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Gradient of the single layer potential and quantitative rectifiability for general Radon measures

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ABSTRACT

We identify a set of sufficient local conditions under which a significant portion of a Radon measure μ on $\mathbb{R}^{n+1}$ with compact support can be covered by an uniformly n-rectifiable set, at the level of a ball $B \subset \mathbb{R}^{n+1}$ such that $\mu(B) \approx r(B)^n$. This result involves a flatness condition, formulated in terms of the so-called β_1-number of B, and the $L^2(\mu|_B)$-boundedness, as well as a control on the mean oscillation on the ball, of the operator

$$T_\mu f(x) = \int \nabla_x \mathcal{E}(x,y) f(y)\, d\mu(y).$$

Here $\mathcal{E}(\cdot,\cdot)$ is the fundamental solution for a uniformly elliptic operator in divergence form associated with an $(n+1)\times(n+1)$ matrix with Hölder continuous coefficients. This generalizes a work by Girela-Sarrión and Tolsa for the n-Riesz transform.

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The motivation for our result stems from a two-phase problem for the elliptic harmonic measure.

Contents

1. Introduction

Singular integral operators are deeply related to geometric properties of measures and, in particular, with the concepts of rectifiability and uniform rectifiability. An operator that has been profusely studied in this field is the codimension 1 Riesz transform, which has a wide relevance because it arises as the gradient of the single layer potential associated with the laplacian. In particular, it has also been used to investigate the properties of harmonic measure. For this reason, it is interesting to understand if the results which are known for the Riesz transform generalize to operators defined by gradients of single layer potentials associated with suitable elliptic PDEs. In this spirit, the aim of the present article is to establish an elliptic equivalent of a quantitative rectifiability theorem that Girela-Sarrión and Tolsa proved in [21] for the Riesz transform, which is also a crucial tool in the study of certain non-variational free-boundary problems for harmonic measure.

Given a Radon measure μ on $\mathbb{R}^{n+1}$, its associated n-dimensional Riesz transform is

$$\mathcal{R}^n_\mu f(x) = \int \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mu(y), \qquad f \in L^1_{\mathrm{loc}}(\mu),$$

whenever the integral makes sense. Given $x \in \mathbb{R}^{n+1}$ and $r > 0$, we denote by $B(x,r)$ the open ball of center x and radius r. A Radon measure μ has growth of degree n if there exists a constant $C > 0$ such that

$$\mu\big(B(x,r)\big) \leq Cr^n \qquad \text{for all } x \in \mathbb{R}^{\mathrm{n}+1},\ r > 0. \tag{1.1}$$

We call μ n-Ahlfors-David regular (also abbreviated by n-AD-regular or just AD-regular) if there exists $C > 0$, which is referred to as AD-regularity constant, such that

$$C^{-1}r^n \leq \mu\big(B(x,r)\big) \leq Cr^n \qquad \text{for all } x \in \operatorname{supp}\mu,\ 0 < r < \operatorname{diam}(\operatorname{supp}\mu).$$

A set $E \subset \mathbb{R}^{\mathrm{n}+1}$ is said n-AD-regular if $\mathcal{H}^n|_E$ is a n-AD-regular measure, $\mathcal{H}^n$ denoting the n-dimensional Hausdorff measure in $\mathbb{R}^{\mathrm{n}+1}$. Note that the support of an n-AD-regular measure is n-AD-regular.

A set $E \subset \mathbb{R}^{\mathrm{n}+1}$ is called n-rectifiable if there exists a countable family of Lipschitz functions $f_j \colon \mathbb{R}^n \to \mathbb{R}^{\mathrm{n}+1}$ such that

$$\mathcal{H}^n\Big(E \setminus \bigcup_j f_j(\mathbb{R}^n)\Big) = 0.$$

A measure μ is n-rectifiable if it vanishes outside a rectifiable set E and, moreover, it is absolutely continuous with respect to $\mathcal{H}^n|_E$.

A set E is said uniformly n-rectifiable (or just uniformly rectifiable) if it is n-AD regular and there exist $\theta, M > 0$ such that for all $x \in E$ and all $r > 0$ there is a Lipschitz mapping g from the ball $B_n(0,r) \subset \mathbb{R}^n$ to $\mathbb{R}^{\mathrm{n}+1}$ with $\operatorname{Lip}(g) \leq M$ such that

$$\mathcal{H}^n\big(E \cap B(x,r) \cap g(B_n(0,r))\big) \geq \theta r^n.$$

We say that a measure μ is uniformly n-rectifiable if it is n-AD-regular and it vanishes outside of a uniformly n-rectifiable set.

Many characterizations of uniformly rectifiable measures are available in the literature (see e.g. the monograph [18]). To the scopes of the present paper, we are particularly interested in those which are formulated in terms of singular integrals.

David and Semmes showed in [17] that, under the background n-AD-regularity assumption, a measure μ is uniformly n-rectifiable if and only if all the singular integral operators associated to μ with smooth antisymmetric convolution-type kernels are $L^2(\mu)$-bounded. This class includes the n-Riesz transform and it is interesting to understand if the $L^2(\mu)$-boundedness of $\mathcal{R}^n_\mu$ alone suffices to characterize uniform rectifiability.

This deep question is usually referred to as *David-Semmes problem in codimension 1*. It has been solved for $n = 1$ (or, equivalently, for the Cauchy transform) by Mattila, Melnikov and Verdera in [35] using the so-called Menger curvature of a measure, and by Nazarov, Tolsa and Volverg in [40] for $n > 1$ via a set of techniques that includes a variational argument and rely on the harmonicity of the codimension 1 Riesz kernel. We remark that the David-Semmes problem is formulated, in its more general form, also for higher codimensions, and its full solution is not known yet.

The codimension 1 case has various applications, and plays a crucial role in the study of the geometric properties of harmonic measure. In particular, it was used in [6] to prove that mutual absolute continuity of the harmonic measure associated with an open set $\Omega \subset \mathbb{R}^{n+1}$ with respect to surface measure $\mathcal{H}^n$ in a subset of $\partial\Omega$ implies the n-rectifiability of that subset. This answered a problem raised by Bishop (see [13]).

The analogous result for elliptic measure relative to an operator in divergence form associated with a uniformly elliptic matrix with Hölder coefficients has been proved in [43], following the ideas of [6], as an application of the characterization of uniform rectifiability via the boundedness of the gradient of single layer potential. The same problem was also tackled in [50] and [8] via alternative techniques, under more restrictive assumptions on the domains and different hypotheses on the coefficients of the uniformly elliptic matrix.

Another question proposed by Bishop asks whether, given two disjoint domains $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$, mutual absolute continuity of their respective harmonic measures implies absolute continuity with respect to surface measure in $\partial\Omega_1 \cap \partial\Omega_2$ and rectifiability.

This is a so-called *two-phase problem* for harmonic measure and was solved in its full generality in [11]. This work relies on three main tools: a blow-up argument for harmonic measure (see [29] and [9]), a monotonicity formula ([2]) and a quantitative rectifiability criterion (see [21]).

In particular, we point out that the theorem by Girela-Sarrión and Tolsa can be interpreted as a higher-dimensional version of previous results by David and Léger, which were formulated in terms of the Menger curvature (see [15] and [31]). Their theorems are of fundamental importance also in other two-phase problems examined in the works [10] and [44]. The goal of the present paper is to identify an analogous criterion in the context of elliptic PDEs in divergence form with Hölder continuous coefficients.

Let $A = (a_{ij})_{1\le i,j\le n+1}$ be an $(n+1)\times(n+1)$ matrix whose entries $a_{ij}\colon \mathbb{R}^{n+1} \to \mathbb{R}$ are measurable functions in $L^\infty(\mathbb{R}^{n+1})$. Assume also that there exists $\Lambda > 0$ such that

$$\Lambda^{-1}|\xi|^2 \le \langle A(x)\xi, \xi\rangle, \qquad \text{for all } \xi \in \mathbb{R}^{n+1} \text{ and a.e. } x \in \mathbb{R}^{n+1}, \tag{1.2}$$

$$\langle A(x)\xi, \eta\rangle \le \Lambda|\xi||\eta|, \qquad \text{for all } \xi, \eta \in \mathbb{R}^{n+1} \text{ and a.e. } x \in \mathbb{R}^{n+1}. \tag{1.3}$$

We consider the elliptic equation

$$L_A u(x) := -\mathrm{div}\left(A(\cdot)\nabla u(\cdot)\right)(x) = 0, \tag{1.4}$$

which should be understood in the distributional sense: we say that a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a *solution* of (1.4), or L_A*-harmonic*, in an open set $\Omega \subset \mathbb{R}^{n+1}$ if

$$\int A\nabla u \cdot \nabla\varphi = 0, \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

We denote by $\mathcal{E}_A(x,y)$, or just by $\mathcal{E}(x,y)$ when the matrix A is clear from the context, the *fundamental solution* for L_A in $\mathbb{R}^{n+1}$, so that $L_x\mathcal{E}_A(x,y) = \delta_y$ in the distributional sense, where δ_y is the Dirac mass at the point $y \in \mathbb{R}^{n+1}$. For a construction of the fundamental solution under the assumptions (1.2) and (1.3) on the matrix A we refer to [24]. Given a measure μ, the function $f(x) = \int \mathcal{E}_A(x,y)\,d\mu(y)$ is usually known as the *single layer potential* of μ. We define

$$K(x,y) = \nabla_1 \mathcal{E}_A(x,y), \tag{1.5}$$

the subscript 1 indicating that we take the gradient with respect to the first variable, and we consider (1.5) as the kernel of the singular integral operator

$$T\mu(x) = \int K(x,y)\,d\mu(y),$$

for x away from $\operatorname{supp}(\mu)$. Observe that $T\mu$ is the gradient of the single layer potential of μ.

Given a function $f \in L^1_{loc}(\mu)$, we set also

$$T_\mu f(x) = T(f\,\mu)(x) = \int K(x,y) f(y)\,d\mu(y),$$

and, for $\varepsilon > 0$, we consider the ε-truncated version

$$T_\varepsilon\mu(x) = \int\limits_{|x-y|>\varepsilon} K(x,y)\,d\mu(y).$$

We also write $T_{\mu,\varepsilon}f(x) = T_\varepsilon(f\mu)(x)$. We say that the operator T_μ is bounded on $L^2(\mu)$ if the operators $T_{\mu,\varepsilon}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$.

In the specific case when A is the identity matrix, we have that $-L_A = \Delta$ and T_μ is the n-dimensional Riesz transform up to a dimensional constant factor. We say that the matrix A is Hölder continuous with exponent $\alpha \in (0,1)$ (or briefly C^α continuous), if there exists $C_h > 0$ such that

$$|a_{ij}(x) - a_{ij}(y)| \le C_h|x-y|^\alpha \quad \text{for all } x,y \in \mathbb{R}^{n+1} \text{ and } 1 \le i,j \le n+1. \tag{1.6}$$

Under this assumption on the coefficients, the kernel $K(\cdot,\cdot)$ turns out to be locally of Calderón-Zygmund type (see Lemma 2.1). However we remark that, contrarily to what happens in the case of the kernel of the Riesz transform, in general $K(\cdot,\cdot)$ is neither homogeneous nor antisymmetric.

Under the assumption (1.6) together with uniform ellipticity, it has been shown by Conde-Alonso, Mourgoglou and Tolsa in [14, Theorem 2.5] that T_μ is bounded on $L^2(\mu)$ if μ is a uniformly n-rectifiable measure with compact support. Moreover, they also proved that, if μ is a non-zero Borel measure whose upper n-density is positive μ-a.e. and the lower n-density vanishes μ-a.e. in $\mathbb{R}^{n+1}$, then T_μ is *not* bounded on $L^2(\mu)$. This result was proved for $\mathcal{R}^n_\mu$ by Eiderman, Nazarov and Volverg in [20] and it inspired the variational argument of [40]. We remark that the results of [14] have been recently further extended by Bailey, Morris and Reguera (see [12]) to Schrödinger operators of the form $L_A^V = -\mathrm{div} A\nabla + V$, for A Hölder continuous and the potential V belonging to the reverse Hölder class RH_{n+1}.

Furthermore, Prat, Puliatti, and Tolsa proved in [43] that, under the same assumptions of [14] on L_A, an elliptic analogue of the codimension 1 David-Semmes problem holds: if the measure μ is n-AD-regular, has compact support and T_μ is bounded on $L^2(\mu)$, then μ is uniformly n-rectifiable.

For our applications, it is essential to determine whether $T_{\mu,\varepsilon} f$ converges pointwise μ-almost everywhere for $\varepsilon \to 0$. In case it does, we denote the limit as

$$\mathrm{pv}\, T_\mu f(x) = \lim_{\varepsilon\to 0} T_{\mu,\varepsilon} f(x)$$

and we refer to it as the *principal value* of the integral $T_\mu f(x)$. We prove that, analogously to the n-Riesz transform (see [48, Chapter 8] and the references therein), the $L^2(\mu)$-boundedness of T_μ entails the existence of the principal values for general Radon measures with compact support and growth of degree n. In the following statement, $M(\mathbb{R}^{n+1})$ indicates the vector space of Borel real finite measure on $\mathbb{R}^{n+1}$, which is a Banach space when endowed with the total variation norm.

Theorem 1.1. *Let μ be a Radon measure on $\mathbb{R}^{n+1}$ with compact support and with growth of degree n, i.e. suppose that there is $C > 0$ such that*

$$\mu(B(x,r)) \leq Cr^n \quad \textit{for all } \; x \in \mathbb{R}^{n+1}\,.$$

Let A be a matrix that satisfies (1.2), (1.3) and (1.6) and assume, moreover, that the gradient of the single layer potential T_μ associated with L_A is bounded on $L^2(\mu)$. Then:

(1) for $1 \leq p < \infty$ and all $f \in L^p(\mu)$, $\mathrm{pv}\, T_\mu f(x)$ exists for μ-a.e. $x \in \mathbb{R}^{n+1}$;
(2) for all $\nu \in M(\mathbb{R}^{n+1})$, $\mathrm{pv}\, T\nu(x)$ exists for μ-a.e. $x \in \mathbb{R}^{n+1}$.

In light of this result, in the rest of the paper we will often denote the principal value operator simply as $T\nu$, with a slight abuse of notation. We remark that the previous theorem was first proved in the case of the Cauchy transform by Tolsa in [47].

Given a ball $B = B(x,r) \subset \mathbb{R}^{n+1}$, we denote by $r(B)$ its radius and, for $a > 0$, by aB its dilation $B(x,ar)$. Multiple notions of density come into play in this paper. For a ball B, we denote

$$\Theta_\mu(B) = \frac{\mu(B)}{r(B)^n}$$

and, for $\gamma > 0$, its smoothened version

$$P_{\mu,\gamma}(B) := \sum_{j\geq 0} 2^{-j\gamma}\Theta_\mu(2^j B). \tag{1.7}$$

We remark that if $\gamma_1 \leq \gamma_2$, then

$$P_{\mu,\gamma_2}(B) = \sum_{j\geq 0} 2^{-j\gamma_2}\Theta_\mu(2^j B) \leq \sum_{j\geq 0} 2^{-j\gamma_1}\Theta_\mu(2^j B) = P_{\mu,\gamma_1}(B).$$

Another notion of density that we need is the pointwise one. In particular, we denote the upper and lower n-densities of μ at x respectively as

$$\Theta^*_\mu(x) := \limsup_{r\to 0} \frac{\mu\big(B(x,r)\big)}{(2r)^n} \quad \text{and} \quad \Theta_{*,\mu}(x) := \liminf_{r\to 0} \frac{\mu\big(B(x,r)\big)}{(2r)^n}.$$

A way to quantify the flatness of a measure at the level of a ball B is in terms of the β_1-coefficients. For an n-plane L we denote

$$\beta^L_{\mu,1}(B) = \frac{1}{r(B)^n}\int_B \frac{\operatorname{dist}(x,L)}{r(B)} d\mu(x) \qquad \text{and} \qquad \beta_{\mu,1}(B) = \inf_L \beta^L_{\mu,1}(B),$$

the infimum being taken over all hyperplanes in $\mathbb{R}^{n+1}$. Using a standard notation, given $E \subset \mathbb{R}^{n+1}$ with $\mu(E) > 0$ and $f \in L^1_{loc}(\mu)$ we write

$$m_{\mu,E}(f) = \frac{1}{\mu(E)}\int_E f\, d\mu$$

for the mean of f with respect to the measure μ on the set E. The main result of the paper is the following.

Theorem 1.2. *Let $n > 1$, let μ be a Radon measure on $\mathbb{R}^{n+1}$ with compact support and consider an open ball $B \subset \mathbb{R}^{n+1}$. Let $C_0, C_1 > 0$ and let A be a matrix satisfying (1.2), (1.3) and (1.6). Denote by T_μ the gradient of the single layer potential associated with L_A and μ. Suppose that μ and B are such that, for some positive λ, δ and ε and some $\widetilde{\alpha} \in (0,1)$, the following properties hold*

(1) $r(B) \leq \lambda$.
(2) $C_0^{-1} r(B)^n \leq \mu(B) \leq C_0 r(B)^n$.
(3) $P_{\mu,\widetilde{\alpha}}(B) \leq C_0$ and for all $x \in B$ and $0 < r \leq r(B)$ we have $\mu\big(B(x,r)\big) \leq C_0 r^n$.
(4) $T_{\mu|_B}$ is bounded on $L^2\big(\mu|_B\big)$ with $\|T_{\mu|_B}\|_{L^2(\mu|_B)\to L^2(\mu|_B)} \leq C_1$ and $T\big(\chi_{2B}\mu\big) \in L^2\big(\mu|_B\big)$.
(5) $\beta_{\mu,1}(B) \leq \delta$.

(6) We have

$$\int_B \left|T\mu(x) - m_{\mu,B}(T\mu)\right|^2 d\mu(x) \leq \varepsilon\mu(B).$$

There exists a choice of λ, δ and ε small enough and a proper choice of $\widetilde{\alpha} = \widetilde{\alpha}(\alpha, n)$, all possibly depending on C_0 and C_1, such that if μ satisfies (1)–$\cdots$–(6), there exists a uniformly n-rectifiable set Γ that covers a big portion of the support of μ inside B. That is to say, there exists $\tau > 0$ such that

$$\mu(B \cap \Gamma) \geq \tau\mu(B).$$

Notice that Theorem 1.2 readily implies that a big piece of $\mu|_B$ is mutually absolutely continuous with a big piece of $\mathcal{H}^n|_\Gamma$. This is a relevant feature in light of possible applications. At the moment, very few rectifiability criteria for general measures in terms of singular integrals are available in the literature. Our result is a non trivial extension of [21, Theorem 1.1] to more general operators. A prototype for these results can be found in a quantitative version David-Léger theorem [31, Proposition 1.2]. Given $x, y, z \in \mathbb{R}^2$, define $c(x, y, z) = 0$ if the three points are aligned and, otherwise, $c(x, y, z) = R(x, y, z)^{-1}$ where $R(x, y, z)$ stands for the radius of the circumference containing x, y and z. The theorem asserts that, given a measure μ on $\mathbb{R}^2$ with growth of degree 1, and a ball $B(x, r)$ such that $\mu(B) \approx r$, there exists $\varepsilon > 0$ such that the following holds: if $\nu := \mu|_B$ is such that

$$c^2(\nu) := \iiint c(x, y, z)^2 \, d\nu(x) \, d\nu(y) \, d\nu(z) \leq \varepsilon\mu(B), \tag{1.8}$$

then there exists a possibly rotated Lipschitz graph Γ on the plane such that $\mu(B \cap \Gamma) \geq \frac{99}{100}\mu(B)$. The quantity in the left hand side of (1.8) is the so-called Menger curvature of ν and it was introduced in this field by Melnikov in [36], who also showed together with Verdera in [37] that $c(\nu)$ is (modulo an error term) comparable to the $L^2(\nu)$-norm of the Cauchy transform of ν. However, Menger curvature cannot be used to link the L^2-boundedness of $\mathcal{R}^n_\mu$ to uniform n-rectifiability, if $n > 1$. Hence, in a sense, the property (6) in Theorem 1.2 can be interpreted as a suitable substitute of (1.8).

The formulation of Theorem 1.2 involves a relatively long list of hypotheses. On the other hand we remark that those assumptions are necessary, natural and, most importantly, optimal for the application to elliptic measure.

The assumption (1), namely requiring the ball B to be small enough, represents a relevant conceptual difference with respect to the analogous theorem for the Riesz transform. The locality of our result reflects the non-scale invariant character of the Hölder regularity assumption for the coefficients of the matrix A. This issue is evident also in [43], where the elliptic analogue of the codimension 1 David-Semmes problem requires an additional compactness assumption on the support of the measure. We remark that the assumption (1) is particularly relevant for a localization estimate (see Lemma 9.3), and it allows to bound efficiently the error term in the variational argument of Section 11 (see Subsection 11.2). It is also important to mention that the locality of (1) does not affect the applicability of Theorem 1.2 to the two-phase problem for the elliptic measure, which is of qualitative nature.

The second part of the requirement (3) excludes a class of measures like, for instance, the area measure. It is verified, for example, if μ has growth of degree n in the sense of (1.1). On the other hand, the assumption $P_{\mu,\widetilde{\alpha}}(B) \leq C_0$ plays a different role and is useful to deal with technical problems. For example, as shown at the end of Section 3, it is important to ensure that $T\mu(x) - m_{\mu,B}(T\mu)$ is defined in a BMO-sense, and it is also essential to obtain proper localization estimates for the gradient of the single layer potential (see Lemma 6.1 and its proof). Furthermore we remark that, in order to show that the integral in the left-hand side of the assumption (6) is well-defined, we also use the existence of principal values.

As in the main result of Girela-Sarrión and Tolsa, the hypothesis (5) is important for technical reasons. However, it is not known if is necessary for the result to hold. Indeed, in the case $n = 1$ covered by the David-Léger theorem, an analogous flatness assumption is not needed: under the assumptions discussed before (1.8), the inequality $c^2(\mu|_B) \leq \varepsilon\mu(B)$ implies the existence of a line L such that $\beta^L_{\mu,1}(B) \leq \delta(\varepsilon)\mu(B)$, with $\delta \to 0$ as $\varepsilon \to 0$. However, having to ask $\beta_{\mu,1}(B) \ll 1$ does not constitute a problem when applying the theorem to the study of the two-phase problem for elliptic measure, see Section 12.

Another difference with respect to [21] is that we could not formulate the theorem in terms of $P_{\mu,1}$. Our proof of the theorem shows that a good choice for $\widetilde{\alpha}$ is $\widetilde{\alpha} = \alpha/2^{n+1}$. It is not clear whether Theorem 1.2 holds with a condition on $P_{\mu,\alpha}(B)$, that seems a more natural homogeneity to assume. The proofs of the rectifiability results for the harmonic measure in [9] and [11] actually rely on the fact that the theorem of Girela-Sarrión and Tolsa holds for $\widetilde{\alpha} = 1$. However, a slight variation on their arguments allows to overcome this technical obstacle.

Let us now present an application of Theorem 1.2, which is, in fact, its main motivation. Before stating it, recall that if Ω is a Wiener regular set, the elliptic measure $\omega^p_{L_A}$ with pole at p associated with the elliptic operator L_A is the probability measure supported on $\partial\Omega$ such that, for $f \in C_0(\partial\Omega)$,

$$\int f \, d\omega^p_{L_A} = \widetilde{f}(p),$$

where $\widetilde{f}$ denotes the L_A-harmonic extension of f. A large literature is available on the subject. For example, we refer to [22] and [27] (and the references therein) for its definition and basic properties.

Theorem 1.3. *Let $n \geq 2$ and let A be an elliptic matrix satisfying (1.2), (1.3) and (1.6). Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ be two disjoint Wiener-regular domains and, for $p_i \in \Omega_i$, $i \in \{1, 2\}$,* *let $\omega^{p_i}_{L_A,i}$ be the respective elliptic measures in Ω_i associated with L_A and with pole p_i. Suppose that E is a Borel set such that $\omega^{p_1}_{L_A,1}|_E \ll \omega^{p_2}_{L_A,2}|_E \ll \omega^{p_1}_{L_A,1}|_E$. Then there exists an n-rectifiable set $F \subset E$ with $\omega^{p_1}_{L_A,1}(E \setminus F) = 0$, and such that $\omega^{p_1}_{L_A,1}|_F$ and $\omega^{p_2}_{L_A,2}|_F$ are mutually absolutely continuous with respect to $\mathcal{H}^n|_F$.*

We remark that the generalization of the blow-up methods for the harmonic measure to our elliptic context is contained in the work [8]. Also, the proof of Theorem 1.3 follows closely the path of the work [11]. However, some relevant variations are needed so we decided to sketch the proof at the end of the paper, where we also provide precise references for the reader's convenience.

We finally remark that recently several studies have appeared concerning the connection between the geometry of a domain and the properties of its associated elliptic measure, among which we list [1], [5], [23], [25], [26] and [28].

Discussion of the proofs. For the proof of the main theorem we follow the elaborated scheme of [21]. However, there are many delicate obstacles which are not present when dealing with the Riesz transform and that require original approaches. Section 2 is devoted to settle our notation and to make an overview of the results in PDEs relevant for our work. In particular, we need a rescaled version of some estimate for the gradient of the fundamental solution first proved in the context of homogenization theory, which are indispensable to estimate the behavior of $\mathcal{E}_A(x, y)$ for big values of $|x - y|$.

In *Section 3* we prove Theorem 1.1 separating the case in which μ is rectifiable to the one in which it has zero n-density. This is possible because Prat, Puliatti, and Tolsa proved in [43] that the $L^2(\mathcal{H}^n|_\Gamma)$-boundedness of $T_{\mathcal{H}^n|_\Gamma}$, $\Gamma \subset \mathbb{R}^{n+1}$ compact with $\mathcal{H}^n(\Gamma) < \infty$, implies the rectifiability of Γ, generalizing a result first proved in [41].

Then, in *Section 4* we proceed to state the Main Lemma that we use to prove Theorem 1.2. The biggest advantage of this lemma is that the flatness condition on the β_1-number in Theorem 1.2 is replaced by a smallness hypothesis on the α-numbers of Tolsa. The latter are more powerful tools when trying to transfer the flatness estimates to the integrals. Furthermore, we have to show that we can consider the matrix A of the elliptic operator which defines T_μ to be symmetric.

We then proceed to discuss, in *Section 5*, an equivalent formulation of the Main Lemma in terms of an auxiliary elliptic operator which shares more symmetries than L_A. In particular, we discuss the construction of a particular auxiliary periodic matrix via a sequence of reflections.

In *Sections 6, 7, 8 and 9* we recall the definition of the dyadic cells associated with μ as constructed by David and Mattila, and, in an attempt to balance brevity and clarity, we mainly follow the path of the original work for the Riesz transform. However, we remark that these sections are necessary for the sake of the exposition; indeed, they present the core of the contradiction argument for the proof of the Main Lemma and the construction of a periodic auxiliary measure which is needed for the arguments of the remaining sections of the paper.

The starting point for the crucial variational argument, whose proof occupies Section 11, is a localization estimate for the potential at the level of a small cube. This is proved in *Section 10*, along with the existence of the limit of proper smooth truncates of the potential of bounded periodic functions. We emphasize that, again, these proofs rely heavily on the periodicity of the modification of the elliptic matrix.

In *Section 11* we complete the proof of the Main Lemma via a variational technique. We highlight that one of the most delicate point consists in finding an appropriate variant of a maximum principle in an infinite strip in our elliptic setting. Moreover, the proof in [21] exploits the fact that the map $x = (x_1, \dots, x_{n+1}) \mapsto x_{n+1}$ is harmonic. The fact that this function, in general, is not a solution of L_A requires a more technical method based on the additional symmetries provided by the modified matrix.

In the final *Section 12* we sketch the proof of Theorem 1.3, with particular care of highlighting the points which require additional explanations with respect to its harmonic counterpart.

2. Preliminaries and notation

We write $a \lesssim b$ to denote that there is a constant $C > 0$ such that $a \leq Cb$. To make the dependence of the constant on a parameter t explicit, we write $a \lesssim_t b$. Also, we say that $b \gtrsim a$ if $a \lesssim b$ and $a \approx b$ if both $a \lesssim b$ and $b \lesssim a$.

All the cubes, unless specified, will be considered with their sides parallel to the coordinate axes. Given a cube Q, we denote its side length as $\ell(Q)$ and, for $a > 0$, we understand aQ as the cube with side length $a\ell(Q)$ and sharing the center with Q.

We say that a cube Q has t-thin boundary if

$$\mu\big\{x \in 2Q : \operatorname{dist}(x, \partial Q) \leq \lambda \ell(Q)\big\} \leq t\lambda\mu(2Q)$$

for every $\lambda > 0$. Analogously to (1.7), we define

$$P_{\mu,\gamma}(Q) = \sum_{j \geq 0} 2^{-j\gamma} \Theta_\mu(2^j Q) = \sum_{j \geq 0} 2^{-j\gamma} \frac{\mu(2^j Q)}{\ell(2^j Q)^n}.$$

Given a measure μ and a measurable set E, we denote as $\mu|_E$ the restriction of μ to E and, for $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, we use the notation $\phi\sharp\mu(E) := \mu(\phi^{-1}(E))$. An important tool in the study of rectifiability is the so-called α-number introduced by Tolsa in [49]. Let us fix a cube $Q \subset \mathbb{R}^{n+1}$ and consider two Radon measures μ and ν on $\mathbb{R}^{n+1}$. A natural way to define a distance between μ and ν is to consider the supremum

$$d_Q(\mu, \nu) := \sup_f \int f d(\mu - \nu),$$

where $f \in \operatorname{Lip}(\mathbb{R}^{n+1})$, $\|f\|_{\operatorname{Lip}} \leq 1$ and $\operatorname{supp} f \subseteq Q$. For a n-plane L in $\mathbb{R}^{n+1}$, we define

$$\alpha_\mu^L(Q) \coloneqq \frac{1}{\ell(Q)^{n+1}} \inf_{c\geq 0} d_Q(\mu, c\mathcal{H}^n|_L). \tag{2.1}$$

Given a matrix $A(\cdot)$, possibly with variable coefficients, $A^T(\cdot)$ indicates its transpose. Also, we write $\mathcal{L}^{n+1}$ for the Lebesgue measure on $\mathbb{R}^{n+1}$.

Partial Differential Equations. For any uniformly elliptic matrix A with Hölder continuous coefficients, one can show that $K(x,y) = \nabla_1 \mathcal{E}(x,y)$ is locally a Calderón-Zygmund kernel.

Lemma 2.1. *Let A be an elliptic matrix with Hölder continuous coefficients satisfying (1.2), (1.3) and (1.6). If $K(\cdot,\cdot)$ is given by (1.5), then it is locally a Calderón-Zygmund kernel. That is, for any given $R > 0$,*

(a) $|K(x,y)| \lesssim |x-y|^{-n}$ *for all* $x, y \in \mathbb{R}^{n+1}$ *with* $x \neq y$ *and* $|x-y| \leq R$.
(b) $|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)| \lesssim |y-y'|^\alpha |x-y|^{-n-\alpha}$ *for all* $y, y' \in B(x,R)$ *with* $2|y-y'| \leq |x-y|$.
(c) $|K(x,y)| \lesssim |x-y|^{(1-n)/2}$ *for all* $x, y \in \mathbb{R}^{n+1}$ *with* $|x-y| \geq 1$.

All the implicit constants in (a), (b) and (c) depend on Λ and $\|A\|_\alpha$, while the ones in (a) and (b) depend also on R.

The statements above are rather standard. For more details, see [14, Lemma 2.1]. Let ω_n denote the surface measure of the unit sphere of $\mathbb{R}^{n+1}$. For any elliptic matrix A_0 with constant coefficients, we have an explicit expression for the fundamental solution of L_{A_0}, which we denote by $\Theta(x,y;A_0)$. More precisely, $\Theta(x,y;A_0) = \Theta(x-y;A_0)$ with

$$\Theta(z;A_0) = \Theta(z;A_{0,s}) = \begin{cases} \dfrac{-1}{(n-1)\omega_n\sqrt{\det A_{0,s}}} \dfrac{1}{(A_{0,s}^{-1}z\cdot z)^{(n-1)/2}} & \text{for } n \geq 2, \\ \dfrac{1}{4\pi\sqrt{\det A_{0,s}}} \log\left(A_{0,s}^{-1}z\cdot z\right) & \text{for } n = 1, \end{cases} \tag{2.2}$$

where $A_{0,s}$ is the symmetric part of A_0, that is, $A_{0,s} = \frac{1}{2}(A_0 + A_0^T)$. For more details we refer to [38].

The reason why only the symmetric part of A_0 enters (2.2) it that, using Schwarz's theorem to exchange the order of partial derivatives writing $A_0 = \{a_{ij}\}_{i,j}$, for every appropriate function u we have

$$\begin{aligned} L_{A_0}u &= -\sum_{i,j} \partial_i(a_{ij}\partial_j u) = -\frac{1}{2}\sum_{i,j} a_{ij}\partial_i\partial_j u - \frac{1}{2}\sum_{i,j} a_{ij}\partial_j\partial_i u \\ &= -\sum_{i,j} \frac{a_{ij}+a_{ji}}{2}\partial_i\partial_j u = L_{A_{0,s}}u. \end{aligned} \tag{2.3}$$

These formal considerations can be made rigorous by standard arguments.

Differentiating (2.2) we obtain

$$\nabla\Theta(z;A_0)=\frac{1}{\omega_n\sqrt{\det A_{0,s}}}\frac{A_{0,s}^{-1}z}{(A_{0,s}^{-1}z\cdot z)^{(n+1)/2}}.$$

The next result is proven in [30, Lemma 2.2].

Lemma 2.2. *Let A be an elliptic matrix with Hölder continuous coefficients satisfying (1.2), (1.3) and (1.6). Let also $\Theta(\cdot,\cdot;\cdot)$ be given by (2.2). Then, for $x,y\in\mathbb{R}^{n+1}$, $0<|x-y|\le R$,*

(1) $|\mathcal{E}_A(x,y)-\Theta(x,y;A(x))|\lesssim|x-y|^{\alpha-n+1}$,
(2) $|\nabla_1\mathcal{E}_A(x,y)-\nabla_1\Theta(x,y;A(x))|\lesssim|x-y|^{\alpha-n}$,
(3) $|\nabla_1\mathcal{E}_A(x,y)-\nabla_1\Theta(x,y;A(y))|\lesssim|x-y|^{\alpha-n}$.

Similar inequalities hold if we reverse the roles of x and y and we replace ∇_1 by ∇_2. All the implicit constants depend on Λ, $\|A\|_\alpha$, and R.

The gradient of the fundamental solution in the periodic case. We denote as Λ_α the set of matrices such that (1.2), (1.3) hold and with α-Hölder coefficients. We say that the matrix $A\in\Lambda_\alpha$ is ℓ-periodic, $\ell>0$, if

$$A(x+\ell z)=A(x)\quad\text{for every } z\in\mathbb{Z}^{n+1}.$$

For periodic matrices the estimates in Lemma 2.1 turn out to be global.

Lemma 2.3 *([30]). Let $A\in\Lambda_\alpha$ be 1-periodic and let $\mathcal{E}_A$ be the fundamental solution of L_A. Let $K(\cdot,\cdot)$ is given by (1.5). Then*

(1) $|\nabla_1\mathcal{E}_A(x,y)|\le c_1|x-y|^{-n}$ *for every* $x,y\in\mathbb{R}^{n+1}$ *with* $x\neq y$.
(2) $|\nabla_1\mathcal{E}_A(x,y)-\nabla_1\mathcal{E}_A(x',y)|+|\nabla_1\mathcal{E}_A(y,x)-\nabla_1\mathcal{E}_A(y,x')|\le c_2|x-x'|^\alpha|x-y|^{-(n+\alpha)}$ *for every* $x,x',y\in\mathbb{R}^{n+1}$ *such that* $2|x-x'|\le|x-y|$.

The constants appearing in (1) and (2) are such that $c_1\approx_{n,\Lambda}c_2\approx_{n,\Lambda}\|A\|_\alpha$.

The period of the matrix plays an important role in our construction, so it is useful to rephrase the previous lemma for matrices with a period different from 1. We are interested in studying matrices with small period, so we only consider the case in which it is strictly smaller than 1.

Lemma 2.4. *Let $0<\ell<1$. Let $A\in\Lambda_\alpha$ be ℓ-periodic and let $\mathcal{E}_A$ be the fundamental solution associated with L_A. Then*

(1) $|\nabla_1 \mathcal{E}_A(x,y)| \le c_1'|x-y|^{-n}$ *for every* $x, y \in \mathbb{R}^{n+1}$ *with* $x \neq y$.
(2) $|\nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \mathcal{E}_A(x',y)| + |\nabla_1 \mathcal{E}_A(y,x) - \nabla_1 \mathcal{E}_A(y,x')| \le c_2'|x-x'|^\alpha |x-y|^{-n-\alpha}$ *for every* $x, x', y \in \mathbb{R}^{n+1}$ *such that* $2|x-x'| \le |x-y|$.

The constants appearing in (1) and (2) are such that $c_1' \approx_{n,\Lambda} c_2' \approx_{n,\Lambda} \|A\|_\alpha$.

Proof. For $\ell \in (0,1)$ and all $x \in \mathbb{R}^{n+1}$ we define the rescaled matrix

$$\widetilde{A}(x) := A(\ell x)$$

and we denote by $\widetilde{\mathcal{E}}$ the fundamental solution of $L_{\widetilde{A}}$. By the definition of fundamental solution, it is not difficult to see that

$$\nabla_1 \widetilde{\mathcal{E}}(x,y) = \ell^n \nabla_1 \mathcal{E}_A(\ell x, \ell y) \qquad \text{for } x, y \in \mathbb{R}^{n+1}. \tag{2.4}$$

Moreover,

$$|\widetilde{A}(x) - \widetilde{A}(y)| = |A(\ell x) - A(\ell y)| \le \ell^\alpha \|A\|_\alpha |x-y|^\alpha \le \|A\|_\alpha |x-y|^\alpha,$$

so that $\|\widetilde{A}\|_\alpha \le \|A\|_\alpha$. Applying Lemma 2.3 together with (2.4) we get

$$|\nabla_1 \mathcal{E}_A(x,y)| = \ell^{-n} |\nabla_1 \widetilde{\mathcal{E}}(\ell^{-1}x, \ell^{-1}y)| \lesssim \ell^{-n} |\ell^{-1}x - \ell^{-1}y|^{-n} = |x-y|^{-n}$$

for any x, y and

$$\begin{aligned} |\nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \mathcal{E}_A(x',y)| &= \ell^{-n} |\nabla_1 \widetilde{\mathcal{E}}(\ell^{-1}x, \ell^{-1}y) - \nabla_1 \widetilde{\mathcal{E}}(\ell^{-1}x', \ell^{-1}y)| \\ &\lesssim \ell^{-n} \frac{|\ell^{-1}x - \ell^{-1}x'|^\alpha}{|\ell^{-1}x - \ell^{-1}y|^{n+\alpha}} = \frac{|x-x'|^\alpha}{|x-y|^{n+\alpha}}, \end{aligned}$$

for $2|x-x'| \le |x-y|$. The same estimate holds for $|\nabla_1 \mathcal{E}_A(y,x) - \nabla_1 \mathcal{E}_A(y,x')|$. $\square$

The following is the (global) analogue of Lemma 2.2 in the 1-periodic setting.

Lemma 2.5. *Let* $A \in \Lambda_\alpha$ *be* 1*-periodic. Then for every* $x, y \in \mathbb{R}^{n+1}$, $x \neq y$, *we have*

$$\begin{aligned} \left| \mathcal{E}_A(x,y) - \Theta(x,y;A(x)) \right| &\lesssim |x-y|^{\alpha-n+1} \\ \left| \nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \Theta(x,y;A(x)) \right| &\lesssim |x-y|^{\alpha-n} \\ \left| \nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \Theta(x,y;A(y)) \right| &\lesssim |x-y|^{\alpha-n}, \end{aligned}$$

the implicit constants depending on $\|A\|_\alpha$ *and* Λ. *Similar estimates hold if we replace* ∇_1 *by* ∇_2.

Let us now recall some result from elliptic homogenization. For more details we refer to the work by Avellaneda and Lin [3]. For this purpose, we need to recall the definition of vector of correctors χ and homogenized matrix A_0. Let $\ell > 0$ and let $A \in \Lambda_\alpha$ be a 1-periodic matrix, i.e.

$$A(x+z) = A(x) \qquad \text{for every} \quad z \in \mathbb{Z}^{n+1}.$$

We will denote by $\chi(x) = (\chi^i(x))$, for $i \in \{1, \dots, n+1\}$ the vector of correctors, which is defined as the solution of the following cell problem

$$\begin{cases} L\chi = \operatorname{div} A, \\ \chi \text{ is 1-periodic}, \\ \int_{[0,1]^{n+1}} \chi(x) dx = 0, \end{cases} \tag{2.5}$$

where the first condition in (2.5) has to be understood in coordinates as

$$\sum_{i,j} \partial_{x^i} \left[a_{ij} \partial_{x^j} \chi^h \right](x) = -\sum_i \partial_{x^i} a_{ih}(x),$$

$(a_{ij})_{i,j}$ being the coefficients of the matrix A. An important fact is that

$$\|\nabla \chi\|_\infty \leq C, \tag{2.6}$$

the bound C depending only on n, α and $\|A\|_{C^\alpha}$. We remark that $\nabla\chi$ denotes the matrix with variable coefficients whose entries are $\partial_i \chi^j$ for $i, j = 1, \dots, n+1$. Now, if we consider the following family of elliptic operators

$$L_\varepsilon := \operatorname{div} \big(A(x/\varepsilon) \nabla \cdot \big)$$

depending on the parameter $\varepsilon > 0$, it can be proved that for any $f \in L^2(\mathbb{R}^{n+1})$, the solutions $u_\varepsilon \in W^{1,2}(\mathbb{R}^{n+1})$ of

$$L_\varepsilon u_\varepsilon = \operatorname{div} f$$

converge weakly in $W^{1,2}(\mathbb{R}^{n+1})$ to a function u_0 as $\varepsilon \to 0$. This function solves the equation

$$L_0 u_0 := \operatorname{div}(A_0 \nabla u_0) = \operatorname{div} f,$$

where A_0 is an elliptic matrix with constant coefficients usually called *homogenized matrix* (see, for example, [45]).

Homogenization is a powerful tool to study the fundamental solution of an elliptic equation in divergence form whose associated matrix is periodic and has C^α coefficients. The main result that we will use is the following (see [3, Lemma 2] and [30, Lemma 2.5]).

Lemma 2.6. *Let $A \in \Lambda_\alpha$. Let us assume that A is 1-periodic. There exists $\gamma \in (0,1)$ depending on $\alpha, \|A\|_{C^\alpha}$ and n such that*

$$\left| \mathcal{E}_A(x,y) - (Id + \nabla\chi(x))\Theta(x,y;A_0)\right| \lesssim \frac{c}{|x-y|^{n+\gamma-1}} \tag{2.7}$$

and

$$\left|\nabla_1 \mathcal{E}_A(x,y) - (Id + \nabla\chi(x))\nabla_1\Theta(x,y;A_0)\right| \lesssim \frac{c}{|x-y|^{n+\gamma}}, \tag{2.8}$$

where Id denotes the identity matrix and the implicit constants in (2.7) and (2.8) depend just on n, α and $\|A\|_\alpha$.

The period of the coefficients of A plays a crucial role in these estimates. We will be dealing with matrices with periodicity different from 1, so we need a suitably adapted version of the previous lemma. Let $A \in \Lambda_\alpha$ be a ℓ-periodic matrix. Let us define the 1-periodic matrix

$$\widetilde{A}(x) := A(\ell x)$$

for $x \in \mathbb{R}^{n+1}$ and let $\widetilde{\chi}$ denote the vector of correctors associated with $\widetilde{A}$ defined according to (2.5). For $\ell > 0$ we define

$$\chi_\ell(x) := \ell\,\widetilde{\chi}\Big(\frac{x}{\ell}\Big).$$

Observe that, because of (2.6) there exists $C > 0$ depending on the n, α and $\|A\|_{C^\alpha}$ such that

$$\|\nabla\chi_\ell\|_\infty \leq C. \tag{2.9}$$

Lemma 2.7. *Let $0 < \ell < 1$. Let $A \in \Lambda_\alpha$ be an ℓ-periodic matrix. Then there exists $\gamma \in (0,1)$ and $c > 0$, both depending just on n, α and $\|A\|_\alpha$ such that*

$$\left|\nabla_1 \mathcal{E}_A(x,y) - \nabla_1\Theta(x,y;A(x))\right| \leq c\ell^\alpha |x-y|^{\alpha-n}, \tag{2.10}$$

$$\left|\nabla_2 \mathcal{E}_A(x,y) - \nabla_2\Theta(x,y;A(y))\right| \leq c\ell^\alpha |x-y|^{\alpha-n}. \tag{2.11}$$

$$\left|\nabla_1 \mathcal{E}_A(x,y) - (Id + \nabla\chi_\ell(x))\nabla_1\Theta(x,y;A_0)\right| \leq c\ell^\gamma |x-y|^{-n-\gamma},$$

for every $x \neq y$.

Proof. Let $\widetilde{\mathcal{E}}$ denote the fundamental solution of the operator $L_{\widetilde{A}}$. As in (2.4), we have

$$\nabla_1 \mathcal{E}_A(x,y) = \ell^{-n}\nabla_1\widetilde{\mathcal{E}}(x/\ell, y/\ell), \tag{2.12}$$

so an application of Lemma 2.5 gives

$$\begin{aligned}&\big|\nabla_1 \mathcal{E}_A(x,y) - \nabla_1\Theta(x,y;A(x))\big|\\&\quad = \ell^{-n}|\nabla_1 \mathcal{E}_{\widetilde{A}}(\ell^{-1}x,\ell^{-1}y) - \nabla_1\Theta(x,y;\widetilde{A}(\ell^{-1}x))| \le c\ell^{\alpha}|x-y|^{\alpha-n}.\end{aligned}$$

Using (2.8) and (2.12), we get

$$\begin{aligned}&|\nabla_1 \mathcal{E}_A(x,y) - (Id+\nabla\chi_\ell(x))\nabla_1\Theta(x,y;A_0)|\\&\quad = \ell^{-n}|\nabla_1\widetilde{\mathcal{E}}(x/\ell,y/\ell) - (Id+\nabla\widetilde{\chi}(x/\ell))\nabla_1\Theta(x/\ell,y/\ell;A_0)|\\&\quad \lesssim \frac{c\ell^{n+\gamma}}{\ell^n|x-y|^{n+\gamma}} = \frac{c\ell^{\gamma}}{|x-y|^{n+\gamma}},\end{aligned}$$

where c depends on n,α and $\|\widetilde{A}\|_\alpha$, $\|\widetilde{A}\|_\alpha \le \|A\|_\alpha$. Inequality (2.11) follows as (2.10). □

3. The existence of principal values

The proof of the existence of principal values can be divided into the study of two different cases: the case in which μ is a rectifiable measure and the one in which μ has zero n-density, i.e.

$$\lim_{r\to 0}\frac{\mu(B(x,r))}{r^n} = 0 \qquad \text{for } \mu\text{-a.e. } x\in\mathbb{R}^{n+1}.$$

Indeed, if μ is a measure on $\mathbb{R}^{n+1}$ with no point masses and T_μ is bounded on $L^2(\mu)$, [43, Theorem 1.2] allows us to write $\mu = \mu_0 + \mu_1$, where μ_0 has vanishing upper n-density μ_0-almost everywhere and μ_1 is n-rectifiable. See also the argument in [48, Chapter 8] for the case of the Cauchy transform.

3.1. Principal values for rectifiable measures with compact support

This subsection follows the scheme of [14, Section 2.2]. The proof of the existence of principal values for T_μ if the measure μ is rectifiable and has compact support relies on the following result.

Theorem 3.1. *Let μ be a rectifiable measure. Let $K\in C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$ be an odd kernel and homogeneous of degree $-n$, i.e. $K(x) = -K(-x)$ and $K(\lambda x) = \lambda^{-n}K(x)$. Assume, for some $M = M(n)$, the further regularity condition*

$$|\nabla_j K(x)| \lesssim_n C(j)|x|^{-n-j} \quad \text{for all } 0\le j\le M \quad \text{and } x\in\mathbb{R}^{n+1}\setminus\{0\}.$$

Then the operator $T_{K,\mu}$ is bounded on $L^2(\mu)$ with operator norm

$$\|T_{K,\mu}\|_{L^2(\mu)\to L^2(\mu)} \lesssim_n \|K|_{\mathbb{S}^n}\|_{C^M(\mathbb{R}^{n+1})}.$$

Moreover, the principal value

$$T_{K,\mu}f(x) = \lim_{\varepsilon\to 0} \int\limits_{|x-y|\geq\varepsilon} K(x-y)f(y)d\mu(y)$$

exists μ-almost everywhere.

The proof of the boundedness of $T_{K,\mu}$ is due to David and Semmes. The result on principal values was first proved imposing an analogous condition for all $j = 0, 1, 2, \ldots$ (for a more detailed exposition we refer, for example, to [33, Chapter 20]). We remark that it has been more recently improved by Mas (see [32, Corollary 1.6]).

The previous theorem together with a spherical harmonics expansion of the kernel is the key tool to prove the following result.

Lemma 3.1. *Let μ be an n-rectifiable measure. There exists $M = M(n)$ such that the following holds. Let $b(x,z)$ be odd in z and homogeneous of degree $-n$ in z, and assume $D_z^\alpha b(x,z)$ is continuous and bounded on $\mathbb{R}^{n+1}\times\mathbb{S}^n$, for any multi-index $|\alpha|\leq M$. Then for every $f\in L^2(\mu)$, the limit*

$$Bf(x) = \lim_{\varepsilon\to 0} \int\limits_{|x-y|>\varepsilon} b(x,x-y)f(y)d\mu(y)$$

exists for μ-almost every x.

Proof. This result is used in [39] (see for example [39, (1.14)]). The proof is a variation of the argument in [39, Proposition 1.2]. For the reader's convenience we discuss the details below.

Let $\{\varphi_{j,l}\}_{j\geq 1, 1\leq l\leq N_j}$ be an orthonormal basis of $L^2(\mathbb{S}^n)$ consisting of surface spherical harmonics of degree j. Recall that (see [4, (2.12)])

$$N_j = O(j^{n-1}), \qquad \text{for } j \gg 1. \tag{3.1}$$

Using the homogeneity assumption for $b(x,\cdot)$ and the orthonormal expansion, we write

$$\begin{aligned} b(x,z) = b\Big(x,\frac{z}{|z|}\Big)|z|^{-n} &= \sum_{j\geq 1}\sum_{l=1}^{N_j} \langle b(x,\cdot),\varphi_{j,l}\rangle_{L^2(\mathbb{S}^n)}\varphi_{j,l}\Big(\frac{z}{|z|}\Big)|z|^{-n} \\ &= \sum_{j,l} b_{j,l}(x)\varphi_{j,l}\Big(\frac{z}{|z|}\Big)|z|^{-n}, \end{aligned} \tag{3.2}$$

where $b_{j,l}(x) := \langle b(x,\cdot),\varphi_{j,l}\rangle_{L^2(\mathbb{S}^n)}$. Since $b(x,\cdot)$ is an odd function and $\varphi_{2j,l}$ is even for every j, $b_{j,l}(x)\equiv 0$ for j even. Being b in $L^\infty(\mathbb{R}^{n+1}\times\mathbb{S}^n)$ by hypothesis and Hölder's inequality, we have

$$|b_{j,l}(x)| \le C(n)\|b(x,\cdot)\|_{L^\infty(\mathbb{S}^n)}\|\varphi_{j,l}\|_{L^2(\mathbb{S}^n)} \le C(n)\|b\|_{L^\infty(\mathbb{R}^{n+1}\times\mathbb{S}^n)} \le C(n). \tag{3.3}$$

Moreover, recalling that we can suppose j odd, the function $\widetilde{K}_{j,l}(z) := \varphi_{j,l}\big(z/|z|\big)|z|^{-n}$ satisfies the hypothesis in Theorem 3.1: there exists an harmonic polynomial $P_{j,l}$ of odd degree j such that $\varphi_{j,l}(z/|z|) = P_{j,l}(z)/|z|^j$, so

$$\Big|\nabla\varphi_{j,l}\Big(\frac{z}{|z|}\Big)\Big| \lesssim \frac{1}{|z|}$$

and

$$\big|\nabla\widetilde{K}_{j,l}(z)\big| \lesssim \Big|\nabla\varphi_{j,l}\Big(\frac{z}{|z|}\Big)\Big|\frac{1}{|z|^n} + \Big|\varphi_{j,l}\Big(\frac{z}{|z|}\Big)\Big|\frac{1}{|z|^{n+1}} \lesssim \frac{1}{|z|^{n+1}}.$$

Analogous estimates hold for higher order derivatives. So, Theorem 3.1 ensures that

$$T_{\widetilde{K}_{j,l},\mu}f(x) = \lim_{\varepsilon\to 0}\int\limits_{|x-y|>\varepsilon}\widetilde{K}_{j,l}(x-y)f(y)d\mu(y) \equiv \lim_{\varepsilon\to 0}T_{\widetilde{K}_{j,l},\mu,\varepsilon}f(x) \tag{3.4}$$

exists for μ-a.e x. Recall also that by the Theorem 3.1 there exists $M = M(n)$ such that $T_{\widetilde{K}_{j,l},\mu}$ is bounded on $L^2(\mu)$ with operator norm

$$\|T_{\widetilde{K}_{j,l},\mu}\|_{L^2(\mu)\to L^2(\mu)} \lesssim \|\widetilde{K}_{j,l}|_{\mathbb{S}^n}\|_{C^M(\mathbb{S}^n)} = \|\varphi_{j,l}\|_{C^M(\mathbb{S}^n)}. \tag{3.5}$$

Gathering (3.2), (3.3) and (3.4), to prove the lemma it is enough to show that the dominated convergence theorem applies and, in particular, that

$$\sum_{j,l}\big|b_j(x)T_{\widetilde{K}_{j,l},\mu,\varepsilon}f(x)\big| \le C(x) < \infty, \tag{3.6}$$

where $C(x)$ does not depend on ε. By Lebesgue differentiation theorem, to prove (3.6) it suffices to show that for every ball $B_0 \subset \mathbb{R}^{n+1}$ we have

$$\begin{aligned}\sum_{j,l}\int\limits_{B_0}\big|b_{j,l}(x)T_{\widetilde{K}_{j,l},\mu,\varepsilon}f(x)\big|d\mu(x) &\lesssim_{B_0,n}\sum_{j,l,m}\|b_{j,l}\|_\infty\|\varphi_{j,l}\|_{C^m(\mathbb{S}^n)}\|f\|_{L^2(\mu)}\\ &\le C\|f\|_{L^2(\mu)}\end{aligned}$$

for some $C > 0$, where the first inequality above uses the L^2-boundedness (3.5).

The smoothness of b implies that (see [46, 3.1.5])

$$\|b_{j,l}\|_\infty \lesssim \frac{1}{j^{\frac{3}{2}n+2+M}},$$

where the exponent on the right hand side is chosen accordingly to what we need next. Now, recall that the Sobolev space $H^s(\mathbb{S}^n)$, $s \in \mathbb{R}$ can be defined via spherical harmonics expansion. In particular, it is the completion of $C^\infty(\mathbb{S}^n)$ with respect to the norm

$$\|v\|_{H^s(\mathbb{S}^n)} := \Big(\sum_{j,l} \Big(j + \frac{n-1}{2} \Big)^{2s} |v_{j,l}|^2 \Big)^{1/2}, \tag{3.7}$$

where $v_{j,l} = \langle v, \varphi_{j,l} \rangle_{L^2(\mathbb{S}^n)}$. For the definition and the properties of this space, we refer for example to [4, Section 3.8] and to [4, Section 6.3] for the relation of (3.7) with that via the restriction of the gradient to the unit sphere. By Sobolev embedding theorem, $H^s(\mathbb{S}^n)$ continuously embeds into $C(\mathbb{S}^n)$ for $s > n/2$. So, choosing $s = \frac{n}{2} + 1$ and using (3.7) we can estimate

$$\|D^m \varphi_{j,l}\|_{C(\mathbb{S}^n)} \lesssim_n \|\varphi_{j,l}\|_{H^{s+m}(\mathbb{S}^n)} = \Big(\frac{2j+n-1}{2} \Big)^{\frac{n}{2}+m+1}.$$

Hence, using (3.1)

$$\sum_{j,l} \|b_{j,l}\|_\infty \|\varphi_{j,l}\|_{C^M(\mathbb{S}^n)} \lesssim_n \sum_{m=0}^{M} \sum_{j \geq 1} N_j j^{-\frac{3}{2}n-2-M} j^{\frac{n}{2}+m+1} \lesssim \sum_{j\geq 1} \frac{1}{j^2} < \infty,$$

which concludes the proof. □

Theorem 3.2. *Let μ be an n-rectifiable measure on $\mathbb{R}^{n+1}$ with compact support. Let A be a matrix having the properties (1.2), (1.3) and (1.6). Then for every $f \in L^2(\mu)$ the principal value*

$$T_\mu f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \nabla_1 \mathcal{E}(x,y) f(y) d\mu(y)$$

exists for μ-almost every x.

Proof. Let $\varepsilon > 0$ and denote $b(x,z) := \nabla_1 \Theta(z,0;A(x))$. As a consequence of the explicit formula (2.2), it is not difficult to see that each component of b verifies the hypothesis of Lemma 3.1. So, split $T_{\mu,\varepsilon}$ as

$$\begin{aligned} T_{\mu,\varepsilon} f(x) = & \int_{|x-y|>\varepsilon} b(x, x-y) f(y) d\mu(y) \\ & + \int_{|x-y|>\varepsilon} \Big(\nabla_1 \mathcal{E}(x,y) - \nabla_1 \Theta(x,y;A(x)) \Big) f(y) d\mu(y). \end{aligned} \tag{3.8}$$

The limit for $\varepsilon \to 0$ of the first integral in the right hand side of (3.8) exists μ-a.e. because of Lemma 3.1. On the other hand, $\nabla_1 \mathcal{E}(x,y) - \nabla_1 \Theta(x,y;A(x))$ defines an operator which is compact on $L^p(\mu)$ because of Lemma 2.2, which guarantees that the limit for $\varepsilon \to 0$ exists for μ-a.e. x and concludes the proof. □

3.2. Principal values for measures with zero density

Again, suppose that μ has compact support.

A combination of the proof of [34, Theorem 1.4] (see also [48, Theorem 8.10]) and Lemma 2.2 (which has to be used instead of antisymmetry) makes possible to prove that if μ is a Radon measure in $\mathbb{R}^{n+1}$ with growth of degree n, then for every $1 < p < \infty$ and $f \in L^p(\mu)$, $\{T_{\mu,\varepsilon} f\}_\varepsilon$ admits a weak limit $T_\mu^w f$ in $L^p(\mu)$ as $\varepsilon \to 0$. Moreover, the representation formula

$$T_\mu^w f(x) = \lim_{r\to 0} \fint_{B(x,r)} T_\mu\big(f\chi_{B(x,r)^c}\big)(y)d\mu(y) \tag{3.9}$$

holds for μ-almost every $x \in \mathbb{R}^{\mathrm{n}+1}$, giving an explicit way of computing the weak limit. We remark that, in general, we can only infer that formula (3.9) holds if T_μ has an antisymmetric kernel.

Let us recall the following theorem by Mattila and Verdera (see [34]), here reported in the formulation of [48, Theorem 8.11].

Theorem 3.3. *Let μ be a Radon measure in $\mathbb{R}^d$ that has growth of degree n and zero n-dimensional density μ-a.e. Let $\mathcal{T}_\mu$ be an n-dimensional antisymmetric Calderón-Zygmund operator. Then, for all $1 < p < \infty$ and $f \in L^p(\mu)$, $\operatorname{pv} \mathcal{T}_\mu f(x)$ exists for μ-a.e. $x \in \mathbb{R}^d$ and coincides with $\mathcal{T}_\mu^w f(x)$. Also, for all $\nu \in M(\mathbb{C})$, $\operatorname{pv} \mathcal{T}\nu(x)$ exists for μ-a.e. $x \in \mathbb{R}^d$.*

This result can be transferred to the gradients of the single layer potential T_μ.

Theorem 3.4. *Let μ be a Radon measure in $\mathbb{R}^{n+1}$ that has growth of degree n, zero n-dimensional density and compact support. Suppose that T_μ is a bounded operator from $L^2(\mu)$ to $L^2(\mu)$. Then, for all $1 < p < \infty$ and $f \in L^p(\mu)$, $\operatorname{pv} T_\mu f(x)$ exists for μ-a.e. $x \in \mathbb{R}^{\mathrm{n}+1}$ and coincides with $T_\mu^w f(x)$. Also, for all $\nu \in M(\mathbb{C})$, $\operatorname{pv} T\nu(x)$ exists for μ-a.e. $x \in \mathbb{R}^{\mathrm{n}+1}$.*

Proof. Let $1 < p < \infty$ and $f \in L^p(\mu)$. We decompose $T_\mu f$ into its symmetric and antisymmetric part. That is to say,

$$T_\mu f(x) = T_\mu^{(a)} f(x) + T_\mu^{(s)} f(x),$$

where $T_\mu^{(a)}$ is the integral operator with kernel $(\nabla_1 \mathcal{E}(x,y) - \nabla_1 \mathcal{E}(y,x))/2$ and $T_\mu^{(s)}$ whose kernel is $(\nabla_1 \mathcal{E}(x,y) + \nabla_1 \mathcal{E}(y,x))/2$. We can apply Theorem 3.3 to antisymmetric part $T_\mu^{(a)}$, obtaining that $\operatorname{pv} T_\mu^{(a)} f(x)$ exists for μ-a.e. x.

On the other hand, $T_\mu^{(s)}$ defines a compact operator on $L^p(\mu)$ since

$$\int |\nabla_1 \mathcal{E}(x,y) + \nabla_1 \mathcal{E}(y,x)| d\mu(y) \lesssim \operatorname{diam}(\operatorname{supp}\mu)^\alpha,$$

so that the principal values exist.

The fact that $T_\mu^w f$ coincides with $\operatorname{pv} T_\mu f$ a.e. follows from the definition of weak limit together with dominated convergence theorem:

$$\int T_\mu^w f g\, d\mu = \lim_{\varepsilon\to 0} \int T_{\mu,\varepsilon} f g\, d\mu = \int \operatorname{pv} T_\mu f g\, d\mu \quad \text{for all} \quad g \in L^{p'}(\mu),$$

p' being the Hölder conjugate exponent of p. □

A remark on the well-posedness of the assumption (6) of Theorem 1.2. Let T, μ and B be as in Theorem 1.2. Let $x, y \in B$ and $\varepsilon > 0$ and write

$$T_\varepsilon\mu(x) - T_\varepsilon\mu(y) = T_{\mu,\varepsilon}\chi_{2B}(x) - T_{\mu,\varepsilon}\chi_{2B}(y) + \big[T_{\mu,\varepsilon}\chi_{\mathbb{R}^{n+1}\setminus 2B}(x) - T_{\mu,\varepsilon}\chi_{\mathbb{R}^{n+1}\setminus 2B}(y)\big]. \tag{3.10}$$

Now observe that, being the operator $T_{\mu|_B}$ bounded on $L^2(\mu|_B)$, Theorem 1.1 (2) applies with $\nu = \chi_{2B}\mu$. So, the first two summands on the right hand side of (3.10) admit a limit as $\varepsilon \to 0$ for almost every $x, y \in B$. The limit for $\varepsilon \to 0$ of the last summand exists, too. Indeed, since x, y do not belong to $\mathbb{R}^{n+1}\setminus 2B$, for $\varepsilon < r(B)$,

$$T_{\mu,\varepsilon}\chi_{\mathbb{R}^{n+1}\setminus 2B}(x) - T_{\mu,\varepsilon}\chi_{\mathbb{R}^{n+1}\setminus 2B}(y) = \int_{\mathbb{R}^{n+1}\setminus 2B} \big(\nabla_1 \mathcal{E}(x,z) - \nabla_1 \mathcal{E}(y,z)\big) d\mu(y).$$

If we assume $\widetilde{\alpha} \le \alpha$ in the statement of the main theorem, an application of the Calderón-Zygmund property of the kernel combined with a dyadic decomposition of the domain of integration gives

$$\begin{aligned}\Big|\int_{\mathbb{R}^{n+1}\setminus 2B} \big(\nabla_1 \mathcal{E}(x,z) - \nabla_1 \mathcal{E}(y,z)\big) d\mu(z)\Big| &\lesssim |x-y|^\alpha \sum_{j=1}^{+\infty} \int_{2^{j+1}B\setminus 2^j B} \frac{1}{|x-z|^{n+\alpha}} d\mu(z)\\ &\le P_{\mu,\alpha}(B) \le P_{\mu,\widetilde{\alpha}}(B) < +\infty.\end{aligned} \tag{3.11}$$

In particular, this tells that $T\mu(x) - T\mu(y)$ exists in the principal value sense for almost every $x, y \in B$.

We also want to point out that $T\mu - m_{\mu,B}(T\mu)$ defines an $L^2(\mu|_B)$-function. Indeed, for $x \in B$ and using (3.11),

$$|T\mu(x) - m_{\mu,B}(T\mu)| \le \frac{1}{\mu(B)} \int_B |T\mu(x) - T\mu(y)| d\mu(y)$$
$$\le |T(\chi_{2B}\mu)(x)| + \big(m_{\mu,B}|T(\chi_{2B}\mu)|^2\big)^{1/2} + P_{\mu,\widetilde{\alpha}}(B).$$

The right hand side of the previous majorization defines an $L^2(\mu|_B)$ function because of the assumptions $T(\chi_{2B}\mu) \in L^2(\mu|_B)$ and $P_{\mu,\widetilde{\alpha}}(B) < +\infty$ in Theorem 1.2.

4. The Main Lemma

As we mentioned in the Introduction, it is convenient to formulate the statement of the main Theorem 1.2 in terms of α-numbers (see (2.1)). This is made possible by a mostly geometric argument: a careful read shows that the same arguments of [21, Section 3] apply to our case. More specifically, in order to prove Theorem 1.2, it suffices to prove the following result, whose proof we shortly outline after the statement for the reader's convenience.

Lemma 4.1 *(Main Lemma). Let $n > 1$ and let $C_0, C_1 > 0$ be some arbitrary constants. There exist $M = M(C_0, C_1, n) > 0$ big enough, $\lambda(C_0, C_1, n) > 0$ and $\varepsilon = \varepsilon(C_0, C_1, M, n) > 0$ small enough such that if $\delta = \delta(M, C_0, C_1, n) > 0$ is sufficiently small, then the following holds. Let μ be a Radon measure in $\mathbb{R}^{n+1}$ with compact support and $Q_0 \subset \mathbb{R}^{n+1}$ a cube centered at the origin satisfying the properties:*

(1) $\ell(MQ_0) \le \lambda$.
(2) $\mu(Q_0) = \ell(Q_0)^n$.
(3) $P_{\mu,\widetilde{\alpha}}(MQ_0) \le C_0$.
(4) For all $x \in 2Q_0$ and $0 < r \le \ell(Q_0)$, $\Theta_\mu(B(x,r)) \le C_0$.
(5) Q_0 has C_0-thin boundary.
(6) $\alpha_\mu^L(3MQ_0) \le \delta$, for some hyperplane L through the origin.
(7) $T_{\mu|_{2Q_0}}$ is bounded on $L^2(\mu|_{2Q_0})$ with $\|T_{\mu|_{2Q_0}}\|_{L^2(\mu|_{2Q_0}) \to L^2(\mu|_{2Q_0})} \le C_1$.
(8) We have

$$\int_{Q_0} |T\mu(x) - m_{\mu,Q_0}(T\mu)|^2 d\mu(x) \le \varepsilon \mu(Q_0). \tag{4.1}$$

Then there exists some constant $\tau > 0$ and a uniformly n-rectifiable set $\Gamma \subset \mathbb{R}^{n+1}$ such that

$$\mu(Q_0 \cap \Gamma) \ge \tau \mu(Q_0), \tag{4.2}$$

where the constant τ and the uniform rectifiability constants of Γ depend on all the constants above.

Sketch of the proof of Theorem 1.2 via the Main Lemma 4.1. Let σ be a Radon measure on $\mathbb{R}^n$ and, given a cube $Q \subset \mathbb{R}^n$, we denote

$$\alpha_\sigma^{\mathbb{R}^n}(Q) := \frac{1}{\ell(Q_0)^n} \inf_{c \geq 0} d_Q\big(\sigma, c\mathcal{H}^n|_{\mathbb{R}^n}\big).$$

Let $\mathcal{D}(\mathbb{R}^n)$ be the family of dyadic cubes in $\mathbb{R}^n$. Let us assume that σ is a finite Radon measure such that, for some $\widetilde{C} > 0$ and $\ell_0 > 0$ we have $\sigma(Q) \leq \widetilde{C}\ell(Q)^n$ for all $Q \in \mathcal{D}(\mathbb{R}^n)$ and $\ell(Q) \geq \ell_0$. A general geometric argument (see [21, Lemma 3.4]) gives

$$\sum_{Q \in \mathcal{D}(\mathbb{R}^n) : Q \subset R, \ell(Q) \geq \ell_0} \alpha_\sigma^{\mathbb{R}^n}(3Q)^2 \ell(Q)^n \lesssim \widetilde{C}^2 \ell(R)^n.$$

This formula and another purely geometric proof allow us, given B and μ as in the statement of Theorem 1.2, for all $M' > 10$ and $\delta', \varepsilon' > 0$ to find a cube Q_0 with thin boundary such that $3M'Q_0 \subset B$ and $\operatorname{dist}(3M'Q_0, \partial B) \geq C_0' r(B)$ for $C_0' = C_0'(C_0, n)$. The cube Q_0 can be constructed in such a way that it satisfies $\mu(Q_0) \geq C_0'\ell(Q_0)^n$ and, for δ small enough in the statement of the Main Theorem, $\alpha_\mu(3M'Q_0) \leq \delta'$. Furthermore,

$$\begin{aligned}\int_{Q_0} |T\mu(x) - m_{\mu,Q_0}(T\mu)|^2 \, d\mu(x) &\leq 2 \int_{Q_0} |T\mu(x) - m_{\mu,B}(T\mu)|^2 \, d\mu(x) \\ &\leq 2\varepsilon\mu(B) \approx_{C_0,\delta} \varepsilon\mu(Q_0).\end{aligned}$$

This gives the analogue of [21, Lemma 3.2]. Hence, it is not hard to prove that the construction of Q_0 implies that the measure $\widetilde{\mu} := \frac{\ell(Q_0)^n}{\mu(Q_0)}\mu$ satisfies the hypotheses (1)-...-(8) of Lemma 5.5. The details, which we omit for brevity, follow verbatim Section 3 of the paper of Girela-Sarrión and Tolsa, to which we also refer for the whole discussion of the construction sketched here. □

The matrix A may have a very general form. In particular, we need some additional argument to overcome the lack of "symmetries" of the matrix with respect to reflections and to periodization (the exact meaning of this sentence will be clear after the reading of Section 5, where we recall how second order PDE's in divergence form are affected by a change of variable). Indeed, this is a crucial point for our proof to work. A similar problem has been faced in [43]. First, in order to be able to argue via a change of variables, we have to show that we can assume the matrix A to be symmetric.

We recall Schur's test for integral operators with a reproducing kernel. The proof is a standard application of Cauchy-Schwarz's inequality.

Lemma 4.2. *Let $K \colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a function such that, for a constant $C > 0$, we have*

$$\int |K(x,y)| d\mu(x) \leq C$$

and

$$\int |K(x,y)|d\mu(y) \le C.$$

Then the operator $Tf = K * f$ *is a continuous operator from* $L^2(\mu)$ *to* $L^2(\mu)$ *and*

$$\|T\|_{L^2(\mu)\to L^2(\mu)} \le C.$$

Let A be a matrix as before. We denote by $A_s = (A + A^T)/2$ its symmetric part and by $T_\mu^{A_s}$ its correspondent gradient of the single layer potential.

Lemma 4.3. *Let* Q *be a cube in* $\mathbb{R}^{n+1}$ *such that, for* $M > 1$, $P_{\mu,\alpha}(MQ) \le C_1$. *The operator* $T^{(s)}_{\mu|2Q}$ *is bounded on* $L^2(\mu|2Q)$ *if and only if* $T_{\mu|2Q}$ *is bounded on* $L^2(\mu|2Q)$. *In particular*

$$\left\|T_{\mu|_{2Q}}\right\|_{L^2(\mu|2Q)\to L^2(\mu|2Q)} = \left\|T^{A_s}_{\mu|_{2Q}}\right\|_{L^2(\mu|2Q)\to L^2(\mu|2Q)} + O(\ell(Q)^\alpha). \tag{4.3}$$

Moreover

$$\begin{aligned}\int_Q &\left|T^{A_s}\mu(x) - m_{\mu,Q}(T^{A_s}\mu)\right|^2 d\mu(x) \\ &\lesssim_{\Lambda,\|A\|_\alpha} \int_Q \left|T\mu(x) - m_{\mu,Q}(T\mu)\right|^2 d\mu(x) + \left(M^\alpha \ell(Q)^\alpha + M^{-\alpha}\right)^2 \mu(Q).\end{aligned} \tag{4.4}$$

Proof. Let us first prove (4.3). The identity (2.3) for matrices with constant coefficients leads to

$$\begin{aligned} T^{A_s}_{\mu|_{2Q}} f(x) &= \int_{2Q} \nabla_1 \mathcal{E}_{A_s}(x,y) f(y) d\mu(y) \\ &= \int_{2Q} \left(\nabla_1 \mathcal{E}_{A_s}(x,y) - \nabla_1 \Theta(x,y;A_s(x))\right) f(y) d\mu(y) \\ &+ \int_{2Q} \left(\nabla_1 \Theta(x,y;A(x)) - \nabla_1 \mathcal{E}(x,y)\right) f(y) d\mu(y) + \int_{2Q} \nabla_1 \mathcal{E}(x,y) f(y) d\mu(y) \\ &\equiv I + II + T_{\mu|_{2Q}} f(x). \end{aligned} \tag{4.5}$$

To estimate I and II in (4.5) it suffices, then, to invoke Lemma 2.7 and Schur's test. This finishes the proof of the first part of the lemma.

Let us now prove (4.4). We split

$$
\begin{aligned}
&T\mu(x) - m_{\mu,Q}(T\mu)\\
&\quad = \Big(T(\chi_{MQ}\mu)(x) - m_{\mu,Q}\big(T(\chi_{MQ}\mu)\big)\Big) + \Big(T(\chi_{(MQ)^c}\mu)(x) - m_{\mu,Q}\big(T(\chi_{(MQ)^c}\mu)\big)\Big).
\end{aligned}
\tag{4.6}
$$

Let us estimate the two terms in the right hand side separately. Again, as a consequence of (4.5) and Lemma 2.2 we can write

$$
\Big|T(\chi_{MQ}\mu) - m_{\mu,Q}(T(\chi_{MQ}\mu)) - \Big(T^{A_s}(\chi_{MQ}\mu) + m_{\mu,Q}\big(T^{A_s}(\chi_{MQ}\mu)\big)\Big)\Big| \lesssim M^\alpha \ell(Q)^\alpha.
$$

To bound the second term in the right hand side of (4.6), notice that for $x, y \in Q$ standard estimates together with Lemma 2.7 give

$$
\begin{aligned}
\big|T_\mu\chi_{(MQ)^c}(x) - T_\mu\chi_{(MQ)^c}(y)\big| &\lesssim \int\limits_{(MQ)^c} \frac{|x-y|^\alpha}{|x-z|^{n+\alpha}}\,d\mu(z)\\
&\lesssim \frac{|x-y|^\alpha}{\ell(MQ)^\alpha} P_{\mu,\alpha}(MQ) \lesssim \frac{1}{M^\alpha} P_{\mu,\alpha}(MQ) \lesssim \frac{1}{M^\alpha},
\end{aligned}
$$

thus, averaging over y in Q we have

$$
\big|T(\chi_{(MQ)^c}\mu)(x) - m_{\mu,Q}\big(T(\chi_{(MQ)^c}\mu)\big)\big| \lesssim M^{-\alpha}
$$

The same calculations lead to

$$
\Big|T^{A_s}(\chi_{(MQ)^c}\mu)(x) - m_{\mu,Q}\big(T^{A_s}(\chi_{(MQ)^c}\mu)\big)\Big| \lesssim M^{-\alpha},
$$

so the inequality (4.4) in the statement of the lemma follows by gathering all the previous considerations. □

As an immediate application of Lemma 4.3, we can assume in Lemma 4.1 (and hence in Theorem 1.2) that the matrix A is symmetric. In order to see this, let us suppose that we can construct a uniformly n-rectifiable set Γ as in the statement of the Main Lemma 4.1 if the conditions (7) and (8) hold for T^{A_s}. More specifically, assume that there exists $\widetilde{C}_1 > 0$ such that $T^{A_s}_{\mu|_{2Q_0}}$ is bounded on $L^2(\mu|_{2Q_0})$ with

$$
\|T^{A_s}_{\mu|_{2Q_0}}\|_{L^2(\mu|_{2Q_0})\to L^2(\mu|_{2Q_0})} \le \widetilde{C}_1.
$$

Then, by (4.3) and the condition (1) in the Main Lemma, we have that for λ small enough it holds

$$
\|T_{\mu|_{2Q_0}}\|_{L^2(\mu|_{2Q_0})\to L^2(\mu|_{2Q_0})} \le \widetilde{C}_1 + \lambda^{\alpha/2}.
$$

By (4.4), an analogous consideration holds for the assumption (8) in the Main Lemma. Hence, recalling that $\lambda = \lambda(C_0, \widetilde{C}_1, n)$, Γ is the desired uniformly n-rectifiable that satisfies (4.2).

Thus, in the rest of the paper *we will understand that A is symmetric.*

Remark 4.1. Arguing as in Lemma 4.3, one could prove that

$$\|T_{\mu|_{2Q}}\|_{L^2(\mu|_{2Q})\to L^2(\mu|_{2Q})} = \|T^a_{\mu|_{2Q}}\|_{L^2(\mu|_{2Q})\to L^2(\mu|_{2Q})} + O(\ell(Q)^\alpha), \tag{4.7}$$

where T^a is the operator corresponding to the antisymmetric part of the kernel $K(\cdot,\cdot)$, that is to say $K^a(x,y) = (K(x,y) - K(y,x))/2$. Although for the purposes of many technical parts of the paper it may look natural to work with $T^a_{\mu|_{2Q}}$, we prefer not to make this reduction. Indeed, it would create problems at some crucial stages of the main variational argument. In particular, it would be an obstacle to the application of the maximum principle: it is not clear how to adapt the argument in (11.26) to show that the adjoint operator $(T^a)^*$ (see (11.1) for the definition) solves a proper elliptic equation. We remark that an analogous problem had to be dealt in the papers [14] and [43], where T^a_μ was not used for the same reasons. Finally, since the inequality (4.7) is never used in the rest of the paper, we prefer to omit its proof.

5. The modification of the matrix

5.1. The change of variable

The following lemma deals with how the fundamental solution and its gradient are affected by a change of variable.

Lemma 5.1 *(see [43], Lemma 5.2). Let $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a locally bilipschitz map and let $A \in \Lambda_\alpha$. Let $\mathcal{E}_A$ be the fundamental solution of $L_A = -\operatorname{div}(A\nabla\cdot)$. Set $A_\phi := |\det\phi| D(\phi^{-1})(A\circ\phi)D(\phi^{-1})^T$. Then*

$$\mathcal{E}_{A_\phi}(x,y) = \mathcal{E}_A(\phi(x),\phi(y)) \text{ for } x,y \in \mathbb{R}^{n+1},$$

and

$$\nabla_1 \mathcal{E}_{A_\phi}(x,y) = D(\phi)^T(x)\,\nabla_1 \mathcal{E}_A(\phi(x),\phi(y)) \quad \text{for} \quad x \in \mathbb{R}^{n+1}.$$

Let us state a lemma concerning how the gradient of the fundamental solution transforms under a change of variable ϕ as in Lemma 5.1. We use the notation

$$T_\phi\mu(x) := \int \nabla_1 \mathcal{E}_{A_\phi}(x,y)d\mu(y).$$

Lemma 5.2 *(see [43], Lemma 5.3). Let $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a bilipschitz change of variables. For every $x \in \mathbb{R}^{n+1}$ we have*

$$T_{\phi}\mu(x) = D(\phi)^{T}(x) T\phi\sharp\mu(\phi(x)).$$

A particularly useful change of variable is the one that turns the symmetric part of the matrix at a given point into the identity. For the following statement we refer to [5].

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and assume that A is a uniformly elliptic matrix with real entries. Let $A_s = (A + A^T)/2$ be the symmetric part of A and for a fixed point $y_0 \in \Omega$ define $S = \sqrt{A_s(y_0)}$. If*

$$\widetilde{A}(\cdot) = S^{-1}(A \circ S)(\cdot)S^{-1},$$

then $\widetilde{A}$ is uniformly elliptic, $\widetilde{A}_s(z_0) = Id$ if $z_0 = S^{-1}y_0$ and u is a weak solution of $L_A u = 0$ in Ω if and only if $\widetilde{u} = u \circ S$ is a weak solution of $L_{\widetilde{A}}\widetilde{u} = 0$ in $S^{-1}(\Omega)$.

We point out that the change of variables $\varphi(x) := Sx$ that we defined in Lemma 5.3 is a linear map and, in particular, a bilipschitz map of $\mathbb{R}^{n+1}$ to itself, namely there exists $C \geq 1$ such that

$$C^{-1}|x-y| \leq |\varphi(x) - \varphi(y)| \leq C|x-y| \qquad \text{for } x, y \in \mathbb{R}^{n+1}.$$

The bilipschitz constant of φ depends on the ellipticity of the matrix A. We need to quantify flatness of *images of cubes* via maps of the aforementioned type. For a set $E \subset \mathbb{R}^{n+1}$, we define the α-number in an analogous ways as for cubes. In particular, for any hyperplane L and any measure ν, we denote

$$\alpha_{\nu}^{L}(E) := \frac{1}{\operatorname{diam}(E)^{n+1}} \inf_{c \geq 0} d_E(\nu, c\mathcal{H}^n|_L).$$

This particular notation will be used only in this section.

Lemma 5.4. *Let φ be an affine, bilipschitz change of variables of $\mathbb{R}^{n+1}$ with bilipschitz constant $C \geq 1$. Let L be a hyperplane in $\mathbb{R}^{n+1}$. Let $J_{\varphi} > 0$ be the Jacobian of φ. Then, for any Radon measure ν, for any cube $Q \subset \mathbb{R}^{n+1}$ and any constant $c \geq 0$ we have that*

$$d_Q(\nu, c\mathcal{H}^n|_L) \approx_{n,C} d_{\varphi(Q)}\big(\varphi\sharp\nu, c\mathcal{H}^n|_{\varphi(L)}\big). \tag{5.1}$$

Hence,

$$\alpha_{\nu}^{L}(Q) \approx_{n,C} \alpha_{\varphi\sharp\nu}^{\varphi(L)}\big(\varphi(Q)\big). \tag{5.2}$$

Proof. Formula (5.2) is an immediate consequence of (5.1) and the fact that $\ell(Q) \approx_C \mathrm{diam}(\varphi(Q))$.

Let us prove (5.1). For every $c \geq 0$

$$\varphi\sharp\big(c\,\mathcal{H}^n\,|_L\big) = c(\varphi\sharp\,\mathcal{H}^n)|_{\varphi(L)}.$$

Indeed for any $\varphi\sharp\,\mathcal{H}^n\,|_L$-measurable set E we have

$$\varphi\sharp(c\,\mathcal{H}^n\,|_L)(E) = c\,\mathcal{H}^n\left(\varphi^{-1}(E)\cap L\right) = c\,\mathcal{H}^n\left(\varphi^{-1}(E\cap\varphi(L))\right) = c(\varphi\sharp\,\mathcal{H}^n)|_{\varphi(L)}(E).$$

Moreover, as a consequence of the Radon-Nikodym differentiation theorem (see [19, Lemma 1, p. 92]), we have

$$\mathcal{H}^n\left(\varphi^{-1}(E)\right) = J_\varphi\,\mathcal{H}^n(E).$$

So,

$$d_Q(\nu, c\,\mathcal{H}^n\,|_L) \approx_C d_{\varphi(Q)}\big(\varphi\sharp\nu, \varphi\sharp c\,\mathcal{H}^n\,|_L\big) \approx_{n,C} d_{\varphi(Q)}\big(\varphi\sharp\nu, c\,\mathcal{H}^n\,|_{\varphi(L)}\big),$$

which proves the lemma. □

5.2. Reduction of the Main Lemma to the case $A(0) = Id$

Recall that by Lemma 4.3 we can assume A to be a symmetric matrix.

Let us begin with a preliminary observation. Let $Q_0 \subset \mathbb{R}^{n+1}$ be a cube as in the Main Lemma 4.1 and let us denote $S := A_s(z_{Q_0})^{1/2}$, where z_{Q_0} is the center of Q_0. We choose the map φ so that $\varphi(x) = Sx$. By Lemma 5.3 we have that $A_\varphi(\varphi^{-1}(z_{Q_0})) = Id$. Denoting $\nu = \varphi^{-1}\sharp\mu$, the change of variables and Lemma 5.2 give

$$\begin{aligned} m_{\nu,\varphi^{-1}(Q_0)}(T_\varphi\nu) &= \frac{1}{\nu(\varphi^{-1}(Q_0))}\int\limits_{\varphi^{-1}(Q_0)} T_\varphi\nu\, d(\varphi^{-1}\sharp\mu) = \frac{1}{\mu(Q_0)}\int\limits_{Q_0} T_\varphi\nu\big(\varphi^{-1}(x)\big)\,d\mu(x) \\ &= \frac{1}{\mu(Q_0)}\int\limits_{Q_0} S\cdot T\nu(x)\,d\mu(x) = S\cdot m_{\mu,Q_0}(T\mu). \end{aligned}$$

Thus,

$$\begin{aligned} \int\limits_{Q_0} \big|T\mu(x) - m_{\mu,Q_0}(T\mu)\big|^2 d\mu(x) &= \int\limits_{\varphi^{-1}(Q_0)} \big|S\cdot(T_\varphi\nu - m_{\nu,\varphi^{-1}(Q_0)}(T_\varphi\nu))\big|^2 d\nu \\ &\approx \int\limits_{\varphi^{-1}(Q_0)} \big|T_\varphi\nu(x) - m_{\nu,\varphi^{-1}(Q_0)}(T_\varphi\nu)\big|^2 d\nu(x). \end{aligned}$$

The implicit constant above depends only on φ and, hence, on the ellipticity of the matrix A. Analogously, it is not hard to see that we have

$$\|T_\varphi \nu\|_{L^2(\nu|_{\varphi^{-1}(2Q_0)})} \approx \|T\mu\|_{L^2(\mu|_{(2Q_0)})}.$$

Using these facts and Lemma 5.4, we have that the Main Lemma 4.1 can be restated as follows.

Lemma 5.5. *Let $n > 1$ and let $C_0, C_1 > 0$ be some arbitrary constants. There exists $M = M(C_0, C_1, n) > 1$ big enough, $\lambda(C_0, C_1, n) > 0$ small enough and $\widetilde{\varepsilon} = \widetilde{\varepsilon}(C_0, C_1, M, n) > 0$ small enough such that if $\delta = \delta(M, C_0, C_1, n) > 0$ is small enough, then the following holds. Let μ be a Radon measure in $\mathbb{R}^{n+1}$, $Q_0 \subset \mathbb{R}^{n+1}$ a cube centered at the origin and $\nu := \varphi^{-1}\sharp\mu$, φ being as in the comments before the lemma, satisfying the following properties:*

(1) $A_\varphi\big(\varphi^{-1}(0)\big) = Id$.
(2) $\ell(MQ_0) \leq \lambda$.
(3) $\nu\big(\varphi^{-1}(Q_0)\big) = \ell(Q_0)^n$.
(4) $P_{\nu,\alpha/2}\big(\varphi^{-1}(MQ_0)\big) \leq C_0$.
(5) For all $x \in 2Q_0$ *and* $0 < r \leq \ell(\widetilde{Q})$, $\Theta_\mu(B(x,r)) \leq C_0$.
(6) Q_0 *has* C_0*-thin boundary.*
(7) $\alpha_\nu^{\varphi^{-1}(H)}\big(\varphi^{-1}(3MQ_0)\big) \leq \delta$, *where* $H = \{x \in \mathbb{R}^{n+1} \colon x_{n+1} = 0\}$.
(8) $T_{\varphi,\nu|_{\varphi^{-1}(2Q_0)}}$ *is bounded on* $L^2(\nu|_{\varphi^{-1}(2Q_0)})$ *with*

$$\big\|T_{\varphi,\nu|_{\varphi^{-1}(2Q_0)}}\big\|_{L^2(\nu|_{\varphi^{-1}(2Q_0)})\to L^2(\nu|_{\varphi^{-1}(2Q_0)})} \leq C_1.$$

(9) we have

$$\int_{\varphi^{-1}(Q_0)} \big|T_\varphi\nu(x) - m_{\nu,\varphi^{-1}(Q_0)}(T_\varphi\nu)\big|^2 d\nu(x) \leq \widetilde{\varepsilon}\nu\big(\varphi^{-1}(Q_0)\big).$$

Then there exists some constant $\tau > 0$ and a uniformly n-rectifiable set $\Gamma \subset \mathbb{R}^{n+1}$ such that

$$\mu(Q_0 \cap \Gamma) \geq \tau\mu(Q_0),$$

where the constant τ and the UR constants of Γ depend on all the constants above.

The aim of most of the rest of the paper is to provide the proof of this result.

In what follows, for the sake of simplicity of the notation, we will assume that $A(0) = A(z_{Q_0}) = Id$, which in particular gives that $\varphi = Id$, $\mu = \nu$ and $T_{\varphi,\mu} = T_\mu$. Indeed, if this is not the case, we should carry the following proofs for the image of cubes via φ^{-1}, periodize with respect to the image of a lattice of standard cubes and work with T_φ instead of T. This would be a merely notational complication that we prefer to avoid to make the arguments more accessible.

Reduction to a periodic matrix. The forthcoming lemma shows, roughly speaking, that the local structure of A close to Q_0 is what matters to the purposes of Lemma 4.1. An immediate consequence of this fact is that, without loss of generality, we can replace the matrix A with a periodic matrix, provided that the new matrix coincides with A in a suitable neighborhood of the cube Q_0.

In what follows, we assume the matrix $\bar{A}$ to have Hölder continuous coefficients of exponent $\widetilde{\alpha} < \alpha$ for technical reasons that will result clearer later on.

Lemma 5.6. *Let $\bar{A} \in \Lambda_{\widetilde{\alpha}}$ be such that $\bar{A}(x) = A(x)$ for every $x \in 2Q_0$. Let $\bar{T}$ denote the gradient of the single layer potential associated with $\bar{A}$. The operator $T_{\mu|2Q_0}$ is bounded in $L^2(\mu|_{2Q_0})$ if and only if $\bar{T}_{\mu|_{2Q_0}}$ is bounded in $L^2(\mu|_{2Q_0})$ and*

$$\|T_{\mu|_{2Q_0}}\|_{L^2(\mu|_{2Q_0})\to L^2(\mu|_{2Q_0})} = \|\bar{T}_{\mu|_{2Q_0}}\|_{L^2(\mu|_{2Q_0})\to L^2(\mu|_{2Q_0})} + O\big(\ell(Q_0)^{\widetilde{\alpha}}\big).$$

Moreover we have

$$\int_{Q_0} |T\mu(x) - m_{\mu,Q_0}(T\mu)|^2 d\mu(x) \lesssim \int_{Q_0} |\bar{T}\mu(x) - m_{\mu,Q_0}(\bar{T}\mu)|^2 d\mu(x) + \big(\ell(MQ_0)^{2\widetilde{\alpha}} + M^{-\widetilde{\alpha}}\big)^2 \mu(Q_0), \tag{5.3}$$

where M is as in the statement of Lemma 4.1 and the implicit constant in (5.3) depends on $\operatorname{diam}(\operatorname{supp}\mu)$.

The proof of Lemma 5.6 relies on the fact that $\Theta(\cdot,\cdot;A(x)) = \Theta(\cdot,\cdot;\bar{A}(x))$ for every $x \in 2Q_0$ and it is very similar to the one of Lemma 4.3, so that we omit it.

In the rest of the paper, without additional specifications, we will deal with a matrix $\bar{A}$ periodic with period ℓ, $2\ell(Q_0) < \ell \lesssim \ell(Q_0)$.

The definition of the matrix $\bar{A}$. The construction in the present subsection is dictated by the necessity of having an auxiliary matrix which agrees with A on $2Q_0$ and has the further properties of being periodic (which is crucial to use the estimates of the theory of homogenization) and of presenting ‘additional symmetries’ with respect to reflections (see the forthcoming Lemma 5.8). For a scheme of this construction we also refer to Fig. 1.

Let e_j denote the j-th element of the canonical basis of $\mathbb{R}^{n+1}$. We denote by $\psi_j\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ the map

$$\psi_j(x) := x + (3\ell(Q_0) - 2x_j)e_j, \tag{5.4}$$

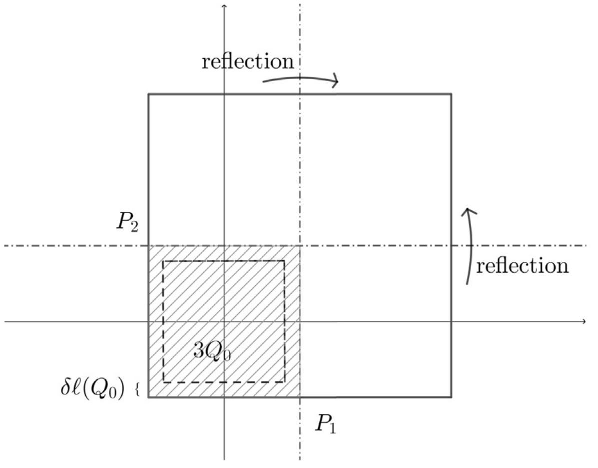

Fig. 1. A schematization of the construction of $\bar{A}$ at the level of the periodic unit.

which corresponds to the reflection across the hyperplane P_j orthogonal to e_j and which passes through the point $\frac{3}{2}\ell(Q_0)e_j$. Let $0 < \delta < 1/10$.

Given a matrix $B(x)$ with variable coefficients, we define B_j as

$$B_j = B_{\psi_j} = D(\psi_j^{-1})(B \circ \psi_j)D(\psi_j^{-1})^T. \tag{5.5}$$

Moreover, we define the matrix $\widetilde{B}$ as

$$\widetilde{B}(x) = \begin{cases} B(x) \\ \quad \text{for } \operatorname{dist}\big(x, \partial(3Q_0)\big) \geq \delta\ell(Q_0), \\ \frac{\operatorname{dist}(x,\partial(3Q_0))}{\delta\ell(Q_0)} B(x) + \Big(1 - \frac{\operatorname{dist}(x,\partial(3Q_0))}{\delta\ell(Q_0)}\Big) Id \\ \quad \text{for } \operatorname{dist}\big(x, \partial(3Q_0)\big) < \delta\ell(Q_0). \end{cases} \tag{5.6}$$

It is also useful to introduce the notation

$$\widehat{B}_j(x) = \begin{cases} B(x) & \text{for } x_j \leq \frac{3}{2}\ell(Q_0), \\ B_j(x) & \text{for } x_j > \frac{3}{2}\ell(Q_0). \end{cases} \tag{5.7}$$

Let us apply the previous constructions to the matrix A. First, observe that the matrix $\hat{A}_j$ is not necessarily continuous. However, $\widehat{(\widetilde{A})}_j$ is continuous because $Id_j = Id$ and $\widetilde{A}|_{\partial(3Q_0)} \equiv Id$. Our aim, now, is to define the final auxiliary matrix $\bar{A}$ by an iteration of the construction in (5.7) along every direction and which is followed by a periodization. Before doing so, let us observe that for $i, j \in \{1, \ldots, n+1\}$,

$$(\widetilde{A}_i)_j(x) = (\widetilde{A}_j)_i(x), \qquad x \in \mathbb{R}^{n+1}\,.$$

This follows directly from (5.5) using the facts that $\psi_i\big(\psi_j(x)\big) = \psi_j\big(\psi_i(x)\big)$ and that the matrices $D(\psi_i^{-1}), D(\psi_j^{-1})$ are diagonal. Thus by the linearity of the interpolation in (5.6) we have that

$$\widehat{\big(\widetilde{(\widehat{A})_i}\big)_j} = \widehat{\big(\widetilde{(\widehat{A})_j}\big)_i} =: \widetilde{(\widehat{A})}_{i,j}, \tag{5.8}$$

so the order of the modifications is not relevant.

Let us now construct the matrix $\bar{A}$ in two steps:

- For x belonging to the cube of side length $6\ell(Q_0)$ centered at the point with coordinates $\frac{3}{2}\ell(Q_0)(1,\ldots,1)$ we define

$$\bar{A}(x) := \widetilde{(\widehat{A})}_{1,\ldots,n+1}.$$

- By (5.6), the matrix $\bar{A}$ defined in the first step coincide with Id for x belonging to the boundary of the cube with side length $6\ell(Q_0)$ and centered at $\frac{3}{2}\ell(Q_0)(1,\ldots,1)$. Hence, $\bar{A}$ admits a continuous and $6\ell(Q_0)$-periodic extension to $\mathbb{R}^{n+1}$ so that

$$\bar{A}(x) = \bar{A}\big(x + 6\vec{k}\ell(Q_0)\big)$$

  for every $\vec{k} \in \mathbb{Z}^{n+1}$.

The following holds.

Lemma 5.7. *The matrix $\bar{A}$ is well-defined and periodic with period $6\ell(Q_0)$. Furthermore, for $\ell(Q_0)$ small enough it is Hölder continuous with exponent $\alpha/2^{n+1}$ and constant not depending on $\ell(Q_0)$.*

Proof. The well-definition of $\bar{A}$ follows from (5.8), and the periodicity holds by construction. We are left with the proof of Hölder continuity. As a first step, we prove that there exists $C > 0$ such that

$$|\bar{A}(x) - \bar{A}(y)| \le C|x-y|^{\alpha/2}, \qquad \text{for } x, y \in 3Q_0. \tag{5.9}$$

By the construction of $\bar{A}$, it suffices to verify the condition (5.9) for $x, y \in \{z \in 3Q_0 : \operatorname{dist}(z, \partial(3Q_0)) \le \delta\ell(Q_0)\}$ with $|x-y| \le \delta\ell(Q_0)$. In this case, we denote $t := \operatorname{dist}(x, \partial(3Q_0))$, $s := \operatorname{dist}(y, \partial(3Q_0))$ and we have

$$|\bar{A}(x) - \bar{A}(y)| = \Big|A(x)\frac{t}{\delta\ell(Q_0)} + \Big(1 - \frac{t}{\delta\ell(Q_0)}\Big)Id - A(y)\frac{s}{\delta\ell(Q_0)} + \Big(1 - \frac{s}{\delta\ell(Q_0)}\Big)Id\Big|$$

$$\leq \left|(A(x)-Id)\frac{t}{\delta\ell(Q_0)}-(A(y)-Id)\frac{s}{\delta\ell(Q_0)}\right|$$
$$\leq |A(x)-Id|\frac{|t-s|}{\delta\ell(Q_0)}+|A(x)-A(y)|\frac{s}{\delta\ell(Q_0)} =: I+II.$$

The α-Hölder continuity of A and the choice $s\leq\delta\ell(Q_0)$ yield

$$II\lesssim |x-y|^\alpha.$$

On the other hand, the choice $|x-y|\leq\delta\ell(Q_0)$, the identity $A(0)=Id$, and the Lipschitz character of the map $\operatorname{dist}(\cdot,\partial(3Q_0))$ give that, for $\ell(Q_0)^{\alpha/2}/\delta^\alpha\leq 1$ we have

$$I\leq |x|^\alpha\frac{|t-s|}{\delta\ell(Q_0)}\lesssim |x|^\alpha\frac{|t-s|^\alpha}{(\delta\ell(Q_0))^\alpha}\lesssim\frac{|x-y|^\alpha}{\delta^\alpha}\lesssim\frac{\ell(Q_0)^{\alpha/2}}{\delta^\alpha}|x-y|^{\alpha/2}\leq |x-y|^{\alpha/2}.$$

Gathering the above estimates for I and II, we obtain (5.9). An analogous calculation shows that (5.9) holds for $x,y\in 3Q_0\cup(3Q_0+3\ell(Q_0)e_{n+1})$ (see also [43, Lemma 8.1]). By periodicity of the matrix, this is enough to conclude the proof of the lemma. In particular, the exponent $\alpha/2^{n+1}$ is given by the fact that the proof has to be iterated for all the $(n+1)$ different directions. $\square$

We remark that, being $\bar{A}$ periodic, in the previous lemma there is no need to introduce a radial cut-off for the matrix as in [43]. For the rest of the paper we use the notation $\widetilde{\alpha}:=\alpha/2^{n+1}$.

Properties of $\mathcal{E}_{\bar{A}}$. As a consequence of the definition of $\bar{A}$ and, more specifically, of its periodicity and the fact that by construction

$$\bar{A}_j(x)=\bar{A}(x)$$

for every $x\in\mathbb{R}^{n+1}$ and $j=1,\dots,n+1$, we have the following.

Lemma 5.8.

$$\mathcal{E}_{\bar{A}}(x,y)=\mathcal{E}_{\bar{A}}(\psi_j(x),\psi_j(y))\qquad \textit{for}\qquad j=1,\dots,n+1 \tag{5.10}$$

and

$$\mathcal{E}_{\bar{A}}(x,y)=\mathcal{E}_{\bar{A}}\left(x+6\vec{k}\ell(Q_0),y+6\vec{k}\ell(Q_0)\right)\qquad \textit{for}\qquad \vec{k}\in\mathbb{Z}^{n+1}.$$

By Lemma 2.3, the function $\bar{K}=\nabla_1\mathcal{E}_{\bar{A}}(\cdot,\cdot)$ is (globally) a Calderón-Zygmund kernel. In particular

(a) $|\bar{K}(x,y)|\lesssim|x-y|^{-n}$ for all $x,y\in\mathbb{R}^{n+1}$ with $x\neq y$.

(b) $|\bar{K}(x,y)-\bar{K}(x,y')|+|\bar{K}(y,x)-\bar{K}(y',x)| \lesssim |y-y'|^{\tilde{\alpha}}|x-y|^{-n-\tilde{\alpha}}$ for $2|y-y'| \leq |x-y|$.

Let $\bar{T}_\mu$ denote the singular integral operator associated with $\bar{K}$,

$$\bar{T}_\mu f(x) = \int \bar{K}(x,y)f(y)d\mu(y).$$

Lemma 5.6 tells that we can prove the Main Lemma for $\bar{T}$ instead of T, possibly by slightly worsening the parameters involved.

6. A first localization lemma

It is useful to provide a local analogue of the BMO-type estimate (4.1). This is possible because of the smallness of the α-number and the bound for the $P_{\mu,\tilde{\alpha}}$-density. Also, recall that because of the assumptions in Lemma 4.1, we have $\mu(MQ_0) \lesssim M^n\mu(Q_0)$. In what follows we sketch the proof of the localization of (4.1) for $\bar{T}_\mu$, highlighting the differences with respect to the case of the Riesz transform (see [21, Lemma 4.2]).

In the rest of the paper we omit to indicate the dependence of the implicit constants in our estimates on C_0 and C_1.

Lemma 6.1. *For δ small enough depending on M, the following inequality holds*

$$\int_{Q_0} |\bar{T}_\mu \chi_{MQ_0}|^2 d\mu \lesssim \left(\varepsilon + \frac{1}{M^{2\tilde{\alpha}}} + M^{4n+2}\delta^{1/(4n+4)} + (M\ell(Q_0))^{2\tilde{\alpha}}\right)\mu(Q_0). \tag{6.1}$$

Proof. First, observe that

$$\begin{aligned}\int_{Q_0} &|\bar{T}_\mu(\chi_{MQ_0})|^2 d\mu \\ &\leq 2\int_{Q_0} |\bar{T}_\mu(\chi_{MQ_0}) - m_{\mu,Q_0}(\bar{T}_\mu\chi_{MQ_0})|^2 d\mu + 2|m_{\mu,Q_0}(\bar{T}_\mu\chi_{MQ_0})|^2\mu(Q_0).\end{aligned} \tag{6.2}$$

Let us estimate the two summands on the right hand side of (6.2) separately. To study the first one, we write

$$\begin{aligned}\int_{Q_0} &\left|\bar{T}_\mu\chi_{MQ_0} - m_{\mu,Q_0}(\bar{T}_\mu\chi_{MQ_0})\right|^2 d\mu \\ &\leq 2\int_{Q_0} \left|\bar{T}_\mu\chi_{(MQ_0)^c}(x) - m_{\mu,Q_0}(\bar{T}_\mu\chi_{(MQ_0)^c})\right|^2 d\mu(x) + 2\int_{Q_0} |\bar{T}\mu - m_{\mu,Q_0}(\bar{T}\mu)|^2 d\mu.\end{aligned} \tag{6.3}$$

Applying Lemma 2.1, it follows that for $x, y \in Q_0$

$$
\begin{aligned}
|\bar{T}_\mu \chi_{(MQ_0)^c}(x) - \bar{T}_\mu \chi_{(MQ_0)^c}(y)| &\leq \int_{(MQ_0)^c} |\bar{K}(x,z) - \bar{K}(y,z)| d\mu(z) \\
&\lesssim |x-y|^{\widetilde{\alpha}} \int_{(MQ_0)^c} \frac{1}{|x-z|^{n+\widetilde{\alpha}}} d\mu(z) \\
&\lesssim |x-y|^{\widetilde{\alpha}} \sum_{j=1}^{\infty} \int_{2^{j+1} MQ_0 \setminus 2^j MQ_0} \frac{1}{|x-z|^{n+\widetilde{\alpha}}} d\mu(z) \lesssim \frac{|x-y|^{\widetilde{\alpha}}}{\ell(MQ_0)^{\widetilde{\alpha}}} P_{\mu,\widetilde{\alpha}}(MQ_0) \lesssim \frac{1}{M^{\widetilde{\alpha}}},
\end{aligned}
$$

being $P_{\mu,\widetilde{\alpha}}(MQ_0) \lesssim 1$. Then, averaging the previous inequality over the variable y, we get

$$
\left|\bar{T}_\mu \chi_{(MQ_0)^c}(x) - m_{\mu,Q_0}(\bar{T}_\mu \chi_{(MQ_0)^c})\right| \lesssim \frac{1}{M^{\widetilde{\alpha}}}
$$

and

$$
\int_{Q_0} |\bar{T}_\mu \chi_{(MQ_0)^c}(x) - m_{\mu,Q_0}(\bar{T}_\mu \chi_{(MQ_0)^c})|^2 d\mu(x) \lesssim \frac{1}{M^{2\widetilde{\alpha}}} \mu(Q_0).
$$

Recalling that by hypothesis we have

$$
\int_{Q_0} \left|\bar{T}\mu - m_{\mu,Q_0}(\bar{T}\mu)\right|^2 d\mu \leq \varepsilon \mu(Q_0),
$$

we can estimate (6.3) as

$$
\int_{Q_0} \left|\bar{T}_\mu(\chi_{(MQ_0)^c}) - m_{\mu,Q_0}(\bar{T}_\mu \chi_{MQ_0})\right|^2 d\mu \lesssim \left(\varepsilon + \frac{1}{M^{2\widetilde{\alpha}}}\right) \mu(Q_0). \tag{6.4}
$$

An application of Lemma 2.7 together with the antisymmetry of $\nabla_1 \Theta(\cdot,\cdot;\bar{A}(x))$ also gives

$$
\left|m_{\mu,Q_0}(\bar{T}_\mu \chi_{Q_0})\right| \lesssim \frac{1}{\mu(Q_0)} \int_{Q_0} \int_{Q_0} |x-y|^{-n+\widetilde{\alpha}} d\mu(x) d\mu(y) \lesssim \ell(Q_0)^{\widetilde{\alpha}}. \tag{6.5}
$$

Minor variations of the arguments which prove [21, (4.2)] show that

$$
\begin{aligned}
|m_{\mu,Q_0}(\bar{T}_\mu \chi_{MQ_0})| &\overset{(6.5)}{\lesssim} |m_{\mu,Q_0}(\bar{T}_\mu \chi_{MQ_0 \setminus Q_0})| + \ell(Q_0)^{\widetilde{\alpha}} \\
&\lesssim M^{2n+1} \delta^{1/8(n+1)} + \left(M\ell(Q_0)\right)^{\widetilde{\alpha}} + \ell(Q_0)^{\widetilde{\alpha}} \\
&\lesssim M^{2n+1} \delta^{1/8(n+1)} + \left(M\ell(Q_0)\right)^{\widetilde{\alpha}}.
\end{aligned} \tag{6.6}
$$

For the sake of brevity we omit the details and we just point out that the presence of the second summand on the right hand side comes from the estimate

$$\Big|\int_{Q_0} \bar{T}\big(\varphi\mathcal{H}^n|_H\big)\, d\mathcal{H}^n|_H\Big| \lesssim \big(M\ell(Q_0)\big)^{\widetilde{\alpha}}\ell(Q_0)^n, \tag{6.7}$$

where φ is a proper even C^1 function with $0 \le \varphi \le 1$ and supported on $MQ_0 \setminus Q_0$. To get the estimate (6.7), we just write

$$\begin{aligned}
&\Big|\int_{Q_0} \bar{T}\big(\varphi\mathcal{H}^n|_H\big)\, d\mathcal{H}^n|_H\Big| \\
&\quad\le \int_{Q_0}\int_{MQ_0} \Big|\frac{1}{2}\bar{K}(x,y) - \frac{1}{2}\nabla_1\Theta(x,y;\bar{A}(x))\Big|\, d\mathcal{H}^n|_H(x)d\mathcal{H}^n|_H(y) \\
&\qquad+ \int_{Q_0}\int_{MQ_0} \Big|\frac{1}{2}\bar{K}(x,y) - \frac{1}{2}\nabla_1\Theta(x,y;\bar{A}(y))\Big|\, d\mathcal{H}^n|_H(x)d\mathcal{H}^n|_H(y) \\
&\qquad+ \frac{1}{2}\Big|\int_{Q_0}\int_{MQ_0} \big(\nabla_1\Theta(x,y;\bar{A}(x)) + \nabla_1\Theta(x,y;\bar{A}(y))\big)\, d\mathcal{H}^n|_H(x)d\mathcal{H}^n|_H(y)\Big|.
\end{aligned}$$

Then, the third summand is null because of the antisymmetry of its integrand and the first two terms can be estimated via Lemma 2.2.

Gathering (6.2), (6.4) and (6.6) we are able to conclude the proof of the lemma. □

7. The David and Mattila lattice associated with μ and its properties

The dyadic lattice constructed by David and Mattila [16, Theorem 3.2] is a powerful tool in the study of the geometry of Radon measures. Its main properties are listed in the following lemma, that we state for a general Radon measure with compact support.

Lemma 7.1 *(David and Mattila). Let σ be a compactly supported Radon measure in $\mathbb{R}^{n+1}$. Consider two constants $K_0 > 1$ and $A_0 > 5000K_0$ and denote $W = \operatorname{supp}\sigma$. Then there exists a sequence of partitions of W into Borel subsets Q, $Q \in \mathcal{D}_{\sigma,k}$, which we will refer to as cells, with the following properties:*

- *For each integer $k \ge 0$, W is the disjoint union of the cells Q, $Q \in \mathcal{D}_{\sigma,k}$. If $k < l$, $Q \in \mathcal{D}_{\sigma,l}$, and $R \in \mathcal{D}_{\sigma,k}$, then either $Q \cap R = \emptyset$ or $Q \subset R$.*
- *For each $k \ge 0$ and each cell $Q \in \mathcal{D}_{\sigma,k}$, there is a ball $B(Q) = B\big(z_Q, r(Q)\big)$ such that*

$$z_Q \in W,\ A_0^{-k} \le r(Q) \le K_0A_0^{-k}$$
$$W \cap B(Q) \subset Q \subset W\cap 28B(Q) = W \cap B\big(z_Q, 28r(Q)\big),$$

and the balls $5B(Q), Q \in \mathcal{D}_{\sigma,k}$ are disjoint.

- *The cells $Q \in \mathcal{D}_{\sigma,k}$ have small boundaries. By this, we mean that for each $Q \in \mathcal{D}_{\sigma,k}$ and each integer $l \geq 0$, if we set*

$$N_l^{\text{int}} := \left\{x \in Q : \operatorname{dist}(x, W \setminus Q) < A_0^{-k-l}\right\}$$
$$N_l^{\text{ext}}(Q) := \left\{x \in W \setminus Q : \operatorname{dist}(x, Q) < A_0^{-k-l}\right\}$$

and

$$N_l(Q) := N_l^{\text{int}}(Q) \cup N_l^{\text{ext}}(Q),$$

we get

$$\sigma\big(N_l(Q)\big) \leq \big(C^{-1} K_0^{-3(n+1)-1} A_0\big)^{-l} \sigma\big(90B(Q)\big)$$

- *Denote by $\mathcal{D}_{\sigma,k}^{\text{db}}$ the family of cells $Q \in \mathcal{D}_{\sigma,k}$ for which*

$$\sigma\big(100B(Q)\big) \leq K_0 \sigma\big(B(Q)\big).$$

We have that $r(Q) = A_0^{-k}$ when $Q \in \mathcal{D}_{\sigma,k} \setminus \mathcal{D}_{\sigma,k}^{\text{db}}$ and

$$\sigma\big(100B(Q)\big) \leq K_0^{-1} \sigma\big(100^{l+1} B(Q)\big) \tag{7.1}$$

for all $l \geq 1$ with $100^l \leq K_0$ and $Q \in \mathcal{D}_{\sigma,k} \setminus \mathcal{D}_{\sigma,k}^{\text{db}}$.

Let us denote $\mathcal{D}_\sigma := \bigcup_k \mathcal{D}_{\sigma,k}$. Let us choose A_0 big enough so that

$$C^{-1} K_0^{-3(n+1)-1} A_0 > A_0^{1/2} > 10. \tag{7.2}$$

Here we list some useful quantities associated with each cell $Q \in \mathcal{D}_{\sigma,k}$:

- $J(Q) := k$, which may be interpreted as the *generation* of Q.
- $\ell(Q) := 56 K_0 A_0^{-k}$, that we also call *side length*. Notice that

$$\frac{1}{28} K_0^{-1} \ell(Q) \leq \operatorname{diam}(28B(Q)) \leq \ell(Q)$$

  and $r(Q) \approx \operatorname{diam}(Q) \approx \ell(Q)$.
- calling z_Q the *center* of Q, we denote $B_Q := 28B(Q) = B(z_Q, 28r(Q))$, which in particular gives

$$Q \cap \frac{1}{28} B_Q \subset Q \subset B_Q.$$

We recall, now, some of the properties of the cells in the David and Mattila lattice.

The choice in (7.2) implies, for $0 < \lambda \le 1$, the estimate

$$\sigma\big(\{x \in Q : \operatorname{dist}(x, W \setminus Q) \le \lambda \ell(Q)\}\big) + \sigma\big(\{x \in 3.5B_Q \setminus Q : \operatorname{dist}(x, Q) \le \lambda \ell(Q)\}\big) \\ \le c\lambda^{1/2}\sigma(3.5B_Q).$$

We denote $\mathcal{D}_\sigma^{\mathrm{db}} \coloneqq \bigcup_{k\ge 0} \mathcal{D}_{\sigma,k}^{\mathrm{db}}$ and we say that it is the lattice of *doubling cells*. This notation is justified by the fact that, for $Q \in \mathcal{D}_\sigma^{\mathrm{db}}$, we have

$$\sigma(3.5B_Q) \le \sigma(100B(Q)) \le K_0\sigma(B(Q)) \le K_0\sigma(Q).$$

An important feature of the David and Mattila lattice is that every cell $Q \in \mathcal{D}_\sigma$ can be covered by doubling cells up to a set of σ-measure zero ([16, Lemma 5.28]). Moreover, if we have two cells $R, Q \in D_\sigma$ with $Q \subset R$ and such that every intermediate cell $Q \subsetneq S \subsetneq R$ belongs to $\mathcal{D}_\sigma \setminus \mathcal{D}_\sigma^{\mathrm{db}}$, we have the control

$$\sigma(100B(Q)) \le A_0^{-10n(J(Q)-J(R)-1)}\sigma(100B(R)) \tag{7.3}$$

on the decay of the measure. The estimate (7.3) is proved via an iterated application of the inequality

$$\sigma(100B(Q)) \le A_0^{-10n}\sigma(100B(\hat{Q})), \tag{7.4}$$

where $\hat{Q}$ is the cell from $\mathcal{D}_{\sigma,J(Q)-1}$ containing Q (also called *parent* of Q). We remark that (7.4) follows by (7.1) and a proper choice of A_0 and K_0 (see [16, Lemma 5.31]).

For $Q \in \mathcal{D}_\sigma$, we denote by $\mathcal{D}_\sigma(Q)$ the cells in $\mathcal{D}_\sigma$ which are contained in Q and $\mathcal{D}_\sigma^{\mathrm{db}}(Q) \coloneqq \mathcal{D}_\sigma(Q) \cap \mathcal{D}_\sigma^{\mathrm{db}}$.

8. The Key Lemma, the stopping time condition and a first modification of the measure

The heart of the proof of Lemma 5.5 consists in providing a control on the abundance of cells with low density (in some sense that we clarify below). The whole construction that we are about to discuss depends on some auxiliary parameter to be chosen properly later in the proof.

Definition 8.1 *(Low density cells).* Let $0 < \theta_0 \ll 1$. A cell $Q \in \mathcal{D}_\mu$ is said to be of low density if

$$\Theta_\mu(3.5B_Q) \le \theta_0$$

and it is maximal with respect to the set inclusion. We denote by LD the family of low density cells.

Most of the rest of the paper deals with the proof of the fact that the low density cells fail to cover a significant portion of Q_0.

Lemma 8.1 *(Key Lemma). Let ε, δ and M be as in Lemma 4.1. There exists $\varepsilon_0 > 0$ such that if M is big enough and θ_0, δ and ε are small enough, then*

$$\mu\left(Q_0 \setminus \bigcup_{Q \in \mathsf{LD}} Q\right) \geq \varepsilon_0 \mu(Q_0). \tag{8.1}$$

Sketch of the proof of the Main Lemma 5.5 using the Key Lemma (8.1). The proof is analogous to the argument of [21, Section 10]. For the reader's convenience, we briefly outline the construction of the uniformly n-rectifiable set Γ in the statement of the Main Lemma. Let us define $F := Q_0 \cap \operatorname{supp} \mu \setminus \bigcup_{Q \in \mathsf{LD}} Q$, the Key Lemma 8.1 gives $\mu(F) \geq \varepsilon_0 \mu(Q_0)$. Let $\sigma := \mu|_{Q_0}$, let $\mathcal{D}_\sigma$ be its associated David-Mattila lattice, and let us define the maximal dyadic operator

$$\mathcal{M}_{\mathcal{D}_\sigma} f(x) := \sup_{Q \in \mathcal{D}_\sigma : x \in Q} \frac{1}{\sigma(Q)} \int_Q |f| d\sigma.$$

Let $\widetilde{F} := \{x \in F : \mathcal{M}_{\mathcal{D}_\sigma}(\chi_{F^c})(x) \leq 1 - (\varepsilon_0/2)\}$. It is not hard to prove that $\sigma(\widetilde{F}) \geq \sigma(F)/2$. We consider a family $\{Q_j\}_{j \in J} \subset \mathcal{D}_\sigma$ of maximal cells (with respect to the inclusion) such that $\sigma(Q_j \setminus F) > \big(1 - (\varepsilon_0/2)\big)\sigma(Q_j)$. For $j \in J$, we denote as $\mathcal{A}_j$ the family of maximal doubling cells that cover Q_j, and by $\mathcal{A}_0$ the family $\{P \in \cup_{j \in J} \mathcal{A}_j : \sigma(P \cap F) > 0\}$. We can now define the main auxiliary measure

$$\zeta = \sigma|_{\widetilde{F}} + \sum_{Q \in \mathcal{A}_0} \mathcal{H}^n|_{S(Q)},$$

where $S(Q)$ indicates an n-dimensional sphere with the same center as $B(Q)$ and radius $B(Q)/4$. Lemma 10.2 in [21] gives that the measure ζ is AD-regular with constant depending only on C_0, θ_0 and ε_0. Furthermore, it is routine to check that the proof of [21, Lemma 10.3] yields that $\bar{T}_\zeta$ is bounded on $L^2(\zeta)$ with norm depending on the same parameters and C_1. Hence, we can apply [43, Theorem 1.1], which gives that ζ is uniformly n-rectifiable. By construction and using the fact that $\sigma(\widetilde{F}) \geq \sigma(F)/2$, $\Gamma := \operatorname{supp} \zeta$ satisfies

$$\mu(\Gamma) \geq \mu(\widetilde{F}) = \sigma(\widetilde{F}) \geq \varepsilon_0 \mu(Q_0)/2,$$

which proves the Main Lemma 5.5. $\square$

The rest of the present article (a part from the last section) is devoted to the proof of Lemma 8.1.

We argue by contradiction: assume that (8.1) does not hold, that is to say

$$\mu\left(\bigcup_{Q\in\mathsf{LD}} Q\right) > (1-\varepsilon_0)\mu(Q_0). \tag{8.2}$$

More specifically, we want to show that a choice of ε_0 small enough leads to an absurd. The proof is based on a stopping time argument. Roughly speaking, for $Q \in \mathsf{LD}$, we say that a cell R belongs to its associated stopping family if it is a descendant of Q (i.e. $R \subset Q$) and it is sufficiently small. The definition of stopping cells depends on a parameter t, which has to be thought small and that will be appropriately chosen later.

Definition 8.2 *(Stopping cells).* Let $Q \in \mathsf{LD}$. Let $0 < t < 1$. We say that $R \in \mathsf{Stop}(Q)$ if the following two conditions are verified and they are maximal with respect to containment:

- $R \in \mathcal{D}_\mu^{\mathrm{db}}$, $R \subset Q$.
- $\ell(R) \leq t\ell(Q)$.

We also denote $\mathsf{Stop} := \bigcup_{Q\in\mathsf{LD}} \mathsf{Stop}(Q)$ the family of all the stopping cells.

Assuming that the stopping cells in $\mathsf{Stop}(Q)$ are doubling makes sense in light of the fact that doubling cells cover Q up to a set of μ-measure zero. In particular, this implies that (8.2) is equivalent to

$$\mu\left(\bigcup_{Q\in\mathsf{Stop}} Q\right) > (1-\varepsilon_0)\mu(Q_0).$$

The proof of the Key Lemma 8.1 involves a periodization of the measure μ, which is essentially carried out by replicating $\mu|Q_0$ on the horizontal plane according to the periodicity of the matrix $\bar{A}$.

The stopping cells which are close to the boundary of Q_0 (in a proper sense) are problematic for our analysis, so we need to rule them out of the construction. In order to do this, we have to show that their contribution to the measure of Q_0 is negligible (see (8.4)). We say that $P \in \mathsf{Bad}$ if $P \in \mathsf{Stop}$ and $1.1B_P \cap \partial Q_0 \neq \emptyset$.

Another technical problem is that Stop may contain infinitely many cells. This second difficulty can be easily overcome considering a finite family of cells, named Stop_0, which contains a big portion of the measure of Stop, e.g.

$$\mu\left(\bigcup_{Q\in\mathsf{Stop}_0} Q\right) > (1-2\varepsilon_0)\mu(Q_0). \tag{8.3}$$

The rest of the section is devoted to a justification of the last affirmations concerning Bad and the first modification of the measure μ. It is essentially a rewriting of [21, Lemma

6.2, Lemma 6.3, Lemma 6.4] in our context, in which we highlight the right homogeneity coming from our elliptic setting.

The following lemma contains an estimate of the density $P_{\mu,\widetilde{\alpha}}$ of the stopping cells in terms of the low density parameter θ_0.

Lemma 8.2. *Let $Q \in \mathsf{Stop}$ and let $t = \theta_0^{1/(n+\widetilde{\alpha})}$. We have*

$$\Theta_\mu(2B_Q) \leq P_{\mu,\widetilde{\alpha}}(2B_Q) \lesssim \theta_0^{\frac{\widetilde{\alpha}}{n+\widetilde{\alpha}}}.$$

Proof. The first inequality is an immediate consequence of the definition of $P_{\mu,\widetilde{\alpha}}$. To prove the second inequality, we consider the maximal cell $R' \in \mathcal{D}_\mu$ such that $Q \subset R' \subset R$ and $\ell(R') \leq t\ell(R)$ and write

$$\begin{aligned} P_{\mu,\widetilde{\alpha}}(2B_Q) \lesssim & \sum_{P\in\mathcal{D}_\sigma:Q\subset P\subset R'} \Theta_\mu(2B_P)\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\widetilde{\alpha}} + \sum_{P\in\mathcal{D}_\sigma:R'\subset P\subset R} \Theta_\mu(2B_P)\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\widetilde{\alpha}} \\ & + \sum_{P\in\mathcal{D}_\sigma:R\subset P\subset Q_0} \Theta_\mu(2B_P)\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\widetilde{\alpha}} + \sum_{k\geq 1} 2^{-k\widetilde{\alpha}}\Theta_\mu(2^kB_Q) \\ =& I + II + III + IV. \end{aligned}$$

Then, the estimates work as in the case of the work of Girela-Sarrión and Tolsa. In particular, the calculations for [21, Lemma 6.2, estimate of $S_1 + S_2$] work verbatim and give

$$I + II \lesssim \frac{\theta_0}{t^n}$$

Furthermore [21, Lemma 6.2, estimate of $S_3 + S_4$] gives

$$III + IV \lesssim t^{\widetilde{\alpha}},$$

which justifies the choice of t in the statement of the lemma. $\square$

For the rest of the paper we assume $t = \theta_0^{1/(n+\widetilde{\alpha})}$ in Definition 8.2.

Using the estimates in Lemma 8.2, one can prove that

$$\mu\Big(\bigcup_{\mathsf{Bad}} Q\Big) \lesssim \theta_0^{\frac{\widetilde{\alpha}}{n+\widetilde{\alpha}}}\mu(Q_0). \tag{8.4}$$

Its proof (see [21, Lemma 6.3] for all the details) relies on a covering argument: let I be an arbitrary finite subset of Bad. An application of Vitali's covering theorem to the family $\{1.15B_Q\}_{Q\in I}$ gives that there exists a family $J \subset I$ such that the cells $\{1.15B_Q\}_{Q\in J}$ are pairwise disjoint and $\bigcup_{Q\in J} 3(1.15B_Q)$ covers $\bigcup_{Q\in I} 1.15B_Q$. Furthermore $Q \in J \subset \mathsf{Bad}$ implies that

$$\mathcal{H}^n(1.15B_Q \cap \partial Q_0) \gtrsim r(B_Q)^n.$$

Lemma 8.2 and the above estimate give

$$\mu\Big(\bigcup_{P\in I} B_P\Big) \leq \sum_{P\in J} \mu(3.45B_P) \lesssim \theta_0^{\frac{\tilde{\alpha}}{n+\tilde{\alpha}}} \sum_{Q\in J} r(B_Q)^n \lesssim \theta_0^{\frac{\tilde{\alpha}}{n+\tilde{\alpha}}} \mathcal{H}^n(\partial Q_0) \approx \theta_0^{\frac{\tilde{\alpha}}{n+\tilde{\alpha}}} \mu(Q_0),$$

which readily implies (8.4).

First modification of the measure. As already mentioned, for technical purposes it is useful to modify the measure inside Q_0 by taking just finitely many stopping cells and getting rid of the cells in Bad. To make the previous statement rigorous, we choose a small parameter $0 < \kappa_0 \ll 1$ to be fixed later and, after denoting

$$I_{\kappa_0}(Q) := \{x \in Q : \operatorname{dist}(x, \operatorname{supp}\sigma \setminus Q) \geq \kappa_0 \ell(Q)\},$$

we define the modified measure

$$\mu_0 := \mu|_{Q_0^c} + \sum_{Q\in \mathsf{Stop}_0 \setminus \mathsf{Bad}} \mu|_{I_{\kappa_0}(Q)}.$$

Using (8.3) and (8.4), it is not difficult to prove that μ_0 differs from μ, in the sense of the total mass, possibly by a very small quantity. Indeed,

$$\|\mu - \mu_0\| \leq \Big(2\varepsilon_0 + C\theta_0^{\tilde{\alpha}/(n+\tilde{\alpha})} + \kappa_0^{1/2}\Big)\mu(Q_0). \tag{8.5}$$

For this modification to be valid for our purposes, we need the gradient of the single layer potential associated with this measure to satisfy a localization estimate analogous to (6.1). This is easily proved by gathering the $L^2(\mu|_{Q_0})$-boundedness of $\bar{T}_{\mu|_{Q_0}}$, the estimate (8.5) and the localization estimate (6.1) for μ (see [21, Lemma 6.4]).

Lemma 8.3. *If δ is chosen small enough (depending on M), then*

$$\begin{aligned}&\int_{Q_0} |\bar{T}(\chi_{MQ_0}\mu_0)|^2 d\mu_0 \\ &\lesssim \Big(\varepsilon + \frac{1}{M^{2\tilde{\alpha}}} + M^{4n+2}\delta^{1/(4n+4)} + (M\ell(Q_0))^{2\tilde{\alpha}} + \varepsilon_0 + \theta_0^{\tilde{\alpha}/(n+\tilde{\alpha})} + \kappa_0^{1/2}\Big)\mu(Q_0).\end{aligned}$$

9. Periodization and smoothing of the measure

The present technical section contains the construction of a suitable auxiliary measure. This is done in two steps: first we need to get rid of the truncation at the level of $M\ell(Q_0)$ present in the estimate of Lemma 8.3. For this purpose, we replicate the measure *periodically* by means of horizontal translations. The localization of the gradient of the single layer potential associated with this auxiliary measure will make us able to implement a variational argument in Section 11. We remark that a similar localization problem had to be dealt with for the horizontal component of the Riesz transform in [40]. In that case, it was possible to argue introducing an auxiliary measure defined by reflections with respect to the horizontal hyperplane. However, it is not know how to adapt that construction when having to estimate *all* the components of the transform, so Girela-Sarrión and Tolsa introduced the periodization of the measure. The purpose of the first part of this section is to show how to adapt their construction to our context.

Moreover, a priori, the measure μ_0 may not be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{n+1}$. This would constitute a problem when trying to implement the variational techniques in Lemma 11.1, as the existence of a minimizer of the functional here introduced relies on the boundedness of the density with respect to the measure $\mathcal{L}^{n+1}$. This issue can be overcome by introducing a *smoothing* of the measure. For this reason, we define a second auxiliary measure in (9.6).

The periodization. We denote by

$$\mathcal{M} := \big\{ Q_0 + z_P : z_P \in 6\ell(Q_0)\mathbb{Z}^n \times \{0\} \big\}$$

the family of disjoint cubes covering H and obtained translating Q_0 along the coordinate (horizontal) axes. The factor 6 is chosen in order for this periodization to be coherent with the period of the matrix $\bar{A}$. Given $P \in \mathcal{M}$ we denote by z_P its center and by $T_P \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ the translation

$$T_P(x) := x + z_P,$$

so that the periodization of the measure reads

$$\widetilde{\mu} := \sum_{P \in \mathcal{M}} T_P \sharp \mu_0|_{Q_0}.$$

Observe that $\mu_0(\partial Q_0) = 0$, which implies $\chi_{Q_0}\widetilde{\mu} = \mu_0$. As for the first modification of the measure, we have to prove the equivalent of the localization Lemma 8.3.

Lemma 9.1. *Let κ_0, θ_0 and ε_0 be as in Section 8 and δ as in the Main Lemma. Letting*

$$\widetilde{\delta} := M^{n+1}\Big(\varepsilon_0 + \theta_0^{\widetilde{\alpha}/(n+\widetilde{\alpha})} + \kappa_0^{1/2} + \delta^{1/2}\Big),$$

we have

$$\alpha^H_{\widetilde{\mu}}(3MQ_0) \lesssim \widetilde{\delta}. \tag{9.1}$$

Moreover, for

$$\widetilde{\varepsilon} := \varepsilon + \frac{1}{M^{2\widetilde{\alpha}}} + M^{4n+2}\delta^{1/(4n+4)} + \varepsilon_0 + \theta_0^{\widetilde{\alpha}/(n+\widetilde{\alpha})} + \kappa_0^{1/2} + M^{2n+2}\widetilde{\delta}^{1/(4n+5)} + (M\ell(Q_0))^{2\widetilde{\alpha}}$$

we have

$$\int_{Q_0} \left|\bar{T}(\chi_{MQ_0}\widetilde{\mu})\right|^2 d\widetilde{\mu} \lesssim \widetilde{\varepsilon}\widetilde{\mu}(Q_0). \tag{9.2}$$

Idea of the proof. The geometric inequality (9.1) was proved in [21, Lemma 7.1]. An estimate analogous to (9.2) was proved in [21, Lemma 7.2]. In order to prove it, one first observes that the construction of the measure $\widetilde{\mu}$ yields $\widetilde{\mu}|_{Q_0} = \mu_0|_{Q_0}$, which in turn implies

$$\int_{Q_0} \left|\bar{T}\big(\chi_{MQ_0}\widetilde{\mu}\big)\right|^2 d\widetilde{\mu} = \int_{Q_0} \left|\bar{T}\big(\chi_{MQ_0}\widetilde{\mu}\big)\right|^2 d\mu_0$$
$$\lesssim \int_{Q_0} \left|\bar{T}\big(\chi_{MQ_0}\mu_0\big)\right|^2 d\widetilde{\mu} + \int_{Q_0} \left|\bar{T}\big(\chi_{MQ_0}(\widetilde{\mu} - \mu_0)\big)\right|^2 d\widetilde{\mu} =: I + II$$

The first summand can be readily estimated via Lemma 8.3. In order to bound the term II let us observe that since, by construction, $\widetilde{\mu}_0|_{Q_0} = \widetilde{\mu}|_{Q_0}$ and $\mu_0|_{(Q_0)^c} = \widetilde{\mu}|_{(Q_0)^c}$, we can write

$$\bar{T}\big(\chi_{MQ_0}(\widetilde{\mu} - \mu_0)\big) = \bar{T}\big(\chi_{MQ_0\setminus Q_0}(\widetilde{\mu} - \mu)\big).$$

Thus, the integral II can be estimated by an approximation of the characteristic function $\chi_{MQ_0\setminus Q_0}$ via a C^1 cut-off which is compactly supported in $MQ_0 \setminus Q_0$ and using the flatness of $\widetilde{\mu}$ at the level of $3MQ_0$. We omit further details. □

It is not difficult to see that the measure $\widetilde{\mu}$ has polynomial growth:

$$\widetilde{\mu}(B(x,r)) \lesssim r^n \quad \text{for every } x \in \mathbb{R}^{n+1} \text{ and } r > 0.$$

The following lemma contains a technical estimate for a suitably modified version of the density $P_{\widetilde{\mu},\widetilde{\alpha}}(2B_Q)$.

Lemma 9.2. *For every $Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}$ the inequality*

$$\int_{1.1B_Q\setminus Q} \int_Q \frac{1}{|x-y|^n} d\widetilde{\mu}(x)d\widetilde{\mu}(y) \lesssim \theta_0^{\frac{\widetilde{\alpha}}{(n+\widetilde{\alpha})(1+2n)}}\widetilde{\mu}(Q) \tag{9.3}$$

holds. Moreover, the function

$$p_{\widetilde{\mu},\widetilde{\alpha}}(x) := \sum_{Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}} \chi_Q P_{\widetilde{\mu},\widetilde{\alpha}}(2B_Q)$$

satisfies

$$\int_{Q_0} p^2_{\widetilde{\mu},\widetilde{\alpha}} d\widetilde{\mu} \lesssim \theta_0^{\frac{2\widetilde{\alpha}}{(n+\widetilde{\alpha})(1+2\widetilde{\alpha})}} \widetilde{\mu}(Q_0). \tag{9.4}$$

Remark on the proof. For (9.3) we refer to [21, Lemma 7.3]. In order to prove (9.4) it suffices to follow verbatim the path of (geometric) [21, Lemma 7.4] taking into consideration the right homogeneity given by $\widetilde{\alpha}$, which leads to

$$\int_{Q_0} p^2_{\widetilde{\mu},\widetilde{\alpha}} d\widetilde{\mu} \lesssim \Big(\bar{\kappa} + \frac{\theta_0^{\frac{2\widetilde{\alpha}}{(n+\widetilde{\alpha})}}}{\bar{\kappa}^{2\widetilde{\alpha}}} + \theta_0^{\frac{\widetilde{\alpha}}{n+\widetilde{\alpha}}}\Big)\widetilde{\mu}(Q_0), \tag{9.5}$$

where $0 < \bar{\kappa} < 1$ is a small constant. Inequality (9.5) gives the desired estimate after making the choice $\bar{\kappa} = \theta_0^{2\widetilde{\alpha}/[(n+\widetilde{\alpha})(1+2\widetilde{\alpha})]}$. □

The smoothing. Let us define

$$\eta_0 := \sum_{Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}} \frac{\mu_0(Q)}{\mathcal{H}^{n+1}\big(\frac{1}{4}B(Q)\big)} \mathcal{H}^{n+1}|_{\frac{1}{4}B(Q)} \tag{9.6}$$

and its periodization

$$\eta := \sum_{P \in \mathcal{M}} T_P \sharp \eta_0.$$

We remark that, since Stop_0 is a finite family, the measures η_0 and η both have bounded density with respect to $\mathcal{H}^{n+1}$. As specific control on the density is not relevant to the purposes of our proof. The following lemma contains a localization estimate for the potential associated with η.

Lemma 9.3. *Denoting*

$$\varepsilon' := \widetilde{\varepsilon} + \ell(Q_0)^{2\widetilde{\alpha}} + M^n \kappa_0^{-2n-2\widetilde{\alpha}} \theta_0^{\frac{2\widetilde{\alpha}}{(n+\widetilde{\alpha})(1+2\widetilde{\alpha})}} + \theta_0^{\frac{2\widetilde{\alpha}}{(n+\widetilde{\alpha})(1+2n)}},$$

we have

$$\int_{Q_0} |\bar{T}(\chi_{MQ_0}\eta)|^2 d\eta \lesssim \varepsilon' \eta(Q_0).$$

The presence of the summand $\ell(Q_0)^{2\widetilde{\alpha}}$ in ε' (already taken into account in $\widetilde{\varepsilon}$) to point out that, as in (6.6), the lack of antisymmetry of $\bar{K}(\cdot,\cdot)$ gives the error term

$$\left|m_{\widetilde{\mu},Q}\left(\bar{T}_{\widetilde{\mu}}\chi_Q\right)\right| \lesssim \ell(Q)^{\widetilde{\alpha}} \lesssim \ell(Q_0)^{\widetilde{\alpha}}$$

for every $Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}$. This contribution is not present in the case of an elliptic matrix with constant coefficients. The rest of the proof is analogous to the one of [21, Lemma 8.1] and all is needed is a careful check that Lemma 9.2 applies and the new homogeneity does not affect the final result. We omit further details.

Remark 9.1. Observe that the expressions of $\widetilde{\delta}, \widetilde{\varepsilon}$ and ε' all include a summand which depends on ε_0. In particular, the quantities in question are small if ε_0 and $M\ell(Q_0)$ are chosen small enough. Then the choice of $\varepsilon_0 \ll 1$ (which is possible because we assumed (8.2) to hold) gives the localization for the potentials associated with the auxiliary measures.

10. The localization of $\bar{T}\eta$

Let $L^\infty_{\mathcal{M}}$ denote the set of functions $f \in L^\infty(\eta)$ such that

$$f(x + z_P) = f(x)$$

for every $x \in \mathbb{R}^{n+1}$ and $P \in \mathcal{M}$.

Let $\varphi \in C^1(\mathbb{R}^{n+1})$ be a non-negative radial function whose support is contained in $B(0,2)$ and that equals 1 on $B(0,1)$. For $r > 0$ and $x \in \mathbb{R}^{n+1}$ let us set $\varphi_r(x) := \varphi(x/r)$. Observe that $\|\nabla\varphi\|_\infty \lesssim 1$. For $x, y \in \mathbb{R}^{n+1}$ we define the regularized kernel

$$\widetilde{K}_r(x,y) = \bar{K}(x,y)\varphi_r(x-y)$$

and its associated operator

$$\widetilde{T}_r(f\eta)(x) := \int \widetilde{K}_r(x,y)f(y)d\eta(y) \qquad \text{for } f \in L^\infty_{\mathcal{M}}(\eta),$$

where the integral above is absolutely convergent.

We are interested in getting an existence result for the limit

$$\operatorname{pv}\bar{T}(f\eta)(x) = \lim_{r\to\infty} \widetilde{T}_r(f\eta)(x). \tag{10.1}$$

For simplicity, we denote the principal value in (10.1) just as $\bar{T}(f\eta)(x)$.

Lemma 10.1. *Let $f \in L^\infty_{\mathcal{M}}$. The principal value $\bar{T}(f\eta)(x)$ exists for every $x \in \mathbb{R}^{n+1}$. Moreover, given any compact set $F \subset \mathbb{R}^{n+1}$, there exist $r_0 = r_0(F) > 0$ and a constant c_F depending on F such that for $s > r \geq r_0$*

$$\left\|\widetilde{T}_r(f\eta) - \widetilde{T}_s(f\eta)\right\|_{\infty,F} \lesssim \frac{c_F}{r^\gamma}\|f\|_\infty,$$

where $\gamma \in (0,1)$ is as in Lemma 2.7.

Remark 10.1. Lemma 10.1 implies that the limit in (10.1) converges uniformly on compact sets and in $\operatorname{supp}\eta$.

Proof. Recall that we can assume $\ell(Q_0) < 1$. Let $s > r$. Let us denote $\nu := f\eta$, $\varphi_{r,s}(x) := \varphi_r(x) - \varphi_s(x)$ for every $x \in \mathbb{R}^{n+1}$ and $\widetilde{K}_{r,s}(x,y) := \bar{K}(x,y)\varphi_{r,s}(x-y)$. Because of the periodicity of f and the definition of η, we have

$$\nu = \sum_{P\in\mathcal{M}} (T_P)_\sharp(\chi_{Q_0}\nu)$$

so that

$$\begin{aligned}\widetilde{T}_r(f\eta)(x) - \widetilde{T}_s(f\eta)(x) &= \int \widetilde{K}_{r,s}(x,y)d\Big(\sum_{P\in\mathcal{M}} (T_P)_\sharp(\chi_{Q_0}\nu)\Big)(y) \\ &= \sum_{P\in\mathcal{M}}\int_{Q_0} \widetilde{K}_{r,s}(x,y+z_p)\,d\nu(y),\end{aligned} \tag{10.2}$$

the last equality being a consequence of $\widetilde{K}_{r,s}$ having compact support, which implies that the sum has only finitely many non-zero terms.

Let A_0 be the homogenized matrix associated with $\{L_\varepsilon\}_\varepsilon > 0$ and χ_ℓ be as in Section 2, with $\ell = 6\ell(Q_0)$. Recall that

$$\|\nabla\chi_\ell\|_\infty \lesssim 1$$

(see (2.9)). The matrix A_0 is an elliptic matrix whose coefficients are constants and can be expressed in terms of χ and those of A. We denote by $\Theta(\cdot,\cdot;A_0)$ the fundamental solution of the operator $L_0 = -\operatorname{div}(A_0\nabla)$. We decompose the right hand side of (10.2) as

$$\begin{aligned}&\sum_{P\in\mathcal{M}}\int_{Q_0} \widetilde{K}_{r,s}(x,y+z_p)d\nu(y) \\ &= \sum_{P\in\mathcal{M}}\int_{Q_0} \big(\bar{K}(x,y+z_P) - (Id+\nabla\chi_\ell(x))\nabla_1\Theta(x,y+z_P;A_0)\big)\varphi_{r,s}(x-y-z_P)d\nu(y) \\ &\qquad + \sum_{P\in\mathcal{M}}\int_{Q_0} \big(Id+\nabla\chi_\ell(x)\big)\nabla_1\Theta(x,y+z_P;A_0)\varphi_{r,s}(x-y-z_P)d\nu(y) \\ &\equiv I_{r,s}(x) + II_{r,s}(x).\end{aligned}$$

Let us observe that since F is compact and $y \in Q_0$, there exists a compact set $\widetilde{F}$ such that $\pm(x-y) \in \widetilde{F}$, so that if we choose $r_0 \geq 2\operatorname{diam}\big(\widetilde{F}\big)$, both $\varphi_{r,s}(x-y-z_P)$ and $\varphi_{r,s}(x-y+z_P)$ vanish for $|z_P|<r$. Moreover, $|x-y| \leq \operatorname{diam}\big(\widetilde{F}\big) \leq r/2 \leq |z_P|$ and

$$|(x-y)-z_P| \approx |(x-y)+z_P| \approx |z_P|.$$

Let us now estimate $I_{r,s}(x)$. As stated in Lemma 2.7, there exist $C>0$ and $\gamma \in (0,1)$ depending only on n and α such that

$$\left|\nabla_1 \mathcal{E}_{\bar{A}}(x,y+z_P) - \big(Id + \nabla\chi_\ell(x)\big)\nabla_1\Theta(x,y+z_P;A_0)\right| \leq C\ell(Q_0)^\gamma |x-y-z_P|^{-(n+\gamma)}$$

for every $x, y \in \mathbb{R}^{n+1}$. Then, exploiting the linear growth of η and the considerations on the support of $\varphi_{r,s}$, we get

$$\begin{aligned} |I_{r,s}(x)| &\lesssim \sum_{P\in\mathcal{M}, |z_P|\geq r} \int_{Q_0} \frac{\ell(Q_0)^\gamma d|\nu|(y)}{|x-y-z_P|^{n+\gamma}} \lesssim \|f\|_\infty \sum_{P\in\mathcal{M}, |z_P|\geq r} \frac{\ell(P)^{n+\gamma}}{|z_P|^{n+\gamma}} \\ &\lesssim \frac{\|f\|_\infty \ell(Q_0)^\gamma}{r^\gamma}. \end{aligned} \tag{10.3}$$

In the last inequality of (10.3) we used the convergence of $\sum_{P\in\mathcal{M}} \ell(P)^n |z_P|^{-n}$.

We are left with the estimate of $II_{r,s}(x)$. First, we observe that there exists a compact set $\widetilde{F}$ such that both $(x-y)$ and $-(x-y)$ belong to $\widetilde{F}$ for all $x \in F$ and $y \in Q_0$. Furthermore, $P \in \mathcal{M}$ if and only if $-P \in \mathcal{M}$, by [21, Lemma 8.2, p. 41] we have

$$\begin{aligned} &\left|\nabla\Theta(x-y-z_P;A_0)\varphi_{r,s}(x-y-z_P) - \nabla\Theta(x-y+z_P;A_0)\varphi_{r,s}(x-y+z_P)\right| \\ &\lesssim \frac{|x-y|}{|z_P|^{n+1}} \leq \frac{\operatorname{diam}\widetilde{(F)}}{|z_P|^{n+1}}, \end{aligned} \tag{10.4}$$

for all $P \in \mathcal{M}$ with $|z_P| \geq r$. Moreover, the quantity in the left-hand side of the display above vanishes for $|z_P|<r$.

Hence, using the antisymmetry of $\nabla\Theta(\cdot;A_0)$ and the facts highlighted above, there exists a constant $c_F>0$ such that

$$\begin{aligned} |II_{r,s}(x)| &\leq \|Id + \nabla\chi_\ell\|_\infty \left|\sum_{P\in\mathcal{M}} \int_{Q_0} \nabla_1\Theta(x,y+z_P;A_0)\varphi_{r,s}(x-y-z_P)d\nu(y)\right| \\ &\underset{(2.9)}{\lesssim} \left|\sum_{P\in\mathcal{M}} \int_{Q_0} \nabla_1\Theta(x,y+z_P;A_0)\varphi_{r,s}(x-y-z_P)d\nu(y)\right| \\ &\leq \frac{1}{2}\sum_{P\in\mathcal{M}} \int_{Q_0} \big|\nabla\Theta(x-y-z_P;A_0)\varphi_{r,s}(x-y-z_P) \\ &\qquad - \nabla\Theta(x-y+z_P;A_0)\varphi_{r,s}(x-y+z_P)\big|\, d|\nu|(y) \end{aligned}$$

$$\underset{(10.4)}{\lesssim} \sum_{P\in\mathcal{M}:|z_P|\geq r} \int_{Q_0} \frac{|x-y|}{|z_P|^{n+1}}\, d|\nu|(y)$$

$$\lesssim \operatorname{diam}(\widetilde{F}) \sum_{P\in\mathcal{M}:|z_P|\geq r} \frac{|\nu|(Q_0)}{|z_P|^{n+1}} \leq \frac{c_F\|f\|_\infty}{r}. \tag{10.5}$$

We conclude the proof of the lemma gathering (10.3), (10.5) and observing that, being $\gamma\in(0,1)$ and $r>1$, $r^{-1}<r^{-\gamma}$. $\square$

The measure η is $\mathcal{M}$-periodic and the matrix $\bar{A}$, by construction, is $6\ell(Q_0)$-periodic. This implies that for every $f\in L^\infty_{\mathcal{M}}(\eta)$ and $r>0$, the function $\widetilde{T}_r(f\eta)$ is $\mathcal{M}$-periodic, too. The same holds for $\operatorname{pv} T(f\eta)$. Using Lemma 10.1, the following result is immediate.

Corollary 10.1. *$\bar{T}_\eta$ is a bounded operator from $L^\infty_{\mathcal{M}}$ to $L^\infty_{\mathcal{M}}$. For $r>0$ big enough and for every $f\in L^\infty_{\mathcal{M}}(\eta)$ we have*

$$\left\|\bar{T}(f\eta)-\widetilde{T}_r(f\eta)\right\|_{\infty,F} \lesssim_F \frac{\|f\|_\infty}{r^\gamma}.$$

Our next intent is to prove the final localization estimate

$$\int_{Q_0} \left|\bar{T}\eta\right|^2 d\eta \ll \eta(Q_0). \tag{10.6}$$

We have already proved that for M big enough there exists $\varepsilon'\ll 1$ such that

$$\int_{Q_0} \left|\bar{T}(\chi_{MQ_0}\eta)\right|^2 d\eta \lesssim \varepsilon'\eta(Q_0). \tag{10.7}$$

Then, in order to prove (10.6), it suffices to use the estimate in the following lemma.

Lemma 10.2. *Let $f\in L^1_{loc}(\eta)$ be a $\mathcal{M}$-periodic function and let $\widetilde{M}=6\widetilde{N}$, where $\widetilde{N}\geq 3$ is an odd number. For all $x\in 2Q_0$ we have*

$$\left|\bar{T}\big(\chi_{(\widetilde{M}Q_0)^c} f\eta\big)(x)\right| \lesssim \frac{1}{\widetilde{M}^\gamma \ell(Q_0)^n} \int_{Q_0} |f|\, d\eta. \tag{10.8}$$

Proof. Being $\widetilde{N}$ odd, there exists a subfamily $\widetilde{\mathcal{M}}\subset\mathcal{M}$ such that

$$\chi_{(\widetilde{M}Q_0)^c}\eta = \sum_{P\in\widetilde{\mathcal{M}}} T_P\sharp\eta$$

and whose elements $P\in\widetilde{\mathcal{M}}$ satisfy $|z_P|\gtrsim \widetilde{M}\ell(Q_0)$. In particular

$$|x-y-z_P| \approx |z_p| \quad \text{for } x,y \in 2Q_0. \tag{10.9}$$

Let $r>0$ and $x \in 2Q_0$. Denote $\nu := f\eta$ and observe that there are just finitely many cubes $P \in \widetilde{\mathcal{M}}$ such that $|z_P|<r$. Arguing as in the proof of Lemma 10.1, this justifies the following writings.

$$\begin{aligned}
&\widetilde{T}_r\big(\chi_{(\widetilde{M}Q_0)^c} f\eta\big)(x) = \int \bar{K}(x,y)\varphi_r(x-y)d\nu(y)\\
&= \sum_{P\in \widetilde{M}Q_0} \int \bar{K}(x,y+z_P)\varphi_r(x-y-z_P)d\nu(y)\\
&= \sum_{P\in \widetilde{\mathcal{M}}Q_0} \int \big(\bar{K}(x,y+z_P)-(Id+\nabla\chi_\ell(x))\nabla_1\Theta(x,y+z_P;A_0)\big)\varphi_r(x-y-z_P)d\nu(y)\\
&\quad + \sum_{P\in \widetilde{\mathcal{M}}Q_0} \int \big(Id+\nabla\chi_\ell(x)\big)\nabla_1\Theta(x,y+z_P;A_0)\varphi_r(x-y-z_P)d\nu(y)\\
&\equiv I_r(x)+II_r(x)
\end{aligned}$$

Let us estimate $I_r(x)$. Using (10.9) together with Lemma 2.7 and the estimate $|z_P| \gtrsim \widetilde{M}\ell(Q_0)$ for $P \in \widetilde{\mathcal{M}}$, we can write

$$\begin{aligned}
|I_r(x)| &\lesssim \sum_{P\in \widetilde{\mathcal{M}}Q_0} \int \frac{\ell(Q_0)^\gamma}{|x-y-z_P|^{n+\gamma}}d\nu(y) \approx \sum_{P\in \widetilde{\mathcal{M}}Q_0} \int \frac{\ell(Q_0)^\gamma}{|z_P|^{n+\gamma}}d\nu(y)\\
&= \sum_{P\in\widetilde{\mathcal{M}}} \frac{\ell(Q_0)^\gamma}{|z_P|^{n+\gamma}}|\nu|(Q_0) \lesssim \frac{|\nu|(Q_0)}{\widetilde{M}^\gamma \ell(Q_0)^n}\Bigg(\sum_{P\in\widetilde{\mathcal{M}}}\frac{\ell(Q_0)^n}{|z_P|^n}\Bigg)\\
&\lesssim \frac{1}{\widetilde{M}^\gamma\ell(Q_0)^n}\int_{Q_0}|f|d\eta.
\end{aligned} \tag{10.10}$$

We claim that

$$|II_r(x)| \lesssim \frac{1}{\widetilde{M}\ell(Q_0)^n}\int_{Q_0}|f|d\eta. \tag{10.11}$$

In order to prove the claim, we observe that $\check{K}_r(\cdot) := \nabla\Theta(\cdot;A_0)\varphi_r(\cdot)$ is an antisymmetric kernel and that $P \in \widetilde{\mathcal{M}}$ if and only if $-P \in \widetilde{\mathcal{M}}$. Thus, we can write

$$II_r = \frac{1}{2}\sum_{P\in\widetilde{\mathcal{M}}}(Id+\nabla\chi_\ell(x))\int_{Q_0}\big[\check{K}_r(z_P+(x-y))-\check{K}_r(z_P-(x-y))\big]d\nu(y).$$

Let us observe that for $x \in 2Q_0$ and $P \in \widetilde{\mathcal{M}}$ we have $|z_P+(x-y)| \approx |z_P-(x-y)| \approx |z_P|$. Hence, by (2.9) and the Calderón-Zygmund properties of the kernel $\check{K}_r$ the following inequalities hold:

$$|II_r| \lesssim \sum_{P\in\widetilde{\mathcal{M}}Q_0} \int \frac{|x-y|}{|z_P|^{n+1}} d|\nu|(y) \lesssim \sum_{P\in\widetilde{\mathcal{M}}} \frac{\ell(Q_0)}{|z_P|^n} |\nu|(Q_0) \lesssim \frac{|\nu|(Q_0)}{\widetilde{M}\ell(Q_0)}.$$

This proves the claim (10.11).

The estimates (10.10) and (10.11), together with the observation that $\widetilde{M}^{-1} \leq \widetilde{M}^{-\gamma}$, conclude the proof of the lemma after taking the limit for $r \to \infty$. □

Corollary 10.2 *(Final localization estimate). We have*

$$\int_{Q_0} |\bar{T}\eta|^2 d\eta \lesssim \Big(\frac{1}{M^{2\gamma}} + \varepsilon'\Big)\eta(Q_0).$$

Proof. Inequality (10.8) in the case $f \equiv 1$ reads

$$|\bar{T}(\chi_{MQ_0}\eta)| \lesssim \frac{1}{M^\gamma},$$

so that applying it together with (10.7), we have

$$\int_{Q_0} |\bar{T}\eta|^2 d\eta \lesssim \int_{Q_0} |\bar{T}(\chi_{MQ_0}\eta)|^2 d\eta + \int_{Q_0} |\bar{T}(\chi_{(MQ_0)^c}\eta)|^2 d\eta \lesssim \Big(\frac{1}{M^{2\gamma}} + \varepsilon'\Big)\eta(Q_0),$$

which finishes the proof. □

11. A pointwise inequality and the conclusion of the proof

The following lemma implements a variational technique inspired by potential theory that allows to obtain a pointwise inequality for the potential of a proper auxiliary measure. We denote as $\bar{T}^*\vec{\xi}$ the operator that, given a vector-valued measure $\vec{\xi}$, is defined by

$$\bar{T}^*\vec{\xi}(x) := \int \nabla_1 \bar{\mathcal{E}}(y,x) \cdot d\vec{\xi}(y) \tag{11.1}$$

and which corresponds to the adjoint of $\bar{T}$.

Lemma 11.1. *Suppose that for some $0 < \lambda \leq 1$ the inequality*

$$\int_{Q_0} |\bar{T}\eta|^2 d\eta \leq \lambda\eta(Q_0) \tag{11.2}$$

holds. Then there is a function $b \in L^\infty(\eta)$ such that

- $0 \leq b \leq 2$,

- *b is $\mathcal{M}$-periodic,*
- $\int_{Q_0} b\,d\eta = \eta(Q_0)$,

and such that the measure $\nu = b\eta$ satisfies

$$\int_{Q_0} |\bar{T}\nu|^2 d\nu \le \lambda \nu(Q_0) \tag{11.3}$$

and

$$|\bar{T}\nu|^2(x) + 2\bar{T}^*\big((\bar{T}\nu)\nu\big)(x) \le 6\lambda \text{ for } \nu\text{-a.e. } x \in \mathbb{R}^{n+1}. \tag{11.4}$$

Proof. The proof works almost verbatim as that of [21, Lemma 9.1]. We briefly outline it for the reader's convenience. In particular, we recall that the way to prove (11.4) consists in defining an adapted energy functional

$$J(a) = \lambda \|a\|_{L^\infty(\eta)} \eta(Q_0) + \int_{Q_0} |\bar{T}(a\eta)|^2 d\eta,$$

where a ranges in

$$\mathcal{A} = \Big\{ a \in L^\infty(\eta) : a \ge 0, a \text{ is } \mathcal{M}\text{-periodic, and } \int_{Q_0} a\,d\eta = \eta(Q_0) \Big\}.$$

Then, observe that $1 \in \mathcal{A}$ and

$$\inf_{a\in\mathcal{A}} J(a) \le J(1) \overset{(11.2)}{\le} 2\lambda\eta(Q_0).$$

Thus, since $J(a) \ge \lambda \|a\|_{L^2\infty(\eta)} \eta(Q_0)$, we have that

$$\inf_{a\in\mathcal{A}} J(a) = \inf_{a\in\mathcal{A}, \|a\|_{L^\infty(\eta)}\le 2} J(a).$$

Hence Banach-Alaoglu theorem implies that, possibly passing to a subsequence, a_j converges weakly * in $L^\infty(\eta)$ to some $b \in L^\infty(\eta)$. It is not hard to see that $b \in \mathcal{A}$. Then (using the fact that η has bounded density with respect to the Lebesgue measure) one proves that b minimizes J in $\mathcal{A}$ and, finally, tests J on the competitors

$$b_t := \big(1 - t\chi_{P_{\mathcal{M}}(B)}\big) b + t\frac{\nu(B)}{\nu(Q_0)} b, \qquad 0 \le t < 1,$$

where B is a ball centered at $\operatorname{supp}\nu \cap Q_0$ which is contained in Q_0, and $P_{\mathcal{M}}(B) := \bigcup_{R\in\mathcal{M}} (B + z_R)$. In this way, we obtain

$$\int_B |\bar{T}\nu|^2 d\nu + 2\int_B \bar{T}^*\big((T\nu)\nu\big) d\nu \le 6\lambda\nu(B),$$

so (11.4) follows by Lebesgue's differentiation theorem as $\nu(B) \to 0$. We further remark that Girela-Sarrión and Tolsa's calculations *do not use the antisymmetry of the kernel* of $\bar{T}$ but just its $\mathcal{M}$-periodicity, which follows by the construction of $\bar{A}$. $\square$

11.1. A maximum principle

Let λ, b and ν be as in Lemma 11.1. In order to be able to perform the final argument to get the contradiction, we need to extend the inequality (11.4) out of the support of ν. More precisely, the next step consists in proving that a inequality similar to that provided by Lemma 11.1 holds in a suitable strip. To this purpose, some version of maximum principle is needed. The elliptic setting of the problem makes this procedure slightly more technical then the one adopted by Girela-Sarrión and Tolsa in the case of the Riesz transform.

Before presenting the main result of the section, we introduce some notation. We denote by $\widetilde{H}$ the hyperplane

$$\widetilde{H} := \{x \in \mathbb{R}^{\mathrm{n}+1} : x_{n+1} = 3\ell(Q_0)/2\},$$

which corresponds to the translate of H that contains the upper face of $3Q_0$. Let $K_S \gg 1$ to be chosen later and let S denote the strip

$$S := \{x \in \mathbb{R}^{\mathrm{n}+1} : \operatorname{dist}(x, \widetilde{H}) < K_S\ell(Q_0)\}.$$

Its boundary ∂S is given by the union of two hyperplanes ∂S_+ and ∂S_- which lay in the upper and lower half spaces respectively. Let

$$x_{S\pm} = \frac{3}{2}\ell(Q_0)(1,\dots,1,1) \pm \big(0,\dots,0,K_S\ell(Q_0)\big). \tag{11.5}$$

For the proof of our next lemma we need to invoke a result on elliptic measure. Suppose that $\Omega \subsetneq \mathbb{R}^{\mathrm{n}+1}$ is an open set with n-AD-regular boundary and consider a point $p \in \Omega$. Let ω^p_Ω denote the elliptic measure on $\partial\Omega$ associated with the operator $L_{\bar{A}}$ with pole at p. For the proof of the following standard result we refer to [7, Lemma 2.3].

Lemma 11.2. *Let $\Omega \subsetneq \mathbb{R}^{\mathrm{n}+1}$ be open with n-AD-regular boundary with constant C_{AD}. There exists $\vartheta = \vartheta(n, A, C_{AD}) \in (0,1)$ such that for every $x \in \partial\Omega$ and $0 < r < \operatorname{diam}\Omega$, we have*

$$\omega^y_\Omega\big(B(x,r)^c\big) \le C\Big(\frac{|x-y|}{r}\Big)^\vartheta \qquad \textit{for } y \in \Omega \cap B(x,r). \tag{11.6}$$

An application of (11.6) gives a boundary regularity result for $L_{\bar{A}}$-harmonic functions, see e.g Lemma 2.10 in [5].

Lemma 11.3. *Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be open with n-AD-regular boundary with constant C_{AD}. Let $u \geq 0$ be $L_{\bar{A}}$-harmonic function in $B(x,4r) \cap \Omega$ and continuous in $B(x,4r) \cap \bar{\Omega}$. Suppose, moreover, that $u \equiv 0$ in $\partial\Omega \cap B(x,4r)$. Then, extending u by zero in $B(x,4r) \setminus \bar{\Omega}$, there exists $\vartheta = \vartheta(n, A, C_{AD}) \in (0,1)$ such that u is ϑ-Hölder continuous in $B(x,4r)$ and, in particular,*

$$u(y) \lesssim_{n,A,C_{AD}} \left(\frac{\operatorname{dist}(y,\partial\Omega)}{r}\right)^{\vartheta} \sup_{B(x,2r)} u \qquad \text{for all } y \in B(x,r).$$

Lemma 11.4 *(Maximum principle on the strip). Let S be the strip as before and let f be a bounded continuous $L_{\bar{A}}$-harmonic function on S so that $f|_{\partial S} \equiv 0$. Then $f \equiv 0$ on S.*

Proof. Choose $R > 100K_S$ and set $S_R := S \cap [-R,R]^{n+1}$. For $p \in S$, denote $h_p := \operatorname{dist}(p, \partial S)$ and let x_p be a point that realizes the distance. We choose p far from the "vertical" parts $\partial S_R \setminus (\partial S_+ \cup \partial S_-)$ of ∂S_R, in particular such that $B(x_p, R/10) \cap (\partial S_R \setminus \partial S) = \emptyset$. Let ω_R^p denote the elliptic measure with pole at p associated with $L_{\bar{A}}$ on S_R. The family $\{S_R\}_R$ is a collection of AD-regular sets whose AD-regularity constants do not depend on R. Then inequality (11.6) implies that there exit two constants C and ϑ, both independent on R, such that

$$\omega_R^p(\partial S_R \setminus \partial S) \leq \omega_R^p\big(B(x_p, R/10)^c\big) \leq C\Big(\frac{h_p}{R}\Big)^{\vartheta}.$$

By hypothesis we may assume that $f \leq 1$ on $\partial S_R \setminus \partial S$. Thus, we have

$$|f(p)| = \Big| \int f d\omega_R^p \Big| \leq \|f|_{\partial S_R \setminus \partial S}\|_\infty \omega_R^p(\partial S_R \setminus \partial S) \leq C\Big(\frac{h_P}{R}\Big)^{\vartheta}. \tag{11.7}$$

The results stated in the lemma follows by passing to the limit in (11.7) for $R \to \infty$. □

Now, we prove an existence result on the infinite strip S.

Lemma 11.5. *There exists a function $f_S \colon \bar{S} \to \mathbb{R}$ such that:*

(1) f_S is $L_{\bar{A}}$-harmonic in the strip S and continuous in $\bar{S}$.
(2) f_S is $\mathcal{M}$-periodic.
(3) $f_S(x) = \pm 1$ on $\partial S_\pm$ and $f_S(x) = 0$ for $x \in \widetilde{H}$.

Proof. Let $k \in \mathbb{N}$, $k \geq 100K_S$ and denote $S_k = S \cap [-k,k]^{n+1}$. We define the continuous functions f_k on ∂S_k as

$$f_k(x) = \frac{x_{n+1} - \frac{3}{2}\ell(Q_0)}{K_S \ell(Q_0)}.$$

In particular, observe that $f_k(x) = \pm 1$ for $x \in \partial S_\pm$ and

$$f_k(x) = -f_k\Big(\big(x_1, \dots, x_n, -x_{n+1} + 3\ell(Q_0)\big)\Big),$$

i.e. it is antisymmetric with respect to $\widetilde{H}$.

Define u_k be the $L_{\bar{A}}$-harmonic function such that $u_k|_{\partial S_k} = f_k$, whose existence is guaranteed by the continuity of f_k and the AD-regularity of S_k. Our aim is to prove that, a part from possibly considering a proper subsequence, u_j converges uniformly in the compact subsets of S_k, for every k to an $L_{\bar{A}}$-harmonic function in S.

We claim that there exist $\gamma \in (0, 1)$ and $C_k > 0$ such that

$$|u_j(x) - u_j(y)| \le C_k |x - y|^\gamma \qquad \text{for} \qquad x, y \in \bar{S}_k, \quad j \ge k + 2. \tag{11.8}$$

Assume that (11.8) holds. As a consequence of Ascoli-Arzelà's theorem together with a standard diagonalization argument, there is a function f_S so that u_k converges to f_S uniformly on the compact subsets of S. The $L_{\bar{A}}$-harmonicity of f_S is a consequence of Caccioppoli's estimate (cfr. [22, Theorem 3.77]).

To prove (2), define $\vec{v} = (6\ell(Q_0), 0, \dots, 0)$ and observe that, being the matrix $\bar{A}$ $\mathcal{M}$-periodic and since f_S is constant on $\partial S_\pm$, the function $f(x) = f_S(x) - f_S(x + \vec{v})$ satisfies the hypothesis of Lemma 11.4. So, $f \equiv 0$ and f_S is $\mathcal{M}$-periodic.

To prove (3), first observe that $A(x) = A_\phi\Big(\big(x_1, \dots, x_n, -x_{n+1} + 3\ell(Q_0)\big)\Big)$, where ϕ is the function that maps a point to its reflected with respect to $\widetilde{H}$ and A_ϕ is defined as in (5.5). Then we can apply again Lemma 11.4 to

$$\widetilde{f}(x) = f_S(x) + f_S\Big(\big(x_1, \dots, x_n, -x_{n+1} + 3\ell(Q_0)\big)\Big),$$

which is $L_{\bar{A}}$-harmonic and vanishes on ∂S.

We are left with the proof of the claim (11.8). By Lemma 11.3, there exists $\vartheta \in (0, 1)$ depending only on n, $\bar{A}$ and the AD-regularity of $\partial\Omega$ (hence independent both on j and k) such that u_j is ϑ-Hölder continuous in the set $\{x \in \bar{\Omega} : \operatorname{dist}(x, \partial\Omega) \le 2\ell(Q_0)\}$. Being $\|u_j\|_\infty \le 2$ for every j, by De Giorgi-Nash interior estimates we can infer that there exists γ_k independent on j such that, for every $j \ge k + 2$, u_j is γ_k-Hölder continuous in $\{x \in \Omega_{k+1} : \operatorname{dist}(x, \partial\Omega) > \ell(Q_0)\}$. Gathering the interior and the boundary regularity of u_j proves (11.8). $\square$

By the previous lemma, Lemma 11.3 and the fact that $f_S \equiv 0$ on $\widetilde{H}$, we have the estimate

$$|f_S(y)| \lesssim \left(\frac{\operatorname{dist}(y, \widetilde{H})}{K_S \ell(Q_0)}\right)^\vartheta, \qquad \text{for} \quad y \in S \text{ with } \operatorname{dist}(y, \widetilde{H}) \le 10\ell(Q_0).$$

Let us define the auxiliary function

$$F_S(x) := f_S(x)\bar{T}\nu(x_{S+}).$$

Observe that $F_S|_{\partial S_\pm} \equiv \pm\bar{T}\nu(x_{S+})$. The rest of the present section is devoted to the proof of the following, which may be regarded as an approximated maximum principle on S.

Lemma 11.6 *(Pointwise bound for the potential on the strip). For $x \in S$ we have*

$$|\bar{T}\nu(x) - F_S(x)|^2 + 4\bar{T}^*\big((\bar{T}\nu)\nu\big)(x) \lesssim \lambda^{1/2} + \frac{1}{K_S^{2\widetilde{\alpha}}} + \frac{1}{K_S^{\vartheta}} + (C_S\ell(Q_0))^{\widetilde{\alpha}} + \Big(\frac{K_S}{C_S}\Big)^{\widetilde{\alpha}},$$

where C_S is a constant chosen so that $C_S \gg K_S$.

Before proving this lemma, we need some auxiliary result.

Lemma 11.7. *Let x_{S+} and x_{S-} as in (11.5). Then:*

(1) For $x \in \partial S_+$, $\operatorname{dist}(x, x_{S+}) \lesssim \ell(Q_0)$ we have the estimate

$$|\bar{T}\nu(x) - \bar{T}\nu(x_{S+})| \lesssim \frac{1}{K_S^{\widetilde{\alpha}}}. \tag{11.9}$$

The analogous estimate holds in $x \in \partial S_-$ replacing x_{S+} with x_{S-}.

(2) The difference of $-\bar{T}\nu(x_{S+})$ and $\bar{T}\nu(x_{S-})$ can be estimated as

$$|\bar{T}\nu(x_+) + \bar{T}\nu(x_{S-})| \lesssim \frac{1}{K_S^{\widetilde{\alpha}}}.$$

(3) For x with $\operatorname{dist}(x, \widetilde{H}) \geq 2\ell(Q_0)$ we have

$$\bar{T}^*\big((\bar{T}\nu)\nu\big)(x) \lesssim \lambda^{1/2}. \tag{11.10}$$

Proof. Let us begin with the proof of (1). Because of the $\mathcal{M}$-periodicity of $\bar{T}\nu$, we can assume without loss of generality that $x_H \in [-3\ell(Q_0), 3\ell(Q_0)]^n \times \{0\}$, x_H denoting the projection of x on H. We claim that for $P \in \mathcal{M}$ and $y \in Q_0$ we have

$$|\bar{K}(x, y + z_P) - \bar{K}(x_{S+}, y + z_P)| \lesssim \frac{\ell(Q_0)^{\widetilde{\alpha}}}{(K_S\ell(Q_0))^{n+\widetilde{\alpha}} + |z_P|^{n+\widetilde{\alpha}}}.$$

This follows from the (global) Calderón-Zygmund estimates for $\bar{K}(\cdot,\cdot)$ once we observe that $|x - x_{S+}| \lesssim |x - y - z_P| \approx K_S\ell(Q_0) + |z_P|$. So, for $r > 0$, standard calculations give

$$\begin{aligned}\big|\widetilde{T}_r\nu(x)-\widetilde{T}_r\nu(x_{S+})\big| &\lesssim \sum_{P\in\mathcal{M}}\int_{Q_0}\frac{\ell(Q_0)^{\widetilde{\alpha}}}{(K_S\ell(Q_0))^{n+\widetilde{\alpha}}+|z_P|^{n+\widetilde{\alpha}}}d\nu(y)\\ &=\sum_{P\in\mathcal{M}}\frac{\ell(Q_0)^{n+\widetilde{\alpha}}}{(K_S\ell(Q_0))^{n+\widetilde{\alpha}}+|z_P|^{n+\widetilde{\alpha}}}\lesssim\frac{\ell(Q_0)^{n+\widetilde{\alpha}}}{(K_S\ell(Q_0))^{n+\widetilde{\alpha}}}=\frac{1}{K_S^{\widetilde{\alpha}}}.\end{aligned}$$

Being this estimate independent on the choice of r, in the limit for $r\to\infty$ we have (11.9). The proof of the analogous estimate for x_{S-} is identical, so we omit it and we go to the proof of (2).

Denote by x^* the reflection of the point x across $x_0=\frac{3}{2}\ell(Q_0)(1,\dots,1)$, i.e.

$$x^*=2x_0-x.$$

By the specific choice of x_0, this transformation can be obtained via a composition of the reflections ψ_j's with respect to the hyperplanes passing through x_0 which we defined in (5.4):

$$x^*=\psi_1\circ\cdots\circ\psi_{n+1}(x).$$

Moreover,

$$(x_{S+})^*=3\ell(Q_0)(1,\dots,1)-\frac{3}{2}\ell(Q_0)(1,\dots,1)-\big(0,\dots,0,K_S\ell(Q_0)\big)=x_{S_-}. \tag{11.11}$$

Thus, an immediate application of Lemma 5.1 and (5.10) gives that, for $y\in Q_0$,

$$\bar{K}(x_{S_+},y+z_P)=-\bar{K}(x_{S_-},y^*+z_P^*),\qquad\qquad P\in\mathcal{M}. \tag{11.12}$$

Observe that

$$|y+z_P-(y^*-z_P^*)|\le|y-y^*|+|z_P-(-z_P^*)|\lesssim\ell(Q_0)$$

which, combined with Lemma 2.4, gives

$$\begin{aligned}&\big|\bar{K}(x_{S_+},y+z_P)+\bar{K}(x_{S_-},y-z_P)\big|\\ &\overset{(11.12)}{=}|\bar{K}(x_{S_+},y+z_P)-\bar{K}(x^*_{S_-},y^*-z_P^*)|\\ &=|\bar{K}(x_{S_+},y+z_P)-\bar{K}(x_{S_+},y^*-z_P^*)|\lesssim\frac{\ell(Q_0)^{\widetilde{\alpha}}}{(K_S\ell(Q_0))^{n+\widetilde{\alpha}}+|z_P|^{n+\widetilde{\alpha}}},\end{aligned} \tag{11.13}$$

where the second equality uses (11.11) (with $-z_P=z_{-P}$ replacing z_P). Taking $r>0$ and applying (11.13), we have

$$\big|\widetilde{T}_r\nu(x_{S^+})+\widetilde{T}_r\nu(x_{S^-})\big|\lesssim\sum_{P\in\mathcal{M}}\frac{\ell(Q_0)^{n+\widetilde{\alpha}}}{(K_S\ell(Q_0))^{n+\widetilde{\alpha}}+|z_P|^{n+\widetilde{\alpha}}}\lesssim\frac{1}{K_S^{\widetilde{\alpha}}}$$

which, taking the limit for $r \to \infty$, proves (2).

We are left with the proof of (3). Set $\sigma = (\bar{T}\nu)\nu$ and observe that this measure is $\mathcal{M}$-periodic. So, without loss of generality, we can assume that $x_H \in [-3\ell(Q_0), 3\ell(Q_0)]^n \times \{0\}$. Let $r > 0$ and, denoting by A_0 the homogenized matrix associated with $\bar{A}$, by χ the vector of correctors and $\ell = 6\ell(Q_0)$, write

$$\begin{aligned}\widetilde{T}_r\sigma(x) &= \sum_{P\in\mathcal{M}}\int_{Q_0} \widetilde{K}_r(y+z_P,x)d\sigma(y)\\ &= \sum_{P\in\mathcal{M}}\int_{Q_0} (\bar{K}(y+z_P,x) - (Id+\nabla\chi_\ell(y+z_P))\nabla_1\Theta(y+z_P,x;A_0))\varphi_r(x-y-z_P)d\sigma(y)\\ &\quad+ \sum_{P\in\mathcal{M}}\int_{Q_0} (Id+\nabla\chi_\ell(y+z_P))\nabla_1\Theta(y+z_P,x;A_0)\varphi_r(x-y-z_P)d\sigma(y)\\ &\equiv I_r + II_r.\end{aligned}$$

Recalling that $\|\nabla\chi_\ell\|_\infty \lesssim 1$ and using Lemma 2.7, we can proceed with the following estimates

$$\begin{aligned}|I_r| &\lesssim \sum_{P\in\mathcal{M}}\int_{Q_0} \frac{\ell(Q_0)^\gamma}{|x-y-z_P|^{n+\gamma}}d|\sigma|(y) \lesssim \sum_{P\in\mathcal{M}}\int_{Q_0} \frac{\ell(Q_0)^\gamma}{(\operatorname{dist}(x,\widetilde{H})+|z_P|)^{n+\gamma}}d|\sigma|(y)\\ &\lesssim \frac{\ell(Q_0)^{n+\gamma}}{(\operatorname{dist}(x,\widetilde{H})+|z_P|)^{n+\gamma}}\frac{|\sigma|(Q_0)}{\ell(Q_0)^n} \lesssim \frac{\ell(Q_0)^{n+\gamma}}{\operatorname{dist}(x,\widetilde{H})^\gamma}\frac{|\sigma|(Q_0)}{\ell(Q_0)^n} \lesssim \frac{|\sigma|(Q_0)}{\ell(Q_0)^n},\end{aligned} \tag{11.14}$$

where the last inequality holds because we assumed $\operatorname{dist}(x,\widetilde{H}) \geq 2\ell(Q_0)$. We claim that

$$|II_r| \lesssim \frac{|\sigma|(Q_0)}{\ell(Q_0)}. \tag{11.15}$$

In order to prove this, let us first observe that the antisymmetry of $\nabla\Theta(\cdot;A_0)$ and the periodicity of χ_ℓ yield

$$\begin{aligned}II_r &= \frac{1}{2}\sum_{P\in\mathcal{M}}\int_{Q_0} \big[\nabla\chi_\ell(z_P+y)\nabla\Theta(z_P+(y-x);A_0)\varphi_r(x-y+z_P)\\ &\qquad - \nabla\chi_\ell(-z_P+y)\nabla\Theta(z_P-(y-x);A_0)\varphi_r(x-y-z_P)\big]d\nu(y)\\ &= \frac{1}{2}\sum_{P\in\mathcal{M}}\int_{Q_0} \nabla\chi_\ell(z_P+y)\big[\nabla\Theta(z_P+(y-x);A_0)\varphi_r(x-y+z_P)\\ &\qquad - \nabla\Theta(z_P-(y-x);A_0)\varphi_r(x-y-z_P)\big]d\nu(y).\end{aligned} \tag{11.16}$$

Let us define $\check{K}_r(\cdot) := \nabla\Theta(\cdot;A_0)\varphi_r(\cdot)$. We claim that, for $x \in Q_0 \cap H$ such that $\operatorname{dist}(x,H) \geq \ell(Q_0)$

$$|\check{K}_r(z_P+(x-y))-\check{K}_r(z_P-(x-y))| \lesssim \frac{\operatorname{dist}(x,H)}{(\operatorname{dist}(x,H)+|z_P|)^{n+1}} \qquad \text{for } y\in Q_0. \quad (11.17)$$

The proof of the formula above can be conducted splitting the cases $|z_P| \leq 2|x-y|$ and $|z_P| > 2|x-y|$, and coupling the Calderón-Zygmund properties of $\check{K}_r$ with simple geometric considerations. The details can be found in [21, (8.25)]. Gathering (11.16), (11.17) and (2.9) we obtain

$$\begin{aligned}|II_r| &\lesssim \|\nabla\chi_\ell\|_\infty \sum_{P\in\mathcal{M}} \int_{Q_0} |\check{K}_r(z_P+(x-y))-\check{K}_r(z_P-(x-y))|d|\nu|(y) \\ &\lesssim \sum_{P\in\mathcal{M}} \int_{Q_0} \frac{\operatorname{dist}(x,H)}{(\operatorname{dist}(x,H)+|z_P|)^{n+1}} d|\nu|(y) \\ &= |\nu|(Q_0)\frac{\operatorname{dist}(x,H)}{\ell(P)^n} \sum_{P\in\mathcal{M}} \frac{\ell(P)^n}{\big(\operatorname{dist}(x,H)+|z_P|\big)^{n+1}} \lesssim \frac{|\nu|(Q_0)}{\ell(P)^n},\end{aligned}$$

which concludes the proof of (11.15) and, thus, the lemma.

It is possible to prove this estimate analogously to the case of the Riesz transform. We omit its proof in order not to make the presentation too lengthy. We remark that the calculations that lead to (11.15) solely relies on the Calderón-Zygmung property of the kernel and some geometric considerations that are independent on its specific expression. We refer to [21, (8.20)] for more details.

Hence, gathering (11.14), (11.15) and passing to the limit on r, we get

$$\bar{T}^*\big((\bar{T}\nu)\nu\big)(x) \lesssim \frac{1}{\ell(Q_0)^n}\int_{Q_0} |\bar{T}\nu| d\nu.$$

Then, recalling (11.3), the growth of ν and using Cauchy-Schwarz's inequality,

$$\bar{T}^*\big((\bar{T}\nu)\nu\big)(x) \lesssim \frac{1}{\ell(Q_0)^n}\Big(\int_{Q_0} |\bar{T}\nu|^2 d\nu\Big)^{1/2}\nu(Q_0) \lesssim \lambda^{1/2},$$

which finishes the proof of (3). □

The following result is a direct consequence of Lemma 11.7.

Corollary 11.1. *For $x\in\partial S$*

$$|\bar{T}\nu(x)-F_S(x)|^2 \lesssim \frac{1}{K_S^{2\tilde{\alpha}}}, \tag{11.18}$$

where the implicit constant does not depend on S.

Another result which is needed for the application of the maximum principle is the estimate of $|F_S(x)|$ for x close to the support of the measure ν.

Lemma 11.8. *Let K_S be an odd natural number, $K_S \geq 3$. For $x \in \mathbb{R}^{n+1}$ with* $\operatorname{dist}(x, \widetilde{H}) \leq 10\ell(Q_0)$ *we have*

$$|F_S(x)| \lesssim \frac{1}{K_S^{\vartheta}}.$$

Proof. Because of the Hölder continuity of f_S, we can write

$$|F_S(x)| \lesssim \Big(\frac{\operatorname{dist}(x, \widetilde{H})}{K_S\ell(Q_0)}\Big)^{\vartheta} |\bar{T}\nu(x_{S+})| \lesssim \frac{|\bar{T}\nu(x_{S+})|}{K_S^{\vartheta}}.$$

So, to prove the lemma, it suffices to show that

$$|\bar{T}\nu(x_{S+})| \leq C$$

for some constant $C > 0$ not depending on K_S. Recall now that $\nu = b\,\eta$. Applying Lemma 10.2 with $\widetilde{M} = 6K_S$ and $f = b$ to the point $0 \in 2Q_0$, we have the estimate

$$\big|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(0)\big| \lesssim \frac{1}{(6K_S)^{\gamma}\ell(Q_0)^n} \int_{Q_0} |b|\, d\eta \lesssim 1, \tag{11.19}$$

where the implicit constant in the last inequality does not depend on K_S. Now, we observe that the (global) Calderón-Zygmund properties of $\bar{K}$ and the fact that $|x_{S+}| \lesssim K_S\ell(Q_0)$ imply

$$\begin{aligned} \big|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(0) - \bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(x_{S+})\big| &\lesssim \int_{(6K_SQ_0)^c} \big|\bar{K}(0,y) - \bar{K}(x_{S+},y)\big| d\nu(y) \\ &\lesssim \int_{(6K_SQ_0)^c} \frac{|x_{S+}|^{\widetilde{\alpha}}}{(|y| + |x_{S+}|)^{n+\widetilde{\alpha}}}\, d\nu(y) \lesssim 1. \end{aligned} \tag{11.20}$$

Then, by (11.19), (11.20) and the triangle inequality, we have

$$\begin{aligned} &\big|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu(x_{S+})\big)\big| \\ &\quad \leq \big|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(0)\big| + \big|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(0) - \bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(x_{S+})\big| \lesssim 1. \end{aligned} \tag{11.21}$$

Moreover, since $\operatorname{dist}(x_{S+}, \operatorname{supp}\nu)^n \gtrsim K_S\ell(Q_0)$ and estimating the kernel via Lemma 2.4,

$$\begin{aligned}\left|\bar{T}\big(\chi_{6K_SQ_0}\nu\big)(x_{S+})\right| &\leq \int_{6K_SQ_0} |\bar{K}(x_{S+},y)|d\nu(y) \lesssim \int_{6K_SQ_0} \frac{1}{|x_{S+}-y|^n}d\nu(y)\\ &\lesssim \frac{\nu(6K_SQ_0)}{\operatorname{dist}(x_{S+},\operatorname{supp}\nu)^n} \lesssim \frac{K_S^n\ell(Q_0)^n}{\operatorname{dist}(x_{S+},\operatorname{supp}\nu)^n} \lesssim 1. \qquad (11.22)\end{aligned}$$

Thus, gathering (11.21) and (11.22) we obtain

$$|\bar{T}\nu(x_{S+})| \leq \left|\bar{T}\big(\chi_{6K_SQ_0}\nu\big)(x_{S+})\right| + \left|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu(x_{S+})\big)\right| \lesssim 1,$$

which proves the lemma. □

In order to be able to use the previous lemma, from now on we assume without loss of generality $K_S \geq 3$ and we suppose it to be an odd number. Observe that for $x \in \operatorname{supp}\nu$, Lemma 11.8 and (11.4) give

$$\begin{aligned}&\sup_{x\in\operatorname{supp}\nu} \left|\bar{T}\nu(x) - F_S(x)\right|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(x)\\ &\quad\leq \sup_{x\in\operatorname{supp}\nu} 2\left|\bar{T}\nu(x)\right|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(x) + 2|F_S(x)|^2 \qquad (11.23)\\ &\quad\leq 12\lambda + 2|F_S(x)|^2 \lesssim \lambda + \frac{1}{K_S^{\vartheta}}.\end{aligned}$$

Moreover, by (11.10) and (11.18),

$$\sup_{x\in\partial S} \left|\bar{T}\nu(x) - F_S(x)\right|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(x) \lesssim \frac{1}{K_S^{2\tilde{\alpha}}} + \lambda^{1/2}$$

which, together with (11.23) brings us to

$$\sup_{x\in\partial S\cup\operatorname{supp}\nu} \left|\bar{T}\nu(x) - F_S(x)\right|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(x) \lesssim \lambda^{1/2} + \frac{1}{K_S^{2\tilde{\alpha}}} + \frac{1}{K_S^{\vartheta}}. \qquad (11.24)$$

Finally, we provide the proof of Lemma 11.6.

Proof. We recall that $\bar{A} = \bar{A}^T$. Let $\vec{g} \in L^\infty(S;\mathbb{R}^{n+1})$. We claim that $\bar{T}^*\big(\vec{g}\mathcal{L}^{n+1}\big)$ is a $L_{\bar{A}^T}$-harmonic (vector valued) function. This would imply the maximum principle

$$\sup_{x\in S} \bar{T}^*\big(\vec{g}\mathcal{L}^{n+1}\big)(x) = \sup_{x\in\partial S\cap\operatorname{supp}\nu} \bar{T}^*\big(\vec{g}\mathcal{L}^{n+1}\big)(x). \qquad (11.25)$$

Observe that, because of Lemma 11.5, the same equality holds with $F_S(x)$ in place of $\bar{T}^*(\vec{g}\mathcal{L}^{n+1})(x)$. Let $\varphi \in C_c^\infty(S \setminus \operatorname{supp}\vec{g})$ be a test function. To prove the claim, apply the definition of $\bar{T}^*$ together with Fubini's theorem together with the fact that $\bar{\mathcal{E}}(x,y) = \mathcal{E}_{A^T}(y,x)$:

$$
\begin{aligned}
\int \bar{A}^T \nabla \bar{T}^*(\vec{g}\mathcal{L}^{n+1}) \cdot \nabla \varphi &= \int \bar{A}^T \nabla_x \Big(\int \nabla_y \bar{\mathcal{E}}(y,x) \cdot \vec{g}(y) dy \Big) \cdot \nabla \varphi(x) dx \\
&= \int \nabla_y \Big(\int \bar{A}^T \nabla_x \bar{\mathcal{E}}(y,x) \cdot \nabla \varphi(x) dx \Big) \cdot \vec{g}(y) dy \\
&= \int \nabla_y \Big(\int \bar{A}^T \nabla_x \, \mathcal{E}_{\bar{A}^T}(x,y) \cdot \nabla \varphi(x) dx \Big) \cdot \vec{g}(y) dy \\
&= \int \nabla \varphi \cdot \vec{g} = 0.
\end{aligned} \tag{11.26}
$$

Notice that for every $z \in \mathbb{R}^{n+1}$ we have the elementary relation

$$
|z|^2 = \sup_{\beta \geq 0, e \in \mathbb{S}^n} 2\langle e, z \rangle - \beta^2,
$$

so that, choosing $z = \bar{T}\nu(x) - F_S(x)$, it reads

$$
\big| \bar{T}\nu(x) - F_S(x) \big|^2 = \sup_{\beta \geq 0, e \in \mathbb{S}^n} 2\langle e, \bar{T}\nu(x) \rangle - 2\langle e, F_S(x) \rangle - \beta^2. \tag{11.27}
$$

We want to show that the argument of the supremum in the right hand side of (11.27) differs from a $L_{\bar{A}}$-harmonic function possibly by a small term. This will allow to apply the maximum principle on the strip and to finish the proof.

For a fixed $e \in \mathbb{S}^n$ and $x \in \operatorname{supp} \nu$, we split

$$
\langle e, \bar{T}\nu(x) \rangle = -\bar{T}^*(\nu e)(x) + \big(\bar{T}^*(\nu e)(x) + \langle e, \bar{T}\nu(x) \rangle \big)
$$

and consider that, claiming that the dominated convergence theorem applies,

$$
\bar{T}^*(\nu e)(x) + \langle e, \bar{T}\nu(x) \rangle = \lim_{r \to \infty} \int \big(\widetilde{K}_r(x,y) + \widetilde{K}_r(y,x) \big) \cdot e \, d\nu(y). \tag{11.28}
$$

To prove that the previous identity holds, set $C_S \gg K_S$ to be chosen later. By the triangle inequality, the antisymmetry of $\nabla_1 \Theta(x,y;\bar{A}(x))$ and the linear growth of ν, we have

$$
\begin{aligned}
&\int_{|x-y|<C_S \ell(Q_0)} \big| \widetilde{K}_r(x,y) + \widetilde{K}_r(y,x) \big| d\nu(y) \\
&\quad \lesssim \int_{|x-y|<C_S \ell(Q_0)} \big| \bar{K}(x,y) - \nabla_1 \Theta(x,y;\bar{A}(x)) \big| d\nu(y) \\
&\qquad + \int_{|x-y|<C_S \ell(Q_0)} \big| \bar{K}(y,x) - \nabla_1 \Theta(y,x;\bar{A}(x)) \big| d\nu(y) \\
&\quad \lesssim \int_{|x-y|<C_S \ell(Q_0)} \frac{1}{|x-y|^{n-\widetilde{\alpha}}} d\nu(y) \lesssim \big(C_S \ell(Q_0) \big)^{\widetilde{\alpha}}.
\end{aligned} \tag{11.29}
$$

So, to bound (11.28) we have to estimate the integral on its rights hand side for $|x-y| > C_S\ell(Q_0)$. As before, by the periodicity of M_S we can assume that $x_H \in [-3\ell(Q_0), 3\ell(Q_0)]^n \times \{0\}$. Hence, using arguments analogous to the ones in Lemma 11.7, it is possible to prove that for $y \in Q_0$ and z_P such that $|x-y-z_P| > C_S\ell(Q_0)$, we have

$$\left|\bar{K}(x, y+z_P) + \bar{K}(x, y-z_P)\right| \lesssim \frac{(K_S\ell(Q_0))^{\widetilde{\alpha}}}{|z_P|^{n+\widetilde{\alpha}} + |x|^{n+\widetilde{\alpha}}},$$

hence, calling $\mathcal{M}_S$ the subset of $P \in \mathcal{M}$ such that $|x-y-z_P| > C_S\ell(Q_0)$ for every $y \in Q_0$, we have

$$\sum_{P\in\mathcal{M}_S}\int_{Q_0} |\widetilde{K}_r(x, y+z_P) + \widetilde{K}_r(x, y-z_P)|d\nu(y) \lesssim \left(\frac{K_S}{C_S}\right)^{\widetilde{\alpha}}. \tag{11.30}$$

Analogously, one can prove

$$\sum_{P\in\mathcal{M}_S}\int_{Q_0} |\widetilde{K}_r(y+z_P, x) + \widetilde{K}_r(y-z_P, x)|d\nu(y) \lesssim \left(\frac{K_S}{C_S}\right)^{\widetilde{\alpha}}, \tag{11.31}$$

so, gathering (11.29), (11.30) and (11.31) and letting $r \to \infty$, we can use the dominated convergence theorem and we estimate (11.28) as

$$\left|\bar{T}^*(\nu e)(x) + \langle e, \bar{T}\nu(x)\rangle\right| \lesssim (C_S\ell(Q_0))^{\widetilde{\alpha}} + \left(\frac{K_S}{C_S}\right)^{\widetilde{\alpha}}. \tag{11.32}$$

We are now ready to proceed with the calculations for the maximum principle. Indeed, taking $x \in S$, an application of (11.27) and (11.32) gives

$$\begin{aligned}
&\left|\bar{T}\nu(x) - F_S(x)\right|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(x) \\
&= \sup_{\beta\geq 0, e\in\mathbb{S}^n} 2\langle e, \bar{T}\nu(x)\rangle - 2\langle e, F_S(x)\rangle - \beta^2 + \bar{T}^*((\bar{T}\nu)\nu)(x) \\
&\lesssim \sup_{\beta\geq 0, e\in\mathbb{S}^n} -2\bar{T}^*(\nu e)(x) - 2\langle e, F_S(x)\rangle - \beta^2 + \bar{T}^*((\bar{T}\nu)\nu)(x) \\
&\quad + (C_S\ell(Q_0))^{\widetilde{\alpha}} + \left(\frac{K_S}{C_S}\right)^{\widetilde{\alpha}}.
\end{aligned}$$

Then, using the maximum principle (11.25) we have

$$\begin{aligned}
&|\bar{T}\nu(x) - F_S(x)|^2 + 4\bar{T}^*((T\nu)\nu)(x) \\
&\lesssim \sup_{\beta\geq 0, e\in\mathbb{S}^n} 2 - \bar{T}^*\big(\nu e + (\bar{T}\nu)\nu\big)(x) - 2\langle e, F_S(x)\rangle - \beta^2 + (C_S\ell(Q_0))^{\widetilde{\alpha}} + \left(\frac{K_S}{C_S}\right)^{\widetilde{\alpha}} \\
&\leq \sup_{z\in\partial S\cup\operatorname{supp}\nu}\ \sup_{\beta\geq 0, e\in\mathbb{S}^n} -2\bar{T}^*\big(\nu e + (\bar{T}\nu)\nu\big)(z) - 2\langle e, F_S(z)\rangle - \beta^2
\end{aligned}$$

$$+ (C_S \ell(Q_0))^{\widetilde{\alpha}} + \Big(\frac{K_S}{C_S}\Big)^{\widetilde{\alpha}}.$$

So, another application of (11.27) and (11.32) concludes the proof of the lemma. Indeed, recalling the estimate (11.24) on $\partial S \cup \operatorname{supp} \nu$,

$$\begin{aligned}
&|\bar{T}\nu(x) - F_S(x)|^2 + 4\bar{T}^*((T\nu)\nu)(x) \\
&\lesssim \sup_{z\in\partial S\cup\operatorname{supp}\nu} \ \sup_{\beta\geq 0, e\in\mathbb{S}^n} 2\langle e, \bar{T}\nu(x)\rangle - 2\langle e, F_S(x)\rangle - \beta^2 \\
&\qquad\qquad + \bar{T}^*((\bar{T}\nu)\nu)(x) + (C_S\ell(Q_0))^{\widetilde{\alpha}} + \Big(\frac{K_S}{C_S}\Big)^{\widetilde{\alpha}} \\
&\lesssim \sup_{z\in\partial S\cup\operatorname{supp}\nu} |\bar{T}\nu(z) - F_S(z)|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(z) + (C_S\ell(Q_0))^{\widetilde{\alpha}} + \Big(\frac{K_S}{C_S}\Big)^{\widetilde{\alpha}} \\
&\lesssim \lambda^{1/2} + \frac{1}{K_S^{2\widetilde{\alpha}}} + \frac{1}{K_S^{\vartheta}} + (C_S\ell(Q_0))^{\widetilde{\alpha}} + \Big(\frac{K_S}{C_S}\Big)^{\widetilde{\alpha}}. \quad \square
\end{aligned}$$

11.2. The conclusion of the proof of the Key Lemma

To simplify the notation, set

$$\mathtt{Err}(K_S, C_S, \ell(Q_0)) := \frac{1}{K_S^{2\widetilde{\alpha}}} + \frac{1}{K_S^{\vartheta}} + (C_S\ell(Q_0))^{\widetilde{\alpha}} + \Big(\frac{K_S}{C_S}\Big)^{\widetilde{\alpha}}.$$

Notice that if $x \in 2Q_0$, Lemma 11.6 together with Lemma 11.8 allows to majorize $|\bar{T}\nu(x)|^2$ as

$$\begin{aligned}
\big|\bar{T}\nu(x)\big|^2 &\lesssim |\bar{T}\nu(x) - F_S(x)|^2 + |F_S(x)|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(x) - 4\bar{T}^*((\bar{T}\nu)\nu)(x) \\
&\lesssim \lambda^{1/2} + \mathtt{Err}(K_S, C_S, \ell(Q_0)) + |F_S(x)|^2 - \bar{T}^*((\bar{T}\nu)\nu)(x) \\
&\lesssim \lambda^{1/2} + \mathtt{Err}(K_S, C_S, \ell(Q_0)) - \bar{T}^*((\bar{T}\nu)\nu)(x).
\end{aligned} \tag{11.33}$$

Let φ be a smooth function such that $\chi_{Q_0} \leq \varphi \leq \chi_{2Q_0}$ and $\|\nabla\varphi\|_\infty \lesssim \ell(Q_0)^{-1}$. Set $\psi := \bar{A}^T\nabla\varphi$ and observe that it verifies

$$\begin{aligned}
\bar{T}^*[\psi\,\mathcal{L}^{n+1}](x) = \bar{T}^*[\bar{A}^T\nabla\varphi\,\mathcal{L}^{n+1}](x) &= \int \nabla_1\,\mathcal{E}_{\bar{A}}(y,x)\cdot\bar{A}^T(y)\nabla\varphi(y)dy \\
&= \int \bar{A}(y)\,\nabla_1\,\mathcal{E}_{\bar{A}}(y,x)\cdot\nabla\varphi(y)dy = \varphi(x),
\end{aligned}$$

the last equality being a consequence of the definition of fundamental solution.

The choice of $\varphi \geq \chi_{Q_0}$, together with Cauchy-Schwarz's inequality, gives

$$\begin{aligned}
\nu(Q_0) \leq \int \varphi d\nu &= \int \bar{T}^*(\psi\mathcal{L}^{n+1})d\nu = \int \bar{T}\nu\cdot\psi d\mathcal{L}^{n+1} \\
&\leq \Big(\int |\bar{T}\nu|^2|\psi|d\mathcal{L}^{n+1}\Big)^{1/2}\Big(\int |\psi|d\mathcal{L}^{n+1}\Big)^{1/2}.
\end{aligned} \tag{11.34}$$

Now, observe that

$$\|\psi\|_\infty \leq \|\bar{A}^T\|_\infty \|\nabla\varphi\|_\infty \lesssim \ell(Q_0)^{-1} \tag{11.35}$$

and

$$\int |\psi| d\mathcal{L}^{n+1} \lesssim \frac{1}{\ell(Q_0)} \mathcal{L}^{n+1}(2Q_0) \lesssim \ell(Q_0)^n. \tag{11.36}$$

We claim that

$$\int |\bar{T}\nu|^2 |\psi| d\mathcal{L}^{n+1} \ll \ell(Q_0)^n.$$

Applying (11.33) and (11.36), we can write

$$\begin{aligned}
&\int |\bar{T}\nu|^2 |\psi| d\mathcal{L}^{n+1} \\
&\lesssim \big(\lambda^{1/2} + \mathtt{Err}(K_S, C_S, \ell(Q_0))\big) \int |\psi| d\mathcal{L}^{n+1} + \Big| \int \bar{T}^*\big((\bar{T}\nu)\nu\big) |\psi| d\mathcal{L}^{n+1} \Big| \\
&\lesssim \big(\lambda^{1/2} + \mathtt{Err}(K_S, C_S, \ell(Q_0))\big) \int |\psi| d\mathcal{L}^{n+1} \\
&\quad + \Big| \int \bar{T}^*\big(\chi_{(30Q_0)^c}(\bar{T}\nu)\nu\big) |\psi| d\mathcal{L}^{n+1} \Big| + \Big| \int \bar{T}^*\big(\chi_{30Q_0}(\bar{T}\nu)\nu\big) |\psi| d\mathcal{L}^{n+1} \Big| \\
&\lesssim \big(\lambda^{1/2} + \mathtt{Err}(K_S, C_S, \ell(Q_0))\big) \ell(Q_0)^n + \Big| \int \bar{T}^*\big(\chi_{(30Q_0)^c}(\bar{T}\nu)\nu\big) |\psi| d\mathcal{L}^{n+1} \Big| \\
&\quad + \Big| \int \bar{T}^*\big(\chi_{30Q_0}(\bar{T}\nu)\nu\big) |\psi| d\mathcal{L}^{n+1} \Big| \\
&= \big(\lambda^{1/2} + \mathtt{Err}(K_S, C_S, \ell(Q_0))\big) \ell(Q_0)^n + I + II,
\end{aligned} \tag{11.37}$$

where I and II are defined by the last equality.
The estimate for I is an application of (10.8) with $\widetilde{M} = 30$. In particular, recalling (11.3),

$$\begin{aligned}
\big|\bar{T}^*\big(\chi_{(30Q_0)^c}(\bar{T}\nu)b\eta\big)\big)(x)\big| &\lesssim \frac{1}{\ell(Q_0)^n} \int_{Q_0} |(\bar{T}\nu)b| d\eta \\
&\leq \frac{\nu(Q_0)^{1/2}}{\ell(Q_0)^n} \Big(\int_{Q_0} |\bar{T}\nu|^2 d\nu \Big)^{1/2} \leq \lambda^{1/2} \frac{\nu(Q_0)}{\ell(Q_0)^n},
\end{aligned}$$

which, together with (11.36), implies

$$I \lesssim \lambda^{1/2} \nu(Q_0). \tag{11.38}$$

For the estimate of II, recall that $|\bar{K}(x,y)| \lesssim |x-y|^{-n}$. This and (11.35) imply

$$\left|\bar{T}\big(|\psi|\mathcal{L}^{n+1}\big)(x)\right| = \left|\int \bar{K}(x,y)|\psi|(y)dy\right| \lesssim \frac{1}{\ell(Q_0)} \int_{2Q_0} \frac{1}{|x-y|^n}dy \lesssim \frac{\ell(Q_0)}{\ell(Q_0)} = 1.$$

Then, by Cauchy-Schwarz's inequality, the periodicity of $\bar{T}\nu$ and the localization (11.3),

$$II \leq \left| \int_{30Q_0} \bar{T}(|\psi|\mathcal{L}^{n+1}) \cdot \bar{T}\nu d\nu \right| \lesssim \nu(Q_0)^{1/2} \Big(\int_{30Q_0} |\bar{T}\nu|^2 d\nu \Big)^{1/2} \lesssim \lambda^{1/2}\nu(Q_0). \tag{11.39}$$

So, gathering (11.34), (11.37), (11.38) and (11.39), we have

$$\nu(Q_0) \lesssim \big(\mathtt{Err}(K_S, C_S, \ell(Q_0)) + \lambda^{1/2}\big)^{1/2}\nu(Q_0). \tag{11.40}$$

Choosing K_S big enough, K_S/C_S small enough, $C_S\ell(Q_0)$ and λ small enough, we have

$$\mathtt{Err}(K_S, C_S, \ell(Q_0)) + \lambda^{1/2} \ll 1,$$

so (11.40) brings us to the contradiction

$$\nu(Q_0) \ll \nu(Q_0).$$

This proves the Key Lemma and, hence, completes the proof of Theorem 1.2.

12. The two-phase problem for the elliptic measure

To the purpose of the application to the study of the elliptic measure, it is useful to reformulate Theorem 1.2 under slightly different hypothesis. The proof of the following closely resembles that of [9, Theorem 3.3].

Theorem 12.1. *Let μ be a Radon measure in $\mathbb{R}^{n+1}$ and let $B \subset \mathbb{R}^{n+1}$ be a ball with $\mu(B) > 0$. Assume that, for some constants $C_0, C_1 > 0$ and $0 < \lambda, \delta, \tau \ll 1$ the following conditions hold:*

(1) $r(B) \leq \lambda$.
(2) $P_{\mu,\widetilde{\alpha}}(B) \leq C_0\Theta_\mu(B)$.
(3) There is some n-plane L through the center of B such that $\beta^L_{\mu,1}(B) \leq \delta\Theta_\mu(B)$.
(4) There is $G_B \subset B$ such that for all $x \in G_B$

$$\sup_{0<r\leq 2r(B)} \frac{\mu\big(B(x,r)\big)}{r^n} + T_*(\chi_{2B}\mu)(x) \leq C_1\Theta_\mu(B).$$

(5) $\int_{G_B} |T\mu(x) - m_{\mu,G_B}(T\mu)|^2 d\mu(x) \leq \tau\Theta_\mu(B)^2\mu(B)$.

There exists $\vartheta > 0$ such that, if δ, τ and λ are small enough (depending on C_0 and C_1), there is a uniformly n-rectifiable set Γ such that

$$\mu(B \cap \Gamma) \geq \vartheta \mu(B).$$

The proof in the case $A \equiv Id$ is based on a Tb theorem for *suppressed kernels* by Nazarov, Treil and Volberg. To replicate the proof of Azzam, Mourgoglou and Tolsa in the elliptic context, we define the suppressed kernel associated with $K(\cdot, \cdot)$ as

$$\widetilde{K}_\Phi(x, y) = \widetilde{\chi}\Big(\frac{|x-y|^2}{\Phi(x)\Phi(y)}\Big) K(x, y),$$

where $\widetilde{\chi}$: $[0, +\infty) \to [0, 1]$ is a smooth, vanishes identically in $[0, 1/2]$ and equals 1 in $[1, +\infty)$ and Φ is a 1-Lipschitz function to be chosen as in the proof of [9]. Then, one can split

$$K(x, y) = \frac{1}{2}\big(K(x, y) + K(y, x)\big) + \frac{1}{2}\big(K(x, y) - K(y, x)\big) = K^{(s)}(x, y) + K^{(a)}(x, y),$$

apply the Tb theorem for suppressed kernels (see also [48, Section 5.12] and the references therein) to the antisymmetric part of K and exploit the L^2-boundedness of the symmetric part guaranteed by the freezing technique of Lemma 2.2. We leave to the interested reader to check that there is no further difficulty in the proof Theorem 12.1.

The rest of the present section is devoted to show how to apply Theorem 12.1 to prove the two-phase problem for the elliptic measure.

After possibly splitting the set E, we can assume $\operatorname{diam} E \leq \frac{1}{10} \min\big(\operatorname{diam} \Omega_1, \operatorname{diam} \Omega_2\big)$. We choose the poles p_i, $i = 1, 2$ such that $p_i \in \Omega_i \cap 2\widetilde{B} \setminus \widetilde{B}$, where $\widetilde{B}$ is a ball centered at E with radius $r(\widetilde{B}) = 2 \operatorname{diam} E$.

We are going to apply Theorem 12.1 to the measure ω_1: we are going to prove that we can find an n-rectifiable set $F \subset E$ such that $\omega_1|_F \ll \mathcal{H}^n|_F \ll \omega_1|_F$. In particular, we can suppose that Ω_1 is such that

$$\mathcal{H}^{n+1}\big(\widetilde{B} \cap \Omega_1\big) \approx r(\widetilde{B}). \tag{12.1}$$

By the so-called Bourgain's estimates (see [43, Section 12] for the statement in the elliptic case and [6] for a proof in the case $A \equiv Id$) together with (12.1), we can infer that there exists δ_0 such that

$$\omega_1\big(2\delta^{-1}\widetilde{B}\big) \approx 1, \qquad \text{for } 0 < \delta < \delta_0.$$

Let $a, \widetilde{\gamma} > 0$ and $i = 1, 2$. We say that a ball B is a-$P_{\omega_i,\widetilde{\gamma}}$-doubling if

$$P_{\omega_i,\widetilde{\gamma}}(B) \leq a\Theta_{\omega_i}(B).$$

The following lemma is important for the applicability of the doubling condition.

Lemma 12.1. *Let $\widetilde{\gamma} \in (0,1)$. Let Ω_1, Ω_2 be Wiener regular domains in $\mathbb{R}^{n+1}$ and let $E \subset \partial\Omega_1 \cap \partial\Omega_2$ be a set on which $\omega_1|_E \ll \omega_2|_E \ll \omega_1|_E$. Then there exists a constant $a = a(\widetilde{\gamma}, n)$ big enough such that for $\omega_1|_E$-almost every $x \in \mathbb{R}^{n+1}$ we can find a sequence of a-$P_{\omega_i,\widetilde{\gamma}}$-doubling balls $B(x, r_i)$ with $r_i \to 0$ as $i \to \infty$.*

Proof. Let $i = 1, 2$. Let $m \in \mathbb{Z}, m \geq 1$ and denoting

$$Z_m := \left\{x \in \partial\Omega_i : \text{ for all } j \geq m,\ B(x, 2^{-j}) \text{ is not } a\text{-}P_{\omega_i,\widetilde{\gamma}}\text{-doubling}\right\}$$

it suffices to prove that $\omega_i|_E(Z_m) = 0$ for every m. Arguing as in [11, Lemma 6.1] we have that, for $x \in Z_m$, we can estimate the elliptic measure of $B(x, r)$ as

$$\omega_i(B(x, r)) \leq C(m) r^{n+\widetilde{\gamma}} \qquad \text{for } r \leq 2^{-m}.$$

Then

$$\omega|_E(A) \leq \omega(A) \leq C(m)\mathcal{H}^{n+\widetilde{\gamma}}(A) \qquad \text{for any } A \subset Z_m.$$

We recall that the dimension of $\omega|_E$ can be defined as

$$\begin{aligned}\dim \omega|_E := \inf \big\{ s \colon \exists\, F \subset \partial\Omega \text{ s.t. } \mathcal{H}^s(F) = 0 \\ \text{and } \omega|_E(F \cap K) = \omega|_E(\partial\Omega \cap K)\, \forall K \subset \mathbb{R}^{n+1} \text{ compact}\big\}\end{aligned}$$

First let us bound $\dim \omega|_E$ from below. Let $F \subset \partial\Omega$ be such that $\mathcal{H}^{n+\widetilde{\gamma}}(F) = 0$. For $K \subset Z_m$ compact and such that $\omega|_E(K) > 0$, we have $\omega|_E(F \cap K) \leq C(m)\mathcal{H}^{n+\widetilde{\gamma}}(F \cap K) = 0$. This in turn implies

$$\dim \omega|_E \geq n + \widetilde{\gamma}. \tag{12.2}$$

Conversely, [8] gives that $\dim \omega|_E = n$, which gathered with (12.2) tells that

$$n \geq n + \widetilde{\gamma}.$$

Being $\widetilde{\gamma} > 0$, this brings to a contradiction and, in particular, this proves that $\omega(Z_m) = 0$ for every m. $\square$

Let $i = 1, 2$. Denote by $u_i(\cdot) = G_i(p_i, \cdot)$ the Green function associated with Ω_i with pole at p_i. We understand that u_i is extended by zero to Ω_i^c. As a corollary of [5, Theorem 1.5], which was formulated under weaker assumptions on the regularity of the matrix A, we can state the following monotonicity formula.

Lemma 12.2 *(Monotonicity formula). Let Ω_i and u_i be as above and let $R > 0$. Suppose that $A_s(\xi) = Id$ for $\xi \in \partial\Omega_1 \cap \partial\Omega_2$. Then, setting*

$$\gamma(\xi, r) = \left(\frac{1}{r^2} \int\limits_{B(\xi,2r)} \frac{|\nabla u_1(y)|^2}{|y-\xi|^{n-1}} dy\right) \cdot \left(\frac{1}{r^2} \int\limits_{B(\xi,2r)} \frac{|\nabla u_2(y)|^2}{|y-\xi|^{n-1}} dy\right),$$

we have that, for some $c > 0$,

$$\gamma(\xi, r) \leq \gamma(\xi, s) e^{c(s^\alpha - r^\alpha)} < \infty \qquad \text{for } 0 < r \leq s < R.$$

We remark that Azzam, Garnett, Mourgoglou and Tolsa proved their result under the hypothesis $A(\xi) = Id$. However, the same proof works under our assumption.[3]

The following lemma is crucial to prove the elliptic variant version of the blowups.

Lemma 12.3. *Let Ω_1 be a Wiener regular domain and denote by $\omega_1 = \omega_1^{p_1}$ its associated elliptic measure with pole at $p_1 \in \Omega_1$. Let B be a ball centered at $\partial\Omega_1$ and such that $p_1 \notin 10B$. Assuming that $\omega_1(8B) \leq C\omega_1(\delta_0 B)$ and $\mathcal{H}^{n+1}(B \setminus \Omega_1) \geq C^{-1} r(B)^{-1}$, we have*

$$\mathcal{H}^{n+1}(\Omega_1 \cap 2\delta_0 B) \gtrsim r(B)^{n+1}. \tag{12.3}$$

Moreover

$$\mathcal{H}^{n+1}(2\delta_0 B \setminus \Omega_1) \approx \mathcal{H}^{n+1}(2\delta_0 B \setminus \Omega_2) \approx r(B)^{n+1}. \tag{12.4}$$

Proof. Denote $r = r(B)$. Let us first prove (12.3). Consider a smooth function $\varphi \geq 0$ such that $\varphi \equiv 1$ on $\delta_0 B$ and $\operatorname{supp} \varphi \subset 2\delta_0 B$. In particular, suppose that $\|\varphi\|_\infty \lesssim (\delta_0 r)^{-1}$. Then, recalling that, by the properties of Green's function and being x_1 outside of the support of φ,

$$\int \varphi d\omega_1 = -\int A^T \nabla u_1 \cdot \nabla \varphi,$$

we use the ellipticity of the matrix A and write

$$\begin{aligned} \omega_1(2\delta_0 B) \leq \int \varphi d\omega_1 &\leq \int |\nabla u_1 \cdot A \nabla \varphi| \\ &\lesssim \int |\nabla u_1||\nabla \varphi| = \int\limits_{\Omega_1 \cap 2\delta_0 B} |\nabla u_1||\nabla \varphi| \lesssim \frac{1}{\delta_0 r} \int\limits_{\Omega_1 \cap 2\delta_0 B} |\nabla u_1|. \end{aligned}$$

Then applying, in order, Hölder's and Caccioppoli's inequalities,

[3] It suffices to define the matrix D in [5, Appendix A.1] as $D = A(\xi) - A$ and observe that $L_{A(\xi)} = L_{A_s(\xi)} = Id$.

$$\frac{1}{\delta_0 r}\int\limits_{\Omega_1\cap 2\delta_0 B}|\nabla u_1| \leq \frac{\mathcal{H}^{n+1}(\Omega_1\cap 2\delta_0 B)^{1/2}}{\delta_0 r}\Big(\int\limits_{2\delta_0 B}|\nabla u_1|^2\Big)^{1/2}$$

$$\lesssim \frac{\mathcal{H}^{n+1}(\Omega_1\cap 2\delta_0 B)^{1/2}}{\delta_0 r}\frac{1}{\delta_0 r}\Big(\int\limits_{4\delta_0 B}|u_1|^2\Big)^{1/2},$$

so

$$\omega_1(2\delta_0 B) \lesssim \mathcal{H}^{n+1}(\Omega_1\cap 2\delta_0 B)^{1/2}\frac{(\delta_0 r)^{(n+1)/2}}{(\delta_0 r)^2}\sup_{4\delta_0 B}|u_1|.$$

At this point, recalling that (see [43, Section 12])

$$\sup_{y\in 4\delta_0 B} u_1(y) \lesssim \frac{\omega_1(8B)}{r^{n-1}},$$

we have

$$\omega_1(\delta_0 B) \lesssim \mathcal{H}^{n+1}(\Omega_1\cap 2\delta_0 B)^{1/2}\frac{(\delta_0 r)^{(n-3)/2}}{(\delta_0 r)^{n-1}}\omega_1(8B)$$

which, since we suppose $\omega_1(8B) \leq C\omega_1(\delta_0 B)$, concludes the proof of (12.3).

The second estimate in the statement of the lemma is a direct application of the first one (see also [11, Lemma 3.4]). □

The following lemma provides the connection between the function γ in Lemma 12.2 and elliptic measure.

Lemma 12.4. *Let $i = 1, 2$ and Ω_i, p_i be as above. Let $0 < R < \min_i \operatorname{dist}(p_i, \partial\Omega_i)$. Then, for $0 < r < R/4$ and $\xi \in \partial\Omega_1 \cap \partial\Omega_2$ we have*

$$\frac{\omega_1(B(\xi,r))}{r^n}\frac{\omega_2(B(\xi,r)}{r^n} \lesssim \gamma(\xi,2r)^{1/2}. \tag{12.5}$$

Moreover, if $r < \delta_0 R/8$ and $\omega_i(B(\xi,8r)) \lesssim \omega_i(B(\xi,\delta_o r))$,

$$\gamma(\xi,r)^{1/2} \lesssim \frac{\omega_1(B(\xi,16\delta_0^{-1}))}{r^n}\cdot\frac{\omega_2(B(\xi,16\delta_0^{-1}))}{r^n}. \tag{12.6}$$

The proof of (12.5) is analogous to that for the harmonic measure in [29]. The proof of (12.6) is an application of Caccioppoli's inequality together with Lemma 12.3 (see also [11, Lemma 3.5]).

The blowup technique for the elliptic measure developed in [8] is crucial to prove the next lemma. We remark that the authors formulated this result under more general assumptions on the matrix A then the ones of the present work.

Lemma 12.5. *Let Ω_1, Ω_2 and E be as above. Let $\varepsilon < 1/100$ and, for $m \geq 1$, define E_m as the set of $\xi \in E$ such that for all $\xi \in E$, $0 < r < 1/m$ and $i = 1, 2$ the following properties hold:*

(E1) $\omega_i(B(\xi, 2r)) \leq m\, \omega_i(B(\xi, r))$.
(E2) $\mathcal{H}^{n+1}(B(\xi, r) \cap \Omega_i) \geq \frac{1}{m} r^{n+1}$.
(E3) $\beta_{\omega_1,1}(B(\xi, r)) < \varepsilon r^{-n} \omega_1(B(\xi, r))$.

The sets E_m cover E up to a set of ω_1-measure 0, i.e.

$$\omega_1\Big(E \setminus \bigcup_{m \geq 1} E_m\Big) = 0.$$

The proof follows by known results in the literature. However, we think that it may be useful to the reader to dispose of precise references.

Sketch of the proof. Set

$$E^* = \Big\{\xi \in E : \lim_{r \to 0} \frac{\omega_1(E \cap B(\xi, r))}{\omega_1(B(\xi, r))} = \lim_{r \to 0} \frac{\omega_2(E \cap B(\xi, r))}{\omega_2(B(\xi, r))} = 1\Big\}.$$

One can see that $\omega_i(E \setminus E^*) = 0$, $i = 1, 2$. Now, for $\xi \in E^*$, set $h(\xi) = \frac{d\omega_1}{d\omega_2}(\xi)$,

$$\Lambda = \big\{\xi \in E^* : 0 < h(\xi) < \infty\big\}$$

and

$$\Gamma = \{\xi \in \Lambda : \xi \text{ is a Lebesgue point for } h \text{ with respect to } \omega_1\}.$$

By Lebesgue differentiation theorem, $\omega_i(E \setminus \Gamma) = \omega_i(E^* \setminus \Gamma)$ for $i = 1, 2$. Then, in order to prove the lemma it suffices to show that for ω_1-almost every $\xi \in \Gamma$:

(P1) ω_1 is locally doubling, i.e.

$$\limsup_{r \to 0} \frac{\omega_1(B(\xi, 2r))}{\omega_1(B(\xi, r))} < \infty.$$

(P2) For $i = 1, 2$

$$\liminf_{r \to 0} \frac{\mathcal{H}^{n+1}(B(\xi, r) \cap \Omega_i)}{r^{n+1}} > 0$$

(P3) We have the flatness estimate

$$\lim_{r \to 0} \beta_{\omega_1,1}(B(\xi, r)) \frac{r^n}{\omega_1(B(\xi, r))} = 0.$$

The condition (P1) holds because of the flatness of the tangents $\mathrm{Tan}(\omega_i, \xi)$, see [8, Theorem 1.3], which is known to imply the locally doubling condition ([42, Corollary 2.7]).

The property (P2) follows by the arguments in [11] together with (12.4).

To prove (P3), it suffices to argue as in the end of [11, Section 5]. $\square$

Now consider $m \geq 1$ such that $\omega_i(E_m)$.

Lemma 12.6. *Let $\delta > 0$. For ω_1-almost every $x \in E_m$ there is $r_x > 0$ such that, given an a-$P_{\widetilde{\gamma},\omega_1}$-doubling ball $B(x,r)$ with $r \leq r_x$, there exits a set $G_m(x,r) \subset E_m \cap B(x,r)$ such that*

$$\frac{\omega_1(B(z,t))}{t^n} \lesssim \frac{\omega_1(B(x,r))}{r^n} \text{ for every } z \in G_m(x,r),\ 0 < t \leq 2r.$$

In particular,

$$\omega_1(B(x,r) \setminus G_m(x,r)) \leq \delta\omega_1(B(x,r)), \tag{12.7}$$

and, if we denote by $\widetilde{E}_{m\delta}$ the set of points where (12.7) is verified, we have

$$\omega_1\big(E_m \setminus \widetilde{E}_{m,\delta}\big) = 0.$$

This lemma can be proved arguing as in [11, Lemma 6.2] and more precisely combining the locally doubling property of the elliptic measure ensured by the blowup argument together with Lemma 12.4.

We also point out that their argument relies on the monotonicity formula of Alt, Caffarelli and Friedman. So, to prove it in the elliptic case we have to invoke Lemma 12.2, whose hypothesis include the assumption $A_s(x) = Id$. This, of course, is not true in general. However, one can argue via the change of variable in Lemma 5.3 to achieve this property. For a more detailed treatment of how the elliptic measure varies under that transformation we refer to [5, Corollary 2.5]. We omit further details.

From now on fix $\widetilde{\gamma} = \widetilde{\alpha}$. The following lemma contains an estimate of the potential of ω_1 which is needed to recollect the property (4) in Theorem 12.1.

Lemma 12.7 *(cfr. [11, Lemma 6.3]). Let $0 < c \ll 1$ to be chosen small enough. For $m \geq 1$ and $\delta > 0$, let $\widetilde{E}_{m,\delta}$ and r_{x_0} be as in the previous lemma. Consider $x_0 \in \widetilde{E}_{m,\delta}$ and take*

$$0 < r_0 < \min\big(r_{x_0}, 1/m, \mathrm{dist}(p_1, \partial\Omega_1)\big).$$

Assume, moreover, that $B_0 = B(x_0, r)$ is an a-$P_{\omega_1,\widetilde{\alpha}}$-doubling ball. Then, for all $x \in G_m(x_0, r_0)$ we have

$$T_*(\chi_{2B_0}\omega_1)(x) \lesssim \Theta_{\omega_1}(B_0).$$

Proof. Suppose $A_s(x_0) = Id$. Indeed, if this is not the case, one can argue via a change of variable as mentioned before. Also, without loss of generality, we can consider only the case $r \leq r_0/4$.

Let $\varepsilon > 0$. The proof relies on the estimates for the smoothened potential

$$\widetilde{T}_\varepsilon \omega_1(z) := \int K(z,y)\varphi_\varepsilon(z-y)d\omega_1(y), \qquad z \in \mathbb{R}^{n+1},$$

where $\varphi\colon \mathbb{R}^{n+1} \to [0,1]$ is a smooth radial function whose support is contained in $\mathbb{R}^{n+1} \setminus B(0,1)$, equals 1 on $\mathbb{R}^{n+1} \setminus B(0,2)$ and φ_ε denotes the dilate $\varphi_\varepsilon(z) = \varphi(\varepsilon^{-1}z)$.

Now take $x \in G_m(x_0, r_0)$ and considering $r \leq r_0/4$ and define

$$\begin{aligned} &v_r(z) = \mathcal{E}(p_1, z) - \int \mathcal{E}(z,y)\varphi_r(x-y)d\omega_1(y), \\ &z \in \mathbb{R}^{n+1} \setminus [\mathrm{supp}(\varphi_r(x-\cdot)\omega_1) \cup \{p_1\}]. \end{aligned} \tag{12.8}$$

Recall that $A_s(x_0) = Id$ and that $\Theta(\cdot; A(x_0)) = \Theta(\cdot; A_s(x_0))$. On the same range of z of (12.8) we consider

$$\bar{v}_r(z) = \Theta(p_1 - z; Id) - \int \Theta(z-y; Id)\varphi_r(x-y)d\omega_1(y).$$

As in [11, Lemma 6.3], to prove the lemma it suffices to show the validity of the estimate

$$|\widetilde{T}_r \omega_1(x) - \widetilde{T}_{r_0/4}\omega_1(x)| \lesssim \Theta_{\omega_1}(B_0).$$

To this purpose, observe that

$$\begin{aligned} |\widetilde{T}_r \omega_1(x) - \widetilde{T}_{r_0/4}\omega_1(x)| &= |\nabla v_r(x) - \nabla v_{r_0/4}(x)| \\ &= \Big| \int \nabla_1 \mathcal{E}(x,y)\big(\varphi_r(x-y) - \varphi_{r_0/4}(x-y)\big)d\omega_1(y)\Big|. \end{aligned}$$

Now, using Lemma 2.2 and the Hölder continuity of A, it is not difficult (recall that $r_0 \leq 1$) to prove that

$$|\nabla_1 \mathcal{E}(x,y) - \nabla_1 \Theta(x-y; Id)| \lesssim \frac{r_0^{\widetilde{\alpha}}}{|x-y|^n} \leq \frac{1}{|x-y|^n},$$

which in turn implies

$$\begin{aligned} &|\widetilde{T}_r \omega_1(x) - \widetilde{T}_{r_0/4}\omega_1(x)| \\ &\lesssim \Theta_{\omega_1}(B_0) + \Big| \int \nabla_1 \Theta(x-y; Id)\big(\varphi_r(x-y) - \varphi_{r_0/4}(x-y)\big)d\omega_1(y)\Big| \\ &= |\nabla \bar{v}_r(x) - \nabla \bar{v}_{r_0/4}(x)| + \Theta_{\omega_1}(B_0). \end{aligned} \tag{12.9}$$

We claim that $|\bar{v}_r(x) - \bar{v}_{r_0/4}(x)| \lesssim \Theta_{\omega_1}(B_0)$, which would conclude the proof. To show this, notice that functions $\bar{v}_r$ and $\bar{v}_{r_0/4}$ are harmonic outside $\operatorname{supp}(\varphi_r(x-\cdot)\omega_1) \cup \{p_1\}$, hence in particular in $B(x,r)$. Then, an application of the mean value property gives

$$|\nabla \bar{v}_r(x) - \nabla \bar{v}_{r_0/4}(x)| \lesssim \frac{1}{r} \fint_{B(x,r)} |\bar{v}_r(z) - \bar{v}_{r_0/4}(z)| dz. \tag{12.10}$$

Another application of the freezing argument together with the $C^{\widetilde{\alpha}}$-continuity of A proves

$$|\bar{v}_r(z) - \bar{v}_{r_0/4}(z) - v_r(z) - v_{r_0/4}(z)| \lesssim r_0^{\widetilde{\alpha}} r\, \Theta_{\omega_1}(B_0), \qquad z \in B(x,r)$$

that, gathered with (12.9) and (12.10) gives

$$\begin{aligned} |\widetilde{T}_r\omega_1(x) - \widetilde{T}_{r_0/4}\omega_1(x)| &\lesssim \Theta_{\omega_1}(B_0) + \frac{1}{r} \fint_{B(x,r)} |v_r(z) - v_{r_0/4}(z)| dz \\ &\leq \Theta_{\omega_1}(B_0) + \frac{1}{r} \fint_{B(x,r)} |v_r(z)| dz + \frac{1}{r} \fint_{B(x,r)} |v_{r_0/4}(z)| dz. \end{aligned}$$

From this point on, the proof is analogous to that in [11]. $\square$

The proof of Theorem 1.3 follows verbatim the footprints of that of [9] and [11]. More precisely, taking $x_0 \in \widetilde{E}_{m,\delta}$ and r_0 as in Lemma 12.7, we split the set $G_m(x_0,r_0)$ as a union of

$$G_m^{zd}(x_0,r_0) = \left\{x \in G_m(x_0,r_0) : \lim_{r\to 0} \Theta_{\omega_1}\big(B(x,r)\big) = 0\right\}$$

and

$$G_m^{pd}(x_0,r_0) = G_m(x_0,r_0) \setminus G_m^{zd}(x_0,r_0).$$

Then, using Lemma 12.7, the elliptic analogue of [11, Lemma 6.5] and Theorem 12.1, it is possible to infer that

$$\omega_1(G_m^{zd}(x_0,r_0)) = 0.$$

On the other side, [43, Theorem 1.3] ensures the existence of an n-rectifiable set $F(x_0,r_0) \subset G_m^{pd}(x_0,r_0)$ of mutual absolute continuity of the elliptic measure $\omega_1|_{F(x_0,r_0)}$ and the Hausdorff measure $\mathcal{H}^n|_{F(x_0,r_0)}$ that covers $G_m(x_0,r_0)$ up to a ω_1-null set. This concludes the proof of Theorem 1.3.

Declaration of competing interest

None.

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