ORIGINAL PAPER

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On a mixture of an MGT viscous material and an elastic solid

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Abstract A lot of attention has been paid recently to the study of mixtures and also to the Moore–Gibson–Thompson (MGT) type equations or systems. In fact, the MGT proposition can be used to describe viscoelastic materials. In this paper, we analyze a problem involving a mixture composed by a MGT viscoelastic type material and an elastic solid. To this end, we first derive the system of equations governing the deformations of such material. We give the suitable assumptions to obtain an existence and uniqueness result. The semigroups theory of linear operators is used. The paper concludes by proving the exponential decay of solutions with the help of a characterization of the exponentially stable semigroups of contractions and introducing an extra assumption. The impossibility of location is also shown.

1 Introduction

In the second half of the last century, different generalized models were proposed to study the behavior of solids and fluids. One of them corresponds to mixtures where two (or more) components interact to form another material. Metallic alloys are well-known examples of mixtures, but there are many others. It is worth recalling several references where this kind of materials was firstly described (see, for instance, [4,7–11,15,17,18,20,21,24,29]). The first theory, where the Lagrangian description was proposed and where the independent variables were the gradients of each displacement and the relative displacement, was presented in the articles by Bedford and Stern [9,10]. These models have been used by Tiersten and Jahanmir [32] to derive the theory of composites where the relative displacement of the individual constituents is infinitesimal. The theory of mixtures is well accepted in the scientific community.

For a mixture of two interacting continua occupying a domain, the displacements of each component of typical particles at time t are denoted by u and w, respectively, depending on the material point and the time. We assume that the particles under consideration are in the same position at the initial time.

On the other hand, in the last decade great interest has been developed to understand the so-called Moore–Gibson–Thompson equation which was first used in fluid mechanics. Recently, this equation has

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been considered as a heat equation (and then, to analyze the Moore–Gibson–Thompson thermoelasticity) [1–3,5,6,13,14,16,22,23,28,31] and a new kind of viscous elastic materials [12,13,27]. In this work, we want to consider a mixture of a viscous solid of Moore–Gibson–Thompson type and an elastic material.

It has been deeply commented the fact that the thermal waves obtained from the Fourier's thermal law (and also the type III thermoelasticity introduced by Green and Naghdi [19]) propagate instantaneously, and therefore, they violate the well-known "causality principle." This fact has led to propose different theories for the heat conduction like, for instance, the Maxwell–Cattaneo law, the type II thermoelastic theory of Green and Naghdi [19] or some dual-phase-lag theories, where we can find again the Moore–Gibson–Thompson theory. All these theories avoid this drawback and they allow the propagation of the thermal waves at a finite speed. However, to our knowledge there is not a similar criticism in the literature for the mechanical waves in the Kelvin–Voigt viscoelasticity. This theory is also affected by the same paradox (see [30, p.39]), and therefore, it would be reasonable to assume an alternative theory which would help to overcome it as in the heat conduction case. In this way, the viscoelastic theory based on the MGT equation eliminates this phenomenon [26]. We can conclude that, in this aspect, the viscoelastic theory of MGT is more realistic than the classical Kelvin–Voigt viscoelastic theory. As a consequence, it is a good candidate to describe the viscoelastic effects by using partial differential equations in a similar way as the alternative theories for the heat conduction.

Once we have clarified this issue, we can ask ourselves which can be the system of equations for a mixture of a viscoelastic material and an elastic material. In this paper, we derive these equations from the theory of the viscoelastic materials with memory [21]. To be precise, the plan of this work is the following. In the next section, we propose the basic equations describing these solids as well as the assumptions we will need in this paper. In Sect. 3, we transform our problem into a Cauchy problem on a suitable Hilbert space. Existence and uniqueness of the solutions are obtained in Sect. 4, and we conclude our work by proving the exponential decay of the solutions.

2 Basic equations

In this section, we obtain the field equations for a mixture of a Moore–Gibson–Thompson viscous solid with an elastic solid from the theory of mixtures of materials with memory.

We denote by B a three-dimensional region such that its boundary is smooth enough to apply the divergence theorem.

The evolution equations for a mixture are given by

$$\rho_1 \ddot{u}_i = \tau_{ij,j} - p_i, \quad \rho_2 \ddot{w}_i = \sigma_{ij,j} + p_i,$$

where u_i and w_i are the displacements of each component, ρ_1 and ρ_2 are their mass densities, τ_{ij} and σ_{ij} are the partial stress tensors and p_i is the internal body force.

We assume that the constitutive equations have the following form (see [21]):

$$\tau_{ij} = \int_{-\infty}^{t} \left(A_{ijrs}^*(t-s)\dot{u}_{r,s}(s) + B_{ijrs}^*(t-s)\dot{w}_{r,s}(s) \right) ds,$$

$$\sigma_{ij} = \int_{-\infty}^{t} \left(B_{rsij}^*(t-s)\dot{u}_{r,s}(s) + C_{ijrs}^*(t-s)\dot{w}_{r,s}(s) \right) ds,$$

$$p_i = \int_{-\infty}^{t} a_{ij}^*(t-s)(\dot{u}_j(s) - \dot{w}_j(s)) \, ds.$$

Here, the constitutive tensors satisfy $A_{ijrs}^* = A_{rsij}^*$, $C_{ijrs}^* = C_{rsij}^*$ and $a_{ij}^* = a_{ji}^*$, and we assume that

$$A_{ijrs}^*(x, s) = A_{ijrs}^*(x) + (\tau^{-1}A_{ijrs}(x) - A_{ijrs}^*(x))e^{-\tau^{-1}s},$$
$$B_{ijrs}^*(x, s) = B_{ijrs}^*(x), \quad C_{ijrs}^*(x, s) = C_{ijrs}^*(x),$$
$$a_{ij}^*(x, s) = a_{ij}^*(x),$$

where τ is a positive constant.

If we use the notation $\hat{w}_i = w_i + \tau \dot{w}_i$ and we also assume that the deformations vanish at time $t = -\infty$, our system becomes (see [13]):

$$\rho_1(\tau \dddot{u}_i + \ddot{u}_i) = \left(A^*_{ijrs} u_{r,s} + A_{ijrs} \dot{u}_{r,s} + B^*_{ijrs} \hat{w}_{r,s} \right)_{,j} - a^*_{ij}(u_j + \tau \dot{u}_j - \hat{w}_j),$$

$$\rho_2 \ddot{\hat{w}}_i = \left(B^*_{rsij}(u_{r,s} + \tau \dot{u}_{r,s}) + C^*_{ijrs} \hat{w}_{r,s} \right)_{,j} + a^*_{ij}(u_j + \tau \dot{u}_j - \hat{w}_j),$$

but, from now on, we omit the hat for the sake of simplicity. In view of the kind of functions we have chosen, we are assuming that the first component (variable u_i) corresponds to a viscoelastic solid and the second one (variable w_i) to the elastic material.

To define a problem based on our system we need to impose the boundary conditions:

$$u_i(x, t) = w_i(x, t) = 0 \quad \forall x \in \partial B,$$

and the initial conditions, for all $x \in B$,

$$u_i(x, 0) = u_i^0(x), \quad \dot{u}_i(x, 0) = v_i^0(x), \quad \ddot{u}_i(x, 0) = z_i^0(x),$$
$$w_i(x, 0) = w_i^0(x), \quad \dot{w}_i(x, 0) = y_i^0(x).$$

We will make the following assumptions in the whole paper:

(i) $\rho_1(x)$ and $\rho_2(x)$ are strictly positive.
(ii) There exists a positive constant C such that

$$A^*_{ijrs}\xi_{ij}\xi_{rs} + 2B^*_{ijrs}\xi_{ij}\eta_{rs} + C^*_{ijrs}\eta_{ij}\eta_{rs} \geq C \left(\xi_{ij}\xi_{ij} + \eta_{ij}\eta_{ij} \right)$$

for every tensors ξ_{ij} and η_{ij}.
(iii) There exists a positive constant C^* such that

$$a^*_{ij}\xi_i\xi_j \geq C^*\xi_i\xi_j$$

for every vector ξ_i.
(iv) There exists a constant greater than one C^{**} such that

$$A_{ijrs}\xi_{ij}\xi_{rs} \geq C^{**}\tau A^*_{ijrs}\xi_{ij}\xi_{rs}$$

for every tensor ξ_{ij}.

3 A Cauchy problem

In this section, we transform our problem into a Cauchy problem written on a suitable Hilbert space. To this end, we consider the space

$$\mathcal{H} = W_0^{1,2}(B) \times W_0^{1,2}(B) \times L^2(B) \times W_0^{1,2}(B) \times L^2(B),$$

where $W_0^{1,2}(B) = [W_0^{1,2}(B)]^3$ and $L^2(B) = [L^2(B)]^3$, and $W_0^{1,2}(B)$ and $L^2(B)$ are the usual Sobolev spaces.

Let us denote an element $U \in \mathcal{H}$ by $U = (u, v, z, w, y)$, and define the operators:

$$A_i u = \frac{1}{\tau \rho_1} \left[(A^*_{ijrs} u_{r,s})_{,j} - a^*_{ij} u_j \right], \quad A = (A_i),$$

$$B_i v = \frac{1}{\tau \rho_1} \left[(A_{ijrs} v_{r,s})_{,j} - \tau a^*_{ij} v_j \right], \quad B = (B_i),$$

$$C_i w = \frac{1}{\tau \rho_1} \left[(B^*_{ijrs} w_{r,s})_{,j} + a^*_{ij} w_j \right], \quad C = (C_i),$$

$$D_i u = \frac{1}{\rho_2} \left[(B^*_{rsij} u_{r,s})_{,j} + a^*_{ij} u_j \right], \quad D = (D_i),$$

$$E_i v = \frac{1}{\rho_2} \left[(\tau B^*_{rsij} v_{r,s}),_j + \tau a^*_{ij} v_j \right], \quad E = (E_i),$$

$$F_i w = \frac{1}{\rho_2} \left[(C^*_{ijrs} w_{r,s}),_j - a^*_{ij} w_j \right], \quad F = (F_i),$$

$$G_i z = \frac{1}{\tau} z_i, \quad G = (G_i).$$

If we construct the matrix operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ A & B & G & C & 0 \\ 0 & 0 & 0 & 0 & I \\ D & E & 0 & F & 0 \end{pmatrix},$$

we can write our problem as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = U^0, \tag{1}$$

where $U^0 = (u^0, v^0, z^0, w^0, y^0)$.

In this Hilbert space $\mathcal{H}$, we can define the inner product

$$\langle (u, v, z, w, y), (u^*, v^*, z^*, w^*, y^*) \rangle = \frac{1}{2} \int_B \left(\rho_2 \overline{y_i^*} y_i + \rho_1 (\tau z_i + v_i) \overline{(\tau z_i^* + v_i^*)} \right.$$

$$+ \tau \overline{A}_{ijrs} v_{i,j} \overline{v_{r,s}^*} + A^*_{ijrs} (u_{i,j} + \tau v_{i,j}) \overline{(u_{i,j}^* + \tau v_{i,j}^*)} + C^*_{ijrs} w_{i,j} \overline{w_{r,s}^*}$$

$$+ B^*_{ijrs} \left((u_{i,j} + \tau v_{i,j}) \overline{w_{r,s}^*} + \overline{(u_{i,j}^* + \tau v_{i,j}^*)} w_{r,s} \right)$$

$$\left. + a^*_{ij} ((u_i + \tau v_i) - w_i) \overline{((u_i^* + \tau v_i^*) - w_i^*)} \right) dv,$$

where the bar over the variables denotes the complex conjugate and $\overline{A}_{ijrs} = A_{ijrs} - \tau A^*_{ijrs}$.

It is important to note that this inner product defines a norm which is equivalent to the usual one in the Hilbert space $\mathcal{H}$ defined as

$$\|(u, v, z, w, y)\|^2 = \frac{1}{2} \int_B \left(\rho_1 (\tau z_i + v_i) \overline{(\tau z_i + v_i)} + A^*_{ijrs} (u_{i,j} + \tau v_{i,j}) \overline{(u_{i,j} + \tau v_{i,j})} \right.$$

$$+ \tau \overline{A}_{ijrs} v_{i,j} \overline{v_{r,s}} + C^*_{ijrs} w_{i,j} \overline{w_{r,s}} + B^*_{ijrs} ((u_{i,j} + \tau v_{i,j}) \overline{w_{r,s}} + \overline{(u_{i,j} + \tau v_{i,j})} w_{r,s})$$

$$\left. + \rho_2 \overline{y_i} y_i + a^*_{ij} ((u_i + \tau v_i) - w_i) \overline{((u_j + \tau v_j) - w_j)} \right) dv.$$

4 Existence of solutions

In this section, we give an existence and uniqueness result to the problem determined by (1). To this end, we will use the Lumer–Phillips corollary to the Hille–Yosida theorem. In this sense, we note that the domain of the operator $\mathcal{A}$ is given by (u, v, z, w, y) such that $u, z, y \in W_0^{1,2}(B)$, $Au + Bv + Cw \in L^2(B)$ and $Du + Ev + Fw \in L^2(B)$. It is clear that it is a dense subspace of $\mathcal{H}$.

If we assume that $U = (u, v, z, w, y)$ belongs to the domain of the operator $\mathcal{A}$, it follows that

$$Re \langle \mathcal{A}U, U \rangle = -\frac{1}{2} \int_B \overline{A}_{ijrs} v_{ij} \overline{v}_{rs} \, dv \le 0.$$

Therefore, in order to prove the existence, it will be enough to show that zero belongs to the resolvent of the operator $\mathcal{A}$. Let $(f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$, then we have to solve the system:

$$v = f_1, \quad z = f_3, \quad y = f_4,$$
$$Au + Bv + Gz + Cw = f_3,$$
$$Du + Ev + Fw = f_5.$$

It is clear that we can find v, z and y. Therefore, we should solve the remaining two equations:

$$Au + Cw = f_3 - Bf_1 - Gf_2, \quad Du + Fw = f_5 - Ef_1.$$

Obviously, the right-hand side of this system belongs to $W^{-1,2}(B) \times W^{-1,2}(B)$. Moreover, in view of the assumptions (i)-(iv) and the use of the Lax–Milgram lemma, we can guarantee the existence of $(u, w) \in W_0^{1,2}(B) \times W_0^{1,2}(B)$ satisfying the system, and we can also obtain the existence of a positive constant K such that $\|(u, v, z, w, y)\| \leq K \|(f_1, f_2, f_3, f_4, f_5)\|$.

Hence, we can conclude the existence and uniqueness of solutions.

Theorem 1 *The operator $\mathcal{A}$ generates a contractive semigroup. Then, for every $U^0 \in Dom(\mathcal{A})$, there exists a unique solution to problem (1).*

In fact, the usual arguments of the semigroups theory allow us to prove the continuous dependence of the solutions with respect to the initial data (and supply terms in the case that we impose them).

5 Exponential decay of solutions

In this section, we will prove that the solutions to our problem decay in an exponential way. That is, there exist two constants $M > 0$ and $\alpha > 0$ such that

$$\|U(t)\| \leq M \|U(0)\| e^{-\alpha t} \tag{2}$$

for every $t \geq 0$.

Here, we also assume the existence of a positive constant C_1 such that either

$$\int_B B_{ijrs} \xi_{ij} \xi_{rs} \, dv \geq C_1 \xi_{ij} \xi_{ij},$$
$$\text{or} \tag{3}$$
$$\int_B B_{ijrs} \xi_{ij} \xi_{rs} \, dv \leq -C_1 \xi_{ij} \xi_{ij},$$

for every tensor ξ_{ij}.

In view of the well-known results of the semigroups theory it is sufficient to show that (see [25]):

1. The imaginary axis is contained in the resolvent of the operator $\mathcal{A}$.
2. $\varlimsup_{|\lambda| \to \infty} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

We will prove both conditions by using a similar argument. Let us assume that one of the conditions does not hold. Then, there exist a sequence of real parameters $\lambda_n \to \lambda^* \neq 0$ (or ∞) and a sequence of elements in the domain of the operator $\mathcal{A}$, with unit norm, denoted as $(u_n, v_n, z_n, w_n, y_n)$ such that

$$i\lambda_n u_n - v_n \to 0 \text{ in } W_0^{1,2}(B), \tag{4}$$
$$i\lambda_n v_n - z_n \to 0 \text{ in } W_0^{1,2}(B), \tag{5}$$
$$i\lambda_n z_n - Au_n - Bv_n - Gz_n - Cw_n \to 0 \text{ in } L^2(B), \tag{6}$$
$$i\lambda_n w_n - y_n \to 0 \text{ in } W_0^{1,2}(B), \tag{7}$$
$$i\lambda_n y_n - Du_n - Ev_n - Fw_n \to 0 \text{ in } L^2(B). \tag{8}$$

We first note that, in view of the dissipation inequality, we see that $v_n \to 0$ in $W_0^{1,2}(B)$, and therefore, it follows that $\lambda_n u_n \to 0$ in $W_0^{1,2}(B)$. If we multiply convergence (6) by v_n, we find that $z_n \to 0$ in $L^2(B)$. If we multiply again convergence (6) by w_n, we see that $\int_B B_{ijrs}^* w_{r,s} \overline{w_{i,j}} \, dv \to 0$, and so, we obtain $w_n \to 0$ in $W_0^{1,2}(B)$. Now, if we multiply convergence (8) by w_n we also have $y_n \to 0$ in $L^2(B)$. However, this is not possible because we assumed that the sequence had unit norm. Therefore, we have proved the following result.

Theorem 2 *If we assume that (3) holds, then the semigroup generated by the operator $\mathcal{A}$ is exponentially stable; that is, there exist two positive constants M and α such that (2) is satisfied.*

We can conclude this section by asking if this exponential decay of the solutions is faster enough to obtain the localization of the solutions. That is to say: is the decay of the solutions so fast that they can be zero after a finite period of time? We will provide a negative answer, i.e. the only solution which can be zero after a finite period of time is the null solution. We note that to show this fact is equivalent to prove the uniqueness of the solutions to the backward in time problem. That is, we need to show that the only solution to the problem determined by the system:

$$\rho_1(\ddot{u}_i - \tau\dddot{u}_i) = \left(A^*_{ijrs}u_{r,s} - A_{ijrs}\dot{u}_{r,s} + B^*_{ijrs}w_{r,s}\right)_{,j} - a^*_{ij}(u_j - \tau\dot{u}_j - w_j),$$

$$\rho_2\ddot{w}_i = \left(B^*_{rsij}(u_{r,s} - \tau\dot{u}_{r,s}) + C^*_{ijrs}w_{r,s}\right)_{,j} + a^*_{ij}(u_j - \tau\dot{u}_j - w_j),$$

with null initial and boundary conditions is the null solution. In order to prove it, we consider the function

$$F(t) = \frac{1}{2}\int_B \Big[\rho_1(\dot{u}_i - \tau\ddot{u}_i)(\dot{u}_i - \tau\ddot{u}_i) + \rho_2\dot{w}_i\dot{w}_i + A^*_{ijrs}(u_{i,j} - \tau\dot{u}_{i,j})(u_{r,s} - \tau\dot{u}_{r,s})$$

$$+ \tau\overline{A}_{ijrs}\dot{u}_{i,j}\dot{u}_{r,s} + C^*_{ijrs}w_{i,j}w_{r,s} + 2B^*_{ijrs}(u_{i,j} - \tau\dot{u}_{i,j})w_{r,s}$$

$$+ a^*_{ij}(u_i - \tau\dot{u}_i - w_i)(u_j - \tau\dot{u}_j - w_j)\Big]\,dv.$$

We have

$$\dot{F}(t) = \int_B \overline{A}_{ijrs}\dot{u}_{i,j}\dot{u}_{r,s}\,dv.$$

However, we also can show that $\dot{F}(t) \leq CF(t)$ for a suitable constant and so, $F(t) \leq F(0)e^{Ct}$. If we assume null initial conditions, we conclude that $F(t) = 0$ for every $t \geq 0$, and then we obtain the null solution.

6 Conclusions

We highlight the main aspects of this paper.

1. We have obtained the system of equations governing the isothermal deformations of a mixture composed by a Moore–Gibson–Thompson viscoelastic material and an elastic material with the help of the theory of viscoelastic solids satisfying the invariance of the entropy under time reversal.
2. We have derived the suitable framework to prove an existence and uniqueness result to the above-commented system. We have used the semigroup theory of linear operators.
3. We have also showed the exponential decay of solutions, under adequate conditions on a coupling term. We have used a result involving the spectrum of the generator of a contractive linear semigroup and a known asymptotic condition. Impossibility of localization has been also proved.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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