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Note on an elementary inequality and its application to the regularity of p-harmonic functions

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Abstract. We study the Sobolev regularity of p-harmonic functions. We show that $|Du|^{\frac{p-2+s}{2}}Du$ belongs to the Sobolev space $W^{1,2}_{\mathrm{loc}}$, $s > -1 - \frac{p-1}{n-1}$, for any p-harmonic function u. The proof is based on an elementary inequality.

Huomautus eräästä alkeellisesta epäyhtälöstä ja sen sovellus p-harmonisten funktioiden säännöllisyyteen

Tiivistelmä. Tutkimme p-harmonisten funktioiden Sobolev-säännöllisyyttä. Osoitamme, että $|Du|^{\frac{p-2+s}{2}}Du$ kuuluu Sobolev-avaruuteen $W^{1,2}_{\mathrm{loc}}$, kun $s > -1 - \frac{p-1}{n-1}$ ja u on p-harmoninen funktio. Todistus perustuu alkeelliseen epäyhtälöön.

1. Introduction

In [7] Dong, Peng, Zhang and Zhou established the following inequality. Let v be a smooth real-valued function defined on a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Let $Dv := (v_{x_1}, \dots, v_{x_n})$ denote its gradient and $D^2v := (v_{x_ix_j})_{i,j=1}^n$ its Hessian. The Laplacian of v is denoted as

$$\Delta v := \operatorname{tr}(D^2v) = \sum_{i=1}^n v_{x_ix_i}$$

and the infinity Laplacian of v as

$$\Delta_\infty v := \langle Dv, D^2vDv\rangle = \sum_{i,j=1}^n v_{x_i}v_{x_ix_j}v_{x_j}.$$

Then

$$\begin{aligned} &\Big||D^2vDv|^2 - \Delta v\Delta_\infty v - \frac{1}{2}\big(|D^2v|^2 - (\Delta v)^2\big)|Dv|^2\Big| \\ &\leq \frac{n-2}{2}\big(|D^2v|^2|Dv|^2 - |D^2vDv|^2\big) \end{aligned} \tag{1.1}$$

holds everywhere in Ω. The authors derived (1.1) as a direct consequence of the inequality

$$\begin{aligned} &\Big|\sum_{i=1}^n(\lambda_ia_i)^2 - \Big(\sum_{i=1}^n\lambda_i\Big)\Big(\sum_{i=1}^n\lambda_ia_i^2\Big) - \frac{1}{2}\Big(|\lambda|^2 - \Big(\sum_{i=1}^n\lambda_i\Big)^2\Big)\Big| \\ &\leq \frac{n-2}{2}\Big(|\lambda|^2 - \sum_{i=1}^n(\lambda_ia_i)^2\Big) \end{aligned} \tag{1.2}$$

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that holds for any vectors $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ such that $|a| = 1$. For the proof of (1.2), see the proof of Lemma 2.2 in [7]. The inequality (1.1) is applied to study the regularity of solutions to p-Laplacian equation (see the equation (1.5) below) and its parabolic counterparts. For further details, we refer the reader to Theorems 1.1, 1.3 and 1.5 in [7].

The inequality (1.1) in the case $n = 2$ (when it is sharp) has been used to prove Sobolev regularity for planar infinity harmonic functions, see [15]. See also [18].

In this paper we show that (1.1) can be derived as a consequence of another elementary inequality that has been used before by Colding [3] to prove monotonicity formulas for solutions to certain elliptic partial differential equations. See for instance the proof of Theorem 2.4 in [3]. This elementary inequality says that for any symmetric matrix $A \in \mathbb{R}^{n\times n}$ and for any vector $e \in \mathbb{R}$ we have

$$|e|^4|A|^2 \geq 2|e|^2|Ae|^2 + \frac{\big(|e|^2 \operatorname{tr}(A) - \langle e, Ae\rangle\big)^2}{n-1} - \langle e, Ae\rangle^2. \tag{1.3}$$

If $n = 2$, we have equality instead of inequality in (1.3).

For a smooth function v, we apply the inequality (1.3) with $A = D^2v$ and $e = Dv$ to obtain a lower bound for the Hilbert–Schmidt norm of the Hessian D^2v with respect to the gradient Dv. More precisely, we obtain

$$|Dv|^4|D^2v|^2 \geq 2|Dv|^2|D^2vDv|^2 + \frac{\big(|Dv|^2\Delta v - \Delta_\infty v\big)^2}{n-1} - (\Delta_\infty v)^2. \tag{1.4}$$

The main point is that (1.4) implies (1.1) but not vice versa, apart from the case $n = 2$ where both inequalities reduce to equality. See Section 2 for details. Consequently, we are able to improve Theorem 1.1 in [7], which concerns regularity of p-harmonic functions.

Let $1 < p < \infty$. A function $u \in W^{1,p}(\Omega)$ is called p-harmonic, if it solves the p-Laplacian equation

$$\Delta_p u := \operatorname{div}\big(|Du|^{p-2}Du\big) = 0 \tag{1.5}$$

in the weak sense, that is, if

$$\int_\Omega |Du|^{p-2}\langle Du, D\varphi\rangle\, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$.

Let u denote a p-harmonic function in $\Omega \subset \mathbb{R}^n$, $n \geq 2$. For $s \in \mathbb{R}$, we define the vector field $V_s\colon \mathbb{R}^n \to \mathbb{R}^n$ as

$$V_s(z) := \begin{cases} |z|^{\frac{p-2+s}{2}} z & \text{for } z \in \mathbb{R}^n \setminus \{0\}; \\ 0 & \text{for } z = 0. \end{cases} \tag{1.6}$$

We study the Sobolev regularity of the vector field $V_s(Du)\colon \Omega \to \mathbb{R}^n$. The letter V refers to the notation used in [1, 12, 21]. The subscript s is a perturbation parameter that describes the deviation from the "natural" vector field $V(Du) := V_0(Du)$. We may call the vector field $V(Du)$ "natural" in this setting, because its Sobolev regularity arises more naturally than the one of the gradient Du alone. See for instance Proposition 2 in [2], where the authors apply the difference quotient characterization of Sobolev functions to show that $V(Du) \in W^{1,2}_{\text{loc}}(\Omega)$. For similar results, see for instance [23, Lemma 3.1], [11, Remark 8.4] and [21, Lemma 3.2].

In fact, on the contrary to the $W^{1,2}_{\text{loc}}$-regularity of $V(Du)$, it is not certain if the weak Hessian D^2u necessarily exists. Manfredi and Weitsman have shown in

[20, Lemma 5.1] that p-harmonic functions belong to $W^{2,2}_{\mathrm{loc}}$, provided that $1 < p < 3 + \frac{2}{n-2}$. This restriction for the range of p arises from so-called Cordes condition [4].

In this paper we are interested in the $W^{1,2}_{\mathrm{loc}}$-regularity of $V_s(Du)$ when $s \neq 0$. Dong, Peng, Zhang and Zhou apply (1.1) to prove that $V_s(Du) \in W^{1,2}_{\mathrm{loc}}$ whenever

$$s > 2 - \min\Big\{p + \frac{n}{n-1}, 3 + \frac{p-1}{n-1}\Big\}, \tag{1.7}$$

see [7, Theorem 1.1]. We improve this bound to

$$s > -1 - \frac{p-1}{n-1}. \tag{1.8}$$

In other words, we show that the condition $s > 2 - p - \frac{n}{n-1}$ is redundant and obtain nontrivial improvement in the case $1 < p < 2$ and $n \geq 3$.

The following theorem is an application of (1.4) and the main result of this paper. In the statement of the theorem, and throughout the paper, a generic ball in $\mathbb{R}^n$ with radius $r > 0$ is denoted briefly as B_r.

Theorem 1.1. *Let $n \geq 2$, $1 < p < \infty$, and $s > -1 - \frac{p-1}{n-1}$. If u is p-harmonic in $\Omega \subset \mathbb{R}^n$, then $V_s(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Moreover, there exists a constant $C = C(n,p,s) > 0$ such that*

$$\int_{B_r} |D(V_s(Du))|^2 \, dx \leq \frac{C}{r^2} \int_{B_{2r}} |V_s(Du) - z|^2 \, dx \tag{1.9}$$

for all vectors $z \in \mathbb{R}^n$ and all concentric balls $B_r \subset B_{2r} \subset\subset \Omega$.

Proof of Theorem 1.1 follows from establishing the case $z = 0$ in Section 3 and applying known results of p-harmonic functions in Section 4. Note that the right hand side of (1.9) is finite due to the well-known $C^{1,\alpha}_{\mathrm{loc}}$-regularity of p-harmonic functions for some $\alpha = \alpha(n,p) \in (0,1)$. For this classical result, we refer the reader to [6, 8, 17, 22–24]. For results concerning optimal regularity of p-harmonic functions, see [16] and [2, 13].

Using Sobolev–Poincaré inequality and Gehring's Lemma [9] with the estimate (1.1) leads to a higher integrability result for $D(V_s(Du))$. Here and subsequently, we denote the integral average of a locally integrable function v as

$$(v)_{B_r} := \fint_{B_r} v \, dx = \frac{1}{|B_r|} \int_{B_r} v \, dx.$$

Corollary 1.2. *Under the same hypothesis as Theorem 1.1, there exists a constant $\delta = \delta(n,p,s) > 0$ such that $D(V_s(Du)) \in L^q_{\mathrm{loc}}(\Omega)$ for every $1 \leq q < 2 + \delta$. Moreover, there exists a constant $C = C(n,p,s,q) > 0$ such that*

$$\Big(\fint_{B_r} |D(V_s(Du))|^q \, dx\Big)^{1/q} \leq C\Big(\fint_{B_{2r}} |D(V_s(Du))|^2 \, dx\Big)^{1/2} \tag{1.10}$$

for all concentric balls $B_r \subset B_{2r} \subset\subset \Omega$.

Proof. Combination of Sobolev–Poincare inequality and (1.9) with $z = \big(V_s(Du)\big)_{B_{2r}}$ yields

$$\begin{aligned}\Big(\fint_{B_r} |D(V_s(Du))|^2 dx\Big)^{1/2} &\leq \frac{C}{r}\Big(\fint_{B_{2r}} |V_s(Du) - \big(V_s(Du)\big)_{B_{2r}}|^2 dx\Big)^{1/2} \\ &\leq C\Big(\fint_{B_{2r}} |D(V_s(Du))|^{\frac{2n}{n+2}} dx\Big)^{\frac{n+2}{2n}}.\end{aligned}$$

Now Gehring's Lemma is applicable. The estimate (1.10) follows immediately. □

We finish the introduction by mentioning some interesting values of the parameter s. If $1 < p < 3+\frac{n}{n-2}$, then we can select $s = 2-p$. This reproves the $W^{2,2}_{\text{loc}}$-regularity of p-harmonic functions discussed above. The same conclusion can be drawn also from the stronger restriction (1.7) due to Dong, Peng, Zhang and Zhou. Our weakening (1.8) allows us to select $s = p - 2$, which reproves the known $W^{1,2}_{\text{loc}}$-regularity of the weakly divergence free vector field $|Du|^{p-2}Du$, see [5, 19] and [1, Theorem 4.1].

2. An elementary inequality

In this section we explain in detail how we improve the inequality (1.1).

Lemma 2.1. *Let $A \in \mathbb{R}^{n\times n}$, $n \geq 2$, be a symmetric matrix and $e \in \mathbb{R}^n$ a vector. Then we have*

$$|e|^4|A|^2 \geq 2|e|^2|Ae|^2 + \frac{\big(|e|^2 \operatorname{tr}(A) - \langle e, Ae\rangle\big)^2}{n-1} - \langle e, Ae\rangle^2. \tag{2.1}$$

If $n = 2$, equality holds in place of the inequality in (2.1).

Proof. If $e = 0$, then (2.1) is trivially true, thus we prove (2.1) for $e \neq 0$. Since (2.1) is homogeneous, we may assume without loss of generality that $|e| = 1$. We fix an orthogonal coordinate system $\{e_1, \dots, e_n\}$ in $\mathbb{R}^n$, such that $e_n = e$. Let $O := (e_1, \dots, e_n)$ be the corresponding orthogonal rotation matrix, where $e_1, \dots, e_n$ are interpreted as column vectors.

Denote $B := O^{\intercal}AO = \big(\langle e_i, Ae_j\rangle\big)_{i,j=1}^n$. Let $B_{n-1} := (B_{ij})_{i,j=1}^{n-1}$ be the submatrix given by the first $n-1$ rows and $n-1$ columns of B. We may decompose

$$|B|^2 = |B_{n-1}|^2 + 2\sum_{i=1}^{n-1}\langle e_i, Ae_n\rangle^2 + \langle e_n, Ae_n\rangle^2. \tag{2.2}$$

Consider the submatrix B_{n-1} as an element of the Hilbert space $\mathbb{R}^{(n-1)\times(n-1)}$ with the Hilbert–Schmidt matrix inner product. Apply Pythagoras's theorem to obtain

$$\begin{aligned} |B_{n-1}|^2 &= \frac{(\operatorname{tr}(B_{n-1}))^2}{n-1} + \left|B_{n-1} - \frac{\operatorname{tr}(B_{n-1})}{n-1}I\right|^2 \\ &\geq \frac{(\operatorname{tr}(B) - \langle e_n, Ae_n\rangle)^2}{n-1}, \end{aligned} \tag{2.3}$$

where I stands for the identity matrix in $\mathbb{R}^{(n-1)\times(n-1)}$. Note that if $n = 2$, we have equality in place of inequality in the above display (2.3). Rewrite the middle term on the right hand side of (2.2) as

$$2\sum_{i=1}^{n-1}\langle e_i, Ae_n\rangle^2 = 2|Ae_n|^2 - 2\langle e_n, Ae_n\rangle^2. \tag{2.4}$$

As we plug (2.3) and (2.4) into (2.2), we obtain

$$|B|^2 \geq \frac{(\operatorname{tr}(B) - \langle e_n, Ae_n\rangle)^2}{n-1} + 2|Ae_n|^2 - \langle e_n, Ae_n\rangle^2.$$

The desired estimate now follows, since by the cyclic property of trace we have $\operatorname{tr}(B) = \operatorname{tr}(A)$, and $|B|^2 = \operatorname{tr}(B^{\intercal}B) = \operatorname{tr}(A^{\intercal}A) = |A|^2$. □

Corollary 2.2. *If v is a smooth function in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, then we have*

$$|Dv|^4|D^2v|^2 \geq 2|Dv|^2|D^2vDv|^2 + \frac{\big(|Dv|^2\Delta v - \Delta_\infty v\big)^2}{n-1} - (\Delta_\infty v)^2. \tag{2.5}$$

everywhere in Ω. If $n = 2$, equality holds in the place of the inequality in (2.5).

Proof. Let $A = D^2v$ and $e = Dv$ in (2.1). □

2.1. Comparison between Corollary 2.2 and the inequality (1.1). We rewrite the two inequalities given by (1.1) as two lower bounds for the quantity $|Dv|^2|D^2v|^2$. Thus (1.1) is equivalent with the two inequalities

$$(n-3)|Dv|^2|D^2v|^2 \geq (n-4)|D^2vDv|^2 - |Dv|^2(\Delta v)^2 + 2\Delta v\Delta_\infty v \tag{2.6}$$

and

$$|Dv|^2|D^2v|^2 \geq \frac{n}{n-1}|D^2vDv|^2 + \frac{1}{n-1}|Dv|^2(\Delta v)^2 - \frac{2}{n-1}\Delta v\Delta_\infty v. \tag{2.7}$$

It is easy to show that the bound (2.6) is trivial. We now compare (2.7) with (2.5), and show that (2.5) is slightly sharper. Namely, we rewrite (2.5) as

$$\begin{aligned}|Dv|^2|D^2v|^2 \geq{}& \frac{n}{n-1}|D^2vDv|^2 + \frac{1}{n-1}|Dv|^2(\Delta v)^2 - \frac{2}{n-1}\Delta v\Delta_\infty v\\ &+ \frac{n-2}{n-1}\Big(|D^2vDv|^2 - \frac{(\Delta_\infty v)^2}{|Dv|^2}\Big).\end{aligned}$$

By Cauchy–Schwartz inequality

$$(\Delta_\infty v)^2 = \langle Dv, D^2vDv\rangle^2 \leq |Dv|^2|D^2vDv|^2.$$

Hence (2.5) implies (1.1).

3. Application of the inequality

The following Theorem is an improved version of Theorem 1.1 in [7].

Theorem 3.1. *Let $n \geq 2$, $1 < p < \infty$ and $s > -1 - \frac{p-1}{n-1}$. If u is p-harmonic in $\Omega \subset \mathbb{R}^n$, then $V_s(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Moreover, there exists a constant $C = C(n,p,s) > 0$ such that*

$$\int_{B_r} |D(V_s(Du))|^2\,dx \leq \frac{C}{r^2}\int_{B_{2r}} |V_s(Du)|^2\,dx \tag{3.1}$$

for any concentric balls $B_r \subset B_{2r} \subset\subset \Omega$.

To prove Theorem 3.1, we use essentially the same proof as in [7]. The only significant difference is that we apply the sharper inequality (2.5) in Corollary 2.2 instead of the inequality (1.1). For the reader's convenience, we provide a detailed proof of Theorem 3.1.

Let u be p-harmonic in $\Omega \subset \mathbb{R}^n$ and $U \subset\subset \Omega$ be a smooth subdomain of Ω. For $\epsilon > 0$ small, consider the regularized Dirichlet problem

$$\begin{cases} \operatorname{div}\Big((|Du^\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} Du^\epsilon\Big) = 0 & \text{in } U;\\ u^\epsilon = u & \text{on } \partial U.\end{cases} \tag{3.2}$$

By the standard elliptic regularity theory [10], there exists a unique solution $u^\epsilon \in C^\infty(U) \cap C^0(\overline{U})$. Furthermore, the family $\{u^\epsilon\}_\epsilon$ is uniformly bounded in $C^{1,\alpha}_{\mathrm{loc}}(U)$

for some $\alpha = \alpha(n,p) \in (0,1)$. That is, for any subdomain $V \subset\subset U$ there exists a constant $C = C(n, p, \operatorname{dist}(V, \partial U), \|u\|_{L^\infty(U)}) > 0$ such that

$$\|u^\epsilon\|_{C^{1,\alpha}(V)} \le C, \tag{3.3}$$

see for instance [25]. The Arzelà–Ascoli compactness theorem implies that

$$Du^\epsilon \xrightarrow{\epsilon\to 0} Du \quad \text{locally uniformly in } U, \tag{3.4}$$

up to a subsequence. Hereafter, we always consider appropriate subsequences of the family $\{u^\epsilon\}_\epsilon$.

For notational convenience, we introduce the regularized version of the vector field V_s. Let us define $V_s^\epsilon\colon \mathbb{R}^n \to \mathbb{R}^n$ as

$$V_s^\epsilon(z) := (|z|^2+\epsilon)^{\frac{p-2+s}{4}} z \quad \text{for } z \in \mathbb{R}^n.$$

We aim to show a bound similar to (3.1) for $V_s^\epsilon(Du^\epsilon)$. Namely, we show that there exists a constant $C = C(n,p,s) > 0$ such that

$$\int_U |D(V_s^\epsilon(Du^\epsilon))|^2 \phi^2\,dx \le C \int_U (|Du^\epsilon|^2+\epsilon)^{\frac{p+s}{2}} |D\phi|^2\,dx \tag{3.5}$$

for any $\phi \in C_0^\infty(U)$.

The estimate (3.1) can be derived from (3.5) as follows. Let us fix the concentric balls $B_r \subset B_{2r} \subset\subset \Omega$ and select a subdomain $U \subset\subset \Omega$ such that $B_{2r} \subset\subset U$. Let $\phi \in C_0^\infty(U)$ be a cutoff function such that

$$\phi = 1 \quad \text{in } B_r, \quad \operatorname{spt}\phi = \overline{B}_{2r} \quad \text{and} \quad |D\phi| \le \frac{10}{r}.$$

The estimate (3.5) implies that

$$\int_{B_r} |D(V_s^\epsilon(Du^\epsilon))|^2\,dx \le \frac{C}{r^2} \int_{B_{2r}} (|Du^\epsilon|^2+\epsilon)^{\frac{p+s}{2}}\,dx \tag{3.6}$$

for $C = C(n,p,s) > 0$. If $s > -p$, we can apply (3.3) to conclude that the right hand side of the above display (3.6) is bounded from above by a constant independent of ϵ. Thus $\{V_s^\epsilon(Du^\epsilon)\}_\epsilon$ is bounded in $W^{1,2}(B_r)$, and consequently we may extract a subsequence that converges weakly in $W^{1,2}(B_r)$ and strongly in $L^q(B_r)$ for any $1 \le q < \frac{2n}{n-2}$. By (3.4) and Dominated convergence theorem

$$\int_{B_{2r}} (|Du^\epsilon|^2+\epsilon)^{\frac{p+s}{2}}\,dx \xrightarrow{\epsilon\to 0} \int_{B_{2r}} |V_s(Du)|^2\,dx \tag{3.7}$$

and

$$V_s^\epsilon(Du^\epsilon) \xrightarrow{\epsilon\to 0} V_s(Du) \quad \text{in } L^2(B_r). \tag{3.8}$$

Finally, recalling that norm is lower semicontinuous with respect to the weak convergence, we can let $\epsilon \to 0$ in (3.6) to obtain (3.1).

3.1. Caccioppoli type estimates. Let us henceforth denote

$$\mu := (|Du^\epsilon|^2+\epsilon)^{1/2}$$

and

$$A := I + (p-2)\frac{Du^\epsilon \otimes Du^\epsilon}{|Du^\epsilon|^2+\epsilon},$$

where I stands for the identity matrix in $\mathbb{R}^{n\times n}$ and $\otimes$ stands for the tensor product (or outer product) of two vectors in $\mathbb{R}^n$, resulting a matrix in $\mathbb{R}^{n\times n}$. Note that

$$\min\{1,p-1\} I \le A \le \max\{1,p-1\} I \tag{3.9}$$

uniformly in U. Differentiating the PDE in (3.2) yields that the partial derivatives $u^\epsilon_{x_k}$, $k = 1, \ldots, n$ solve the linear, degenerate elliptic equation

$$\operatorname{div}\left(\mu^{p-2} A D u^\epsilon_{x_k}\right) = 0. \tag{3.10}$$

In this subsection we test the equation (3.10) with various test functions.

The following Lemma is the basic Caccioppoli type estimate related to the equation (3.10). It will not be needed to prove Theorem 3.1. Instead, it will be employed in Section 4.

Lemma 3.2. *Let u^ϵ solve (3.2). Then we have for any $\phi \in C_0^\infty(U)$ and $z \in \mathbb{R}^n$ that*

$$\int_U \mu^{p-2} |D^2 u^\epsilon|^2 \phi^2 \, dx \le C \int_U \mu^{p-2} |Du^\epsilon - z|^2 |D\phi|^2 \, dx, \tag{3.11}$$

where $C = C(p) > 0$ is independent of ϵ.

Proof. Let $\phi \in C_0^\infty(U)$ and $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ and put

$$\varphi = \phi^2 (u^\epsilon_{x_k} - z_k).$$

We have

$$D\varphi = 2\phi (u^\epsilon_{x_k} - z_k) D\phi + \phi^2 D u^\epsilon_{x_k},$$

and hence

$$\begin{aligned}
\int_U \mu^{p-2} \langle A D u^\epsilon_{x_k}, D u^\epsilon_{x_k} \rangle \phi^2 \, dx &= -2 \int_U \mu^{p-2} \langle A D u^\epsilon_{x_k}, D\phi \rangle (u^\epsilon_{x_k} - z_k) \phi \, dx \\
\le 2 \int_U \mu^{p-2} \sqrt{\langle A D u^\epsilon_{x_k}, D u^\epsilon_{x_k} \rangle} & \sqrt{\langle A D\phi, D\phi \rangle} |u^\epsilon_{x_k} - z_k| |\phi| \, dx.
\end{aligned}$$

Application of Young's inequality together with the uniform ellipticity of A, (3.9), yields

$$\int_U \mu^{p-2} |D u^\epsilon_{x_k}|^2 \phi^2 \, dx \le C \int_U \mu^{p-2} |D\phi|^2 |u^\epsilon_{x_k} - z_k|^2 \, dx,$$

where $C = C(p) > 0$. Finally sum over $k = 1, \ldots, n$ to conclude (3.11). □

The following Lemma is analogous to Lemma 3.1 in [7].

Lemma 3.3. *Let u^ϵ solve (3.2) and let $s \in \mathbb{R}$. Then we have for any $\eta > 0$ and for any $\phi \in C_0^\infty(U)$ that*

$$\begin{aligned}
&\int_U |D^2 u^\epsilon|^2 \mu^{p-2+s} \phi^2 \, dx + (p - 2 + s - \eta) \int_U |D^2 u^\epsilon D u^\epsilon|^2 \mu^{p-4+s} \phi^2 \, dx \\
&+ (s(p-2) - \eta) \int_U (\Delta_\infty u^\epsilon)^2 \mu^{p-6+s} \phi^2 \, dx \le \frac{C}{\eta} \int_U \mu^{p+s} |D\phi|^2 \, dx,
\end{aligned} \tag{3.12}$$

where $C = C(p) > 0$ is independent of ϵ.

Proof. Let $\phi \in C_0^\infty(U)$ and $s \in \mathbb{R}$, and put

$$\varphi = \phi^2 \mu^s u^\epsilon_{x_k}.$$

We have

$$D\varphi = 2\phi \mu^s u^\epsilon_{x_k} D\phi + s \mu^{s-2} \phi^2 u^\epsilon_{x_k} D^2 u^\epsilon D u^\epsilon + \phi^2 \mu^s D u^\epsilon_{x_k}.$$

To ease the notation, let

$$w := \mu^{p-2+s} \phi^2.$$

We obtain

$$\begin{aligned}(3.13)\qquad &\int_U \langle ADu^\epsilon_{x_k}, Du^\epsilon_{x_k}\rangle w\,dx + s\int_U \mu^{-2}\langle ADu^\epsilon_{x_k}, D^2u^\epsilon Du^\epsilon\rangle u^\epsilon_{x_k} w\,dx\\ &= -2\int_U \langle ADu^\epsilon_{x_k}, D\phi\rangle u^\epsilon_{x_k}\phi^{-1} w\,dx.\end{aligned}$$

Note that

$$\langle ADu^\epsilon_{x_k}, Du^\epsilon_{x_k}\rangle = |Du^\epsilon_{x_k}|^2 + (p-2)\frac{\langle Du^\epsilon, Du^\epsilon_{x_k}\rangle^2}{\mu^2}$$

and

$$\langle ADu^\epsilon_{x_k}, D^2u^\epsilon Du^\epsilon\rangle = \langle Du^\epsilon_{x_k}, D^2u^\epsilon Du^\epsilon\rangle + (p-2)\frac{\langle Du^\epsilon_{x_k}, Du^\epsilon\rangle \Delta_\infty u^\epsilon}{\mu^2}.$$

Summing over $k = 1, \dots, n$ yields

$$\begin{aligned}(3.14)\qquad &\int_U |D^2u^\epsilon|^2 w\,dx + (p-2+s)\int_U \mu^{-2}|D^2u^\epsilon Du^\epsilon|^2 w\,dx\\ &+ s(p-2)\int_U \mu^{-4}(\Delta_\infty u^\epsilon)^2 w\,dx = -2\int_U \langle AD^2u^\epsilon Du^\epsilon, D\phi\rangle\phi^{-1} w\,dx.\end{aligned}$$

The proof follows from the identity (3.14) via an application of Young's inequality. For any $\eta > 0$, we can estimate the integrand on the right hand side of (3.14) as follows:

$$\begin{aligned}-2\langle AD^2u^\epsilon Du^\epsilon, D\phi\rangle\phi^{-1}w &\le 2|D^2u^\epsilon Du^\epsilon||D\phi|\phi^{-1}w + 2|p-2|\frac{|\Delta_\infty u^\epsilon||Du^\epsilon||D\phi|}{\mu^2}\phi^{-1}w\\ &\le \eta|D^2u^\epsilon Du^\epsilon|^2\mu^{-2}w + \frac{C}{\eta}|D\phi|^2\mu^2\phi^{-2}w\\ &\quad + \eta(\Delta_\infty u^\epsilon)^2\mu^{-4}w + \frac{C(p-2)^2}{\eta}|Du^\epsilon|^2|D\phi|^2\phi^{-2}w,\end{aligned}$$

where $C > 0$ is an absolute constant. The proof is complete. □

The following Corollary gives, roughly speaking, an L^2-estimate for the Hessian D^2u^ϵ in terms of the second order derivative quantity $D^2u^\epsilon Du^\epsilon$ and the gradient Du^ϵ.

Corollary 3.4. *Let u^ϵ solve (3.2) and let $s \in \mathbb{R}$. Then we have for any $\phi \in C_0^\infty(U)$ that*

$$(3.15)\qquad \int_U |D^2u^\epsilon|^2\mu^{p-2+s}\phi^2\,dx \le C\Big(\int_U |D^2u^\epsilon Du^\epsilon|^2\mu^{p-4+s}\phi^2\,dx + \int_U \mu^{p+s}|D\phi|^2\,dx\Big)$$

where $C = C(p,s) > 0$ is independent of ϵ.

Proof. Move the second and third integral on the left hand side of (3.12) to the right hand side of the inequality. Estimate

$$(\Delta_\infty u^\epsilon)^2 \le |Du^\epsilon|^2|D^2u^\epsilon Du^\epsilon|^2 \le \mu^2|D^2u^\epsilon Du^\epsilon|^2$$

to conclude the proof. □

3.2. Lower bound for $|D^2u^\epsilon|^2$ and proof of Theorem 3.1. We begin with observing that by the smoothness of u^ϵ, $|Du^\epsilon|$ is locally Lipschitz continuous, and thus, by Rademacher theorem, differentiable almost everywhere. Moreover, if

$Du^\epsilon = 0$ at a point where $|Du^\epsilon|$ is differentiable, we must have $D|Du^\epsilon| = 0$ at that point. This allows us to define the normalized infinity Laplacian

$$\Delta_\infty^{\mathrm{N}} u^\epsilon := \langle \frac{Du^\epsilon}{|Du^\epsilon|}, D|Du^\epsilon|\rangle$$

almost everywhere in U. Note that if $Du^\epsilon \neq 0$, we have

$$\Delta_\infty^{\mathrm{N}} u^\epsilon = \frac{\Delta_\infty u^\epsilon}{|Du^\epsilon|^2}.$$

We can therefore rewrite

$$|D^2u^\epsilon Du^\epsilon|^2 = |Du^\epsilon|^2 |D|Du^\epsilon||^2 \quad \text{and} \quad (\Delta_\infty u^\epsilon)^2 = |Du^\epsilon|^4 (\Delta_\infty^{\mathrm{N}} u^\epsilon)^2 \tag{3.16}$$

almost everywhere in U.

Lemma 3.5. *Let $n \geq 2$ and u^ϵ solve* (3.2). *Then*

$$|D^2 u^\epsilon|^2 \geq 2|D|Du^\epsilon||^2 + \Phi(\Delta_\infty^{\mathrm{N}} u^\epsilon)^2 \tag{3.17}$$

almost everywhere in U, where

$$\Phi := \frac{(p-1)^2}{n-1} - 1 - \frac{\epsilon}{\mu^2} \cdot \frac{2(p-1)(p-2)}{n-1} + \frac{\epsilon^2}{\mu^4} \cdot \frac{(p-2)^2}{n-1}.$$

If $n = 2$, equality holds in the place of inequality in (3.17).

Proof. By the smoothness of u^ϵ, the non-divergence form of the PDE in (3.2),

$$\Delta u^\epsilon + (p-2)\frac{\Delta_\infty u^\epsilon}{|Du^\epsilon|^2 + \epsilon} = 0, \tag{3.18}$$

is equivalent with the original one. The proof follows now immediately from Corollary 2.2, by plugging the non-divergence form (3.18) into (2.5). □

Finally we gather together the above estimates to prove Theorem 3.1.

Proof of Theorem 3.1. Recall that to prove Theorem 3.1 it suffices to show that the estimate (3.5), that is,

$$\int_U |D(\mu^{\frac{p-2+s}{2}} Du^\epsilon)|^2 \phi^2 \, dx \leq C \int_U \mu^{p+s} |D\phi|^2 \, dx,$$

holds for any $\phi \in C_0^\infty(U)$ with a constant $C = C(n,p,s) > 0$ independent of ϵ. We start with

$$\begin{aligned}\int_U |D(\mu^{\frac{p-2+s}{2}} Du^\epsilon)|^2 \phi^2 \, dx = \int_U \mu^{p-2+s} \Big(|D^2 u^\epsilon|^2 + (p-2+s)\frac{|D^2u^\epsilon Du^\epsilon|^2}{\mu^2} \\ + \frac{(p-2+s)^2}{4} \frac{|Du^\epsilon|^2 |D^2 u^\epsilon Du^\epsilon|^2}{\mu^4} \Big) \phi^2 \, dx \\ \leq \Big(1 + |p-2+s| + \frac{(p-2+s)^2}{4} \Big) \int_U |D^2 u^\epsilon|^2 \mu^{p-2+s} \phi^2 \, dx.\end{aligned}$$

We apply Corollary 3.4 to obtain

$$\begin{aligned}\int_U |D(\mu^{\frac{p-2+s}{2}} Du^\epsilon)|^2 \phi^2 \, dx \leq C(p,s) \Big(\int_U |D^2 u^\epsilon Du^\epsilon|^2 \mu^{p-4+s} \phi^2 \, dx \\ + \int_U \mu^{p+s} |D\phi|^2 \, dx \Big).\end{aligned} \tag{3.19}$$

This estimate holds for any $s \in \mathbb{R}$.

In the remaining part of the proof, we estimate the first integral on the right hand side of (3.19) by combining Lemma 3.3 and Lemma 3.5. We estimate the first integral of the left hand side of (3.12) from below by (3.17). In addition we rewrite $|D^2u^\epsilon Du^\epsilon|^2$ and $\Delta_\infty u^\epsilon$ on the left hand side of (3.12) according to (3.16). We conclude that for any $\eta > 0$ and for any $\phi \in C_0^\infty(U)$

$$\begin{aligned}&\int_U \Big((p-2+s-\eta)\frac{|Du^\epsilon|^2}{\mu^2}+2\Big)|D|Du^\epsilon||^2 w\,dx\\ &+\int_U \Big(\Phi+(s(p-2)-\eta)\frac{|Du^\epsilon|^4}{\mu^4}\Big)(\Delta_\infty^{\mathrm{N}}u^\epsilon)^2 w\,dx \le \frac{C}{\eta}\int_U \mu^{p+s}|D\phi|^2\,dx,\end{aligned}$$

where $C = C(p) > 0$ and $w := \mu^{p-2+s}\phi^2$.

Writing

$$1 = \frac{|Du^\epsilon|^2}{\mu^2}+\frac{\epsilon}{\mu^2}$$

yields

$$\begin{aligned}&\int_U \Big((p+s-\eta)\frac{|Du^\epsilon|^2}{\mu^2}+\frac{2\epsilon}{\mu^2}\Big)|D|Du^\epsilon||^2 wdx+\int_U \Psi(\Delta_\infty^{\mathrm{N}}u^\epsilon)^2 wdx\\ &\le \frac{C}{\eta}\int_U \mu^{p+s}|D\phi|^2dx,\end{aligned}$$

where

$$\Psi := \Phi+(s(p-2)-\eta)\frac{|Du^\epsilon|^4}{\mu^4}.$$

Observe that, if $s > -1-\frac{p-1}{n-1}$ then also $s > -p$, and we may choose $\eta = \eta(p,s) > 0$ so small that we can estimate

$$\begin{aligned}&\eta\int_U \frac{|Du^\epsilon|^2}{\mu^2}|D|Du^\epsilon||^2 w\,dx+\int_U \Big((p+s-2\eta)\frac{|Du^\epsilon|^2}{\mu^2}+\frac{2\epsilon}{\mu^2}+\Psi\Big)(\Delta_\infty^{\mathrm{N}}u^\epsilon)^2 w\,dx\\ &\le \frac{C}{\eta}\int_U \mu^{p+s}|D\phi|^2\,dx.\end{aligned}$$

Now it remains to show that the condition $s > -1-\frac{p-1}{n-1}$ guarantees that we can adjust $\eta > 0$ even further so that

$$\int_U \Big((p+s-2\eta)\frac{|Du^\epsilon|^2}{\mu^2}+\frac{2\epsilon}{\mu^2}+\Psi\Big)(\Delta_\infty^{\mathrm{N}}u^\epsilon)^2 w\,dx \ge 0.$$

Note that

$$(p+s-2\eta)\frac{|Du^\epsilon|^2}{\mu^2}+\frac{2\epsilon}{\mu^2}+\Psi = a\frac{|Du^\epsilon|^4}{\mu^4}+b\frac{\epsilon|Du^\epsilon|^2}{\mu^4}+c\frac{\epsilon^2}{\mu^4}$$

where

$$\begin{aligned}a &= (p-1)\Big(s+1+\frac{p-1}{n-1}\Big)-3\eta,\\ b &= p+s+\frac{2(p-1)^2}{n-1}-\frac{2(p-1)(p-2)}{n-1}-2\eta = p+s+\frac{2(p-1)}{n-1}-2\eta\end{aligned}$$

and

$$c = 1+\frac{(p-1)^2}{n-1}-\frac{2(p-1)(p-2)}{n-1}+\frac{(p-2)^2}{n-1} = 1+\frac{1}{n-1}.$$

We can now easily see that the restrictive condition for s is indeed $s > -1-\frac{p-1}{n-1}$. $\square$

4. Proof of Theorem 1.1

In this section we explain how to conclude the estimate (1.9) from the estimate (3.1) and the known regularity results of p-harmonic functions. First, note that it suffices to find $C = C(n,p,s) > 0$ and $M = M(n,p,s) \geq 4$ such that

$$\int_{B_r} |D(V_s(Du))|^2\,dx \leq \frac{C}{r^2}\int_{B_{Mr}} |V_s(Du) - z|^2\,dx \tag{4.1}$$

for all vectors $z \in \mathbb{R}^n$ and all concentric balls $B_r \subset B_{Mr} \subset\subset \Omega$. Indeed, fix $B_r \subset B_{2r} \subset\subset \Omega$ concentric and let $\rho := M^{-1}r$. There exists an integer $N = N(n,M) > 0$ such that B_r may be covered with a family $\{B_\rho(x_i)\}_{i=1}^N$, where the center points $x_i \in B_r$. Then

$$\begin{aligned}\int_{B_r} |D(V_s(Du))|^2\,dx &\leq \sum_{i=1}^N \int_{B_\rho(x_i)} |D(V_s(Du))|^2\,dx \leq \sum_{i=1}^N \frac{C}{\rho^2}\int_{B_{M\rho}(x_i)} |V_s(Du) - z|^2\,dx \\ &\leq \frac{CNM^2}{r^2}\int_{B_{2r}} |V_s(Du) - z|^2\,dx.\end{aligned}$$

Also, note that it suffices to show (4.1) for $z = \big(V_s(Du)\big)_{B_{Mr}}$.

To show (4.1) for some $M \geq 4$ to be selected later, we divide the sufficiently small balls B_r inside Ω into two categories. By 'sufficiently', we mean that $B_{Mr} \subset\subset \Omega$. In our setting, we say a ball $B_r \subset\subset \Omega$ is degenerate if

$$\int_{B_{2r}} |V_s(Du)|^2 \leq \int_{B_{Mr}} |V_s(Du) - \big(V_s(Du)\big)_{B_{Mr}}|^2\,dx; \tag{4.2}$$

and non-degenerate if

$$\int_{B_{2r}} |V_s(Du)|^2 > \int_{B_{Mr}} |V_s(Du) - \big(V_s(Du)\big)_{B_{Mr}}|^2\,dx. \tag{4.3}$$

In this section such balls B_r, B_{2r} and B_{Mr} are always assumed to be concentric unless otherwise stated.

Let us fix a ball B_r such that $B_{Mr} \subset\subset \Omega$. The ball B_r must be either degenerate of non-degenerate. If B_r is degenerate, then (4.1) follows directly from (3.1). In this case we need to restrict $s > -1 - \frac{p-1}{n-1}$. If B_r is non-degenerate, we apply a method from the proof of Proposition 5.1 in [1]. The main consequence of the non-degeneracy condition (4.3) is that we can select M so large that Du is approximately a nonzero constant vector in B_{2r}. To prove this we use the known $C^{1,\alpha}_{\text{loc}}$-regularity of p-harmonic functions. We remark that in the non-degenerate case it suffices to restrict $s > -p$. If $n = 2$, the degenerate and non-degenerate conditions for s are the same.

The following Theorem summarizes the basic regularity of p-harmonic functions that we need to prove Theorem 1.1. For the proof we refer to [17] and [14, Theorem 2], [21, Lemma 3.1].

Theorem 4.1. *Let $n \geq 2$ and $1 < p < \infty$. There exists $\alpha = \alpha(n,p) \in (0,1)$ such that any p-harmonic function u in $\Omega \subset \mathbb{R}^n$ belongs to $C^{1,\alpha}_{\text{loc}}(\Omega)$. Moreover, for any fixed $t > 0$, there exists a constant $C = C(n,p,t) > 0$ such that*

$$\operatorname*{osc}_{B_r} Du \leq C\Big(\frac{r}{R}\Big)^\alpha \Big(\fint_{B_R} |Du|^t\,dx\Big)^{1/t} \tag{4.4}$$

holds for all concentric balls $B_r \subset B_{2r} \subset B_R \subset\subset \Omega$.

The following lemma is a straightforward generalization of Lemma 5.3 in [1].

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ and $v \in L^2_{\mathrm{loc}}(\Omega)$ be such that*

$$\int_{B_{mr}} |v|^2\,dx > \int_{B_{Mr}} |v-(v)_{B_{Mr}}|^2\,dx \tag{4.5}$$

for some concentric balls $B_{mr} \subset B_{Mr} \subset\subset \Omega$, where $0 < m < M < \infty$. Then for any $\kappa \in [m, M]$ we have

$$\fint_{B_{\kappa r}} |v|^2\,dx \le 9 \fint_{B_{mr}} |v|^2\,dx.$$

Proof. Apply Minkowski inequality and then Hölder inequality to obtain

$$\begin{aligned}\left(\fint_{B_{\kappa r}} |v|^2\right)^{1/2} &\le \left(\fint_{B_{\kappa r}} |v-(v)_{B_{Mr}}|^2\,dx\right)^{1/2} + \fint_{B_{mr}} |v-(v)_{B_{Mr}}|\,dx + \fint_{B_{mr}} |v|\,dx\\ &\le \left(\fint_{B_{\kappa r}} |v-(v)_{B_{Mr}}|^2\,dx\right)^{1/2} + \left(\fint_{B_{mr}} |v-(v)_{B_{Mr}}|^2\,dx\right)^{1/2} + \left(\fint_{B_{mr}} |v|^2\,dx\right)^{1/2}.\end{aligned}$$

Enlarging the integral domains in the first two items on the bottom row of the above display yields

$$\left(\fint_{B_{\kappa r}} |v|^2\right)^{1/2} \le 2\left(|B_{mr}|^{-1}\int_{B_{Mr}} |v-(v)_{B_{Mr}}|^2\,dx\right)^{1/2} + \left(\fint_{B_{mr}} |v|^2\,dx\right)^{1/2}.$$

Now the assumption (4.5) is applicable on the first item on the right hand side of the above inequality. The desired estimate follows and the proof is complete. □

For the proof of the following algebraic inequalities, see [12, Lemma 2.1].

Lemma 4.3. *Let $1 < p < \infty$ and $s > -p$. There exist constants $c_1 = c_1(p, s) > 0$ and $c_2 = c_2(p, s) > 0$ such that*

$$c_1\big(\epsilon + |z|^2 + |w|^2\big)^{\frac{p-2+s}{2}}|z-w|^2 \le \big|V_s^\epsilon(z) - V_s^\epsilon(w)\big|^2 \le c_2\big(\epsilon + |z|^2 + |w|^2\big)^{\frac{p-2+s}{2}}|z-w|^2$$

for any two vectors $z, w \in \mathbb{R}^n$.

Let us introduce the notation

$$\lambda := \left(\fint_{B_{2r}} |Du|^{p+s}\,dx\right)^{\frac{1}{p+s}} = \left(\fint_{B_{2r}} |V_s(Du)|^2\,dx\right)^{\frac{1}{p+s}}.$$

Note that if $\lambda = 0$, then the desired estimate (1.9) is trivial. Hence we may assume that $\lambda > 0$.

The following lemma is an adapted version of Lemma 5.5 in [1].

Lemma 4.4. *Let $n \ge 2$, $1 < p < \infty$ and $s > -p$. Suppose that u is p-harmonic in $\Omega \subset \mathbb{R}^n$. Given any $\sigma > 0$, there exists a constant $M = M(n, p, s, \sigma) \ge 4$ such that for any ball $B_r \subset\subset \Omega$ the non-degeneracy condition (4.3) implies that*

$$|Du - Du(x_0)| \le \sigma\lambda \quad \text{in } B_{2r}, \tag{4.6}$$

where $x_0 \in B_{2r}$ is a point such that $|Du(x_0)| = \lambda$.

Proof. By mean value theorem, we can fix a point $x_0 \in B_{2r}$ such that $|Du(x_0)| = \lambda$. Let $x \in B_{2r}$. We apply Theorem 4.1 with $t = p + s > 0$ to estimate

$$|Du(x) - Du(x_0)| \le \operatorname*{osc}_{B_{2r}} Du \le C\Big(\frac{2}{M}\Big)^\alpha \left(\fint_{B_{Mr}} |Du|^{p+s}\,dx\right)^{\frac{1}{p+s}},$$

where $C = C(n,p,s) > 0$. The non-degeneracy condition (4.3) allows us to employ Lemma 4.2 with $v = V_s(Du)$ and $m = 2$ to obtain

$$\left(\fint_{B_{Mr}} |Du|^{p+s}\,dx\right)^{\frac{1}{p+s}} \le 9^{\frac{1}{p+s}}\lambda.$$

We can now adjust $M = M(n,p,s,\sigma) \ge 4$ such that $C\left(\frac{2}{M}\right)^{\alpha} 9^{\frac{1}{p+s}} \le \sigma$. This completes the proof. $\square$

We are finally ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\sigma = \sigma(p,s) > 0$ be a very small constant to be selected later, and accordingly let $M = M(n,p,s,\sigma) \ge 4$ be given by Lemma 4.4. Fix a ball $B_r \subset\subset \Omega$ such that $B_{Mr} \subset\subset \Omega$. Recall that, in view of Theorem 3.1, it suffices to study the case when B_r is non-degenerate (4.3). To run the computations, we consider the regularization (3.2) in a subdomain $U \subset\subset \Omega$ such that $B_{Mr} \subset\subset U$. By (3.4) and Lemma 4.4, we may henceforth consider $0 < \epsilon < \sigma\lambda^2$ so small that

$$|Du^\epsilon - Du(x_0)| \le 2\sigma\lambda \quad \text{and} \quad \frac{3}{4}\lambda \le \mu \le \frac{5}{4}\lambda \quad \text{in } B_{2r}, \tag{4.7}$$

where $x_0 \in B_{2r}$ is a point such that $|Du(x_0)| = \lambda$, and $\mu = (|Du^\epsilon|^2 + \epsilon)^{1/2}$.

In what follows, the constants $C = C(n,p,s) > 0$ and $c = c(p,s) > 0$ may vary from line to line. By (4.7)

$$|D(V_s^\epsilon(Du^\epsilon))|^2 \le C\mu^{p-2+s}|D^2u^\epsilon|^2 \le C\lambda^s\mu^{p-2}|D^2u^\epsilon|^2 \quad \text{in } B_{2r}. \tag{4.8}$$

We employ Lemma 3.2 with a cutoff function $\phi \in C_0^\infty(U)$ such that

$$\phi = 1 \quad \text{in } B_r, \quad \operatorname{spt}\phi = \overline{B}_{2r} \quad \text{and} \quad |D\phi| \le \frac{10}{r},$$

and use the estimates (4.8) and (4.7) to arrive at

$$\begin{aligned}\int_{B_r} |D(V_s^\epsilon(Du^\epsilon))|^2\,dx &\le \frac{C\lambda^s}{r^2}\int_{B_{2r}} \mu^{p-2}|Du^\epsilon - z|^2\,dx \\ &\le \frac{C}{r^2}\int_{B_{2r}} \mu^{p-2+s}|Du^\epsilon - z|^2\,dx\end{aligned} \tag{4.9}$$

for any $z \in \mathbb{R}^n$. In particular, since $V_s^\epsilon\colon \mathbb{R}^n \to \mathbb{R}^n$ is bijective, we may select $z = z^\epsilon \in \mathbb{R}^n$ such that

$$V_s^\epsilon(z^\epsilon) = \left(V_s^\epsilon(Du^\epsilon)\right)_{B_{2r}}. \tag{4.10}$$

Observe that, by Lemma 4.3 and (4.7),

$$\begin{aligned}|V_s^\epsilon(z^\epsilon) - V_s^\epsilon(Du(x_0))| &\le \fint_{B_{2r}} |V_s^\epsilon(Du^\epsilon) - V^\epsilon(Du(x_0))|\,dx \\ &\le c_2 \fint_{B_{2r}} (\mu^2 + \lambda^2)^{\frac{p-2+s}{4}}|Du^\epsilon - Du(x_0)|\,dx \\ &\le c\sigma\lambda^{\frac{p+s}{2}}.\end{aligned} \tag{4.11}$$

We employ the above estimate (4.11) to estimate $|z^\epsilon|$ from above and below. If $p - 2 + s \ge 0$, we have

$$(1 - c\sigma)\lambda^{\frac{p+s}{2}} \le (|z^\epsilon|^2 + \epsilon)^{\frac{p-2+s}{4}}|z^\epsilon| \le ((1+\sigma)^{\frac{p-2+s}{4}} + c\sigma)\lambda^{\frac{p+s}{2}}.$$

If $p-2+s<0$, we have similarly

$$((1+\sigma)^{\frac{p-2+s}{4}} - c\sigma)\lambda^{\frac{p+s}{2}} \le (|z^\epsilon|^2+\epsilon)^{\frac{p-2+s}{4}}|z^\epsilon| \le (1+c\sigma)\lambda^{\frac{p+s}{2}}.$$

Consequently, we may select $\sigma=\sigma(p,s)>0$ such that

$$\frac{1}{2}\lambda^{\frac{p+s}{2}} \le (|z^\epsilon|^2+\epsilon)^{\frac{p-2+s}{4}}|z^\epsilon| \le 2\lambda^{\frac{p+s}{2}}.$$

We can now restrict ϵ so small, depending on λ, p and s, that

$$c^{-1}\lambda < |z^\epsilon| \le c\lambda \tag{4.12}$$

for some $c=c(p,s)>0$.

We apply (4.7) and (4.12), together with Lemma 4.3, to estimate the integrand on the bottom row of (4.9) with $z=z^\epsilon$ as follows;

$$\begin{aligned}\mu^{p-2+s}|Du^\epsilon - z^\epsilon|^2 &\le c(\mu^2+|z^\epsilon|^2)^{\frac{p-2+s}{2}}|Du^\epsilon - z^\epsilon|^2 \\ &\le c|V_s^\epsilon(Du^\epsilon) - V^\epsilon(z^\epsilon)|^2 \quad \text{in } B_{2r}.\end{aligned} \tag{4.13}$$

Combination of (4.9) and (4.13) yields that

$$\int_{B_r} |D(V_s^\epsilon(Du^\epsilon))|^2 dx \le \frac{C}{r^2}\int_{B_{2r}} |V_s^\epsilon(Du^\epsilon) - \big(V_s^\epsilon(Du^\epsilon)\big)_{B_{2r}}|^2\,dx, \tag{4.14}$$

where $C=C(n,p,s)>0$ is independent of ϵ. Therefore, as explained in Section 3, we can let $\epsilon\to 0$ in (4.14) to obtain

$$\int_{B_r} |D(V_s(Du))|^2\,dx \le \frac{C}{r^2}\int_{B_{2r}} |V_s(Du) - \big(V_s(Du)\big)_{B_{2r}}|^2\,dx. \tag{4.15}$$

Note that this implies (4.1). The proof is complete. □

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