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The Φ^4_3 measure via Girsanov's theorem*

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Abstract

We construct the Φ^4_3 measure on a periodic three dimensional box as an absolutely continuous perturbation of a random translation of the Gaussian free field. The shifted measure is constructed via Girsanov's theorem and the relevant filtration is the one generated by a scale parameter. As a byproduct we give a self-contained proof that the Φ^4_3 measure is singular wrt. the Gaussian free field.

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1 Introduction

The Φ^4_3 measure on the three dimensional torus $\Lambda = \mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ is the probability measure ν on distributions $\mathscr{S}'(\Lambda)$ corresponding to the formal functional integral

$$\nu(\mathrm{d}\varphi) = \text{``} \quad \frac{1}{Z} \exp\left[-\lambda \int_\Lambda (\varphi^4 - \infty\varphi^2)\mathrm{d}x\right] \mu(\mathrm{d}\varphi) \quad \text{''} \tag{1.1}$$

where μ is the law of the Gaussian free field with covariance $(1-\Delta)^{-1}$ on Λ, Z a normalization constant and λ the coupling constant. The ∞ appearing in this expression reminds us that many things are wrong with this recipe. The key difficulty can be traced to the fact that the measure we are looking for it is not absolutely continuous wrt. the reference measure μ. This fact seems part of the folklore even if we could not find a rigorous proof for it in the available literature apart from a work of Albeverio and Liang [1], which however refers to the Euclidean fields at time zero, and the work of Feldman and Osterwalder [11] in infinite volume. The singularity of the Φ^4_3 measure is indeed a major technical difficulty in a rigorous study of (1.1). Obtaining a complete

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construction of this formal object (both in finite and infinite volume) has been one of the main achievements of the constructive quantum field theory program [12, 10, 26, 11, 22, 5, 8].

In recent years the rigorous study of the Φ^4_3 model has been pursued from the point of view of *stochastic quantization*. In the original formulation of Parisi–Wu [25], stochastic quantization is a way to introduce additional degrees of freedom (in particular a dependence on a fictious time) in order to obtain an *equation* whose solutions describe a measure of interest, in this case the Φ^4_3 measure on Λ as in (1.1) or its counterpart in the full space. Rigorous analysis of stochastic quantization for simpler models like Φ^4_2 (the two-dimensional analog of eq. (1.1)) started with the work [20]. It has been only with the fundamental work of Hairer on regularity structures [19] that the three dimensional model could be successfully attacked, see also [9, 21]. This new perspective on this and related problems led to a series of new results on the global space-time control of the stochastic dynamics [24, 15, 2, 23] and to a novel proof of the construction of non-Gaussian Euclidean quantum field theories in three dimensions [14].

A conceptual advantage of stochastic quantization is that it is a method which is insensitive to questions of absolute continuity wrt. to a reference measure. This, on the other hand, is the main difficulty of the Gibbisan point of view as expressed in eq. (1.1). In order to explore further the tradeoffs of different approaches we have recently developed a variational method [4] for the construction and *description* of Φ^4_3. We were able to provide an explicit formula for the Laplace transform of Φ^4_3 in terms of a stochastic control problem in which the controlled process represents the scale-by-scale evolution of the interacting random field.

The present paper is the occasion to explore further this point of view by constructing a novel measure via a random translation of the Gaussian free field and by proving that the Φ^4_3 measure can be obtained as an absolutely continuous perturbation thereof. Without entering into technical details right now, let us give the broad outline of this construction. We consider a Brownian martingale $(W_t)_{t\geqslant 0}$ with values in $\mathscr{S}'(\Lambda)$ and such that W_t is a regularization of the Gaussian free field μ at (Fourier) scale t. Let us denote $\mathbb{P}$ its law and $\mathbb{E}$ the corresponding expectation. In particular, $W_t \to W_\infty$ in law as $t \to \infty$ and W_∞ has law μ. We can identify the Φ^4_3 measure ν as the weak limit $\nu^T \to \nu$ as $T \to \infty$ of the family of probability measures $(\nu^T)_{T\geqslant 0}$ on $\mathscr{S}'(\Lambda)$ defined as

$$\nu^T(\cdot) = \mathbb{P}^T(W_T \in \cdot),$$

where $\mathbb{P}^T$ is the measure on paths $(W_t)_{t\geqslant 0}$ with density

$$\frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\mathbb{P}} = \frac{1}{Z_T} e^{-V_T(W_T)},$$

and

$$V_T(\varphi) := \lambda \int_\Lambda (\varphi(x)^4 - a_T \varphi(x)^2 + b_T)\mathrm{d}x,$$

is a quartic polynomial in the field φ with $(a_T, b_T)_T$ a family of (suitably diverging) renormalization constants. The scale parameter $t \in \mathbb{R}_+$ allows to introduce a filtration and a measure $\mathbb{Q}^v$ defined as the Girsanov transformation

$$\left.\frac{\mathrm{d}\mathbb{Q}^v}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_T} = \exp\left(L^v_T - \frac{1}{2}\langle L^v\rangle_T\right), \qquad L^v_t = \int_0^t \langle v_s, \mathrm{d}W_s\rangle_{L^2(\Lambda)} \tag{1.2}$$

where $(\langle L^v\rangle_t)_{t\geqslant 0}$ is the quadratic variation of the (scalar) local martingale $(L^v_t)_{t\geqslant 0}$ and $(v_t)_{t\geqslant 0}$ is a progressively measurable process with values in $L^2(\Lambda)$. Let

$$D_T := \frac{1}{Z_T} e^{-V_T(W_T)} \left(\frac{\mathrm{d}\mathbb{Q}^v}{\mathrm{d}\mathbb{P}}\right)^{-1},$$

be the density of $\mathbb{P}^T$ wrt. $\mathbb{Q}^v$. We will show that it is possible to choose v in such a way that the family $(D_T)_{T\geqslant 0}$ is uniformly integrable under $\mathbb{Q}^v$ and that $D_T \to D_\infty$ weakly in $L^1(\mathbb{Q}^v)$. With particular choice of v we call $\mathbb{Q}^v$ the *drift* measure: it is the central object of this paper. By Girsanov's theorem the canonical process $(W_t)_{t\geqslant 0}$ satisfies the equation

$$\mathrm{d}W_t = v_t \mathrm{d}t + \mathrm{d}\tilde{W}_t, \qquad t \geqslant 0,$$

where $(\tilde{W}_t)_{t\geqslant 0}$ is a Gaussian martingale under $\mathbb{Q}^v$ (and has law equal to that of $(W_t)_{t\geqslant 0}$ under $\mathbb{P}$, i.e. it is a regularized Gaussian free field). We will show also that the drift v_t can be written as a (polynomial) function of $(\tilde{W}_s)_{s\in[0,t]}$, that is $v_t = \tilde{V}_t((\tilde{W}_s)_{s\in[0,t]})$. Therefore we have an explicit description of the process $(W_t)_{t\geqslant 0}$ under the drift measure $\mathbb{Q}^v$ as the unique solution of the path-dependent SDE

$$\mathrm{d}W_t = \tilde{V}_t((\tilde{W}_s)_{s\in[0,t]})\mathrm{d}t + \mathrm{d}\tilde{W}_t, \qquad t \geqslant 0. \tag{1.3}$$

Let us note that this formula expresses the "interacting" random field $(W_t)_t$ as a function of the "free" field $(\tilde{W}_t)_t$. It is a formula which shares very similar technical merits with the stochastic quantization approach.

The drift measure $\mathbb{Q}^v$ is half way between the variational description in [4] and the (formal) Gibbsian description of eq. (1.1). It constitutes a measure which is relatively explicit, easy to construct and analyze and which can be used as reference measure for Φ^4_3, very much like the Gaussian free field can be used as reference measure for Φ^4_2 [13, 17].

Let us remark that from eq. (1.3), after specifying $\tilde{V}_t$, (see equations (3.1) and (3.2) below) we can deduce a decomposition of the form

$$W_T = \tilde{W}_T + \tilde{\mathbb{W}}^{[3]}_T + R_T. \tag{1.4}$$

Here $R \in C\left([0,\infty], \mathscr{C}^{1-\delta}\right)$ $\mathbb{Q}^v$-almost surely and $\tilde{\mathbb{W}}^{[3]} \in C\left([0,\infty], \mathscr{C}^{1/2-\delta}\right)$ $\mathbb{Q}^v$-almost surely, can both be constructed form $\tilde{W}$, and have given law under $\mathbb{Q}^v$. This decomposition allows one to reduce many almost sure properer ties of $\mathbb{Q}^v$ and so of Φ^4_3 to properties of the "free field" $\tilde{W}$. For the slightly less singular Hartree nonlinearity this has been exploited in [6, 7] to prove local wellposedness almost surely for initial data distributed according to the Gibbs measure. Let us briefly sketch another application of (1.4): we can use it prove that the Wick square is well defined almost surely with respect to the Φ^4_3 measure. Indeed we can write

$$\begin{aligned} [\![W^2_\infty]\!] &:= \lim_{T\to\infty} (W^2_T - \mathbb{E}_{\mathbb{P}}[W^2_T(0)]) \\ &= \lim_{T\to\infty} (\tilde{W}^2_T - \mathbb{E}_{\mathbb{Q}^v}[\tilde{W}^2_T(0)]) + \tilde{W}_T\tilde{\mathbb{W}}^{[3]}_T + \tilde{W}_T R_T. \end{aligned}$$

Now it is well known that $\tilde{W}^2_T - \mathbb{E}_{\mathbb{Q}^v}[\tilde{W}^2_T(0)]$ converges to a well defined random distribution as $T\to\infty$, which is the Wick square of $\tilde{W}_T$. It has been shown in [4] Lemma 4 and Lemma 25 that $\tilde{W}_T\tilde{\mathbb{W}}^{[3]}_T$ converges to a well defined random distribution. Finally since $R \in C\left([0,\infty], \mathscr{C}^{1-\delta}\right)$ and $\tilde{W} \in C\left([0,\infty], \mathscr{C}^{-1/2-\delta}\right)$ we can deduce that also $\lim_{T\to\infty}\tilde{W}_T R_T$ is well defined.

As another application of (1.4) we provide also a self-contained proof of the singularity of the Φ^4_3 measure ν wrt. the Gaussian free field μ. We have already remarked that the singularity of Φ^4_3 seems to belongs to the folklore and we were not able to trace any written proof of that. However, M. Hairer, during a conference at Imperial College in 2019, showed us an unpublished proof of his of singularity using the stochastic quantization equation. Our proof and his are very similar and we do not claim any essential novelty in this respect. Albeit the proof is quite straightforward we wrote down all the details in order to provide a reference for this fact. The basic idea is to consider the observable $\int_\Lambda [\![W_T^4]\!]$ where the brackets denotes Wick products and prove that it diverge with different speed as $T \to \infty$ under the measure $\mathbb{P}$ and $\mathbb{Q}^v$ because in the first case the process $(W_t)_t$ is a Brownian martingale and therefore by the properties of Wick products also the process $\left(\int_\Lambda [\![W_t^4]\!]\right)_t$ is a martingale with variance growing like T. Under the measure $\mathbb{Q}^v$ however the presence of the drift $(V_t)_{t\geqslant 0}$ produces a deterministic contribution whose size is also T and which dominates the fluctuations of the observable. Therefore the singularity of Φ^4_3 can be directly linked with the pathwise properties of the scale-by-scale process $(W_t)_{t\geqslant 0}$ in the ultraviolet region and our proof of singularity shows also that the drift measure $\mathbb{Q}^v$ is singular wrt. $\mathbb{P}$. Intuitively, the drift $(V_t)_{t\geqslant 0}$ in the SDE (1.3) is not regular enough (as $t \to \infty$) to be along Cameron–Martin directions for the law $\mathbb{P}$ of the process $(W_t)_{t\geqslant 0}$ and therefore the Girsanov transform (1.2) gives a singular measure when extended all the way to $T = +\infty$.

Let us stress that the main contribution of the present paper remains that of describing the drift measure as a novel object in the context of Φ^4_3 and similar measures and pursuing the study of Euclidean quantum fields from the point of view of stochastic analysis.

Notations. Let us fix some notations and objects.

- For $a \in \mathbb{R}^d$ we let $\langle a \rangle := (1 + |a|^2)^{1/2}$. $B(x, r) \subseteq \mathbb{R}$ denotes the open ball of center $x \in \mathbb{R}$ and radius $r > 0$. We write $A \lesssim B$ for $A \leqslant CB$ for some constant C and $A \asymp B$ for $A \lesssim B$ and $B \lesssim A$.
- The constant $\kappa > 0$ represents a small positive number which can be different from line to line.
- Denote with $\mathscr{S}(\Lambda)$ the space of Schwartz functions on Λ and with $\mathscr{S}'(\Lambda)$ the dual space of tempered distributions. The notation $\hat{f}$ or $\mathscr{F} f$ stands for the space Fourier transform of f and we will write $g(\mathrm{D})$ to denote the Fourier multiplier operator with symbol $g : \mathbb{R}^n \to \mathbb{R}$, i.e. $\mathscr{F}(g(\mathrm{D})f) = g\mathscr{F} f$.
- $B^\alpha_{p,q} = B^\alpha_{p,q}(\Lambda)$ denotes the Besov spaces of regularity α and integrability indices p, q as usual. $\mathscr{C}^\alpha = \mathscr{C}^\alpha(\Lambda)$ is the Hölder–Besov space $B^\alpha_{\infty,\infty}$, $W^{\alpha,p} = W^{\alpha,p}(\Lambda)$ denote the standard fractional Sobolev spaces defined by the norm $\|f\|_{W^{s,q}} := \|\langle \mathrm{D} \rangle^s f\|_{L^q}$ and $H^\alpha = W^{\alpha,2}$. The symbols $\prec, \succ, \circ$ denotes spatial paraproducts wrt. a standard Littlewood–Paley decomposition. The reader is referred to Appendix A for an overview of the various functional spaces and paraproducts.

2 The setting

The setting of this paper is the same of that in our variational study [4]. In this section we will briefly recall it and also state some results from that paper which will be needed below. They concern the Boué–Dupuis formula and certain estimates which are relevant to the analysis of absolute continuity.

Let $\Omega := C\left(\mathbb{R}_+; \mathscr{C}^{-3/2-\kappa}(\Lambda)\right)$ and $\mathscr{F}$ be the Borel σ–algebra of Ω. On $(\Omega, \mathscr{F})$ consider the probability measure $\mathbb{P}$ which makes the canonical process $(X_t)_{t\geqslant 0}$ a cylindrical Brownian motion on $L^2(\Lambda)$ and let $(\mathscr{F}_t)_{t\geqslant 0}$ the associated filtration completed with respect to sets of $\mathbb{P}$-measure 0. In the following $\mathbb{E}$ without any qualifiers will denote expectations wrt. $\mathbb{P}$ and $\mathbb{E}_{\mathbb{Q}}$ will denote expectations wrt. some other measure $\mathbb{Q}$.

On the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ there exists a collection $(B^n_t)_{n\in\mathbb{Z}^3}$ of complex (2-dimensional) Brownian motions, such that $\overline{B^n_t} = B^{-n}_t$, B^n_t, B^m_t independent for $m \neq \pm n$ and $X_t = \sum_{n\in\mathbb{Z}^3} e^{i\langle n,\cdot\rangle} B^n_t$, for example in $\mathscr{S}'(\Lambda)$.

Fix some decreasing $\rho \in C_c^\infty(\mathbb{R}_+, \mathbb{R}_+)$ such that $\rho = 1$ on $B(0, 9/10)$ and $\operatorname{supp}\rho \subset B(0,1)$. For $x \in \mathbb{R}^3$ let $\rho_t(x) := \rho(\langle x\rangle/t)$ and

$$\sigma_t(x) := \left[\frac{\mathrm{d}}{\mathrm{d}t}(\rho_t^2(x))\right]^{1/2} = (-2(\langle x\rangle/t)\rho(\langle x\rangle/t)\rho'(\langle x\rangle/t))^{1/2}/t^{1/2}.$$

Denote $J_s = \sigma_s(\mathrm{D})\langle \mathrm{D}\rangle^{-1}$ and consider the process $(W_t)_{t\geqslant 0}$ defined by

$$W_t := \int_0^t J_s \mathrm{d}X_s = \sum_{n\in\mathbb{Z}^3} e^{i\langle n,\cdot\rangle} \int_0^t \frac{\sigma_s(n)}{\langle n\rangle} dB_s^n, \qquad t \geqslant 0. \tag{2.1}$$

It is a centered Gaussian process with covariance

$$\mathbb{E}[\langle W_t, \varphi\rangle \langle W_s, \psi\rangle] \;=\; \sum_{n\in\mathbb{Z}^3} \frac{\rho^2_{\min(s,t)}(n)}{\langle n\rangle^2} \hat{\varphi}(n)\overline{\hat{\psi}(n)},$$

for any $\varphi, \psi \in \mathscr{S}(\Lambda)$ and $t, s \geqslant 0$, by Fubini theorem and Ito isometry. By dominated convergence $\lim_{t\to\infty} \mathbb{E}[\langle W_t, \varphi\rangle\langle W_t, \psi\rangle] = \sum_{n\in\mathbb{Z}^3} \langle n\rangle^{-2}\hat{\varphi}(n)\overline{\hat{\psi}(n)}$ for any $\varphi, \psi \in L^2(\Lambda)$. For any finite "time" T the random field W_T on Λ has a bounded spectral support and the stopped process $W_t^T = W_{t\wedge T}$ for any fixed $T > 0$, is in $C(\mathbb{R}_+, C^\infty(\Lambda))$. Furthermore $(W_t^T)_t$ only depends on a finite subset of the Brownian motions $(B^n)_{n\in\mathbb{Z}^3}$.

Observe that J_t satisfies the following bound

$$\|J_t f\|_{B^{s+1-\alpha}_{p,p}} \lesssim \langle t\rangle^{-\alpha-1/2} \|f\|_{B^s_{p,p}}$$

for any function $f \in B^s_{p,p}$ or $f \in W^{s,p}$ with $p \in [1,\infty]$ and $s \in \mathbb{R}$ and for any $\alpha \in \mathbb{R}$.

We will denote by $[\![W_t^n]\!]$, $n = 1, 2, 3$, the n-th Wick-power of the Gaussian random variable W_t (under $\mathbb{P}$) and introduce the convenient notations $\mathbb{W}_t^2 := 12[\![W_t^2]\!]$, $\mathbb{W}_t^3 := 4[\![W_t^3]\!]$. Furthermore we will write $[\![(\langle \mathrm{D}\rangle^{-1/2} W_t)^n]\!], n \in \mathbb{N}$ for the n-th Wick-power of $\langle \mathrm{D}\rangle^{-1/2} W_t$. It exists for any $0 < t < \infty$ and any $n \geqslant 1$ since it is easy to see that $\langle \mathrm{D}\rangle^{-1/2} W_t$ has a covariance with a diagonal behavior which can be controlled by $\log\langle t\rangle$. These Wick powers converge as $T \to \infty$ in spaces of distributions with regularities given in the following table:

Table 1: Regularities of the various stochastic objects. The domain of the time variable is understood to be $[0,\infty]$, $C\mathscr{C}^\alpha = C([0,\infty]; \mathscr{C}^\alpha)$ and $L^2\mathscr{C}^\alpha = L^2(\mathbb{R}_+; \mathscr{C}^\alpha)$. Estimates in these norms holds a.s. and in $L^p(\mathbb{P})$ for all $p \geqslant 1$ (see [4]).

W	$\mathbb{W}^2$	$s \mapsto J_s \mathbb{W}_s^3$	$[\![(\langle \mathrm{D}\rangle^{-1/2} W)^n]\!]$
$C\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{-1-}$	$C\mathscr{C}^{-1/2-} \cap L^2\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{0-}$

We denote by $\mathbb{H}_a$ the space of $(\mathscr{F}_t)_{t\geqslant 0}$-progressively measurable processes which are $\mathbb{P}$-almost surely in $\mathcal{H} := L^2(\mathbb{R}_+ \times \Lambda)$. We say that an element v of $\mathbb{H}_a$ is a *drift*. Below we will also need drifts belonging to $\mathcal{H}^\alpha := L^2(\mathbb{R}_+; H^\alpha(\Lambda))$ for some $\alpha \in \mathbb{R}$ where $H^\alpha(\Lambda)$ is the Sobolev space of regularity $\alpha \in \mathbb{R}$ and we will denote the corresponding space with $\mathbb{H}_a^\alpha$. For any $v \in \mathbb{H}_a$ define the measure $\mathbb{Q}^v$ on Ω by

$$\frac{\mathrm{d}\mathbb{Q}^v}{\mathrm{d}\mathbb{P}} = \exp\left[\int_0^\infty v_s \mathrm{d}X_s - \frac{1}{2}\int_0^\infty \|v_s\|^2_{L^2}\mathrm{d}s\right].$$

Denote with $\mathbb{H}_c \subseteq \mathbb{H}_a$ the set of drifts $v \in \mathbb{H}_a$ for which $\mathbb{Q}^v(\Omega) = 1$, and set $W^v := W - I(v)$, where

$$I_t(v) = \int_0^t J_s v_s \mathrm{d}s.$$

We will need also the following objects. For all $t \geqslant 0$ let $\theta_t : \mathbb{R}^3 \to [0,1]$ be a smooth function such that

$$\begin{array}{lll} \theta_t(\xi)\sigma_s(\xi) & = & 0 \text{ for } s \geqslant t, \\ \theta_t(\xi) & = & 1 \text{ for } |\xi| \leqslant t/2 \text{ provided that } t \geqslant T_0 \end{array} \tag{2.2}$$

for some $T_0 > 0$. For example one can fix smooth functions $\tilde{\theta}, \eta : \mathbb{R}^3 \to \mathbb{R}_+$ such that $\tilde{\theta}(\xi) = 1$ if $|\xi| \leqslant 1/2$ and $\tilde{\theta}(\xi) = 0$ if $|\xi| \geqslant 2/3$, $\eta(\xi) = 1$ if $|\xi| \leqslant 1$ and $\eta(\xi) = 0$ if $|\xi| \geqslant 2$. Then let $\tilde{\theta}_t(\xi) := \tilde{\theta}(\xi/t)$ and define

$$\theta_t(\xi) := (1 - \eta(\xi))\tilde{\theta}_t(\xi) + \zeta(t)\eta(\xi)\tilde{\theta}_t(\xi)$$

where $\zeta(t) : \mathbb{R}_+ \to \mathbb{R}$ is a smooth function such that $\zeta(t) = 0$ for $t \leqslant 10$ and $\zeta(t) = 1$ for $t \geqslant 3$. Then eq. (2.2) holds with $T_0 = 3$. Let

$$f^\flat := \theta(\mathrm{D})f \tag{2.3}$$

for any $f \in \mathscr{S}'(\Lambda)$.

Our aim here to study the measures μ_T defined on $\mathscr{C}^{-1/2-\kappa}$ as

$$\frac{\mathrm{d}\mu_T}{\mathrm{d}\mathbb{P}} = e^{-V_T(W_T)},$$

with

$$V_T(\varphi) := \lambda \int_\Lambda (\varphi^4 - a_T\varphi^2 + b_T)\mathrm{d}x, \qquad \varphi \in C^\infty(\Lambda), \tag{2.4}$$

and suitable $a_T, b_T \to \infty$. For convenience the measure μ^T is not normalized and, wrt. to the notations in the introduction we have

$$\frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\mu^T} = \frac{1}{\mu^T(\Omega)}.$$

Recall the following results of [4].

Theorem 2.1. *For any* $a_T, b_T \in \mathbb{R}$*, and* $f : \mathscr{C}^{-1/2-\kappa}(\Lambda) \to \mathbb{R}$ *with linear growth let*

$$V_T^f(\varphi) := f(\varphi) + V_T(\varphi),$$

where V_T *is given by* (2.4)*. Then the variational formula*

$$\begin{array}{ll} & -\log \int_{\mathscr{S}'(\Lambda)} e^{-V_T^f(\varphi)}\mu_T(\mathrm{d}\varphi) \\ = & -\log \mathbb{E}[e^{-V_T^f(W_T)}] \\ = & \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[V_T^f(W_T + I_T(u)) + \frac{1}{2}\int_0^T \|u_t\|^2_{L^2(\Lambda)}\mathrm{d}t\right] \end{array} \tag{2.5}$$

holds for any finite T*.*

This is a consequence of the more general Boué–Dupuis formula.

Theorem 2.2 (BD formula)**.** *Assume* $F : C([0,T], C^\infty(\Lambda)) \to \mathbb{R}$*, be Borel measurable and such that there exist* $p, q \in (1,\infty)$*, with* $1/p + 1/q = 1$*,* $\mathbb{E}[|F(W)|^p] < \infty$ *and* $\mathbb{E}[|e^{-F(W)}|^q] < \infty$ *(where we can regard* W *as an element of* $C([0,T], C^\infty(\Lambda))$ *by restricting to* $[0,T]$*). Then*

$$-\log \mathbb{E}[e^{-F(W)}] = \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[F(W + I(u)) + \frac{1}{2}\int_0^T \|u_s\|^2_{L^2(\Lambda)}\right]. \tag{2.6}$$

We will use several times below eq. (2.6) in order to control exponential integrability of various functionals. By a suitable choice of renormalization and a change of variables in the control problem (2.5) we were able in [4] to control the functional in Theorem 2.1 uniformly up to infinity.

Theorem 2.3. *There exist a sequence $(a_T, b_T)_T$ with $a_T, b_T \to \infty$ as $T \to \infty$, such that*

$$\begin{aligned} & \mathbb{E}\left[V_T^f(W_T + I_T(u)) + \frac{1}{2}\int_0^T \|u_t\|_{L^2(\Lambda)}\mathrm{d}t\right] \\ = \quad & \mathbb{E}\left[\Psi_T^f(W, I(u)) + \lambda\int (I_T(u))^4 + \frac{1}{2}\|l^T(u)\|^2_{\mathcal{H}}\right] \end{aligned}$$

where (recall that $I_t^\flat(u) = \theta(\mathrm{D})I_t(u)$ by (2.3)*)*

$$l_t^T(u) := u_t + \lambda \mathbb{1}_{t \leqslant T}\mathbb{W}_t^{\langle 3\rangle} + \lambda \mathbb{1}_{t\leqslant T} J_t(\mathbb{W}_t^2 \succ I_t^\flat(u)) \tag{2.7}$$

and the functionals $\Psi_T^f : C([0,T], C^\infty(\Lambda)) \times C([0,T], C^\infty(\Lambda)) \to \mathbb{R}$ satisfy the following bound

$$|\Psi_T^f(W, I(u))| \leqslant Q_T(W) + \frac{1}{4}(\|I_T(u)\|^4_{L^4} + \|l^T(u)\|^2_{\mathcal{H}})$$

where $Q_T(W)$ is a function of W independent of u and such that $\sup_T \mathbb{E}[|Q_T(W)|] < \infty$.

As a consequence we obtain the following corollary (cfr. Corollary 1 and Lemma 6 in [4])

Corollary 2.4. *For $f : \mathscr{C}^{-1/2-\kappa}(\Lambda) \to \mathbb{R}$ with linear growth the bound*

$$-C \leqslant \mathbb{E}_{\mu_T}[e^f] \leqslant C,$$

holds, with a constant C independent of T. In particular μ_T is tight on $\mathscr{C}^{-1/2-\kappa}$.

3 Construction of the drift measure

We start now to implement the strategy discussed in the introduction: identify a translated measure sufficiently similar to Φ^4_3. Intuitively the Φ^4_3 measure should give rise to a canonical process which is a shift of the Gaussian free field with a drift of the form given by eq. (2.7). Indeed this drift u should be the optimal drift in the variational formula. A small twist is given by the fact that the relevant Gaussian free field entering these considerations is not the process $W = W(X)$ but that obtained from the shifted canonical process $X_t^u = X_t - \int_0^t u_s \mathrm{d}s$ which we denote by

$$W^u := W(X^u) = W - I(u).$$

Moreover, to prevent explosion at finite time, we have to modify the drift in large scales and add a coercive term. This will also allow later to prove some useful estimates. As a consequence, we define the functional

$$\Xi_s(W, u := -\lambda J_s \mathbb{W}_s^3 - \lambda \mathbb{1}_{\{s \geqslant \bar{T}\}} J_s(\mathbb{W}_s^2 \succ I_s^\flat(u)) + J_s \langle \mathrm{D}\rangle^{-1/2}(\llbracket(\langle \mathrm{D}\rangle^{-1/2} W_s)^n\rrbracket) \tag{3.1}$$

where $\bar{T} > 0, n \in \mathbb{N}$ are constants which will be fixed later on and where we understand all the Wick renormalizations to be given functions of W, i.e. polynomials in W where the constants are determined according to the law of W under $\mathbb{P}$. We look now for the solution u of the equation

$$u = \Xi(W^u, u) = \Xi(W - I(u), u). \tag{3.2}$$

Expanding the Wick polynomials appearing in $\Xi(W - I(u), u)$ we obtain the equation

$$\begin{aligned} u_s \quad = \quad & \Xi(W - I(u), u) \\ = \quad & -\lambda J_s[\mathbb{W}_s^3 - \mathbb{W}_s^2 I_s(u) + 12 W_s (I_s(u))^2 - 4(I_s(u))^3] \\ & -\lambda \mathbb{1}_{\{s \geqslant \bar{T}\}} J_s[((\mathbb{W}_s^2 - 24 W_s I_s(u) + 12(I_s(u))^2)) \succ I_s^\flat(u)] \\ & + \textstyle\sum_{i=0}^n \binom{n}{i} J_s \langle \mathrm{D}\rangle^{-1/2}[\llbracket(\langle \mathrm{D}\rangle^{-1/2} W_s)^i\rrbracket(-\langle \mathrm{D}\rangle^{-1/2} I_s(u))^{n-i}] \end{aligned} \tag{3.3}$$

for all $s \geqslant 0$. This is an integral equation for $t \mapsto u_t$ with smooth coefficients depending smoothly on W and can be solved via standard methods. Since the coefficients are of polynomial growth the solution could explode in finite time. Note that for any finite time the process $(u_s)_{s>0}$ has bounded spectral support. As a consequence we can solve the equation in L^2 and as long as $\int_0^t \|u\|^2_{L^2}\mathrm{d}s$ is finite we can see from the equation that $\sup_{s\leqslant t}\|u_s\|^2_{L^2}$ is finite. By the existence of local solutions we have that, for all $N \geqslant 0$, the stopping time

$$\tau_N := \inf\left\{ t \geqslant 0 \middle| \int_0^t \|u_s\|^2_{L^2}\mathrm{d}s \geqslant N \right\},$$

is strictly positive $\mathbb{P}$-almost surely and u exists up to the (explosion) time $T_{\exp} := \sup_{N\in\mathbb{N}} \tau_N$. The following lemma will help to show that $\mathbb{P}$-almost surely $T_{\exp} = +\infty$ and will also be very useful below.

Lemma 3.1. *Let*

$$\mathrm{Aux}_s(W,w) := \sum_{i=0}^n \binom{n}{i} J_s \langle \mathrm{D}\rangle^{-1/2} (\llbracket (\langle \mathrm{D}\rangle^{-1/2} W_s)^i \rrbracket (\langle \mathrm{D}\rangle^{-1/2} I_s(w))^{n-i}).$$

then we have

$$\begin{aligned}&\mathbb{E}\int_0^t \|w_s\|^2_{L^2}\mathrm{d}s + \sup_{s\leqslant t}\mathbb{E}\|I_s(w)\|^{n+1}_{W^{-1/2,n+1}} \\ \lesssim & 1 + \int_0^t (2\mathbb{E}\|w_s + g_s\|^2_{L^2} + 4\mathbb{E}\|g_s - \mathrm{Aux}_s(W,w)\|^2_{L^2})\mathrm{d}s,\end{aligned}$$

uniformly in $t \geqslant 0$, for any pair of adapted processes $w, g \in L^2(\mathbb{P}, \mathcal{H})$ such that

$$\mathbb{E}\int_0^t \|g_s - \mathrm{Aux}_s(W,w)\|^2_{L^2}\mathrm{d}s < \infty.$$

Proof. Take $\iota_n = \inf\left\{ t > 0 : \int_0^t \|w_s\|^2_{L^2}\mathrm{d}s > N \right\}$. By Ito's formula we have

$$\int_0^{t\wedge\iota_N}\int_\Lambda \mathrm{Aux}_s(W,w) w_s \,\mathrm{d}s = \overline{\mathrm{Aux}}_{t\wedge\iota_N}(W,w) + \text{martingale}$$

where

$$\overline{\mathrm{Aux}}_t(W,w) := \sum_{i=0}^n \frac{1}{n+1-i}\binom{n}{i}\int_\Lambda (\llbracket (\langle \mathrm{D}\rangle^{-1/2} W_t)^i \rrbracket (\langle \mathrm{D}\rangle^{-1/2} I_t(w))^{n+1-i}). \tag{3.4}$$

Integrating over the probability space and using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}&\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left(\int_0^{t\wedge\iota_N}\|w_s\|^2\mathrm{d}s + 4\overline{\mathrm{Aux}}_{t\wedge\iota_N}(W,w)\right)\\ =\;& \mathbb{E}\left[\int_\Lambda \mathbb{1}_{\{t\leqslant\iota_N\}}(w_t^2 + 4\,\mathrm{Aux}_t(W,w)w_t)\right]\\ \leqslant\;& \mathbb{E}\left[\mathbb{1}_{\{t\leqslant\iota_N\}}\left(2\|w_t+g_t\|^2_{L^2} + 4\int_\Lambda(\mathrm{Aux}_t(W,w)-g_t)w_t - \|w_t\|^2_{L^2}\right)\right]\\ \leqslant\;& 2\mathbb{E}\mathbb{1}_{\{t\leqslant\iota_N\}}\|w_t+g_t\|^2_{L^2} + 4\mathbb{E}\mathbb{1}_{\{t\leqslant\iota_N\}}\|g_t - \mathrm{Aux}_t(W,w)\|^2_{L^2}.\end{aligned}$$

where g_t is an arbitrary function. By Lemma 3.7 below, we have constants c, C and a random variable $Q_T(W)$ such that

$$\sup_{t\in\mathbb{R}}\sup_{N\in\mathbb{N}}\mathbb{E}[|Q_{t\wedge\iota_N}(W)|] < \infty,$$

and

$$c\int_0^t \|w_s\|^2_{L^2}\mathrm{d}s + c\|I_t(w)\|^{n+1}_{W^{-1/2,n+1}} - Q_t(W) \leqslant \int_0^t \|w_s\|^2_{L^2}\mathrm{d}s + \overline{\mathrm{Aux}}_t(W,w)$$

$$\leqslant C\|I_t(w)\|^{n+1}_{W^{-1/2,n+1}}+C\int_0^t\|w_s\|^2_{L^2}\mathrm{d}s+Q_t(W).$$

As a consequence, we deduce

$$\begin{aligned}&\mathbb{E}\int_0^t\mathbb{1}_{\{s\leqslant\iota_N\}}\|w_s\|^2_{L^2}\mathrm{d}s+\mathbb{E}\mathbb{1}_{\{s\leqslant\iota_N\}}\|I_s(w)\|^{n+1}_{W^{-1/2,n+1}}\\ \lesssim\;& 1+\int_0^t(2\mathbb{E}\mathbb{1}_{\{s\leqslant\iota_N\}}\|w_s+g_s\|^2_{L^2}+4\mathbb{1}_{\{s\leqslant\iota_N\}}\mathbb{E}\|g_s-\mathrm{Aux}_s(W,w)\|^2_{L^2})\mathrm{d}s.\\ \leqslant\;& 1+\int_0^t(2\mathbb{E}\|w_s+g_s\|^2_{L^2}+4\mathbb{E}\|g_s-\mathrm{Aux}_s(W,w)\|^2_{L^2})\mathrm{d}s.\end{aligned}$$

And we can conclude by sending $N\to\infty$ and using monotone convergence. □

In particular, taking $w=-\mathbb{1}_{t\leqslant\tau_N}u$ and $g=-w$, we have

$$\begin{aligned}&\mathbb{E}\int_0^t\|\mathbb{1}_{s\leqslant\tau_N}u_s\|^2_{L^2}\mathrm{d}s+\sup_{s\leqslant t}\mathbb{E}\|I_s(\mathbb{1}_{\cdot\leqslant\tau_N}u)\|^{n+1}_{W^{-1/2,n+1}}\\ \lesssim& 1+\int_0^t\mathbb{E}(\mathbb{1}_{s\leqslant\tau_N}\|u_s-\mathrm{Aux}_s(W,-u)\|^2_{L^2})\mathrm{d}s,\end{aligned}$$

for all $t\leqslant T$, where, using (3.3),

$$\begin{aligned}u_s-\mathrm{Aux}_s(W,-u)\;=\;&-\lambda J_s[\mathbb{W}^3_s-\mathbb{W}^2_sI_s(u)+12W_s(I_s(u))^2-4(I_s(u))^3]\\ &-\lambda\mathbb{1}_{\{s\geqslant\bar{T}\}}J_s[((\mathbb{W}^2_s-24W_sI_s(u)+12(I_s(u))^2))\succ I^\flat_s(u)].\end{aligned}\tag{3.5}$$

Then, for any $s\leqslant T$ we have

$$\mathbb{E}(\mathbb{1}_{s\leqslant\tau_N}\|u_s-\mathrm{Aux}_s(W,-u)\|^2_{L^2})\leqslant C_T+\mathbb{E}\|I_s(\mathbb{1}_{s\leqslant\tau_N}u)\|^{n+1}_{W^{-1/2,n+1}},$$

provided n is chosen sufficiently large. Using Gronwall's inequality this gives

$$\mathbb{E}\int_0^T\|\mathbb{1}_{s\leqslant\tau_N}u_s\|^2_{L^2}\mathrm{d}s\lesssim C_T,$$

and we can let $N\to\infty$ to obtain

$$\mathbb{E}\int_0^T\|u_s\|^2_{L^2}\mathrm{d}s\lesssim C_T,$$

which implies $T_{\exp}=+\infty$. In addition and by construction, the process $u^N_t:=\mathbb{1}_{\{t\leqslant\tau_N\}}u_t$ satisfies Novikov's condition, so it is in $\mathbb{H}_c$ and Girsanov's transformation allows us to define the probability measure $\mathbb{Q}^{u^N}$ on $C\left(\mathbb{R}_+,\mathscr{C}^{-1/2-\kappa}(\Lambda)\right)$ given by

$$\mathrm{d}\mathbb{Q}^{u^N}:=\exp\left[\int_0^\infty u^N_s\mathrm{d}X_s-\frac{1}{2}\int_0^\infty\|u^N_s\|^2_{L^2(\Lambda)}\mathrm{d}s\right]\mathrm{d}\mathbb{P},$$

under which $X^{u^N}_t=X_t-\int_0^t u^N_s\mathrm{d}s$ is a cylindrical Brownian motion. Moreover, under $\mathbb{Q}^{u^N}$ the process $\left(W^{u^N}_t:=\int_0^t J_s\mathrm{d}X^{u^N}_s\right)_{t\geqslant0}$ has the same law as $(W_t)_{t\geqslant0}$ under $\mathbb{P}$. We observe also that $W^{u^N}_s=W^u_s$ for $0\leqslant s\leqslant\tau_N$ and that u satisfies the equation

$$u_s=-\lambda J_s\mathbb{W}^{u,3}_s-\lambda\mathbb{1}_{\{s\geqslant\bar{T}\}}J_s(\mathbb{W}^{u,2}_s\succ I^\flat_t(u))+J_s\langle\mathrm{D}\rangle^{-1/2}([\![(\langle\mathrm{D}\rangle^{-1/2}W^u_s)^n]\!]),\tag{3.6}$$

where we introduced the notations $\mathbb{W}^{u,3}_s:=4[\![(W^u_s)^3]\!]$ and $\mathbb{W}^{u,2}_s:=12[\![(W^u_s)^2]\!]$. Note that here the Wick powers are still taken to be given functions of W, i.e we are still taking the Wick ordering with respect to the law of W under $\mathbb{P}$ (or, equivalently, the law of W^{u^N} under $\mathbb{Q}^{u^N}$).

If we think of the terms containing W^u as given (that is, we ignore their dependence on u), eq. (3.6) is a linear integral equation in u which can be estimated via Gronwall-type arguments. In order to do so, let us denote by $U : H \mapsto \hat{u}$ the solution map of the equation

$$\hat{u} = \Xi(H, \hat{u}). \tag{3.7}$$

This last equation is linear and therefore has nice global solutions (let's say in $C(\mathbb{R}_+, L^2)$) and by uniqueness and eq. (3.6) we have $u_t = U_t(W^u)$ for $t \in [0, T_{\exp})$. From this perspective the residual dependence on u will not play any role since under the shifted measure the law of the process W^u does not depend on u. By standard paraproduct estimates (see Appendix A) we have

$$\begin{aligned} \|I_t(u)\|_{L^\infty} &\lesssim \tilde{H}_t + \int_0^t \mathbb{1}_{\{s \geqslant \bar{T}\}} \|J_s^2(\mathbb{W}_s^{u,2} \succ I_s^\flat(u))\|_{L^\infty} \mathrm{d}s \\ &\lesssim \tilde{H}_t + \bar{T}^{-\kappa} \int_0^t \langle s \rangle^{-3/2} \|\mathbb{W}_s^{u,2}\|_{\mathscr{C}^{-1-\kappa}} \|I_s^\flat(u)\|_{L^\infty} \mathrm{d}s, \end{aligned}$$

where we have crucially exploited the presence of the cutoff $\mathbb{1}_{\{s \geqslant \bar{T}\}}$ to introduce the small factor $\bar{T}^{-\kappa}$ and we have employed the notation

$$\tilde{H}_t := \int_0^t [\|J_s^2 \mathbb{W}_s^{u,3}\|_{L^\infty} + \|J_s \langle \mathrm{D} \rangle^{-1/2} (\llbracket (\langle \mathrm{D} \rangle^{-1/2} W_s^u)^n \rrbracket)\|_{L^\infty}] \mathrm{d}s$$

$$\lesssim \int_0^t \frac{1}{\langle s \rangle^{1/2+\kappa}} \|J_s \mathbb{W}_s^{u,3}\|_{\mathscr{C}^{-1/2-\kappa}} \mathrm{d}s + \int_0^t \frac{1}{\langle s \rangle^{3/2}} \|\llbracket (\langle \mathrm{D} \rangle^{-1/2} W_s^u)^n \rrbracket\|_{H^{-1/2}} \mathrm{d}s.$$

By Gronwall's lemma,

$$\sup_{t \leqslant \tau_N} \|I_t(u)\|_{L^\infty} \lesssim \tilde{H}_{\tau_N} \exp\left[C \bar{T}^{-\kappa} \int_0^{\tau_N} \|\mathbb{W}_s^{u,2}\|_{\mathscr{C}^{-1-\kappa}} \frac{\mathrm{d}s}{\langle s \rangle^{1+\kappa}}\right]. \tag{3.8}$$

Under $\mathbb{Q}^{u^N}$, the terms in $\tilde{H}_{\tau_N}$ are in all the L^p spaces by hypercontractivity and moreover for any $p \geqslant 1$ one can choose $\bar{T}$ large enough so that also the exponential term is in L^p. Using eq. (3.6) it is then not difficult to show that $\mathbb{E}_{\mathbb{Q}^{u^{N_1}}}[\|u^{N_2}\|^p_{\mathcal{H}^{-1/2-\kappa}}] < \infty$ for any $p > 1$ (again provided we take $\bar{T}$ large enough depending on p) as long as $N_1 > N_2$. By the spectral properties of J and the equation for u, the process $t \mapsto \mathbb{1}_{\{t \leqslant T\}} u_t$ is spectrally supported in a ball of radius T, so we get in particular that

$$\mathbb{E}_{\mathbb{Q}^{u^{N_1}}} \left[\int_0^{\tau_{N_2} \wedge T} \|u_s\|_{L^2}^2 \mathrm{d}s\right] \lesssim T^{1+\kappa},$$

uniformly for any choice of $N_1 \geqslant N_2 \geqslant 0$.

Lemma 3.2. *The family* $(\mathbb{Q}^{u^N})_N$ *weakly converges to a limit* $\mathbb{Q}^u$ *on* $C\left(\mathbb{R}_+, \mathscr{C}^{-3/2-\kappa}\right)$. *Under* $\mathbb{Q}^u$ *it holds* $T_{\exp} = \infty$ *almost surely and* $\mathrm{Law}_{\mathbb{Q}^u}(X^u) = \mathrm{Law}_{\mathbb{P}}(X)$. *Moreover for any finite* T

$$\frac{\mathrm{d}\mathbb{Q}^u|_{\mathscr{F}_T}}{\mathrm{d}\mathbb{P}|_{\mathscr{F}_T}} = \exp\left[\int_0^T u_s \mathrm{d}X_s - \frac{1}{2} \int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s\right].$$

Proof. Consider the filtration $(\mathscr{G}_N = \mathscr{F}_{\tau_N})_N$ and observe that $\left(\mathbb{Q}^{u^N}|_{\mathscr{G}_N}\right)_N$ is a consistent family of inner regular probability distributions and therefore there exists a unique extension $\mathbb{Q}^u$ to $\mathscr{G}_\infty = \vee_N \mathscr{G}_N$. Next observe that $\{T_{\exp} < \infty\} = \bigcup_{T\in\mathbb{N}}\{T_{\exp} < T\} \subset \bigcup_{T\in\mathbb{N}}\bigcap_{N\in\mathbb{N}}\{\tau_N < T\}$ and that for any $N, T < \infty$, we have

$$\mathbb{E}_{\mathbb{Q}^u}\left[\int_0^{\tau_N\wedge T}\|u_s\|_{L^2}^2 \mathrm{d}s\right] = \mathbb{E}_{\mathbb{Q}^{u^N}}\left[\int_0^{\tau_N\wedge T}\|u_s\|_{L^2}^2 \mathrm{d}s\right] \lesssim T^{1+\kappa}.$$

On the event $\{\tau_N \leqslant T\}$ we have

$$\int_0^{\tau_N\wedge T}\|u_s\|_{L^2}^2 \mathrm{d}s = N,$$

and therefore we also have $\mathbb{Q}^u(\{\tau_N \leqslant T\}) \leqslant CT^{1+\kappa}N^{-1}$ which in turn implies $\mathbb{Q}^u(T_{\exp} < T) = 0$. This proves that $T_{\exp} = +\infty$ under $\mathbb{Q}^u$, almost surely. As a consequence we can extend $\mathbb{Q}^u$ to all of $\mathscr{F} = \vee_T CF_T$ since for any $A \in \mathscr{F}_T$ we can set

$$\begin{aligned}\mathbb{Q}^u(A) =& \mathbb{Q}^u(A \cap \{T_{\exp} = +\infty\})\\ =& \lim_N \mathbb{Q}^u(A \cap \{T_{\exp} = +\infty, \tau_N \geqslant T\})\\ =& \lim_N \mathbb{Q}^{u^N}(A \cap \{\tau_N \geqslant T\}).\end{aligned}$$

If $A \in \mathscr{F}_T$ then by monotone convergence

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^u}[\mathbb{1}_A(X^u)] &= \lim_{N\to\infty}\mathbb{E}_{\mathbb{Q}^u}[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}(X^u)] = \lim_{N\to\infty}\mathbb{E}_{\mathbb{Q}^{u^N}}[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}(X^{u^N})]\\ &= \lim_{N\to\infty}\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}(X)] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A(X)].\end{aligned}$$

This establishes that $\mathrm{Law}_{\mathbb{Q}^u}(X^u) = \mathrm{Law}_{\mathbb{P}}(X)$. On the other hand if $A \in \mathscr{F}_T$ we have, using the martingale property of the Girsanov density,

$$\mathbb{E}_{\mathbb{Q}^u}[\mathbb{1}_A] = \lim_{N\to\infty}\mathbb{E}_{\mathbb{Q}^u}[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}] = \lim_{N\to\infty}\mathbb{E}_{\mathbb{Q}^{u^N}}[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}]$$

$$= \lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}e^{\int_0^{\tau_N}u_s\mathrm{d}X_s - \frac{1}{2}\int_0^{\tau_N}\|u_s\|_{L^2}^2\mathrm{d}s}\right]$$

$$= \lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}e^{\int_0^{T}u_s\mathrm{d}X_s - \frac{1}{2}\int_0^{T}\|u_s\|_{L^2}^2\mathrm{d}s}\right].$$

$$= \mathbb{E}\left[\mathbb{1}_A e^{\int_0^{T}u_s\mathrm{d}X_s - \frac{1}{2}\int_0^{T}\|u_s\|_{L^2}^2\mathrm{d}s}\right]$$

by monotone convergence and the fact that $T_{\exp} = +\infty$ $\mathbb{P}$-almost surely. Therefore

$$\frac{\mathrm{d}\mathbb{Q}^u|_{\mathscr{F}_T}}{\mathrm{d}\mathbb{P}|_{\mathscr{F}_T}} = e^{\int_0^{T}u_s\mathrm{d}X_s - \frac{1}{2}\int_0^{T}\|u_s\|_{L^2}^2\mathrm{d}s},$$

as claimed. □

The following lemma will also be useful in the sequel and it is a consequence of the above discussion:

Lemma 3.3. *For any $p > 1$ there exists a suitable choice of $\bar{T}$ such that*

$$\mathbb{E}_{\mathbb{Q}^u}[\sup_{t\geqslant 0}\|I_t(u)\|_{L^\infty}^p] < \infty.$$

Proof. This follows from the bound (3.8), after choosing $\bar{T}$ large enough, where we recall that $\bar{T}$ has been introduce in the definition of the drift through eq. (3.1). □

3.1 Proof of absolute continuity

In this section we prove that the measure μ_T is absolutely continuous with respect to the measure $\mathbb{Q}^u$ which we constructed in Lemma 3.2. First recall that the measures μ_T defined on Ω as

$$\frac{\mathrm{d}\mu_T}{\mathrm{d}\mathbb{P}} = e^{-V_T(W_T)}$$

can be described, using Lemma 3.2, as a perturbation of $\mathbb{Q}^u$ with density D_T given by

$$D_T := \left.\frac{\mathrm{d}\mu_T}{\mathrm{d}\mathbb{Q}^u}\right|_{\mathscr{F}_T} = \left.\frac{\mathrm{d}\mu_T}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_T} \left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}^u}\right|_{\mathscr{F}_T} = e^{-V_T(W_T) - \int_0^T u \mathrm{d}X + \frac{1}{2}\int_0^T \|u_t\|^2_{L^2}\mathrm{d}t},$$

at least on $\mathscr{F}_T$.

Lemma 3.4. *There exists a $p > 1$, such that for any $K > 0$,*

$$\sup_T \mathbb{E}_{\mathbb{Q}^u}\left[|D_T|^p \mathbb{1}_{\left\{\|W_\infty\|_{\mathscr{C}^{-1/2-\kappa}} \leqslant K\right\}}\right] < \infty. \tag{3.9}$$

in particular, the family $(D_T)_T$ is uniformly integrable under $\mathbb{Q}^u$.

Proof. The proof eq. (3.9) is given in Section 3.2 below. For the uniform integrability fix $\varepsilon > 0$. Our aim is to show that there there exists $\delta > 0$ such that $\mathbb{Q}^u(A) < \delta$ implies $\int_A D_T \mathrm{d}\mathbb{Q}^u < \varepsilon$. From Corollary 2.4, for any $\varepsilon > 0$ there exists a $K > 0$ such that

$$\varepsilon/2 > \mu_T\left(\{\|W_\infty\|_{\mathscr{C}^{-1/2-\kappa}} \geqslant K\}\right) = \int_{\left\{\|W_\infty\|_{\mathscr{C}^{-1/2-\kappa}} \geqslant K\right\}} D_T \mathrm{d}\mathbb{Q}^u.$$

Then for any $A \in \mathscr{F}$ such that $\mathbb{Q}^u(A)^{(p-1)/p} < \varepsilon / \left(2 \sup_T \mathbb{E}_{\mathbb{Q}^u}\left[|D_T|^p \mathbb{1}_{\left\{\|W_\infty\|_{\mathscr{C}^{-1/2-\kappa}} \leqslant K\right\}}\right]\right)$

$$\begin{aligned}\int_A D_T \mathrm{d}\mathbb{Q}^u &= \int_{A \cap \left\{\|W_\infty\|_{\mathscr{C}^{-1/2-\kappa}} \geqslant K\right\}} D_T \mathrm{d}\mathbb{Q}^u + \int_{A \cap \left\{\|W_\infty\|_{\mathscr{C}^{-1/2-\kappa}} \leqslant K\right\}} D_T \mathrm{d}\mathbb{Q}^u \\ &\leqslant \varepsilon/2 + \sup_T \mathbb{E}_{\mathbb{Q}^u}\left[|D_T|^p \mathbb{1}_{\left\{\|W_\infty\|_{\mathscr{C}^{-1/2-\kappa}} \leqslant K\right\}}\right] \mathbb{Q}^u(A)^{(p-1)/p} \\ &\leqslant \varepsilon\end{aligned}$$

□

Corollary 3.5. *The family of measures $(\mu_T)_{T \geqslant 0}$ is sequentially compact w.r.t. strong convergence on $(\Omega, \mathscr{F})$. Furthermore any accumulation point is absolutely continuous with respect to $\mathbb{Q}^u$.*

Proof. We choose a sub-sequence (not relabeled) such that $D_T \to D_\infty$ weakly in $L^1(\mathbb{Q}^u)$, for some $D_\infty \in L^1(\mathbb{Q}^u)$. It always exists by uniform integrability. We now claim that for any $A \in \mathscr{F}$

$$\lim_{T \to \infty} \mu_T(A) = \int_A D_\infty \mathrm{d}\mathbb{Q}^u.$$

It is enough to check this for $A \in \mathscr{F}_S$ for any $S \in \mathbb{R}_+$ since these generate $\mathscr{F}$. But there we have for $T \geqslant S$,

$$\mu_T(A) = \int_A D_T \mathrm{d}\mathbb{Q}^u \to \int_A D_\infty \mathrm{d}\mathbb{Q}^u$$

by weak L^1 convergence. □

Recall that the Φ^4_3 measure can be defined as a weak limit of the measures $\tilde{\mu}_T$ on $\mathscr{C}^{-1/2-\kappa}$ given by

$$\int f(\varphi)\tilde{\mu}_T(\mathrm{d}\varphi) = \int f(\varphi) e^{-V_T(\varphi)} \theta_T(\mathrm{d}\varphi) = \mathbb{E}_{\mathbb{P}}[f(W_T) e^{-V_T(W_T)}]$$

where θ_T is the Gaussian measure with covariance $\rho_T^2(\mathrm{D})\langle\mathrm{D}\rangle^{-2}$. From this, together with the above considerations, we see that any accumulation point $\tilde{\mu}_\infty$ of $\tilde{\mu}_T$ satisfies

$$\tilde{\mu}_\infty(A) = \mathbb{E}_{\mathbb{Q}^u}[\mathbb{1}_A(W_\infty)D_\infty], \tag{3.10}$$

for some $D_\infty \in L^1(\mathbb{Q}^u)$.

3.2 L^p bounds

Now we will prove local L^p-bounds on the density D_T. In the sequel we will denote $\tilde{W} = W^u$, with u satisfying (3.3), namely $u = U(\tilde{W})$. Before we proceed let us study how the functional $U(\tilde{W})$ behaves under shifts of $\tilde{W}$, since later we will want to apply the Boué–Dupuis formula and this kind of behavior will be crucial. Let $w \in L^2([0,\infty)\times\Lambda)$ and denote

$$u^w := U(\tilde{W} + I(w)) \quad \text{and} \qquad h^w := U(\tilde{W} + I(w)) + w = u^w + w.$$

The process h^w satisfies

$$h^w - w = u^w = \Xi(\tilde{W} + I(w), u^w).$$

More explicitly, for all $s \geqslant 0$ we have

$$\begin{aligned} h_s^w - w_s \quad = \quad & -4\lambda J_s[\![\tilde{W}_s^3]\!] - 12\lambda J_s[\![\tilde{W}_s^2]\!]I_s(w) - 12\lambda J_s\tilde{W}_s(I_s(w))^2 - 4\lambda J_s(I_s(w))^3 \\ & -12\lambda\mathbb{1}_{\{s\geqslant\bar{T}\}}J_s([\![\tilde{W}_s^2]\!] \succ I_s^\flat(u^w)) - 24\lambda\mathbb{1}_{\{s\geqslant\bar{T}\}}(J_s(\tilde{W}_sI_s(w) \succ I_s^\flat(u^w))) \\ & -12\lambda\mathbb{1}_{\{s\geqslant\bar{T}\}}J_s((I_s(w))^2 \succ I_s^\flat(u^w)) \\ & +\textstyle\sum_{i=0}^n \binom{n}{i} J_s[\![(\langle\mathrm{D}\rangle^{-1/2}\tilde{W}_s)^i]\!](\langle\mathrm{D}\rangle^{-1/2}I_s(w))^{n-i}. \end{aligned}$$

Decomposing

$$[\![\tilde{W}_s^2]\!]I_s(w) = [\![\tilde{W}_s^2]\!] \succ \theta_sI_s(w) + [\![\tilde{W}_s^2]\!] \succ (1-\theta_s)I_s(w) + [\![\tilde{W}_s^2]\!] \circ I_s(w) + [\![\tilde{W}_s^2]\!] \prec I_s(w),$$

we can write

$$u^w = U(\tilde{W} + I(w)) = -4\lambda J_s[\![\tilde{W}_s^3]\!] - 12\lambda J_s([\![\tilde{W}_s^2]\!] \succ I_s^\flat(h^w)) + r_s^w, \tag{3.11}$$

with

$$\begin{aligned} r_s^w \quad = \quad & -12\lambda J_s[\![\tilde{W}_s^2]\!] \succ (1-\theta_s)I_s(w) - 12\lambda J_s([\![\tilde{W}_s^2]\!] \circ I_s(w)) - 12\lambda J_s[\![\tilde{W}_s^2]\!] \prec I_s(w) \\ & -12\lambda J_s\tilde{W}_s(I_s(w))^2 - 4\lambda J_s(I_s(w))^3 - 24\lambda\mathbb{1}_{\{s\geqslant\bar{T}\}}(J_s(\tilde{W}_sI_s(w) \succ \theta_sI_s^\flat(u^w))) \\ & -12\lambda\mathbb{1}_{\{s\geqslant\bar{T}\}}J_s((I_s(w))^2 \succ I_s^\flat(u^w)) + 12\lambda\mathbb{1}_{\{s<\bar{T}\}}J_s([\![\tilde{W}_s^2]\!] \succ I_s^\flat(u^w)) \\ & +\textstyle\sum_{i=0}^n \binom{n}{i} J_s\langle\mathrm{D}\rangle^{-1/2}[[\![(\langle\mathrm{D}\rangle^{-1/2}W_s)^i]\!](\langle\mathrm{D}\rangle^{-1/2}I_s(w))^{n-i}]. \end{aligned} \tag{3.12}$$

The first two terms in (3.11) will be used for renormalization while the remainder r^w contains terms of higher regularity which will have to be estimated in the sequel.

Proof. Proof of eq. (3.9) Observe that

$$\mathbb{1}_{\{\|W_\infty\|_{\mathscr{C}^{-1/2-\kappa}}\leqslant K\}} \lesssim_{K,n} \exp\left(-\|W_\infty\|^n_{\mathscr{C}^{-1/2-\kappa}}\right) = \exp\left(-\|\tilde{W}_\infty + I_\infty(U(\tilde{W}))\|^n_{\mathscr{C}^{-1/2-\kappa}}\right)$$

and

$$|D_T|^p = e^{-p\left[V_T(\tilde{W}_T + I(U(\tilde{W}))) + \int_0^T U(\tilde{W})\mathrm{d}\tilde{X} + \frac{1}{2}\int_0^T \|U_t(\tilde{W})\|^2_{L^2}\mathrm{d}t\right]}.$$

Combining these two facts we have

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}^u}\left[|D_T|^p\mathbb{1}_{\{\|W\|_{\mathscr{C}^{-1/2-\kappa}}\leqslant K\}}\right] \\ \lesssim \;& \mathbb{E}_{\mathbb{Q}^u}\left[\exp\left(-p\left(V_T(\tilde{W}_T + I_T(U(\tilde{W}))) + \int_0^T U_t(\tilde{W})\mathrm{d}\widetilde{X_t} + \tfrac{1}{2}\int_0^T \|U_t(\tilde{W})\|^2_{L^2}\mathrm{d}t\right)\right.\right. \\ & \qquad\qquad \left.\left. -\|\tilde{W}_\infty + I_\infty(U(\tilde{W}))\|^n_{\mathscr{C}^{-1/2-\kappa}}\right)\right] \\ = \;& \mathbb{E}\left[\exp\left(-p\left(V_T(W_T + I_T(U(W))) + \int_0^T U_t(W)\mathrm{d}X_t + \tfrac{1}{2}\int_0^T \|U_t(W)\|^2_{L^2}\mathrm{d}t\right)\right.\right. \\ & \qquad\qquad \left.\left. -\|W_\infty + I_\infty(U(W))\|^n_{\mathscr{C}^{-1/2-\kappa}}\right)\right]. \end{aligned}$$

The Boué–Dupuis formula (2.6) provides the variational bound

$$
\begin{aligned}
&-\log \mathbb{E}_{\mathbb{Q}^u}\left[|D_T|^p \mathbb{1}_{\{\|W\|_{\mathscr{C}^{-1/2-\kappa}}\leqslant K\}}\right]\\
&\gtrsim \inf_{w\in\mathbb{H}_a} \mathbb{E}\Big[p\Big(V_T(W_T+I_T(h^w))+\tfrac{1}{2}\textstyle\int_0^T\|h^w\|_{L^2}^2\mathrm{d}t\Big)\\
&\qquad\qquad +\tfrac{1-p}{2}\textstyle\int_0^T\|w_t\|_{L^2}^2\mathrm{d}t+\|W_\infty+I_\infty(h^w)\|^n_{\mathscr{C}^{-1/2-\kappa}}+\tfrac{1}{2}\int_T^\infty\|w_t\|_{L^2}^2\mathrm{d}t\Big]
\end{aligned}
$$

where we have set $h^w = w + U(W + I(w))$ as above. Recall now that from Theorem 2.3 there exists a constant C, independent of T, such that for each h^w,

$$
\begin{aligned}
&\mathbb{E}\left[p\left(V_T(W_T+I_T(h^w))+\frac{1}{2}\int_0^T\|h^w\|_{L^2}^2\mathrm{d}t\right)\right]\\
&\geqslant -C+\frac{1}{4}\mathbb{E}_{\mathbb{P}}\left[\lambda\|I_T(h^w)\|_{L^4}^4+\int_0^T\|l^T(h^w)\|_{L^2}^2\right]
\end{aligned}
$$

where

$$
l_t^T(h^w)=h_t^w+\lambda\mathbb{1}_{t\leqslant T}\mathbb{W}_t^{\langle 3\rangle}+\lambda\mathbb{1}_{t\leqslant T}J_t(\mathbb{W}_t^2\succ I_t^\flat(h^w)).
$$

Using eq. (3.11) we compute

$$
\begin{aligned}
\mathbb{1}_{t\leqslant T}l_t^T(h^w) &= \mathbb{1}_{t\leqslant T}h_t^w+\lambda\mathbb{1}_{t\leqslant T}\mathbb{W}_t^{\langle 3\rangle}+\lambda\mathbb{1}_{t\leqslant T}J_t(\mathbb{W}_t^2\succ I_t^\flat(h^w))\\
&= \mathbb{1}_{t\leqslant T}(u_t^w+w_t)+\lambda\mathbb{1}_{t\leqslant T}\mathbb{W}_t^{\langle 3\rangle}+\lambda\mathbb{1}_{t\leqslant T}J_t(\mathbb{W}_t^2\succ I_t^\flat(h^w))\\
&= \mathbb{1}_{t\leqslant T}(r_t^w+w_t).
\end{aligned}
$$

At this point we need a lower bound for

$$
\begin{aligned}
&\mathbb{E}\Big[\tfrac{1}{4}\Big(\lambda\|I_T(h^w)\|_{L^4}^4+\textstyle\int_0^T\|r_t^w+w_t\|_{L^2}^2\mathrm{d}t\Big)+\tfrac{1-p}{2}\int_0^T\|w_t\|_{L^2}^2\mathrm{d}t\\
&\qquad +\|W_\infty+I_\infty(h^w)\|^n_{\mathscr{C}^{-1/2-\kappa}}+\tfrac{1}{2}\textstyle\int_T^\infty\|w_t\|_{L^2}^2\mathrm{d}t\Big]-C.
\end{aligned}
$$

Given that we need to take $p>1$, this expression present a difficulty in the fact that the term $\int_0^T\|w_t\|_{L^2}^2\mathrm{d}t$ appears with a negative coefficient. Note that this term cannot easily be controlled via $\int_0^T\|r_t^w+w_t\|_{L^2}^2\mathrm{d}t$ since the contribution r^w, see eq. (3.12), contains factors which are homogeneous in w of order up to 3. This is the reason we had to localize the estimate, introduce the "good" term $\|W_\infty+I_\infty(h^w)\|^n_{\mathscr{C}^{-1/2-\kappa}}$, and introduce the term $J_s\langle \mathrm{D}\rangle^{-1/2}(\llbracket(\langle \mathrm{D}\rangle^{-1/2}W_s)^n\rrbracket)$ in (3.1) which will help us to control the growth of r^w. Indeed in Lemma 3.6 below, a Gronwall argument will allow us to show that $\int_0^T\|w_t\|_{L^2}^2\mathrm{d}t$ can be bounded by a combination of the other "good" terms as

$$
\mathbb{E}\left[\int_0^T\|w\|_{L^2}^2\mathrm{d}t\right]\lesssim\mathbb{E}\left[\|I_T^\flat(h)\|_{L^4}^4+\|I_T^\flat(h)\|^n_{\mathscr{C}^{-1/2-\kappa}}+\int_0^T\|w_t+r_t^w\|_{L^2}^2\mathrm{d}t+1\right].
$$

This implies that for $1<p\ll 2$,

$$
\begin{aligned}
&-\log\mathbb{E}_{\mathbb{Q}^u}\left[|D_T|^p\mathbb{1}_{\{\|W\|_{\mathscr{C}^{-1/2-\kappa}}\leqslant K\}}\right]\\
&\geqslant \inf_{w\in\mathbb{H}_a}\mathbb{E}\Big\{\tfrac{1}{4}\Big[\lambda\|I_T(h^w)\|_{L^4}^4+\textstyle\int_0^T\|l_t^T(h^w)\|_{L^2}^2\mathrm{d}t\Big]\\
&\qquad +(1-p)C\Big[\|I_T^\flat(h^w)\|_{L^4}^4+\|I_T^\flat(h^w)\|^n_{\mathscr{C}^{-1/2-\kappa}}+\textstyle\int_0^T\|l_t^T(h^w)\|_{L^2}^2\mathrm{d}t\Big]\\
&\qquad +\|W_\infty+I_\infty(h^w)\|^n_{\mathscr{C}^{-1/2-\kappa}}\Big\}-C\\
&\geqslant -C
\end{aligned}
$$

which gives the claim. Note that here we used the bound

$$
\begin{aligned}
\mathbb{E}\|I_\infty(h^w)\|^n_{\mathscr{C}^{-1/2-\kappa}} &\lesssim \mathbb{E}\|W_\infty\|^n_{\mathscr{C}^{-1/2-\kappa}}+\mathbb{E}\|W_\infty+I_\infty(h^w)\|^n_{\mathscr{C}^{-1/2-\kappa}}\\
&\lesssim C+\mathbb{E}\|W_\infty+I_\infty(h^w)\|^n_{\mathscr{C}^{-1/2-\kappa}}
\end{aligned}
$$

as well as the fact that $\|I_t^\flat(h^w)\|_{\mathscr{C}^{-1/2-\kappa}}\lesssim\|I_\infty(h^w)\|_{\mathscr{C}^{-1/2-\kappa}}$ to conclude. □

The following lemmas complete the proof.

Lemma 3.6. *For $n \in \mathbb{N}$ odd and large enough*

$$\mathbb{E}\int_0^T \|w_s\|_{L^2}^2 \mathrm{d}s \lesssim \mathbb{E}\int_0^T \|w_s + r_s^w\|^2 \mathrm{d}s + \mathbb{E}\|I_T^\flat(h^w)\|_{\mathscr{C}^{-1/2-\kappa}}^{n+1} + \|I_T^\flat(h^w)\|_{L^4}^4 + 1.$$

Proof. Let us recall the notation

$$\mathrm{Aux}_s(W,w) := \sum_{i=0}^n \binom{n}{i} J_s \langle \mathrm{D}\rangle^{-1/2} (\llbracket (\langle \mathrm{D}\rangle^{-1/2} W_s)^i \rrbracket (\langle \mathrm{D}\rangle^{-1/2} I_s(w))^{n-i}).$$

Write $r_s^w = \tilde{r}_s^w + \mathrm{Aux}_s(W,w)$ and observe that by Lemma 3.1 we with $g = r^w$ we have

$$\mathbb{E}\int_0^t \|w_s\|_{L^2}^2 \mathrm{d}s + \sup_{s \leqslant t} \mathbb{E}\|I_s(w)\|_{W^{-1/2,n+1}}^{n+1} \lesssim 1 + \int_0^t (2\mathbb{E}\|w_s + r_s\|_{L^2}^2 + 4\mathbb{E}\|\tilde{r}_s^w\|_{L^2}^2) \mathrm{d}s,$$

Now by Lemma 3.8 below

$$\begin{aligned} &\langle t\rangle^{1+\kappa} \|\tilde{r}_t^w\|_{L^2}^2 \\ &\lesssim \int_0^t \|w_s\|_{L^2}^2 \mathrm{d}s + \|I_t(w)\|_{W^{-1/2,n+1}}^{n+1} + \|I_t^\flat(h^w)\|_{\mathscr{C}^{-1/2-\kappa}}^{n+1} + \|I_t^\flat(h^w)\|_{L^4}^4 + Q_t(W) \end{aligned}$$

for a random variable $Q_t(W)$ such that $\sup_{t \in \mathbb{R}} \mathbb{E}[Q_t(W)] < \infty$. Gronwall inequality allows to conclude. □

Lemma 3.7. *There exists constants c, C and a random variable $Q_T(W)$ such that for any stopping time τ*

$$\sup_T \mathbb{E}[|Q_{\tau \wedge T}(W)|] < \infty,$$

and

$$\begin{aligned} & -Q_T(W) + c\int_0^T \|w_s\|_{L^2}^2 \mathrm{d}s + c\|I_T(w)\|_{W^{-1/2,n+1}}^{n+1} \\ \leqslant\ & \int_0^T \|w_s\|_{L^2}^2 \mathrm{d}s + \overline{\mathrm{Aux}}_T(W,w) \\ \leqslant\ & C\|I_T(w)\|_{W^{-1/2,n+1}}^{n+1} + C\int_0^T \|w_s\|_{L^2}^2 \mathrm{d}s + Q_T(W) \end{aligned}$$

Proof. We recall that (see eq. (3.4))

$$\begin{aligned} \overline{\mathrm{Aux}}_T(W,w) &= \sum_{i=0}^n \frac{1}{n+1-i}\binom{n}{i} \int (\llbracket (\langle \mathrm{D}\rangle^{-1/2} W_T)^i \rrbracket (\langle \mathrm{D}\rangle^{-1/2} I_T(w))^{n+1-i}) \\ &= \sum_{i=1}^n \frac{1}{n+1-i}\binom{n}{i} \int (\llbracket (\langle \mathrm{D}\rangle^{-1/2} W_T)^i \rrbracket (\langle \mathrm{D}\rangle^{-1/2} I_T(w))^{n+1-i}) \\ &\quad + \frac{1}{n+1}\|I_T(w)\|_{W^{-1/2,n+1}}^{n+1} \end{aligned}$$

and since $\mathbb{E}\left[\sup_{T<\infty} \|\llbracket (\langle \mathrm{D}\rangle^{-1/2} W_T)^i \rrbracket\|_{\mathscr{C}^{-\kappa}}^p\right] < \infty$ for any $p < \infty$ and any $\varepsilon > 0$ it is enough to bound $\|(\langle \mathrm{D}\rangle^{-1/2} I_T(w))^{n+1-i}\|_{B_{1,1}^\varepsilon}^q$ for some $q > 1$ by the terms $\|I_T(w)\|_{W^{-1/2,n+1}}^{n+1}$ and $\|I_T(w)\|_{H^1}^2 \lesssim \int_0^T \|w_s\|_{L^2}^2 \mathrm{d}s$. By interpolation we can estimate, for $i \geqslant 1$,

$$\begin{aligned} \|(\langle \mathrm{D}\rangle^{-1/2} I_T(w))^{n+1-i}\|_{B_{1,1}^\varepsilon} &\lesssim \|\langle \mathrm{D}\rangle^{-1/2} I_T(w)\|_{B_{n,1}^\varepsilon}^n + C \\ &\lesssim \|I_T(w)\|_{W^{-1/2,n+1}}^{n-\frac{1}{(n-1)}} \|I_T(w)\|_{H^1}^{\frac{1}{n-1}} + C \qquad (\text{let } \varepsilon = \tfrac{1}{n(n-1)}) \end{aligned}$$

Choosing $q = n/\left(n - \frac{1}{(n-1)}\right) > 1$, we have

$$\left(\|I_T(w)\|_{W^{-1/2,n+1}}^{n-\frac{1}{(n-1)}} \|I_T(w)\|_{H^1}^{\frac{1}{n-1}}\right)^q = \|I_T(w)\|_{W^{-1/2,n+1}}^n \|I_T(w)\|_{H^1}^{\frac{n}{(n-1)n-1}}.$$

Now for n large enough $\frac{n}{(n-1)n-1} \leqslant \frac{2}{n+1}$ and using Young's inequality we can estimate

$$\begin{aligned}\|I_T(w)\|_{W^{-1/2,n+1}}^n \|I_T(w)\|_{H^1}^{\frac{n}{(n-1)n-1}} &\lesssim \|I_T(w)\|_{W^{-1/2,n+1}}^n \left(\|I_T(w)\|_{H^1}^{\frac{2}{n+1}} + 1\right) \\ &\lesssim \|I_T(w)\|_{W^{-1/2,n+1}}^{n+1} + \|I_T(w)\|_{H^1}^2 + 1 \end{aligned}$$ □

Lemma 3.8. *Let*

$$\begin{aligned}\tilde{r}_s^w \;=\; & -12\lambda J_s[\![W_s^2]\!] \succ (1-\theta_s) I_s(w) + 12\lambda J_s([\![W_s^2]\!] \circ I_s(w)) + 12\lambda J_s [\![W_s^2]\!] \prec I_s(w) \\ & -12\lambda J_s W_s (I_s(w))^2 - 4\lambda J_s (I_s(w))^3 - 24\lambda (J_s(W_s I_s(w) \succ \theta_s I_s^\flat(u^w))) \\ & -12\lambda J_s((I_s(w))^2 \succ \theta_s I_s^\flat(u^w)) + \lambda \mathbb{1}_{\{s<\bar{T}\}} J_s(\mathbb{W}_s^2 \succ I_s^\flat(u^w)).\end{aligned}$$

Setting $h^w = u + w$, there exists a random variable $Q_t(W)$ such that $\sup_t \mathbb{E}[|Q_t(W)|] < \infty$ and

$$\langle t\rangle^{1+\kappa} \|\tilde{r}_t^w\|^2 \lesssim \int_0^t \|w_s\|_{L^2}^2 \mathrm{d}s + \|I_t(w)\|_{W^{-1/2,n+1}}^{n+1} + \|I_t^\flat(h^w)\|_{\mathscr{C}^{-1/2-\kappa}}^{n+1} + \|I_t^\flat(h^w)\|_{L^4}^4 + Q_t(W).$$

Proof. Note that

$$\begin{aligned}\|\mathbb{1}_{\{s<\bar{T}\}} J_s(\mathbb{W}_s^2 \succ I_s^\flat(u^w))\|_{L^2}^2 &\lesssim_{\bar{T}} \frac{1}{\langle s\rangle^2} \|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\kappa}}^2 \|I_s^\flat(u^w)\|_{L^4}^2 \\ &\lesssim \frac{1}{\langle s\rangle^2}\left(\|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\kappa}}^4 + \|I_s^\flat(u^w)\|_{L^4}^4\right).\end{aligned}$$

Moreover $h^w = u^w + w$ implies

$$\|I_t^\flat(u^w)\|_{\mathscr{C}^{-1/2-\kappa}}^{n+1} \lesssim \|I_t^\flat(w)\|_{\mathscr{C}^{-1/2-\kappa}}^{n+1} + \|I_t^\flat(h^w)\|_{\mathscr{C}^{-1/2-\kappa}}^{n+1},$$

and $\|I_t^\flat(u^w)\|_{L^4}^4 \lesssim \|I_t^\flat(h^w)\|_{L^4}^4 + \|I_t^\flat(w)\|_{L^4}^4$. From Lemma 5.2 we get

$$\|I_t^\flat(w)\|_{L^4}^4 \lesssim C + \int_0^t \|w_s\|_{L^2}^2 \mathrm{d}s + \|I_t(w)\|_{W^{-1/2,n+1}}^{n+1}.$$

The estimation for the other terms is easy but technical and postponed until Section 5. □

4 Singularity of Φ^4_3 w.r.t. the free field

The goal of this section is to prove that the Φ^4_3 measure is singular with respect to the Gaussian free field. For this we have to find a set $S \subseteq \mathscr{C}^{-1/2-\kappa}(\Lambda)$ such that $\mathbb{P}(W_\infty \in S) = 1$ and $\mathbb{Q}^u(W_\infty \in S) = 0$. Together with (3.10), this will imply singularity. We claim that setting

$$S := \left\{ f \in \mathscr{C}^{-1/2-\kappa}(\Lambda) : \frac{1}{T_n^{1/2+\delta}} \int_\Lambda [\![(\theta_{T_n} f)^4]\!] \to 0 \right\}$$

for some suitable sequence $(T_n)_n$ such that $T_n \to \infty$, does the job. Here

$$[\![(\theta_T f)^4]\!] = (\theta_T f)^4 - 6\mathbb{E}[(\theta_T W_\infty(0))^2](\theta_T f)^2 + 3\mathbb{E}[(\theta_T W_\infty(0))^2]^2$$

denotes the Wick ordering with respect to the Gaussian free field. For later use we define

$$\mathbb{W}_t^{\theta_T,3} = 4(\theta_T W_t)^3 - 12\mathbb{E}[(\theta_T W_t(0))^2](\theta_T W_t)$$

and

$$\mathbb{W}_t^{\theta_T,2} = 12((\theta_T W_t)^2 - \mathbb{E}[(\theta_T W_t(0))^2]).$$

Before we proceed with the proof let us briefly motivate the choice of the event S and give a sketch of the proof below. By Ito's formula one can show that

$$\int_\Lambda [\![\theta_T W_\infty^4]\!] = \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t \tag{4.1}$$

where we recall that under $\mathbb{P}$ X_t is a cylindrical Wiener process. From this formula using the properties of $\mathbb{W}_t^{\theta_T,3}$ and Ito isometry we will deduce that

$$\mathbb{E}\left[\left|\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t\right|^2\right] \asymp T, \tag{4.2}$$

and extracting a subsequence we get $\mathbb{P}(W_\infty \in S) = 1$. On the other hand with R denoting a regular remainder we have

$$\int_\Lambda [\![\theta_T W_\infty^4]\!] = \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t^u - \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \mathbb{W}_t^{u,3} \mathrm{d}t + R, \tag{4.3}$$

and under $\mathbb{Q}^u$ the process X^u is a again a cylindrical Wiener process. Therefore, as in (4.2) we have a martingale whose variance is estimated as

$$\mathbb{E}_{\mathbb{Q}^u}\left[\left|\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t^u\right|^2\right] \asymp T.$$

However now the additional drift term in (4.3) grows faster than $T^{1/2+\delta}$ since it behaves as the positive term

$$\int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3})^2 \mathrm{d}t,$$

whose average can be estimated exactly as in (4.3), that is proportional to T. To make the argument rigorous we need only to show that all the neglected terms cannot compensate for this divergence, this will be done by estimating their average size and then using Borel–Cantelli.

Let us start by proving that $\mathbb{P}(W_\infty \in S) = 1$ for some sequence $(T_n)_n$.

Lemma 4.1. *For any $\delta > 0$*

$$\lim_{T\to\infty} \mathbb{E}\left[\left(\frac{1}{T^{(1+\delta)/2}} \int_\Lambda [\![(\theta_T W_\infty)^4]\!]\right)^2\right] = 0.$$

Proof. Wick products corresponds to iterated Ito integrals. Introducing the notation

$$\mathrm{d}w_t^{\theta_T} = \theta_T J_t \mathrm{d}X_t,$$

we can verify by Ito formula that

$$\int_\Lambda [\![\theta_T W_\infty^4]\!] = \int_0^\infty \int_\Lambda \mathbb{W}_t^{\theta_T,3} \mathrm{d}w_t^{\theta_T} = \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t.$$

Since $\theta_T J_t = 0$ for $t \geqslant T$, Ito isometry gives

$$\mathbb{E}\left|\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t\right|^2 = \mathbb{E}\int_0^T \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3})^2 \mathrm{d}t.$$

Then, again by Ito formula the expectation on the r.h.s. can be estimated as

$$
\begin{aligned}
& \mathbb{E}\left[\int_\Lambda (\mathbb{W}_t^{\theta_T,3})^2\right] \\
= \; & 4\mathbb{E}\left[\left|\sum_{k_1,k_2,k_3}\int_0^t\int_0^{s_1}\int_0^{s_2} \mathrm{d}w_{s_1}^{\theta_T}(k_1)\mathrm{d}w_{s_2}^{\theta_T}(k_2)\mathrm{d}w_{s_3}^{\theta_T}(k_3)\right|^2\right] \\
= \; & 24\mathbb{E}\left[\sum_{k_1,k_2,k_3}\int_0^t\int_0^{s_1}\int_0^{s_2} \frac{\theta_T^2(k_1)\sigma_{s_1}^2(k_1)}{\langle k_1\rangle^2}\frac{\theta_T^2(k_2)\sigma_{s_2}^2(k_2)}{\langle k_2\rangle^2}\frac{\theta_T^2(k_3)\sigma_{s_3}^2(k_3)}{\langle k_3\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\right] \\
\leqslant \; & 24\mathbb{E}\left[\sum_{k_1,k_2,k_3}\int_0^t\int_0^t\int_0^t \frac{\sigma_{s_1}^2(k_1)}{\langle k_1\rangle^2}\frac{\sigma_{s_2}^2(k_2)}{\langle k_2\rangle^2}\frac{\sigma_{s_3}^2(k_3)}{\langle k_3\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\right] \\
\lesssim \; & t^3
\end{aligned}
$$

Now recall that $\|J_t f\|_{L^2(\Lambda)} \lesssim \langle t\rangle^{-3/2}\|f\|_{L^2(\Lambda)}$ to conclude:

$$
\mathbb{E}\left[\frac{1}{T^{1+\delta}}\int_0^T\int_\Lambda(\theta_T J_t \mathbb{W}_t^{\theta_T,3})^2\mathrm{d}t\right] \leqslant \frac{1}{T^{1+\delta}}\int_0^T \frac{1}{t^3}\mathbb{E}[\|(\theta_T\mathbb{W}_t^{\theta_T,3})\|^2_{L^2(\Lambda)}]\mathrm{d}t \to 0. \qquad \square
$$

The lemma implies that $T^{-(1+\delta)/2}\int_\Lambda [\![(\theta_T W_\infty)^4]\!] \to 0$ in $L^2(\mathbb{P})$. So there exists a sequence $(T_n)_n$ such that $T_n^{-(1+\delta)/2}\int_\Lambda [\![(\theta_{T_n} W_\infty)^4]\!] \to 0$ almost surely.

The next step of the proof is to check that $\mathbb{Q}^u(W_\infty \in S) = 0$. More concretely we will show that for a sub-sequence of $(T_n)_n$ (not relabeled)

$$
\frac{1}{T_n^{1-\delta}}\int_\Lambda [\![(\theta_{T_n} W_\infty)^4]\!] \to -\infty,
$$

$\mathbb{Q}^u$ almost surely. Observe that

$$
\begin{aligned}
\int_\Lambda [\![(\theta_T W_\infty)^4]\!] \;&=\; \int_0^\infty\int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3}\mathrm{d}X_t \\
&=\; \int_0^\infty\int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3}\mathrm{d}X_t^u + \int_0^\infty\int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3}u_t\mathrm{d}t \\
&=\; \int_0^\infty\int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3}\mathrm{d}X_t^u - \lambda\int_0^\infty\int_\Lambda(\theta_T J_t\mathbb{W}_t^{\theta_T,3})J_t\mathbb{W}_t^{u,3}\mathrm{d}t \\
&\quad -\lambda\int_{\bar{T}}^\infty\int_\Lambda(\theta_T J_t\mathbb{W}_t^{\theta_T,3})J_t(\mathbb{W}_t^{u,2}\succ I_t^\flat(u))\mathrm{d}t \\
&\quad +\int_0^\infty\int_\Lambda(\theta_T J_t\mathbb{W}_t^{\theta_T,3})J_t\langle D\rangle^{-1/2}[\![(\langle D\rangle^{-1/2}W_t^u)^n]\!]\mathrm{d}t.
\end{aligned}
$$

We expect the term

$$
\int_0^\infty\int_\Lambda(\theta_T J_t\mathbb{W}_t^{\theta_T,3})J_t\mathbb{W}_t^{u,3}\mathrm{d}t
$$

to go to infinity faster than $T^{1-\delta}$, $\mathbb{Q}^u$-almost surely. To actually prove it, we start by a computation in average.

Lemma 4.2. *It holds*

$$
\mathbb{E}\left[\int_0^\infty\int_\Lambda(\theta_T J_t\mathbb{W}_t^{\theta_T,3})J_t\mathbb{W}_t^3\mathrm{d}t\right] \asymp T.
$$

Proof. Recall that $\mathrm{d}w_t^{\theta_T} = \theta_T J_t \mathrm{d}X_t$. With a slight abuse of notation we can write

$$
\begin{aligned}
& \int_0^\infty\int_\Lambda(\theta_T J_t\mathbb{W}_t^{\theta_T,3})J_t\mathbb{W}_t^3\mathrm{d}t \\
= \; & 16\int_0^\infty\sum_k \frac{\theta_T(k)\sigma_t^2(k)}{\langle k\rangle^2}\Big(\sum_{k_1+k_2+k_3=k}\int_0^t\int_0^{s_1}\int_0^{s_2}\mathrm{d}w_{s_1}^{\theta_T}(k_1)\mathrm{d}w_{s_2}^{\theta_T}(k_2)\mathrm{d}w_{s_3}^{\theta_T}(k_3) \\
& \times\sum_{k_1+k_2+k_3=k}\int_0^t\int_0^{s_1}\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3)\Big)\mathrm{d}t
\end{aligned}
$$

and by Ito isometry

$$\begin{aligned} & \mathbb{E}\Big[\textstyle\sum_{k_1+k_2+k_3=k}\int_0^t\int_0^{s_1}\int_0^{s_2}\mathrm{d}w^{\theta_T}_{s_1}(k_1)\mathrm{d}w^{\theta_T}_{s_2}(k_2)\mathrm{d}w^{\theta_T}_{s_3}(k_3) \\ & \times\textstyle\sum_{k_1+k_2+k_3=k}\int_0^t\int_0^{s_1}\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3)\Big] \\ = \quad & 6\textstyle\sum_{k_1+k_2+k_3=k}\int_0^t\int_0^{s_1}\int_0^{s_2}\frac{\theta_T(k_1)\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2}\frac{\theta_T(k_2)\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2}\frac{\theta_T(k_3)\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\end{aligned}$$

For T large enough and since σ^2 and θ are positive, we have denoting $\tilde{\sigma}_s = \theta_T\sigma^2_s$

$$\begin{aligned} & \int_0^\infty\sum_k\frac{\theta_T(k)\sigma^2_t(k)}{\langle k\rangle^2}\sum_{k_1+k_2+k_3=k}\int_0^t\int_0^{s_1}\int_0^{s_2}\frac{\tilde{\sigma}_{s_1}(k_1)}{\langle k_1\rangle^2}\frac{\tilde{\sigma}_{s_2}(k_2)}{\langle k_2\rangle^2}\frac{\tilde{\sigma}_{s_3}(k_3)}{\langle k_3\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}t \\ \geqslant \quad & \int_{T/8}^{T/2}\sum_k\frac{\sigma^2_t(k)}{\langle k\rangle^2}\sum_{k_1+k_2+k_3=k}\int_0^{T/8}\int_0^{s_1}\int_0^{s_2}\frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2}\frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2}\frac{\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}t\end{aligned}$$

Introduce the notation $\mathbb{Z}^3_+ = \{n\in\mathbb{Z}^3 : n = (n_1,n_2,n_3)\,\mathrm{with}\,n_i\geqslant 0\}$. Furthermore we denote $\tilde{\rho}_T(k) = \rho_{T/2}(k) - \rho_{T/8}(k)$ After restricting the sum to $(\mathbb{Z}^3_+)^3$ we get the bound

$$\begin{aligned} \geqslant \quad & \textstyle\int_{T/8}^{T/2}\sum_k\frac{\sigma^2_t(k)}{\langle k\rangle^2}\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^3_+\\ k_1+k_2+k_3=k}}\int_{3T/32}^{T/8}\int_{3T/32}^{s_1}\int_{3T/32}^{s_2}\frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2}\frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2}\frac{\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}t \\ \gtrsim \quad & \textstyle\frac{1}{T^2}\sum_{k\in\mathbb{Z}^3_+}\tilde{\rho}_T(k)\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^3_+\\ k_1+k_2+k_3=k}}\int_{3T/32}^{T/8}\int_{3T/32}^{s_1}\int_{3T/32}^{s_2}\frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2}\frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2}\frac{\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\end{aligned}$$

Now, for large enough T if $k_1+k_2+k_3 = k$ and $\langle k_i\rangle\leqslant T/8$ then $\langle k\rangle\leqslant T/2\times 0.9$. Furthermore if T large enough and $k_1,k_2,k_3\in\mathbb{Z}^3_+$ and $k_1+k_2+k_3 = k$, while $\langle k_i\rangle\geqslant(3T/32)\times 0.9$ (recall that if $\langle k_i\rangle < (3T/32)\times 0.9$ and $s > 3T/32$ then $\sigma_s(k_1) = 0$) we have $\langle k\rangle\geqslant T/8$. So for any k for which the integral is nonzero we have $\rho_{T/2}(k) - \rho_{T/8}(k) = 1$ (recall that $\rho = 1$ on $B(0,9/10)$ and $\rho = 0$ outside of $B(0,1)$). This implies

$$\begin{aligned} & \frac{1}{T^2}\sum_{k\in\mathbb{Z}^3_+}\tilde{\rho}_T(k)\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^3_+\\ k_1+k_2+k_3=k}}\int_{3T/32}^{T/8}\int_{3T/32}^{s_1}\int_{3T/32}^{s_2}\frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2}\frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2}\frac{\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3 \\ = \quad & \frac{1}{T^2}\sum_{k_1,k_2,k_3\in\mathbb{Z}^3_+}\int_{3T/32}^{T/8}\int_{3T/32}^{s_1}\int_{3T/32}^{s_2}\frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2}\frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2}\frac{\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3 \\ \gtrsim \quad & T\end{aligned}$$

The upper bound $\lesssim T$ it is easier to obtain, essentially as in Lemma 4.1. □

Next we upgrade this average bound to almost sure divergence of the random variable at least as $T^{1-\delta}$ for some δ small.

Lemma 4.3. *There exists a* $\delta_0 > 0$ *such that for any* $\delta_0\geqslant\delta > 0$*, there exists a sequence* $(T_n)_n$ *such that* $\mathbb{P}-\mathrm{almost\,surely}$

$$\frac{1}{T_n^{1-\delta}}\int_0^\infty\int_\Lambda\left(\theta_{T_n}J_t\mathbb{W}^{\theta_{T_n},3}_t\right)J_t\mathbb{W}^3_t\mathrm{d}t\to\infty.$$

Proof. Define

$$G_T := \frac{1}{T^{1-\delta}}\int_0^\infty\int_\Lambda(\theta_T J_t\mathbb{W}^{\theta_T,3}_t)J_t\mathbb{W}^3_t\mathrm{d}t + \sup_{t<\infty}\|W_t\|^K_{\mathscr{C}^{-1/2-\kappa}}.$$

We will show that $e^{-G_T} \to 0$ in $L^1(\mathbb{P})$, which implies that there exists a sub-sequence $(T_n)_n$ such that $e^{-G_{T_n}} \to 0$ almost surely. From this our statement follows. By the Boué–Dupuis formula

$$\begin{aligned}
-\log \mathbb{E}[e^{-G_T}] &= \inf_{v\in\mathbb{H}_a} \mathbb{E}\left[\frac{1}{T^{1-\delta}}\int_0^\infty \int_\Lambda (\theta_T J_t [\![\theta_T((W_t+I_t(v))^3)]\!]) J_t [\![(W_t+I_t(v))^3]\!] \mathrm{d}t + \right.\\
&\qquad \left. + \sup_{t<\infty} \|W_t+I_t(v)\|^K_{\mathscr{C}^{-1/2-\kappa}} + \frac{1}{2}\int_0^\infty \|v_t\|^2_{L^2}\mathrm{d}t\right]\\
&= \inf_{v\in\mathbb{H}_a} \mathbb{E}\left[\frac{1}{T^{1-\delta}}\int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}^{\theta_T,3}_t) J_t \mathbb{W}^3_t \mathrm{d}t + \right.\\
&\qquad + \frac{1}{T^{1-\delta}} \sum_{(i,j)\in\{0,1,2,3\}^2\setminus(0,0)} \int_0^T \int_\Lambda A^i_t B^j_t \mathrm{d}t\\
&\qquad \left. + \sup_{t<\infty} \|W_t+I_t(v)\|^K_{\mathscr{C}^{-1/2-\kappa}} + \frac{1}{2}\int_0^\infty \|v_t\|^2_{L^2}\mathrm{d}t\right]\\
&\geqslant \inf_{v\in\mathbb{H}_a} \mathbb{E}\left[\frac{1}{T^{1-\delta}}\int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}^{\theta_T,3}_t) J_t \mathbb{W}^3_t \mathrm{d}t \right.\\
&\qquad + \frac{1}{T^{1-\delta}} \sum_{(i,j)\in\{0,1,2,3\}^2\setminus(0,0)} \int_0^T \int_\Lambda A^i_t B^j_t \mathrm{d}t\\
&\qquad \left. + \frac{1}{2}\sup_{t<\infty} \|I_t(v)\|^K_{\mathscr{C}^{-1/2-\kappa}} - C\sup_{t<\infty} \|W_t\|^K_{\mathscr{C}^{-1/2-\kappa}} + \frac{1}{2}\int_0^\infty \|v_t\|^2_{L^2}\mathrm{d}t\right]
\end{aligned}$$

where where have used that $\theta_T J_t = 0$ for $t \geqslant T$ and introduced the notations, for $0 \leqslant i \leqslant 3$,

$$A^i_t := 4\binom{3}{i} J_t \theta_T([\![(\theta_T W_t)^{3-i}]\!](\theta_T I_t(v))^i),$$

and

$$B^i_t := 4\binom{3}{i} J_t([\![W^{3-i}_t]\!](I_t(v))^i).$$

Our aim now to prove that the last three terms are bounded below uniformly as $T \to \infty$ (while we already know that the first one diverges). For $i \in \{1,2,3\}$

$$\|A^i_t\|^2_{L^2} + \|B^i_t\|^2_{L^2} \lesssim \langle t\rangle^{-1+\delta}\left(\|I_t(u)\|^K_{\mathscr{C}^{-1/2-\kappa}} + \|I_t(u)\|^2_{H^1} + Q_t(W)\right),$$

by Lemmas 5.4 and 5.6. Here $Q_t(W)$ is a random variable only depending on W such that $\sup_t \mathbb{E}[|Q_t(W)|^p] < \infty$ for any $p < \infty$. Then

$$\begin{aligned}
& \tfrac{1}{T^{1-\delta}} \textstyle\sum_{(i,j)\in\{0,1,2,3\}^2\setminus(0,0)} \int_0^T \int_\Lambda |A^i_t B^j_t| \mathrm{d}t\\
\leqslant\ & \tfrac{1}{T^{1-\delta}} \textstyle\sum_{(i,j)\in\{1,2,3\}^2} \int_0^T \|A^i_t\|^2_{L^2} + \|B^j_t\|^2_{L^2}\mathrm{d}t\\
& + \tfrac{1}{T^{1-\delta}} \textstyle\sum_{i\in\{1,2,3\}} \int_0^T \|A^0_t\|_{L^2}\|B^i_t\|_{L^2}\mathrm{d}t + \tfrac{1}{T^{1-\delta}} \textstyle\sum_{i\in\{1,2,3\}} \int_0^T \|A^i_t\|_{L^2}\|B^0_t\|_{L^2}\mathrm{d}t.
\end{aligned}$$

Now for the first term we obtain

$$\begin{aligned}
& \mathbb{E}\left[\tfrac{1}{T^{1-\delta}} \textstyle\sum_{(i,j)\in\{1,2,3\}^2} \int_0^T \|A^i_t\|^2_{L^2} + \|B^j_t\|^2_{L^2}\mathrm{d}t\right]\\
=\ & \mathbb{E}\left[\tfrac{1}{T^{1-\delta}} \textstyle\sum_{(i,j)\in\{1,2,3\}^2} \int_0^T \langle t\rangle^{-1+\delta}\left(\|I_t(v)\|^K_{\mathscr{C}^{-1/2-\kappa}} + \|I_t(v)\|^2_{H^1} + Q_t(W)\right)\mathrm{d}t\right]\\
=\ & \tfrac{C}{T^{1-2\delta}} \mathbb{E}\left[\sup_t \left(\|I_t(v)\|^K_{\mathscr{C}^{-1/2-\kappa}} + \|I_t(v)\|^2_{H^1}\right)\right] + \tfrac{C}{T^{1-2\delta}}.
\end{aligned}$$

For the second term we use that $\|A^0_t\|_{L^2} \leqslant Q_t(W)$ so

$$
\begin{aligned}
& \frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T \|A^0_t\|_{L^2}\|B^i_t\|_{L^2}\mathrm{d}t\right] \\
\leqslant\ & \frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T \langle t\rangle^{-1/2}\|A^0_t\|^2_{L^2}\mathrm{d}t + \int_0^T \langle t\rangle^{1/2}\|B^i_t\|^2_{L^2}\mathrm{d}t\right] \\
\lesssim\ & \frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T \langle t\rangle^{-1/2}\|A^0_t\|^2_{L^2}\mathrm{d}t\right] \\
& + \frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T \langle t\rangle^{-1/2+\delta}\left(\|I_t(v)\|^K_{\mathscr{C}^{-1/2-\kappa}} + \|I_t(v)\|^2_{H^1} + Q_t(W)\right)\mathrm{d}t\right] \\
\lesssim\ & \frac{C}{T^{1/2-2\delta}}\mathbb{E}\left[\sup_t\left(\|I_t(v)\|^K_{\mathscr{C}^{-1/2-\kappa}} + \|I_t(v)\|^2_{H^1}\right)\right] + \frac{C}{T^{1/2-2\delta}}
\end{aligned}
$$

Since $\sup_t \|I_t(v)\|^2_{H^1} \lesssim \int_0^\infty \|v_t\|^2_{L^2}\mathrm{d}t$ in total we obtain for T large enough. The third term is estimated analogously. Then

$$
\begin{aligned}
& -\log \mathbb{E}[e^{-G_T}] \\
\geqslant\ & \inf_{v\in\mathbb{H}_a}\mathbb{E}\left[\frac{1}{T^{1-\delta}}\int_0^\infty\int_\Lambda(\theta_T J_t \mathbb{W}^{\theta_T,3}_t)J_t\mathbb{W}^3_t\mathrm{d}t + \left(\frac{1}{2} - \frac{C}{T^{1/2-2\delta}}\right)\sup_{t<\infty}\|I_t(v)\|^K_{\mathscr{C}^{-1/2-\kappa}}\right. \\
& \left. -C\sup_{t<\infty}\|W_t\|^K_{\mathscr{C}^{-1/2-\kappa}} + \left(\frac{1}{2} - \frac{C}{T^{1/2-2\delta}}\right)\int_0^\infty \|v_t\|^2_{L^2}\mathrm{d}t - \frac{C}{T^{1/2-2\delta}}\right] \\
\geqslant\ & \mathbb{E}\left[\frac{1}{T^{1-\delta}}\int_0^\infty\int_\Lambda(\theta_T J_t\mathbb{W}^{\theta_T,3}_t)J_t\mathbb{W}^3_t\mathrm{d}t\right] - C \to \infty
\end{aligned}
$$

as claimed. □

Next we obtain an estimate which will help with the proof of the main theorem.

Lemma 4.4. *We have*

$$\sup_T \mathbb{E}_{\mathbb{Q}^u}\left[\int_0^\infty\int_\Lambda \frac{1}{t^{1+\delta}}(\theta_T J_t\mathbb{W}^{\theta_T,3}_t)^2\mathrm{d}t\right] < \infty.$$

Furthermore, there exists a (deterministic) sub-sequence $(T_n)_n$ such that

$$\frac{1}{T_n^{1/2+\delta}}\left|\int_0^\infty\int_\Lambda \theta_{T_n}J_t\mathbb{W}^{\theta_{T_n},3}_t\mathrm{d}X^u_t\right| \to 0$$

$\mathbb{Q}^u$ *almost surely.*

Proof. Recall that under $\mathbb{Q}^u$ we have $W_t = W^u_t + I_t(u)$ where u is defined above by (3.3) and $\mathrm{Law}_{\mathbb{Q}^u}(W^u) = \mathrm{Law}_{\mathbb{P}}(W)$. With this in mind we compute

$$\int_0^T\int_\Lambda \frac{1}{t^{1+\delta}}(\theta_T J_t\mathbb{W}^{\theta_T,3}_t)^2\mathrm{d}t = \sum_{i,j\leqslant 3}\int_0^T\int_\Lambda\frac{1}{t^{1+\delta}}A^i_tA^j_t\mathrm{d}t,$$

where, as above,

$$A^i_t = 4\binom{3}{i}J_t\theta_T([\![(\theta_T W^u_t)^{3-i}]\!](\theta_T I_t(u))^i).$$

By Lemmas 5.4 and 5.6 we have that $\mathbb{E}_{\mathbb{Q}^u}[\|A^i_t\|^2_{L^2}] \leqslant C$ so the Cauchy–Schwartz inequality gives the result. □

Theorem 4.5. *There exists a sequence $(T_n)_n$ such that, $\mathbb{Q}^u$-almost surely,*

$$\frac{1}{T_n^{1-\delta}}\int_\Lambda [\![(\theta_{T_n} W_\infty)^4]\!] \to -\infty.$$

Proof. We have

$$\int_\Lambda [\![(\theta_T W_\infty)^4]\!] = \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t.$$

Since $\mathrm{d}X_t = \mathrm{d}X_t^u + u_t \mathrm{d}t$ we have also

$$\begin{array}{rl} & \frac{1}{T^{1-\delta}}\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3}\mathrm{d}X_t \\ = & \frac{1}{T^{1-\delta}}\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3}\mathrm{d}X_t^u + \frac{1}{T^{1-\delta}}\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} u_t\mathrm{d}t \\ = & \frac{1}{T^{1-\delta}}\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3}\mathrm{d}X_t^u - \frac{\lambda}{T^{1-\delta}}\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \mathbb{W}_t^{u,3}\mathrm{d}t \\ & -\frac{\lambda}{T^{1-\delta}}\int_{\bar{T}}^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t(\mathbb{W}_t^{u,2} \succ I_t^\flat(u))\mathrm{d}t \\ & +\frac{1}{T^{1-\delta}}\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \langle D\rangle^{-1/2}[\![(\langle D\rangle^{-1/2} W_t^u)^n]\!]\mathrm{d}t. \end{array}$$

The first term goes to 0 $\mathbb{Q}^u$-almost surely by Lemma 4.4. To analyze the third term we estimate

$$\begin{array}{rl} & \frac{1}{T^{1-\delta}}\int_{\bar{T}}^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t(\mathbb{W}_t^{u,2} \succ I_t^\flat(u))\mathrm{d}t \\ = & \frac{1}{T^{1-\delta}}\int_{\bar{T}}^T \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t(\mathbb{W}_t^{u,2} \succ I_t^\flat(u))\mathrm{d}t \\ \leqslant & \frac{1}{T^{1-\delta}}\int_{\bar{T}}^T \|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2} \|J_t(\mathbb{W}_t^{u,2} \succ I_t^\flat(u))\|_{L^2}\mathrm{d}t \\ \lesssim & \frac{1}{T^{1-\delta}}\int_{\bar{T}}^T t^{-1/2+\delta/2}\|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2} \|\mathbb{W}_t^{u,2}\|_{\mathcal{C}^{-1-\delta/2}} \|I_t(u)\|_{L^2}\mathrm{d}t \\ \leqslant & T^{-1/2-2\delta}\left(\int_{\bar{T}}^T \|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2}^2 \mathrm{d}t\right)^{1/2} \\ & \times T^{-1/2+2\delta}\left(\int_{\bar{T}}^T t^{-1+\delta}(\|\mathbb{W}_t^{u,2}\|_{\mathcal{C}^{-1-\delta/2}}\|I_t(u)\|_{L^2})^2\right)^{1/2}. \end{array} \tag{4.4}$$

By the computation from Lemma 4.4 we have then

$$\mathbb{E}_{\mathbb{Q}^u}\left[T^{-1/2-2\delta}\left(\int_{\bar{T}}^T \|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2}^2 \mathrm{d}t\right)^{1/2}\right] \to 0,$$

and $\sup_t \mathbb{E}_{\mathbb{Q}^u}[(\|\mathbb{W}_t^{u,2}\|_{\mathcal{C}^{-1-\delta/2}}\|I_t(u)\|_{L^2})^2] < \infty$. Therefore (4.4) converges to 0 in $L^1(\mathbb{Q}^u)$. For the fourth term we proceed in the same way:

$$\begin{array}{rl} & \left|\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \langle D\rangle^{-1/2}[\![(\langle D\rangle^{-1/2} W_t^u)^n]\!]\mathrm{d}t\right| \\ = & \left|\int_0^T \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \langle D\rangle^{-1/2}[\![(\langle D\rangle^{-1/2} W_t^u)^n]\!]\mathrm{d}t\right| \\ \leqslant & \int_0^T \|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2} \|J_t \langle D\rangle^{-1/2}[\![(\langle D\rangle^{-1/2} W_t^u)^n]\!]\|_{L^2}\mathrm{d}t \\ \lesssim & \int_0^T (\|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2}) t^{-2+\delta} \|[\![(\langle D\rangle^{-1/2} W_t^u)^n]\!]\|_{H^{-\delta}}\mathrm{d}t \\ \leqslant & \left(\int_0^T t^{-2(1-\delta)}(\|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2})^2\mathrm{d}t\right)^{1/2}\left(\int_0^T t^{-2(1-\delta)}\|[\![(\langle D\rangle^{-1/2} W_t^u)^n]\!]\|_{H^{-\delta}}^2\mathrm{d}t\right)^{1/2} \end{array}$$

which is bounded in expectation uniformly in T, so the fourth term goes to 0 in $L^1(\mathbb{Q}^u)$ as well. It remains to analyze the second term. Again introducing the notation

$$A_t^i = 4\binom{3}{i} J_t\theta_T([\![(\theta_T W_t^u)^{3-i}]\!](\theta_T I_t(u))^i), \qquad \mathbb{W}_t^{\theta_T,u,3} = 4[\![(\theta_T W_t^u)^3]\!],$$

we have

$$\begin{aligned}&\frac{1}{T^{1-\delta}}\int_0^\infty\int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \mathbb{W}_t^{u,3}\mathrm{d}t\\ =\;&\frac{1}{T^{1-\delta}}\int_0^T\int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,u,3} J_t \mathbb{W}_t^{u,3}\mathrm{d}t+\sum_{1\leqslant i\leqslant 3}\frac{1}{T^{1-\delta}}\int_0^T\int_\Lambda A_t^i J_t \mathbb{W}_t^{u,3}\mathrm{d}t.\end{aligned}$$

Now observe that

$$\frac{1}{T^{1-\delta}}\int_0^T\int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,u,3} J_t \mathbb{W}_t^{u,3}\mathrm{d}t_{\mathbb{Q}^u}\sim_{\mathbb{P}}\quad\frac{1}{T^{1-\delta}}\int_0^T\int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \mathbb{W}_t^{3}\mathrm{d}t,$$

so the $\limsup$ of this is ∞ almost surely. To estimate the sum we again observe that for $i\geqslant 3$ $\mathbb{E}_{\mathbb{Q}^u}[\|A_t^i\|_{L^2}^2]\lesssim\langle t\rangle^{-1+\delta}$ and by Young's inequality

$$\begin{aligned}\textstyle\int_0^T\int_\Lambda A_t^i J_t \mathbb{W}_t^{u,3}\mathrm{d}t &\leqslant \textstyle\int_0^T\int_\Lambda \|A_t^i\|_{L^2}\|J_t \mathbb{W}_t^{u,3}\|_{L^2}\mathrm{d}t\\ &\leqslant \textstyle\int_0^T\int_\Lambda \langle t\rangle^{1/3}\|A_t^i\|_{L^2}\langle t\rangle^{-1/3}\|J_t \mathbb{W}_t^{u,3}\|_{L^2}\mathrm{d}t\\ &\leqslant \textstyle\int_0^T\int_\Lambda \langle t\rangle^{2/3}\|A_t^i\|_{L^2}^2+\int_0^T\int_\Lambda\langle t\rangle^{-2/3}\|J_t \mathbb{W}_t^{u,3}\|_{L^2}^2\mathrm{d}t.\end{aligned}$$

Taking expectation we obtain

$$\begin{aligned}&\frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T\int_\Lambda A_t^i J_t \mathbb{W}_t^{u,3}\mathrm{d}t\right]\\ \leqslant\;&\frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T\int_\Lambda \langle t\rangle^{2/3}\|A_t^i\|_{L^2}^2\right]+\frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T\int_\Lambda \langle t\rangle^{-2/3}\|J_t \mathbb{W}_t^{u,3}\|_{L^2}^2\mathrm{d}t\right]\\ \lesssim\;&\frac{1}{T^{1-\delta}}\int_0^T\int_\Lambda\langle t\rangle^{-1/3+\delta}+\frac{1}{T^{1-\delta}}\int_0^T\int_\Lambda\langle t\rangle^{-2/3}\mathrm{d}t\to 0.\end{aligned}$$

We have deduced that

$$\frac{1}{T^{1-\delta}}\int_\Lambda [\![(\theta_T W_\infty)^4]\!]=-\frac{1}{T^{1-\delta}}\int_0^T\int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,u,3} J_t \mathbb{W}_t^{u,3}\mathrm{d}t+R_T,$$

where $R_T\to 0$ in $L^1(\mathbb{Q}^u)$. We can conclude by selecting a sub-sequence $(T_n)_n$ such that

$$\frac{1}{T_n^{1-\delta}}\int_0^{T_n}\int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_{T_n},u,3} J_t \mathbb{W}_t^{u,3}\mathrm{d}t\to\infty$$

$\mathbb{Q}^u$-almost surely and $R_{T_n}\to 0$, $\mathbb{Q}^u$-almost surely. □

5 Some analytic estimates

We collect in this final section various technical estimates needed to complete the proof of Lemma 3.8.

Proposition 5.1. *Let $1<p<\infty$ and $p_1,p_2,p_1',p_2'>1$ such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_1'}+\frac{1}{p_2'}=\frac{1}{p}$. Then for every $s,\alpha\geqslant 0$*

$$\|\langle\mathrm{D}\rangle^s(fg)\|_{L^p}\lesssim\|\langle\mathrm{D}\rangle^{s+\alpha}f\|_{L^{p_2}}\|\langle\mathrm{D}\rangle^{-\alpha}g\|_{L^{p_1}}+\|\langle\mathrm{D}\rangle^{s+\alpha}g\|_{L^{p_1'}}\|\langle\mathrm{D}\rangle^{-\alpha}f\|_{L^{p_2'}}.$$

Proof. See [18]. □

Lemma 5.2. *There exists $\varepsilon>0,n\in\mathbb{N}$ such that for any $\delta>0$ there exists $C_\delta<\infty$ for which the following inequality holds for any $\phi\in H^1(\Lambda)$*

$$\|\phi\|_{L^4}^{4+\varepsilon}\leqslant C\|\phi\|_{W^{-1/2,n+1}}^{n+1}+\delta\|\phi\|_{H^1}^2+C_\delta.$$

Proof.

$$\begin{aligned}\int \phi^4 \mathrm{d}x &\leqslant \|\langle \mathrm{D}\rangle^{-1/2}\phi\|_{L^8}\|\langle \mathrm{D}\rangle^{1/2}\phi^3\|_{L^{8/7}}\\ &\leqslant \|\langle \mathrm{D}\rangle^{-1/2}\phi\|_{L^8}\|\langle \mathrm{D}\rangle^{1/2}\phi\|_{L^{8/3}}\|\phi\|^2_{L^4}\\ &\leqslant \|\langle \mathrm{D}\rangle^{-1/2}\phi\|_{L^8}\|\phi\|^{1/2}_{H^1}\|\phi\|^{5/2}_{L^4}\end{aligned}$$

So

$$(\|\phi\|^4_{L^4})^{21/20} \leqslant \|\langle \mathrm{D}\rangle^{-1/2}\phi\|^{21/20}_{L^8}\|\phi\|^{21/40}_{H^1}\|\phi\|^{104/40}_{L^4}$$

and applying Young's inequality with the exponents $(32, 32/9, 32/22)$, we obtain

$$\begin{aligned}\|\langle \mathrm{D}\rangle^{-1/2}\phi\|^{21/20}_{L^8}\|\phi\|^{21/40}_{H^1}\|\phi\|^{104/40}_{L^4} &\leqslant C_\delta\|\langle \mathrm{D}\rangle^{-1/2}\phi\|^{168/5}_{L^8} + \delta\|\phi\|^{16/9}_{H^1} + \delta\|\phi\|^{208/55}_{L^4}\\ &\leqslant \|\langle \mathrm{D}\rangle^{-1/2}\phi\|^{34}_{L^8} + \delta\|\phi\|^2_{H^1} + \delta(\|\phi\|^4_{L^4})^{21/20} + C_\delta\end{aligned}$$

and subtracting $\delta(\|\phi\|^4_{L^4})^{21/20}$ on both sides of the inequality gives the result. □

Lemma 5.3. *The following estimates hold with $\varepsilon > 0$ small enough*

$$\|J_t(⟦W_t^2⟧ \succ (1-\theta_t)I_t(w))\|^2_{L^2} \lesssim \frac{1}{\langle t\rangle^{1+\varepsilon}}\left(\int_0^t \|w_s\|^2 \mathrm{d}s + \|I_t(w)\|^n_{W^{-1/2,n+1}} + \|⟦W_t^2⟧\|^n_{\mathscr{C}^{-1-\varepsilon}}\right)$$

$$\|J_t(⟦W_t^2⟧ \circ I_t(w))\|^2_{L^2} \lesssim \frac{1}{\langle t\rangle^{1+\varepsilon}}\left(\int_0^t \|w_s\|^2 \mathrm{d}s + \|I_t(w)\|^n_{W^{-1/2,n+1}} + \|⟦W_t^2⟧\|^n_{\mathscr{C}^{-1-\varepsilon}}\right)$$

$$\|J_t⟦W_t^2⟧ \prec I_t(w)\|^2_{L^2} \lesssim \frac{1}{\langle t\rangle^{1+\varepsilon}}\left(\int_0^t \|w_s\|^2 \mathrm{d}s + \|I_t(w)\|^n_{W^{-1/2,n+1}} + \|⟦W_t^2⟧\|^n_{\mathscr{C}^{-1-\varepsilon}}\right)$$

Proof. We observe that since $⟦W_t^2⟧$ is spectrally supported in a ball or radius $\sim t$

$$\|⟦W_t^2⟧\|_{\mathscr{C}^{-1+\varepsilon}} \lesssim \langle t\rangle^{2\varepsilon}\|⟦W_t^2⟧\|_{\mathscr{C}^{-1-\varepsilon}}.$$

For the first estimate we know that $(1-\theta_t)I_t(w)$ is supported in an annulus of radius $\sim t$, so $\|(1-\theta_t)I_t(w)\|_{L^2} \lesssim \langle t\rangle^{-1+\varepsilon}\|I_t(w)\|_{H^{1-\varepsilon}}$ and furthermore by interpolation $\|I_t(w)\|_{H^{1-\varepsilon}} \lesssim \|I_t(w)\|^{1-\varepsilon}_{H^1}\|I_t(w)\|^{\varepsilon}_{L^2} \lesssim \|I_t(w)\|^{1-\varepsilon}_{H^1}\|I_t(w)\|^{\varepsilon}_{L^4}$. By definition $\langle t\rangle^{1/2}J_t$ is a uniformly bounded Fourier multiplier regularizing by 1, and putting everything together, by paraproduct estimates

$$\begin{aligned} & \|J_t(⟦W_t^2⟧ \succ (1-\theta_t)I_t(w))\|^2_{L^2}\\ \lesssim\ & \langle t\rangle^{-1}\langle t\rangle^{2\varepsilon}\langle t\rangle^{-2+2\varepsilon}\|⟦W_t^2⟧\|^2_{\mathscr{C}^{-1-\varepsilon}}\|I_t(w)\|^2_{H^{1-\varepsilon}}\\ \lesssim\ & \langle t\rangle^{-1}\langle t\rangle^{2\varepsilon}\langle t\rangle^{-2+2\varepsilon}\|⟦W_t^2⟧\|^2_{\mathscr{C}^{-1-\varepsilon}}\|I_t(w)\|^{2-2\varepsilon}_{H^1}\|I_t(w)\|^{2\varepsilon}_{L^4}\\ (\varepsilon = 2/7) \qquad \lesssim\ & \langle t\rangle^{-3/2}\left(\|⟦W_t^2⟧\|^{14}_{\mathscr{C}^{-1-\varepsilon}} + \|I_t(w)\|^2_{H^1} + \|I_t(w)\|^4_{L^4}\right)\\ \lesssim\ & \langle t\rangle^{-3/2}\left(\int_0^t \|w\|^2 \mathrm{d}s + \|I_t(w)\|^n_{W^{-1/2,n+1}} + \|⟦W_t^2⟧\|^{14}_{\mathscr{C}^{-1-\varepsilon}}\right)\end{aligned}$$

For the second term in addition observe that the function $\langle t\rangle^{1/2}J_t$ is spectrally supported in an annulus of radius $\sim t$, and regularizes by 1 so again by estimates for the resonant product

$$\begin{aligned}\|J_t(⟦W_t^2⟧ \circ I_t(w))\|^2_{L^2} &\lesssim \langle t\rangle^{-3}\|⟦W_t^2⟧\|^2_{\mathscr{C}^{-1+2\varepsilon}}\|I_t(w)\|^2_{H^{1-\varepsilon}}\\ &\lesssim \langle t\rangle^{-3}\langle t\rangle^{6\varepsilon}\|⟦W_t^2⟧\|^2_{\mathscr{C}^{-1-\varepsilon}}\|I_t(w)\|^2_{H^{1-\varepsilon}}\end{aligned}$$

For the third estimate again applying paraproduct estimates and the properties of J,

$$\|J_t(⟦W_s^2⟧ \prec I_t(w))\|^2_{L^2} \lesssim \langle t\rangle^{-3+4\varepsilon}\|⟦W_s^2⟧\|^2_{\mathscr{C}^{-1-\varepsilon}}\|I_t(w)\|^2_{H^{1-\varepsilon}}.$$

Now, the claim follows from interpolation and Young's inequality

$$\begin{aligned}
& \|[\![W_t^2]\!]\|^2_{\mathscr{C}^{-1-\varepsilon}}\|I_t(w)\|^2_{H^{1-\varepsilon}} \\
\lesssim\ & \|[\![W_t^2]\!]\|^2_{\mathscr{C}^{-1-\varepsilon}}\|I_t(w)\|^{2-2\varepsilon}_{H^1}\|I_t(w)\|^{2\varepsilon}_{L^4} \\
(\varepsilon = 2/7) \qquad \lesssim\ & \|[\![W_t^2]\!]\|^{14}_{\mathscr{C}^{-1-\varepsilon}} + \|I_t(w)\|^2_{H^1} + \|I_t(w)\|^4_{L^4} \\
\lesssim\ & \left(\int_0^t \|w_s\|^2_{L^2}\mathrm{d}s + \|I_t(w)\|^n_{W^{-1/2,n+1}} + \|[\![W_t^2]\!]\|^{14}_{\mathscr{C}^{-1-\varepsilon}}\right).
\end{aligned}$$

□

Lemma 5.4. *Let $f \in C\left([0,\infty], \mathscr{C}^{-1/2-\varepsilon}\right)$ and $g \in C([0,\infty], H^1)$ such that f_t, g_t have spectral support in a ball of radius proportional to t. There exists $n \in \mathbb{N}$ such that the following estimates hold:*

$$\|J_t(f_t g_t^2)\|^2_{L^2} \lesssim \langle t\rangle^{-3/2}\|f_t\|^2_{\mathscr{C}^{-1/2-\delta}}\|g_t\|^4_{L^4},$$

$$\|J_t(f_t g_t^2)\|^2_{L^2} \lesssim \langle t\rangle^{-3/2}\left(\|f_t\|^n_{\mathscr{C}^{-1/2-\delta}} + \|g_t\|^2_{H^1} + \|g_t\|^n_{W^{-1/2,n}}\right),$$

and

$$\|J_t(g_t^3)\|^2_{L^2} \lesssim \langle t\rangle^{-3/2}(\|g_t\|^2_{H^1} + \|g_t\|^n_{W^{-1/2,n}}).$$

Proof. By the spectral properties of J_t,

$$\|J_t(f_t g_t^2)\|^2_{L^2} \lesssim \langle t\rangle^{-3}\|f_t\|^2_{L^\infty}\|g_t\|^4_{L^4} \lesssim \langle t\rangle^{-3/2}\|f_t\|^2_{\mathscr{C}^{-1/2-\delta}}\|g_t\|^4_{L^4}.$$

Applying Young's inequality with exponents $\left(\frac{n}{2}, \frac{n/2}{(n/2-1)}\right)$ with n such that $\frac{2n}{(n/2-1)} \leqslant 4+\varepsilon$ where ε is chosen as in Lemma 5.2 we have

$$\begin{aligned}
\langle t\rangle^{-3/2}\|f_t\|^2_{\mathscr{C}^{-1/2-\delta}}\|g_t\|^4_{L^4} &\leqslant \langle t\rangle^{-3/2}\left(\|f_t\|^n_{\mathscr{C}^{-1/2-\delta}} + \|g_t\|^{4+\varepsilon}_{L^4}\right) \\
&\leqslant \langle t\rangle^{-3/2}\left(\|f_t\|^n_{\mathscr{C}^{-1/2-\delta}} + \|g_t\|^n_{W^{-1/2,n}} + \|g_t\|^2_{H^1}\right).
\end{aligned}$$

Now the second estimate follows from chosing n large enough (depending on δ) and using Besov embedding after taking $f = g$. □

Lemma 5.5. *The following estimates hold*

$$\langle t\rangle^{1+\kappa}\|J_s(W_s I_t(w) \succ I_t^\flat(u))\|^2_{L^2} \lesssim \|I_t(w)\|^{4+\kappa}_{L^4} + \|I_t^\flat(u)\|^4_{L^4} + \|W_t\|^n_{\mathscr{C}^{-1/2-\kappa}},$$

$$\langle t\rangle^{1+\kappa}\|J_s((I_s(w))^2 \succ I_s^\flat(u))\|^2_{L^2} \lesssim \|I_t(w)\|^{4+\kappa}_{L^4} + \|I_t^\flat(u)\|^n_{\mathscr{C}^{-1/2-\kappa}}.$$

Proof. For the first estimate we again use the spectral properties of W, I, and J and obtain by paraproduct estimate

$$\begin{aligned}
\|J_s(W_t I_t(w) \succ I_t^\flat(u))\|^2_{L^2} &\lesssim \langle t\rangle^{-3}\|W_t\|^2_{L^\infty}\|I_t(w)\|^2_{L^4}\|I_t^\flat(u)\|^2_{L^4} \\
&\lesssim \langle t\rangle^{-3}\langle t\rangle^{1+4\kappa}\|W_t\|^2_{\mathscr{C}^{-1/2-\kappa}}\|I_t(w)\|^2_{L^4}\|I_t^\flat(u)\|^2_{L^4}
\end{aligned}$$

and the claim follows by Young's inequality. For the second

$$\|J_s((I_s(w))^2 \succ I_s^\flat(u))\|^2_{L^2} \lesssim \langle t\rangle^{2-2\kappa}\|(I_s(w))\|^4_{L^4}\|I_t^\flat(u)\|^2_{\mathscr{C}^{-1/2-\kappa}},$$

and the claim follows again by Young's inequality. □

Lemma 5.6. *Let $f_t \in C\left([0,\infty], \mathscr{C}^{-1/2-\delta}\right)$ and $g_t \in C([0,\infty], H^1)$ such that f_t, g_t have spectral support in a ball of radius proportional to t. Then the following estimates hold*

$$\|(J_t(f_t g_t))\|^2_{L^2} \lesssim \langle t\rangle^{-1+2\delta}\|f_t\|^2_{\mathscr{C}^{-1-\delta}}\|g_t\|^2_{L^2}$$

$$\|(J_t(f_t g_t))\|^2_{L^2} \lesssim \langle t\rangle^{-1+2\delta}\left(\|f_t\|^8_{\mathscr{C}^{-1-\delta}} + \|g_t\|^4_{H^{-1}} + \|g_t\|^2_{H^1}\right)$$

Proof.
$$\|(J_t(f_t g_t))\|^2_{L^2} \lesssim \langle t\rangle^{-3}\|f_t\|^2_{L^\infty}\|g_t\|^2_{L^2} \lesssim \langle t\rangle^{-1+2\delta}\|f_t\|^2_{\mathscr{C}^{-1-\delta}}\|g_t\|^2_{L^2}.$$
This proves the first estimate. For the second we continue
$$\begin{aligned}\langle t\rangle^{-1+2\delta}\|f_t\|^2_{\mathscr{C}^{-1-\delta}}\|g_t\|^2_{L^2} &\lesssim \langle t\rangle^{-1+2\delta}\|f_t\|^2_{\mathscr{C}^{-1-\delta}}\|g_t\|_{H^1}\|g_t\|_{H^{-1}} \\ &\lesssim \langle t\rangle^{-1+2\delta}\left(\|f_t\|^8_{\mathscr{C}^{-1-\delta}} + \|g_t\|^4_{H^{-1}} + \|g_t\|^2_{H^{-1}}\right).\end{aligned}$$ □

Lemma 5.7. *It holds*
$$\int_0^T\int_\Lambda (J_t(\mathbb{W}^2_t \succ I^\flat_t(w)))^2 \lesssim T^{3\delta}\left(\sup_t \|\mathbb{W}^2_t\|^2_{\mathscr{C}^{-1-\delta}}\right)(\sup_t \|I_t(w)\|^2_{L^2}),$$
and
$$\int_0^T\int_\Lambda (J_t(\mathbb{W}^2_t \succ I^\flat_t(w)))^2 \lesssim T^{3\delta}\left(\sup_t \|I_t(w)\|^4_{H^{-1}} + \int_0^T \|w_t\|^2_{L^2}\mathrm{d}t + \sup_t \|\mathbb{W}^2_t\|^8_{\mathscr{C}^{-1-\delta}}\right).$$

Proof. This follows in the same fashion as Lemma 5.6. □

A Besov spaces and paraproducts

In this section we will recall some well known results about Besov spaces, embeddings, Fourier multipliers and paraproducts. The reader can find full details and proofs in [3, 16].

First recall the definition of Littlewood–Paley blocks. Let χ, φ be smooth radial functions $\mathbb{R}^d \to \mathbb{R}$ such that

- $\operatorname{supp}\chi \subseteq B(0,R)$, $\operatorname{supp}\varphi \subseteq B(0,2R)\setminus B(0,R)$;
- $0 \leqslant \chi, \varphi \leqslant 1$, $\chi(\xi) + \sum_{j\geq 0}\varphi(2^{-j}\xi) = 1$ for any $\xi \in \mathbb{R}^d$;
- $\operatorname{supp}\varphi(2^{-j}\cdot) \cap \operatorname{supp}\varphi(2^{-i}\cdot) = \varnothing$ if $|i-j| > 1$.

Introduce the notations $\varphi_{-1} = \chi$, $\varphi_j = \varphi(2^{-j}\cdot)$ for $j \geqslant 0$. For any $f \in \mathscr{S}'(\Lambda)$ we define the operators $\Delta_j f := \mathscr{F}^{-1}_\xi(\varphi_j(\xi)\hat{f}(\xi))$, $j \geqslant -1$.

Definition A.1. *Let $s \in \mathbb{R}, p, q \in [1,\infty]$. For a Schwarz distribution $f \in \mathscr{S}'(\Lambda)$ define the norm*
$$\|f\|_{B^s_{p,q}} := \|(2^{js}\|\Delta_j f\|_{L^p})_{j\geqslant -1}\|_{\ell^q}.$$
Then the space $B^s_{p,q}$ is the closure of Schwarz distributions under this norm. We denote $\mathscr{C}^\alpha = B^\alpha_{\infty,\infty}$ the Besov–Hölder space and $H^\alpha = B^\alpha_{2,2}$ the Sobolev spaces.

Proposition A.2. *Let $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$ and $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$. Then $B^s_{p_1,q_1}$ is continuously embedded in $B^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,q_2}$.*

Proposition A.3. *For any $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$, any $p, q \in [1,\infty]$ the Besov space $B^{s_1}_{p,q}$ is compactly embedded into $B^{s_2}_{p,q}$.*

Definition A.4. *Let $f, g \in \mathscr{S}(\Lambda)$. We define the paraproducts*
$$f \succ g := \sum_{j<i-1}\Delta_i f \Delta_j g, \qquad \text{and} \qquad f \prec g := \sum_{j>i+1}\Delta_i f\Delta_j g = g \succ f.$$
Moreover we introduce the resonant product
$$f \circ g := \sum_{|i-j|\leqslant 1}\Delta_i f\Delta_j g.$$
Then $fg = f \prec g + f \circ g + f \succ g$.

Proposition A.5. *Let $\alpha < 0, \beta \in \mathbb{R}$. For $f, g \in \mathscr{S}(\Lambda)$ we have the estimates*

$$\begin{array}{llllll} \|f \succ g\|_{H^{\beta-\delta}} & \lesssim & \|f\|_{\mathscr{C}^\beta}\|g\|_{L^2}, & \|f \succ g\|_{\mathscr{C}^\beta} & \lesssim & \|f\|_{\mathscr{C}^\beta}\|g\|_{L^\infty}, \\ \|f \succ g\|_{H^{\beta-\alpha}} & \lesssim & \|f\|_{\mathscr{C}^\beta}\|g\|_{H^\alpha}, & \|f \succ g\|_{\mathscr{C}^\beta} & \lesssim & \|f\|_{\mathscr{C}^\beta}\|g\|_{\mathscr{C}^\alpha}. \end{array}$$

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$. Then

$$\|f \circ g\|_{H^{\alpha+\beta}} \lesssim \|f\|_{\mathscr{C}^\beta}\|g\|_{H^\alpha}, \qquad \|f \circ g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathscr{C}^\beta}\|g\|_{\mathscr{C}^\alpha}.$$

By density the paraproduct and resonant product also extend to bilinear operators on the respective spaces.

Proposition A.6. *Let $\alpha \in (0,1)$ $\beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma < 0$, $\alpha + \beta + \gamma > 0$. Then for $f, g, h \in \mathscr{S}$, and for any $\delta > 0$,*

$$\|(f \succ g) \circ h - g(f \circ h)\|_{H^{\alpha+\beta+\gamma-\delta}} \lesssim \|f\|_{\mathscr{C}^\gamma}\|h\|_{\mathscr{C}^\beta}\|g\|_{H^\alpha},$$

$$\|(f \succ g) \circ h - g(f \circ h)\|_{\mathscr{C}^{\alpha+\beta+\gamma}} \lesssim \|f\|_{\mathscr{C}^\gamma}\|h\|_{\mathscr{C}^\beta}\|g\|_{\mathscr{C}^\alpha}.$$

Proposition A.7. *Assume $f \in \mathscr{C}^\alpha, g \in H^\beta, h \in H^\gamma$ and $\alpha + \beta + \gamma = 0$. Then*

$$\int_{\mathbb{T}^d} [(f \succ g)h - (f \circ h)g] \lesssim \|f\|_{\mathscr{C}^\alpha}\|g\|_{H^\beta}\|h\|_{H^\gamma}.$$

Remark A.8. Proposition A.7 is not proven in the above references but is quite easy and the reader can fill out a proof.

Definition A.9. *A smooth function $\eta : \mathbb{R}^d \to \mathbb{R}$ is said to be an S^m-multiplier if for every multi-index α there exists a constant C_α such that*

$$\left| \frac{\partial^\alpha}{\partial \xi^\alpha} f(\xi) \right| \lesssim_\alpha (1 + |\xi|)^{m-|\alpha|}, \qquad \forall \xi \in \mathbb{R}^d. \tag{A.1}$$

We say that a family $(\eta_t)_{t \geqslant 0}$ is a uniform S^m-multiplier if (A.1) is satisfied for every η_t with C_α independent of $t \geqslant 0$.

Proposition A.10. *Let η be an S^m-multiplier, $s \in \mathbb{R}$, $p, q \in [1, \infty]$, and $f \in B^s_{p,q}(\mathbb{T}^d)$, then*

$$\|\eta(\mathrm{D})f\|_{B^{s-m}_{p,q}} \lesssim \|f\|_{B^s_{p,q}}.$$

Furthermore the constant depends only on s, p, q, d and the constants C_α in (A.1).

Proposition A.11. *Assume $m \leqslant 0$, $\alpha \in (0,1), \beta \in \mathbb{R}$. Let η be an S^m-multiplier, $f \in \mathscr{C}^\beta$, $g \in H^\alpha$. Then for any $\delta > 0$.*

$$\|\eta(\mathrm{D})(f \succ g) - (\eta(\mathrm{D})f \succ g)\|_{H^{\alpha+\beta-m-\delta}} \lesssim \|f\|_{\mathscr{C}^\beta}\|g\|_{H^\alpha}.$$

Again the constant depends only on α, β, δ and the constants in (A.1).

Proposition A.12. *Let $\delta > 0$.We have for any $q_1, q_2 \in [1, \infty], q_1 < q_2$*

$$\|f\|_{B^s_{p,q_2}} \leqslant \|f\|_{B^s_{p,q_1}} \leqslant \|f\|_{B^{s+\delta}_{p,\infty}}.$$

Furthermore, if we denote by $W^{s,p}$, $s \in \mathbb{R}, p \in [1, \infty]$ the fractional Sobolev spaces defined by the norm $\|f\|_{W^{s,q}} := \|\langle \mathrm{D} \rangle^s f\|_{L^q}$, then, for any $q \in [1, \infty]$,

$$\|f\|_{B^s_{p,q}} \leqslant \|f\|_{W^{s+\delta,p}} \leqslant \|f\|_{B^{s+2\delta}_{p,\infty}}.$$

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