Pathology of essential spectra of elliptic problems in periodic family of beads threaded by a spoke thinning at infinity. [1]

Sergei A. Nazarov
Saint-Petersburg State University, Universitetskaya nab., 7–9, St. Petersburg, 199034, Russia, and
Institute of Problems of Mechanical Engineering RAS, V.O., Bolshoj pr., 61, St. Petersburg, 199178, Russia
e-mail: s.nazarov@spbu.ru, srgnazarov@yahoo.co.uk

Jari Taskinen[2]
Department of Mathematics and Statistics, University of Helsinki, P.O.Box 68, FI-00014 Helsinki, Finland
e-mail: jari.taskinen@helsinki.fi

Abstract. We construct "almost periodic" unbounded domains, where a large class of elliptic spectral problems have essential spectra possessing peculiar structure: they consist of monotone, non-negative sequences of isolated points and thus have infinitely many gaps.

§1. Formulation of the spectral problem.

1.1. Domain and boundary-value problem. The purpose of this paper is to construct unbounded domains $\Pi \subset \mathbb{R}^d$, $d \geqslant 2$, where a large class of elliptic spectral problems (1.11)–(1.12) has essential spectrum $\wp_{es}$ with infinitely many spectral gaps: as stated in our main result Theorem 2.1, $\wp_{es}$ equals a countable set without finite accumulation points, $\Sigma \subset \overline{\mathbb{R}^+} = [0, +\infty)$, (2.6), which is the set of

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[2]Corresponding author

eigenvalues of an elliptic spectral problem on a bounded domain. The domain consists of an infinite row of translates of a bounded cell ϖ such that the neighboring ones have common tangential boundary components and the cells are connected via a sequence of thinning apertures. While further discussion of the ideas of our construction is postponed to Section 1.3, we start here by a detailed definition of the domain and the boundary value problems.

Let $\varpi \subset \mathbb{R}^d$, be a domain with Lipschitz boundary $\partial\varpi$ and compact closure $\overline{\varpi} = \varpi \cup \partial\varpi$. We assume that ϖ belongs to the layer $\{(y, z) : y \in \mathbb{R}^{d-1}, z \in (0, 1)\}$ and the surface $\partial\varpi$ has two planar parts $\gamma^0 = \mathbb{B}_R^{d-1}(0) \times \{0\}$ and $\gamma^1 = \mathbb{B}_R^{d-1}(0) \times \{1\}$ where $\mathbb{B}_R^{d-1}(0) = \{y \in \mathbb{R}^{d-1} : |y| < R\}$ is a ball in $\mathbb{R}^{d-1}$ of radius $R > 0$. We also introduce a $(d-1)$-dimensional domain $\omega \subset \mathbb{B}_{R/2}^{d-1}(0)$, a positive infinitesimal sequence $\{\alpha_j\}_{j\in\mathbb{Z}} \subset (0, 1)$ and the sets

$$\omega_j = \{(y, z) : \alpha_j^{-1} y \in \omega, z = j\} \subset \mathbb{R}^d, \quad j \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}. \tag{1.1}$$

The domain Π is the union of the periodicity cells

$$\varpi_j = \{(y, z) : (y, z - j) \in \varpi\}, \quad j \in \mathbb{Z}, \tag{1.2}$$

connected through apertures (1.1), Fig. 1.a, namely

$$\Pi = \bigcup_{j\in\mathbb{Z}} (\varpi_j \cup \omega_j). \tag{1.3}$$

Let $\mathcal{A}$ be a function matrix of size $N \times N$, which is 1-periodic in z, Hermitian, and positive definite for all $x \in \overline{\varpi}$, and let $D(\nabla)$ be a $N \times n$-matrix of first order differential operators with constant coefficients, which are in general complex. The matrix D is *algebraically complete*, see [24, § 3.7.4], which means that there exists a natural number $\varrho_D \in \mathbb{N} = \{1, 2, 3, \dots\}$ such that, for any row $p = (p_1, \dots, p_n)$ of homogeneous polynomials of order $\deg p_j = \varrho \geqslant \varrho_D$ in the variables $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, one can find a row of polynomials $q = (q_1, \dots, q_N)$ such that there holds the relation

$$p(\xi) = q(\xi) D(\xi), \quad \xi \in \mathbb{R}^d. \tag{1.4}$$

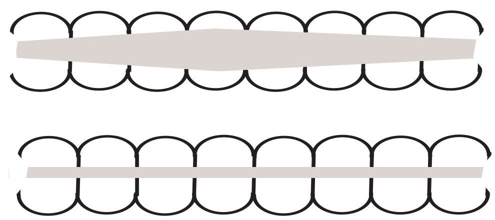

Figure 1.1: Periodic beads connected with converging (a) and straight (b) spoke (shaded).

We assume for a while that the surface $\partial\varpi$ and the entries of the matrix $\mathcal{A}$ are smooth enough and introduce the $n\times n$-matrix of second order differential operators in divergence form

$$\mathcal{L}(x,\nabla)=\overline{D(-\nabla)}^{\top}\mathcal{A}(x)D(\nabla). \tag{1.5}$$

We write the Green formula

$$(\mathcal{L}u,v)_{\varpi}+(\mathcal{N}u,v)_{\partial\varpi}=a(u,v;\varpi):=(\mathcal{A}(x)D(\nabla)u,D(\nabla)v)_{\varpi}, \tag{1.6}$$

where $u=(u_1,\dots,u_n)^{\top}$ and $v=(v_1,\dots,v_n)^{\top}$ are any smooth vector functions. Here, ∇ denotes the gradient, $\top$ stands for transposition, $(\,,\,)_{\varpi}$ is the natural scalar product in the (scalar or vector) Lebesgue space $L^2(\varpi)$, and $\mathcal{N}$ is the Neumann boundary condition operator

$$\mathcal{N}(x,\nabla)=\overline{D(\nu(x))}^{\top}\mathcal{A}(x)D(\nabla), \tag{1.7}$$

where $\nu=(\nu_1,\dots,\nu_d)^{\top}$ is the outward normal unit vector on $\partial\varpi$.

Owing to the algebraic completeness (1.4) of D, operator (1.5) can be called *formally positive* [24, § 3.7.4] and the positive Hermitian sesquilinear form $a(u,v;\varpi)$ in (1.6) satisfies the Korn inequality [24, Theorem 3.7.7]

$$\|u;H^1(\varpi)\|^2\leqslant c(\mathcal{A},D,\varpi)(a(u,u;\varpi)+\|u;L^2(\varpi)\|^2), \tag{1.8}$$

where $c(\mathcal{A},D,\varpi)>0$ is a constant and u is any function belonging to the Sobolev space $H^1(\varpi)^n$. (The latter superscript n indicates the number of components of the vector function u, but we do not display this number in the notation of norms and scalar products.) Moreover, the couple $\{\mathcal{L}, a\}$ possesses the *polynomial property* [15, 16]: for any domain $G \subset \mathbb{R}^d$,

$$a(u, u; G), u \in H^1(G)^n \quad \Leftrightarrow \quad u \in \mathcal{P}\big|_G, \tag{1.9}$$

where $\mathcal{P}$ is some finite-dimensional subspace of vector polynomials $p(x) = (p_1(x), \dots, p_n(x))^\top$ independent of the domain G. It is straightforward to deduce from (1.4) that $\deg p_j < \varrho_D$ for all $p \in \mathcal{P}$. Furthermore, $\mathcal{L}p = 0$ and $\mathcal{N}p = 0$ for $p \in \mathcal{P}$.

Let Θ be a unitary matrix function of size $n \times n$ on the surface $\partial\varpi$. Denoting by the subindex q the qth row of a matrix, we compose the $n \times n$-matrices $\mathcal{B}(x, \nabla)$ and $\mathcal{T}(x, \nabla)$ by defining the qth rows, $q = 1, \dots, n$, of them in one of the following ways:

$$\begin{gathered} \mathcal{B}_q(x, \nabla) = (\Theta(x)\mathcal{N}(x, \nabla))_q\ , \quad \mathcal{T}_q(x, \nabla) = \Theta_q(x) \\ \text{or} \\ \mathcal{B}_q(x, \nabla) = \Theta_q(x)\ , \quad \mathcal{T}_q(x, \nabla) = -(\Theta(x)\mathcal{N}(x, \nabla))_q. \end{gathered} \tag{1.10}$$

Notice that here, the index q must be independent of x; however, $\mathcal{B}(x, \nabla)$ and therefore also $\mathcal{T}(x, \nabla)$) may contain both types of rows.

In view of (1.10) and (1.7), applying the Green formula (1.6) twice yields the identity

$$(\mathcal{L}u, v)_\varpi + (\mathcal{B}u, \mathcal{T}v)_{\partial\varpi} = (u, \mathcal{L}v)_\varpi + (\mathcal{T}u, \mathcal{B}v)_{\partial\varpi}.$$

1.2. The spectral problem and its variational and operator formulations. We consider the boundary-value problem in the unbounded domain (1.3)

$$\mathcal{L}(x, \nabla)u(x) = \lambda\mathcal{M}(x)u(x), \quad x \in \Pi, \tag{1.11}$$

$$\mathcal{B}(x, \nabla)u(x) = 0, \quad x \in \partial\Pi, \tag{1.12}$$

where $\mathcal{M}$ is a smooth Hermitian positive non-zero matrix function of size $n \times n$ and λ is a spectral parameter. The differential operators $\mathcal{L}$ and $\mathcal{B}$, which were originally defined in ϖ and $\partial\varpi$, respectively, are extended 1-periodically over the "almost periodic" set Π; recall that $\partial\Pi \subset \bigcup_{j\in\mathbb{Z}} \partial\varpi_j$. The entries of the matrix $\mathcal{M}$ are assumed to be 1-periodic in the variable z, too. Since the coefficients are smooth, the polynomial property (1.9) assures that the matrix operator $\mathcal{L}(x,\nabla)$ is elliptic, cf., [16]; moreover, the boundary condition operator $\mathcal{B}(x,\nabla)$ covers it on $\partial\Pi$, namely, the Shapiro–Lopatinsky conditions hold true everywhere in $\partial\Pi$ except for the $(d-2)$-dimensional edges $\partial\Pi \cap \overline{\omega_j}$, $j \in \mathbb{Z}$, of the connecting apertures (1.1).

The variational formulation of the problem (1.11), (1.12) reads as the integral identity [10, 11]

$$a(u,v;\Pi) = \lambda(\mathcal{M}u,v)_\Pi \quad \forall v \in \mathcal{H}(\Pi). \tag{1.13}$$

Here, $\mathcal{H}(\Pi)$ is the subspace of vector functions v in the Sobolev space $H^1(\Pi)^n$ which satisfy the *stable* boundary conditions of (1.10), i.e.,

$$\Theta_q(x)u(x) = 0,\ x \in \partial\Pi,\ \text{ corresponding to } \mathcal{B}_q(x,\nabla) = \Theta_q(x). \tag{1.14}$$

The indices q, for which (1.14) holds, form the set $Q \subset \{1,\dots,n\}$.

The integral identity (1.13) makes sense even when the entries of the matrices $\mathcal{A}$ and $\mathcal{M}$ are only measurable and bounded, and the matrices are positive almost everywhere in ϖ. These properties are assumed to hold in the sequel, although in addition the local smoothness near the flat surfaces γ^0 and γ^1 will also be needed so that in particular the periodicity of the matrices $\mathcal{A}$ and $\mathcal{M}$ will be preserved.

Let us suppose that there holds the condition

$$\begin{aligned} p \in \mathcal{P},\ q \in Q,\quad (\mathcal{M}p,p)_\varpi = 0,\ \Theta_q(x)p(x) = 0\ \forall\, x \in \partial\varpi \\ \Rightarrow \quad p = 0\ ; \end{aligned} \tag{1.15}$$

here, $\mathcal{P}$ is the subspace of polynomials in (1.9). Then the lemma on equivalent norms guarantees that the Korn inequality (1.8) can be written on periodicity cells as

$$\|u; H^1(\varpi_j)\|^2 \leqslant c\,(\mathcal{A},\mathcal{B},\mathcal{M},D,\varpi)(a(u,u;\varpi_j) + (\mathcal{M}u,u)_{\varpi_j}), \tag{1.16}$$

where $c(\mathcal{A},\mathcal{B},\mathcal{M},D,\varpi)$ is a constant independent of j and $u \in \mathcal{H}(\Pi)$, see [24, Theorem 3.7.7], [3, Theorem 2.2.2] and others. Summing the inequalities (1.16) over $j \in \mathbb{Z}$ shows that the sesquilinear Hermitian form

$$\langle u, v \rangle = a(u, v; \Pi) + (\mathcal{M}u, v)_\Pi \tag{1.17}$$

can be taken as an inner product in the Hilbert space $\mathcal{H}(\Pi)$.

We introduce the operator M in $\mathcal{H}(\Pi)$ by the identity

$$\langle Mu, v \rangle = (\mathcal{M}u, v)_\Pi \quad \forall u, v \in \mathcal{H}(\Pi) \tag{1.18}$$

and turn (1.13) into the abstract equation

$$Mu = \mu u \quad \text{in} \quad \mathcal{H}(\Pi) \tag{1.19}$$

where, according to (1.17) and (1.13), the spectral parameters are in the relationship

$$\mu = (1 + \lambda)^{-1}. \tag{1.20}$$

The operator M in (1.18) is continuous, positive and symmetric, therefore, self-adjoint. Its norm does not exceed 1; compare (1.17) and (1.18). Thus, the μ-spectrum of the operator M is located in the segment $[0, 1] \subset \mathbb{R} \subset \mathbb{C}$. Since the domain (1.9) is not bounded, the embedding $H^1(\Pi) \subset L^2(\Pi)$ is not compact and, by [2, Theorem 10.1.5], the essential μ-spectrum $\wp_{es}(M)$ cannot consist only of the point $\mu = 0$. The relation (1.20) sends this point to infinity. Hence, the λ-spectrum of the problem (1.13) (or (1.11), (1.12))

$$\wp = \{\lambda \in \mathbb{C} : \, (1 + \lambda)^{-1} \in \wp(M)\} \subset \overline{\mathbb{R}_+} \tag{1.21}$$

has a non-trivial essential component $\wp_{es}$. As it was already announced, the main goal of the paper is to describe the essential spectrum $\wp_{es}$. It should be emphasized that the question if $\wp_{es}$ may include an eigenvalue of infinite multiplicity, remains open.

1.3. Motivation. From the point of view of modelling, our results are related to wave processes in almost periodic media. For example, consider acoustic or elastic waves in an infinite $3d$-packing of deformable metal balls in vacuum, subject to a gravity center, where, due to diminishing compression, the contact zones between neighboring balls decrease with the distance from this center. Similar examples can be considered for acoustic waves in an ordered cloud of gas bubbles in liquids. Our results demonstrate that the stopping zones for waves in these infinite media are small so that at almost all frequencies waves can propagate through.

As for the mathematical context, the structure (1.3) of the waveguide Π, consisting of the periodic family of identical cells connected through thinning apertures, emerges from the previous works [18, 20, 22, 23] of the authors and their attempts to prove or disprove the existence of infinite number of spectral gaps in the spectrum of periodic elastic and piezoelectric waveguides; we also refer to [26] for a treatment of the scalar case. This question is related to the classical Bethe–Sommerfeld conjecture, see [32], which has been until now solved only for scalar problems, [25, 30, 31]. The waveguide Π^ε of the papers [18, 20, 22, 23] consists of the union of cells (1.2) and the thin cylinder $\Omega^\varepsilon = \{(y,z) : \varepsilon^{-1}y \in \omega, z \in \mathbb{R}\}$, where the domain $\omega \subset \mathbb{R}^{d-1}$ is as above, but $\varepsilon > 0$ is a small parameter. In other words, the cells are connected through small but *fixed* apertures $\omega_j^\varepsilon = \{(y,z) : \varepsilon^{-1}y \in \omega, z = j\}$; compare Fig. 1.a and b, where Π and Π^ε are depicted. The main results of the citations state that for any $N \in \mathbb{N} = \{1,2,3,\dots\}$ there exists $\varepsilon_N > 0$ such that the spectrum of the problem in Π^ε with $\varepsilon \in (0, \varepsilon_N]$ has at least N opened spectral gaps. This was proven by constructing asymptotics of eigenvalues of a model problem in ϖ, which is obtained from the system (1.11)–(1.12) in Π^ε using the Floquet–Bloch–Gelfand transform [5]. However, since $\varepsilon_N \to +0$ as $N \to +\infty$, this asymptotic analysis appears not to be adequate to solve, if an infinite number of gaps can occur.

Using a different argument from the asymptotic theory of elliptic problems in singularly perturbed domains, see [12], we will show in Section 2 that the essential spectrum of the problem (1.11), (1.12) coincides with an unbounded countable set Σ in $\overline{\mathbb{R}_+} = [0, +\infty)$, which is the discrete spectrum of the model problem (2.1), (2.2) in the periodicity cell ϖ, see Theorem 2.1. Thus, we have an infinite number of spectral gaps for sure. Since (1.3) is not a periodic set, the Floquet–Bloch–Gelfand technique does not apply. To identify the essential spectrum we construct for any point (1.20) with $\lambda \in \Sigma \subset \wp_{es}$ a Weyl singular sequence for the operator M, and on the other hand in the case $\lambda \notin \Sigma$ we construct a right parametrix for the problem operator (1.13); this implies the Fredholm property for M and thus assures that $\lambda \notin \wp_{es}$. These facts together lead to the equality $\Sigma = \wp_{es}$.

For the sake of shortness of the paper, we will mostly consider the problem (1.11), (1.12) in dimension $d \geqslant 3$ and outline modifications needed for the planar case $d = 2$ only in Section 3.2. The procedures are quite similar in both cases, however, since the fundamental solutions and resolvent kernels (i.e. the integral kernels of the inverse operators) behave logarithmically for $d = 2$, all related formulas would look quite different; a direct transformation algorithm between the two cases could be found in [6, 12].

1.4. Concrete spectral problems. $1°$. *Scalar case.* Let $n = 1$, $N = d$ and $D(\nabla) = \nabla$. Then, (1.5) is a scalar elliptic second-order differential operator in the divergence form. Clearly, $\varrho_D = 1$ in (1.4). If in addition $\mathcal{A} = \mathbb{I}_N$ is the unit $N \times N$-matrix, then $\mathcal{L}(\nabla) = -\Delta$ is the Laplacian. In the case $\mathcal{B}(x, \nabla) = \nu(x)^\top \nabla$ with the outward normal unit vector ν, we have the Neumann problem, which describes for example the propagation of waves in a homogeneous acoustic media. For a more general real, symmetric and positive definite matrix function $\mathcal{A}$, the medium can be anisotropic and inhomogeneous, in particular, stratified. The case of the Dirichlet boundary condition corresponds to quantum waveguides. In Section 3.4 we will also discuss the spectral Steklov problem

$$-\Delta u(x) = 0,\ x \in \Pi, \quad \partial_\nu u(x) := \nu(x)^\top \nabla u(x) = \lambda u(x) = 0,\ x \in \partial\Pi, \tag{1.22}$$

with the spectral parameter λ in the boundary condition. This is related to the linear theory of water waves, cf., [9].

$2°$. *Elasticity system.* Let $d = n = 3$, $N = 6$ and

$$D(\nabla)^\top = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 2^{-1/2}\partial_3 & 2^{-1/2}\partial_2 \\ 0 & \partial_2 & 0 & 2^{-1/2}\partial_3 & 0 & 2^{-1/2}\partial_1 \\ 0 & 0 & \partial_3 & 2^{-1/2}\partial_2 & 2^{-1/2}\partial_1 & 0 \end{pmatrix}. \tag{1.23}$$

Regarding the column $u(x) = (u_1(x), u_2(x), u_3(x))^\top$ as the displacement vector of point x in a solid, we observe that

$$D(\nabla)u = (\epsilon_{11}(u), \epsilon_{22}(u), \epsilon_{33}(u), \sqrt{2}\epsilon_{23}(u), \sqrt{2}\epsilon_{31}(u), \sqrt{2}\epsilon_{12}(u))^\top \tag{1.24}$$

is the corresponding strain column with the Cartesian components $\epsilon_{jk}(u)$ of the strain tensor; the components are defined by the linearized Cauchy formulas

$$\epsilon_{jk}(u) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right), \quad j, k = 1, 2, 3.$$

Denoting by $\mathcal{A}(x)$ the real, symmetric and positive definite 6×6-matrix of elastic moduli of the waveguide solid material at point x, we remark that Hooke's law

$$\sigma(u; x) = \mathcal{A}(x)\epsilon(u; x) = \mathcal{A}(x)D(\nabla)u(x)$$

gives the stress column the same structure as in (1.24). Furthermore, $\mathcal{M}(x) = \rho(x)\mathbb{I}_d$, where $\rho > 0$ is the material density. We emphasize that the factors $2^{-1/2}$ and $\sqrt{2}$ in (1.23) and (1.24) make the intrinsic norms of the strain tensor $(\epsilon_{jk}(u))_{j,k=1,2,3}$ and the strain column $\epsilon(u)$ equal to each other. In the case of a homogeneous isotropic material the numerical matrix $\mathcal{A}$ has the block-diagonal form

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{(11)} & \mathbb{O}_{3\times 3} \\ \mathbb{O}_{3\times 3} & \mathcal{A}_{(22)} \end{pmatrix}, \ \mathcal{A}_{(11)} = \begin{pmatrix} 2\boldsymbol{\mu} + \boldsymbol{\lambda} & \boldsymbol{\lambda} & \boldsymbol{\lambda} \\ \boldsymbol{\lambda} & 2\boldsymbol{\mu} + \boldsymbol{\lambda} & \boldsymbol{\lambda} \\ \boldsymbol{\lambda} & \boldsymbol{\lambda} & 2\boldsymbol{\mu} + \boldsymbol{\lambda} \end{pmatrix},$$

$$\mathcal{A}_{(22)} = \begin{pmatrix} 2\boldsymbol{\mu} & 0 & 0 \\ 0 & 2\boldsymbol{\mu} & 0 \\ 0 & 0 & 2\boldsymbol{\mu} \end{pmatrix},$$

where $\mathbb{O}_{n\times m}$ is the null $n \times m$-matrix and $\boldsymbol{\lambda} \geqslant 0$ and $\boldsymbol{\mu} > 0$ are the Lamé constants.

The matrix (1.23) is algebraically complete with $\varrho_D = 2$ in (1.4), see [24, § 3.7.5]. [16, Example 1.12], and possesses the polynomial property (1.9), where the 6-dimensional subspace $\mathcal{P}$ consists of rigid motions $d(x)a$ with arbitrary $a \in \mathbb{R}^6$ (three translations plus three rotations) and the 3×6-matrix

$$d(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 2^{-1/2}x_3 & -2^{-1/2}x_2 \\ 0 & 1 & 0 & -2^{-1/2}x_3 & 0 & 2^{-1/2}x_1 \\ 0 & 0 & 1 & 2^{-1/2}x_2 & -2^{-1/2}x_1 & 0 \end{pmatrix}.$$

If $\mathcal{B} = \mathcal{N}$, see (1.7) and (1.10), then the boundary condition (1.12) corresponds to a traction-free surface and involves the traction vector

$$D(\nu(x))^\top \sigma(u;x) = D(\nu(x))^\top \mathcal{A}(x) D(\nabla) u(x) = 0, \quad x \in \partial\Pi. \quad (1.25)$$

The Dirichlet condition in (1.12) is related to $\mathcal{B}u = u$ and $\mathcal{T}u = -\mathcal{N}u$ in (1.10), meaning that the surface $\partial\Pi$ is clamped and the displacement vector u vanishes there. Also, choosing in (1.10) the orthogonal 3×3-matrix $\Theta = (\nu, s^1, s^2)$ which is composed from the unit normal vector ν and two tangential unit vectors s^1 and s^2 perpendicular to each other, leads to the linearized Signorini conditions

$$u_\nu = 0, \ \sigma_{s^i\nu}(u;x) = 0, \quad x \in \partial\Pi, \ i = 1, 2. \quad (1.26)$$

Here, $u_\nu = \nu^\top u$ and $\sigma_{s^i\nu}(u) = (s^i)^\top \sigma(u)$. Hence, in (1.26) there are the normal component of the displacement vector as well as two tangential components of the traction vector (1.25).

3° *Piezoelectricity system.* Let us denote the differential operator matrix (1.23) by $D^{\mathsf{M}}(\nabla)$ and let $D^{\mathsf{E}}(\nabla) = \nabla$, where the superscripts M and E stand for "mechanical" and "electric", respectively. Setting $d = 3$, $n = 4$ and $N = 9$, we introduce the matrices

$$D(\nabla) = \begin{pmatrix} D^{\mathsf{M}}(\nabla) & \mathbb{O}_{6\times 1} \\ \mathbb{O}_{3\times 3} & D^{\mathsf{E}}(\nabla) \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \mathcal{A}^{\mathsf{MM}} & -\mathcal{A}^{\mathsf{ME}} \\ \mathcal{A}^{\mathsf{EM}} & \mathcal{A}^{\mathsf{EE}} \end{pmatrix} \quad (1.27)$$

of sizes 9×4 and 9×9 respectively. Here, $\mathcal{A}^{\mathsf{MM}}$ and $\mathcal{A}^{\mathsf{EE}}$ are the elastic and dielectric matrices (tensors), which are real, symmetric, and positive definite, and of sizes 6×6 and 3×3 respectively, while no restriction is exposed on the piezoelectric matrix $\mathcal{A}^{\mathsf{ME}} = (\mathcal{A}^{\mathsf{EM}})^\top \neq \mathbb{O}_{6\times 3}$. Although the matrix $\mathcal{A}$ in (1.27) is not symmetric, the spectrum of the piezoelectricity system (1.11) with appropriate boundary conditions

(1.12) is contained in $\overline{\mathbb{R}_+}$. This is due to the specific structure of the diagonal matrix

$$\mathcal{M}(x) = \rho^{\mathsf{M}}(x)\,\mathrm{diag}\,\{1,1,1,0\}. \tag{1.28}$$

on the right of (1.11), where $\rho^{\mathsf{M}} > 0$ is the same mechanical material density as in the previous example of elasticity. Moreover, the vector function $u = (u_1^{\mathsf{M}}, u_2^{\mathsf{M}}, u_3^{\mathsf{M}}, u^{\mathsf{E}})^\top$ is composed of the displacement vector $u^{\mathsf{M}} = (u_1^{\mathsf{M}}, u_2^{\mathsf{M}}, u_3^{\mathsf{M}})^\top$ and the electric potential u^{E}. The Dirichlet condition for u^{E},

$$u^{\mathsf{E}}(x) = 0,\ x \in \partial\Pi, \tag{1.29}$$

means contact with an absolute conductor, and this case matches logically with the mechanically clamped surface

$$u^{\mathsf{M}}(x) = 0,\ x \in \partial\Pi. \tag{1.30}$$

The Neumann boundary condition

$$D(\nu(x))^\top \mathcal{A}(x) D(\nabla) u(x) = 0,\ x \in \partial\Pi, \tag{1.31}$$

for both elastic and electric fields appears, if the piezoelectric solid waveguide is in vacuum, which is an insulator and also corresponds to a traction-free boundary.

The operator (1.5) of the piezoelectric system (1.11) possesses the polynomial property (1.9) with $\varrho_D = 2$, see [16, Example 1.13], because, according to (1.27), the quadratic form

$$\begin{aligned} &a(u,u;G) = (\mathcal{A}D(\nabla)u, D(\nabla)u)_G \\ &= (\mathcal{A}^{\mathsf{MM}} D^{\mathsf{M}}(\nabla)u^{\mathsf{M}}, D^{\mathsf{M}}(\nabla)u^{\mathsf{M}})_G + (\mathcal{A}^{\mathsf{EE}} D^{\mathsf{E}}(\nabla)u^{\mathsf{E}}, D^{\mathsf{E}}(\nabla)u^{\mathsf{E}})_G \end{aligned}$$

is positive and degenerates on the subspace

$$\mathcal{P} = \{u : \ u^{\mathsf{M}} = d(x)a, a \in \mathbb{R}^6, u^{\mathsf{E}} = a_0 \in \mathbb{R}\}, \quad \dim \mathcal{P} = 7.$$

In the case (1.29), formula (1.28) shows that condition (1.15) is met and therefore all our considerations in Section 2 only require cosmetic modifications, so that they can be applied to the piezoelectricity system with either the Dirichlet conditions (1.30), (1.29), or mixed boundary conditions (1.25) for u^{M} and (1.29) for u^{E}. However, in the case of the Neumann conditions (1.31), implication (1.15) fails, since $u = (0,0,0,a_0)^\top$ satisfies the relations on the left-hand side of (1.15). In Section 3.5 we however will adapt the whole scheme to problem (1.11), (1.31) in piezoelectricity, too.

§2. Essential spectrum.

2.1. Limit problem and formula for $\wp_{es}$. The spectral problem

$$\mathcal{L}(x,\nabla)U(x) = \Lambda\mathcal{M}(x)U(x), \quad x \in \varpi, \tag{2.1}$$

$$\mathcal{B}(x,\nabla)U(x) = 0, \quad x \in \partial\varpi, \tag{2.2}$$

posed in the finite periodicity cell ϖ, has the variational formulation

$$a(U,V;\varpi) = \Lambda(\mathcal{M}U,V)_\varpi \quad \forall V \in \mathcal{H}(\varpi) \tag{2.3}$$

and its spectrum consists of the unbounded monotone sequence of nonnegative eigenvalues

$$0 \leqslant \Lambda_1 \leqslant \Lambda_2 \leqslant \dots \leqslant \Lambda_m \leqslant \dots \to +\infty, \tag{2.4}$$

where multiplicities are taken into account. Here, $\mathcal{H}(\varpi)$ is the subspace of vector functions $U \in H^1(\varpi)^n$ satisfying the stable boundary conditions in (2.2), see (1.14). The corresponding eigenvectors $U^1, U^2, \dots, U^m, \dots \in \mathcal{H}(\varpi)$ can be subject to the normalization and orthogonality conditions

$$a(U^p,U^q;\varpi) + (\mathcal{M}U^p,U^q)_\varpi = \delta_{p,q} \tag{2.5}$$

where $\delta_{p,q}$ is the Kronecker symbol.

These well-known facts can be proved by introducing in the same way as in (1.18), (1.19) a positive and self-adjoint operator in $\mathcal{H}(\varpi)$, which is also is compact due to the compact embedding $H^1(\varpi) \subset L^2(\varpi)$ in the bounded Lipschitz domain ϖ. According to [2, Theorems 10.1.5 and 10.2.2] the spectrum of this operator consists of the essential spectrum $\{M_\infty = 0\}$ and the infinitesimal positive sequence of eigenvalues $\{M_m\}_{m\in\mathbb{N}}$, which turns into (2.4) via the relation (1.20) between the spectral parameters M and Λ. We will also need the countable set

$$\Sigma = \{\Lambda_m : m \in \mathbb{N}\}, \tag{2.6}$$

which does not take into account the multiplicities of eigenvalues.

Let us assume that

($\bigstar$) the matrices $\mathcal{A}$ and Θ, which are 1-periodic in z, belong to the classes $C^{1,\delta}$ and $C^{0,\delta}$ for some $\delta \in (0,1)$ in the d-dimensional half-balls $\overline{\varpi} \cap \mathbb{B}^d_R(P^q) = \{x = (y,z) : |x - P^q| < R, (-1)^q(z-q) > 0\}$ and the $(d-1)$-dimensional balls $\mathbb{B}^{d-1}_R(0) \times \{q\} \subset \partial\varpi$, respectively.

Here, $q = 0, 1$, $R > 0$ is the same radius as in Section 1.1 and $P^q = (0, \dots, 0, q)$ are the opposite points on the surface $\partial\varpi$, around which the apertures have been created.

Assumption ($\bigstar$) allows us to apply the local elliptic estimates [1], [11, Ch. 2] and to conclude the inequality

$$\begin{aligned} &\|U^m; H^2(\varpi \cap \mathbb{B}^d_{3R/4}(P^q)\| \\ &\leqslant c(\Lambda_m \|U^m; L^2(\varpi \cap \mathbb{B}^d_R(P^q)\| + \|U^m; L^2(\varpi \cap \mathbb{B}^d_R(P^q)\|) \\ &\leqslant c(\Lambda_m + 1)\|U^m; L^2(\varpi \cap \mathbb{B}^d_R(P^q)\| \leqslant C_m. \end{aligned} \tag{2.7}$$

The last estimate is due to to the normalization condition (2.5) and the Korn inequality (1.16).

In the next sections we will prove

Theorem 2.1. *Under condition* ($\bigstar$)*, the essential spectrum* $\wp_{es}$ *of the problem* (1.13) *(or problem* (1.11)–(1.12)*) coincides with set* (2.6)*.*

2.2. Singular Weyl sequence. For any $\Lambda_m \in \Sigma$, we will construct a sequence $\{u^m_{(j)}\}_{j\in\mathbb{N}} \subset \mathcal{H}(\Pi)$ such that

1°. $\|u^m_{(j)}; \mathcal{H}(\Pi)\| = 1$,

2°. $u^m_{(j)} \rightharpoonup 0$ weakly in $\mathcal{H}(\Pi)$,

3°. $\|Mu^m_{(j)} + (1+\Lambda_m)^{-1}u^m_{(j)}; \mathcal{H}(\Pi)\| \to 0$.

These three properties make $\{u^m_{(j)}\}_{j\in\mathbb{N}}$ into a singular sequence for the operator M at the point $\mu_m = (1+\Lambda_m)^{-1}$ and thus the Weyl criterion, see, e.g., [2, Thm 9.1.2] and [28, Thm. VII.12] implies that $\mu_m \in \wp_{es}(M)$. By the definition of $\wp$, see (1.21), this means that $\Lambda_m \in \wp_{es}$.

For each $j \in \mathbb{Z}$, we set

$$u^m_{(j)} = \|\mathcal{U}^m_{(j)}; \mathcal{H}(\Pi)\|^{-1} \mathcal{U}^m_{(j)}, \quad \mathcal{U}^m_{(j)}(x) = \mathcal{X}_j(x) U^m(y, z-j), \ x \in \varpi_j, \tag{2.8}$$

where U^m is an eigenvector of the problem (2.3) and $\mathcal{X}_j \in C^\infty(\overline{\varpi_j})$ is a cut-off function such that

$$\begin{aligned}
&\mathcal{X}_j(x) = 1 \text{ for } x \in \varpi_j \setminus (\mathbb{B}^d_{3R\alpha_j/4}(P^j) \cup \mathbb{B}^d_{3R\alpha_j/4}(P^{j+1})), \\
&\mathcal{X}_j(x) = 0 \text{ for } x \in \varpi_j \cap (\mathbb{B}^d_{R\alpha_j/2}(P^j) \cup \mathbb{B}^d_{R\alpha_j/2}(P^{j+1})), \\
&0 \leqslant \mathcal{X}_j \leqslant 1, \\
&|\nabla^k \mathcal{X}_j(x)| \leqslant c_k (\min\{\alpha_j, \alpha_{j+1}\})^{-k}, \ k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}.
\end{aligned} \tag{2.9}$$

Here, $\mathcal{U}^m_{(j)}$ can be extended as null over $\Pi \setminus \varpi_j$, because $\mathcal{X}_j = 0$ on the apertures $\overline{\omega_j} \cup \overline{\omega_{j+1}} = \Pi \cap \partial\varpi_j$.

Obviously, $u^m_{(j)}$ satisfies 1°. For any test function v belonging to the space $C^\infty_c(\overline{\Pi})^n$ of infinitely differentiable, compactly supported vector functions, we have

$$\langle u^m_{(j)}, v \rangle = 0 \quad \text{for all large } j \in \mathbb{N}.$$

Since $C^\infty_c(\overline{\Pi})^n$ is dense in the space $\mathcal{H}(\Pi)$, condition 2° holds as well. Moreover, in view of (1.17), (1.18) and (2.4), we obtain

$$\begin{aligned}
&\|M u^m_{(j)} - (1+\Lambda_m)^{-1} u^m_{(j)}; \mathcal{H}(\Pi)\| \\
&= \sup |\langle M u^m_{(j)} - (1+\Lambda_m)^{-1} u^m_{(j)}, v \rangle| \\
&= (1+\Lambda_m)^{-1} \|\mathcal{U}^m_{(j)}; \mathcal{H}(\Pi)\|^{-1} \\
&\qquad \times \sup \left| a(\mathcal{U}^m_{(j)}, v; \Pi) - (1+\Lambda_m)(\mathcal{M}\mathcal{U}^m_{(j)}, v)_\Pi \right|.
\end{aligned} \tag{2.10}$$

Here, according to the definition of the norm of a Hilbert space, supremum is calculated over all $v \in \mathcal{H}(\Pi)$ such that $\|v; \mathcal{H}(\Pi)\| = 1$.

The last expression inside the modulus sign in (2.10) is equal to

$$a(U^m, \mathcal{X}_j v; \varpi_j) - (1+\Lambda_m)(\mathcal{M}U^m, \mathcal{X}_j v)_{\varpi_j}$$

$$+ (\mathcal{A}(D(\nabla)\mathcal{X}_j)U^m, D(\nabla)v)_{\varpi_j} - (\mathcal{A}D(\nabla)U^m, (D(\nabla)\mathcal{X}_j)v)_{\varpi_j} \quad (2.11)$$

The first couple of terms in (2.11) cancel each other owing to the integral identity (2.3) with $U = U^m$, $\Lambda = \Lambda_m$ and $V = \mathcal{X}_j v$. To estimate the last two terms, we need the following Hardy type inequalities in dimensions $d \geqslant 3$; see Section 3.2 for $d = 2$.

Lemma 2.2. *Given an arbitrary number $\delta > 0$, let $\kappa = 1$ for $d > 4$, $\kappa = 1 - \delta$ for $d = 4$ and $\kappa = \frac{1}{2} - \delta$ for $d = 3$. Then,*

$$\|\mathbf{r}^{-1}U; L^2(\varpi)\| \leqslant c\|U; H^1(\varpi)\|, \quad (2.12)$$
$$\|\mathbf{r}^{-1-\kappa}U; L^2(\varpi \cap \mathbb{B}^d_R(P^q))\| \leqslant c\|U; H^2(\varpi \cap \mathbb{B}^d_R(P^q))\|, \quad (2.13)$$

where $\mathbf{r} = \mathrm{dist}(x, P^0 \cup P^1)$ and the constant $c = c_\delta$ does not depend on U.

Proof. Let us recall the standard one-dimensional Hardy inequality

$$\int_0^{+\infty} \mathbf{r}^{-1+t}|W(\mathbf{r})|^2 d\mathbf{r} \leqslant \frac{4}{t^2}\int_0^{+\infty} \mathbf{r}^{1+t}\left|\frac{dW}{d\mathbf{r}}(\mathbf{r})\right|^2 d\mathbf{r} \quad (2.14)$$

where $t > 0$ and $W \in C^1_c[0, +\infty)$. To prove (2.12), we put $t = d-2 > 0$ and $W(\mathbf{r}) = U(x)\chi(\mathbf{r})$, where χ is a smooth cut-off function, $\chi(\mathbf{r}) = 1$ for $0 < \mathbf{r} < 3R/4$ and $\chi(\mathbf{r}) = 0$ for $\mathbf{r} > R$ (this χ is needed to satisfy the condition $W(\mathbf{r}) = 0$ for $\mathbf{r} > \mathbf{r}_W$ in (2.14)). Integrating the obtained inequality in the angular variables ϕ over the unit semi-sphere and using the relation $dx = \mathbf{r}^{d-1}d\mathbf{r}ds_\phi$ we conclude that

$$\begin{aligned}
&\|\mathbf{r}^{-1}U; L^2(\varpi \cap \mathbb{B}^d_{3R/4}(P^q))\|^2 \leqslant \|\mathbf{r}^{-1}\chi U; L^2(\varpi \cap \mathbb{B}^d_R(P^q))\|^2 \\
&\leqslant \frac{4}{(d-2)^2}\left\|\frac{\partial}{\partial \mathbf{r}}(\chi U); L^2(\varpi \cap \mathbb{B}^d_R(P^q))\right\|^2 \\
&\leqslant c(\|\nabla U; L^2(\varpi \cap \mathbb{B}^d_R(P^q))\|^2 + \|U; L^2(\varpi \cap (\mathbb{B}^d_R(P^q) \setminus \mathbb{B}^d_{3R/4}(P^q))\|^2) \\
&\leqslant c\|U; H^1(\varpi \cap \mathbb{B}^d_R(P^q))\|^2. \quad (2.15)
\end{aligned}$$

Hence, (2.12) is indeed true.

To verify (2.13) it suffices to prove the weighted inequality

$$\begin{aligned} &\|\mathbf{r}^{-1-\kappa}U; L^2(\varpi\cap\mathbb{B}^d_{3R/4}(P^q))\|^2 \\ \leqslant\ & c(\|\mathbf{r}^{-\kappa}\nabla U; L^2(\varpi\cap\mathbb{B}^d_R(P^q))\|^2 \\ &+\|U; L^2(\varpi\cap(\mathbb{B}^d_R(P^q)\setminus\mathbb{B}^d_{3R/4}(P^q)))\|^2). \end{aligned} \tag{2.16}$$

We may still use the one-dimensional Hardy inequality (2.14) in the same way as in (2.15), although we now take $t=d-2-2\kappa>0$; this restriction just implies the conditions on κ formulated in the lemma. To treat the weight $\mathbf{r}^{-\kappa}$ multiplying ∇U in (2.16) we recall that $\kappa\leqslant 1$ and, hence, $\|\mathbf{r}^{-\kappa}\nabla U; L^2(\varpi\cap\mathbb{B}^d_R(P^q))\|\leqslant\|\mathbf{r}^{-1}\nabla U; L^2(\varpi\cap\mathbb{B}^d_R(P^q))\|\leqslant c\|\nabla U; H^1(\varpi\cap\mathbb{B}^d_R(P^q))\|$, by (2.12). $\boxtimes$

Since the matrix function $D(\nabla)\mathcal{X}_j$ is supported in the union $\mathbb{A}_j$ of the semi-annuli $\mathbb{A}_{j,0}$ and $\mathbb{A}_{j+1,1}$, where

$$\mathbb{A}_{j,q}=\{x\in\overline{\mathbb{B}^d_{3R\alpha_j/4}(P^j)}\setminus\mathbb{B}^d_{R\alpha_j/2}(P^j)\,:\,(-1)^q z>j\}\ ,\ q=0,1, \tag{2.17}$$

we have $|D(\nabla)\mathcal{X}_j(x)|\leqslant c\alpha^{-1}_{j+q}$ due to (2.9) and $\mathbf{r}_j(x)=\min\{r_j,r_{j+1}\}\leqslant R\min\{\alpha_j,\alpha_{j+1}\}$ for $x\in\mathbb{A}^d_{R\alpha_j}(P^j)\cup\mathbb{A}^d_{R\alpha_{j+1}}(P^{j+1})$; here $r_j=|x-P^j|$. We now obtain that

$$\begin{aligned} &|(\mathcal{A}(D(\nabla)\mathcal{X}_j)U^m, D(\nabla)v)_{\varpi_j}| \\ \leqslant\ & c\sum_{q=0,1}\alpha^{-1}_{j+q}\|U^m; L^2(\mathbb{A}_{j+q,q})\|\,\|D(\nabla)v; L^2(\varpi_j)\| \\ \leqslant\ & c(\alpha_j^{-1}\alpha_j^{1+\kappa}+\alpha^{-1}_{j+1}\alpha^{1+\kappa}_{j+1})\|U^m; H^2(\varpi\cap(\mathbb{B}^d_{3R/4}(P^0)\cup\mathbb{B}^d_{3R/4}(P^1))\| \\ \leqslant\ & c_m(\min\{\alpha_j,\alpha_{j+1}\})^\kappa, |(\mathcal{A}D(\nabla)U^m,(D(\nabla)\mathcal{X}_j)v)_{\varpi_j}| \\ \leqslant\ & c\sum_{q=0,1}\|U^m; L^2(\mathbb{A}_{j+q,q})\|\,\alpha^{-1}_{j+q}\|v; L^2(\mathbb{A}_{j+q,q})\| \\ \leqslant\ & c_m\min\{\alpha_j,\alpha_{j+1}\}. \end{aligned} \tag{2.18}$$

Here, we have applied inequalities (2.7) and (2.12), (2.13). Hence, expression (2.13) tends to zero as $j\to+\infty$. In a similar way, it is straightforward to verify that

$$\|\mathcal{U}^m_{(j)};\mathcal{H}(\Pi)\|^2\ \to\ a(U^m,U^m;\varpi)+(\mathcal{M}U^m,U^m)_\varpi=\Lambda_m+1 \text{ as } j\to+\infty.$$

We have checked up condition 3°, and therefore $\mu_m \in \wp_{es}(M)$ as well as $\Lambda_m \in \wp_{es}$.

2.3. Parametrix. We are now going to present the most complicated part of the proof of Theorem 2.1. Let us assume that $\Lambda_m < \Lambda_{m+1}$ and

$$\lambda \in (\Lambda_m, \Lambda_{m+1}). \tag{2.19}$$

We aim to construct a continuous operator

$$R(\lambda) : \mathcal{H}(\Pi) \to \mathcal{H}(\Pi) \tag{2.20}$$

such that the mapping

$$(M - (1+\lambda)^{-1})R(\lambda) - 1 : \mathcal{H}(\Pi) \to \mathcal{H}(\Pi) \tag{2.21}$$

is compact. Operator (2.20) is called a *(right) parametrix* and, as known, properties of (2.19) and (2.20) assure that the operator $M - (1+\lambda)^{-1}$ in $\mathcal{H}(\Pi)$ is Fredholm and, therefore, the point $(1+\lambda)^{-1}$ belongs to the regularity field of M. In view of definitions (1.17), (1.18) and (1.20) we observe that either λ is an isolated normal eigenvalue of problem (1.11)–(1.12), see also (1.13), or for any $f \in \mathcal{H}(\Pi)^*$, the inhomogeneous problem

$$a(u, v; \Pi) - \lambda(\mathcal{M}u, v)_\Pi = f(v) \quad \forall v \in \mathcal{H}(\Pi) \tag{2.22}$$

has a unique solution $u \in \mathcal{H}(\Pi)$ and there holds the estimate

$$\|u; \mathcal{H}(\Pi)\| \leqslant c(\lambda)\|f; \mathcal{H}(\Pi)^*\|$$

with a constant $c(\lambda)$ which depends on λ but not on f.

We now construct the parametrix $\mathcal{R}(\lambda)$ for the variational problem in Π. To determine the vector $\mathcal{R}(\lambda)f \in \mathcal{H}(\Pi)$, we first proceed with the inhomogeneous limit problem in the cell,

$$a(U_{(j)}, V_{(j)}; \varpi_j) - \lambda(\mathcal{M}U_{(j)}, V_{(j)})_{\varpi_j} = f^j(V_{(j)}) \quad \forall V_{(j)} \in \mathcal{H}(\varpi_j), \tag{2.23}$$

where $j \in \mathbb{Z}$ and $f^j(V_{(j)}) = f(\mathcal{X}_j V_{(j)})$ with the cut-off function $\mathcal{X}_j$, (2.9). We emphasize that

$$\sum_{j\in\mathbb{Z}} |f^j(V_{(j)})|^2 \leqslant c\|f; \mathcal{H}(\Pi)^*\|^2 \sum_{j\in\mathbb{Z}} \|\mathcal{X}_j V_{(j)}; \mathcal{H}(\varpi_j)\|^2$$

$$\leqslant c\|f;\mathcal{H}(\Pi)^*\|^2\sum_{j\in\mathbb{Z}}(\|\mathcal{X}_j\nabla V_{(j)};L^2(\varpi_j)\|^2+\|(\nabla\mathcal{X}_j)V_{(j)};L^2(\varpi_j)\|^2)$$
$$\leqslant c\|f;\mathcal{H}(\Pi)^*\|^2\sum_{j\in\mathbb{Z}}\|V_{(j)};\mathcal{H}(\varpi_j)\|^2, \tag{2.24}$$

where we used the remarks above the calculation (2.18) and the following consequence of the Hardy inequality (2.12) ,

$$\|(\nabla\mathcal{X}_j)V_{(j)};L^2(\varpi_j)\|^2\leqslant c\|\mathbf{r}_j^{-1}V_{(j)};L^2(\varpi_j)\|^2\leqslant c\|V_{(j)};\mathcal{H}(\varpi_j)\|^2. \tag{2.25}$$

Since λ is not an eigenvalue of the spectrum (2.4) of the problem (2.3), there exists a unique solution $U_{(j)}\in\mathcal{H}(\varpi_j)$ of the problem (2.11), which satisfies the estimate

$$\|U_{(j)};\mathcal{H}(\varpi_j)\|^2\leqslant c(\lambda)\|f^j;\mathcal{H}(\varpi_j)^*\|^2; \tag{2.26}$$

notice that the constant $c(\lambda)$ is independent of $j\in\mathbb{Z}$ and f^j due to the periodicity of coefficients in differential operators.

We now set

$$\mathcal{R}(\lambda)f=\sum_{j\in\mathbb{Z}}\mathcal{X}_jU_{(j)}+\mathcal{R}^\sharp(\lambda)f=:\mathcal{R}^0(\lambda)f+\mathcal{R}^\sharp(\lambda)f, \tag{2.27}$$

where the second term is still to be determined and the first term can be estimated as follows:

$$\Big\|\sum_{j\in\mathbb{Z}}\mathcal{X}_jU_{(j)};\mathcal{H}(\Pi)\Big\|^2=\sum_{j\in\mathbb{Z}}\|\mathcal{X}_jU_{(j)};\mathcal{H}(\varpi_j)\|^2$$
$$\leqslant c\sum_{j\in\mathbb{Z}}(\|U_{(j)};\mathcal{H}(\varpi_j)\|^2+\|(\nabla\mathcal{X}_j)U_{(j)};\mathcal{H}(\varpi_j)\|^2)$$
$$\leqslant c\sum_{j\in\mathbb{Z}}\|U_{(j)};\mathcal{H}(\varpi_j)\|^2\leqslant c_\lambda\sum_{j\in\mathbb{Z}}\|f^j;\mathcal{H}(\varpi_j)^*\|^2. \tag{2.28}$$

Here, we took into account that the supports of $\mathcal{X}_j$ and $\mathcal{X}_k$ are disjoint for $j\neq k$, then applied the relation (2.25) with $V_{(j)}=U_{(j)}$ as well as the inequalities (2.26) and (2.24), and finally summed with respect to $j\in\mathbb{Z}$. This yields the upper bound $C_\lambda\|f;\mathcal{H}(\Pi)^*\|^2$ for the expression (2.28).

Let us now describe discrepancies left by $\mathcal{R}^0(\lambda)f$ in problem (2.22). To this end, we need the cut-off function

$$\chi_j(x) = \begin{cases} 1 - \mathcal{X}_j(x), & x \in \varpi_j \cap \overline{\mathbb{B}^d_{3R\alpha_j/4}(P^j)}, \\ 1 - \mathcal{X}_{j-1}(x), & x \in \varpi_{j-1} \cap \overline{\mathbb{B}^d_{3R\alpha_j/4}(P^j)}, \end{cases} \tag{2.29}$$
$$\text{equivalently, } \chi_j(x) = \chi(\alpha_j^{-1}|x - P^j|).$$

To have the latter identity with some standard cut-off function $\chi \in C_c^\infty(\mathbb{R})$ we assume that the choice in (2.9) has been done in a proper, translation invariant, way. Then, using the definition $f^j(v) = f(\mathcal{X}_j v)$ of the right-hand side of (2.23) and formulas (2.9), (2.29), we obtain that

$$\begin{aligned} & a(\mathcal{R}^0(\lambda)f, v; \Pi) - \lambda(\mathcal{M}\mathcal{R}^0(\lambda)f, v)_\Pi - f(v) \qquad (2.30) \\ = & \sum_{j\in\mathbb{Z}} ((\mathcal{A}[D(\nabla), \mathcal{X}_j]U^m, D(\nabla)v)_{\varpi_j} - (\mathcal{A}D(\nabla)U^m, [D(\nabla), \mathcal{X}_j]v)_{\varpi_j}) \\ + & \sum_{j\in\mathbb{Z}} \big((\mathcal{A}D(\nabla)U^m, D(\nabla)(\mathcal{X}_j v))_{\varpi_j} - \lambda(\mathcal{M}U^m, \mathcal{X}_j v)_{\varpi_j}\big) - f(v) \\ = & \sum_{j\in\mathbb{Z}} (\mathcal{F}^{j0}(v) + \mathcal{F}^{j1}(v)) + \sum_{j\in\mathbb{Z}} f(\mathcal{X}_j v) - f(v) \\ = & \sum_{j\in\mathbb{Z}} (\mathcal{F}^{j0}(v) + \mathcal{F}^{j1}(v) - f(\chi_j v)). \end{aligned}$$

Here, $[D(\nabla), \chi_j]$ is the commutator of the differential matrix operator $D(\nabla)$ and the function χ_j, which is an operator of multiplication with the matrix function $D(\nabla)\chi_j$. Furthermore, $\mathcal{F}^{j0}$ and $\mathcal{F}^{j1}$ are functionals supported in the semi-annulus (2.17). In this way, discrepancy (2.30) can be indeed represented as the sum of the functionals

$$v \mapsto \mathcal{F}^j(v) = \mathcal{F}^{j0}(v) + \mathcal{F}^{j1}(v) - f(\chi_j v) \tag{2.31}$$

with support in the closed ball $\overline{\mathbb{B}^d_{3R\alpha_j/4}(P^j)}$. We define the sets

$$\begin{aligned} & \Xi = \mathbb{R}^d \setminus \Omega, \Omega = \{(y, z) : \, y \notin \overline{\omega}, z = 0\}, \\ & \Omega_j = \{(y, z) : \, \xi := \alpha_j^{-1}(y, z - j) \in \Omega\}. \end{aligned} \tag{2.32}$$

Then, owing to estimates (2.24) and (2.26), we have

$$\sum_{j\in\mathbb{Z}} \|\mathcal{F}^j; \mathcal{H}(\mathbb{R}^d \setminus \Omega_j)^*\|^2 \leqslant c\|f; \mathcal{H}(\Pi)^*\|^2. \tag{2.33}$$

2.4. Boundary layer phenomenon. In order to compensate the functionals $\mathcal{F}^j$ we employ a general asymptotic procedure of [12, Ch.4] and construct boundary layers around apertures (1.1). We fix a point $P^j = (0, \dots, 0, j)$ and make the coordinate dilation

$$x \mapsto \xi^j = (\eta^j, \zeta^j) := (\alpha_j^{-1} y, \alpha_j^{-1}(z - j)). \tag{2.34}$$

In view of (2.32) and (1.1), letting $\alpha_j \to +0$ formally transforms the union $\varpi_j \cup \varpi_{j+1} \cup \omega_j = \{x \in \Pi : j - 1 < z < j + 1\}$ into the space $\mathbb{R}^d$ with an infinite cut, namely the domain Ξ in (2.32); this is the union of two half-spaces connected through the aperture ω in the wall $\{\xi = (\eta, \zeta) : \zeta = 0\}$. Furthermore, we freeze coefficients of the differential operators $\mathcal{L}$ and $\mathcal{B}$ at the points P^0 and P^1, recall the smoothness and periodicity assumptions ($\bigstar$), and thus obtain the differential operator matrices with constant coefficients

$$\mathcal{L}^0(\nabla) = \overline{D(-\nabla)}^\top \mathcal{A}(P^q) D(\nabla), \quad q = 0, 1,$$

and $\mathcal{B}^q(\nabla)$, $q = 0, 1$. Notice that $\mathcal{L}^0(\nabla)$, the principal part of $\mathcal{L}(x, \nabla)$, consists of a matrix of homogeneous second-order differential operators,

$$\mathcal{L}^0(\nabla_x) = \alpha_j^{-2} \mathcal{L}^0(\nabla_{\xi^j}). \tag{2.35}$$

Recalling our choice of the boundary condition operator $\mathcal{B}$, we see that $\mathcal{B}^+ := \mathcal{B}^0$ and $\mathcal{B}^- := \mathcal{B}^1$ may differ from each other.

Performing the coordinate change (2.34) and setting $\alpha_j = 0$ turn the boundary-value problem (1.11), (1.12) locally into

$$\mathcal{L}^0(\nabla_\xi) W(\xi) = g(\xi), \quad \xi \in \Xi, \tag{2.36}$$

$$\mathcal{B}^\pm(\nabla_\xi) W(\eta, \pm 0) = 0, \quad \eta \in \Omega := \mathbb{R}^{d-1} \setminus \overline{\varpi}. \tag{2.37}$$

We remark that, according to (2.35),

$$\mathcal{L}^0(\nabla_x) - \lambda \mathcal{M}^0 = \alpha_j^{-2} \mathcal{L}^0(\nabla_\xi) - \lambda \mathcal{M}^0, \tag{2.38}$$

where the last term is of higher order in α_j and thus has been neglected when composing the equation (2.36). The differential operators have been replaced by their principal parts for the same reason; see also Corollary 2.4. The boundary conditions (2.37) are posed on the two surfaces $\Omega^\pm$ of Ω. The variational formulation of problem (2.36), (2.37) reads as

$$(\mathcal{A}^0 D(\nabla_\xi)W, D(\nabla_\xi)V)_\Xi = G(V), \tag{2.39}$$

where $G(V) = (g, V)_\Xi$ and, at first, the test functions $V = (V_1, \dots, V_n)^\top$ are compactly supported and infinitely differentiable in Ξ up to the surfaces $\overline{\Omega^\pm} = (\mathbb{R}^{d-1} \setminus \overline{\omega}) \times \{\pm 0\}$ of the wall, and satisfy the stable boundary conditions (2.37), cf. (1.14). By a completion argument, we may extend the integral identity (2.39) to the space $\mathcal{V}(\Xi)$, which consists of vector functions $V \in H^1_{\mathrm{loc}}(\Xi \cup \Omega^\pm)^n$ decaying at infinity (see below), satisfying the above mentioned stable boundary conditions on $\Omega^\pm$, and having the finite Dirichlet norm

$$\|V; \mathcal{V}(\Xi)\| = (\|\nabla_\xi V; L^2(\Xi)\|^2 + \|V; L^2(\Xi \cap \mathbb{B}^d_R(0))\|^2)^{1/2} \tag{2.40}$$

where $R > 0$ is a radius such that $\overline{\omega} \times \{0\} \subset \mathbb{B}^d_R(0)$. The Hardy type inequality

$$\|(1 + |\xi|)^{-1} V; L^2(\Xi)\| \leqslant c_d \|\nabla_\xi V; L^2(\Xi)\|, \quad d > 2$$

can be easily proved in the same way as in Lemma 2.2, by changing $\mathbf{r} \mapsto \rho = |\xi|$, and this implies that the weighted Kondratiev norm [7]

$$\Big(\int_\Xi |\nabla_\xi V(\xi)|^2 d\xi + \int_\Xi (1 + |\xi|^2)^{-1} |V(\xi)|^2 d\xi \Big)^{1/2} \tag{2.41}$$

is equivalent to the norm (2.40) in the space $\mathcal{V}(\Xi)$ The above mentioned decay properties of V emerge from the last integral.

We look for a solution W of the problem (2.39) in the space $\mathcal{V}(\Xi)$ and also assume that G belongs to the dual space $\mathcal{V}(\Xi)^*$. All terms in (2.39) are now properly defined.

The Kondratiev theory [7] (see also papers [27, 13, 14, 16] and, e.g., monographs [21, 8]) can be applied to problem (2.36), (2.37). It is straightforward to see that all results of the theory, which usually deal with the classical formulation of elliptic boundary-value problems, can be passed over to integral identities of type (2.39), cf. the review paper [17]. Repeating an argument in [16, § 5], we observe that, first, the operator

$$\mathbf{A} : \mathcal{V}(\Xi) \ \rightarrow \ \mathcal{V}(\Xi)^* \tag{2.42}$$

of problem (2.39) is self-adjoint, and, second, any solution $W \in \mathcal{V}(\Xi)$ of the homogeneousproblem (2.39) ($G = 0$) must be a polynomial $W \in \mathcal{V}(\Xi)$, by the polynomial property (1.9). However, due to the convergence of integrals in (2.41) we must have $W = 0$; this argument requires d to be at least 3, cf., Section 3.2.. By the Kondratiev theory, these facts lead to the following assertion.

Proposition 2.3. *Mapping* (2.42) *is an isomorphism, that is, for any* $G \in \mathcal{V}(\Xi)^*$*, problem* (2.39) *has a unique solution* $W \in \mathcal{V}(\Xi)$ *and the following estimate is valid:*

$$\|W; \mathcal{V}(\Xi)\| \leqslant c\|G; \mathcal{V}(\Xi)^*\|. \tag{2.43}$$

We will actually need the following, a bit more general version of the proposition.

Corollary 2.4. *Proposition* 2.3 *remains true, with the constant* c *of* (2.43) *independent of* j*, for the problem*

$$(\mathcal{A}^{0j} D(\nabla_\xi)W, D(\nabla_\xi)V)_\Xi = G(V) \quad \forall V \in \mathcal{V}(\Xi), \tag{2.44}$$

which is obtained by perturbing the constant matrix $\mathcal{A}^0$ *of* (2.39) *on a compact set as* $\mathcal{A}^{0j}(\xi) = \mathcal{A}^0 + \chi^0(\xi)(\mathcal{A}(\alpha_j\xi) - \mathcal{A}^0)$,*where*

$$\chi^0 \in C^\infty(\mathbb{R}^d), \quad \chi^0(\xi) = 1 \text{ for } \rho < R, \quad \chi^0(\xi) = 0 \text{ for } \rho < 5R/4. \tag{2.45}$$

Proof. The assertion follows, since the operator (1.5) with the matrix $\mathcal{A}^{0j}$ in place of $\mathcal{A}$ is still formally positive; the independence of c on j is a consequence of the assumption (★) which makes the perturbation $\mathcal{A}^0 - \mathcal{A}^{0j}$ small when $j \to \pm\infty$. ⊠

The properties of the solutions of the problem (2.39) can be improved by using again the Kondratiev theory, when the functional $G \in \mathcal{V}(\Xi)^*$ has a compact support, i.e.

$$G(V) = 0 \text{ for all } V \in \mathcal{V}(\Xi) \text{ such that } V(\xi) = 0 \text{ for } \xi \in \Xi \cap \mathbb{B}^d_{3R/4}(0). \tag{2.46}$$

Proposition 2.5. *If* (2.46) *holds for* G*, then the solution* $W \in \mathcal{V}(\Xi)$ *of the problem* (2.39) *is smooth outside a larger ball* $\mathbb{B}^d_{\mathbf{R}}(0)$, $\mathbf{R} > 3R/4$, *and has the bound*

$$\int\limits_{\Xi\setminus\mathbb{B}^d_R(0)} (1+\rho^2)^{\varkappa}\big((1+\rho^2)|\nabla^2_\xi W(\xi)|^2 + |\nabla_\xi W(\xi)|^2$$
$$+((1+\rho^2)^{-1}|W(\xi)|^2\big)\, d\xi \leqslant c_\varkappa(\mathbf{R})\|G;\mathcal{V}(\Xi)^*\| \tag{2.47}$$

where $\rho = |\xi|$ *and the constant* $c_\varkappa(\mathbf{R})$ *depends on the fixed radius* $\mathbf{R}$ *and the choice of the exponent* $\varkappa < (d-2)/2$*. In particular,* $\|W;\mathcal{V}^1_\varkappa(\Xi)\| \leqslant c_\varkappa(\mathbf{R})\|G;\mathcal{V}(\Xi)^*\|$.

2.5. Parametrix (continuation). The discrepancies (2.31), which were left to the problem (2.22) by the first term $\mathcal{R}^0(\lambda)f$ in (2.27), are compensated by the sum

$$\mathcal{R}^\sharp(\lambda)f(x) = \sum_{j\in\mathbb{Z}} \alpha_j^{-d} X_j(x) W^j(\xi^j), \tag{2.48}$$

where X_j is a cut-off function

$$X_j(x) = 0 \text{ outside } \mathbb{B}^d_R(P^j) \text{ and } X_j(x) = 1 \text{ inside } \mathbb{B}^d_{3R/4}(P^j). \tag{2.49}$$

The boundary layer terms W^j are determined as solutions of the problem (2.44), where the right-hand sides are taken from (2.33) and written in the stretched coordinates (2.34),

$$G(V) = \mathcal{G}^j(V), \quad \mathcal{G}^j(V) = \alpha_j^2 \mathcal{F}^j(V^j), \quad V^j = V(\alpha_j^{-1}y, \alpha_j^{-1}(z-j)).$$

The factor α_j^2 comes from (2.35), (2.38) and (2.48). Furthermore, we obtain

$$|\mathcal{G}^j(V)|^2 = \alpha_j^4 |\mathcal{F}^j(V^j)|^2$$

$$
\begin{aligned}
&\leqslant c\alpha_j^4 \|\mathcal{F}^j; \mathcal{H}(\mathbb{R}^d \setminus \Omega_j)\|^2 (\|\nabla_x V^j; L^2(\Pi \cap \mathbb{B}^d_{R\alpha_j})\|^2 \\
&\quad + \|V^j; L^2(\Pi \cap \mathbb{B}^d_{R\alpha_j})\|^2) \\
&\leqslant c\alpha_j^4 \|\mathcal{F}^j; \mathcal{H}(\mathbb{R}^d \setminus \Omega_j)\|^2 \alpha_j^{d-2} (\|\nabla_\xi V; L^2(\Xi \cap \mathbb{B}^d_R)\|^2 \\
&\quad + \alpha_j^2 \|V; L^2(\Xi \cap \mathbb{B}^d_R)\|^2) \\
&\leqslant c\alpha_j^{d+2} \|\mathcal{F}^j; \mathcal{V}(\mathbb{R}^d \setminus \Omega_j)\|^2 \|V; \mathcal{V}(\Xi)\|^2 \qquad (2.50)
\end{aligned}
$$

so that

$$
\|\mathcal{G}^j; \mathcal{V}(\Xi)^*\|^2 \leqslant c\alpha_j^{d+2} \|\mathcal{F}^j; \mathcal{V}(\mathbb{R}^d \setminus \Omega_j)\|^2. \qquad (2.51)
$$

According to estimate (2.43) in Proposition 2.3,

$$
\|W^j; \mathcal{V}(\Xi)\|^2 \leqslant c\alpha_j^{d+2} \|\mathcal{F}^j; \mathcal{V}(\mathbb{R}^d \setminus \Omega_j)\|^2. \qquad (2.52)
$$

Note that

$$
\begin{aligned}
&\|X_j W^j; H^1(\Pi \cap \mathbb{B}^d_R(P^j))\|^2 \\
&\leqslant c(\|\nabla_x W^j; L^2(\Pi \cap \mathbb{B}^d_R(P^j))\|^2 + \|W^j; L^2(\Pi \cap \mathbb{B}^d_R(P^j))\|^2) \\
&\leqslant c\alpha_j^{d-2} (\|\nabla_\xi W^j; L^2(\Xi \cap \mathbb{B}^d_{R/\alpha_j}(0))\|^2 + \alpha_j^2 \|W^j; L^2(\Xi \cap \mathbb{B}^d_{R/\alpha_j}(0))\|^2) \\
&\leqslant c\alpha_j^{d-2} (\|\nabla_\xi W^j; L^2(\Xi \cap \mathbb{B}^d_{R/\alpha_j}(0))\|^2 \\
&\quad + \|(1+\rho)^{-1} W^j; L^2(\Xi \cap \mathbb{B}^d_{R/\alpha_j}(0))\|^2) \\
&\leqslant c\alpha_j^{d-2} \|W^j; \mathcal{V}(\Xi)\|^2; \qquad (2.53)
\end{aligned}
$$

here we took into account the equivalence of the norms (2.40) and (2.41). From (2.52), (2.53) and (2.33) we derive that the expression (2.48) has the bound

$$
\begin{aligned}
&\|\mathcal{R}^\sharp(\lambda) f; \mathcal{V}(\Xi)\|^2 \leqslant c \sum_{j\in\mathbb{Z}} \alpha_j^{-2d} \|X_j W^j; H^1(\Pi \cap \mathbb{B}^d_R(P^j))\|^2 \\
&\leqslant c \sum_{j\in\mathbb{Z}} \alpha_j^{-d-2} \|W^j; \mathcal{H}(\Pi \cap \mathbb{B}^d_R(P^j))\|^2 \\
&\leqslant c \sum_{j\in\mathbb{Z}} \|\mathcal{F}^j; \mathcal{V}(\mathbb{R}^d \setminus \Omega_j)^*\|^2 \leqslant c\|f; \mathcal{H}(\Pi)^*\|^2.
\end{aligned}
$$

Let $\mathbf{f} \in \mathcal{H}(\Pi)$ and let $f \in \mathcal{H}(\Pi)$ be defined as

$$
f(v) = -(1+\lambda)^{-1} \langle \mathbf{f}, v \rangle \qquad (2.54)
$$

where $\langle\, ,\rangle$ is the scalar product (1.17) in the Hilbert space $\mathcal{H}(\Pi)$. We now define the operator (2.20) by the formula

$$R(\lambda)\mathbf{f} = \mathcal{R}(\lambda)f$$

and show that mapping (2.21) becomes compact. In view of (1.17), (1.18) and (2.54), the abstract equation

$$Mu - (1+\lambda)^{-1}u = \mathbf{f} \text{ in } \mathcal{H}(\Pi) \tag{2.55}$$

is equivalent to the variational problem (2.22). Thus, we need to consider the expression

$$I(f,v) = a(\mathcal{R}(\lambda)f, v; \Pi) - \lambda(\mathcal{M}\mathcal{R}(\lambda)f, v)_\Pi - f(v) \tag{2.56}$$

which by (2.30), (2.31), (2.27), and (2.48) can be written as

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}(a(\mathcal{X}_j U_{(j)}, v; \varpi_j) - \lambda(\mathcal{M}\mathcal{X}_j U_{(j)}, v)_{\varpi_j} \\
&+\alpha_j^{-d}\big(a(X_j W^j, v; \Pi\cap\mathbb{B}_R^d(P^j)) - \lambda(\mathcal{M}X_j W^j, v)_{\Pi\cap\mathbb{B}_R^d(P^j)})\big) - f(v) \\
=&\sum_{j\in\mathbb{Z}}\alpha_j^{-d}(a(X_j W^j, v; \Pi\cap\mathbb{B}_R^d(P^j)) \\
&-\alpha_j^d\mathcal{F}^j(v) - \lambda(\mathcal{M}X_j W^j, v)_{\Pi\cap\mathbb{B}_R^d(P^j)}).
\end{aligned}$$

First, we have $\mathcal{F}^j(v) = \mathcal{F}^j(X^j v)$, and using (2.52) and (2.47) we write the simple estimate

$$\begin{aligned}
&\alpha_j^{-d}\lambda\left|(\mathcal{M}X_j W^j, v)_{\Pi\cap\mathbb{B}_R^d(P^j)}\right| \\
\leqslant\ & c\alpha_j^{-d}\|v; L^2(\Pi\cap\mathbb{B}_R^d(P^j))\|\left(\alpha_j^d\int\limits_{\Pi\cap\mathbb{B}_R^d(P^j)}|W^j(\xi)|^2 d\xi\right)^{1/2} \\
\leqslant\ & c\alpha_j^{-d/2}\|v; L^2(\Pi\cap\mathbb{B}_R^d(P^j))\|\Bigg(\|W^j; \mathcal{V}(\Xi)\|^2 \\
&+\alpha_j^{-\min\{0,2\varkappa-2\}}\int\limits_{\Xi\cap(\mathbb{B}_{R\alpha_j}^d(0)\setminus\mathbb{B}_R^d(0))}(1+\rho^2)^{-1+\varkappa}|W^j(\xi)|^2 d\xi\Bigg)^{1/2}
\end{aligned}$$

$$\leqslant c\alpha_j^{-d/2}\alpha_j^{-\min\{0,\varkappa-1\}}\|W^j;\mathcal{V}(\Xi)\|\,\|v;L^2(\Pi\cap\mathbb{B}^d_R(P^j))\|$$
$$\leqslant c\alpha_j^{1+\max\{0,\varkappa-1\}}\|\mathcal{F}^j;\mathcal{V}(\mathbb{R}^d\setminus\Omega_j)\|\,\|v;L^2(\Pi\cap\mathbb{B}^d_R(P^j))\|, \quad (2.57)$$

where we may choose $\varkappa\in(0,1/2)\subset(0,(d-2)/2)$ such that the final exponent of α_j in (2.57) becomes positive, see Proposition 2.5.

Furthermore, we have

$$\begin{aligned}
&\alpha_j^{-d}a(X_jW^j,v;\Pi\cap\mathbb{B}^d_R(P^j))-\mathcal{F}^j(X_jv)\\
=\ &\alpha_j^{-d}a(W^j,X_jv;\Pi\cap\mathbb{B}^d_R(P^j))-\mathcal{F}^j(X_jv)\\
&+\alpha_j^{-d}(\mathcal{A}[D(\nabla),X_j]W^j,D(\nabla)v)_{\Pi\cap\mathbb{B}^d_R(P^j)}\\
&-\alpha_j^{-d}(\mathcal{A}D(\nabla)W^j,[D(\nabla),X_j]v)_{\Pi\cap\mathbb{B}^d_R(P^j)}\\
=\ &((\mathcal{A}^0+\chi^0(\mathcal{A}-\mathcal{A}^0))D(\nabla_\xi)W^j,D(\nabla_\xi)(X_jv))_\Xi-\mathcal{G}^j(X_jv)\\
&+\alpha_j^{-d}((1-\chi^0)(\mathcal{A}-\mathcal{A}^0))D(\nabla)W^j,D(\nabla)(X_jv))_{\Pi\cap\mathbb{B}^d_R(P^j)}\\
&+\alpha_j^{-d}(\mathcal{A}^0[D(\nabla),X_j]W^j,D(\nabla)v)_{\Pi\cap\mathbb{B}^d_R(P^j)}\\
&-\alpha_j^{-d}(\mathcal{A}^0D(\nabla)W^j,[D(\nabla),X_j]v)_{\Pi\cap\mathbb{B}^d_R(P^j)}. \qquad (2.58)
\end{aligned}$$

The first couple of terms on the right vanishes due to the definition of W^j as a solution of problem (2.44) and the last one obeys the estimate

$$\begin{aligned}
&\alpha_j^{-d}\big|(\mathcal{A}D(\nabla)W^j,[D(\nabla),X_j]v)_{\Pi\cap\mathbb{B}^d_R(P^j)}\big|\\
\leqslant\ & c\alpha_j^{-d}\|v;L^2(\Pi\cap(\mathbb{B}^d_R(P^j)\setminus\mathbb{B}^d_{3R/4}(P^j)))\|\left(\alpha_j^{d-2}\int\limits_{\mathbb{A}_j}|\nabla_\xi W^j(\xi)|^2d\xi\right)^{1/2}\\
\leqslant\ & c\alpha_j^{-1-d/2}\|v;L^2(\Pi\cap\mathbb{B}^d_R(P^j))\|\left(\alpha_j^{2\varkappa}\int\limits_{\mathbb{A}_j}(1+\rho^2)^\varkappa|\nabla_\xi W^j(\xi)|^2d\xi\right)^{1/2}\\
\leqslant\ & c\alpha_j^{-1+\varkappa-d/2}\|v;L^2(\Pi\cap\mathbb{B}^d_R(P^j))\|\,\|\mathcal{G}^j;\mathcal{V}(\Pi)^*\|\\
\leqslant\ & c\alpha_j^\varkappa\|\mathcal{F}^j;\mathcal{V}(\mathbb{R}^d\setminus\Omega_j)^*\|\,\|v;L^2(\Pi\cap\mathbb{B}^d_R(P^j))\|, \qquad (2.59)
\end{aligned}$$

where the integration domain is as above, $\mathbb{A}_j=\Xi\cap(\mathbb{B}^d_{R/\alpha_j}(0)\setminus\mathbb{B}^d_{3R/(4\alpha_j)}(0))$. In the calculation (2.59) we have taken into account that, by (2.49) and (2.34), the coefficients of the commutator are non-vanishing only in the annulus $\mathbb{B}^d_R(0)\setminus\mathbb{B}^d_{3R/4}(0)$ where $\rho=|\xi|\sim\alpha_j^{-1}$, and finally we also have applied relation (2.51).

Integration by parts in the second and third terms on the right hand side of (2.58) makes them into

$$\alpha_j^{-d}\left(\overline{D(-\nabla)}^\top(1-\chi^0)(\mathcal{A}-\mathcal{A}^0)D(\nabla)W^j, X_j v\right)_{\Pi\cap\mathbb{B}^d_R(P^j)}$$
$$+\ \alpha_j^{-d}\left(\overline{D(-\nabla)}^\top\mathcal{A}[D(\nabla),X_j]W^j, v\right)_{\Pi\cap\mathbb{B}^d_R(P^j)}. \qquad (2.60)$$

The last scalar product contains the factors W^j and ∇W^j, and due to factor $[D(\nabla), X_j]$, this term can be restricted to the above-mentioned annulus. Hence, it can be estimated similarly to the calculation (2.59) and thus has the same bound. To process the first scalar product in (2.60) we need to use a different argument. We observe that

$$|\mathcal{A}(x)-\mathcal{A}^0| \leqslant c r_j = c\,|x-P^j| \qquad (2.61)$$

and, by definitions (2.45) and (2.49), the scalar product can be restricted to the annulus $\mathbb{B}^d_R(P^j)\setminus\mathbb{B}^d_{3R\alpha_j/4}(P^j)$, where $\mathcal{A}-\mathcal{A}^0$ (assumption ($\bigstar$)) and W^j (Proposition 2.5) are smooth. Finally, fixing some $\varkappa\in(0,1/2)$ in (2.47), we first write:

$$\alpha_j^{-d}\left|(\overline{D(-\nabla)}^\top(1-\chi^0)(\mathcal{A}-\mathcal{A}^0)D(\nabla)W^j, X_j v)_{\Pi\cap\mathbb{B}^d_R(P^j)}\right|$$
$$\leqslant \alpha_j^{-d}\|r_j^{-\varkappa}v; L^2(\Pi\cap\mathbb{B}^d_R(P^j))\|$$
$$\times\left(\alpha_j^{d-2+2\varkappa}\int\limits_{\Xi\cap\mathbb{B}^d_{R/\alpha_J}(0)}\rho^{2\varkappa}(\rho^2|\nabla^2_\xi W^j(\xi)|^2+|\nabla_\xi W^j(\xi)|^2)d\xi\right)^{1/2}. \qquad (2.62)$$

In the last integral, the factor $|\xi|^2$ is obtained from (2.61), and $\alpha_j^{2\varkappa}$, $|\xi|^{2\varkappa}$ from the factor $r_j^{-\varkappa}$ in the norm of v and from the Cauchy–Schwartz–Bunyakovski inequality. Now we apply (2.47) and (2.51) to bound (2.62) by

$$c\alpha_j^{-1+\varkappa-d/2}\|r_j^{-\varkappa}v; L^2(\Pi\cap\mathbb{B}^d_R(P^j))\|\,\|\mathcal{G}^j;\mathcal{H}(\Pi)^*\|$$
$$\leqslant c\alpha_j^{\varkappa}\|\mathcal{F}^j;\mathcal{V}(\mathbb{R}^d\setminus\Omega_j)^*\|\,\|v; L^2(\Pi\cap\mathbb{B}^d_R(P^j))\|. \qquad (2.63)$$

Summarizing our calculations, we observe that inequalities (2.57), (2.59) and (2.63) involve factors α_j^δ with some positive exponents δ.

Hence, taking into account (2.33), we can estimate the expression (2.56) by

$$|I(f,v)| \leqslant c \sum_{j\in\mathbb{Z}} \|\mathcal{F}^j; \mathcal{V}(\mathbb{R}^d \setminus \Omega_j)^*\| \, \alpha_j^\delta \|(1+r_j^{-\varkappa})v; L^2(\Pi \cap \mathbb{B}_R^d(P^j))\|$$

$$\leqslant \|f; \mathcal{H}(\Pi)^*\| \Big(\sum_{j\in\mathbb{Z}} \alpha_j^{2\delta} \|(1+r_j^{-\varkappa})v; L^2(\Pi \cap \mathbb{B}_R^d(P^j))\| \Big)^{1/2}. \qquad (2.64)$$

Furthermore, the Hölder inequality and the Hardy inequality (2.12) with $\varkappa \in (0,1/2)$ yield

$$\|(1+r_j^{-\varkappa})v; L^2(\Pi \cap \mathbb{B}_R^d(P^j))\|^2 \leqslant 2\|v; L^2(\Pi \cap \mathbb{B}_R^d(P^j))\|$$
$$+2\Big(\int\limits_{\Pi\cap\mathbb{B}_R^d(P^j)} |v(x)|^2 dx \Big)^{\varkappa} \Big(\int\limits_{\Pi\cap\mathbb{B}_R^d(P^j)} r_j^{-2} |v(x)|^2 dx \Big)^{1-\varkappa}$$
$$\leqslant \|v; L^2(\Pi \cap \mathbb{B}_R^d(P^j))\|^{\varkappa} \, \|v; H^1(\Pi \cap \mathbb{B}_R^d(P^j))\|^{1-\varkappa}.$$

For the final step of the proof we still define the auxiliary function space $\mathbf{H}_\alpha(\Pi)$, which consists of functions f on the domain

$$\Omega_R := \Pi \cap \bigcup_{j\in\mathbb{Z}} \mathbb{B}_R^d(P^j)$$

satisfying

$$\|f; \mathbf{H}_\alpha(\Pi)\|^2 := \sum_{j\in\mathbb{Z}} \|(1+r_j^{-\varkappa})f; L^2(\Pi \cap \mathbb{B}_R^d(P^j))\|^2 < \infty.$$

The following properties hold true: (i) $\mathcal{H}(\Pi)$ is contained in $\mathbf{H}_\alpha(\Pi)$; (ii) the embedding $\mathcal{H}(\Pi) \subset \mathbf{H}_\alpha(\Pi)$ is compact; (iii) expression (2.56) is a continuous linear functional in $\mathbf{H}_\alpha(\Pi) \ni v$. This claim follows from the factors $\alpha_j^{2\delta}$ and $\|v; L^2(\Pi \cap \mathbb{B}_R^d(P^j))\|^{\varkappa}$ in the bound of (2.64), together with facts that $\{\alpha_j\}_{j\in\mathbb{Z}}$ is an infinitesimal sequence and the embedding $L^2(\Pi \cap \mathbb{B}_R^d(P^j)) \subset H^1(\varpi_j \cup \varpi_{j-1})$ is compact.

The equality

$$I(f,v) = -(1+\lambda)(\langle (M-(1+\lambda)^{-1})\mathcal{R}(\lambda)\mathbf{f}, v\rangle - \langle \mathbf{f}, v\rangle)$$

follows from (2.54)–(2.55) and (1.17), (1.18), and we thus deduce from (i)–(iii) the compactness of the mapping (2.21), which completes our proof of Theorem 2.1.

§3. Possible generalizations.

3.1. Eigenvalues with infinite multiplicity. The essential spectrum of problem (1.11), (1.12) is characterized by Theorem 2.1 as $\wp_{es} = \Sigma$, and thus there are two possibilities for each entry Λ_m of the eigenvalue sequence (2.4):

1°. Λ_m is an eigenvalue with infinite multiplicity for the problem (1.11), (1.12) in Π;

2°. the discrete spectrum $\wp_{di}$ of the problem (1.11), (1.12) contains a sequence $\{\lambda_{(mk)}\}_{k\in\mathbb{N}}$ such that

$$\lambda_{(mk)} \ \to \ \Lambda_m \text{ as } m \ \to \ \infty.$$

At the moment we are unfortunately not able to disprove the case 1°, not even for scalar problems, although this hypothesis is very probably true. However, here we present a simple, elementary argument to find a concentration of eigenvalues around a point Λ_m in (2.4).

Let $dE_M(t)$ be the spectral measure of the positive self-adjoint operator M defined in Section 2.1 and let $d\mu_{U,U}(t)$ be the corresponding real scalar measure generated by $U \in \mathcal{H}(\Pi)$, see, e.g., [2, Ch. 6], [29, Th. 12.23]. Given an interval

$$\Upsilon_m(\delta) = [(1+\Lambda_m)^{-1} - \delta, (1+\Lambda_m)^{-1} + \delta]$$

with $\delta > 0$, we define the orthogonal projection $P_m(\delta) = \int_{\Upsilon_m(\delta)} dE_M(t)$ in $\mathcal{H}(\Pi)$. Let the vector function $u^m_{(j)}$, normalized in $\mathcal{H}(\Pi)$, be as in (2.8).

Lemma 3.6. *There exists c_m such that, for any $\delta > 0$ and $j \in \mathbb{Z}$,*

$$\|u^m_{(j)} - P_m(\delta)u^m_{(j)}; \mathcal{H}(\Pi)\| \leqslant c_m\delta^{-1}\min\{\alpha_j, \alpha_{j+1}\}. \qquad (3.1)$$

Proof. We have

$$\|u^m_{(j)} - P_m(\delta)u^m_{(j)}; \mathcal{H}(\Pi)\|^2 = \int\limits_{\mathbb{R}\setminus\Upsilon_m(\delta)} d\mu_{u^m_{(j)},u^m_{(j)}}(t)$$

$$\leqslant \frac{1}{\delta^2} \int\limits_{\mathbb{R}\setminus\Upsilon_m(\delta)} (t-(1+\Lambda_m)^{-1})^2 d\mu_{u^m_{(j)},u^m_{(j)}}(t)$$

$$\leqslant \frac{1}{\delta^2}\|Mu^m_{(j)} - (1+\Lambda_m)^{-1}u^m_{(j)}; \mathcal{H}(\Pi)\|^2.$$

It suffices to recall the estimate (2.18) with the constant c_m. ⊠

Since $\operatorname{supp} u^m_{(j)} \cap \operatorname{supp} u^m_{(k)} = \varnothing$ for $j \neq k$, the test vector functions $\{u^m_{(j)}\}_{j\in\mathbb{Z}}$ are orthonormal in $\mathcal{H}(\Pi)$. Hence, estimate (3.1) and the assumption $\alpha_{\pm m} \to +0$ as $m \to +\infty$ prove that the subspace $P(\delta)\mathcal{H}(\Pi)$ cannot be finite-dimensional for any δ.

3.2. The planar case $d = 2$. Our considerations in Section 2 do not require essential changes for the Dirichlet problem (1.11) in dimension 2. Clearly, if $\tau = 0$, the Hardy inequality (2.14) includes a logarithmic factor and can be written as

$$\int\limits_0^1 r^{-1}|\ln r|^{-2}|W(r)|^2 dr \leqslant 4\int\limits_0^1 r\left|\frac{dW}{dr}(r)\right|^2 dr,\ W\in C^1_c(0,1],\ W(1)=0,$$

so that neither (2.12) nor (2.13) can be deduced from it. However, inequality (2.12) readily follows from the Friedrichs inequality on the arc $(0,\pi) \ni \phi$

$$\int\limits_0^\pi |W(\phi)|^2 d\phi \leqslant \frac{1}{\pi^2}\int\limits_0^\pi \left|\frac{dW}{d\phi}(\phi)\right|^2 d\phi,\ \ W\in H^1(0,\pi]\, W(0)=W(\pi)=0$$

by integrating in the radial variable r. Thus the material of Section 2 remains unchanged for the Dirichlet problem in dimension 2.

In the case of the Neumann or mixed boundary value problem, several steps in our proof fail, because problems (2.39) and (2.44) are no longer solvable in the energy space $\mathcal{H}(\Xi)$. Moreover, the operators of these problems are no more isomorphisms in any Kondratiev spaces $\mathcal{V}^1_\beta(\Xi)$ with weight index $\beta \in \mathbb{R}$ (cf. (2.41) and [21, Ch. 2, 6]).

Asymptotic procedures, see [6, Ch. 3], [12, Ch. 2,5] and others, allow us to improve the situation. Knowing these, it is no longer a hard job to adapt the whole Section 2 to dimension 2. However, due to the logarithmic terms of fundamental solutions and resolvent kernels, all formulas must be modified and rewritten. For shortness, we only outline the necessary modifications needed for two-dimensional Neumann and mixed boundary-value problems.

First of all, the resolvent kernels $T^j_{(\pm)}$, i.e., the solutions of the homogeneous problem (2.36), (2.37) in $\mathbb{R}^2_\pm$, have logarithmic growth at infinity, see, e.g., [21, Ch. 6] and [16, § 2]. For $\Lambda \notin \Sigma$, also the problem (2.1), (2.2) in the cell ϖ has solutions with logarithmic behavior near the points P^q. These solutions compose the Green matrices $G(x; P^q)$ with singularities at P^q, $q = 0, 1$. Accordingly, we replace the solution $U_{(j)} \in H^1(\varpi)^n$ of the limit problem problem (2.23) by the linear combination

$$U_{(j)} + G(x; P^j)b^j + G(x; P^{j+1})b^{j+1} \tag{3.2}$$

with coefficient columns $b^j, b^{j+1} \in \mathbb{C}^n$. The method of matched asymptotic expansions (see [4, 6] and others) allows us to construct a global asymptotic approximation involving inner and outer expansions and in particular to find the coefficients b^j and b^{j+1}. A simple trick of [12, Ch. 2] helps to reorganize the obtained asymptotic stucture as the sum of the expression (3.2) and the boundary layer terms of type (2.48) with decay rate $O(|\xi|^{-1})$ for $|\xi| \to +\infty$.

3.3. Similar shapes. Our approach applies for other geometric shapes, for example, the cells can be deformed slightly and connected by thin and short ligaments, Fig. 2. Thus, our results remain valid for the waveguide

$$\widehat{\Pi} = \widehat{\Omega} \cup \bigcup_{j \in \mathbb{Z}} \widehat{\varpi}_j,$$

where

$$\widehat{\Omega} = \{(y, z) : \alpha_0(z)^{-1} y \in \omega, z \in \mathbb{R}\},$$
$$\widehat{\omega}_j = \{(y, z) : \alpha_1(j)^{-1}(y, z - j) \in \varpi\},$$

$$0 < \alpha_p \in C^1(\mathbb{R}), \quad \alpha_p(z) \to +p \text{ as } z \to \pm\infty, \quad p = 0, 1. \quad (3.3)$$

Another type of generalization is drawn in Fig. 3: the domain is composed of the cells

$$\varpi_\varsigma = \{(y, z) \in \mathbb{R}^{d-d_0} \times \mathbb{R}^{d_0} : (y, z - \varsigma) \in \varpi\} \subset \mathbb{R}^d \quad (3.4)$$

where $d_0 \in \mathbb{N}$, $1 \leqslant d_0 \leqslant d$, $\varsigma = (\varsigma_1, \dots, \varsigma_{d_0}) \in \mathbb{Z}^{d_0}$ and ϖ is a bounded Lipschitz domain in the prism $\{(y, z) : y \in \mathbb{R}^{d-d_0}, z_k \in (0, 1), k = 1, \dots, d_0\}$ such that the boundary $\partial\varpi$ includes the planar parts $\mathbb{B}_R^{d-d_0}(0) \times \{P^{kq}\}$ with $q = 0, 1$, $k = 1, \dots, d_0$ and $P^{kq} = qe_{(k)}$; $e_{(k)} \in \mathbb{R}^{d_0}$ is the unit vector of the z_k-axis, $z = (z_1, \dots, z_{d_0})$. To describe the apertures connecting cells (3.4), we define the rank of the prism $\mathbb{P}_\varsigma$ and the cell ϖ_ς to be $r_\varsigma = \max\{|\varsigma_1|, \dots, |\varsigma_{d_0}|\}$. We also define bounded Lipschitz domains $\omega^k \subset \mathbb{B}_R^{d-d_0} \subset \mathbb{R}^{d-d_0}$, $k = 1, \dots, d_0$, and an infinitesimal sequence $\{\alpha_j\}_{j \in \mathbb{N} \cup \{0\}} \subset (0, 1)$. Assume that two cells ϖ_{ς^1} and ϖ_{ς^2} have the common planar part $\mathbb{B}_R^{d-d_0} \times \{P^{kq}\}$ and that ϖ_{ς^p} is of rank r_p and $r = \min\{r_1, r_2\}$. Then the aperture between ϖ_{ς^1} and ϖ_{ς^2} is defined as

$$\{(y, z) : \alpha_r^{-1} y \in \omega^k, z = P^{kq}\}. \quad (3.5)$$

As a result, we obtain the domain Π, which is contained in the whole space ($d_0 = d$) or in a hyperplane ($1 < d_0 < d$). The cells (3.4) connected by the converging zones (3.5) pave the domain Π.

The geometric structure of the domain Π allows us to repeat word-by-word our consideration in Section 2 and to derive formula (2.6) for the essential spectrum of the problem (1.13) in Π. Since no new idea nor technicality is needed to make this conclusion, we skip the reformulation of Theorem 2.1.

3.4. Steklov problem. Let us next focus on the applications listed in Section 1.4. In the linear water-wave equation the spectral parameter is contained in the boundary condition, see (1.22) and, e.g., [9]. The variational formulation of the problem can be written as

$$(\nabla u, \nabla v)_\Pi = \lambda (u, v)_{\partial\Pi} \qquad \forall v \in H^1(\Pi). \quad (3.6)$$

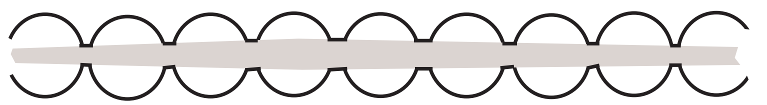

Figure 3.2: Beads connected with this and short ligaments.

However, the scalar product

$$\langle u, v \rangle = (\nabla u, \nabla v)_\Pi + (u, v)_{\partial\Pi} \qquad \forall u, v \in H^1(\Pi),$$

and the new, self-adjoint, positive operator $M : H^1(\Pi) \to H^1(\Pi)$,

$$\langle Mu, v \rangle = (u, v)_{\partial\Pi} \qquad \forall u, v \in H^1(\Pi),$$

reduces problem (3.6) to the abstract equation (1.19) with the new spectral parameter (1.20). In this way all of our results can be adapted to problem (1.22). This reduction of (1.22) to the abstract equation (1.19) was proposed in [19], and it has become very useful in various problems of the linear theory of water-waves.

3.5. Piezoelectric waveguides. As mentioned in Section 1.3, 3°, the piezoelectricity system defined by (1.11) and (1.27), (1.28), and supplied with the Dirichlet conditions (1.29), (1.30), satisfies requirements (1.9), (1.15), and thus the results obtained in Sections 2 apply. Let us next show how to treat the other physically relevant case of the Neumann conditions (1.31).

By $\mathcal{H}^{\mathsf{E}}(\Pi)$ we understand the function space obtained as the completion of $C_c^\infty(\overline{\Pi})$ with respect to the Dirichlet norm

$$\left(\|\nabla u^{\mathsf{E}}; L^2(\Pi)\|^2 + \|u^{\mathsf{E}}; L^2(\varpi_0)\|^2\right)^{1/2},$$

cf. (2.40). It is straightforward to see that $\mathcal{H}^{\mathsf{E}}(\Pi)$ contains constant functions. Hence, the problem

$$(\mathcal{A}^{\mathsf{EE}} D^{\mathsf{E}}(\nabla) u^{\mathsf{E}}, D^{\mathsf{E}}(\nabla) v^{\mathsf{E}})_\Pi = \mathcal{F}^{\mathsf{E}}(v^{\mathsf{E}}) \qquad \forall v^{\mathsf{E}} \in \mathcal{H}^{\mathsf{E}}(\Pi), \qquad (3.7)$$

where $\mathcal{A}^{\mathsf{EE}}$ is a real symmetric and positive definite 3×3-matrix and $D^{\mathsf{E}}(\nabla) = \nabla$, has a solution $u^{\mathsf{E}} \in \mathcal{H}^{\mathsf{E}}(\Pi)$ if and only if $\mathcal{F}^{\mathsf{E}}(1) = 0$. This

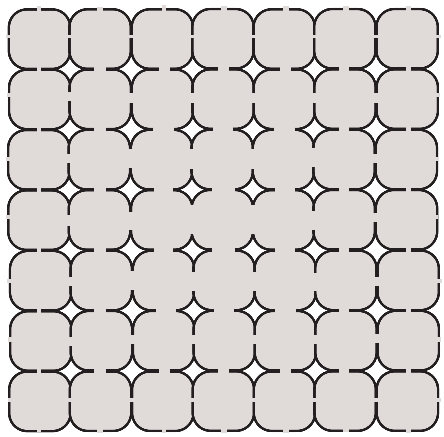

Figure 3.3: Double-periodic family of identical cells connected with ligaments converging at infinity.

is the case for the problem (3.7) with the particular right-hand side

$$\mathcal{F}^{\mathsf{E}}(v^{\mathsf{E}}) = -(\mathcal{A}^{\mathsf{EM}} D^{\mathsf{M}}(\nabla) u^{\mathsf{M}}, D^{\mathsf{E}}(\nabla) v^{\mathsf{E}})_{\Pi}, \quad D^{\mathsf{M}}(\nabla) u^{\mathsf{M}} \in L^2(\Pi)^6, \tag{3.8}$$

where the notation of (1.27), (1.23) is used. This problem is obtained from the piezoelectricity problem (1.13) by restricting to "electric" test functions $v = (0,0,0,v^{\mathsf{E}})^{\top}$ and observing that the lowest line of the matrix $\mathcal{M}$ equals $(0,0,0,0)$, see (1.28).

Following [23], we introduce the continuous mapping

$$L^2(\Pi)^6 \ni D^{\mathsf{M}}(\nabla) u^{\mathsf{M}} \ \mapsto \ J(u^{\mathsf{M}}) = -D^{\mathsf{E}}(\nabla) u^{\mathsf{E}} \in L^2(\Pi)^3.$$

Notice that the solution $u^{\mathsf{E}} \in \mathcal{H}^{\mathsf{E}}(\Pi)$ is defined up to a constant addendum which of course is annihilated by the operator J. Putting

$$\begin{aligned} &-(\mathcal{A}^{\mathsf{ME}} D^{\mathsf{E}}(\nabla) u^{\mathsf{E}}, D^{\mathsf{M}}(\nabla) v^{\mathsf{M}})_{\Pi} \\ &= (\mathcal{A}^{\mathsf{ME}} J(u^{\mathsf{E}}), D^{\mathsf{M}}(\nabla) v^{\mathsf{M}})_{\Pi} =: \mathcal{J}(u^{\mathsf{M}}, v^{\mathsf{M}}), \end{aligned} \tag{3.9}$$

we rewrite the piezoelectricity problem (1.13) with the "mechanical" test function $v = ((v^{\mathsf{M}})^{\top}, 0)^{\top}$ in the form

$$(\mathcal{A}^{\mathsf{MM}} D^{\mathsf{M}}(\nabla) u^{\mathsf{M}}, D^{\mathsf{M}}(\nabla) v^{\mathsf{M}})_{\Pi} + \mathcal{J}(u^{\mathsf{M}}, v^{\mathsf{M}})$$

$$= \lambda(\mathcal{M}^{\mathsf{M}} u^{\mathsf{M}}, v^{\mathsf{M}})_\Pi \quad \forall v^{\mathsf{M}} \in \mathcal{H}^{\mathsf{M}}(\Pi),$$

where $\mathcal{M}^{\mathsf{M}}(x) = \rho^{\mathsf{M}}(x)\mathbb{I}_3$ is the truncated matrix (1.28).

Lemma 3.7. *The bilinear form* (3.9) *is symmetric and positive.*

Proof. Using formulas (3.7), (3.8) as such and then also by exchanging $u \leftrightarrow v$, we conclude the desired properties from the equalities

$$\mathcal{J}(u^{\mathsf{M}}, v^{\mathsf{M}}) = -(\mathcal{A}^{\mathsf{ME}} D^{\mathsf{E}}(\nabla) u^{\mathsf{E}}, D^{\mathsf{M}}(\nabla) v^{\mathsf{M}})_\Pi$$
$$= -(D^{\mathsf{E}}(\nabla) u^{\mathsf{E}}, \mathcal{A}^{\mathsf{EM}} D^{\mathsf{M}}(\nabla) v^{\mathsf{M}})_\Pi = (\mathcal{A}^{\mathsf{EE}} D^{\mathsf{E}}(\nabla) u^{\mathsf{E}}, D^{\mathsf{E}}(\nabla) v^{\mathsf{E}})_\Pi. \qquad \boxtimes$$

We now introduce the specific scalar product

$$\langle u^{\mathsf{M}}, v^{\mathsf{M}} \rangle = (\mathcal{A}^{\mathsf{MM}} D^{\mathsf{M}}(\nabla) u^{\mathsf{M}}, D^{\mathsf{M}}(\nabla) v^{\mathsf{M}})_\Pi + \mathcal{J}(u^{\mathsf{M}}, v^{\mathsf{M}}) + (\mathcal{M}^{\mathsf{M}} u^{\mathsf{M}}, v^{\mathsf{M}})_\Pi$$

and the self-adjoint operator M^{M} in the space $\mathcal{H}^{\mathsf{M}}(\Pi) = H^1(\Pi)^3$,

$$\langle M^{\mathsf{M}} u^{\mathsf{M}}, v^{\mathsf{M}} \rangle = (\mathcal{M}^{\mathsf{M}} u^{\mathsf{M}}, v^{\mathsf{M}})_\Pi \qquad \forall u^{\mathsf{M}}, v^{\mathsf{M}} \in \mathcal{H}^{\mathsf{M}}(\Pi). \tag{3.10}$$

We have thus obtained the abstract equation (1.19) of Section 1.2 with the spectral parameter (1.20). Since this equation is equivalent to the piezoelectricity problem (1.13) with definitions (1.27) and (1.28), the spectrum of the operator M^{M} in (3.10) is to be regarded as the spectrum of the original problem. Moreover, our considerations in Section 2 yield the same conclusions on the essential spectrum.

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