Unique Continuation at the Boundary for Harmonic Functions in C^1 Domains and Lipschitz Domains with Small Constant

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Abstract

Let $\Omega \subset \mathbb{R}^n$ be a C^1 domain, or more generally, a Lipschitz domain with small local Lipschitz constant. In this paper it is shown that if u is a function harmonic in Ω and continuous in $\overline{\Omega}$, which vanishes in a relatively open subset $\Sigma \subset \partial\Omega$; moreover, the normal derivative $\partial_\nu u$ vanishes in a subset of Σ with positive surface measure; then u is identically zero. © 2021 The Authors. *Communications on Pure and Applied Mathematics* published by Wiley Periodicals LLC.

1 Introduction

In $\mathbb{R}^n$, with $n \geq 3$, there are examples of harmonic functions in the half-space $\mathbb{R}^n_+$, C^1 up to the boundary, such that the function and its normal derivative vanish simultaneously on a set of positive measure of $\partial\mathbb{R}^n_+$. This was shown by Bourgain and Wolff in [5]. The same result was generalized later to arbitrary $C^{1,\alpha}$ domains by Wang [19]. A related conjecture that is still open is the following:

Conjecture. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and let $\Sigma \subset \partial\Omega$ be relatively open with respect to $\partial\Omega$. Let u be a function harmonic in Ω and continuous in $\overline{\Omega}$. Suppose that u vanishes in Σ and the normal derivative $\partial_\nu u$ vanishes in a subset of Σ with positive surface measure. Then $u \equiv 0$ in $\overline{\Omega}$.

Note that the assumption that u vanishes continuously in Σ implies that ∇u exists σ-a.e. as a nontangential limit in Σ, and moreover $\nabla u = (\partial_\nu u)\, \nu \in L^2_{\text{loc}}(\sigma|_\Sigma)$. Here σ stands for the $(n-1)$-dimensional surface measure and ν is the outer unit normal. See Appendix A for more details.

The preceding conjecture is an open problem that is already mentioned in Fang-Hua Lin's work [12]. It was later stated explicitly as a conjecture in the works by Adolfsson, Escauriaza, and Kenig [1, 2][1]. The conjecture is known to be true in the plane, and also in higher dimensions if one assumes the function u to be positive. In the first case, this can be deduced from the subharmonicity of $\log|\nabla u|$,

[1] For an accurate historical account, see the recent work [10].

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and in the second one it is possible to use standard techniques in connection with harmonic measure and the comparison principle.

In this paper I show that the conjecture is true for Lipschitz domains with small local Lipschitz constant. The precise result is the following:

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, let B be a ball centered in $\partial\Omega$, and suppose that $\Sigma = B \cap \partial\Omega$ is a Lipschitz graph with slope at most τ_0, where τ_0 is some positive small enough constant depending only on n. Let u be a function harmonic in Ω and continuous in $\bar{\Omega}$. Suppose that u vanishes in Σ and the normal derivative $\partial_\nu u$ vanishes in a subset of Σ with positive surface measure. Then $u \equiv 0$ in $\bar{\Omega}$.*

As an immediate corollary, it follows that the above conjecture holds for C^1 domains.

Note that, up now, the result stated in Theorem 1.1 (and in the conjecture) was only known in the case of Dini domains (i.e., Lipschitz domains whose outer normal is Dini continuous), by results of Adolfsson and Escauriaza [1] and Kukavica and Nyström [11], and also in the case of convex Lipschitz domains, by Adolfsson, Escauriaza, and Kenig [2]. Previously, the case of $C^{1,1}$ domains had been solved by F.-H. Lin [12]. See also [16] for a recent contribution in the particular case of convex domains where the recent geometric techniques introduced by Naber and Valtorta [17] are applied to study the strata of the set where $\partial_\nu u$ vanishes.

The proof of Theorem 1.1 is based on the study of the doubling properties of L^2 averages of the harmonic function u by means of the so-called Almgren's frequency function, which is analogous to the works mentioned in the previous paragraph. The strategy in this paper consists in studying the behavior of the frequency function at points in Ω approaching the boundary. This strategy is closer to the one of Kukavica and Nyström in [11] than to the one of Adolfsson and Escauriaza [1], which is based on the use of a clever change of variables that transforms the Laplace equation into an elliptic PDE in divergence form with nonconstant coefficients and improves the domain, in a sense.

The main novelty in the arguments to prove Theorem 1.1 is the application of some combinatorial techniques developed by Logunov and Malinnikova in the works [13–15] in connection with the nodal sets of harmonic functions and the Nadirashvili and Yau conjectures. In particular, one of the main technical results in this paper, the Key Lemma 3.1, uses some ideas inspired by [13] to bound the set where the frequency function is large. With the Key Lemma 3.1 in hand, in the last section of the paper a probabilistic argument is used to show that

$$\liminf_{r\to 0} \frac{\int_{\partial B(x,12r)} u^2\,d\sigma}{\int_{\partial B(x,r)} u^2\,d\sigma} < \infty$$

for almost all points $x \in \Sigma$. By a lemma due to Adolfsson and Escauriaza [1, lemma 0.2], this suffices to show that $\partial_\nu u$ cannot vanish in a subset of Σ with positive measure.

In the case that Ω is a Dini domain, in [1, 11] it is also proven that if u is harmonic in Ω and vanishes continuously in Σ (where Σ is as in Theorem 1.1), then $|\partial_\nu u|$ is a local B_2 weight in Σ with respect to surface measure (i.e., it satisfies a local reverse Hölder inequality with exponent 2). This follows from the local uniform bound of Almgren's frequency function proven in [1, 11], which in turn implies a local uniform doubling condition for L^2 averages of the function u on surface balls. Then, as shown in [2, theorem 1], this doubling condition ensures that $|\partial_\nu u|$ is a local B_2 weight. In the case of Lipschitz domains with small constant, the proof of Theorem 1.1 in this paper does not ensure that the frequency function is locally uniformly bounded (or even pointwise bounded!) in Σ, and thus one cannot deduce that $|\partial_\nu u|$ is a local B_2 weight.

In a similar vein, under the Dini assumption, in [1] it is shown that the dimension of the set where $\partial_\nu u$ vanishes in Σ has dimension at most $n-2$. This follows by arguments developed previously in [12] in the case of $C^{1,1}$ domains, which are based on the monotonicity of the frequency function in Σ. For C^1 domains or Lipschitz domains with small constant, one cannot derive any bound on the Hausdorff dimension smaller than $n-1$ from the arguments in this paper, as far as I know.

Finally, it is worth mentioning a corollary about harmonic measure that follows easily from Theorem 1.1:

COROLLARY 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, let B be a ball centered in $\partial\Omega$, and suppose that $\Sigma = B \cap \partial\Omega$ is a Lipschitz graph with small enough slope. Let ω^p, ω^q be the harmonic measures for Ω with respective poles in $p, q \in \Omega$. Suppose that there exists some subset $E \subset \Sigma$ with positive harmonic measure such that*

$$\omega^p|_E = \omega^q|_E$$

Then $p = q$.

Note that saying that $\omega^p|_E = \omega^q|_E$ is the same as saying that

$$\omega^p(F) = \omega^q(F) \quad \text{for all Borel sets } F \subset E.$$

The corollary follows by applying the theorem to

$$u = g(\cdot, p) - g(\cdot, q) \quad \text{in } \Omega \setminus (\bar{B}(p,\varepsilon) \cup \bar{B}(q,\varepsilon)),$$

where $g(\cdot,\cdot)$ is the Green function of Ω and $\varepsilon > 0$ is small enough so that $B(p, 2\varepsilon) \cup B(q, 2\varepsilon) \subset \Omega$. Observe that u is harmonic in the Lipschitz domain $\Omega_\varepsilon := \Omega \setminus (\bar{B}(p,\varepsilon) \cup \bar{B}(q,\varepsilon))$, it is continuous in $\overline{\Omega_\varepsilon}$, and it vanishes identically in $\partial\Omega \subset \partial\Omega_\varepsilon$. Further, by Dahlberg's classical theorem [6], it follows that the harmonic measures ω^p and ω^q are mutually absolutely continuous with the surface measure σ on $\partial\Omega$, with

$$\frac{d\omega^p}{d\sigma} = -\partial_\nu g(\cdot, p), \quad \frac{d\omega^q}{d\sigma} = -\partial_\nu g(\cdot, q),$$

so that $\partial_\nu u$ vanishes in the whole E. So Ω_ε and u satisfy the assumptions of Theorem 1.1, and thus $u \equiv 0$ in Ω_ε. This implies that $p = q$. Otherwise, letting $\varepsilon \to 0$ we infer that $u(x) \to \infty$ as $x \to p$.

2 The Frequency Function

As usual in harmonic analysis, in the whole paper, the letters C, c are used to denote positive constants that just depend on the dimension n and whose values may change at different occurrences. On the other hand, constants with subscripts, such as C_0, retain their values in different occurrences. The notation $A \lesssim B$ is equivalent to $A \leq C\,B$, and $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

In the whole paper, unless otherwise stated, we assume that Ω and Σ are as in Theorem 1.1. We consider a function u harmonic in Ω and continuous in $\bar{\Omega}$ which vanishes in Σ, and we assume that u is not constant in Ω. We extend u by 0 out of $\bar{\Omega}$, so that u is continuous across Σ. Without loss of generality we assume that Σ is a Lipschitz graph with respect to the horizontal axes and that $\Omega \cap B$ lies above $\Sigma \cap B$. For $0 < \varepsilon \leq \frac{1}{2} r(B)$, we denote $\Sigma_\varepsilon = \Sigma + \varepsilon e_n$ and $\Omega_\varepsilon = \Omega + \varepsilon\, e_n$, where $e_n = (0, \ldots, 0, 1)$.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote

$$h(x,r) = \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} u^2 \, d\sigma.$$

For a ball $B(x,r)$ that intersects Ω, the Almgren frequency function (or just frequency function) associated with u is defined by

$$F(x,r) = r \partial_r \log h(x,r).$$

LEMMA 2.1. *Let $x \in \mathbb{R}^n$ and let $I \subset (0,\infty)$ be a closed bounded interval. Suppose that $B(x,r) \cap \Omega \neq \varnothing$ and $\overline{B(x,r)} \subset 2B$ for all $r \in I$ (where B is as Theorem 1.1). Then $h(x,\cdot)$ is of class C^1 in I and*

$$\begin{aligned}
\partial_r h(x,r) &= \frac{2}{\sigma(\partial B_r)} \int_{\partial B(x,r)} u(y) \nabla u(y) \cdot \frac{y-x}{r} \, d\sigma(y) \\
&= \frac{2}{\sigma(\partial B_r)} \int_{\partial B(x,r) \cap \Omega} u \partial_\nu u \, d\sigma \qquad (2.1) \\
&= \frac{2}{\sigma(\partial B_r)} \int_{B(x,r) \cap \Omega} |\nabla u|^2 \, dy \quad \text{for a.e. } r \in I.
\end{aligned}$$

Further, the identity $\partial_r h(x,r) = \frac{2}{\sigma(\partial B_r)} \int_{B(x,r)\cap\Omega} |\nabla u|^2 \, dy$ holds for all $r \in I$. Also,

$$F(x,r) = r \, \frac{\partial_r h(x,r)}{h(x,r)} = \frac{2r \int_{B(x,r)} |\nabla u|^2 \, dy}{\int_{\partial B(x,r)} u^2 \, d\sigma} \quad \text{for all } r \in I. \qquad (2.2)$$

Note that, for $B(x,r)$ as in the lemma, we have

$$\int_{B(x,r)\cap\Omega} |\nabla u|^2\,dy < \infty.$$

Indeed, write $u = u^+ - u^-$ and, for any $\varepsilon > 0$, let $u_\varepsilon^+ = \max(u^+,\varepsilon) - \varepsilon$, $u_\varepsilon^- = \max(u^-,\varepsilon) - \varepsilon$, and $v_\varepsilon = u_\varepsilon^+ - u_\varepsilon^-$. It is immediate to check that u_ε^+ and u_ε^- belong to $W^{1,2}(B(x,r'))$ for some $r' > r$, and moreover they are subharmonic in $B(x,r')$. As a consequence, by Caccioppoli's inequality,

$$\int_{B(x,r)} |\nabla u_\varepsilon^\pm|^2\,dy \lesssim C(r,r') \int_{B(x,r')} |u_\varepsilon^\pm|^2\,dy \le C(r,r') \int_{B(x,r')} |u|^2\,dy.$$

Letting $\varepsilon \to 0$, we deduce that

$$\int |\nabla u|^2\,dy < \infty.$$

Further, it follows easily that $v_\varepsilon \to u$ in $W^{1,2}(B(x,r))$, and so $u \in W^{1,2}(B(x,r))$ too.

An immediate corollary of the lemma and, in particular, of the third identity in (2.1), is that $\partial_r h(x,r) \ge 0$ and thus $h(x,r)$ is nondecreasing with respect to r.

PROOF OF LEMMA 2.1. The calculations in the lemma are quite straightforward and well-known in the case when u is sufficiently smooth up to the boundary. In the general case when we only assume u to be continuous up to the boundary, we have to be a little more careful, and so we will show here all the details.

Notice first that the second identity in (2.1) is immediate. Concerning the third one, for a.e. $r \in I$ we have $\int_{\partial B(x,r)} |\nabla u|^2\,d\sigma < \infty$, and so

$$\frac{2}{\sigma(\partial B_r)} \int_{\partial B(x,r)\cap\Omega} u\,\partial_\nu u\,d\sigma = \lim_{\varepsilon\to 0} \frac{2}{\sigma(\partial B_r)} \int_{\partial(B(x,r)\cap\Omega_\varepsilon)} u\,\partial_\nu u\,d\sigma$$

because

$$\lim_{\varepsilon\to 0} \frac{2}{\sigma(\partial B_r)} \int_{B(x,r)\cap\partial\Omega_\varepsilon} u\,\partial_\nu u\,d\sigma = 0.$$

This follows easily from the fact that u vanishes continuously in Σ, while

$$\lim_{\varepsilon\to 0} \nabla u(x+\varepsilon e_n) \to \nabla u(x) \quad \text{in } L^2_{\text{loc}}(\sigma|_\Sigma) \tag{2.3}$$

with $\nabla u = (\partial_\nu u)\nu \in L^2_{\text{loc}}(\sigma|_\Sigma)$ defined as a nontangential limit, as shown in Theorem A.1. Then, by Green's theorem, using that u is C^∞ in a neighborhood of

$\Omega_\varepsilon \cap B(x,r)$ for any $\varepsilon > 0$ sufficiently small and for a.e. $r \in I$, we obtain

$$2\int_{\partial(B(x,r)\cap\Omega_\varepsilon)} u\,\partial_\nu u\,d\sigma = \frac{1}{\sigma(\partial B_r)}\int_{\partial(B(x,r)\cap\Omega_\varepsilon)} \partial_\nu(u^2)d\sigma = \frac{1}{\sigma(\partial B_r)}\int_{B(x,r)\cap\Omega_\varepsilon} \Delta(u^2)dy = \frac{2}{\sigma(\partial B_r)}\int_{B(x,r)\cap\Omega_\varepsilon} |\nabla u|^2\,dy.$$

So letting $\varepsilon \to 0$, taking into account that $u \in W^{1,2}_{\rm loc}(B)$, the third identity in (2.1) follows.

To show the first identity in (2.1), observe that, for all $[a,b] \subset I$, writing $\sigma(\partial B_r) = c_n\, r^{n-1}$, we have

$$\begin{aligned}
&\int_a^b \frac{2}{\sigma(\partial B_r)}\int_{\partial B(x,r)} u(y)\nabla u(y)\cdot\frac{y-x}{r}\,d\sigma(y)dr\\
&= 2c_n^{-1}\int_a^b\int_{\partial B(x,r)} u(y)\,\nabla u(y)\cdot\frac{y-x}{|y-x|^n}\,d\sigma(y)dr\\
&= 2c_n^{-1}\int_{A(x,a,b)\cap\Omega} u(y)\,\nabla u(y)\cdot\frac{y-x}{|y-x|^n}\,dy\\
&= c_n^{-1}\int_{A(x,a,b)\cap\Omega} {\rm div}_y\left(u(y)^2\frac{y-x}{|y-x|^n}\right)dy,
\end{aligned}$$

where $A(x,a,b)$ stands for the open annulus centered at x with inner radius a and outer radius b . Since $u \in W^{1,2}_{\rm loc}(B)$ and is smooth in a neighborhood of $B(x,b) \cap \Omega_\varepsilon$, by the divergence theorem, we have

$$\begin{aligned}
&c_n^{-1}\int_{A(x,a,b)\cap\Omega} {\rm div}_y\left(u(y)^2\frac{y-x}{|y-x|^n}\right)dy\\
&= c_n^{-1}\lim_{\varepsilon\to0}\int_{A(x,a,b)\cap\Omega_\varepsilon} {\rm div}_y\left(u(y)^2\frac{y-x}{|y-x|^n}\right)dy\\
&= \lim_{\varepsilon\to0}\left(\frac{1}{\sigma(\partial B_b)}\int_{\partial B(x,b)\cap\Omega_\varepsilon} u^2\,d\sigma - \frac{1}{\sigma(\partial B_a)}\int_{\partial B(x,a)\cap\Omega_\varepsilon} u^2\,d\sigma\right)\\
&\quad + \lim_{\varepsilon\to0} c_n^{-1}\int_{A(x,a,b)\cap\partial\Omega_\varepsilon} u(y)^2\,\frac{\nu(y)\cdot(y-x)}{|y-x|^n}\,d\sigma(y).
\end{aligned}$$

Since u vanishes continuously up to the boundary Σ, the last limit on the right-hand side above vanishes. Therefore,

$$\int_a^b \frac{2}{\sigma(\partial B_r)}\int_{\partial B(x,r)\cap\Omega} u(y)\,\nabla u(y)\cdot\frac{y-x}{r}\,d\sigma(y)dr = h(x,b)-h(x,a). \tag{2.4}$$

On the other hand, we have already shown that, for a.e. $r \in I$,

$$\frac{2}{\sigma(\partial B_r)} \int_{\partial B(x,r)\cap\Omega} u(y)\nabla u(y) \cdot \frac{y-x}{r}\, d\sigma(y) = \frac{2}{\sigma(\partial B_r)} \int_{B(x,r)\cap\Omega_\varepsilon} |\nabla u|^2\, dy,$$

and so this term is continuous in r, as $u \in W^{1,2}_{\mathrm{loc}}(B)$. Then, the first identity in (2.1) follows from (2.4) and the fundamental theorem of calculus.

The identity (2.2) is an immediate consequence of the definition of $F(x,r)$ and (2.1). $\square$

The following lemma is already known. It is essentially contained (but not stated in this way) in [2]. For the reader's convenience we include the detailed proof here.

LEMMA 2.2. *Let $x \in \mathbb{R}^n$ and let $I \subset (0,\infty)$ be a closed bounded interval. Suppose that $B(x,r)\cap\Omega \neq \varnothing$ and $\overline{B(x,r)} \subset 2B$ for all $r \in I$, where B is as in Theorem* 1.1. *Then $F(x,\cdot)$ is absolutely continuous in I and, for a.e. $r \in I$,*

$$\begin{aligned}
&\partial_r F(x,r) \\
&\quad= \frac{4r}{\mathcal{H}(x,r)^2}\Bigg(\int_{\partial B(x,r)\cap\Omega} |u|^2\, d\sigma \int_{\partial B(x,r)\cap\Omega} \left|\partial_\nu u\right|^2 d\sigma \\
&\qquad\qquad - \left(\int_{\partial B(x,r)} u\, \partial_\nu u\, d\sigma \right)^2 \Bigg) \\
&\qquad + \frac{2}{\mathcal{H}(x,r)} \int_{B(x,r)\cap\partial\Omega} (y-x)\cdot\nu(y) \left|\partial_\nu u(y)\right|^2 d\sigma(y),
\end{aligned} \tag{2.5}$$

where

$$\mathcal{H}(x,r) = \sigma(\partial B_r)\, h(x,r) = \int_{\partial B(x,r)} u^2\, d\sigma.$$

In particular, if $(y-x)\cdot\nu(y) \geq 0$ for σ-a.e. $y \in B(x,r)\cap\partial\Omega$, then $\partial_r F(x,r) \geq 0$.

PROOF. Denote

$$\mathcal{I}(x,r) = \int_{B(x,r)} |\nabla u|^2\, dy.$$

Since

$$\mathcal{I}(x,r) = \int_0^r \int_{\partial B(x,r)} |\nabla u|^2\, d\sigma(y) dt,$$

it follows that $\mathcal{I}(x,\cdot)$ is absolutely continuous with respect to r. As $\mathcal{H}(x,\cdot)$ is of class C^1 and bounded away from 0 in I, we deduce that $F(x,\cdot)$ is also absolutely continuous in I.

By (2.1), we have

$$\begin{aligned}
\partial_r F(x,r) &= \partial_r \frac{2r\,\mathcal{I}(x,r)}{\mathcal{H}(x,r)} \\
&= 2\,\frac{(\mathcal{I}(x,r) + r\,\mathcal{I}'(x,r))\,\mathcal{H}(x,r) - r\,\mathcal{I}(x,r)\,\mathcal{H}'(x,r)}{\mathcal{H}(x,r)^2},
\end{aligned}$$

where the symbol $'$ denotes the derivative with respect to r. Observe that, by (2.1), for a.e. $r \in I$,

$$\begin{aligned}\mathcal{H}'(x,r) &= \partial_r \sigma(\partial B_r)\, h(x,r) + \sigma(\partial B_r)\, h'(x,r)\\ &= \frac{(n-1)\sigma(\partial B_r)}{r}\, h(x,r) + 2\,\mathcal{I}(x,r) = \frac{(n-1)}{r}\,\mathcal{H}(x,r) + 2\,\mathcal{I}(x,r).\end{aligned}$$

Therefore,

$$\begin{aligned}\partial_r F(x,r) = \frac{2}{\mathcal{H}(x,r)^2}\big(& r\,\mathcal{H}(x,r)\,\mathcal{I}'(x,r)\\ &- (n-2)\mathcal{H}(x,r)\,\mathcal{I}(x,r) - 2r\,\mathcal{I}(x,r)^2\big).\end{aligned} \tag{2.6}$$

To calculate $\mathcal{I}'(x,r)$ we take into account that

$$\begin{aligned}\mathcal{I}'(x,r) &= \int_{\partial B(x,r)} |\nabla u|^2\, d\sigma\\ &= \int_{\partial(B(x,r)\cap\Omega)} \frac{y-x}{r}\cdot \nu(y)\,|\nabla u(y)|^2\, d\sigma(y)\\ &\quad - \int_{B(x,r)\cap\partial\Omega} \frac{y-x}{r}\cdot \nu(y)|\nabla u(y)|^2\, d\sigma(y).\end{aligned}$$

By the Rellich-Necas identity with vector field $\beta(y) = y - x$, $y \in \Omega$, we have

$$\operatorname{div}(\beta\,|\nabla u|^2) = 2\operatorname{div}((\beta\cdot\nabla u)\,\nabla u) + (n-2)\,|\nabla u|^2 \quad \text{in } \Omega.$$

Integrating in $B(x,r)\cap\Omega_\varepsilon$ (with Ω_ε as in the proof of Lemma 2.1), applying the divergence theorem in this domain, and then letting $\varepsilon \to 0$, taking into account (2.3), we derive

$$\begin{aligned}&\int_{\partial(B(x,r)\cap\Omega)} (y-x)\cdot\nu(y)\,|\nabla u(y)|^2\, d\sigma(y)\\ &\quad = 2\int_{\partial(B(x,r)\cap\Omega)} (y-x)\cdot\nabla u(y)\,\partial_\nu u(y) d\sigma(y) + (n-2)\,\mathcal{I}(x,r)\\ &\quad = 2r\int_{\partial B(x,r)\cap\Omega} |\partial_\nu u|^2\, d\sigma + 2\int_{B(x,r)\cap\partial\Omega} (y-x)\cdot\nu(y)\,|\partial_\nu u(y)|^2\, d\sigma(y)\\ &\qquad + (n-2)\,\mathcal{I}(x,r).\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{I}'(x,r) = 2&\int_{\partial B(x,r)\cap\Omega} \left|\partial_\nu u\right|^2 d\sigma + \frac{n-2}{r}\,\mathcal{I}(x,r)\\ &+ \frac{1}{r}\int_{B(x,r)\cap\partial\Omega} (y-x)\cdot\nu(y)\left|\partial_\nu u(y)\right|^2 d\sigma(y).\end{aligned}$$

Plugging the last calculation for $\mathcal{I}'(x,r)$ into (2.6), we obtain

$$\begin{aligned}\partial_r F(x,r)\\ &= \frac{4r}{\mathcal{H}(x,r)^2}\Bigg(\mathcal{H}(x,r)\int_{\partial B(x,r)\cap\Omega}\left|\partial_\nu u\right|^2 d\sigma\\ &\qquad + \frac{\mathcal{H}(x,r)}{2r}\int_{B(x,r)\cap\partial\Omega}(y-x)\cdot\nu(y)\left|\partial_\nu u(y)\right|^2 d\sigma(y)\\ &\qquad - \mathcal{I}(x,r)^2\Bigg).\end{aligned} \tag{2.7}$$

Observe now that $\mathcal{I}(x,r)$ can be written in the following way:

$$\mathcal{I}(x,r) = \frac{1}{2}\int_{B(x,r)\cap\Omega}\Delta(u^2)dy = \frac{1}{2}\int_{\partial(B(x,r)\cap\Omega)}\partial_\nu(u^2)d\sigma = \int_{\partial B(x,r)} u\,\partial_\nu u\,d\sigma.$$

Plugging the last identity into (2.7), we get (2.5).

The last assertion in the lemma follows from the fact that, by Cauchy-Schwarz,

$$\int_{\partial B(x,r)\cap\Omega}|u|^2\,d\sigma\int_{\partial B(x,r)\cap\Omega}\left|\partial_\nu u\right|^2 d\sigma - \left(\int_{\partial B(x,r)} u\,\partial_\nu u\,d\sigma\right)^2 \geq 0,$$

and from the condition that $(y-x)\cdot\nu(y)\geq 0$ for σ-a.e. $y\in B(x,r)\cap\partial\Omega$, which implies that

$$\int_{B(x,r)\cap\partial\Omega}(y-x)\cdot\nu(y)\left|\partial_\nu u(y)\right|^2 d\sigma(y)\geq 0. \qquad \square$$

It is immediate to check that saying that $\partial_r F(x,r)\geq 0$ a.e. in an interval is equivalent to saying that the function

$$f(t) = \log h\big(x,e^t\big)$$

is convex in $t=\log r$ for r in that interval, i.e., $f''(\log r)\geq 0$ a.e. in the interval.

LEMMA 2.3. *Given $x\in\mathbb{R}^n$, let $I\subset(0,\infty)$ be an interval such that $h(x,r)>0$ and $\partial_r F(x,r)\geq 0$ for a.e. $r\in I$. Given $a>1$, if both $r, ar\in I$, then*

$$F(x,r)\leq \log_a\frac{h(x,ar)}{h(x,r)}\leq F(x,ar). \tag{2.8}$$

Another way of writing the preceding estimate is the following: for $R=ar$

$$h(x,r)\left(\frac{R}{r}\right)^{F(x,r)}\leq h(x,R)\leq h(x,r)\left(\frac{R}{r}\right)^{F(x,R)}. \tag{2.9}$$

PROOF. By the convexity of the function f defined above, we have

$$f'(\log r)\leq\frac{f(\log ar)-f(\log r)}{\log ar-\log r}\leq f'(\log ar).$$

It is immediate to check that this is equivalent to (2.8). $\square$

From now on, we say that an interval $I \subset (0,\infty)$ is admissible for $x \in \mathbb{R}^n$ if $h(x,r) > 0$ and $\partial_r F(x,r) \geq 0$ for a.e. $r \in I$.

LEMMA 2.4. *Let $x, y \in \mathbb{R}^n$ and $r > 0$, $\gamma \in (0, 1/10)$, such that $|x - y| \leq \gamma r$. Let I be an open interval admissible for x and y such that both r, $2(1 + \gamma^{1/2})r \in I$. Suppose that $B(x, 5r) \cap \partial\Omega \subset \Sigma$. Then*

$$F(y,r) \leq (1 + C\gamma^{1/2})\, F\big(x, 2(1+\gamma^{1/2})r\big) + C\gamma^{1/2}, \tag{2.10}$$

for some absolute constant $C > 0$.

PROOF. Let x, y, r, γ be as above and let $\delta \in (0,1)$ to be chosen below. Since $h(y,\cdot)$ is nondecreasing, we deduce that

$$h(y,r) = \fint_{\partial B(y,r)} u^2\, d\sigma \leq \fint_{A(y,r,(1+\delta)r)} u^2\, dm.$$

Analogously,

$$h(y,r) \geq \fint_{A(y,(1-\delta)r,r)} u^2\, dm.$$

The same estimates are valid interchanging y with x and/or r with $2r$. Then, by (2.8), we have

$$F(y,r) \leq \log_2 \frac{h(y,2r)}{h(y,r)} \leq \log_2 \frac{\fint_{A(y,2r,(2+\delta)r)} u^2\, dm}{\fint_{A(y,(1-\delta)r,r)} u^2\, dm}.$$

Observe now that

$$A_y^2 := A(y, 2r, (2+\delta)r) \subset A(x, (2-\gamma)r, (2+\delta+\gamma)r) =: A_x^2,$$

and

$$A_y^1 := A(y, (1-\delta)r, r) \supset A(x, (1-\delta+\gamma)r, (1-\gamma)r) =: A_x^1.$$

Thus,

$$\begin{aligned} F(y,r) &\leq \log_2 \frac{\fint_{A_y^2} u^2\, dm}{\fint_{A_y^1} u^2\, dm} \leq \log_2 \left(\frac{\fint_{A_x^2} u^2\, dm}{\fint_{A_x^1} u^2\, dm} \cdot \frac{m(A_x^2)\, m(A_y^1)}{m(A_x^1)\, m(A_y^2)} \right) \\ &\leq \log_2 \frac{\fint_{\partial B(x,(2+\delta+\gamma)r)} u^2\, dm}{\fint_{\partial B(x,(1-\delta+\gamma)r)} u^2\, dm} + C_{\delta,\gamma}, \end{aligned}$$

where we denoted

$$C_{\delta,\gamma} = \log_2 \frac{m(A_x^2)\, m(A_y^1)}{m(A_x^1)\, m(A_y^2)}.$$

We choose $\delta = \gamma^{1/2}$. Using just that $\gamma \le \delta$, we get

$$F(y,r) \le \log_2 \frac{\fint_{\partial B(x,(2+2\delta)r)} u^2\,dm}{\fint_{\partial B(x,(1-\delta)r)} u^2\,dm} + C_{\delta,\gamma}$$
$$= \frac{\log \frac{2+2\delta}{1-\delta}}{\log 2} \log_{\frac{2+2\delta}{1-\delta}} \frac{\fint_{\partial B(x,(2+2\delta)r)} u^2\,dm}{\fint_{\partial B(x,(1-\delta)r)} u^2\,dm} + C_{\delta,\gamma}.$$

Since $(2+2\delta)r \in I$ (by assumption), by Lemma 2.3 we have

$$\log_{\frac{2+2\delta}{1-\delta}} \frac{\fint_{\partial B(x,(2+2\delta)r)} u^2\,dm}{\fint_{\partial B(x,(1-\delta)r)} u^2\,dm} \le F(x,(2+2\delta)r).$$

It is also immediate to check that

$$\frac{\log \frac{2+2\delta}{1-\delta}}{\log 2} \le 1 + C\delta = 1 + C\gamma^{1/2}.$$

Hence,

$$F(y,r) \le (1 + C\gamma^{1/2}) F(x,(2+2\gamma^{1/2})r) + C_{\delta,\gamma}.$$

It only remains to show that $C_{\delta,\gamma} \le C\gamma^{1/2}$. To this end, observe that

$$\frac{m(A_x^2)}{m(A_y^2)} = \frac{(2+\delta+\gamma)^n - (2-\gamma)^n}{(2+\delta)^n - 2^n}$$
$$= \frac{(2+\delta)^n + n(2+\delta)^{n-1}\gamma + O(\gamma^2) - 2^n + n2^{n-1}\gamma + O(\gamma^2)}{(2+\delta)^n - 2^n}$$
$$= 1 + \frac{n(2+\delta)^{n-1}\gamma + n2^{n-1}\gamma + O(\gamma^2)}{n2^{n-1}\delta + O(\delta^2)}.$$

It follows that

$$\left| \frac{m(A_x^2)}{m(A_y^2)} - 1 \right| \le C\frac{\gamma}{\delta} = C\gamma^{1/2}.$$

Almost the same arguments show that

$$\left| \frac{m(A_y^1)}{m(A_x^1)} - 1 \right| \le C\frac{\gamma}{\delta} = C\gamma^{1/2}.$$

Therefore,

$$C_{\delta,\gamma} = \log_2 \frac{m(A_x^2)}{m(A_y^2)} + \log_2 \frac{m(A_y^1)}{m(A_x^1)} \lesssim \gamma^{1/2},$$

as wished. □

3 The Key Lemma

To prove Theorem 1.1 we consider an arbitrary ball B_0 centered in Σ such that $M^2 B_0 \subset B$, where B is the ball in Theorem 1.1 and $M \gg 1$ will be fixed below. We denote $\Sigma_0 = \partial\Omega \cap MB_0$. We will show that if u is a non-zero (i.e., not identically zero) function harmonic in Ω and continuous in $\overline{\Omega}$ which vanishes in Σ, then the normal derivative $\partial_\nu u$ cannot vanish in a subset of $\Sigma_0 \cap B_0$ with positive surface measure. Clearly, this suffices to prove Theorem 1.1.

Let H_0 be the horizontal hyperplane through the origin. By the hypotheses in the theorem, we can assume that $\partial\Omega \cap MB_0$ is a Lipschitz graph with respect to the hyperplane H_0 with slope at most $\tau_0 \ll 1$, and that $\Omega \cap MB_0$ is above the graph. We consider the following Whitney decomposition of Ω: we have a family $\mathcal{W}$ of dyadic cubes in $\mathbb{R}^n$ with disjoint interiors such that

$$\bigcup_{Q \in \mathcal{W}} Q = \Omega,$$

and moreover there are some constants $\Lambda > 20$ and $D_0 \geq 1$ such the following holds for every $Q \in \mathcal{W}$:

(i) $10Q \subset \Omega$;
(ii) $\Lambda Q \cap \partial\Omega \neq \varnothing$;
(iii) there are at most D_0 cubes $Q' \in \mathcal{W}$ such that $10Q \cap 10Q' \neq \varnothing$. Further, for such cubes Q', we have $\frac{1}{2}\ell(Q') \leq \ell(Q) \leq 2\ell(Q')$.

Above, we denote by $\ell(Q)$ the side length of Q. From the properties (i) and (ii) it is clear that $\operatorname{dist}(Q, \partial\Omega) \approx \ell(Q)$. We assume that the Whitney cubes are small enough so that

$$\operatorname{diam}(Q) < \frac{1}{20} \operatorname{dist}(Q, \partial\Omega). \tag{3.1}$$

The arguments to construct a Whitney decomposition satisfying the properties above are standard but we include the detailed arguments in Lemma B.1 below for the convenience of the reader.

Let Π denote the orthogonal projection on H_0. By translating the usual dyadic lattice if necessary, we can assume that there exists some cube $R_0 \in \mathcal{W}$ such that $\Pi(B_0) \subset \Pi(R_0)$ and $\ell(R_0) \leq C\, r(B_0)$ and moreover $R_0 \subset \frac{M}{2} B_0$, for M big enough.

Next we need to define some "generations" of cubes in $\mathcal{W}$. We let $\mathcal{D}_{\mathcal{W}}^0(R_0) = \{R_0\}$. For $k \geq 1$ we define $\mathcal{D}_{\mathcal{W}}^k(R_0)$ as follows. Let

$$J(R_0) = \{\Pi(Q) : Q \in \mathcal{W} \text{ such that } \Pi(Q) \subset \Pi(R_0) \text{ and } Q \text{ is below } R_0\}. \tag{3.2}$$

Observe that $J(R_0)$ is a family of $(n-1)$-dimensional dyadic cubes in H_0, all of them contained in $\Pi(R_0)$. Let $J_k(R_0) \subset J(R_0)$ be the subfamily of $(n-1)$-dimensional dyadic cubes in H_0 with side length equal to $2^{-k}\ell(R_0)$. To each $Q' \in J_k(R_0)$ we assign some $Q \in \mathcal{W}$ such that $\Pi(Q) = Q'$, $\Pi(Q) \subset \Pi(R_0)$, and such that Q is below R_0 (see Lemma B.2 for more details), and we write

$s(Q') = Q$. Notice there may be more than one possible choice for Q. However, the choice is irrelevant. Anyway, for definiteness we take the cube Q that is closest to R_0 among all the possible choices. Then we define

$$\mathcal{D}_{\mathcal{W}}^k(R_0) = \{s(Q') : Q' \in J_k(R_0)\}.$$

Next we let

$$\mathcal{D}_{\mathcal{W}}(R_0) = \bigcup_{k\geq 0} \mathcal{D}_{\mathcal{W}}^k(R_0)\}.$$

Notice that, for each k, the family $\{\Pi(Q) : Q \in \mathcal{D}_{\mathcal{W}}^k(R_0)\}$ is a partition of $\Pi(R_0)$. Finally, for each $R \in \mathcal{D}_{\mathcal{W}}^k(R_0)$ and $j \geq 1$ we denote

$$\mathcal{D}_{\mathcal{W}}^j(R) = \{Q \in \mathcal{D}_{\mathcal{W}}^{k+j}(R_0) : \Pi(Q) \subset \Pi(R)\}.$$

By the properties of the Whitney cubes, it is easy to check that

$$Q \in \mathcal{D}_{\mathcal{W}}(R_0) \quad \Rightarrow \quad \operatorname{dist}(Q, \Sigma_0) = \operatorname{dist}(Q, \partial\Omega) \approx \ell(Q).$$

From now on, for any cube Q, we denote by x_Q its center. Further, we denote by m_{n-1} the $(n-1)$-dimensional Lebesgue measure on the hyperplane H_0.

KEY LEMMA 3.1. *Under the assumptions of Theorem* 1.1*, let* R_0 *be as above and let* $N_0 > 1$ *be big enough. There exists some absolute constant* $\delta_0 > 0$ *such that for all* $A \gg 1$ *big enough the following holds, assuming also* τ_0 *small enough and* M *big enough. Let* $R \in \mathcal{D}_{\mathcal{W}}(R_0)$ *satisfy* $F(x_R, A\,\ell(R)) \geq N_0$*. There exists some positive integer* $K = K(A)$ *big enough such that if we let*

$$\mathcal{G}_K(R) = \left\{Q \in \mathcal{D}_{\mathcal{W}}^K(R) : F(x_Q, A\ell(Q)) \leq \tfrac{1}{2}\, F(x_R, A\,\ell(R))\right\},$$

then:

(a) $m_{n-1}\Big(\bigcup_{Q\in\mathcal{G}_K(R)} \Pi(Q)\Big) \geq \delta_0\, m_{n-1}(\Pi(R))$.

(b) *For all* $Q \in \mathcal{D}_{\mathcal{W}}^K(R)$*, it holds*

$$F(x_Q, A\ell(Q)) \leq (1 + CA^{-1/2})\, F(x_R, A\ell(R)).$$

A key point in the lemma is that δ_0 does not depend on A. On the other hand, τ_0, M, and K depend on A. The constant N_0 is also an absolute constant independent of the other parameters.

The general strategy for the proof of the Key Lemma is similar to one of the Hyperplane Lemma 4.1 from [13]. The main differences stem from the fact that in the lemma above we wish to estimate the frequency function in points that are close to $\partial\Omega$, and then we have to be more careful and more precise with the monotonicity properties of the frequency function.

A basic tool for the proof of the Key Lemma 3.1 is the following result on quantitative Cauchy uniqueness:

THEOREM 3.2. *Let v be a function harmonic in the half ball $B_1^+ = \{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$ and C^1 smooth up to the boundary. Let Γ the following subset of ∂B_1^+:*

$$\Gamma = \{x \in \mathbb{R}^n : |x| < 3/4, x_n = 0\}.$$

Suppose that

$$\int_{B_1^+} |v|^2 \, dm \leq 1$$

and

$$\sup_\Gamma |v| + \sup_\Gamma |\nabla v| \leq \varepsilon,$$

for some $\varepsilon \in (0,1)$. Then

$$\sup_{B(1/2,1/4)} |v| \leq C\varepsilon^\alpha,$$

where C, α are positive absolute constants.

This result appears in [12, Lemma 4.3] and it is proven in much greater generality in [3, Theorem 1.7].

Remark 3.3. We claim that, for $T > 0$ and $R \in \mathcal{D}_{\mathcal{W}}(R_0)$, if $x \in \Omega$ satisfies

$$\operatorname{dist}(x, R) \leq T\,\ell(R) \quad \text{and} \quad \operatorname{dist}(x, \partial\Omega) \geq T^{-1}\,\ell(R),$$

then the interval $(0, A\ell(R))$ is admissible for x, assuming $M \gg T, A$ and that τ_0 is small enough, depending on T and A. This property will be essential for the proof of the Key Lemma.

To prove the claim it suffices to check that, in the situation above,

$$(y - x) \cdot \nu(y) \geq 0 \quad \sigma\text{-a.e. } y \in B(x,r) \cap \partial\Omega,\ 0 < r \leq A\ell(R), \tag{3.3}$$

since then Lemma 2.2 ensures that $\partial_r F(x,r) \geq 0$ for $0 < r \leq A\ell(R)$. To prove (3.3), let $x' \in \Sigma$ be the point such that $\Pi(x) = \Pi(x')$, so that $x - x'$ is orthogonal to H_0. Denote by $H_{x'}$ the hyperplane parallel to H_0 through x'. Given y as in (3.3), let y' be the orthogonal projection of y on $H_{x'}$; see Figure 3.1. So $y' - x'$ is orthogonal to $x - x'$. Moreover, since the slope of Σ is at most τ_0,

$$|y - y'| \leq \tau_0\,|y' - x'| \leq \tau_0\,|y - x| \quad \text{and} \quad |\nu(y) - (-e_n)| \leq \tau_0.$$

Then we write

$$\begin{aligned}
(y - x) \cdot \nu(y) &= \big[(y - y') + (y' - x') + (x' - x)\big] \cdot \big[-e_n + (e_n + \nu(y))\big] \\
&\geq |x - x'| - |x' - x|\,|e_n + \nu(y)| - |y - y'| - |y' - x'|\,|e_n + \nu(y)| \\
&\geq |x' - x| - \tau_0\,|x - x'| - \tau_0\,|y - x| - \tau_0\,|y - x| \\
&= (1 - \tau_0)\,|x' - x| - 2\tau_0\,|y - x|.
\end{aligned}$$

Using now that, for $\tau_0 \leq 1/2$, we have $|x' - x| \approx \operatorname{dist}(x, \partial\Omega)$, we get

$$(y - x) \cdot \nu(y) \geq c\ \operatorname{dist}(x, \partial\Omega) - 2\tau_0\,|y - x| \geq c\,T^{-1}\,\ell(R) - 2\,\tau_0\,A\,\ell(R) \geq 0,$$

assuming $\tau_0 \leq \frac{c}{2}\,(AT)^{-1}$, which proves (3.3).

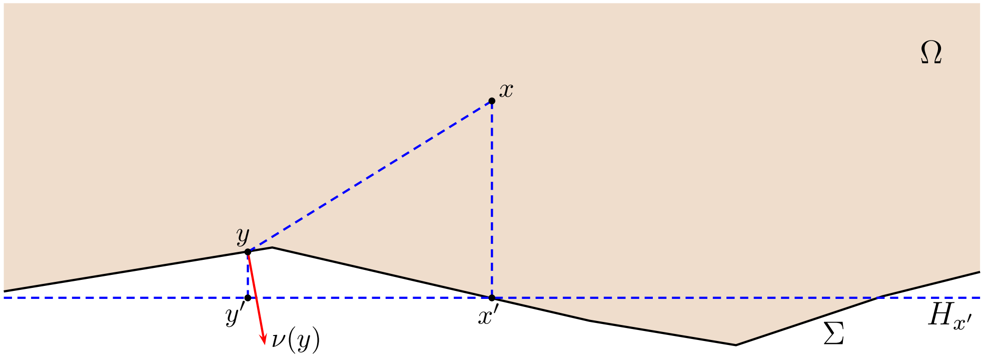

FIGURE 3.1. The domain Ω and the Lipschitz graph Σ. The points x, y satisfy $(y-x)\cdot\nu(y) \geq 0$.

Proof of the Key Lemma 3.1. For any $Q \in \mathcal{W}$, we consider its associated cylinder:

$$\mathcal{C}(Q) = \Pi^{-1}(\Pi(Q)).$$

Let $R \in \mathcal{D}_{\mathcal{W}}(R_0)$ be as in the lemma and let $A \gg 1$. For some $j \gg 1$ to be fixed below (independent of A), let L be a hyperplane parallel to H_0 such that

$$\operatorname{dist}(L, \Sigma_0 \cap \mathcal{C}(R)) = 2^{-j}\,\ell(R).$$

Notice that there are two possible choices for L. If τ_0 is small enough (and so Σ flat enough) depending on j, then

$$\operatorname{dist}(x, \partial\Omega \cap \mathcal{C}(10R)) \approx 2^{-j}\,\ell(R) \quad \text{for all } x \in L \cap \mathcal{C}(10R). \tag{3.4}$$

Then we choose L so that $L \cap \mathcal{C}(10R) \subset \Omega$.

Let J denote the family of cubes from $\mathcal{W}$ which intersect $L \cap \mathcal{C}(\frac{1}{2}R)$. By the properties of Whitney cubes and (3.4), it is clear that

$$\ell(Q) \approx 2^{-j}\,\ell(R) \quad \text{and} \quad \Pi(Q) \subset \Pi(R) \quad \text{for all } Q \in J.$$

Let $\operatorname{Adm}(2\Lambda Q)$ be the set of $x \in \Omega \cap 2\Lambda Q$ such that the interval $(0, \operatorname{diam}(25\Lambda Q))$ is admissible for x. Recall that Λ is one of the constants in the definition of Whitney cubes. We assume τ_0 small enough so that $3Q \subset \operatorname{Adm}(2\Lambda Q)$[2]. Then by Lemma 2.4,

$$\sup_{x \in \operatorname{Adm}(2\Lambda Q)} F(x, \operatorname{diam}(5\Lambda Q)) \leq C_0\, F(x_Q, \operatorname{diam}(20\Lambda Q)) + C_0, \tag{3.5}$$

where C_0 is an absolute constant.

[2] Notice that $2\Lambda Q \not\subset \operatorname{Adm}(2\Lambda Q)$ because $2\Lambda Q$ intersects $\mathbb{R}^n \setminus \overline{\Omega}$. Instead, a big portion of $2\Lambda Q$ is contained in $\operatorname{Adm}(2\Lambda Q)$ if Σ_0 is flat enough.

Claim. There exists some $Q \in J$ such that

$$F(x_Q, \operatorname{diam}(20\Lambda Q)) \leq \frac{F(x_R, A\ell(R))}{4C_0} \tag{3.6}$$

if j is big enough and we assume that τ_0 is small enough depending on j, and also that N_0 is big enough.

Remark again that the choice of j will not depend on the constant A.

To prove the claim we intend to apply a rescaled version of Theorem 3.2 to a suitable half ball B_+ centered at z_R, the orthogonal projection of x_R on L. We take

$$B_+ = \{x \in B(z_R, \ell(R)/4) : x_n > (z_R)_n\},$$

so that $B_+ \subset \Omega$, assuming that $\Sigma_0 \cap \mathcal{C}(R)$ is below L. We also consider the point

$$\tilde{z}_R = z_R + (0, \ldots, 0, \ell(R)/8).$$

Notice that $\tilde{z}_R \in B_+$ (in fact, $B(\tilde{z}_R, \ell(R)/8) \subset B_+$).

Aiming for a contradiction, suppose that $F(x_Q, \operatorname{diam}(20\Lambda Q)) > \frac{N}{4C_0}$ for all $Q \in J$, where $N = F(x_R, A\ell(R))$. For each $Q \in J$, by the subharmonicity of $|u|$ and (2.9), we have

$$\begin{aligned}\sup_{2Q} |u| &\lesssim \fint_{\partial B(x_Q, \operatorname{diam}(3Q))} |u|\, d\sigma \leq h(x_Q, \operatorname{diam}(20\Lambda Q))^{1/2} \\ &\leq h(x_Q, \ell(R))^{1/2} \left(\frac{\operatorname{diam}(20\Lambda Q)}{\ell(R)}\right)^{F(x_Q, \operatorname{diam}(20\Lambda Q))/2}.\end{aligned}$$

Here we applied the property described in Remark 3.3, allowing the smallness of the slope constant τ_0 to depend on j. Below we will make repeated use of this property, often without further reference.

To estimate $h(x_Q, \ell(R))$ we take into account that

$$h(x_Q, \ell(R)) \leq \fint_{A(x_Q, \ell(R), 2\ell(R))} |u|^2\, dm \lesssim \fint_{B(\tilde{z}_R, C_1\ell(R))} |u|^2\, dm \leq h(\tilde{z}_R, C_1\ell(R)),$$

since $A(x_Q, \ell(R), 2\ell(R)) \subset B(\tilde{z}_R, C_1\ell(R))$ for some fixed $C_1 > 1$. Further, by (2.9),

$$\begin{aligned}h(\tilde{z}_R, C_1\ell(R)) &\leq h(\tilde{z}_R, \ell(R)/16)\, (16\, C_1)^{F(\tilde{z}_R, C_1\ell(R))} \\ &\lesssim (16\, C_1)^{F(\tilde{z}_R, C_1\ell(R))} \fint_{B(\tilde{z}_R, \ell(R)/8)} |u|^2\, dm \\ &\lesssim (16\, C_1)^{F(\tilde{z}_R, C_1\ell(R))} \fint_{B_+} |u|^2\, dm,\end{aligned}$$

recalling that $B(\tilde{z}_R, \ell(R)/8) \subset B_+$. Thus,

$$\sup_{2Q} |u|^2 \lesssim (16\, C_1)^{F(\tilde{z}_R, C_1\ell(R))} \left(\frac{\operatorname{diam}(20\Lambda Q)}{\ell(R)}\right)^{F(x_Q, \operatorname{diam}(20\Lambda Q))} \fint_{B_+} |u|^2\, dm. \tag{3.7}$$

Observe now that, by Lemma 2.4,

$$F(\tilde{z}_R, C_1\ell(R)) \leq C\, F(x_R, C\ell(R)) + C,$$

for a suitable absolute constant $C > 2C_1$. So for A and N_0 big enough (both independent of j, just larger than some absolute constant),

$$F(\tilde{z}_R, C_1\ell(R)) \leq C\, F(x_R, A\ell(R)) + C \leq C'\, N. \tag{3.8}$$

Therefore, recalling also the assumption $F(x_Q, \operatorname{diam}(20\Lambda Q)) > \frac{N}{4C_0}$, by (3.7) we get

$$\begin{aligned}\sup_{2Q} |u|^2 &\lesssim (16\, C_1)^{C'N} \left(\frac{\operatorname{diam}(20\Lambda Q)}{\ell(R)}\right)^{N/4C_0} \fint_{B_+} |u|^2\, dm \\ &= 2^{-jcN + C''N} \fint_{B_+} |u|^2\, dm\end{aligned}$$

(here we took into account that $\operatorname{diam}(20\Lambda Q) \leq \ell(R)$ for j larger than some absolute constant). Also, by standard interior estimates for harmonic functions,

$$\sup_{\frac{3}{2}Q} |\nabla u|^2 \lesssim \frac{1}{\ell(Q)^2} \sup_{2Q} |u|^2 \lesssim \frac{2^{2j}}{\ell(R)^2}\, 2^{-jcN + C''N} \fint_{B_+} |u|^2\, dm.$$

From the last two estimates we deduce that if j is big enough and N_0 (and thus N) also big enough, then there exists some $c' > 0$ such that

$$\sup_{\frac{3}{2}Q} \big(|u|^2 + \ell(R)^2\, |\nabla u|^2\big) \lesssim 2^{-jc'N} \fint_{B_+} |u|^2\, dm.$$

Since the cubes $\frac{3}{2}Q$ with $Q \in J$ cover the flat part of the boundary of B_+, which we denote by Γ, it is clear that

$$\sup_{\Gamma} \big(|u|^2 + \ell(R)^2\, |\nabla u|^2\big) \lesssim 2^{-jc'N} \fint_{B_+} |u|^2\, dm.$$

Applying now a rescaled version of Theorem 3.2 to the half ball B_+, we infer that

$$\sup_{B(\tilde{z}_P, \ell(R)/16)} |u|^2 \lesssim 2^{-jc'N\alpha} \fint_{B_+} |u|^2\, dm.$$

Consequently,

$$h(\tilde{z}_R, \ell(R)/16) \lesssim 2^{-2jc'N\alpha} \fint_{B(\tilde{z}_R, \ell(R))} |u|^2\, dm \lesssim 2^{-2jc'N\alpha}\, h(\tilde{z}_R, \ell(R)),$$

for some fixed $\alpha > 0$. By Lemma 2.3, this implies that

$$F(\tilde{z}_R, \ell(R)) \geq \log_{16} \frac{h(\tilde{z}_R, \ell(R))}{h(\tilde{z}_R, \ell(R)/16)} \geq c\, jN\alpha,$$

for some fixed $c > 0$. However, for j big enough this contradicts the fact that $F(\tilde{z}_R, \ell(R)) \lesssim N$, which follows from (3.8). So the proof of the claim is concluded.

Now we are ready to introduce the set $\mathcal{G}_K(R)$. Fix $Q_0 \in J$ such that (3.6) holds for Q_0. Notice that, by (3.5),

(3.9)
$$\sup_{x\in \mathrm{Adm}(2\Lambda Q_0)} F(x, \Lambda\ell(Q_0)) \leq C_0\, F(x_{Q_0}, \mathrm{diam}(20\Lambda Q_0)) + C_0 \leq \frac{N}{4} + C_0 \leq \frac{N}{2},$$

since $N \geq N_0$ and we assume N_0 big enough. Now we just define

$$\mathcal{G}_K(R) = \{Q \in \mathcal{D}_{\mathcal{W}}^{j+k}(R) : \Pi(Q) \subset \Pi(Q_0)\},$$

with $k = \lceil \log_2 A\rceil$. So we have $\mathcal{G}_K(R) \subset \mathcal{D}_{\mathcal{W}}^K(R)$ with $K = j + k$ and it holds $\ell(Q) \approx 2^{-k}\ell(Q_0)$ for every $Q \in \mathcal{G}_K(R)$.

The property (a) in the lemma follows easily from (3.9). Indeed, if $P \in \mathcal{G}_K(R)$, then taking into account that $x_P \in \mathrm{Adm}(2\Lambda Q_0)$ for τ_0 small enough (depending on A),

$$F(x_P, A\ell(P)) \leq F(x_P, \ell(Q_0)) \leq F(x_P, \Lambda\ell(Q_0)) \leq \frac{N}{2}.$$

Notice also that

$$m_{n-1}\Big(\bigcup_{Q\in\mathcal{G}_K(R)} \Pi(Q)\Big) = \ell(Q_0)^{n-1} \approx (2^{-j}\,\ell(R))^{n-1},$$

and recall that j is independent of A. So (a) holds with $\delta_0 \approx 2^{-j(n-1)}$.

The property (b) is an easy consequence of Lemma 2.4. Indeed, for any $P \in \mathcal{D}_{\mathcal{W}}^K(R)$, since $|x_P - x_R| \lesssim \ell(R)$, taking $\gamma \approx A^{-1}$ in (2.10), we deduce

$$\begin{aligned} F(x_P, A\ell(P)) &\leq F(x_P, A\ell(R)/3) \leq (1 + CA^{-1/2})\, F\big(x_R, A\ell(R)\big) + CA^{-1/2} \\ &\leq (1 + 2CA^{-1/2})\, N. \end{aligned}$$

□

4 Proof of Theorem 1.1

Our next objective is to prove the following result:

LEMMA 4.1. *Under the assumptions of Theorem* 1.1*, let $R_0 \in \mathcal{W}$ be as in Section* 3*. Then,*

$$\liminf_{r\to 0} \frac{h(x,12r)}{h(x,r)} < \infty \quad \textit{for } \sigma\textit{-a.e. } x \in \Sigma_0 \cap \mathcal{C}(R_0).$$

Recall that $\mathcal{C}(R_0)$ is the cylindernn

$$\mathcal{C}(R_0) = \Pi^{-1}(\Pi(R_0)),$$

and it contains B_0, by the assumption just after (3.1).

The proof of the preceding lemma will use the following version of the law of large numbers, due to Etemadi [7]:

THEOREM 4.2. *Let $\{X_k\}_{k\geq 1}$ be a sequence of nonnegative random variables with finite second moments such that*:

(a) $\sup_{k\geq 1} \mathbb{E} X_k < \infty$,
(b) $\mathbb{E}(X_j\, X_k) \leq \mathbb{E} X_j\, \mathbb{E} X_k$ *for* $j \neq k$, *and*
(c) $\sum_{k\geq 1} \frac{1}{k^2} \operatorname{Var} X_k < \infty$.

Let $S_m = X_1 + \cdots + X_m$. *Then*

$$\lim_{m\to\infty} \frac{S_m - \mathbb{E} S_m}{m} = 0 \quad \textit{almost surely.}$$

PROOF OF LEMMA 4.1. Let $\Pi_{\Sigma_0} : \mathcal{C}(R_0) \to \mathcal{C}(R_0) \cap \Sigma_0$ denote the projection on $\mathcal{C}(R_0) \cap \Sigma_0$ in the direction orthogonal to the horizontal hyperplane H_0. We consider the measure

$$\mu = \Pi_{\Sigma_0} \#(m_{n-1}|_{\mathcal{C}(R_0)\cap H_0}).$$

This is the image measure (or push-forward measure) of the $(n-1)$-dimensional Lebesgue measure on $\mathcal{C}(R_0) \cap H_0$ to $\mathcal{C}(R_0) \cap \Sigma_0$. Obviously, μ is mutually absolutely continuous with the surface measure σ on $\mathcal{C}(R_0) \cap \Sigma_0$.

Next we consider the families of cubes from $\mathcal{W}$ defined by

$$\mathcal{T}_j' = \bigl\{Q \in \mathcal{D}_{\mathcal{W}}^{jK}(R_0) : F(x_Q, A\ell(Q)) \leq N_0\bigr\}, \quad j \geq 0, \tag{4.1}$$

where the constants K, A, and N_0 are given by the Key Lemma 3.1 (the precise large value of A will be chosen below). We also denote

$$R_{\Sigma_0} = \Pi_{\Sigma_0}(R_0)$$

and consider the following subset of R_{Σ_0}:

$$T_j = \bigcup_{Q\in\mathcal{T}_j'} \Pi_{\Sigma_0}(Q), \quad j \geq 0.$$

We will prove the following:

Claim. We have

$$\mu\Big(R_{\Sigma_0} \setminus \limsup_{j\to\infty} T_j\Big) = 0.$$

Let us see first that the lemma follows from this claim. Indeed, if $x \in T_j$, then there exists some cube $Q \in \mathcal{T}_j'$ such that $x \in \Pi_{\Sigma_0}(Q)$. By construction, $F(x_Q, A\ell(Q)) \leq N_0$ and thus, by (2.9),

$$h(x_Q, A\ell(Q)) \leq 48^{N_0}\, h(x_Q, A\ell(Q)/48).$$

For A big enough, we have

$$B(x, A\ell(Q)/24) \supset \tfrac{3}{2} B(x_Q, A\ell(Q)/48) \quad \text{and}$$
$$\tfrac{3}{2} B(x, A\ell(Q)/2) \subset B(x_Q, A\ell(Q)).$$

Then, by the subharmonicity of $|u|$ in a neighborhood of Σ_0 and standard arguments,

$$h(x, A\ell(Q)/24) \gtrsim h(x_Q, A\ell(Q)/48) \quad \text{and} \quad h(x, A\ell(Q)/2) \lesssim h(x_Q, A\ell(Q)).$$

Hence, for each $x \in T_j$,

$$\frac{h(x, A\ell(Q)/2)}{h(x, A\ell(Q)/24)} \lesssim \frac{h(x_Q, A\ell(Q))}{h(x_Q, A\ell(Q)/48)} \leq 48^{N_0},$$

with $\ell(Q) \approx 2^{-jK}\ell(R_0)$.

Consequently, if $x \in \limsup_{j\to\infty} T_j$, then a sequence of radii $r_j \to 0$ exists such that

$$\frac{h(x, 12r_j)}{h(x, r_j)} \lesssim 48^{N_0},$$

which implies that

$$\liminf_{r\to 0} \frac{h(x, 12r)}{h(x, r)} < \infty,$$

and yields the lemma, assuming the claim.

To prove the claim above we need to introduce some additional notation. For $j \geq 0$ and K as in the Key Lemma 3.1, we denote

$$\widetilde{\mathcal{D}}_j(R_{\Sigma_0}) = \Pi_{\Sigma_0}\big(\mathcal{D}_{\mathcal{W}}^{jK}(R_0)\big),$$

or more precisely,

$$\widetilde{\mathcal{D}}_j(R_{\Sigma_0}) = \big\{\Pi_{\Sigma_0}(Q') : Q' \in \mathcal{D}_{\mathcal{W}}^{jK}(R_0)\big\}.$$

We also set

$$\widetilde{\mathcal{D}}(R_{\Sigma_0}) = \bigcup_{j\geq 0} \widetilde{\mathcal{D}}_j(R_{\Sigma_0}).$$

For any $R \in \widetilde{\mathcal{D}}_j(R_{\Sigma_0})$ such that $R = \Pi_{\Sigma_0}(R')$ for some $R' \in \mathcal{D}_{\mathcal{W}}^{jK}(R_0)$, in case that $F(x_{R'}, A\,\ell(R')) \geq N_0$, we consider the good set

$$G(R) = \bigcup_{Q'\in\mathcal{G}_K(R')} \Pi_{\Sigma_0}(Q'),$$

with $\mathcal{G}_K(R')$ as in the Key Lemma 3.1. In case that $F(x_{R'}, A\,\ell(R')) < N_0$, we let

$$G(R) = \Pi_{\Sigma_0}(R') = R.$$

Finally, we write

$$\mathcal{T}_j = \big\{\Pi_{\Sigma_0}(Q') : Q' \in \mathcal{D}_{\mathcal{W}}^{jK}(R_0), F(x_{Q'}, A\ell(Q')) \leq N_0\big\},$$

or in other words,

$$\mathcal{T}_j = \Pi_{\Sigma_0}(\mathcal{T}_j'),$$

with $\mathcal{T}_j'$ defined in (4.1)

To prove the claim we have to show that $\bigcup_{j\geq h} T_j$ has full μ-measure in R_{Σ_0} for every $h \geq 0$. To this end, for any fixed h we define the following functions f_j, $j \geq h$:

$$f_j = \sum_{Q\in\widetilde{\mathcal{D}}_j(R_{\Sigma_0})} f_Q,$$

where $f_Q = 0$ if Q is contained in some "cube" $\widetilde{Q} \in \bigcup_{j \geq h} \mathcal{T}_j$ and, otherwise,

$$f_Q = \frac{\mu(Q)}{\mu(G(Q))}\, \chi_{G(Q)} - \chi_Q.$$

It is immediate to check that the functions f_j have zero μ-mean, and they are orthogonal, i.e., $\int f_i\, f_j\, d\mu = 0$ if $i \neq j$ (taking into account that f_j has zero μ-mean in each $Q \in \widetilde{\mathcal{D}}_j(R_{\Sigma_0})$ and is constant in each $P \in \widetilde{\mathcal{D}}_{j+1}(R_{\Sigma_0})$). Observe also that the functions f_j are uniformly bounded, due to the fact that $\mu(G(Q)) \geq \delta_0\, \mu(Q)$ in the latter case by the Key Lemma. So their $L^2(\mu)$ norms are uniformly bounded too.

We consider the probability measure $\mu_{|R_{\Sigma_0}}/\mu(R_{\Sigma_0})$ and the random variables $X_j = f_j + 1$, $j \geq h$. Notice that they are nonnegative and the assumptions in Theorem 4.2 are satisfied. Indeed, (a) and (c) follow from the uniform boundedness of the functions f_j, and the zero mean of each f_j and the mutual orthogonality of the f_j's imply that $\mathbb{E}(X_i\, X_j) = \mathbb{E}(X_i)\, \mathbb{E}(X_j)$ if $i \neq j$. Applying the theorem then we infer that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=h+1}^{m} f_j(x) = 0 \quad \text{for } \mu\text{-a.e. } x \in R_{\Sigma_0}, \tag{4.2}$$

using the fact that $\mathbb{E} X_j = 1$ for all j.

We will show that

$$x \in R_{\Sigma_0} \setminus \bigcup_{j\geq h} T_j \qquad \Rightarrow \qquad \lim_{m\to\infty} \frac{1}{m} \sum_{j=h+1}^{m} f_j(x) \neq 0. \tag{4.3}$$

Clearly, by (4.2), this implies that $R_{\Sigma_0} \setminus \bigcup_{j\geq h} T_j$ has null μ-measure and finishes the proof of the claim.

We prove (4.3) by contradiction. Suppose that there exists some point $x \in R_{\Sigma_0} \setminus \bigcup_{j\geq h} T_j$ such that $\lim_{m\to\infty} \frac{1}{m} \sum_{j=h+1}^{m} f_j(x) = 0$. Denote by Q_j the "cube" from $\widetilde{D}_j(R_{\Sigma_0})$ that contains x. Since $Q_i \notin \mathcal{T}_i$ for any $i \geq h$, by definition we have

$$f_j(x) = \frac{\mu(Q_j)}{\mu(G(Q_j))}\, \chi_{G(Q_j)}(x) - 1 \quad \text{for any } j \geq h.$$

Then (4.2) tells us that, for any $\varepsilon > 0$,

$$\left| \sum_{j=h+1}^{m} \frac{\mu(Q_j)}{\mu(G(Q_j))}\, \chi_{G(Q_j)}(x) - m \right| \leq \varepsilon\, m$$

for any m big enough. In particular, choosing $\varepsilon = \frac{1}{2}$ we infer that, for some $m_0 = m_0(x)$,

$$\sum_{j=h+1}^{m} \frac{\mu(Q_j)}{\mu(G(Q_j))}\, \chi_{G(Q_j)}(x) \geq \frac{m}{2} \quad \text{for any } m \geq m_0.$$

Since $\mu(G(Q_j)) \geq \delta_0\, \mu(Q_j)$, we get

$$\sum_{j=h+1}^{m} \chi_{G(Q_j)}(x) \geq \frac{\delta_0\, m}{2} \quad \text{for } m \geq m_0. \tag{4.4}$$

For each $j \geq h$, let $Q_j' \in \mathcal{D}_{\mathcal{W}}(R_0)$ be such that $Q_j = \Pi_{\Sigma_0}(Q_j')$. Recall that the Key Lemma (we can apply this because $F(x_{Q_j'}, A\ell(Q_j')) \geq N_0$) asserts that

$$F(x_{Q_{j+1}'}, A\ell(Q_{j+1}')) \leq \frac{1}{2}\, F(x_{Q_j'}, A\ell(Q_j')) \quad \text{if } x \in G(Q_j)$$

and otherwise just ensures that

$$F(x_{Q_{j+1}'}, A\ell(Q_{j+1}')) \leq (1 + CA^{-1/2})\, F(x_{Q_j'}, A\ell(Q_j')) \quad \text{if } x \in Q_j \setminus G(Q_j).$$

These estimates and (4.4) imply that

$$F(x_{Q_{m+1}'}, A\ell(Q_{m+1}')) \leq \left(\frac{1}{2}\right)^{\delta_0 m/2} (1 + CA^{-1/2})^m \quad \text{for } m \geq m_0.$$

However, if A is chosen big enough (recall that A is independent of δ_0 and can be taken arbitrarily big in the Key Lemma 3.1), this implies that

$$F(x_{Q_m'}, A\ell(Q_m')) \to 0 \quad \text{as } m \to \infty,$$

which cannot happen because $x \notin \bigcup_{j\geq h} T_j$, recalling the definition of T_j. This concludes the proof of the claim and of the lemma. □

The proof of Theorem 1.1 will follow as a straightforward consequence of Lemma 4.1 and the next result of Adolfsson and Escauriaza:

LEMMA 4.3 ([1, lemma 0.2]). *Let $D \subset \mathbb{R}^n$ be a Lipschitz domain and let V be a relatively open subset of ∂D. Let v be a nonzero function harmonic in D and continuous in $\overline{D}$ that vanishes identically in V, and whose normal derivative $\partial_\nu v$ vanishes in a subset $E \subset V$ of positive surface measure. Then, for every point $x \in V$ that is a density point of E (with respect to surface measure), we have*

$$\lim_{r\to 0} \frac{\int_{B(x,r)\cap D} |v|\,dm}{\int_{B(x,6r)\cap D} |v|\,dm} = 0. \tag{4.5}$$

Actually, the identity (4.5) is not stated explicitly in lemma 0.2 in [1]. Instead, it is said that v vanishes to infinite order in x. However, a quick inspection of the proof shows that the authors actually prove (4.5), which in turn implies that v vanishes to infinite order in x. The lemma above relies on [2, lemma 1 and theorem 1]. Though the proof of [2, lemma 1] is not correct—as explained in [4, paragraph before lemma 5]—one can replace that lemma either by [1, lemma 2.2] or by more quantitative arguments involving [4, lemma 4] and well-known properties of harmonic functions.[3] For the reader's convenience I provide an alternative self-contained proof in the Appendix C.

PROOF OF THEOREM 1.1. As explained at the beginning of Section 3, it suffices to show that $\partial_\nu u$ cannot vanish in a subset of positive surface measure of $\Sigma_0 \cap \mathcal{C}(R_0)$ (since this set contains the ball B_0).

For the sake of contradiction, suppose that $\partial_\nu u$ vanishes in a subset $E \subset \Sigma_0 \cap \mathcal{C}(R_0)$ of positive surface measure. By Lemma 4.3, for any $x \in \Sigma_0 \cap \mathcal{C}(R_0)$ that is a density point of E,

$$\lim_{r\to 0} \frac{\int_{B(x,6r)} |u|\,dm}{\int_{B(x,r)} |u|\,dm} = \infty.$$

By the subharmonicity of $|u|$, for r small enough,

$$h(x,r/2)^{1/2} = \left(\fint_{\partial B(x,r/2)} |u|^2\,d\sigma\right)^{1/2} \lesssim \fint_{B(x,r)} |u|\,dm.$$

Also, by Cauchy-Schwarz and the fact that $h(x,\cdot)$ is nondecreasing in r,

$$\fint_{B(x,6r)} |u|\,dm \le \left(\fint_{B(x,6r)} |u|^2\,dm\right)^{1/2} \le h(x,6r)^{1/2}.$$

Therefore,

$$\liminf_{r\to 0} \frac{h(x,6r)^{1/2}}{h(x,r/2)^{1/2}} \gtrsim \liminf_{r\to 0} \frac{\fint_{B(x,6r)} |u|\,dm}{\fint_{B(x,r)} |u|\,dm} = \infty.$$

Consequently,

$$\lim_{r\to 0} \frac{h(x,12r)}{h(x,r)} = \infty,$$

which contradicts Lemma 4.1. □

Appendix A Existence of Nontangential Limits for ∇u

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, and let σ denote the surface measure on $\partial\Omega$. For σ-a.e. x, there exists a tangent hyperplane to $\partial\Omega$ in x and the outer unit normal $\nu(x)$ is well-defined. For an aperture parameter $a \in (0,1)$ we consider the one-sided inner cone with axis in the direction of $-\nu(x)$ defined by

$$X_a^+(x) = \{y \in \mathbb{R}^n : (x-y)\cdot\nu(x) > a|y-x|\}.$$

Analogously, we consider the outer cone

$$X_a^-(x) = \{y \in \mathbb{R}^n : (y-x)\cdot\nu(x) > a|y-x|\}.$$

[3] I thank Luis Escauriaza for informing me about this fact.

For a given function $f : \mathbb{R}^n \setminus \partial\Omega \to \mathbb{R}$ and a fixed parameter $a \in (0,1)$, we define the nontangential limits

$$f_{+,a}(x) = \lim_{X_a^+(x) \ni y \to x} f(y), \qquad f_{-,a}(x) = \lim_{X_a^-(x) \ni y \to x} f(y),$$

whenever they exist.

Although the following result is already known (see [9, theorem 5.19], I have not been able to find an easy argument in the literature and thus I provide a detailed proof based on Dahlberg's theorem on harmonic measure [6].

THEOREM A.1. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, let B be an open ball centered in $\partial\Omega$, and let $\Sigma = B \cap \partial\Omega$ be a Lipschitz graph. Let u be a function harmonic in Ω and continuous in $\overline{\Omega}$. Suppose that u vanishes in Σ. Then, for any $a \in (0,1)$, $(\nabla u)_{+,a}$ exists σ-a.e. and belongs to $L^2_{\text{loc}}(\sigma|_\Sigma)$. Further, $(\nabla u)_{+,a}$ has vanishing tangential component. That is, $(\nabla u)_{+,a} = (\partial_\nu u)\,\nu$. In addition, assuming that $\Omega \cap B$ is above Σ,*

$$\lim_{\varepsilon \to 0} \nabla u(\cdot + \varepsilon\, e_n) \to (\partial_\nu u)\,\nu \quad \text{in } L^2_{\text{loc}}(\sigma|_\Sigma).$$

Also, in the sense of distributions,

$$(\Delta u)|_B = -\partial_\nu u\ \sigma|_\Sigma.$$

PROOF. We extend u by 0 out of Ω and denote

$$u^+ = \max(u,0), \quad u^- = -\min(u,0),$$

so that u^+ and u^- are continuous and subharmonic in B. This implies that, in the sense of distributions, in B, $\Delta u = \Delta u^+ - \Delta u^-$ is a signed Radon measure supported on Σ.

First we claim that

$$(\Delta u)|_B = \rho\,\omega|_\Sigma, \tag{A.1}$$

where $\rho \in L^\infty_{\text{loc}}(\Sigma)$ and ω is the harmonic measure for Ω with respect to some fixed pole $p \in \Omega$. To prove the lemma we may assume B small enough so that $\Omega \setminus 2B \neq \varnothing$ and that $p \in \Omega \setminus 2B$. To prove the claim, let B' be an open ball concentric with B such that $\overline{B}' \subset B$. We will show that that there exists some constant C_2 depending on B' and p such that for any compact set $K \subset \Sigma$, it holds that

$$|\langle \Delta u, \chi_K \rangle| \le C_2\,\omega(K). \tag{A.2}$$

By duality, this implies (A.1).

Given an arbitrary $\varepsilon \in (0, \frac{1}{2}\operatorname{dist}(K, \mathbb{R}^n \setminus B'))$, let $\{Q_i\}_{i\in I}$ be a lattice of cubes that cover $\mathbb{R}^n$, with diameter equal to $\varepsilon/2$. Let $\{\varphi_i\}_{i\in I}$ be a partition of unity of $\mathbb{R}^n$, so that each φ_i is supported in $2Q_i$ and $\|\nabla^j \varphi_i\|_\infty \lesssim \ell(Q_i)^{-j}$ for $j = 0,1,2$. Then we have

$$\langle \Delta u, \chi_K \rangle = \Big\langle \Delta u, \sum_{i\in I'} \varphi_i \Big\rangle - \Big\langle \Delta u, \sum_{i\in I'} \varphi_i - \chi_K \Big\rangle, \tag{A.3}$$

where I' is the collection of indices $i \in I$ such that $2Q_i \cap K \neq \varnothing$. Since $(\Delta u)|_B$ is a signed Radon measure,

$$\Big|\Big\langle \Delta u,\ \sum_{i\in I'} \varphi_i - \chi_K \Big\rangle\Big| \leq \big\langle |\Delta u|,\ \chi_{U_\varepsilon(K)\setminus K}\big\rangle \to 0 \quad \text{as } \varepsilon \to 0,$$

where $U_\varepsilon(K)$ is the ε-neighborhood of K. Concerning the other term in (A.3), we have

$$\Big|\Big\langle \Delta u, \sum_{i\in I'} \varphi_i \Big\rangle\Big| \leq \sum_{i\in I'} \big|\langle u, \Delta\varphi_i\rangle\big| \lesssim \sum_{i\in I'} \frac{1}{\ell(Q_i)^2} \int_{2Q_i} |u|\,dm.$$

Since $|u|$ is subharmonic and continuous in B and vanishes in $B\setminus\Omega$, by the boundary Harnack principle (see theorem 5.1 from [8], for example), we have

$$|u(x)| \leq C_3\, g(x,p) \quad \text{for all } x \in B' \cap \Omega,$$

where $g(\cdot,\cdot)$ is the Green function of Ω and C_3 depends on u, p, and B', but not on K. Thus,

$$\Big|\Big\langle \Delta u,\ \sum_{i\in I'} \varphi_i \Big\rangle\Big| \lesssim \sum_{i\in I'} \frac{1}{\ell(Q_i)^2} \int_{2Q_i} g(x,p)\,dx,$$

with the implicit constant depending on u, p, and B'. By standard estimates for the harmonic measure (see (4.3) and (4.4) from [8], for example), we have

$$g(x,p) \lesssim \frac{\omega(4Q_i)}{\ell(Q_i)^{n-2}} \quad \text{for all } x \in 2Q_i \cap \Omega,\ i \in I.$$

Therefore,

$$\Big|\Big\langle \Delta u,\ \sum_{i\in I'} \varphi_i \Big\rangle\Big| \lesssim \sum_{i\in I'} \omega(4Q_i) \leq \omega(U_{4\varepsilon}(K)).$$

Letting $\varepsilon \to 0$, we have $\omega(U_{4\varepsilon}(K)) \to \omega(K)$ and thus (A.2) follows, which implies the claim (A.1) .

Next recall that by Dahlberg's theorem, harmonic measure on a Lipschitz domain Ω is a B_2 weight with respect to the surface measure σ. In particular, the density function $\frac{d\omega}{d\sigma}$ belongs to $L^2_{\text{loc}}(\sigma)$. Therefore, in the sense of distributions,

$$(\Delta u)|_B = h\,\sigma|_\Sigma \quad \text{for some } h \in L^2_{\text{loc}}(\sigma).$$

Our next objective consists in showing that $(\nabla u)_{+,a}$ exists σ-a.e., and moreover $(\nabla u)_{+,a} = (\partial_\nu u)\,\nu \in L^2_{\text{loc}}(\sigma|_\Sigma)$. To this end, consider an arbitrary open ball $\widetilde{B}$ centered in Σ such that $4\widetilde{B} \subset B$. Let φ be a C^∞ function that equals 1 on $2\widetilde{B}$ and vanishes out of $3\widetilde{B}$, and let $v = \varphi\, u$. Observe that

$$v = \mathcal{E} * \Delta(\varphi\, v) = \mathcal{E} * (\varphi\,\Delta u + u\,\Delta\varphi + 2\,\nabla u\cdot\nabla\varphi), \tag{A.4}$$

where $\mathcal{E}$ is the fundamental solution of the Laplacian. Note also that $\nabla u \in L^2_{\text{loc}}(B)$, by Caccioppoli's inequality.

For a finite Borel measure η, let $R\eta$ be the $(n-1)$-dimensional Riesz transform of η. That is,

$$R\eta(x) = \int \frac{x-y}{|x-y|^n}\, d\eta(y),$$

whenever the integral makes sense. From the identity (A.4), we deduce that, for all $x \notin \Sigma$,

$$\nabla v(x) = c_n \big(R(\varphi\, g\, \sigma|_{\Sigma})(x) + R(u\,\Delta\varphi\, m)(x) + 2\, R(\nabla u \cdot \nabla\varphi\, m)(x) \big)$$

(recall that m is the Lebesgue measure in $\mathbb{R}^n$). Observe that $R(u\,\Delta\varphi\, m)$ and $R(\nabla u\cdot \nabla\varphi m)$ are continuous functions in $\widetilde{B}$. On the other hand, the nontangential limit $(R(\varphi\, g\,\sigma|_{\Sigma}))_{\pm,a}(x)$ exists for σ-a.e. $x \in \partial\Omega$, by the classical jump formulas for the Riesz transforms (see [18], for example). Taking also into account that $\nabla v = \nabla u$ in $\widetilde{B}$, it follows then that $(\nabla u)_{\pm,a}(x)$ exists for σ-a.e. $x \in \Sigma \cap \widetilde{B}$. By the $L^2(\sigma)$ boundedness of the (principal value) Riesz transform operator $R(\cdot\,\sigma)$ on Lipschitz graphs, we deduce that $(\nabla u)_{\pm,a} \in L^2(\sigma|_{\Sigma\cap\widetilde{B}})$.

Since $u \equiv 0$ in Ω^c, it is clear that $(\nabla u)_{-,a} \equiv 0$ in $\Sigma \cap \widetilde{B}$. As the tangential component of $R(\varphi\, g\,\sigma|_{\Sigma})(x)$ is continuous across $\partial\Omega$ for σ-a.e. $x \in \partial\Omega$, again by the jump formulas for the Riesz transforms, we deduce that the tangential component of $(\nabla u)_{+,a}$ coincides with the tangential component of $(\nabla u)_{-,a}$ σ-a.e. in $\Sigma \cap \widetilde{B}$, and thus $(\nabla u)_{+,a} \equiv 0$ in $\Sigma \cap \widetilde{B}$, which is equivalent to saying that $(\nabla u)_{+,a} = (\partial_\nu u)\nu$ in $\Sigma \cap \widetilde{B}$.

It remains to prove that $(\Delta u)|_B = -\partial_\nu u\ \sigma|_{\Sigma}$ in the sense of distributions. To this end, let ψ be a C^∞ function supported in $\widetilde{B}$. Without loss of generality we may assume that $\Sigma \cap \widetilde{B}$ is a Lipschitz graph with respect to the horizontal axes and that $\Omega \cap \widetilde{B}$ lies above $\Sigma \cap \widetilde{B}$. For $0 < \varepsilon \ll r(\widetilde{B})$, denote $\Sigma_\varepsilon = \Sigma + \varepsilon\, e_n$ and $\Omega_\varepsilon = \Omega + \varepsilon\, e_n$, where $e_n = (0, \dots, 0, 1)$. Then we have

$$\begin{aligned}
\langle \Delta u,\ \psi\rangle &= \int u\, \Delta\psi\, dm = \lim_{\varepsilon\to 0} \int_{\widetilde{B}\cap\Omega_\varepsilon} u\,\Delta\psi\, dm \\
&= \lim_{\varepsilon\to 0} \int_{\widetilde{B}\cap\partial\Omega_\varepsilon} u\,\partial_\nu\psi\, d\sigma - \lim_{\varepsilon\to 0} \int_{\widetilde{B}\cap\partial\Omega_\varepsilon} \psi\,\partial_\nu u\, d\sigma \\
&= 0 - \int_{\Sigma} \psi\,\partial_\nu u\, d\sigma.
\end{aligned} \tag{A.5}$$

The last identity follows from the fact that, in a neighborhood of $\Sigma \cap \widetilde{B}$, as $\varepsilon \to 0$, $u(\cdot + \varepsilon\, e_n)$ converges uniformly to 0 and $\nabla u(\cdot + \varepsilon\, e_n)$ converges to $(\nabla u)_{+,a}$ in $L^2(\sigma|_{\Sigma\cap\widetilde{B}})$ (this is proven by arguments analogous to the ones above for the σ-a.e. existence of the limit $(\nabla u)_{+,a}(x)$ in Σ). From (A.5), we deduce that $\Delta u = -\partial_\nu u\ \sigma|_{\Sigma}$ in $\widetilde{B}$, and thus also in B. □

Appendix B Whitney Cubes

In this appendix we prove some of the properties of the Whitney cubes constructed at the beginning of Section 3.

LEMMA B.1. *Let $\Omega \subsetneq \mathbb{R}^n$ be open. Then there exists a family $\mathcal{W}$ of dyadic cubes with disjoint interiors such that*

$$\bigcup_{Q\in\mathcal{W}} Q = \Omega,$$

and moreover there are some constants $\Lambda > 20$ and $D_0 \geq 1$ such the following holds for every $Q \in \mathcal{W}$:

(i) $10Q \subset \Omega$ *and* $\operatorname{diam}(Q) < \frac{1}{20}\operatorname{dist}(Q, \partial\Omega)$;
(ii) $\Lambda Q \cap \partial\Omega \neq \varnothing$;
(iii) *there are at most D_0 cubes $Q' \in \mathcal{W}$ such that $10Q \cap 10Q' \neq \varnothing$. Further, for such cubes Q', we have $\frac{1}{2}\ell(Q') \leq \ell(Q) \leq 2\ell(Q')$.*

PROOF. We assume the dyadic cubes to be half-open and half-closed. Let $c_0 \in (0, \frac{1}{2})$ be some constant to be fixed below. Denote $d(x) = \operatorname{dist}(x, \partial\Omega)$, and let $\mathcal{W}$ be the family of all dyadic cubes $Q \subset \Omega$ that satisfy

$$\ell(Q) \leq c_0 \inf_{x\in Q} d(x) \tag{B.1}$$

and moreover are maximal. It is immediate to check that the cubes from $\mathcal{W}$ cover Ω and are disjoint, because they are maximal.

For all $Q \in \mathcal{W}$, since $c_0\, d(x) \geq \ell(Q)$ for all $x \in Q$, it follows that

$$\operatorname{dist}(Q, \partial\Omega) \geq c_0^{-1}\ell(Q).$$

Taking c_0 small enough, the properties (i) and (ii) follow.

Let $Q, Q' \in \mathcal{W}$ satisfy $10Q \cap 10Q' \neq \varnothing$. Let Q'' the dyadic parent of Q', which is also contained in Ω, by (i). By the definition of $\mathcal{W}$, there exists $x'' \in \overline{Q''}$ such that $\ell(Q'') \geq c_0\, d(x'')$. Fix also any $x \in Q$. From the condition $10Q \cap 10Q' \neq \varnothing$, it follows that $|x - x''| \leq C\,(\ell(Q) + \ell(Q'))$, where C is some constant depending just on n. Then we have

$$\ell(Q) \leq c_0\, d(x) \leq c_0\, d(x'') + c_0\, |x - x''| \leq 2\,\ell(Q') + c_0\, C(\ell(Q) + \ell(Q')).$$

For c_0 small enough we deduce that $\ell(Q) \leq 2.5\,\ell(Q')$, which implies that $\ell(Q) \leq 2\,\ell(Q')$ because $\ell(Q)/\ell(Q') = 2^k$ for some $k \in \mathbb{Z}$. Reversing the roles of Q and Q', we deduce that $\frac{1}{2}\ell(Q') \leq \ell(Q) \leq 2\ell(Q')$. From this property and standard volume considerations it follows easily that there are at most D_0 cubes $Q' \in \mathcal{W}$ such that $10Q \cap 10Q' \neq \varnothing$, with D_0 depending just on n. □

LEMMA B.2. *Let Ω be a Lipschitz domain, and let Σ, B_0, Σ_0, H_0, Π, and R_0 be as in Section 3. Also, let $J(R_0)$ be as in (3.2) and, for $k \geq 1$, let $J_k(R_0) \subset J(R_0)$ be the subfamily of $(n-1)$-dimensional dyadic cubes in H_0 with side length equal to $2^{-k}\ell(R_0)$. Then, for each $Q' \in J_k(R_0)$ there exists some cube $Q \in \mathcal{W}$ such that $\Pi(Q) = Q'$, $\Pi(Q) \subset \Pi(R_0)$, and such that Q is below R_0.*

PROOF. Let $Q' \in J_k(R_0)$ and denote by x' its center, so that $x' \in H_0 \cap \Pi(R_0)$ too. Let $L_{x'}$ be the line orthogonal to H_0 through x'. Let x be the intersection of $L_{x'}$ with the lower face of R_0, and let $x'' = L_{x'} \cap \Sigma_0$. Let S be the segment (x, x''), which lies on $L_{x'}$.

Consider the sequence of dyadic Whitney cubes $\{R_j\}_{j\geq 1} \subset \mathcal{W}$ that intersect S, so that

$$S = L_{x'} \cap \bigcup_{j\geq 1} R_j,$$

and assume that the sequence is ordered in such a way that, for all $j \geq 0$, R_j and R_{j+1} are neighbors and R_{j+1} is below R_j. The sequence of side lengths $\ell(R_j)$ tends to 0 as $j \to \infty$ because $\operatorname{dist}(R_j, \partial\Omega) \to 0$ as $j \to \infty$. Also, for any $j \geq 0$, $\ell(R_j)/\ell(R_{j+1})$ equals 1, $\frac{1}{2}$, or 2, by the property (iii) in Lemma B.1. This implies that, for some $j \geq 1$, $\ell(R_j) = 2^{-k}\ell(R_0)$. Indeed, we claim that the cube R_j such that $\ell(R_j) \leq 2^{-k}\ell(R_0)$ and j is minimal does the job. To check this, notice that, by the minimality of j, $\ell(R_{j-1}) \geq 2^{-k+1}\ell(R_0)$. So property (iii) of Lemma B.1 implies that $\ell(R_j) \geq 2^{-k}\ell(R_0)$ and the claim follows.

Notice now that $\Pi(R_j) = Q'$, because both $\Pi(R_j)$ and Q' are $(n-1)$-dimensional dyadic cubes in H_0 with side length $2^{-k}\ell(R_0)$ and both contain x'. $\square$

Appendix C An Alternative Proof of Lemma 4.3

We assume that we are under the assumptions of Lemma 4.3. So given a Lipschitz domain $D \subset \mathbb{R}^n$ and a relatively open subset V of ∂D, we consider a nonzero function v that is harmonic in D and continuous in $\bar{D}$, vanishing identically in V, and whose normal derivative $\partial_\nu v$ also vanishes in a subset $E \subset V$ of positive surface measure. We have to show that, for every point $x \in V$ that is a density point of E (with respect to surface measure), it holds that

$$\lim_{r\to 0} \frac{\int_{B(x,r)\cap D} |v|\, dm}{\int_{B(x,6r)\cap D} |v|\, dm} = 0. \tag{C.1}$$

To this end, for such point x, given $\varepsilon \in (0, 1)$, let $r_0 > 0$ be small enough so that $B(x, r_0) \cap \partial D \subset V$ and

$$\sigma(\partial D \cap B(x, r) \setminus E) \leq \varepsilon\, \sigma(\partial D \cap B(x, r)) \quad \text{for all } 0 < r \leq r_0.$$

We fix $r \in (0, r_0/3)$. Without loss of generality, we assume that $x = 0$ and denote $B_r = B(0, r)$, We also assume that $\partial D \cap B_{2r}$ is a Lipschitz graph with respect to the horizontal axes, and that D is above the graph. As usual, we understand that v has been extended by 0 in D^c. As shown in Theorem A.1,

$$(\Delta v)|_{B_r} = -\partial_\nu v\ \sigma|_{\partial D\cap B_r} =: \mu.$$

Thus, $g := v - \mathcal{E} * \mu$ is harmonic in B_r.

We intend to apply the three-ball inequality to the function g. In order to do this, first we need to estimate the total variation of the signed measure μ. We apply the

Rellich-Necas identity

$$
\begin{aligned}
&2(\beta\cdot\nabla v)\Delta v\\
&\quad = 2\,\mathrm{div}((\beta\cdot\nabla v)\nabla v) - \mathrm{div}(\beta\,|\nabla v|^2) + |\nabla v|^2\,\mathrm{div}\,\beta - 2\sum_{i,j}\partial_i\beta_j\,\partial_i v\,\partial_j v,
\end{aligned}
$$

with a vector field $\beta = \varphi\, e_n$, where φ is a smooth function supported on $B_{(3/2)r}$ and identically 1 on B_r. Integrating the above identity in $B_{2r}\cap D$ with respect to Lebesgue measure and applying the divergence theorem, we obtain

$$
\begin{aligned}
0 = &\int_{\partial D\cap B_{2r}}\left[2\,\varphi(y)\,\partial_n v(y)\,\partial_\nu v(y) - \varphi(y)\,|\nabla v(y)|^2\,(e_n\cdot\nu(y))\right]d\sigma(y)\\
&+\int_{B_{2r}}\partial_n\varphi\,|\nabla v|^2\,dy - 2\int_{B_{2r}}\sum_i\partial_i\varphi_n\,\partial_i v\,\partial_n v\,dy.
\end{aligned}
$$

In fact, to be more precise, since v need not be smooth up to ∂D, first we apply the divergence theorem in the domain $B_{2r}\cap(\delta e_n + D)$ with $\delta > 0$, and then we let $\delta\to 0$ as in the proof of Lemma 2.2. Taking into account that $\nabla v = (\partial_\nu v)\nu$ on $\partial D\cap B_{2r}$, we get

$$
\begin{aligned}
&\int_{\partial D\cap B_{2r}}\varphi(y)\,|\partial_\nu v(y)|^2\,(e_n\cdot\nu(y))d\sigma(y)\\
&\qquad = -\int_{B_{2r}}\partial_n\varphi\,|\nabla v|^2\,dy + 2\int_{B_{2r}}\sum_i\partial_i\varphi_n\,\partial_i v\,\partial_n v\,dy.
\end{aligned}
$$

Thus, recalling that $\chi_{B_r}\le\varphi\le\chi_{B_{(3/2)r}}$ and applying Caccioppoli's inequality,

$$
\int_{\partial D\cap B_r}|\partial_\nu v(y)|^2\,d\sigma(y)\lesssim\frac{1}{r}\int_{B_{\frac{3}{2}r}}|\nabla v|^2\,dy\lesssim\frac{1}{r^3}\int_{B_{2r}}|v|^2\,dy.
$$

Now, by Cauchy-Schwarz, we derive

$$
\begin{aligned}
\|\mu\| &= \int_{\partial D\cap B_r}|\partial_\nu v(y)|d\sigma(y)\\
&\le\left(\int_{\partial D\cap B_r}|\partial_\nu v(y)|^2\,d\sigma(y)\right)^{1/2}\sigma(B_r\cap\partial D\setminus E)^{1/2}\\
&\lesssim\varepsilon^{\frac{1}{2}}\,r^{\frac{n}{2}-2}\left(\int_{B_{2r}}|v|^2\,dy\right)^{1/2}.
\end{aligned}
$$

Let $x' = -\frac{r}{10}\,e_n$. Then there is some $c_2 > 0$ depending just on the Lipschitz character of D such that $\overline{B(x',2c_2r)}\subset B_r\setminus\bar{D}$. We denote $r' = c_2 r$. Observe now that for any $s\in[r',r]$,

$$
\int_{\partial B(x',s)}|\mathcal{E}*\mu|d\sigma\lesssim\iint_{\partial B(x',s)}\frac{1}{|y-z|^{n-2}}\,d\sigma(y)d|\mu|(z)\lesssim\|\mu\|\,s. \tag{C.2}
$$

Thus, the function[4] $w := \mathcal{E} * \mu$ satisfies

$$\text{(C.3)}\qquad \begin{aligned} ⨍_{\partial B(x',s)} |w|\,d\sigma \lesssim \frac{1}{r^{n-2}}\,\|\mu\| \lesssim \varepsilon^{1/2}\,\frac{1}{r^{n/2}}\left(\int_{B_{2r}} |v|^2\,dy\right)^{1/2} \\ \approx \varepsilon^{1/2}\left(⨍_{B_{2r}} |v|^2\,dy\right)^{1/2}, \end{aligned}$$

with the implicit constants, from now on, possibly depending on c_2 and thus on the Lipschitz character of D.

Since $g = w$ in $\overline{B(x', 2r')}$, and g is harmonic in $B(x', 2r')$ and continuous in $\overline{B(x', 2r')}$, we deduce that, for all $z \in \overline{B(x', r')}$,

$$|g(z)| \lesssim ⨍_{\partial B(x',2r')} |g|\,d\sigma \lesssim \varepsilon^{1/2}\left(⨍_{B_{2r}} |v|^2\,dy\right)^{1/2}.$$

Hence,

$$h_g(x', r') := ⨍_{\partial B(x',r')} |g|^2\,d\sigma \lesssim \varepsilon ⨍_{B_{2r}} |v|^2\,dy.$$

Next we estimate $h_g(x', \frac{3}{4}r)$. We write

$$⨍_{\partial B(x',\frac{4}{5}r)} |g|\,d\sigma \le ⨍_{\partial B(x',\frac{4}{5}r)} |v|\,d\sigma + ⨍_{\partial B(x',\frac{4}{5}r)} |w|\,d\sigma.$$

Since $|v|$ is subharmonic in B_{2r} and continuous in $\overline{B_{2r}}$, we have

$$|v(z)| \lesssim ⨍_{B_{2r}} |v|\,dy \quad \text{for all } y \in B_r.$$

Therefore,

$$⨍_{\partial B(x',\frac{4}{5}r)} |g|\,d\sigma \lesssim ⨍_{B_{2r}} |v|\,dy + \varepsilon^{1/2}\left(⨍_{B_{2r}} |v|^2\,dy\right)^{1/2} \lesssim \left(⨍_{B_{2r}} |v|^2\,dy\right)^{1/2}.$$

Since g is harmonic in $B(x', \frac{4}{5}r)$ and continuous in $\overline{B(x', \frac{4}{5}r)}$, we have, for all $z \in \overline{B(x', \frac{3}{4}r)}$,

$$|g(z)| \lesssim ⨍_{\partial B(x',\frac{4}{5}r)} |g|\,d\sigma \lesssim \left(⨍_{B_{2r}} |v|^2\,dy\right)^{1/2},$$

and so

$$h_g(x', \tfrac{3}{4}r) \lesssim ⨍_{B_{2r}} |v|^2\,dy.$$

Recall now that, by the three-ball inequality, given $\alpha \in (0,1)$ and $r_2 > r_1 > 0$ such that $\overline{B(x', r_2)} \subset B_r$, it holds that

$$h_g\big(x', r_1^{\alpha} r_2^{1-\alpha}\big) \le h_g(x', r_1)^{\alpha}\, h_g(x', r_2)^{1-\alpha}.$$

[4] In the case $n = 2$ it is better to choose $w = \mathcal{E} * \mu - \frac{1}{2\pi}\,\mu(\mathbb{R}^2)\,\log r = \frac{1}{2\pi}\,\log\frac{|x|}{r} * \mu$, and then (C.2) and (C.3) follow analogously.

In fact, this inequality is an immediate consequence of the convexity of the function $\log h_g(x', e^t)$, proven in Lemma 2.2 in a more general situation. Applying this inequality with $r_1 = r'$, $r_2 = \frac{3}{4}r$, and α such that

$$(r')^\alpha \, (\tfrac{3}{4}r)^{1-\alpha} = \tfrac{2}{3}r,$$

i.e.,

$$\alpha = \frac{\log \frac{9}{8}}{\log \frac{3}{4c_2}},$$

we infer that

$$h_g(x', \tfrac{2}{3}r) \lesssim \varepsilon^\alpha \fint_{B_{2r}} |v|^2 \, dy.$$

Hence, using Cauchy-Schwarz and again (C.3),

$$\begin{aligned} \fint_{\partial B(x', \frac{2}{3}r)} |v| d\sigma &\leq \fint_{\partial B(x', \frac{2}{3}r)} |g| d\sigma + \fint_{\partial B(x', \frac{2}{3}r)} |w| d\sigma \\ &\lesssim h_g(x', \tfrac{2}{3}r)^{1/2} + \varepsilon^{1/2} \left(\fint_{B_{2r}} |v|^2 \, dy \right)^{1/2} \\ &\lesssim \varepsilon^{\alpha/2} \left(\fint_{B_{2r}} |v|^2 \, dy \right)^{1/2}. \end{aligned}$$

Since $|v|$ is subharmonic in B_{3r} and continuous in $\overline{B_{3r}}$ and $\overline{B_{(1/2)r}} \subset B(x', \frac{2}{3}r)$, we get

$$\fint_{B_{\frac{1}{2}r}} |v| dy \lesssim \fint_{\partial B(x', \frac{2}{3}r)} |v| d\sigma \lesssim \varepsilon^{\alpha/2} \left(\fint_{B_{2r}} |v|^2 \, dy \right)^{1/2} \lesssim \varepsilon^{\alpha/2} \fint_{B_{3r}} |v| dy.$$

So we have shown that, for any $\varepsilon > 0$, if r is small enough,

$$\frac{\fint_{B_{(1/2)r}} |v| dy}{\fint_{B_{3r}} |v| dy} \lesssim \varepsilon^{\alpha/2},$$

which proves the lemma.

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