(current_strategy)
            new_strategy = [(i, k) for k in resp[0]]
            G1.add_edges_from(new_strategy)
            r.append(resp)
        else:
            r.append(current)
        print(i)
        print(utility, u)
    drawNetwork(G1)
    rounds = 1
    fi = False
    while not fi:
        fi = True
        for i in range(G1.number_of_nodes()):
            print('player ', i)
            u = getutility(G1, i, r[i], alpha, beta)
            G2 = G1.copy()
            resp, utility = bestResponse(G2, i, alpha, beta)
            print(utility,u)
            if utility > u: # compare the utility of current strategy with best response and update graph
                r[i] = resp
                fi = False
                G1.nodes[i]['immunization'] = resp[1]
                current_strategy = [(i, k) for k in G1.adj[i]]
                G1.remove_edges_from(current_strategy)
                new_strategy = [(i, k) for k in resp[0]]
                G1.add_edges_from(new_strategy)
        rounds = rounds + 1
        drawNetwork(G1)
    return rounds, G1

if __name__ == '__main__':
    n = 50
    G = randomGraph(n,0.02)
    drawNetwork(G)
    alpha = 2
    beta = 2
    rounds, G1 = sim(G, alpha, beta)
```

```
    print("rounds:")
    print(rounds)
    drawNetwork(G1)
```