DIVISORS OF EXPECTED JACOBIAN TYPE

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Abstract

Divisors whose Jacobian ideal is of linear type have received a lot of attention recently because of its connections with the theory of D-modules. In this work we are interested on divisors of expected Jacobian type, that is, divisors whose gradient ideal is of linear type and the relation type of its Jacobian ideal coincides with the reduction number with respect to the gradient ideal plus one. We provide conditions in order to be able to describe precisely the equations of the Rees algebra of the Jacobian ideal. We also relate the relation type of the Jacobian ideal to some D-module theoretic invariant given by the degree of the Kashiwara operator.

1. Introduction

Let (X, O) be a germ of a smooth n-dimensional complex variety and $\mathcal{O}_{X,O}$ the ring of germs of holomorphic functions in a neighbourhood of O, which we identify with $R = \mathbb{C}\{x_1, \ldots, x_n\}$ by taking local coordinates. Let $D_R[s]$ be the polynomial ring in an indeterminate s with coefficients in the ring of differential operators $D_R = R\langle\partial_1, \ldots, \partial_n\rangle$ where ∂_i are the partial derivatives with respect to the variables x_i. To any hypersurface defined by $f \in R$ we may attach several invariants coming from the theory of D-modules that measure its singularities. The goal of this work is to get more insight on the *parametric annihilator* $\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s}) := \{P(s) \in D_R[s] \mid P(s)\cdot \boldsymbol{f^s} = 0\}$, where we understand $\boldsymbol{f^s}$ as a formal symbol that takes the obvious meaning f^r when specializing to any integer $r \in \mathbb{Z}$. This is the defining ideal of the $D_R[s]$-modules generated by $\boldsymbol{f^s}$, that we denote as $D_R[s]\boldsymbol{f^s}$, which plays a key role in the theory of *Bernstein-Sato polynomials* as shown by Kashiwara [14]. Among the differential operators annihilating $\boldsymbol{f^s}$ there exists the so-called *Kashiwara operator* [14, Theorem 6.3] that has been used in some of the first algorithmic approaches to the computation of Bernstein-Sato polynomials given by Yano [29] and Briançon et al. [5]. Furthermore, the degree of the Kashiwara operator is an interesting analytic invariant of the singularity, although it is much coarser than the Bernstein-Sato polynomial itself.

A common theme in the study of the parametric annihilator is whether it is generated by operators of degree one. This and some related linearity properties have been used by several authors in a wide range of different problems [7], [26], [20], [8] , [2], [21], [27], [28]. This linearity property of differential operators can be checked using algebraic methods as it was proved by Calderón-Moreno and Narvéz-Macarro in [7]. Namely, this property holds whenever the Jacobian ideal of the hypersurface f is of linear type, that is the Rees algebra and the symmetric algebra of the Jacobian ideal coincide.

The aim of this paper is to get further connections between the Rees algebra of the Jacobian ideal and the parametric annihilator. Building upon work of Muiños and the second author in [17], we introduce in Definition 4.2 the notion of divisors of *expected Jacobian type* as those divisors whose gradient ideal is of linear type and whose Jacobian ideal has relation type equal to its reduction number plus one (see also Definition 3.3). In Remark 4.3, it is easily seen that this is a natural generalization of the divisors of *linear Jacobian type* considered in [7] (see also [20]). For divisors of linear Jacobian type we can find an equation that resembles the initial term or symbol of the Kashiwara operator with respect to a given order, and indeed this is the case under some extra conditions.

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The organization of the paper is as follows: in Section 2 we review the basics on the equations of Rees algebras and recover and extend some of the results of Muiños and the second author in [17]. Section 3 and 4 are devoted to introduce the notion of ideal of expected relation type and its specialization to the case of the Jacobian ideal of a hypersurface. In Section 5 we describe the connection between divisors of expected Jacobian type and the parametric annihilator. We relate the degree of the Kashiwara operator with the relation type of the Jacobian ideal in Proposition 5.3. In Section 6 we present several examples in which we explore the case in which the ideal has the expected relation type. We also study some cases in which this condition is not satisfied.

Any unexplained notation or definition can be found in [6] or [25]. Throghout the paper, $(R,\mathfrak{m})$ is a Noetherian local ring and $\mathfrak{b}\subseteq\mathfrak{a}$ and $J\subseteq I$ are ideals of R.

2. On the equations of Rees algebras

Let $(R,\mathfrak{m})$ be a Noetherian local ring and let $\mathfrak{a}=(f_1,\dots,f_m)$ be an ideal of R, $m\geq 1$. Let $\mathbf{R}(\mathfrak{a})=R[\mathfrak{a}t]=\bigoplus_{d\geq 0}\mathfrak{a}^dt^d\subset R[t]$ be the *Rees algebra of* $\mathfrak{a}$. Let $A=R[\xi_1,\dots,\xi_m]$ be a polynomial ring in a set of variables $\xi_1,\dots,\xi_m$ and coefficients in R. Consider the graded surjective morphism $\varphi:A\to\mathbf{R}(\mathfrak{a})$ sending ξ_i to f_it, for $i=1,\dots,n+1$. The kernel of this morphism is a graded ideal $Q=\bigoplus_{d\geq 1}Q_d$, whose elements will be referred to as the *equations of* $\boldsymbol{R}(\mathfrak{a})$. Let $Q\langle d\rangle$ be the ideal generated by the homogeneous equations of degree at most d. We then have an increasing sequence $Q\langle 1\rangle\subseteq Q\langle 2\rangle\subseteq\cdots\subseteq Q$ that stabilizes at some point. The smallest integer $L\geq 1$ such that $Q\langle L\rangle=Q$ is the *relation type of* $\boldsymbol{R}(\mathfrak{a})$ and will be denoted $\operatorname{rt}(\mathfrak{a})$. We say that $\mathfrak{a}$ is an *ideal of linear type* when $\operatorname{rt}(\mathfrak{a})=1$.

Observe that the ideal Q depends on the polynomial presentation φ. Nevertheless, the quotients $(Q/Q\langle d-1\rangle)_d$, for $d\geq 2$, do not (see [23]). Indeed, let $\alpha:\mathbf{S}(\mathfrak{a})\to\mathbf{R}(\mathfrak{a})$ be the canonical graded surjective morphism between the symmetric algebra $\mathbf{S}(\mathfrak{a})$ of $\mathfrak{a}$ and the Rees algebra $\mathbf{R}(\mathfrak{a})$ of $\mathfrak{a}$. Given $d\geq 2$, the d-th module of effective relations of $\mathfrak{a}$ is defined to be $E(\mathfrak{a})_d=\ker(\alpha_d)/\mathfrak{a}\cdot\ker(\alpha_{d-1})$. One shows that, for $d\geq 2$, $E(\mathfrak{a})_d\cong(Q/Q\langle d-1\rangle)_d$. In particular, the relation type of $\mathfrak{a}$ can be calculated as the least integer $L\geq 1$, such that $E(\mathfrak{a})_d=0$, for all $d\geq L+1$. Moreover, it is known that $E(\mathfrak{a})_d\cong H_1(f_1t,\dots,f_mt;\mathbf{R}(\mathfrak{a}))_d$, where the right-hand module stands for the degree d-component of the first Koszul homology module associated to the sequence of degree one elements $f_1t,\dots,f_mt$ of $\mathbf{R}(\mathfrak{a})$ ([23, Theorem 2.4]).

The characterization of $E(\mathfrak{a})_d$ in terms of the Koszul homology was used in [17] in order to obtain the equations of $\mathbf{R}(\mathfrak{a})$ for equimultiple ideals $\mathfrak{a}$ of deviation one. Our purpose in this section is to rephrase, and extend a little bit, some of those results, but doing more emphasis in the Koszul conditions than in the "regular sequence type conditions". These characterizations will be applied in the next sections to the Jacobian ideal of a hypersurface. For the sake of completeness and self-containment, we outline parts of the line of reasoning in [17]. Let us start by setting our general notations.

Setting 2.1. Let $(R,\mathfrak{m})$ be a Noetherian local ring, $n\geq 2$. Let $f_1,\dots,f_n\in\mathfrak{m}$ and $f=f_{n+1}\in\mathfrak{m}$. Let $J=(f_1,\dots,f_n)$ and $I=(f_1,\dots,f_n,f)=(J,f)$ be ideals of R. For $i=1,\dots,n+1$, let $J_i=(f_1,\dots,f_i)$; set $J_0=0$ and observe that $J_n=J$ and $J_{n+1}=I$.

$$\text{For } i=1,\dots,n+1 \text{ and } d\geq 2,\ \text{set}\qquad T_{i,d}=\frac{(J_{i-1}I^{d-1}:f_i)\cap I^{d-1}}{J_{i-1}I^{d-2}}.$$

$$\text{For } d=1,\ \text{set}\qquad T_{i,1}=J_{i-1}:f_i.$$

Note that for $i=1$ (and any $d\geq 1$), then $T_{1,d}=(0:f_1)\cap I^{d-1}$. For $i=n+1$ and $d\geq 2$, it was shown in [17, Proof of Lemma 3.1] that:

$$T_{n+1,d}=\frac{(JI^{d-1}:f)\cap I^{d-1}}{JI^{d-2}}\cong\frac{JI^{d-1}:f^d}{JI^{d-1}:f^{d-1}}.\tag{2.1}$$

The isomorphism goes as follows. Given $a\in(JI^{d-1}:f)\cap I^{d-1}$, since $I^{d-1}=JI^{d-1}+f^{d-1}R$, write $a=b+cf^{d-1}$ with $b\in JI^{d-1}$ and $c\in R$. The class of an element $a\in(JI^{d-1}:f)\cap I^{d-1}$ is sent to the class of $c\in JI^{d-1}:f^d$.

Notation 2.2. A graded Koszul complex. Let us denote $K(z_1,\dots,z_r;U)$ the Koszul complex of a sequence of elements $z_1,\dots,z_r$ of a ring U. Since U will always be the Rees algebra $\mathbf{R}(I)$ of I, we just skip the letter U. For $i=1,\dots,n+1$, we consider the sequences $\underline{\mathrm{f_i t}} := f_1t,\dots,f_it$ of elements of degree one in $\mathbf{R}(I)$; we highlight the distinct notation with the length one sequence f_it. Set $\underline{\mathrm{ft}} := \underline{\mathrm{f_{n+1}t}} = f_1t,\dots,f_nt,ft$. Thus $K(\underline{\mathrm{f_i t}}) = K(f_1t,\dots,f_it;\mathbf{R}(I))$ stands for the Koszul complex associated to $\underline{\mathrm{f_i t}} = f_1t,\dots,f_it$, with first nonzero zero terms:

$$K(\underline{\mathrm{f_i t}}): \qquad \cdots \to K_2(\underline{\mathrm{f_i t}}) \xrightarrow{\partial_2} K_1(\underline{\mathrm{f_i t}}) \xrightarrow{\partial_1} K_0(\underline{\mathrm{f_i t}}) \to 0.$$

Let $H_j(\underline{\mathrm{f_i t}}) = H_j(K(\underline{\mathrm{f_i t}}))$ be its j-th homology module. Note that, since $\mathbf{R}(I)$ is a graded algebra, $K(\underline{\mathrm{f_i t}})$, and hence its homology, inherit a natural grading. The first nonzero terms of the degree d-component $K(\underline{\mathrm{f_i t}})_d$, $d\geq 2$, (omitting the powers of the variable t) are:

$$\cdots \to K_2(\underline{\mathrm{f_i t}})_d = \wedge_2(R^i)\otimes I^{d-2} \xrightarrow{\partial_{2,d-2}} K_1(\underline{\mathrm{f_i t}})_d = \wedge_1(R^i)\otimes I^{d-1} \xrightarrow{\partial_{1,d-1}} K_0(\underline{\mathrm{f_i t}})_d = I^d \to 0.$$

The Koszul differentials are defined as follows: if $e_1,\dots,e_i$ stands for the canonical basis of R^i and $u\in I^{d-2}$ and $v\in I^{d-1}$, then

$$\partial_{2,d-2}(e_j\wedge e_l\otimes u) = e_l\otimes f_ju - e_j\otimes f_lu \ \text{ and } \ \partial_{1,d-1}(e_j\otimes v) = f_jv.$$

Note that, under the isomorphism $\wedge_1(R^i)\otimes I^{d-1}\cong I^{d-1}\oplus \overset{(i)}{\cdots} \oplus I^{d-1}$, the differential $\partial_{1,d-1}$ sends the i-th tuple $(a_1,\dots,a_i)\in (I^{d-1})^{\oplus i}$ to the element $a_1f_1+\cdots+a_if_i\in I^d$. In particular, for $d=1$, $H_1(\underline{\mathrm{f_i t}})_1 = \{(a_1,\dots,a_i)\in R^i \mid \sum_j^i a_jf_j = 0\} = Z_1(f_1,\dots,f_i)$, the first module of syzygies of $J_i=(f_1,\dots,f_i)$.

Remark 2.3. Equations vs cycles. Let Q be the ideal of equations of $\mathbf{R}(I)$. As said before,

$$E(I)_d \cong \left(\frac{Q}{Q\langle d-1\rangle}\right)_d \cong H_1(\underline{\mathrm{ft}})_d = H_1(K(\underline{\mathrm{ft}}))_d = H_1(f_1t,\dots,f_nt,ft;\mathbf{R(I)})_d, \tag{2.2}$$

i.e., the d-th module of effective relations $E(I)_d$ of I is isomorphic to the degree d-component of the first Koszul homology module $H_1(\underline{\mathrm{ft}})$ of $\underline{\mathrm{ft}}$, where $d\geq 2$. This isomorphism sends the class of an equation $P\in Q_d$ to the class of the cycle $(P_1(\underline{\mathrm{f}}),\dots,P_n(\underline{\mathrm{f}}),P_{n+1}(\underline{\mathrm{f}}))\in \bigoplus_{j=1}^{n+1} I^{d-1}$, where $\underline{\mathrm{f}} = f_1,\dots,f_n,f$, $P=\sum_{j=1}^{n+1}\xi_jP_j$, and $P_j\in A_{d-1}=R[\xi_1,\dots,\xi_{n+1}]_{d-1}$. (See [17, Remark 2.1].)

The next two remarks are devoted to write more explicitly some complexes and morphisms that will be used subsequently.

Remark 2.4. A short exact sequence of Koszul complexes. Let $K(f_it)$ be the Koszul complex associated to the length one sequence $f_it\in\mathbf{R}(I)$. So $K_0(f_it)=\mathbf{R}(I)$, $K_1(f_it)=\wedge_1(R)\otimes\mathbf{R}(I)\cong\mathbf{R}(I)$, and $K_j(f_it)=0$, for $j\neq 0,1$. In degree $d\geq 1$, $K_0(f_it)_d = I^d$, $K_1(f_it) = (\wedge_1(R)\otimes\mathbf{R}(I))_d\cong I^{d-1}$ and, for $a\in I^{d-1}$, then $\partial_{1,d-1}(a)=af_i$.

There is an isomorphism of Koszul complexes $K(\underline{\mathrm{f_i t}})\cong K(\underline{\mathrm{f_{i-1} t}})\otimes K(f_it)$. Concretely,

$$\begin{aligned} K_p(\underline{\mathrm{f_i t}}) \cong \bigoplus_{r+s=p} K_r(\underline{\mathrm{f_{i-1} t}})\otimes K_s(f_it) = K_p(\underline{\mathrm{f_{i-1} t}})\otimes K_0(f_it)\oplus K_{p-1}(\underline{\mathrm{f_{i-1} t}})\otimes K_1(f_it) \cong \\ K_p(\underline{\mathrm{f_{i-1} t}})\otimes\mathbf{R}(I)\oplus K_{p-1}(\underline{\mathrm{f_{i-1} t}})\otimes\mathbf{R}(I)\cong K_p(\underline{\mathrm{f_{i-1} t}})\oplus K_{p-1}(\underline{\mathrm{f_{i-1} t}}), \end{aligned}$$

which induces a short exact sequence of Koszul complexes:

$$0\to K(\underline{\mathrm{f_{i-1} t}})\to K(\underline{\mathrm{f_i t}})\to K(\underline{\mathrm{f_{i-1} t}})(-1)\to 0, \tag{2.3}$$

where $K(\underline{\mathrm{f_{i-1} t}})(-1)$ is the shifted complex by -1, i.e., $K_s(\underline{\mathrm{f_{i-1} t}})(-1) = K_{s-1}(\underline{\mathrm{f_{i-1} t}})$. In particular, for $d\geq 1$, the degree d-component gives rise to the the short exact sequence of complexes:

$$0\to K(\underline{\mathrm{f_{i-1} t}})_d\to K(\underline{\mathrm{f_i t}})_d\to K(\underline{\mathrm{f_{i-1} t}})(-1)_d\to 0.$$

Displaying by columns the first nonzero terms of each complex, we get:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \wedge_2(R^{i-1}) \otimes I^{d-2} \arrow[r] \arrow[d] & \wedge_2(R^{i}) \otimes I^{d-2} \arrow[r] \arrow[d] & R^{i-1} \otimes I^{d-2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^{i-1} \otimes I^{d-1} \arrow[r] \arrow[d] & R^{i} \otimes I^{d-1} \arrow[r] \arrow[d] & I^{d-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{d} \arrow[r] \arrow[d] & I^{d} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & .
\end{tikzcd}$$

The middle row, $0 \to K_1(\underline{\mathrm{f}_{i-1}\mathrm{t}})_d \to K_1(\underline{\mathrm{f}_{i}\mathrm{t}}) \to K_1(\underline{\mathrm{f}_{i-1}\mathrm{t}})(-1)_d \to 0$, is nothing else than:

$$0 \to I^{d-1} \oplus \overset{(i-1)}{\cdots} \oplus I^{d-1} \longrightarrow I^{d-1} \oplus \overset{(i)}{\cdots} \oplus I^{d-1} \to I^{d-1} \to 0,$$

where the first morphism sends $(a_1, \ldots, a_{i-1})$ to $(a_1, \ldots, a_{i-1}, 0)$, the inclusion, and the second morphism sends $(a_1, \ldots, a_i)$ to a_i, the projection to the last component.

Remark 2.5. The long exact sequence in homology. In turn, the short exact sequence (2.3) induces the long exact sequence in homology. We display its degree d-component, $d \geq 1$.

$$\cdots \to H_2(K(\underline{\mathrm{f}_{i-1}\mathrm{t}})(-1))_d \xrightarrow{\delta} H_1(\underline{\mathrm{f}_{i-1}\mathrm{t}})_d \to H_1(\underline{\mathrm{f}_{i}\mathrm{t}})_d \to H_1(K(\underline{\mathrm{f}_{i-1}\mathrm{t}})(-1))_d \xrightarrow{\delta}$$
$$\to H_0(\underline{\mathrm{f}_{i-1}\mathrm{t}})_d \to H_0(\underline{\mathrm{f}_{i}\mathrm{t}}) \to H_0(K(\underline{\mathrm{f}_{i-1}\mathrm{t}})(-1))_d \to 0.$$

Clearly, $H_j(K(\underline{\mathrm{f}_{i-1}\mathrm{t}})(-1))_d = H_{j-1}(\underline{\mathrm{f}_{i-1}\mathrm{t}})_{d-1}$ and $H_0(K(\underline{\mathrm{f}_{i-1}\mathrm{t}})(-1))_d = 0$. The connecting morphism is known to be the multiplication by the element $\pm f_i t$. Thus we get:

$$\cdots \to H_1(\underline{\mathrm{f}_{i-1}\mathrm{t}})_{d-1} \xrightarrow{\cdot \pm f_i t} H_1(\underline{\mathrm{f}_{i-1}\mathrm{t}})_d \to H_1(\underline{\mathrm{f}_{i}\mathrm{t}})_d \to H_0(\underline{\mathrm{f}_{i-1}\mathrm{t}})_{d-1} \xrightarrow{\cdot \pm f_i t} H_0(\underline{\mathrm{f}_{i-1}\mathrm{t}})_d \to H_0(\underline{\mathrm{f}_{i}\mathrm{t}})_d \to 0.$$

If $d = 1$, then $H_1(\underline{\mathrm{f}_{i-1}\mathrm{t}})_0 = 0$, $H_0(\underline{\mathrm{f}_{i-1}\mathrm{t}})_0 = R$ and $H_0(\underline{\mathrm{f}_{i-1}\mathrm{t}})_1 = I/J_{i-1}$. Hence

$$\ker\left(H_0(\underline{\mathrm{f}_{i-1}\mathrm{t}})_0 \xrightarrow{\cdot \pm f_i t} H_0(\underline{\mathrm{f}_{i-1}\mathrm{t}})_1\right) = (J_{i-1} : f_i) = T_{i,1}.$$

In particular, for $i = 1, \ldots, n+1$ and $d = 1$, one deduces the exact sequence:

$$0 \to H_1(\underline{\mathrm{f}_{i-1}\mathrm{t}})_1 \to H_1(\underline{\mathrm{f}_{i}\mathrm{t}})_1 \to T_{i,1} \to 0, \tag{2.4}$$

where $H_1(\underline{\mathrm{f}_{i-1}\mathrm{t}})_1 = Z_1(f_1, \ldots, f_{i-1})$ and $H_1(\underline{\mathrm{f}_{i}\mathrm{t}})_1 = Z_1(f_1, \ldots, f_i)$.

If $d \geq 2$, one can check that $H_0(\underline{\mathrm{f}_{i-1}\mathrm{t}})_{d-1} = I^{d-1}/J_{i-1}I^{d-2}$ and $H_0(\underline{\mathrm{f}_{i-1}\mathrm{t}})_d = I^d/J_{i-1}I^{d-1}$. Thus

$$\ker\left(H_0(\underline{\mathrm{f}_{i-1}\mathrm{t}})_{d-1} \xrightarrow{\cdot \pm f_i t} H_0(\underline{\mathrm{f}_{i-1}\mathrm{t}})_d\right) = (J_{i-1}I^{d-1} : f_i) \cap I^{d-1}/J_{i-1}I^{d-2} = T_{i,d}.$$

(See Setting 2.1.) In particular, for $i = 1, \ldots, n+1$ and $d \geq 2$, we deduce the exact sequence:

$$H_1(\underline{\mathrm{f}_{i-1}\mathrm{t}})_{d-1} \xrightarrow{\cdot \pm f_i t} H_1(\underline{\mathrm{f}_{i-1}\mathrm{t}})_d \to H_1(\underline{\mathrm{f}_{i}\mathrm{t}})_d \to T_{i,d} \to 0. \tag{2.5}$$

Note that the middle morphism in (2.5) is induced by the inclusion. Namely, the class of a cycle $(a_1, \ldots, a_{i-1})$, $a_j \in I^{d-1}$, maps to the class of the cycle $(a_1, \ldots, a_{i-1}, 0)$. Similarly, the right-hand morphism is induced by the projection $(a_1, \ldots, a_i) \mapsto a_i$.

We recover [17, Lemma 3.1]. Keeping the notations as in Setting 2.1 and Notation 2.2:

Corollary 2.6. *For $d \geq 2$, the following sequence is exact.*

$$0 \to \frac{H_1(\underline{\mathrm{f_n t}})_d}{ft \cdot H_1(\underline{\mathrm{f_n t}})_{d-1}} \longrightarrow E(I)_d \longrightarrow \frac{JI^{d-1} : f^d}{JI^{d-1} : f^{d-1}} \to 0. \tag{2.6}$$

The right-hand morphism sends the class of an equation $P \in Q_d$ to the class of $P(0,\dots,0,1)$.

Proof. Take $i = n+1$ and $d \geq 2$ in (2.5). Then $H_1(\underline{\mathrm{f_{n+1} t}})_d = H_1(\underline{\mathrm{ft}})_d$, which by (2.2), is isomorphic to $E(I)_d$. See also the definition of $T_{n+1,d}$ and its isomorphic expression in (2.1). The second part follows from the composition of the morpshims in (2.2), (2.5) and (2.1). Indeed, the class of $P \in Q_d$ is sent to the class of $(P_1(\underline{\mathrm{f}}),\dots,P_{n+1}(\underline{\mathrm{f}})) \in \bigoplus_{j=1}^{n+1} I^{d-1}$ through (2.2), where $P = \sum_{j=1}^{n+1} \xi_j P_j$, $P_j \in A_{d-1}$. By (2.5), $(P_1(\underline{\mathrm{f}}),\dots,P_{n+1}(\underline{\mathrm{f}}))$ is sent to the class of $P_{n+1}(\underline{\mathrm{f}}) \in (JI^{d-1} : f) \cap I^{d-1}$. Write $P_{n+1} = \sum_{j=1}^{n} \xi_j Q_j + c\xi_{n+1}^{d-1}$, with $Q_j \in A_{d-2}$ and $c \in R$. In particular, $P_{n+1}(\underline{\mathrm{f}}) = b + cf^{d-1}$, with $b = \sum_{j=1}^{n} f_j Q_j(\underline{\mathrm{f}}) \in JI^{d-2}$. Then the isomorphism (2.1) sends the class of $P_{n+1}(\underline{\mathrm{f}})$ to the class of $c \in JI^{d-1} : f^d$. Observe that $P(0,\dots,0,1) = P_{n+1}(0,\dots,0,1) = c$.

The first part of the following result is shown in [17, Lema 3.3]. Our proof here is a direct consequence of Remarks 2.4 and 2.5, and the sequences (2.4) and (2.5). We keep the notations as in Setting 2.1 and Notation 2.2.

Theorem 2.7. *Fix $d \geq 1$ and $i = 1,\dots,n+1$.*

(a) *The following two conditions are equivalent:*
  (i) $H_1(f_1 t)_d = 0, H_1(\underline{\mathrm{f_2 t}})_d = 0, \dots, H_1(\underline{\mathrm{f_i t}})_d = 0$;
  (ii) $T_{1,d} = 0, T_{2,d} = 0, \dots, T_{i,d} = 0$.

(b) *Suppose that, for some $i = 1,\dots,n$, $T_{1,d} = 0, T_{2,d} = 0, \dots, T_{i,d} = 0$. Then*
$$H_1(\underline{\mathrm{f_{i+1} t}})_d \cong T_{i+1,d}.$$

(c) *Fix now $d \geq 2$. Suppose that, for some $i = 1,\dots,n-1$,*
$$T_{1,d} = 0, T_{2,d} = 0, \dots, T_{i,d} = 0 \text{ and that } T_{1,d-1} = 0, T_{2,d-1} = 0, \dots, T_{i,d-1} = 0.$$
Then the following sequence is exact.
$$0 \to \frac{(J_i I^{d-1} : f_{i+1}) \cap I^{d-1}}{f_{i+2} \cdot [(J_i I^{d-2} : f_{i+1}) \cap I^{d-2}] + J_i I^{d-2}} \to H_1(\underline{\mathrm{f_{i+2} t}})_d \to T_{i+2,d} \to 0. \tag{2.7}$$

Proof. The implication $(i) \Rightarrow (ii)$ follows directly from the exact sequences (2.4) and (2.5). Since $H_1(\underline{\mathrm{f_0 t}}) = 0$, then, by (2.4) and (2.5), $H_1(f_1 t)_d \cong T_{1,d}$ and $(ii) \Rightarrow (i)$ holds for $i = 1$. Suppose that $(ii) \Rightarrow (i)$ holds for $i - 1 \geq 1$. By the induction hypothesis, $H_1(\underline{\mathrm{f_{i-1} t}})_d = 0$ and, by (2.4) and (2.5), $H_1(\underline{\mathrm{f_i t}})_d \cong T_{i,d} = 0$. This proves (a).

Suppose now that, for some $i = 1,\dots,n-1$, $T_{1,d} = 0, T_{2,d} = 0, \dots, T_{i,d} = 0$. In particular, since $(ii) \Rightarrow (i)$, $H_1(f_1 t)_d = 0, H_1(\underline{\mathrm{f_2 t}})_d = 0, \dots, H_1(\underline{\mathrm{f_i t}})_d = 0$. Using (2.4) and (2.5), for the integer $i+1$, $H_1(\underline{\mathrm{f_{i+1} t}})_d \cong T_{i+1,d}$. This proves (b).

Suppose now that the hypotheses in (c) hold. Then, by (b) applied to $d-1$, we obtain the isomorphism $H_1(\underline{\mathrm{f_{i+1} t}})_{d-1} \cong T_{i+1,d-1}$. Therefore,

$$H_1(\underline{\mathrm{f_{i+1} t}})_{d-1} \cong \frac{(J_i I^{d-2} : f_{i+1}) \cap I^{d-2}}{J_i I^{d-3}} \text{ and } H_1(\underline{\mathrm{f_{i+1} t}})_d \cong \frac{(J_i I^{d-1} : f_{i+1}) \cap I^{d-1}}{J_i I^{d-2}}.$$

Through these isomorphisms,

$$\frac{H_1(\underline{\mathrm{f_{i+1} t}})_d}{f_{i+2} t \cdot H_1(\underline{\mathrm{f_{i+1} t}})_{d-1}} \cong \frac{(J_i I^{d-1} : f_{i+1}) \cap I^{d-1}}{f_{i+2} \cdot [(J_i I^{d-2} : f_{i+1}) \cap I^{d-2}] + J_i I^{d-2}}.$$

The rest follows from the exact sequence (2.5) applied to $i+2$.

Next we specialise Theorem 2.7 (c), to the case $n=2$.

COROLLARY 2.8. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $f_1,f_2,f\in\mathfrak{m}$ and let $J=(f_1,f_2)$ and $I=(f_1,f_2,f)$. Fix $d\geq 2$. Assume that $(0:f_1)\cap I^{d-2}=0$ and $(0:f_1)\cap I^{d-1}=0$. Then the following sequence is exact.*

$$0\to\frac{(f_1I^{d-1}:f_2)\cap I^{d-1}}{f\cdot[(f_1I^{d-2}:f_2)\cap I^{d-2}]+f_1I^{d-2}}\longrightarrow E(I)_d\longrightarrow\frac{(JI^{d-1}:f^d)}{(JI^{d-2}:f^{d-1})}\to 0. \tag{2.8}$$

PROOF. Take $n=2$, $i=1$ and $d\geq 2$ in Theorem 2.7 (c). Then $T_{1,d-1}=(0:f_1)\cap I^{d-2}$ and $T_{1,d}=(0:f_1)\cap I^{d-1}$, which are zero by hypothesis. The rest follows from the sequence (2.7).

COROLLARY 2.9. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $f_1,f_2,f\in\mathfrak{m}$ and let $J=(f_1,f_2)$ and $I=(f_1,f_2,f)$. Fix $L\geq 2$. Suppose that $(0:f_1)\cap I^{d-1}=0$, for all $1\leq d\leq L$, and that $(f_1:f_2)\subseteq(f_1:f)$. Then the following two conditions are equivalent.*

- (a) *$T_{2,d}=(f_1I^{d-1}:f_2)\cap I^{d-1}/f_1I^{d-2}=0$, for all $2\leq d\leq L$;*
- (b) *$E(I)_d\cong(JI^{d-1}:f^2)/(JI^{d-2}:f^{d-1})$, for all $2\leq d\leq L$.*

PROOF. The hypotheses $(0:f_1)\cap I^{d-1}=0$, for all $1\leq d\leq L$, allows us to apply Corollary 2.8, for all $2\leq d\leq L$.

Suppose that (a) holds. The vanishing of $T_{2,d}$ ensures the vanishing of the left-hand side term in the exact sequence (2.8). Thus (b) holds.

Conversely, assume that (b) holds. Let us prove $T_{2,d}=0$, by induction on d, $2\leq d\leq L$. So take $d=2$. Using (b) and (2.8), for $d=2$, then $(f_1I:f_2)\cap I=f\cdot(f_1:f_2)$. Using the hypothesis $(f_1:f_2)\subseteq(f_1:f)$, we get

$$(f_1I:f_2)\cap I=f\cdot(f_1:f_2)\subseteq f\cdot(f_1:f)\subseteq(f_1).$$

Thus $T_{2,2}=0$. Take now $d\geq 3$, $d\leq L$. By the induction hypothesis $T_{2,d-1}=0$, so

$$(f_1I^{d-2}:f_2)\cap I^{d-2}=f_1I^{d-3}.$$

Using (b) and (2.8), for such d, then

$$(f_1I^{d-1}:f_2)\cap I^{d-1}=f\cdot[(f_1I^{d-2}:f_2)\cap I^{d-2}]+f_1I^{d-2}=f\cdot[f_1I^{d-3}\cap I^{d-2}]+f_1I^{d-2}\subseteq f_1I^{d-2}.$$

Hence $T_{2,d}=0$.

REMARK 2.10. In the case that f_1,f_2 is a regular sequence, then clearly $(0:f_1)\cap I^{d-1}=0$ for all $1\leq d\leq L$ and $(f_1:f_2)=f_1R\subseteq(f_1:f)$. However the converse does not always hold as the next example shows.

EXAMPLE 2.11. Let $R=k[[x,y]]$ be the formal power series ring in two variables over a field k of characteristic zero. Take $a,b\geq 2$ and consider the ideals $J=(f_1,f_2)$ and $I=(f_1,f_2,f)$ with $f=x^ay^b$, $f_1=\frac{df}{dx}=ax^{a-1}y^b$ and $f_2=\frac{df}{dy}=bx^ay^{b-1}$. Then $(f_1:f_2)=yR\subseteq(f_1:f)=R$, whereas f_1,f_2 is not a regular sequence. We point out that $J=I$ is an ideal of linear type.

3. Ideals with expected relation type

We recall now a central concept to our purposes.

DEFINITION 3.1. Let $(R,\mathfrak{m})$ be a Noetherian local ring and let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R. The ideal $\mathfrak{b}$ is a *reduction of* $\mathfrak{a}$ if $\mathfrak{b}\subseteq\mathfrak{a}$ and there is an integer $r\geq 0$ such that $\mathfrak{a}^{r+1}=\mathfrak{b}\mathfrak{a}^r$. From the definition it follows that $\operatorname{rad}(\mathfrak{b})=\operatorname{rad}(\mathfrak{a})$, $\operatorname{Min}(R/\mathfrak{b})=\operatorname{Min}(R/\mathfrak{a})$ and $\operatorname{height}(\mathfrak{b})=\operatorname{height}(\mathfrak{a})$ (see, e.g.,[25, Lemma 8.10]). Note that the ideal $\mathfrak{a}$ is always a reduction of itself. An ideal $\mathfrak{a}$ which has no reduction other than itself is called a *basic ideal*. The smallest integer $r\geq 0$ satisfying the equality equality $\mathfrak{a}^{r+1}=\mathfrak{b}\mathfrak{a}^r$ is called the *reduction number* of $\mathfrak{a}$ with respect to $\mathfrak{b}$ and is denoted $\operatorname{rn}_{\mathfrak{b}}(\mathfrak{a})$. For $\mathfrak{b}=\mathfrak{a}$, $\operatorname{rn}_{\mathfrak{b}}(\mathfrak{a})=0$. (see [22]).

The next result is shown in [17, Lemma 3.1]. We deduce it here from our previous remarks.

Proposition 3.2. *Let $(R,\mathfrak{m})$ be Noetherian local ring, $n \geq 2$, and let $J = (f_1,\dots,f_n)$ be a reduction of $I = (f_1,\dots,f_n,f)$. Then*

$$\mathrm{rn}_J(I) + 1 \leq \mathrm{rt}(I).$$

Proof. Let $\mathrm{rt}(I) = L \geq 1$. Hence, $E(I)_d = 0$, for all $d \geq L+1$. By the exact sequence (2.6), $T_{n+1,d} = 0$, for all $d \geq n+1$. Therefore $(JI^{d-1} : f^d) = (JI^{d-2} : f^{d-1})$, for all $d \geq L+1$. Since J is a reduction of I, then $(JI^{m-1} : f^m) = R$, for $m \gg 0$ large enough. Thus $f^L \in JI^{L-1}$, $I^L = JI^{L-1}$ and $\mathrm{rn}_J(I) \leq L-1$.

Definition 3.3. Let $(R,\mathfrak{m})$ be a Noetherian local ring and let $J = (f_1,\dots,f_n)$ be a reduction of $I = (f_1,\dots,f_n,f)$. We say that I has the *expected relation type with respect to* J if

$$\mathrm{rn}_J(I) + 1 = \mathrm{rt}(I).$$

When J is understood by the context, we will skip the locution "with respect to J".

Example 3.4. Let $(R,\mathfrak{m},k)$ be a Noetherian local ring with $k = R/\mathfrak{m}$ an infinite field. Let $\mathfrak{a}$ be an ideal of R.

- (a) If $\mathfrak{a}$ is of linear type, then $\mathfrak{a}$ is basic and has the expected relation type.
- (b) If $\mathfrak{a}$ is a parametric ideal, that is, generated by a system of parameters, then $\mathfrak{a}$ is basic, but it is not necessarily of linear type, nor it has necessarily the expected relation type.

Proof. Let $\mathfrak{b}$ be a reduction of $\mathfrak{a}$. By [22, 2. Theorem 1], there exists an ideal $\mathfrak{c} \subseteq \mathfrak{b} \subseteq \mathfrak{a}$, which is a minimal reduction of $\mathfrak{a}$, that is, no ideal strictly contained in $\mathfrak{c}$ is a reduction of $\mathfrak{a}$. By [22, 2. Lemma 3], every minimal set of generators of $\mathfrak{c} = (x_1,\dots,x_s)$ can be extended to a minimal set of generators of $\mathfrak{a} = (x_1,\dots,x_s,x_{s+1},\dots,x_m)$, with $\mu(\mathfrak{c}) = s \leq \mu(\mathfrak{a}) = m$, where $\mu(\cdot)$ stands for the minimal number of generators. If $\mathfrak{a}$ is of linear type, then $\mathbf{S}(\mathfrak{a}) \cong \mathbf{R}(\mathfrak{a})$. On tensoring by $A/\mathfrak{m}$, $k[T_1,\dots,T_m] \cong \mathbf{F}(\mathfrak{a})$, where $\mathbf{F}(\mathfrak{a}) = \oplus_{d\geq 0}\mathfrak{a}^d/\mathfrak{m}\mathfrak{a}^d$ is the *fiber cone of* $\mathfrak{a}$. On taking Krull dimensions, we get $\mu(\mathfrak{a}) = m = l(\mathfrak{a})$, where $l(\mathfrak{a}) = \dim \mathbf{F}(\mathfrak{a})$ is the *analytic spread of* $\mathfrak{a}$. By [6, Proposition 4.5.8], $l(\mathfrak{a}) \leq \mu(\mathfrak{c})$. Thus $s = m$ and $\mathfrak{c} = \mathfrak{a}$. Therefore $\mathfrak{b} = \mathfrak{a}$ and $\mathfrak{a}$ is basic. In particular, $\mathrm{rn}_{\mathfrak{b}}(\mathfrak{a}) = 0$ and, since $\mathfrak{a}$ is of linear type, $\mathrm{rt}(\mathfrak{a}) = 1 = \mathrm{rn}_{\mathfrak{b}}(\mathfrak{a}) + 1$. This proves (a).

If $\mathfrak{a}$ is a parametric ideal, then height $(\mathfrak{a}) = \mu(\mathfrak{a})$. By [22, 4. Theorem 5], $\mathfrak{a}$ is basic. Take now $R = k[[x,y,z,w]]$, where $w^2 = wz = 0$, and $\mathfrak{a} = (x^{m-1}y + z^m, x^m, y^m)$, $m \geq 2$. Then $\mathfrak{a}$ is a parameter ideal, hence a basic ideal, but its relation type is at least m (see [1, Example 2.1]).

The following result gives a characterization of ideals with expected relation type in terms of the Koszul homology.

Proposition 3.5. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and let $J = (f_1,\dots,f_n)$ be a reduction of $I = (f_1,\dots,f_n,f)$. The following conditions are equivalent.*

- (a) *I has the expected relation type;*
- (b) *$H_1(\underline{\mathrm{f}_\mathrm{n}}\mathrm{t})_d = ft \cdot H_1(\underline{\mathrm{f}_\mathrm{n}}\mathrm{t})_{d-1}$, for all $d \geq \mathrm{rn}_J(I) + 2$.*

In particular, if $T_{i,d} = (J_{i-1}I^{d-1} : f_i) \cap I^{d-1}/J_{i-1}I^{d-1} = 0$, for all $d \geq \mathrm{rn}_J(I) + 2$ and all $i = 1,\dots,n$, then I has the expected relation type.

Proof. Set $r = \mathrm{rn}_J(I)$. If I has the expected relation type, then $E(I)_d = 0$, for all $d \geq r+2$. In particular, using the exact sequence (2.6), we deduce $H_1(\underline{\mathrm{f}_\mathrm{n}}\mathrm{t})_d = ft \cdot H_1(\underline{\mathrm{f}_\mathrm{n}}\mathrm{t})_{d-1}$, for all $d \geq r+2$. Conversely, if $H_1(\underline{\mathrm{f}_\mathrm{n}}\mathrm{t})_d = ft \cdot H_1(\underline{\mathrm{f}_\mathrm{n}}\mathrm{t})_{d-1}$, for all $d \geq r+2$, then by (2.6), $E(I)_d \cong (JI^{d-1} : f^d)/(JI^{d-1} : f^{d-1})$. However, if $d \geq r+2$, then $f^{d-1} \in JI^{d-2}$, so $(JI^{d-1} : f^{d-1}) = R$ and $E(I)_d = 0$. Therefore, $\mathrm{rt}(I) \leq r+1 = \mathrm{rn}_J(I)+1$. The other inequality follows from Proposition 3.2. This shows the equivalence $(a) \Leftrightarrow (b)$.

If $T_{1,d} =, T_{2,d} = 0,\dots,T_{n,d} = 0$, for all $d \geq \mathrm{rn}_J(I) + 2$, by Theorem 2.7, $H_1(\underline{\mathrm{f}_\mathrm{n}}\mathrm{t})_d = 0$, for all $d \geq \mathrm{rn}_J(I) + 2$, and, by $(b) \Rightarrow (a)$, I has the expected relation type.

The main result of Muiños and the second author in [17], gives an instance of ideals of expected relation type. We rephrased it here in terms of our $T_{i,d}$.

Corollary 3.6. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $n \geq 2$. Let $f_1,\dots,f_n \in \mathfrak{m}$ and $f \in \mathfrak{m}$. Let $J=(f_1,\dots,f_n)$ and $I=(f_1,\dots,f_n,f)$ be ideals of R. Assume that for all $d \geq 2$ and all $i=1,\dots,n$,*

$$T_{i,d}=(J_{i-1}I^{d-1}:f_i)\cap I^{d-1}/J_{i-1}I^{d-2}=0.$$

Then, for all $d \geq 2$,

$$E(I)_d \cong \frac{(JI^{d-1}:f^d)}{(JI^{d-2}:f^{d-1})}.$$

In particular, if J is a reduction of I, then I has the expected relation type with respect to J.

Proof. If $T_{1,d}=0, T_{2,d}=0,\dots,T_{n,d}=0$, for all $d \geq 2$, then, by Theorem 2.7 (a), $H_1(\underline{\mathrm{f_n}}\mathrm{t})_d=0$, for all $d \geq 2$. By the exact sequence (2.6), $E(I)_d \cong (JI^{d-1}:f^d)/(JI^{d-2}:f^{d-1})$. The second assertion follows directly from Proposition 3.5.

4. Divisors of expected Jacobian type

Let (X,O) be a germ of a smooth n-dimensional complex variety and $\mathcal{O}_{X,O}$ the ring of germs of holomorphic functions in a neighbourhood of O, which we identify with $R=\mathbb{C}\{x_1,\dots,x_n\}$ by taking local coordinates. Let (D,O) be a germ of divisor defined locally by $f \in R$ and set $f_i=\frac{df}{dx_i}$ for $i=1\dots,n$. From now on, until the end of the paper, we consider the following notations.

Setting 4.1. Let $R=\mathbb{C}\{x_1,\dots,x_n\}$ be the convergent power series ring, which is a Noetherian regular local ring (see, e.g., [25, Lemma 7.1]). Let $f \in \mathfrak{m}$. Set $f_i=\frac{df}{dx_i}$, for $i=1\dots,n$. Let $J=(f_1,\dots,f_n)$ and $I=(f_1,\dots,f_n,f)=(J,f)$. Note that if $f \in \mathfrak{m}^2$, then $f_i \in \mathfrak{m}$ and $J \subseteq I \subseteq \mathfrak{m}$. The ideals J and I are called the *gradient ideal of* f and the *Jacobian ideal of* f, respectively. It is known that, when $f \in \mathfrak{m}$, then $f \in \overline{(x_1f_1,\dots,x_nf_n)} \subseteq \overline{J}$, where $\overline{H}$ stands for the integral closure of the ideal H (see [25, Corollary 7.1.4]). In particular, J is a reduction of I (see, e.g., [25, Proposition 1.1.7]).

Definition 4.2. A germ of divisor (D,O), with reduced equation given by f, is of *linear Jacobian type* if I is an ideal of linear type ([8, Definition 1.11]); f will be said of *expected Jacobian type*, if J is of linear type and I has the expected relation type with respect to J.

Remark 4.3. Divisors of linear Jacobian type are divisors of expected Jacobian type. Indeed, by Example 3.4, (a), if f is of linear Jacobian type, then I is basic, thus $J=I$ is of linear type and I has the expected relation type since $\mathrm{rt}(I)=1$ and $\mathrm{rn}_J(I)=0$.

Example 4.4. Let $R=\mathbb{C}\{x,y\}$ be the convergent power series ring in two variables x,y. Let $f \in \mathbb{C}[x,y]$ be a polynomial such that $f \notin \mathbb{C}[\lambda x+y],\mathbb{C}[x+\mu y]$, for all $\lambda,\mu \in \mathbb{C}$. Then J is an ideal of linear type minimally generated by two elements.

Proof. Since R is local and $J=(f_1,f_2)$, to see that J is minimally generated by two elements it is enough to prove that $f_1 \notin (f_2)$ or $f_2 \notin (f_1)$. Let us see that if $f_1 \in (f_2)$, then f is either in $\mathbb{C}[y]$, or else in $\mathbb{C}[x+\mu y]$, for some $\mu \in \mathbb{C}$ (similarly, one would do the same if $f_2 \in (f_1)$). Since $f \notin \mathbb{C}[y]$, $\deg_x(f)=r \geq 1$. Write $f=\sum_{i=0}^r x^r g_i(y)$ and suppose that $f_1=pf_2$, for some element $p \in \mathbb{C}\{x,y\}$. Since f_1 and f_2 are polynomials, then $p \in \mathbb{C}[x,y]$ must be a polynomial too. Equating the highest degree terms in x in the expression $f_1=pf_2$, one deduces that either $p=0$, or else $g_r'(y)=0$. However, if $p=0$, then $f_1=0$ and $f \in \mathbb{C}[y]$, a contradiction. Thus $g_r'(y)=0$ and $g_r(y)=a_r \in \mathbb{C}$, $a_r \neq 0$, because $\deg_x(f)=r \geq 1$. Substituting $g_r(y)=a_r$ in f and equating again the highest degree term in x in the equality $f_1=pf_2$, one gets $pg_{r-1}'(y)=ra_r \neq 0$. Thus $p \in \mathbb{C}$, $p \neq 0$. Setting $\mu=1/p$, we have $g_{r-1}(y)=ra_r\mu y+a_{r-1}$. Again, substituting this expression in f and equating the $r-2$ degree terms in the equality $f_1 = (1/\mu)f_2$, one gets $g_{r-2}(y) = a_r\binom{r}{2}(\mu y)^2 + a_{r-1}\mu y + a_{r-2}$. Proceeding recursively, one would get the equality $f = \sum_{i=0}^{r} a_i(x+\mu y)^i$ and so f would be an element of $\mathbb{C}[x+\mu y]$, a contradiction.

Therefore, J is minimally generated by two elements. Now apply [13, Proposition 1.5]. Thus J can be generated by two elements which form a d-sequence. In particular J is of linear type (see, e.g., [25, Corollary 5.5.5]).

Example 4.5. Suppose that $J = (f_1, \dots, f_n)$ is generated by an R-regular sequence, for instance, if f has an isolated singularity at O (see, e.g., [24, IV, Remark 2.5]). In particular, J is of linear type ([25, Corollary 5.5.5]). There are three possibilities according to the previous definition. If I is of linear type, then f is a divisor of linear Jacobian type. Suppose that I is not of linear type. Recall that, by the argument in Setting 4.1, J is a reduction of I and, by Proposition 3.2, $\mathrm{rn}_J(I)+1 \leq \mathrm{rt}(I)$. If the equality holds, then f is of expected Jacobian type. The third and last case occurs when J is of linear type, but $\mathrm{rn}_J(I)+1 < \mathrm{rt}(I)$, i.e., f is not of expected Jacobian type.

Remark 4.6. The linear type condition for the Jacobian ideal was investigated by Calderón-Moreno and Narváez-Macarro in [7, 8] (see also [20]). They proved that divisors of linear Jacobian type are Euler homogeneous. Recall that a divisor D is *Euler homogeneous* if there is a vector field χ at O such that $\chi(f) = f$, or in other words, $f \in J$. In particular, $J = I$ and $\mathrm{rn}_J(I) = 0$.

For divisors satisfying the conditions $T_{i,d} = 0$ we can explicitly describe the equations of the Rees algebra of the Jacobian ideal which corresponds to describe the blow-up at the singular locus of the divisor. More precisely, and summarizing several results in a unique statement:

Theorem 4.7. *Let $R = \mathbb{C}\{x_1, \dots, x_n\}$ be the convergent power series ring Let $f \in \mathfrak{m}^2$, J and I be as in Setting 4.1. Suppose that $T_{i,d} = 0$, for all $d \geq 1$ and all $i = 1, \dots, n$. Set $\xi_{i+1} = s$. Let*

$$\varphi : R[\xi_1, \dots, x_n, s] = \mathbb{C}\{x_1, \dots, x_n\}[\xi_1, \dots, x_n, s] \to \boldsymbol{R}(I) \tag{4.1}$$

be a polynomial presentation of $\boldsymbol{R}(I)$, sending ξ_i to $f_i t$ and s to ft. Let $Q = \bigoplus_{d\geq 1} Q_d$ the ideal of equations of $\boldsymbol{R}(I)$. The following conditions hold.

(a) *$f_1, \dots, f_n$ is an R-regular sequence and J is of linear type.*
(b) *J is a reduction of I and $\mathrm{rt}(I) = \mathrm{rn}_J(I)+1$. If $f \in J$, then f is a divisor of linear Jacobian type; otherwise, f is a divisor of expected Jacobian type.*
(c) *For all $d \geq 2$,*

$$E(I)_d \cong (Q/Q\langle d-1\rangle)_d \cong \frac{(JI^{d-1} : f^d)}{(JI^{d-2} : f^{d-1})}; \tag{4.2}$$

the class of $P(\xi_1, \dots, \xi_n, s) \in Q_d$ is sent to the class of $P(0, \dots, 0, 1) \in (JI^{d-1} : f^d)$.

(d) *Set $L = \mathrm{rt}(I)$. A minimal generating set of equations of $\boldsymbol{R}(I)$ can be obtanied from a minimal generating set of Q_1, the first syzygies of I, and representatives of inverse images of a minimal generating set of $(JI^{d-1} : f^d)/(JI^{d-2} : f^{d-1})$, for all $2 \leq d \leq L$.*
(e) *There exists a unique top-degree equation of degree L, which is of the form*

$$s^L + p_1 s^{L-1} + \cdots + p_L, \tag{4.3}$$

where $p_j \in R[\xi_1, \dots, \xi_n]$ are either zero, or else, polynomials of degree j.

Proof. Since $f \in \mathfrak{m}^2$, then $f_i \in \mathfrak{m}$ and so $J \subseteq I \subseteq \mathfrak{m}$. Recall that $T_{i,1} = (J_{i-1} : f_i)$. Therefore, $T_{i,1} = 0$, for all $i = 1, \dots, n$ is equivalent to $f_1, \dots, f_n$ being an R-regular sequence. In particular, I is of linear type. This proves (a); (b) is done in Setting 4.1; (c) and (d) are shown in Corollaries 3.6 and 2.6. Finally, since $L = \mathrm{rt}(I) = \mathrm{rn}_J(I)+1$, then $I^L = JI^{L-1}$ and $f^L \in JI^{L-1}$, which defines an equation of the desired form, namely, $P = \sum_{j=1}^{n} \xi_j P_j + s^L$, with $P_j \in R[\xi_1, \dots, \xi_n, s]_{L-1}$. The image of the class of this equation through the isomorphism (4.2) is precisely the class of $P(0, \dots, 0, 1) = 1$. Note that, for $d = L$, then $(JI^{L-1} : f^L) = R$, and the isomorphism (4.2) is given by $E(I)_L \cong R/(JI^{d-2} : f^{d-1})$, which says, in particular, that there exists a unique top-degree equation.

Remark 4.8. Whenever the conditions $T_{i,d} = 0$, for all $d \geq 1$ and all $i = 1, \dots, n$, do not hold, we need to be more careful when trying to describe the equations of the Rees algebra of the Jacobian ideal of I. Our guide here will be Corollary 2.6. Note that we may have non-zero terms on both sides of the short exact sequence (2.6). In particular, it may happen that $\mathrm{rn}_J(I) + 1 < \mathrm{rt}(I)$. However, even in this case, we will have an equation of the form 4.3, with $L = \mathrm{rn}_J(I) + 1$. The rest of the equations of $\mathbf{R}(I)$ will have degree in s smaller than L, although they may have total degree much bigger.

5. An application to D-module theory

Let X be a smooth n-dimensional complex variety and let D_X be the sheaf of linear differential operators on X with holomorphic coefficients. Taking local coordinates at $O \in X$ we will simply consider $R = \mathbb{C}\{x_1, \dots, x_n\}$ and its associated ring of differential operators $D_R = \mathbb{C}\{x_1, \dots, x_n\}\langle \partial_1, \dots, \partial_n \rangle$ where $\partial_i := \frac{d}{dx_i}$ are the partial derivatives with respect to the variable x_i, $i = 1, \dots, n$. Notice that $\partial_i x_i - x_i \partial_i = 1$ so this is a non-commutative Noetherian ring whose elements can be expressed in its normal form as

$$P := P(x, \partial) = \sum_{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n_{\geq 0}} a_\alpha(x_1, \dots, x_n)\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n},$$

with finitely many $a_\alpha(x_1, \dots, x_n) \in R$ different from zero. The *order* of such a differential operator is $\mathrm{ord}(P) = \max\{|\alpha| \mid a_\alpha \neq 0\}$ and its *symbol* is the element in $\mathbb{C}\{x_1, \dots, x_n\}[\xi_1, \dots, \xi_n]$

$$\sigma(P) = \sum_{|\alpha|=\mathrm{ord}(P)} a_\alpha(x_1, \dots, x_n)\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

Indeed, we have a filtration $F := \{F_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of D_R given by the order, or equivalently setting $\deg(x_i) = 0$ and $\deg(\partial_i) = 1$, whose associated graded ring is

$$gr_F(D_R) \cong \mathbb{C}\{x_1, \dots, x_n\}[\xi_1, \dots, \xi_n],$$

with the isomorphism given by sending $\overline{P} \in gr_F(D_R)$ to the symbol $\sigma(P)$.

More generally we may consider the polynomial ring $D_R[s]$ with coefficients in D_R whose elements are

$$P(s) := P(x, \partial, s) = P_0 s^L + P_1 s^{L-1} + \cdots + P_L,$$

with $P_i \in D_R$. The *total order* of $P(s)$ is $\mathrm{ord}^T(P(s)) = \max\{\mathrm{ord}(P_i) + i \mid i = 0, \dots, L\}$ and its *total symbol* is

$$\sigma^T(P(s)) = \sum_{|\alpha|+i=\mathrm{ord}^T(P(s))} a_\alpha(x_1, \dots, x_n)\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} s^i \in \mathbb{C}\{x_1, \dots, x_n\}[\xi_1, \dots, \xi_n, s].$$

In this case, the filtration $F^T := \{F_i^T\}_{i\in\mathbb{Z}_{\geq 0}}$ of $D_R[s]$ given by the total order, or equivalently setting $\deg(x_i) = 0$, $\deg(\partial_i) = 1$ and $\deg(s) = 1$, provides an isomorphism

$$gr_{F^T}(D_R[s]) \cong \mathbb{C}\{x_1, \dots, x_n\}[\xi_1, \dots, \xi_n, s].$$

Given $f \in R$, there exist $P(s) \in D_R[s]$ and a nonzero polynomial $b(s) \in \mathbb{Q}[s]$ such that

$$P(s) \cdot f\boldsymbol{f^s} = b(s)\boldsymbol{f^s}.$$

The unique monic polynomial of smallest degree satisfying this functional equation is called the *Bernstein-Sato polynomial* of f. This is an important invariant in the theory of singularities (see [11], for further details). The Bernstein-Sato should be understood as an equation in $R_f[s]\boldsymbol{f^s}$, which is the free rank-one $R_f[s]$-module generated by the formal symbol $\boldsymbol{f^s}$. Moreover, $R_f[s]\boldsymbol{f^s}$ has a $D_R[s]$-module structure given by the action of the partial derivatives as follows: for $h \in R_f[s]$ we have

$$\partial_i \cdot h\boldsymbol{f^s} = \left(\frac{dh}{dx_i} + shf^{-1}\frac{df}{dx_i}\right)\boldsymbol{f^s}.$$

Let $D_R[s]\boldsymbol{f^s} \subset R_f[s]\boldsymbol{f^s}$ be the $D_R[s]$-submodule generated by $\boldsymbol{f^s}$. This module has a presentation as

$$D_R[s]\boldsymbol{f^s} \cong \frac{D_R[s]}{\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s})},$$

where $\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s}) := \{P(s) \in D_R[s] \mid P(s) \cdot \boldsymbol{f^s} = 0\}$. The Bernstein-Sato polynomial is the minimal polynomial of the action of s on

$$\frac{D_R[s]\boldsymbol{f^s}}{D_R[s]f\boldsymbol{f^s}} \cong \frac{D_R[s]}{\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s}) + (f)}.$$

In order to study these annihilators we may filter them by the order of the corresponding differential operators

$$\mathrm{Ann}^{(1)}_{D_R[s]}(\boldsymbol{f^s}) \subseteq \mathrm{Ann}^{(2)}_{D_R[s]}(\boldsymbol{f^s}) \subseteq \cdots \subseteq \mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s}).$$

A lot of attention has been paid to the case that this chain stabilizes at the first step.

Definition 5.1. A germ of divisor (D, O), with reduced equation given by $f \in R$, is of *linear differential type* if $\mathrm{Ann}^{(1)}_{D_R[s]}(\boldsymbol{f^s}) = \mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s})$, that is $\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s})$ is generated by total order one differential operators.

Divisors of linear differential type have been considered in relation to several different problems in D-module theory as we mentioned in the Introduction. In [7, Proposition 3.2] the authors proved that divisors of linear Jacobian type are of linear differential type.

Of course, being a divisor of linear differential type is a very restrictive condition since, in general, we will have differential operators of higher total order annihilating $\boldsymbol{f^s}$. Among these higher order operators there exists a monic one of the form

$$P(s) = s^L + P_1 s^{L-1} + \cdots + P_L,$$

with $\mathrm{ord}(P_i) \le i$, which we refer to as the *Kashiwara operator* (cf. [14, Theorem 6.3]). This fact prompted Yano to introduce the following invariants of f (see [29], [30]).

Notation 5.2. The *Kashiwara number* of f is

$$L(f) := \min\{L \mid P(s) = s^L + P_1 s^{L-1} + \cdots + P_L \in \mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s}) \text{ , } \mathrm{ord}(P_i) \le i\}.$$

Moreover, set

- $r(f) := \min\{\, r \ge 1 \mid f^r \in J\}$,
- $id(f) := \min\{\, r \ge 1 \mid f^r \in JI^{r-1}\} =$ integral dependence of f.

Clearly, $id(f) = \mathrm{rn}_J(I) + 1$. Yano proved that

$$r(f) \le id(f) \le L(f). \tag{5.1}$$

Furthermore, if f is quasi-homogeneous we have that $L(f) = 1$.

Yano [29] was able to compute the Bernstein-Sato polynomial of f, when $L(f) = 2, 3$ by giving an explicit free resolution of $\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s})$. This invariant also plays a prominent role in the algorithm presented in [5] to compute Bernstein-Sato polynomials of isolated singularities that are nondegenerate with respect to its Newton polygon.

Notice that the symbol of the Kashiwara operator resembles the equation of the Rees algebra of the Jacobian ideal of top degree in s. So we would like to get deeper insight into this relation. First, since $gr_{F^T}(D_R[s]) \cong \mathbb{C}\{x_1, \ldots, x_n\}[\xi_1, \ldots, \xi_n, s]$, we may interpret the presentation of the Rees algebra of the Jacobian ideal given in Equation 4.1 as a surjective morphism

$$\varphi : gr_{F^T}(D_R[s]) \longrightarrow \mathbf{R}(I).$$

A key result that can be found in [29, §I] (see [8, Lemma 1.9], for more details) states that

$$\sigma^T(\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s})) \subseteq \ker \varphi,$$

where $\sigma^T(\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s})) = \langle \sigma^T(P(s)) \mid P(s) \in \mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s})\rangle$. Indeed, it follows from [14, §5] and [29, Proposition 2.3] that

$$\mathrm{rad}\,(\sigma^T(\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s}))) = \ker\varphi.$$

Therefore there exists some non-negative integer $\ell \in \mathbb{Z}_{\geq 0}$ such that $(\ker\varphi)^\ell \subseteq \sigma^T(\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s}))$. We also point out that $\ker\varphi$ is a prime ideal since $\mathbf{R}(I) \subset R[t]$ is a domain.

Proposition 5.3. *Let (D,O) be a germ of a divisor of expected Jacobian type, with reduced equation given by $f \in R$. Let $\ell \in \mathbb{Z}_{\geq 0}$ be the smallest non-negative integer such that $(\ker\varphi)^\ell$ is contained in $\sigma^T(\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s}))$. Then*

$$\mathrm{rt}(I) \leq L(f) \leq \ell \cdot \mathrm{rt}(I). \tag{5.2}$$

In particular, if $\sigma^T(\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s})) = \ker\varphi$, then $\mathrm{rt}(I) = L(f)$.

Proof. It is readily seen that $id(f) = \mathrm{rn}_J(I) + 1$. Moreover, since f is of expected Jacobian type, then $\mathrm{rt}(I) = \mathrm{rn}_J(I) + 1$, and so, $\mathrm{rt}(I) = id(f) \leq L(f)$ (see (5.1)). Furthermore, the ideal of equations of the Rees algebra of I contains an element of the form

$$s^{r+1} + p_1 s^r + \cdots + p_{r+1} \in \ker\varphi,$$

where $r+1 = \mathrm{rt}(I) = id(f)$, and where each p_j is either zero, or else a polynomial of degree j. Therefore, we have that

$$(s^{r+1} + p_1 s^r + \cdots + p_{r+1})^\ell \in (\ker\varphi)^\ell \subseteq \sigma^T(\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s})),$$

so there exists a Kashiwara operator of degree at most $\ell \cdot \mathrm{rt}(I)$.

The upper bound in (5.2) is far from being sharp as we will see in the examples of Section 6. The issue here is how to lift an equation of the Rees algebra to a differential operator that annihilates $\boldsymbol{f^s}$. We would like to mention that necessary conditions for the existence of such a lifting were already given in [29].

6. Examples

Let $R = \mathbb{C}\{x_1, \ldots, x_n\}$ be the ring of convergent series with coefficients in $\mathbb{C}$ and, for a given $f \in \mathfrak{m}$, let $I = (f_1, \ldots, f_n, f)$ and $J = (f_1, \ldots, f_n)$ be the Jacobian and gradient ideal of f. We know that J is a reduction of I. If moreover J is generated by a regular sequence, then J is of linear type (see Setting 4.1 and Example 4.5).

The aim of this section is to illustrate with some examples the condition of having the expected relation type and compare the relation type of I with the invariant $L(f)$. In order to do so we will use the mathematical software packages `Macaulay2` [12], `Magma` [4], and `Singular` [10], to test the sufficient conditions

$$T_{i,d} = (J_{i-1}I^{d-1} : f_i) \cap I^{d-1}/J_{i-1}I^{d-2} = 0$$

considered in Theorem 4.7. Of course, $T_{1,d} = 0$ is always satisfied since f_1 is a nonzero divisor. In the case of plane curves we will only have to check whether the conditions $T_{2,d} = 0$ hold for $d \geq 2$.

Warning: Notice that we have to deal with an infinite set of conditions, namely, the vanishing of the modules $T_{i,d}$. Up to now, we do not know how to solve this difficulty. In the examples we present we could compute $T_{i,d}$ for all values of d up to a positive integer much larger than the reduction number of I. This suggests that the examples we deal with are of the expected Jacobian type but, in no case, our computations should be considered as formal proofs. We should also point out that the calculations of these colon ideals tend to be extremely costly from the computational point of view.

When computing the invariant $L(f)$ we will use the `Kashiwara.m2` package [3].

6.1. *Some examples of plane curves*
Let $f \in R = \mathbb{C}\{x_1, x_2\}$ be the equation of a germ of plane curve. It is proved in [20, Proposition 2.3.1] that f is a divisor of linear Jacobian type if and only if f is quasi-homogeneous, so this is a very restrictive assumption. Indeed, for irreducible plane curves, this corresponds to the case where $f = x^a + y^b$, with $\gcd(a, b) = 1$, a particular case of irreducible curve with one characteristic exponent. It is well known that the Bernstein-Sato polynomial varies within a deformation with constant Milnor number. We are going to test the behaviour of the expected relation type property with some examples of irreducible plane curves with an isolated singularity at the origin.

• **Reiffen curves:** We consider $f = x^a + y^b + xy^{b-1}$ with $b \geq a+1$, $a \geq 4$. This family of irreducible plane curves with one characteristic exponent has been a recurrent example in the theory of D-modules. In the case $a = 4$, Nakamura [18, 19] gave a a description of $\mathrm{Ann}_{D_R[s]}(\boldsymbol{f^s})$ and showed that $L(f) = 2$.

We have tested many examples of Reiffen curves varying the values of a and b. In all the cases we checked that $T_{2,d} = 0$, for all $d \geq 2$. Thus, by e.g., Corollary 3.6, I has the expected relation type with respect to J, in particular, f is a divisor of expected Jacobian type (see Definition 4.2). By Corollary 2.9 and Remark 2.10, we have $E(I)_d \cong (JI^{d-1} : f^d)/(JI^{d-2} : f^{d-1})$ and, further computations suggest that $(JI^{d-1} : f^d) = ((b-1)x + by, y^{a-1-2d})$, for all $d \leq \lfloor a/2 \rfloor - 1$, and $(JI^{d-1} : f^d) = R$, for $d \geq \lfloor a/2 \rfloor$. In particular, $\mathrm{rt}(I) = \mathrm{rn}_J(I) + 1 = \lfloor a/2 \rfloor$.

• **Irreducible curves with one characteristic exponent:** More generally we consider deformations with constant Milnor number of irreducible curves with one characteristic exponent which have the form

$$f = x^a + y^b - \sum t_{i,j}\, x^i y^j,$$

where $\gcd(a, b) = 1$ and the sum is taken over the monomials $x^i y^j$ such that $0 \leq i \leq a-2$, $0 \leq j \leq b-2$ and $bi + aj > ab$. It is well known that these curves belong to the same equisingularity class but their analytic type varies depending of the parameters $t_{i,j}$. In particular, the Bernstein-Sato polynomial also varies and there exists a stratification of the space of parameters with a specific Bernstein-Sato polynomial at each strata (see [15, 16] and [9]).

We take for example the following case considered by Kato [15]

· Let $f = x^7 + y^5 - t_{3,3}x^3y^3 - t_{5,2}x^5y^2 - t_{4,3}x^4y^3 - t_{5,3}x^5y^3$. The stratification given by the Bernstein-Sato polynomial with its corresponding $L(f)$ invariant is:

$\{t_{5,2} \neq 0, 6t_{5,2} + 175t_{3,3}^4 = 0\}$. We have $L(f) = 2$.
$\{t_{5,2} \neq 0, 6t_{5,2} + 175t_{3,3}^4 \neq 0\}$. We have $L(f) = 3$.
$\{t_{3,3} = 0, t_{5,2}t_{4,3} \neq 0\}$. We have $L(f) = 3$.
$\{t_{3,3} = 0, t_{5,2} \neq 0, t_{4,3} = 0\}$. We have $L(f) = 2$.
$\{t_{3,3} = 0, t_{5,2} = 0, t_{4,3} \neq 0\}$. We have $L(f) = 2$.
$\{t_{3,3} = 0, t_{5,2} = 0, t_{4,3} = 0, t_{5,3} \neq 0\}$. We have $L(f) = 2$.
$\{t_{3,3} = 0, t_{5,2} = 0, t_{4,3} = 0, t_{5,3} = 0\}$. We have $L(f) = 1$.

For any representative in each strata that we considered we checked out that $T_{2,d} = 0$ for all the values of d that we could compute. Moreover, the relation type is always $\mathrm{rt}(I) = 2$ except for the last case which obviously corresponds to the homogeneous case. In particular, there are strata in which we have an strict inequality $\mathrm{rt}(I) < L(f)$.

We have tested several other examples of irreducible plane curves with one characteristic exponent and all of them satisfied the conditions $T_{2,d} = 0$. This suggests that this class of plane curves have the expected relation type and we can describe the module of effective relations using Corollary 3.6.

• **Irreducible curves with two characteristic exponents:** We start considering the simplest example of such a plane curve which is:

· Let $f=(y^2-x^3)^2-x^5y$. We have $L(f)=2$. We checked out that condition $T_{2,d}=0$ is satisfied for all the values of d that we could compute and that the relation type is $\mathrm{rt}(I)=2$ so it seems to have the expected relation type.

However we can find examples with two characteristic exponents not satisfying $T_{2,d}=0$. For example,

· Let $f=(y^2-x^3)^5-x^3y^{10}$. The reduction number of I with respect to J is $\mathrm{rn}_J(I)=4$. We checked that $T_{2,2}=0$, $T_{2,3}\neq 0$, $T_{2,4}\neq 0$ and $T_{2,5}\neq 0$ but $T_{2,d}=0$ for all the values $d\geq 6$ that we could compute. According to Corollary 2.8 we have

$$0\to \frac{(f_1I^{d-1}:f_2)\cap I^{d-1}}{f\cdot[(f_1I^{d-2}:f_2)\cap I^{d-2}]+f_1I^{d-2}}\longrightarrow E(I)_d\longrightarrow \frac{(JI^{d-1}:f^d)}{(JI^{d-2}:f^{d-1})}\to 0.$$

and the right term is zero for $d\geq 6$. Even though the conditions $T_{2,d}=0$ are not satisfied for all $d\geq 2$ we have that the left term of the short exact sequence is zero for $d\geq 6$. Therefore the relation type is $\mathrm{rt}(I)=5$ so f has the expected relation type. Notice that the modules of effective relations are not as easy to describe as in Corollary 2.9.

Our computer runs out of memory before computing the invariant $L(f)$.

6.2. *Some examples which have not the expected relation type*
Narváez-Macarro [20] considered some examples of non-isolated singularities which are not of linear Jacobian type. We will revisit them from our own perspective. We point out that these examples satisfy $J=I$ so the effective relations for $d\geq 2$ are characterized by Corollary 2.6. All these examples satisfy $L(f)=1$ but the relation type is strictly bigger than one.

· $f=xy(x+y)(x+yz)$.

We have that $T_{2,d}=0$ for all the values of d that we could compute. On the other hand, $T_{3,2}\neq 0$ but $T_{3,d}=0$ for all the values of $d\geq 3$ that we could compute. Notice that we are in the situation where

$$E(I)_d\cong\frac{H_1(f_1t,f_2t,f_3t;\mathbf{R}(I))_d}{ftH_1(f_1t,f_2t,f_3t;\mathbf{R}(I))_{d-1}},$$

but in this case $E(I)_d=0$ for all $d\geq 3$, i.e. $\mathrm{rt}(I)=2$, as it was described in [20].

· $f=(xz+y)(x^k-y^k)$.

We have that $T_{2,d}=0$ for all the values of d that we could compute. However:

- $k=4: T_{3,2}\neq 0$, but $T_{3,d}=0$ for $d\geq 3$. This suggests that $\mathrm{rt}(I)=2$.
- $k=7: T_{3,2}\neq 0$, $T_{3,3}\neq 0$ and $T_{3,4}\neq 0$ but $T_{3,d}=0$ for $d\geq 5$. Thus $\mathrm{rt}(I)=4$.

Here we also have

$$E(I)_d\cong\frac{H_1(f_1t,f_2t,f_3t;\mathbf{R}(I))_d}{ftH_1(f_1t,f_2t,f_3t;\mathbf{R}(I))_{d-1}}.$$

For $k=4$ we have $H_1(f_1t,f_2t,f_3t;\mathbf{R}(I))_d=0$ if $d\geq 3$ and, in the case that $k=7$, we have $H_1(f_1t,f_2t,f_3t;\mathbf{R}(I))_d=0$ if $d\geq 5$. This also follows from the computations done in [20].

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References

1. Aberbach, I., Ghezzi, L., and Hà, H. T., *Homology multipliers and the relation type of parameter ideals*, Pacific J. Math. 226 (2006), no. 1, 1-39. 7
2. Arcadias, R., *Minimal resolutions of geometric D-modules*, J. Pure Appl. Algebra 214 (2010), 1477-1496. 1
3. Blanco, G., and Leykin, A., KASHIWARA.M2. A package for Macaulay 2 available at https://github.com/Macaulay2/Workshop-2016-Warwick/tree/master/Dmodules. 12
4. Bosma, W., Cannon, J., and Playoust, C., *The Magma algebra system. I. The user language*, J. Symbolic Comput., 24:235-265, 1997. 12
5. Briançon, J., Granger, M., Maisonobe, Ph., and Miniconi, M., *Algorithme de calcul du polynôme de Bernstein: cas non dégénéré.* Ann. Inst. Fourier (Grenoble) 39 (1989), 553-610. 1, 11
6. Bruns, W., and Herzog, J., *Cohen-Macaulay rings*, Cambridge Studies in advanced mathematics 39, Cambridge University Press, Cambridge 1993. 2, 7
7. Calderón-Moreno, F. J., and Narváez-Macarro, L., *The module $\mathcal{D}f^s$ for locally quasi-homogeneous free divisors,* Compositio Math. 134 (2002), 59–74. 1, 9, 11
8. Calderón-Moreno, F. J., and Narváez-Macarro, L., *On the logarithmic comparison theorem for integrable logarithmic connections*, Proc. London Math. Soc., 98 (2009), 585-606. 1, 8, 9, 11
9. Cassou-Noguès, Pi., *Étude du comportement du polynôme de Bernstein lors d'une déformation à μ-constant de x^a+y^b avec $(a, b) = 1$,* Compositio Math. 63 (1987), 291-313. 13
10. Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H.: SINGULAR 4-1-2 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2019). 12
11. Granger, M., *Bernstein-Sato polynomials and functional equations.* Algebraic approach to differential equations, 225–291, World Sci. Publ., Hackensack, NJ, 2010. 10
12. Grayson, D., and Stillman, M., MACAULAY 2. Available at: http://www.math.uiuc.edu/Macaulay2. 12
13. Huneke, C., *The Theory of d-Sequences and Powers of Ideals.* Adv. in Mathematics 46 (1982), 249-279. 9
14. Kashiwara, M., *B-functions and holonomic systems. Rationality of roots of B-functions.* Invent. Math. 38 (1976/77), 33-53. 1, 11, 12
15. Kato, M., *The b-function of μ-constant deformation of $x^7 + y^5$,* Bull. College Sci. Univ. Ryukyus 32 (1981), 5-10. 13
16. Kato, M., *The b-function of μ-constant deformation of $x^9 + y^4$,* Bull. College Sci. Univ. Ryukyus 33 (1982), 5-8. 13
17. Muiños, F., and Planas-Vilanova, F., *The equations of Rees algebras of equimultiple ideals of deviation one,* Proc. Amer. Math. Soc. 141 (2013), 1241-1254. 1, 2, 3, 5, 7, 8
18. Nakamura, Y., *On invariants of Reiffen's isolated singularity,* Algebraic analysis and the exact WKB analysis for systems of differential equations, 7-13, RIMS Kôkyûroku Bessatsu, B5, Res. Inst. Math. Sci. (RIMS), Kyoto, 2008. 13
19. Nakamura, Y., *The b-function of Reiffen's $(p, 4)$ isolated singularity,* J. Algebra Appl. 15 (2016), no. 6, 1650115, 11 pp. 13
20. Narváez-Macarro, L., *Linearity conditions on the Jacobian ideal and logarithmic-meromorphic comparison for free divisors*, Contemp. Math. 474 (2008) 1, 9, 13, 14
21. Narváez-Macarro, L., *A duality approach to the symmetry of Bernstein-Sato polynomials of free divisors*, Advances in Mathematics 281 (2015), 1242-1273. 1
22. Northcott, D. G., and Rees, D., *Reductions of ideals in local rings*, Mathematical Proceedings of the Cambridge Philosophical Society 50 (1954), 145-158. 6, 7
23. Planas-Vilanova, F., *On the module of effective relations of a standard algebra*, Math. Proc. Camb. Phil. Soc. 124 (1998), 215-229. 2
24. Ruiz, J. M., *The basic theory of power series*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1993. 9
25. Swanson, I., and Huneke, C., *Integral closure of ideals, rings, and modules,* London Mathematical Society Lecture Note Series, 336 Cambridge University Press, 2006. 2, 6, 8, 9
26. Torrelli, T., *Logarithmic comparison theorem and D-modules: an overview.* Singularity theory, 995–1009, World Sci. Publ., Hackensack, NJ, 2007. 1
27. Walther, U., *Survey on the D-module f^s.* With an appendix by Anton Leykin. Math. Sci. Res. Inst. Publ., 67. Vol. I, 391–430, Cambridge Univ. Press, New York, 2015. 1
28. Walther, U., *The Jacobian module, the Milnor fiber, and the D-module generated by f^s,* Invent. Math. 207 (2017), 1239-1287. 1
29. Yano, T., *On the theory of b-functions,* Publ. Res. Inst. Math. Sci. 14 (1978), 111-202. 1, 11, 12
30. Yano, T., *b-functions and exponents of hypersurface isolated singularities,* Singularities, Part 2 (Arcata, Calif., 1981), 641-652, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983. 11

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