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AM-modulus and Hausdorff measure of codimension one in metric measure spaces

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Abstract

Let $\Gamma(E)$ be the family of all paths which meet a set E in the metric measure space X. The set function $E \mapsto AM(\Gamma(E))$ defines the AM-modulus measure in X where AM refers to the approximation modulus [22]. We compare $AM(\Gamma(E))$ to the Hausdorff measure $co\mathcal{H}^1(E)$ of codimension one in X and show that

$$co\mathcal{H}^1(E) \approx AM(\Gamma(E))$$

for Suslin sets E in X. This leads to a new characterization of sets of finite perimeter in X in terms of the AM-modulus. We also study the level sets of BV functions and show that for a.e. t these sets have finite $co\mathcal{H}^1$-measure. Most of the results are new also in $\mathbb{R}^n$.

KEYWORDS

AM-modulus, level sets of *BV*-functions, metric measure spaces, perimeter, sets of co-dimension one

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MC1, MC2, MC3

1 | INTRODUCTION

In a metric measure space X the modulus of a curve family offers a substitute for the Fubini theorem and provides an important tool for analysis in X, see e.g. [26] and [5]. The M_p-modulus, $p \geq 1$, is used to create a space in X similar to the Sobolev space $W^{1,p}(\mathbb{R}^n)$ and the AM-modulus was introduced as a weaker version than the M_1-modulus to study functions of bounded variation in X and in $\mathbb{R}^n$, see [22], [15] and [16].

Let $\Gamma(E)$ be the family of all paths in X which meet the set $E \subset X$. The set function $\phi(E) = AM(\Gamma(E))$ defines a metric outer measure, the AM-modulus measure, in X and satisfies

$$\phi(E) \leq C co\mathcal{H}^1(E) \tag{1.1}$$

provided that the measure μ is doubling, see Theorem 2.1 below. Here $co\mathcal{H}^1$ refers to the Hausdorff measure of codimension one in X. We also present the generalization of (1.1) for all measures $co\mathcal{H}^p$, $p \geq 1$.

In this paper we are interested in the inequalities opposite to (1.1). Such an inequality was obtained in [16] for sets E contained in $(n-1)$-rectifiable sets in $\mathbb{R}^n$. Here we show that this inequality holds for Borel sets in X, and more generally for Suslin sets and for arbitrary sets with σ-finite $co\mathcal{H}^1$-measure, provided that X satisfies standard regularity assumptions, i.e. the measure μ in X is doubling, X is complete and supports the Poincaré inequality. Thus in $\mathbb{R}^n$ the standard $(n-1)$-Hausdorff measure $\mathcal{H}^{n-1}$ satisfies

$$\mathcal{H}^{n-1}(E) \approx \phi(E) \tag{1.2}$$

for all Suslin sets and arbitrary sets of σ-finite $\mathcal{H}^{n-1}$-measure. Note that the ordinary M_p-modulus is more adapted to measure the family $\Gamma(E,\Omega)$ of all curves which join E to the complement of a fixed open set Ω and then the result corresponds to the p-capacity of E. Thus the relation to the $(n-p)$-dimensional Hausdorff measure is mediated through the capacity and does not provide as close a connection as (1.2), see also Remark 2.2.

We apply the above results to study the AM-modulus of path families which are closely associated with sets of finite perimeter in X. Although there is extensive literature on sets of finite perimeter in metric measure spaces, see [1], [2], [19], [20], [21] and [24], the AM-modulus has not yet been used to characterize sets of finite perimeter in X and our results extend the characterizations obtained in [16] in $\mathbb{R}^n$ to X.

We study the level sets of a BV function u in the final section and show that these sets have finite $co\mathcal{H}^1$-measure for a.e. t. In particular, it follows that the ordinary level set $u^{-1}(t)$ of a continuous BV function u has finite $co\mathcal{H}^1$-measure for a.e. t.

2 | PRELIMINARIES

Let (X,d) be a metric space and μ a Borel regular measure in X. The measure μ is *doubling* if there is a constant C_μ such that $\mu(B(x,2r)) \le C_\mu \mu(B(x,r))$ and $0<\mu(B(x,r))<\infty$ for all open balls $B(x,r)$ in X.

A continuous mapping $\gamma : [a,b] \to X$ is called a *curve*. We say that a curve γ is a *path* if it has a finite and non-zero total length; in this case we parametrize γ by its arclength. The *locus* of γ is defined as $\gamma([0,\ell])$ and denoted by $\langle\gamma\rangle$.

We refer to [22] and [15] for the properties of the AM_p-modulus and to [5] and [11] for those of the M_p-modulus. For completeness we recall the definitions.

Let Γ be a family of paths in X. A nonnegative Borel function ρ is *M-admissible*, or simply admissible, for Γ if

$$\int_\gamma \rho\, ds \ge 1$$

for every $\gamma \in \Gamma$. For $p \ge 1$ the M_p*-modulus* of Γ is defined as

$$M_p(\Gamma) = \inf \int_X \rho^p \, d\mu$$

where the infimum is taken over all admissible functions ρ.

A sequence of nonnegative Borel functions ρ_i, $i = 1, 2, \ldots$, is *AM-admissible*, or simply admissible, for Γ if

$$\liminf_{i\to\infty} \int_\gamma \rho_i \, ds \ge 1 \tag{2.1}$$

for every $\gamma \in \Gamma$. The *approximation modulus*, AM_p-modulus for short, of Γ is defined as

$$AM_p(\Gamma) = \inf_{(\rho_i)} \left\{ \liminf_{i\to\infty} \int_X \rho_i^p \, d\mu \right\} \tag{2.2}$$

where the infimum is taken over all AM-admissible sequences (ρ_i) for Γ. We mostly consider the AM_1–modulus and use the abbreviation $AM = AM_1$. Note that for $p>1$, $AM_p(\Gamma) = M_p(\Gamma)$ for every path family Γ in X, see [15, Theorem 1], however, sometimes it is easier to use the AM_p-modulus than the M_p-modulus. Note also that $AM(\Gamma) \le M_1(\Gamma)$ for all path families Γ in X and it could happen that $AM(\Gamma) = 0$ but $M_1(\Gamma) = \infty$ for some family Γ.

We define the AM_c-modulus of Γ with respect to the AM-modulus with the difference that the admissible sequence are now required to consist of continuous functions.

The AM modulus or the AM_c modulus can be also assigned to a family $\mathcal{E}$ of measures, $\int_\gamma \rho_i\, ds$, $\gamma \in \Gamma$, is then replaced by $\int_X \rho_i\, d\nu$, $\nu \in \mathcal{E}$. For a more precise definition we refer to [14].

For $E \subset X$, $\Gamma(E)$ denotes the family of all paths which meet E. From [16, Theorem 1] it follows that the set function $\phi : E \mapsto AM(\Gamma(E))$ is a metric outer measure in X and hence all Borel sets are ϕ measurable. Almost the same proof shows that for $p \geq 1$ the set functions $E \mapsto AM_p(\Gamma(E))$ and $E \mapsto M_p(\Gamma(E))$ also define metric outer measures in X.

We denote by $\mathcal{H}^{n-p}$ the ordinary Hausdorff measure of codimension p in $\mathbb{R}^n$. In metric spaces, the dimension n is not always clearly determined. The right replacement of $\mathcal{H}^{n-p}$ is then the *Hausdorff measure* $co\mathcal{H}^p(E)$ *of codimension* p defined as

$$co\mathcal{H}^p(E) = \sup_{\delta>0} co\mathcal{H}^p_\delta(E)$$

where for $\delta > 0$

$$co\mathcal{H}^p_\delta(E) = \inf\left\{\sum_{j=1}^{\infty} \frac{\mu(B(x_j, r_j))}{r_j^p} : E \subset \bigcup_{j=1}^{\infty} B(x_j, r_j),\ \sup_j r_j < \delta\right\}$$

denotes the δ-content associated with $co\mathcal{H}^p(E)$. It is easily checked that in $\mathbb{R}^n$, $co\mathcal{H}^p$ agrees with the $\mathcal{H}^{n-p}$-measure up to a multiplicative constant.

In the following, we are chiefly interested in $co\mathcal{H}^1(E)$ and its dependence on $AM(\Gamma(E))$ and we first consider upper bounds for $AM(\Gamma(E))$ in terms of $co\mathcal{H}^1(E)$. Such a result was presented in [22, Theorem 3.17] and for completeness we include a proof. For $p > 1$ we present a stronger version in X and extend the implication, see [12, Theorem 2.27] and references therein, that in $\mathbb{R}^n$, $\mathcal{H}^{n-p}(E) < \infty$ implies that the p-capacity of $E \subset \mathbb{R}^n$ is zero.

Theorem 2.1. *Suppose that μ is a doubling measure in X and $E \subset X$. Then*

$$AM(\Gamma(E)) \leq C_\mu co\mathcal{H}^1(E) \tag{2.3}$$

and for $p > 1$, $co\mathcal{H}^p(E) < \infty$ implies $M_p(\Gamma(E)) = 0$.

Proof. First, we prove

$$AM_p(\Gamma(E)) \leq C_\mu co\mathcal{H}^p(E) \tag{2.4}$$

for any $1 \leq p < \infty$. We may assume that $co\mathcal{H}^p(E) < \infty$. For $j = 1, 2,$ choose a covering $B(x_i^j, r_i^j)$, $i = 1, 2, \dots$, of E such that $r_i^j < 1/j$ and

$$\sum_i \frac{\mu\left(B\left(x_i^j, r_i^j\right)\right)}{\left(r_i^j\right)^p} \leq co\mathcal{H}^p_{1/j}(E) + \frac{1}{j}.$$

Set

$$\rho_j(x) = \left\{\sum_i \frac{1}{\left(r_i^j\right)^p} \chi_{B_i^j}(x)\right\}^{1/p}$$

where $B_i^j = B\left(x_i^j, 2r_i^j\right)$. Then ρ_j is a Borel function and we show that the sequence $\left(\rho_j\right)$ is admissible for $\Gamma(E)$. Indeed, if $\gamma \in \Gamma(E)$, then γ meets E and since γ is not a constant path, diam $\langle\gamma\rangle > 4/j$ for large j and hence there is j_0 such that for

$j \geq j_0$ we find $i = i(j)$ such that γ meets $B\left(x_i^j, r_i^j\right)$ and $X \setminus B_i^j$. Thus γ travels in B_i^j at least distance r_i^j. Consequently for $j \geq j_0$

$$\int_\gamma \rho_j\, ds \geq \int_\gamma \frac{\chi_{B^j_{i(j)}}}{r^j_{i(j)}}\, ds \geq 1$$

and hence

$$\liminf_{j\to\infty} \int_\gamma \rho_j\, ds \geq 1.$$

We obtain

$$\begin{aligned} AM_p(\Gamma(E)) &\leq \liminf_{j\to\infty} \int_X \rho_j^p\, d\mu = \liminf_{j\to\infty} \sum_i \frac{\mu\left(B_i^j\right)}{\left(r_i^j\right)^p} \\ &\leq C_\mu \liminf_{j\to\infty} \sum_i \frac{\mu\left(B\left(x_i^j, r_i^j\right)\right)}{\left(r_i^j\right)^p} \leq C_\mu \liminf_{j\to\infty} \left(co\mathcal{H}^p_{1/j}(E) + \frac{1}{j}\right) \\ &= C_\mu co\mathcal{H}^p(E), \end{aligned}$$

which proves (2.4)

Now, for $p = 1$ we are done. If $p > 1$, we know by [15, Theorem 1] that $M_p = AM_p$, therefore we have

$$M_p(\Gamma(E)) \leq C_\mu co\mathcal{H}^p(E). \tag{2.5}$$

To prove that $M_p(\Gamma(E)) = 0$, we first use (2.5) to construct a sequence $\left(\rho_j\right)$ of M-admissible functions for $\Gamma(E)$ such that

$$\int_X \rho_j^p\, d\mu \leq C \text{ with } C = 1 + C_\mu co\mathcal{H}^p(E) \tag{2.6}$$

and $\mu\left(\left\{\rho_j > 0\right\}\right) \to 0$. Note that $\mu\left(\left\{\rho_j > 0\right\}\right)$ can be made arbitrary small. To see this let $\varepsilon > 0$ and since $\mu(E) = 0$ we can choose an open set $G \supset E$ with $\mu(G) < \varepsilon$. If ρ is admissible for $\Gamma(E)$, we set

$$\tilde\rho = \begin{cases} \rho & \text{in } G, \\ 0 & \text{in } X \setminus G. \end{cases}$$

Each path $\gamma \in \Gamma(E)$ has a subpath $\tilde\gamma \in \Gamma(E)$ with locus in G. Then

$$\int_\gamma \tilde\rho\, ds \geq \int_{\tilde\gamma} \rho\, ds \geq 1,$$

and thus $\tilde\rho$ is admissible for $\Gamma(E)$ as well. Moreover, $\mu(\{\tilde\rho > 0\}) < \varepsilon$ and

$$\int_X \tilde\rho^p\, d\mu \leq \int_X \rho^p\, d\mu.$$

Now, we select a special subsequence. We proceed by induction. Set $m_1 = 1$. If $m_1, \dots, m_{j-1}$ are determined, we find m_j such that

$$\int_{E_j} \left(\rho_{m_1} + \cdots + \rho_{m_{j-1}}\right)^p d\mu < 2^{-j} \tag{2.7}$$

holds with $E_j = \{\rho_{m_j} > 0\}$. We claim that

$$\int_X \left(\rho_{m_1} + \cdots + \rho_{m_j}\right)^p d\mu \le 2^{p-1}(Cj+1). \tag{2.8}$$

Indeed, this follows from (2.6) as we prove

$$\int_X \left(\rho_{m_1} + \cdots + \rho_{m_j}\right)^p d\mu \le 2^{p-1}\left(\int_X \left(\rho_{m_1}^p + \cdots + \rho_{m_j}^p\right) d\mu + \sum_{i=1}^{j} 2^{-i}\right) \tag{2.9}$$

by induction. The inequality is trivial for $j = 1$. If it holds for $j-1$, using (2.7) we obtain

$$\begin{aligned}
\int_X \left(\rho_{m_1} + \cdots + \rho_{m_j}\right)^p d\mu &\le \int_{X\setminus E_j} \left(\rho_{m_1} + \cdots + \rho_{m_{j-1}}\right)^p d\mu + \int_{E_j} \left(\rho_{m_1} + \cdots + \rho_{m_j}\right)^p d\mu \\
&\le 2^{p-1}\left(\int_X \left(\rho_{m_1}^p + \cdots + \rho_{m_{j-1}}^p\right) d\mu + \sum_{i=1}^{j-1} 2^{-i}\right) \\
&\quad + 2^{p-1}\left(\int_X \rho_{m_j}^p \, d\mu + \int_{E_j} \left(\rho_{m_1} + \cdots + \rho_{m_{j-1}}\right)^p d\mu\right) \\
&\le 2^{p-1}\left(\int_X \left(\rho_{m_1}^p + \cdots + \rho_{m_j}^p\right) d\mu + \sum_{i=1}^{j} 2^{-i}\right)
\end{aligned}$$

which proves (2.9) for j.

Finally, we test the M_p-modulus of $\Gamma(E)$ by the admissible functions

$$g_k = \frac{1}{k}\sum_{j=1}^{k} \rho_{m_j}.$$

Then it is evident that each g_k is admissible for $\Gamma(E)$ and by (2.8)

$$M_p(\Gamma(E)) \le \int_X g_k^p \, d\mu \le 2^{p-1}k^{-p}(Ck+1).$$

□

Remark 2.2. Consider the inverse implication in Theorem 2.1 for $p > 1$ in $\mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be a Borel set with $M_p(\Gamma(E)) < \infty$, $1 < p \le n$. If $K \subset E$ is compact, then

$$M_p(\Gamma(K)) \le M_p(\Gamma(E)) < \infty$$

and it easily follows that for all open sets $\Omega \supset K$

$$\mathrm{cap}_p(K, \Omega) \le M_p(\Gamma(K))$$

where $\mathrm{cap}_p(K, \Omega)$ stands for the ordinary variational p-capacity of the condenser (K, Ω), see Section 3 and [12, Chapter 2]. From [12, Lemma 2.34] it follows that K has p-capacity zero and hence by the Choquet capacitability theorem E has also capacity zero. This implies, see e.g. [12, Theorem 2.27], that the Hausdorff dimension of E is at most $n - p$ but not that $\mathcal{H}^{n-p}(E) < \infty$.

We also need some properties of functions of bounded variation (BV) in X, see [24] (in metric measure spaces) and [2] (in the Euclidean spaces). Let $\Omega \subset X$ be open and denote by $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ the set of locally Lipschitz functions in Ω. Given

$u \in L^1_{loc}(\Omega)$ and an open set $G \subset \Omega$ we define

$$V(u,G) = \inf \left\{ \liminf_i \int_G |\nabla u_i| \, d\mu \, : \, u_i \to u \text{ in } L^1_{loc}(G) \right\}$$

Here $|\nabla u(x)|$ stands for the local Lipschitz constant for u at x, i.e.

$$|\nabla u(x)| = \liminf_{r\to 0} \sup_{y\in B(x,r)} \frac{|u(y)-u(x)|}{r},$$

see [5, Section 1.3]. A function has *bounded variation* in Ω, $u \in BV(\Omega)$, if $V(u,\Omega) < \infty$.

Let $\Omega \subset X$ be open and let $E \subset X$ be measurable. The *perimeter* of E in Ω is $P(E,\Omega) = V(\chi_E,\Omega)$ and we write $P(E) = P(E,X)$.

The space X supports the (weak) BV-Poincaré inequality, see [24, Remark 3.5], if

$$\int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le C_P \, r \, V\big(u, B\big(x, \lambda_P r\big)\big) \tag{2.10}$$

in each ball $B(x,r)$ and for each $u \in BV(X)$. Here $u_{B(x,r)}$ stands for the mean value of u in $B(x,r)$. The constants $C_P \ge 1$ and $\lambda_P \ge 1$ are independent of $B(x,r)$ and u and called the Poincaré constants of X. Note that (2.10) is a consequence of the standard weak Poincaré inequality for integrable functions with upper gradients, see [5, Chapter 4] and [24].

We use the standard assumptions (A) on the space X:

- X is complete,
- the measure μ is doubling,
- X supports the BV-Poincaré inequality (2.10).

Note that if μ is doubling and X is complete, then X is proper, i.e. closed and bounded subsets of X are compact, see [5, Section 3.1]. Moreover, X is connected [5, Proposition 4.2].

3 | NEWTONIAN AND PERIMETER CAPACITIES IN X

Throughout this and the next section we assume that (X,d) and μ satisfy the assumptions (A).

Let G be a bounded open set in X, let K be a compact subset of G and let $\mathrm{Lip}_0(K,G)$ be the set of all Lipschitz functions u with compact support in G satisfying $u \ge 1$ on K. We define

$$\mathrm{cap}_1(K,G) = \inf \left\{ \int_G |\nabla u| \, d\mu \, : \, u \in \mathrm{Lip}_0(K,G) \right\}. \tag{3.1}$$

Obviously the infimum does not change if restricted to test functions satisfying $0 \le u \le 1$.

It is easy to see that $\mathrm{Lip}_0(K,G) \ne \emptyset$ if $G \ne \emptyset$ and thus $\mathrm{cap}_1(K,G) < \infty$. Note that if G is compact, then the constant function 1 is a competitor and thus $\mathrm{cap}_1(K,G) = 0$.

If $U \subset G$ is open, then we set

$$\mathrm{cap}_1(U,G) = \sup \left\{ \mathrm{cap}_1(K,G) \, : \, K \subset U \text{ compact} \right\}$$

and for an arbitrary set $E \subset G$

$$\mathrm{cap}_1(E,G) = \inf \left\{ \mathrm{cap}_1(U,G) \, : \, U \text{ open }, E \subset U \subset G \right\}.$$

Now there are two definitions for $\mathrm{cap}_1(E,G)$ when E is compact but since the competitors are continuous the next lemma is immediate.

Lemma 3.1. *If $K \subset G$ is compact, then*

$$\operatorname{cap}_1(K,G) = \inf\left\{\operatorname{cap}_1(U,G) \,:\, U \text{ open}, K \subset U \subset G\right\}, \tag{3.2}$$

where the capacity on the left is according to (3.1).

Next we summarize the main properties of the capacity. In particular, we show that $\operatorname{cap}_1(\cdot, G)$ defines a Choquet capacity and thus, by the Choquet capacitability theorem, each Suslin (in particular, a Borel) set $E \subset G$ is capacitable.

We also compare the widely used Newtonian type p-capacity

$$\widetilde{\operatorname{cap}}_p(E,G) = \inf_u \int_G (g_u)^p \, d\mu \tag{3.3}$$

for $p = 1$ to $\operatorname{cap}_1(E,G)$. In (3.3) the infimum is taken over all (precisely defined) $u \in N_0^{1,p}(G)$ such that $u \geq 1$ on E and g_u is the minimal upper gradient of u, see [5, Section 6.3]. This is a Choquet capacity if $p > 1$ but not in the case $p = 1$ because $\widetilde{\operatorname{cap}}_1$ does not satisfy (e) below. For an example see [5, Example 6.18] where it also becomes evident how $\operatorname{cap}_1(E,G)$ differs from $\widetilde{\operatorname{cap}}_1(E,G)$.

Proposition 3.2.

(a) The set function $E \mapsto \operatorname{cap}_1(E,G)$ is monotone, i.e.

$$E_1 \subset E_2 \subset G \Rightarrow \operatorname{cap}_1\left(E_1, G\right) \leq \operatorname{cap}_1\left(E_2, G\right).$$

(b) If $K_1, K_2, \ldots \subset G$ are compact and $K_1 \supset K_2 \supset \ldots$, then

$$\operatorname{cap}_1\left(\bigcap_{j=1}^{\infty} K_j, G\right) = \lim_{j\to\infty} \operatorname{cap}_1\left(K_j, G\right).$$

(c) $\operatorname{cap}_1(E,G) \leq \widetilde{\operatorname{cap}}_1(E,G)$ and $\operatorname{cap}_1(K,G) = \widetilde{\operatorname{cap}}_1(K,G)$ if K is compact.

(d) If K_1, K_2 are compact, then

$$\operatorname{cap}_1\left(K_1 \cup K_1, G\right) + \operatorname{cap}_1\left(K_1 \cap K_2, G\right) \leq \operatorname{cap}_1\left(K_1, G\right) + \operatorname{cap}_1\left(K_2, G\right).$$

(e) $E_1 \subset E_2 \subset \cdots \subset G \Rightarrow \operatorname{cap}_1\left(\bigcup_{j=1}^{\infty} E_j, G\right) = \lim_{j\to\infty} \operatorname{cap}_1\left(E_j, G\right)$.

(f) If $E \subset G$ is Suslin, then

$$\operatorname{cap}_1(E,G) = \sup\left\{\operatorname{cap}_1(K,G) \,:\, K \subset E \text{ compact}\right\}.$$

Proof. The properties (a) and (b) are obvious. The inequality in (c) is obvious if E is open; for the case of E arbitrary we use [5, Theorem 6.19 (vii)] (note that the symbol cap_1 stands for $\widetilde{\operatorname{cap}}_1$ in [5]). The equality for K compact follows from [5, Theorem 6.19 (x)]. The property (d) follows from [5, Theorem 6.17 (iii)] taking into account the equality in (c). Now, the properties (e) and (f) are obtained using the general theory of capacities developed by Choquet in [7], see also [6], [17]. □

If G is a bounded open set in X and $K \subset G$ compact, then we denote by $\Gamma(K,G)$ the family of all paths in X which connect $X \setminus G$ to K.

Lemma 3.3. *If G is a bounded open set in X and $K \subset G$ compact, then*

$$\operatorname{cap}_1(K,G) = M_1(\Gamma(K,G)) = AM(\Gamma(K,G)).$$

Proof. Since for each function $u \in \operatorname{Lip}_0(K,G)$, $|\nabla u|$ is M-admissible for the family $\Gamma(K,G)$, we have $M_1(\Gamma(K,G)) \le \operatorname{cap}_1(K,G)$. For the converse inequality we use the method in [5, Section 5.2]. Let ρ be M-admissible for $\Gamma(K,G)$ and $\varepsilon > 0$. We may assume that ρ is lower semi-continuous. From Lemmata 5.25 and 5.26 in [5] it follows that the function $\rho + \varepsilon$ is an upper gradient of the lower semi-continuous function

$$u(x) = \min\left(1, \inf_{\gamma} \int_{\gamma} (\rho + \varepsilon)\, ds\right)$$

in G. Here the infimum is taken over all paths connecting $X \setminus G$ to $x \in G$. Moreover, $u = 0$ in $X \setminus G$ and $u = 1$ in K. Using Proposition 3.2(c) we obtain

$$\operatorname{cap}_1(K,G) = \widetilde{\operatorname{cap}}_1(K,G) \le \int_G (\rho + \varepsilon)\, d\mu \le \int_G \rho\, d\mu + \varepsilon\, \mu(G)$$

and letting $\varepsilon \to 0$ we obtain the desired inequality.

For the second equality it suffices to show that $M_1(\Gamma(K,G)) \le AM(\Gamma(K,G))$ because $M_1(\Gamma) \ge AM(\Gamma)$ for every path family Γ in X. Let $\Gamma(K,G,L)$ denote the family of all the paths γ in $\Gamma(K,G)$ whose length ℓ satisfies $\ell \le L$. Note that

$$M_1(\Gamma(K,G)) = \sup_L M_1(\Gamma(K,G,L)). \tag{3.4}$$

Indeed, if ρ is admissible for $\Gamma(K,G,L)$, then $\rho + \frac{1}{L}\chi_G$ is admissible for $\Gamma(K,G)$.

Fix L. Each $\gamma \in \Gamma(K,G,L)$ has a reparametrization $\xi : [0,L] \to X$ which is a curve with $\operatorname{Lip} \xi \le 1$; we denote the set of all such reparametrizations by $\Xi(K,G,L)$. For a Borel set $E \subset X$ set

$$\nu_\xi(E) = \int_\xi \chi_E\, ds.$$

Set $\mathcal{E} = \{\nu_\xi : \xi \in \Xi(K,G,L)\}$. Let $\mathcal{K}$ be the weak* closure of $\mathcal{E}$. Then

$$AM_c(\Gamma(K,G,L)) = AM_c(\Xi(K,G,L)) = AM_c(\mathcal{E}) = AM_c(\mathcal{K}). \tag{3.5}$$

Only the last equality is not obvious. Let (ξ_j) be a sequence of curves from $\Xi(K,G,L)$ such that ν_{ξ_j} converge weak* to $\nu \in \mathcal{K}$. By the Arzelà–Ascoli theorem (see [25, p. 169]) there exists a subsequence (not relabelled) which converges uniformly to a limit curve ξ, and, by compactness of K and openness of G, we have $\xi \in \Xi(K,G,L)$. For each non-negative continuous function ρ on X we have

$$\int_\xi \rho\, ds \le \liminf_j \int_{\xi_j} \rho\, ds = \lim_j \int_X \rho\, d\nu_{\xi_j} = \int_X \rho\, d\nu.$$

It follows that each admissible sequence for $AM_c(\mathcal{E})$ is also admissible for $AM_c(\mathcal{K})$ and thus $AM_c(\mathcal{K}) \le AM_c(\mathcal{E})$, whereas the converse inequality is obvious. This proves (3.5). By [14, Theorem 5.5], $AM(\mathcal{K}) = M_1(\mathcal{K})$ (as $\mathcal{K}$ is compact) and by [14, Theorem 3.4], $AM = AM_c$. Hence

$$\begin{aligned} M_1(\Gamma(K,G,L)) &\le M_1(\mathcal{K}) = AM_c(\mathcal{K}) = AM_c(\Gamma(K,G,L)) \\ &= AM(\Gamma(K,G,L)) \le AM(\Gamma(K,G)). \end{aligned}$$

Passing to the supremum over L we obtain the conclusion. □

Lemma 3.4. *If $E \subset G$ is a Suslin set, then* $\operatorname{cap}_1(E,G) \le AM(\Gamma(E))$.

Proof. Since E is a Suslin set, Proposition 3.2(f) implies that there are compact sets $K_1 \subset K_2 \subset \ldots \subset E$ such that $\mathrm{cap}_1(E,G) = \lim_i \mathrm{cap}_1(K_i,G)$. Now by Lemma 3.3

$$\mathrm{cap}_1(K_i,G) = AM(\Gamma(K_i,G)) \leq AM(\Gamma(E))$$

because $\Gamma(K_i,G) \subset \Gamma(E)$. □

Lemma 3.5. *Let $K_1 \subset K_2 \subset \ldots,$ be compact sets in G with*

$$\lim_{i\to\infty} \mathrm{cap}_1(K_i,G) < \infty. \tag{3.6}$$

Then there is a BV function w in X such that $w = 0$ in $X \setminus G$, $w = 1$ on $\bigcup_i K_i$, $0 \leq w \leq 1$ and

$$V(w,X) \leq \lim_{i\to\infty} \mathrm{cap}_1(K_i,G). \tag{3.7}$$

Proof. For each i pick $u_i \in \mathrm{Lip}_0(K_i,G)$ such that $0 \leq u_i \leq 1$ and

$$\int_G |\nabla u_i|\, d\mu \leq \mathrm{cap}_1(K_i,G) + 1/i.$$

By the compact embedding of BV into L^1_{loc}, see [24, Theorem 3.7], there is a limit function w and a subsequence $(v_i)_i$ of $(u_i)_i$ such that $v_i \to w$ in $L^1_{\mathrm{loc}}(X)$ and μ-a.e. In particular, we can assume that $w = 1$ on $\bigcup_i K_i$ and (3.7) holds. □

We recall some measure theoretic notation. Let $E \subset X$ be a (μ-) measurable set. The *measure theoretic boundary* $\partial_* E$ of E consists of points $x \in X$ such that $\Theta(x,E) > 0$ and $\Theta(x,X \setminus E) > 0$ where

$$\Theta(x,A) = \limsup_{r\to 0} \frac{\mu(B(x,r) \cap A)}{\mu(B(x,r))}$$

is the upper μ-density of A at x. The *measure theoretic interior* $\mathrm{int}_* E$ and the *measure theoretic exterior* $\mathrm{ext}_* E$ of E are the sets of points $x \in X$ where $\Theta(x,X \setminus E) = 0$ and $\Theta(x,E) = 0$, respectively. The sets $\partial_* E$, $\mathrm{int}_* E$ and $\mathrm{ext}_* E$ are Borel sets.

For an open bounded set $G \neq X$ and $E \subset G$ we define the *perimeter capacity* of E in G as

$$\mathrm{Cap}(E,G) = \inf\{P(F,X) : E \subset \mathrm{int}_* F,\ F \subset G \text{ measurable}\}.$$

Note that the perimeter of F is relative to X and not relative to G.

Lemma 3.6. *If E is a Suslin set in $G \subset X$ and $AM(\Gamma(E)) < \infty$, then*

$$\mathrm{Cap}(E,G) \leq \mathrm{cap}_1(E,G). \tag{3.8}$$

Proof. Let U be an open set such that $E \subset U \subset G$. By Lemma 3.4 we have $\mathrm{cap}_1(E,G) < \infty$. Next choose compact sets $K_1 \subset K_2 \subset \cdots \subset U$ such that $\bigcup_i K_i = U$; now

$$\mathrm{cap}_1(K_i,G) \leq \mathrm{cap}_1(U,G)$$

for all i.

Let w be the BV function in Lemma 3.5. Note that $w = 1$ in $U = \bigcup_i K_i$. By the co-area formula [24, Proposition 4.2] and Lemma 3.5

$$\int_0^1 P(\{x : w(x) > t\},X)\, dt \leq V(w,X) \leq \lim_i \mathrm{cap}_1(K_i,G) \leq \mathrm{cap}_1(U,G).$$

Thus there is some $t \in (0,1)$ such that the set $A = \{x : w(x) > t\}$ has finite perimeter, $\text{int}_* A \supset E$ and $P(A, X) \leq \text{cap}_1(U, G)$. Note that it is possible that $A = G$. Since

$$\text{Cap}(E, G) \leq P(A, X) \leq \text{cap}_1(U, G)$$

and this holds for all open sets U with $E \subset U \subset G$ we obtain (3.8). □

4 | $\mathbf{AM}(\boldsymbol{\Gamma}(\boldsymbol{E})) \leq \boldsymbol{C}\boldsymbol{co}\boldsymbol{\mathcal{H}}^{1}(\boldsymbol{E})$

Throughout this section we assume that (X, d) and μ satisfy the assumptions (A).

We need the following auxiliary lemma for the main result. Note that the set E below is an arbitrary subset of X.

Lemma 4.1. *If $E \subset X$ and $AM(\Gamma(E)) < \infty$, then $\mu(E) = 0$.*

Proof. By [16, Theorem 2] there is a co-Suslin set $E' \supset E$ such that $AM(\Gamma(E')) = AM(\Gamma(E))$. Since co-Suslin sets are μ-measurable we may assume that E is μ measurable and since we can also assume that E is bounded, it suffices to prove the lemma in the case $\mu(E) < \infty$.

Let $\varepsilon > 0$. Since $\mu\big(\overline{B}(x,r) \setminus B(x,r)\big) = 0$ except for a countable set of $r > 0$ we find by the Vitali covering theorem disjoint balls $\overline{B}\big(x_i, r_i\big)$ such that $r_i < \varepsilon$ and $\bigcup_i B\big(x_i, r_i\big) \supset E \setminus E_0$ where $\mu\big(E_0\big) = 0$. Now we can replace E by $E \setminus E_0$ which we continue to denote by E.

Fix $B_i = B\big(x_i, r_i\big)$ and let $K \subset E \cap B_i$ be compact. For $\delta > 0$ pick $u \in N_0^{1,1}\big(B_i\big)$ such that $u = 1$ on K, $0 \leq u \leq 1$ and

$$\int_{B_i} g_u \, d\mu < \text{cap}_1\big(K, B_i\big) + \delta.$$

By the Poincaré inequality [5, Theorem 5.51] for $N_0^{1,1}\big(B_i\big)$-functions there is a constant C depending only on C_P and C_μ so that

$$\mu(K) \leq \int_{B_i} u \, d\mu \leq C r_i \int_{B_i} g_u \, d\mu < C r_i \big(\text{cap}_1\big(K, B_i\big) + \delta \big)$$

and letting $\delta \to 0$ we obtain from Lemma 3.3

$$\mu(K) \leq C r_i AM\big(\Gamma\big(K, B_i\big)\big) \leq C r_i AM\big(\Gamma\big(E \cap B_i, B_i\big)\big).$$

Since this holds for all compact sets $K \subset E \cap B_i$

$$\mu\big(E \cap B_i\big) \leq C r_i \, AM\big(\Gamma\big(E \cap B_i, B_i\big)\big).$$

The path families $\Gamma\big(E \cap B_i, B_i\big)$ lie in the disjoint sets $\overline{B}_i$ and are subfamilies of $\Gamma(E)$. Summing over i we obtain

$$\mu(E) = \sum_i \mu\big(E \cap B_i\big) \leq C \sum_i r_i AM\big(\Gamma\big(E \cap B_i, B_i\big)\big) \leq C\varepsilon AM(\Gamma(E)),$$

and $\varepsilon \to 0$ completes the proof. □

The comparison of the BV capacity with the $(n-1)$-dimensional Hausdorff content is due to Fleming [10]. It has been generalized to the framework of metric measure spaces by Kinnunen, Korte, Shanmugalingam and Tuominen [18]. Here we need a version for the δ-Hausdorff content related to the $co\mathcal{H}^1$-measure.

Lemma 4.2. *Let M be a bounded open set in X. For $\delta > 0$ there exists $\alpha > 0$ such that for each open set G with $\mu(G) < \alpha$ and $E \subset G \subset M$ we have*

$$co\mathcal{H}^1_\delta(E) \le C\ \mathrm{Cap}(E, G), \tag{4.1}$$

where C depends only on C_P, λ_P and C_μ.

Proof. We write for C a generic constant which depends only on C_P, λ_P and C_μ.

Set $\delta' = \delta/(5\lambda_P)$ and $\varkappa = 4C_P$. Let G be a bounded open set such that $E \subset G \subset M$. We find $\alpha > 0$ such that for each $x \in G$

$$\mu(B(x, \delta') \cap G) \le \frac{1}{\varkappa}\mu(B(x, \delta')) \tag{4.2}$$

provided that $\mu(G) < \alpha$. Suppose that no such α exists. Then there are open sets G_i and $x_i \in G_i$ such that $E \subset G_i \subset M$ and

$$\frac{1}{i} > \mu(G_i) \ge \mu(B(x_i, \delta') \cap G_i) > \frac{1}{\varkappa}\mu(B(x_i, \delta'))$$

but because each x_i belongs to a fixed bounded set M, $\mu\left(B\left(x_i, \delta'\right)\right) > c > 0$ which leads to contradiction.

Fix G as above. To prove (4.1) we may assume that $\mathrm{Cap}(E, G) < \infty$ and for $\varepsilon > 0$ we choose a competitor $F \subset G$ for $\mathrm{Cap}(E, G)$ with $P(F, X) \le \mathrm{Cap}(E, G) + \varepsilon$. Let $x \in E$, $B(r) = B(x, r)$ and define

$$r_x = \inf\left\{ r > 0 :\ \mu(F \cap B(r)) \le \frac{1}{2C_P}\mu(B(r)) \right\}.$$

Now $0 < r_x < \delta'$ because

$$\lim_{r \to 0} \frac{\mu(F \cap B(r))}{\mu(B(r))} = 1$$

and by (4.2)

$$\mu(F \cap B(\delta')) \le \mu(G \cap B(\delta')) \le \frac{1}{4C_P}\mu(B(\delta')) < \frac{1}{2C_P}\mu(B(\delta')).$$

Let $r < r_x$. Then

$$\mu(F \cap B(r_x)) \ge \mu(F \cap B(r)) > \frac{1}{2C_P}\mu(B(r))$$

and letting $r \to r_x$ we obtain

$$\mu(F \cap B(r_x)) \ge \frac{1}{2C_P}\mu(B(r_x)). \tag{4.3}$$

On the other hand we show that

$$\mu(F \cap B(r_x)) \le \frac{1}{2}\mu(B(r_x)). \tag{4.4}$$

If $\mu(F \cap B(r_x)) \le \mu(B(r_x))/(2C_P)$, then equality holds in (4.3) and (4.4) is immediate. If

$$\mu(F \cap B(r_x)) > \frac{1}{2C_P}\mu(B(r_x))$$

then by the definition of r_x there is $r \in (r_x, 2r_x)$ such that

$$\mu(F \cap B(r_x)) \le \mu(F \cap B(r)) \le \frac{1}{2C_P}\mu(B(r)) \le \frac{1}{2}\mu(B(r_x)).$$

Next we use the BV-Poincaré inequality (2.10) for the BV function χ_F in $B(r_x)$. By (4.3) and (4.4)

$$\frac{1}{2C_\mu} \le (\chi_F)_{B(r_x)} = \frac{\mu(F \cap B(r_x))}{\mu(B(r_x))} \le \frac{1}{2}$$

and we obtain

$$\begin{aligned}\frac{\mu(B(r_x))}{4C_\mu} \le \frac{\mu(F \cap B(r_x))}{2} &\le \int_{F\cap B(r_x)} \left(1 - (\chi_F)_{B(r_x)}\right) d\mu \\ &\le \int_{B(r_x)} \left|(\chi_F - (\chi_F)_{B(r_x)}\right| d\mu \le C_P\, r_x P(F, B(\lambda_P\, r_x))\end{aligned}$$

and so

$$\frac{\mu(B(r_x))}{r_x} \le C\, P(F, B(\lambda_P r_x)). \tag{4.5}$$

By the 5-covering lemma we find balls $B_j = B(x_j, \lambda_P r_{x_j})$ from the collection $\{B(x, \lambda_P r_x)\}$ so that the balls B_j are disjoint and the balls $5B_j = B(x_j, 5\lambda_P r_{x_j})$ cover E. Set $D = \bigcup_j 5\,B_j$. Since $5\lambda_P r_{x_j} < 5\,\lambda_P \delta' = \delta$ we obtain from (4.5)

$$\begin{aligned}co\mathcal{H}^1_\delta(E) \le \sum_j \frac{\mu(5B_j)}{5r_{x_j}} &\le C \sum_j \frac{\mu\left(B\left(x_j, r_{x_j}\right)\right)}{r_{x_j}} \\ &\le C\ \sum_j P(F, B_j) \le C\, P(F, X) \le C(\mathrm{Cap}(E, G) + \varepsilon)\end{aligned}$$

where the doubling property of μ and the fact that the balls B_j are disjoint have also been used. Letting $\varepsilon \to 0$ we complete the proof. □

The following lemma combines the achieved results.

Lemma 4.3. *Suppose that $E \subset X$ is a bounded Suslin set such that $AM(\Gamma(E)) < \infty$. Then*

$$co\mathcal{H}^1(E) \le C AM(\Gamma(E)) \tag{4.6}$$

where the constant C depends only on C_P, λ_P and C_μ.

Proof. Lemma 4.1 yields $\mu(E) = 0$. Fix $\delta > 0$ and then, by Lemma 4.2, we find a bounded open set $G \ne X$ containing E with

$$co\mathcal{H}^1_\delta(E) \le C\ \mathrm{Cap}(E, G).$$

Now Lemmata 3.6 and 3.4 imply

$$\mathrm{Cap}(E, G) \le \mathrm{cap}_1(E, G) \le AM(\Gamma(E))$$

and hence $co\mathcal{H}^1_\delta(E) \le C\, AM(\Gamma(E))$. Passing to the supremum w.r.t. $\delta > 0$ we obtain (4.6). □

Theorem 4.4. *Let $E \subset X$ be a Suslin set. Then*

$$C_1\, co\mathcal{H}^1(E) \le AM(\Gamma(E)) \le C_2\, co\mathcal{H}^1(E) \tag{4.7}$$

where the constant $C_1 > 0$ depends only on C_P, λ_P and C_μ and the constant C_2 only on C_μ.

Proof. The second inequality in (4.7) follows from Theorem 2.1. For the first inequality fix $x_0 \in X$ and observe that

$$C_1 co\mathcal{H}^1\big(E \cap B\big(x_0, j\big)\big) \le AM\big(\Gamma\big(E \cap B\big(x_0, j\big)\big)\big) \le AM(\Gamma(E)), \quad j = 1, 2, \dots,$$

by Lemma 4.3. Letting $j \to \infty$ we conclude the proof. □

If $E \subset X$ has σ-finite $co\mathcal{H}^1$-measure, then Theorem 4.4 holds without the assumption that E is a Suslin set.

Theorem 4.5. *Suppose that $E \subset X$ has σ-finite $co\mathcal{H}^1$-measure. Then*

$$C_1 co\mathcal{H}^1(E) \le AM(\Gamma(E)) \le C_2 co\mathcal{H}^1(E) \tag{4.8}$$

where the constants C_1 and C_2 are as in Theorem 4.4.

Proof. The right inequality of (4.8) again follows from Theorem 2.1. For the left inequality suppose first that $co\mathcal{H}^1(E) < \infty$. Then there is a Borel set $F \supset E$ such that $co\mathcal{H}^1(F) = co\mathcal{H}^1(E)$ and a co-Suslin set $E' \supset E$ such that $AM(\Gamma(E')) = AM(\Gamma(E))$, see [16, Theorem 2]. We may assume that $E' \subset F$. Then the set function

$$\nu : A \mapsto co\mathcal{H}^1(A \cap F), \quad A \text{ Borel},$$

is a finite Borel measure. We extend ν to the class of all ν-measurable sets by completion. Then the set E' is ν-measurable as it is co-Suslin [17, Theorem 21.10]. It follows that there is a Borel set $A \subset E'$ such that $\nu(A) = \nu(E')$ [17, Theorem 17.10]. Now,

$$co\mathcal{H}^1(E) \le co\mathcal{H}^1\big(E'\big) = \nu\big(E'\big) = \nu(A) = co\mathcal{H}^1(A)$$

and

$$AM(\Gamma(A)) \le AM\big(\Gamma\big(E'\big)\big) = AM(\Gamma(E)).$$

Since $C_1\, co\mathcal{H}^1(A) \le AM(\Gamma(A))$, we conclude that

$$C_1 co\mathcal{H}^1(E) \le AM(\Gamma(E)).$$

In the general case we find $E_1 \subset E_2 \subset \dots$ such that $co\mathcal{H}^1\big(E_i\big) < \infty$ and $E = \bigcup_i E_i$. Let F_i be Borel set such that $F_i \supset E_i$ and $co\mathcal{H}^1\big(F_i\big) = co\mathcal{H}^1\big(E_i\big)$. Since $E_1 \subset F_1 \cap F_2 \subset F_1$, we have $co\mathcal{H}^1\big(F_1 \setminus F_2\big) = co\mathcal{H}^1\big(F_1\big) - co\mathcal{H}^1\big(F_1 \cap F_2\big) = 0$ and thus $co\mathcal{H}^1\big(F_1 \cup F_2\big) \le co\mathcal{H}^1\big(E_2\big)$. Continuing by induction we may assume that $F_1 \subset F_2 \subset \dots$. Therefore

$$co\mathcal{H}^1(E) \le co\mathcal{H}^1\left(\bigcup_i F_i\right) = \lim_i co\mathcal{H}^1\big(F_i\big) = \lim_i co\mathcal{H}^1\big(E_i\big) \le C_1^{-1} AM(\Gamma(E)).$$ □

In the Euclidean setting, the $co\mathcal{H}^1$ measure satisfies

$$\alpha_{n-1} co\mathcal{H}^1(E) = \alpha_n \mathcal{H}^{n-1}(E),$$

where

$$\mathcal{H}^{n-1}(E) = \sup_{\delta>0} \mathcal{H}^{n-1}_\delta(E)$$

is the spherical Hausdorff measure defined through the spherical Hausdorff δ-content

$$\mathcal{H}^{n-1}_\delta(E) = \inf\left\{ \sum_{i=1}^\infty \alpha_{n-1} r_i^{n-1} \,:\, E \subset \bigcup_{i=1}^\infty B\big(x_i, r_i\big)\,,\, r_i < \delta \right\}$$

and α_m denotes the volume of the m-dimensional unit ball. It is easily seen that the spherical Hausdorff measure is equivalent to the standard Hausdorff measure $\widetilde{\mathcal{H}}^{n-1}$ defined in terms of diameters, namely

$$\widetilde{\mathcal{H}}^{n-1}(E) \le \mathcal{H}^{n-1}(E) \le 2^n \widetilde{\mathcal{H}}^{n-1}(E), \quad E \subset \mathbb{R}^n,$$

see [9, 2.10.2]. Now, Theorems 4.4 and 4.5 yield (with properly modified constants):

Corollary 4.6. *If E is a Suslin set in $\mathbb{R}^n$ or has σ-finite $\widetilde{\mathcal{H}}^{n-1}$-measure, then*

$$C_1\, \widetilde{\mathcal{H}}^{n-1}(E) \le AM(\Gamma(E)) \le C_2\, \widetilde{\mathcal{H}}^{n-1}(E)$$

where the positive constants C_1 and C_2 depend only on n.

5 | PERIMETER AND *AM*-MODULUS IN *X*

We characterize sets E of finite perimeter in X using the AM-modulus of the path family $\Gamma(\partial_* E)$. Such a characterization was presented for $X = \mathbb{R}^n$ in [16].

We also study the connection of the perimeter of E in an open set $\Omega \subset X$ to the family $\Gamma_{\text{cross}}(E, \Omega)$ whose paths lie in an open set Ω and meet both the measure theoretic exterior and interior of E and present a measure theoretic version of the elementary topological fact. Namely, if X is a topological space, $E \subset X$ and $\operatorname{int} E$, $\operatorname{ext} E$ and ∂E are the (topological) interior, exterior and boundary of E, respectively, then every curve $\gamma : [a, b] \to X$ which meets $\operatorname{int} E$ and $\operatorname{ext} E$ also meets ∂E. We show that AM a.e. path $\gamma \in \Gamma_{\text{cross}}(E, \Omega)$ meets the measure theoretic boundary $\partial_* E$ of E provided that E has finite perimeter in Ω. In [20, Theorem 5.3] a closely related result is proved under more restrictive assumptions on E for the M_1-modulus.

We assume that X satisfies (A) and, as before, C is a constant which depends only on C_μ, C_λ and C_P and can change inside a line.

Lemma 5.1. *If Ω be an open set in X and $E \subset X$ measurable, then*

$$AM(\Gamma_{\text{cross}}(E, \Omega)) \le C\, P(E, \Omega).$$

Proof. Let u be the Lebesgue representative of χ_E, i.e.

$$u(x) = \lim_{r\to 0} \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))}$$

whenever the limit exists, then $u(x) = 1,\ x \in \operatorname{int}_* E$, $u(x) = 0,\ x \in \operatorname{ext}_* E$ and $u = \chi_E$ a.e. in Ω.

For the proof we may assume that $P(E, \Omega) < \infty$ and then we can use the special sequence of locally Lipschitz functions constructed in [19, Proposition 4.1]; i.e. there is a sequence $u_k \in \operatorname{Lip}_{\text{loc}}(\Omega)$ such that $u_k \to u$ pointwise $co\mathcal{H}^1$ a.e. in $\Omega \setminus \partial_* E$, $u_k \to u$ in $L^1(\Omega)$ and

$$\liminf_{k\to\infty} \int_\Omega \left|\nabla u_k\right| d\mu \le C P(E, \Omega). \tag{5.1}$$

Let $A \subset \Omega \setminus \partial_* E$ be the set where $\lim_k u_k(x) \neq u(x)$. Now $co\mathcal{H}^1(A) = 0$ and by Theorem 2.1, $AM(\Gamma(A)) = 0$. The sequence of functions $|\nabla u_k|$ is AM-admissible for $\Gamma_{\text{cross}}(E, \Omega) \setminus \Gamma(A)$ since if $\gamma \in \Gamma_{\text{cross}}(E, \Omega) \setminus \Gamma(A)$ then there are points $t_1, t_2 \in [0, \ell]$ such that $\gamma(t_1) \in \text{int}_* E$, $\gamma(t_2) \in \text{ext}_* E$ and

$$1 = \lim_{k\to\infty} \left| u_k(\gamma(t_1)) - u_k(\gamma(t_2)) \right| \leq \liminf_{k\to\infty} \int_\gamma |\nabla u_k|\, ds.$$

By (5.1)

$$AM\left(\Gamma_{\text{cross}}(E, \Omega) \setminus \Gamma(A)\right) \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|\, d\mu \leq CP(E, \Omega)$$

and since $AM(\Gamma(A)) = 0$ we have

$$AM\left(\Gamma_{\text{cross}}(E, \Omega)\right) \leq CP(E, \Omega).$$

□

Theorem 5.2. *If $P(E, \Omega) < \infty$ then AM a.e. path $\gamma \in \Gamma_{\text{cross}}(E, \Omega)$ meets $\partial_* E$.*

Proof. Let Γ be the family of paths in $\Gamma_{\text{cross}}(E, \Omega)$ which do not meet $\partial_* E$. By the subadditivity of the AM-modulus we may assume that Ω is bounded. By [4, Theorem 4.4 and Theorem 4.6] for every open set $G \subset \Omega$

$$P(E, G) = \int_{\partial_* E \cap G} \theta \, dco\mathcal{H}^1$$

where $\theta = \theta_E$ is a Borel function with $1/C \leq \theta \leq C$ in Ω and, moreover, $co\mathcal{H}^1(\partial_* E \cap \Omega) < \infty$. Let $\varepsilon > 0$. Now we find a compact set $K \subset \partial_* E \cap \Omega$ such that $P(E, G) < \varepsilon$ for $G = \Omega \setminus K$.

Next observe that $\Gamma \subset \Gamma_{\text{cross}}(E, G)$ because each $\gamma \in \Gamma$ does not meet K. By Lemma 5.1

$$AM(\Gamma) \leq AM\left(\Gamma_{\text{cross}}(E, G)\right) \leq CP(E, G) \leq C\varepsilon$$

and letting $\varepsilon \to 0$ we complete the proof. □

Theorem 5.3. *Suppose that $E \subset X$ is a (μ-) measurable set. Then for each open set $\Omega \subset X$*

$$C_1 P(E, \Omega) \leq AM(\Gamma(\partial_* E \cap \Omega)) \leq C_2 P(E, \Omega) \tag{5.2}$$

where the constants C_1 and C_2 depend only on C_P, C_λ and C_μ.

Proof. For the right inequality in (5.2) we may assume that $P(E, \Omega) < \infty$ and then by [4, Theorem 4.4],

$$co\mathcal{H}^1(\partial_* E \cap \Omega) \leq CP(E, \Omega)$$

and now Theorem 2.1 gives the required inequality.

For the left side of (5.2) we note that $\partial_* E \cap \Omega$ is a Borel set and thus Theorem 4.4 yields

$$co\mathcal{H}^1(\partial_* E \cap \Omega)) \leq CAM(\Gamma(\partial_* E \cap \Omega)) < \infty.$$

By the recent result of Lahti [21, Theorem 1.1] this implies that $P(E, \Omega) < \infty$ and we can apply again [4, Theorem 4.4] to conclude

$$P(E, \Omega) \leq Cco\mathcal{H}^1(\partial_* E \cap \Omega))$$

and complete the proof. □

6 | GEOMETRY OF LEVEL SETS IN X

The results in the previous sections can be used to study the structure of level sets of BV and continuous functions in X and the latter case together with the results in Section 4 produces a plenitude of open sets in X with $co\mathcal{H}^1$ finite boundaries.

We assume that X satisfies the hypotheses (A) and recall some measure theoretic concepts asociated with BV-functions. For a measurable set E and $x \in X$ we let

$$\overline{D}(E,x) = \limsup_{r\to 0} \frac{\mu(E\cap B(x,r))}{\mu(B(x,r))}, \quad \underline{D}(E,x) = \liminf_{r\to 0} \frac{\mu(E\cap B(x,r))}{\mu(B(x,r))},$$

and $D(E,x) = \overline{D}(E,x)$ if $\overline{D}(E,x) = \underline{D}(E,x)$.

Let Ω be an open set in X and $u \in BV(\Omega)$. The upper and lower *approximate limits* of u at $x \in \Omega$ are

$$u^+(x) = \inf\{s : D(\{u > s\}, x) = 0\} \text{ and } u^-(x) = \sup\{t : D(\{u < t\}, x) = 0\}.$$

Then it is immediate that $u^-(x) \le u^+(x)$. The function u is *approximately continuous* at x if $u^+(x) = u^-(x) = u(x)$. This holds a.e. in Ω by the Lebesgue differentiation theorem. The set $J_u = \{u^- < u^+\}$ is called the jump set of u and it has zero μ-measure, see [19].

For $-\infty \le s, t, \le \infty$ we consider the measure theoretic level sets of $u \in BV(\Omega)$

$$E^t = \{x\in\Omega : u^-(x) \le t\},$$
$$E_s = \{x\in\Omega : u^+(x) \ge s\},$$
$$E_s^t = E_s \cap E^t,$$
$$\Lambda_t = E_t^t.$$

Lemma 6.1. *If $u \in BV(\Omega)$, then*

$$\mu(\Lambda_t) = 0, \tag{6.1}$$

and consequently $P(E_t,\Omega) = P(E^t,\Omega)$, for a.e. $t \in \mathbb{R}$.

If u is (approximately) continuous at x, then $x \in \Lambda_{u(x)}$.

Proof. To prove (6.1) note that $\Lambda_t \subset A_t \cup J_u$, where

$$A_t = \{x\in\Omega : t = u^-(x) = u^+(x)\}.$$

Since $A_t \cap A_{t'} = \emptyset$ for $t \ne t'$ and $\mu(J_u) = 0$, (6.1) follows. If $\mu(\Lambda_t) = 0$, then E_t differs from $\Omega \setminus E^t$ by a μ-null set and thus $P(E_t,\Omega) = P(E^t,\Omega)$.

If u is approximately continuous at x and $t = u(x)$, then $t = u^+(x) = u^-(x)$ and thus $x \in \Lambda_t$. □

Theorem 6.2. *Let $u \in BV(\Omega)$. Then for a.e. $t \in \mathbb{R}$ we have*

$$co\mathcal{H}^1(\Lambda_t) \le C\,P(E^t,\Omega) \tag{6.2}$$

where C depends only on C_P, C_λ and C_μ.

Proof. We first assume that Ω is bounded. Let T be the essential infimum of u. Then (6.2) obviously holds for $t < T$. If $t > T$, then $\mu(E^t) > 0$ and then also $P(E^t,\Omega) > 0$ by the isoperimetric inequality (see e.g. [20]). Denote $\psi(t) = P(E^t,\Omega)$ and note that ψ is integrable, see [1] and [24]. Let $\tau > T$ be a Lebesgue point for ψ such that $\mu(\Lambda_\tau) = 0$. By Lemma 6.1 and the Lebesgue differentiation theorem, a.e. $\tau > T$ has these properties. We show that $t = \tau$ has the required property.

Choose $\delta > 0$. Lemma 4.2 gives $\alpha > 0$ such that for each bounded open set G with $\mu(G) < \alpha$ and $E \subset G$ we have

$$co\mathcal{H}^1_\delta(E) \le C \ \mathrm{Cap}(E, G).$$

Now, using Lemma 6.1 we find $a, b \in \mathbb{R}$ such that $a < \tau < b$, $\psi(a) \le 2\psi(\tau)$, $\psi(b) \le 2\psi(\tau)$, $\mu(\Lambda_a) = \mu(\Lambda_b) = 0$ and $\mu(E^b_a) < \alpha$. We find an open set $G \supset E^b_a$ such that still $\mu(G) < \alpha$. Choose $x \in \Lambda_\tau$. Then $a < u^+(x)$, $u^-(x) < b$, and thus $x \in \partial_* E_a$ (if $\overline{D}(E^a, x) > 0$), or $x \in \partial_* E^b$ (if $\overline{D}(E_b, x) > 0$), or $x \in \mathrm{int}_* E^b_a$ (if $D(E^a, x) = D(E_b, x) = 0$). Summarizing,

$$\Lambda_\tau \subset \partial_* E_a \cup \partial_* E^b \cup \mathrm{int}_* E^b_a.$$

We have

$$co\mathcal{H}^1_\delta(\partial_* E_a) \le C\, P(E_a, \Omega) = C\, P(E^a, \Omega) \le 2CP(E^\tau, \Omega),$$
$$co\mathcal{H}^1_\delta(\partial_* E^b) \le CP(E^b, \Omega) \le 2CP(E^\tau, \Omega)$$

and then

$$\begin{aligned} co\mathcal{H}^1_\delta(\mathrm{int}_* E^b_a) &\le C\,\mathrm{Cap}\,(\mathrm{int}_* E^b_a, G) \le CP(E^b_a, G) \\ &\le C\big(P(E^a, \Omega) + P(E^b, \Omega)\big) \le 4C\, P(E^\tau, \Omega). \end{aligned}$$

Letting $\delta \to 0$ we obtain (6.2).

Suppose that Ω is unbounded. Fix a point $x_0 \in X$ and for each $i = 1, 2, \ldots$ let $\Omega_i = \Omega \cap B(x_0, i)$ and $u_i = u|\Omega_i$. Denote by $E^t(u_i)$ the set E^t associated with u_i and other sets, like $\Lambda_\tau(u_i)$, similarly. Now for a.e. $t \in \mathbb{R}$, $\mu(\Lambda_t(u_i)) = 0$ for every i and so for a.e. $t \in \mathbb{R}$

$$co\mathcal{H}^1(\Lambda_t(u_i)) \le CP(E^t(u_i), \Omega_i) \le CP(E^t(u_i), \Omega) \le CP(E^t, \Omega)$$

for every i and this easily implies (6.2) for u. □

If $u \in BV(\Omega)$ then by the co-area formula [24, Proposition 4.2] for the perimeter $P(E^t, \Omega) < \infty$ for a.e. $t \in \mathbb{R}$. Hence Theorem 6.2 and Lemma 6.1 yield

Corollary 6.3. *If $u \in BV(\Omega)$, then*

$$co\mathcal{H}^1(\Lambda_t) < \infty \text{ for a.e. } t \in \mathbb{R}.$$

If, in addition, u is (approximately) continuous, then $co\mathcal{H}^1(u^{-1}(t)) < \infty$ for a.e. $t \in \mathbb{R}$.

Remark 6.4. The above corollary can be used to construct sets in X whose boundaries have finite $co\mathcal{H}^1$-measure. For example, let $u \in BV(\Omega) \cap C(\Omega)$. Then for a.e. $t \in \mathbb{R}$ the boundary of the open set $\{u > t\}$ has finite $co\mathcal{H}^1$-measure. For a more specific example let $x_0 \in X$ and take $u(x) = d(x, x_0)$. It follows that the topological boundary $\partial B(x_0, r)$ of the ball $B(x_0, r)$ has finite $co\mathcal{H}^1$-measure for a.e. $r > 0$. This is an improvement of the earlier results since it has been only known that $\mu(\partial B(x_0, r)) = 0$ except for a countable number of r and that the reduced boundary $\partial_* B(x_0, r)$ satisfies $co\mathcal{H}^1(\partial_* B(x_0, r)) < \infty$ for a.e. $r > 0$. Note that the reduced boundary can be strictly smaller than the topological boundary. More generally, if $K \subset X$ is a bounded set, then $u(x) = \mathrm{dist}(x, K)$ is a Lipschitz function and thus the boundary of the t-inflation $\{x : \ \mathrm{dist}(x, K) < t\}$ of K has finite $co\mathcal{H}^1$-measure for a.e. $t > 0$. In $\mathbb{R}^n$ this result follows from [9, Lemma 3.2.34].

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