



Sharpness of the Phase Transition for the Orthant Model

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Abstract

The orthant model is a directed percolation model on \mathbb{Z}^d , in which all clusters are infinite. We prove a sharp threshold result for this model: if p is larger than the critical value above which the cluster of 0 is contained in a cone, then the shift from 0 that is required to contain the cluster of 0 in that cone is exponentially small. As a consequence, above this critical threshold, a shape theorem holds for the cluster of 0, as well as ballisticity of the random walk on this cluster.

Keywords Oriented percolation · Phase transitions · Degenerate random environments · Random graphs

Mathematics Subject Classification (2010) 60K35

1 Introduction and Main Result

We consider the orthant model on the directed graph \mathbb{Z}^d , $d \geq 2$, with nearest neighbour edges. This model is informally described as follows. Let e_1, \dots, e_d be the standard unit basis vectors of \mathbb{R}^d . We set $\mathcal{E}_+ := \{e_1, \dots, e_d\}$, and $\mathcal{E}_- := \{-e_1, \dots, -e_d\}$, as well as $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$. A vertex $v \in \mathbb{Z}^d$ is connected to the vertices $v + e$ for all $e \in \mathcal{E}_+$ with a directed edge with probability p , independently of the other vertices. Otherwise, so with probability $1 - p$, the vertex v is connected to the vertices $v + e$ for all $e \in \mathcal{E}_-$. This model was introduced by Holmes and Salisbury [10, 11]. The model is shown in Fig. 1 for $d = 2$.

The random directed graph obtained in this way has the property that every vertex is in an infinite cluster, since, for example, either the edge in the direction e_1 or in the direction $-e_2$ is always available. Therefore, there is no classical percolation phase

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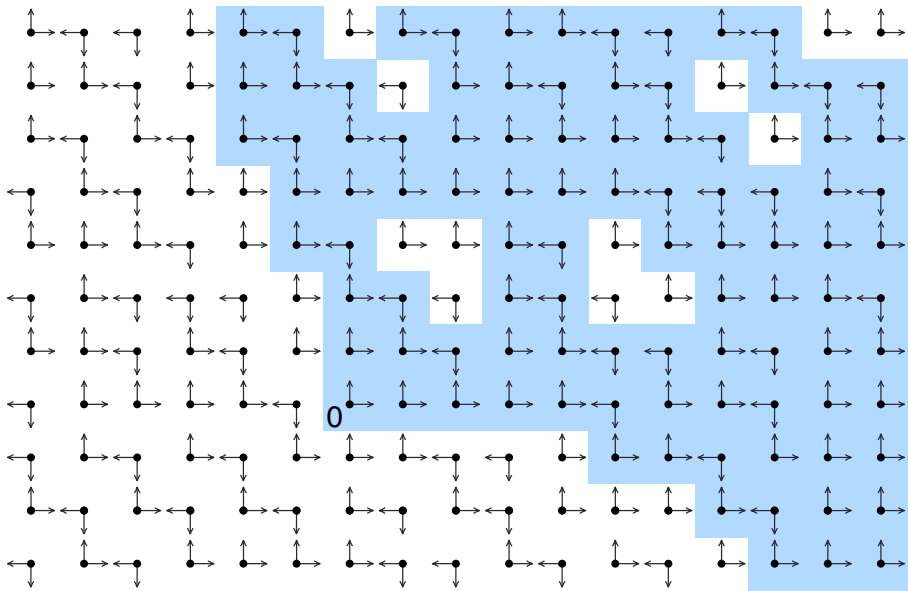


Fig. 1 The orthant model on \mathbb{Z}^2 . The cluster of the origin is shaded blue

transition in this model, where for small p there are only finite clusters, and for large p an infinite cluster exists. Instead, for small p the clusters will have a tendency to move in the directions of \mathcal{E}_- , and for large p a tendency in the directions of \mathcal{E}_+ . In order to make this notion precise, we introduce a cone oriented in the direction $\mathbf{1} := e_1 + \dots + e_d$. For $0 \leq \eta \leq 1$, we define the convex cone

$$\mathcal{K}_\eta = \{x \in \mathbb{R}^d : x \cdot \mathbf{1} \geq \eta \|x\|_1\}.$$

Note that \mathcal{K}_0 is a half-space, and that \mathcal{K}_1 is the positive orthant.

For $v, w \in \mathbb{Z}^d$, we say that $v \rightarrow w$, whenever there is a directed path from v to w . Note that this is not a symmetrical event, since we are working with a directed graph. Furthermore, for $A \subset \mathbb{Z}^d$, we say that $v \rightarrow A$, whenever there exists $w \in A$ such that $v \rightarrow w$. For $v \in \mathbb{Z}^d$, let \mathcal{C}_v denote the forward cluster of v , i.e.,

$$\mathcal{C}_v := \{w \in \mathbb{Z}^d : v \rightarrow w\}.$$

We can define the critical point above which \mathcal{C}_0 is contained in a translated cone with parameter η :

$$\tilde{p}_c(\eta) := \inf \{p : \mathbb{P}_p(\mathcal{C}_0 \subset -n\mathbf{1} + \mathcal{K}_\eta \text{ for some } n \in \mathbb{N}) = 1\}.$$

Note that this critical point is increasing in η , and that $\tilde{p}_c(1) = 1$, so that $\tilde{p}_c(\eta)$ is of interest for $\eta \in [0, 1)$. In fact, Holmes and Salisbury [12] have proven that $p_c^{\text{OSP}} \leq \tilde{p}_c(\eta) < 1$ for all $0 < \eta < 1$, where p_c^{OSP} is the critical parameter for oriented site percolation on the triangular lattice. Furthermore, from considerations later in this section, it follows that $\tilde{p}_c(0) > 0$. Therefore, $\tilde{p}_c(\eta)$ is non-trivial for $0 \leq \eta < 1$. In this paper, we will prove the following result.

Theorem 1 Consider the orthant model on \mathbb{Z}^d with parameter p . Let $0 \leq \eta < 1$ and suppose $p > \tilde{p}_c(\eta)$. Then, there exists a constant $c_p > 0$, such that for all $n \in \mathbb{N}$,

$$\mathbb{P}_p(0 \rightarrow (-n\mathbf{1} + \mathcal{K}_\eta)^c) \leq \exp(-c_p n).$$

The above result for $d = 2$ was proven by Holmes and Salisbury [10] by making a connection with oriented site percolation on the triangular lattice. The result of Theorem 1 is known as a sharp threshold, or as a sharp phase transition. This type of result has been proven in a variety of models, most notably, and initially, for Bernoulli bond percolation in the 80’s by Menshikov [16] and Aizenman and Barsky [1]. More recently, a revolutionary technique using the OSSS inequality was developed by Duminil-Copin, Raoufi and Tassion to prove sharp thresholds in models with more complexities [5–8]. These models include the random cluster model, Voronoi percolation, Boolean percolation, Gaussian fields, the corrupted compass model and the Widom-Rowlinson model [3, 4, 15, 17].

As a consequence of the above sharp threshold result, we can prove a shape theorem for \mathcal{C}_0 above $\tilde{p}_c := \lim_{\eta \downarrow 0} \tilde{p}_c(\eta)$. This critical point can also be written as

$$\tilde{p}_c = \inf \{ p : \exists \eta > 0 \text{ s.t. } \mathbb{P}_p(\mathcal{C}_0 \subset -n\mathbf{1} + \mathcal{K}_\eta \text{ for some } n \in \mathbb{N}) = 1 \}.$$

A shape theorem for the orthant model was first proven by Holmes and Salisbury [14] for large p . Using Theorem 1, we can extend this result to all $p > \tilde{p}_c$. In order to state the shape theorem, we introduce for $u \in \mathbb{Z}^d$

$$\beta_n(u) := \inf \{ k \in \mathbb{Z} : k\mathbf{1} + nu \in \mathcal{C}_0 \}.$$

Furthermore, let $\Lambda_r := \{ v \in \mathbb{Z}^d : \|v\|_\infty \leq r \}$ be the closed ball around 0 with radius r with respect to the L^∞ -norm. Borrowing the notation from [14], the shape theorem for the orthant model can be stated as follows.

Corollary 1 (Shape theorem) Let $p > \tilde{p}_c$. The following hold for the orthant model on \mathbb{Z}^d with parameter p .

- a) For $u \in \mathbb{Z}^d$, there is a deterministic $\gamma(u) \in \mathbb{R}$ such that $\frac{\beta_n(u)}{n} \rightarrow \gamma(u)$, as $n \rightarrow \infty$, \mathbb{P}_p -almost surely.
- b) This limit satisfies $\gamma(u+w) \leq \gamma(u) + \gamma(w)$, $\gamma(ru) = r\gamma(u)$, $\gamma(u+r\mathbf{1}) = \gamma(u) - r$, for $u, w \in \mathbb{Z}^d$, and $r \in \mathbb{N}$. Furthermore, γ is symmetric under permutation of coordinates, $\gamma(u) \geq 0$ if $u \cdot \mathbf{1} \leq 0$, and $\gamma(u) \leq 0$ if u lies in the positive orthant.
- c) The limit γ extends to a Lipschitz map $\mathbb{R}^d \rightarrow \mathbb{R}$ with these same properties, but for $r \in [0, \infty)$ and $u, w \in \mathbb{R}^d$.
- d) The set $C := \{ z \in \mathbb{R}^d : \gamma(z) \leq 0 \}$ is a closed convex cone, which is symmetric under permutations of the coordinates, contains the positive orthant, and is contained in the half-space $\mathcal{K}_0 = \{ z : z \cdot \mathbf{1} \geq 0 \}$.
- e) Let $\mathcal{C}_0^* := \mathcal{C}_0 + e_1\mathbb{N}_0$, i.e., “ \mathcal{C}_0 with its holes filled in”. It holds that $\frac{1}{n}\mathcal{C}_0^* \rightarrow C$, in the sense that for every $\varepsilon > 0$ and every $r > 0$, the following holds \mathbb{P}_p -a.s. for sufficiently large (random) n :

$$\left(\Lambda_r \cap \frac{1}{n}\mathcal{C}_0^* \right) \subset \Lambda_\varepsilon + C, \quad \text{and} \quad (\Lambda_r \cap C) \subset \Lambda_\varepsilon + \frac{1}{n}\mathcal{C}_0^*.$$

To prove this theorem for all $p > \tilde{p}_c$, we modify the proof in [14] by using Theorem 1 in the places where they require p to be large. Another consequence of Theorem 1 is the ballisticity of the random walk on \mathcal{C}_0 .

Corollary 2 (Ballisticity of the Random Walk) *Consider the orthant model on \mathbb{Z}^d with parameter $p > \tilde{p}_c$. Let X_n be a simple random walk on \mathcal{C}_0 and let P be the annealed law of this random walk (i.e., averaged over \mathcal{C}_0). Then there exists $v > 0$ such that $\frac{1}{n}X_n \rightarrow v\mathbf{1}$ P -a.s. as $n \rightarrow \infty$, and*

$$\left(\frac{X_{[nt]} - v\mathbf{1}nt}{\sqrt{n}} \right)_{t \geq 0} \Rightarrow (B_t)_{t \geq 0}, \quad \text{as } n \rightarrow \infty,$$

weakly under P , in the space of càdlàg functions endowed with the Skorohod topology, where $(B_t)_{t \geq 0}$ is a d -dimensional Brownian motion with nonsingular covariance matrix Σ .

This is Theorem 1.4 combined with Corollary 1.9 of [12] by Holmes and Salisbury applied to the orthant model. Their theorem is stated for more general models, and requires two conditions, one of which they show to hold for the orthant model with any value of p . The other condition is the existence of $\eta > 0$ and $c > 0$ such that $\mathbb{P}_p(0 \rightarrow (-n\mathbf{1} + \mathcal{K}_\eta)^c) \leq \exp(-cn^\beta)$, for some $\beta > 0$. By taking $\beta = 1$ and assuming $p > \tilde{p}_c$, it follows from Theorem 1 that this condition holds for the orthant model with parameter p . Corollary 2 is therefore an immediate consequence of combining Theorem 1 with Theorem 1.4 of [12].

Despite the above results, the theoretical picture of the orthant model is still incomplete. We will use the remainder of this section to formulate two open questions for the model. The shape theorem and the ballisticity of the random walk have now been shown to hold for $p > \tilde{p}_c := \lim_{\eta \downarrow 0} \tilde{p}_c(\eta)$. A natural extension would be to prove these results for $p > \tilde{p}_c(0)$. This would follow from the continuity of $\tilde{p}_c(\eta)$.

Open Problem 1 *Consider the orthant model on \mathbb{Z}^d . The function $\eta \mapsto \tilde{p}_c(\eta)$ is continuous on $[0, 1]$.*

A critical value other than \tilde{p}_c can be defined for the orthant model. In order to state this definition, we introduce for $v \in \mathbb{Z}^d$,

$$L_v := \inf\{k \in \mathbb{Z} : v + ke_1 \in \mathcal{C}_0\}.$$

The critical value p_c is defined as

$$p_c := \sup\{p : L_0 = -\infty \text{ a.s.}\}.$$

Holmes and Salisbury [13] have shown that this critical value is nontrivial, i.e., $0 < p_c < 1$. From the definitions of the critical values, it is clear that $p_c \leq \tilde{p}_c(0) \leq \tilde{p}_c$. However, it is as of yet unclear that above p_c there exists a cone with parameter $\eta > 0$ that contains the forward cluster of 0.

Open Problem 2 Consider the orthant model on \mathbb{Z}^d . It holds that

$$p_c = \tilde{p}_c.$$

In order to prove this, perhaps it is most natural to first show that $p_c = \tilde{p}_c(0)$, and subsequently show the continuity of $\tilde{p}_c(\eta)$.

The sharp threshold result of Theorem 1 will be proven in Section 3, while some preliminaries required for this proof are introduced in Section 2. The proof for Corollary 1 is given in Section 4.

2 Preliminaries

We can couple the model for different values of p by considering a coupling similar to the standard coupling in Bernoulli percolation: we consider a family of i.i.d. random variables $(U_v)_{v \in \mathbb{Z}^d}$, and connect v to $v + e$ for all $e \in \mathcal{E}_+$ whenever $U_v < p$, and to $v + e$ for all $e \in \mathcal{E}_-$ if $U_v \geq p$. One difficulty in analysing the orthant model is the lack of monotonicity in p , i.e., a path from v to w might be lost if we increase p . To deal with this issue, we introduce the half-orthant model. In this model a vertex v is always connected to $v + e$ for all $e \in \mathcal{E}_+$, whereas v is connected to $v + e$, for all $e \in \mathcal{E}_-$, with probability $1 - p$. This model is monotone in p , in the sense that $\mathbb{1}\{v \rightarrow w\}$ is monotonically decreasing in p , under the coupling where v is connected to $v + e$, for all $e \in \mathcal{E}_-$, whenever $U_v > p$. Let \mathcal{C}_v^* denote the forward cluster of v in the half-orthant model. The half-orthant model dominates the orthant model, in the sense that $\mathcal{C}_v \subseteq \mathcal{C}_v^*$, almost surely under a suitable coupling between the two models. For $v \in \mathbb{Z}^d$, we further define

$$L_v^* := \inf\{k \in \mathbb{Z} : v + ke_1 \in \mathcal{C}_0^*\}.$$

From the domination it follows that $L_v \geq L_v^*$. However, it turns out that equality holds: $L_v^* = L_v$ for all $v \in \mathbb{Z}^d$ [13, Thm. 1.9]. So, loosely speaking, if we only care about the leftmost boundary of \mathcal{C}_0 , it does not matter if we consider the orthant model or the half-orthant model. This allows us to prove statements for the orthant model by making use of the monotonicity of the half orthant model. In light of this, we remark that the above definition of \mathcal{C}_0^* coincides with the definition stated in Corollary 1. Furthermore, we note that $L_v < \infty$ for all $v \in \mathbb{Z}^d$, since $L_v^* < \infty$, but it might be the case that $L_v = -\infty$ for some $v \in \mathbb{Z}^d$. In fact, Holmes and Salisbury proved that if L_v is finite for some $v \in \mathbb{Z}^d$, then it is finite for all $v \in \mathbb{Z}^d$ (Lemma 2.2 of [13]). For $p < p_c$ it follows that $L_v^* = -\infty$ for all $v \in \mathbb{Z}^d$, and in this case, $\mathcal{C}_0^* = \mathbb{Z}^d$. On the other hand, if $p > p_c$, L_v is finite for all $v \in \mathbb{Z}^d$ using the monotonicity of the half-orthant model.

To prove Theorem 1, it therefore suffices to work with the half-orthant model. We start by giving a formal description of this model. For $p \in [0, 1]$, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$, where

$$\Omega = \{0, 1\}^{\mathbb{Z}^d},$$

the σ -algebra \mathcal{F} is generated by the cylindrical events, and \mathbb{P}_p is the product measure on Ω such that $\mathbb{P}_p(\omega_v = 1) = p$ for all $v \in \mathbb{Z}^d$. From $\omega \in \Omega$ we obtain the edge configuration $\xi \subseteq \{(v, v + e) : v \in \mathbb{Z}^d, e \in \mathcal{E}\}$ by adding the edge $(v, v + e)$ to the graph for all $e \in \mathcal{E}_+$, and for all $e \in \mathcal{E}_-$ whenever $\omega_v = 0$.

For $v, w \in \mathbb{Z}^d$, we say $v \sim w$ when v is a neighbour of w , i.e., whenever $w = v + e$ for some $e \in \mathcal{E}$. Furthermore, we say that $v \rightsquigarrow w$, whenever $(v, w) \in \xi$. For $A \subset \mathbb{Z}^d$, we say that $v \xrightarrow{A} w$, whenever there is a path from v to w using only edges in ξ with starting points in A . Note that w does not have to be an element A for this event to hold. For $A = \mathbb{Z}^d$ we use the shorthand notation $\{v \longrightarrow w\} := \{v \xrightarrow{\mathbb{Z}^d} w\}$. Furthermore, the event $v \xrightarrow{A} v$ trivially holds for all $v \in \mathbb{Z}^d$, and all $A \subset \mathbb{Z}^d$.

The proof will make use of the OSSS inequality for Boolean functions $f : \Omega \rightarrow \{0, 1\}$. In order to state this inequality, we introduce the influence of v on f . This is defined as

$$\text{Inf}_v := \mathbb{P}_p(f(\omega) \neq f(\omega^{v \mapsto p})),$$

where $\omega^{v \mapsto p}$ is given by

$$(\omega^{v \mapsto p})_w = \begin{cases} \omega'_v & \text{if } w = v, \\ \omega_v & \text{if } w \neq v, \end{cases}$$

for a Bernoulli distributed random variable ω'_v with parameter p , which is independent of the other variables. In other words,

$$\text{Inf}_v = 2p(1 - p)\mathbb{P}_p(v \text{ is pivotal for the event } \{f = 1\}).$$

A decision tree T is a random sequence of vertices (v_0, v_1, \dots) that is built sequentially as follows. The tree starts by revealing the value of ω_{v_0} , for the starting vertex v_0 . Then, depending on the value of ω_{v_0} , it chooses a vertex v_1 and reveals the value of ω_{v_1} . This process continues until it has obtained enough information to determine the value of f , i.e., whenever the values of ω_v for unrevealed vertices v , cannot change f any more. This leads to the definition of the revelation of v by T :

$$\text{Rev}_v(T) := \mathbb{P}_p(\omega_v \text{ is revealed by } T).$$

The OSSS inequality states that for a Boolean function f depending on finitely many variables and a decision tree T that determines the value of f we have

$$\text{Var}_p(f) \leq \sum_{v \in \mathbb{Z}^d} \text{Inf}_v \text{Rev}_v(T).$$

This inequality was proven by O’Donnell, Saks, Schramm and Servedio [19]. A detailed exposition of Boolean functions has been written by O’Donnell [18], in which the proof of the OSSS inequality can also be found. Still, we will make use of Boolean functions f that depend on infinitely many variables, so that we require an additional limit argument. The OSSS inequality can be generalized to Boolean functions on Ω , provided f is measurable with respect to the σ -algebra

$$\sigma(\{\omega_v : \omega_v \text{ revealed by } T\}).$$

This is for example the case for decision trees that terminate in a finite number of steps on the set $\{f = 1\}$. This generalisation of the OSSS inequality has been stated by Duminił-Copin, Raoufi and Tassion [7]. A proof can be found in [2].

3 Proof of Theorem 1

For $\eta \geq 0$ and $n \in \mathbb{N}$, we define the Boolean function

$$f_n := \mathbb{1}\{0 \longrightarrow (-n\mathbf{1} + \mathcal{K}_\eta)^c\}.$$

3.1 Exploration Algorithm

We now introduce decision trees that determine the value of f_n . A vital point in the proof is that we can uniformly bound the revealment of the vertices. If we only use one decision tree with a deterministic starting point, then the starting vertex will have revealment 1, so that we cannot find a nontrivial uniform bound on the revealment. Therefore, we will introduce the decision trees T_k , for $1 \leq k \leq n$, which all start at different vertices. In this way, we can average over k and have a meaningful uniform bound on the revealment. The basic idea of the decision tree T_k is that it explores the cluster of the boundary of $-k\mathbf{1} + \mathcal{K}_\eta$. If $0 \longrightarrow (-n\mathbf{1} + \mathcal{K}_\eta)^c$, this path must go through the boundary of the cone $-k\mathbf{1} + \mathcal{K}_\eta$, so that T_k determines f_n . Furthermore, T_k terminates in a finite number of steps when $f_n = 1$.

We will now describe the exploration algorithm of T_k more precisely. We define the boundary and the outer boundary of the cone as

$$\begin{aligned} \partial(-k\mathbf{1} + \mathcal{K}_\eta) &:= \{v \in (-k\mathbf{1} + \mathcal{K}_\eta) \cap \mathbb{Z}^d : \exists w \in (-k\mathbf{1} + \mathcal{K}_\eta)^c \cap \mathbb{Z}^d \text{ with } v \sim w\}, \\ \partial^+(-k\mathbf{1} + \mathcal{K}_\eta) &:= \{v \in (-k\mathbf{1} + \mathcal{K}_\eta)^c \cap \mathbb{Z}^d : \exists w \in (-k\mathbf{1} + \mathcal{K}_\eta) \cap \mathbb{Z}^d \text{ with } v \sim w\}. \end{aligned}$$

The decision tree T_k consists of two phases. In the first phase, T_k explores the backward cluster of $\partial(-k\mathbf{1} + \mathcal{K}_\eta)$ inside the cone, that is, it explores the set $\{v \in -k\mathbf{1} + \mathcal{K}_\eta : v \longrightarrow \partial(-k\mathbf{1} + \mathcal{K}_\eta)\}$. When this is finished, the set of vertices

$$\left\{ v \in \partial^+(-k\mathbf{1} + \mathcal{K}_\eta) : 0 \xrightarrow{-k\mathbf{1} + \mathcal{K}_\eta} v \right\}$$

has been determined. In the second phase, the algorithm explores the forward clusters of these vertices. If for one of these vertices we find that $v \longrightarrow (-n\mathbf{1} + \mathcal{K}_\eta)^c$, then we also have $0 \longrightarrow (-n\mathbf{1} + \mathcal{K}_\eta)^c$. A schematic visualisation of the algorithm is shown in Fig. 2.

There is however one technical issue: since f_n depends on the state of infinitely many vertices, it is possible that the algorithm gets stuck exploring inside $-k\mathbf{1} + \mathcal{K}_\eta$, and never gets to explore the forward clusters outside $-k\mathbf{1} + \mathcal{K}_\eta$. In order to deal with this, the decision tree operates in rounds, denoted by $i \in \mathbb{N}$. Recall that Λ_r is the ball of radius r around 0 with respect to L^∞ -norm. In round i we only explore inside Λ_i , so it is not possible to get stuck in any particular phase. Note that if $0 \longrightarrow (-n\mathbf{1} + \mathcal{K}_\eta)^c$, there exists $i \in \mathbb{N}$ such that $0 \xrightarrow{\Lambda_i} (-n\mathbf{1} + \mathcal{K}_\eta)^c$.

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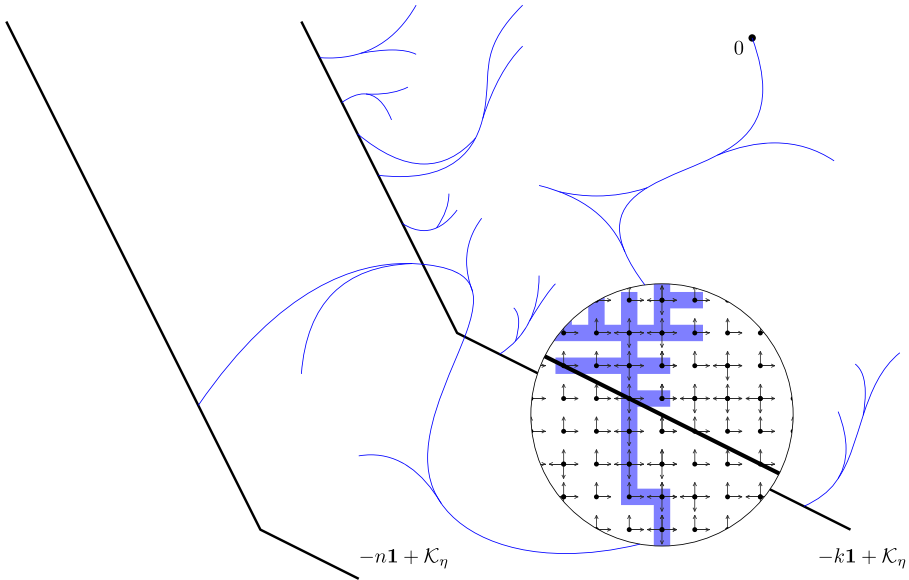


Fig. 2 The algorithm T_k exploring the cluster of $\partial(-k\mathbf{1} + \mathcal{K}_\eta)$ to find a path from 0 to $(-n\mathbf{1} + \mathcal{K}_\eta)^c$. The blue vertices are revealed.

We denote by \mathcal{R} the set of revealed vertices. Furthermore, we denote by \mathcal{A} the set of active vertices for the first phase and by \mathcal{B} the set of active vertices for the second phase. We start the algorithm by setting $\mathcal{A} := \mathcal{A}_0 := \partial^+(-k\mathbf{1} + \mathcal{K}_\eta)$, and $\mathcal{B} := \emptyset$. The pseudocode of T_k is given in Algorithm 1. We have to be careful when updating \mathcal{A} in the first phase: by revealing v it is possible that we create a new path $x \xrightarrow{\mathcal{R}} \partial^+(-k\mathbf{1} + \mathcal{K}_\eta)$ for some $x \neq v$. Therefore, it is not sufficient to only consider $w \sim v$ for the update of \mathcal{A} . Instead, we add w to \mathcal{A} if and only if $w \notin \mathcal{R} \cap \mathcal{B}$, and if there exists $x \in \mathcal{R}$ such that $x \sim w$, and $x \xrightarrow{\mathcal{R}} \partial^+(-k\mathbf{1} + \mathcal{K}_\eta)$. At the start of any iteration of the inner loops of the algorithm, the following hold for the active sets \mathcal{A} and \mathcal{B} :

$$\begin{aligned} \mathcal{A} &\subseteq \mathcal{A}_0 \setminus \mathcal{R} \cup \{v \in (-k\mathbf{1} + \mathcal{K}_\eta) \cap \mathbb{Z}^d : v \notin \mathcal{R}, \exists w \sim v, \text{ s.t. } w \xrightarrow{\mathcal{R}} \partial^+(-k\mathbf{1} + \mathcal{K}_\eta)\}, \\ \mathcal{B} &= \{v \in (-n\mathbf{1} + \mathcal{K}_\eta) \cap \mathbb{Z}^d : v \notin \mathcal{R}, \exists w \in \partial^+(-k\mathbf{1} + \mathcal{K}_\eta) \text{ s.t. } 0 \xrightarrow{\mathcal{R}} w, w \xrightarrow{\mathcal{R}} v\}. \end{aligned} \tag{1}$$

3.2 Bound on the Revelation

Let $\theta_n(p) := \mathbb{P}_p(f_n = 1)$. Summing the OSSS inequality over k gives

$$n\theta_n(p)(1 - \theta_n(p)) \leq \sum_{v \in \mathbb{Z}^d} \text{Inf}_v \sum_{k=1}^n \text{Rev}_v(T_k). \tag{2}$$

We will now bound $\sum_{k=1}^n \text{Rev}_v(T_k)$ uniformly in v . Let k_v be such that $v \in \partial^+(-k_v\mathbf{1} + \mathcal{K}_\eta)$. Note that $k_{v-1} = k_v + 1$. Suppose first that $k > k_v + 1$. If v is

revealed by T_k in the second phase, we have $0 \rightarrow \partial^+(-k\mathbf{1} + \mathcal{K}_\eta)$ by (1). On the other hand, if v is revealed by T_k in the first phase, there exists $w \sim v$ such that $w \rightarrow \partial^+(-k\mathbf{1} + \mathcal{K}_\eta)$. Since $k > k_v + 1$, we know that $v - \mathbf{1} \in -k\mathbf{1} + \mathcal{K}_\eta$, which implies that every neighbour of v is also contained in $-k\mathbf{1} + \mathcal{K}_\eta$. Applying the union bound gives

Algorithm 1: The exploration algorithm T_k .

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i := n;
 $\mathcal{A} := \partial^+(-k\mathbf{1} + \mathcal{K}_\eta) \cap \Lambda_n$ ;
 $\mathcal{B} := \emptyset$ ;
 $\mathcal{R} := \emptyset$ ;
while  $0 \not\stackrel{\mathcal{R}}{\rightarrow} (-n\mathbf{1} + \mathcal{K}_\eta)^c$  do
    while  $\mathcal{A} \cap \Lambda_i \neq \emptyset$  do
        Take lexicographical minimal  $v \in \mathcal{A} \cap \Lambda_i$ ;
        Reveal  $\omega_v$ ;
         $\mathcal{R} := \mathcal{R} \cup \{v\}$ ;
         $\mathcal{A} := \mathcal{A} \setminus \{v\}$ ;
         $\mathcal{A} := \mathcal{A} \cup \{w \in (-k\mathbf{1} + \mathcal{K}_\eta) \cap \mathbb{Z}^d : w \notin \mathcal{R} \cup \mathcal{B}, \exists x \in \mathcal{R}, x \sim w, \text{ s.t. } x \stackrel{\mathcal{R}}{\rightarrow} \partial^+(-k\mathbf{1} + \mathcal{K}_\eta)\}$ ;
         $\mathcal{B} := \mathcal{B} \cup \{w \in \mathbb{Z}^d : w \notin \mathcal{R}, \exists x \in \partial^+(-k\mathbf{1} + \mathcal{K}_\eta) \text{ s.t. } 0 \stackrel{\mathcal{R}}{\rightarrow} x, x \stackrel{\mathcal{R}}{\rightarrow} w\}$ ;
         $\mathcal{A} := \mathcal{A} \setminus \{w \in \mathbb{Z}^d : w \notin \mathcal{R}, \exists x \in \partial^+(-k\mathbf{1} + \mathcal{K}_\eta) \text{ s.t. } 0 \stackrel{\mathcal{R}}{\rightarrow} x, x \stackrel{\mathcal{R}}{\rightarrow} w\}$ ;
    if  $0 \stackrel{\mathcal{R}}{\rightarrow} (-n\mathbf{1} + \mathcal{K}_\eta)^c$  then return I
    while  $\mathcal{B} \cap \Lambda_i \neq \emptyset$  do
        Take lexicographical minimal  $v \in \mathcal{B} \cap \Lambda_i$ ;
        Reveal  $\omega_v$ ;
         $\mathcal{R} := \mathcal{R} \cup \{v\}$ ;
         $\mathcal{B} := \mathcal{B} \setminus \{v\}$ ;
         $\mathcal{B} := \mathcal{B} \cup \{w \in \mathbb{Z}^d : w \notin \mathcal{R}, \exists x \in \partial^+(-k\mathbf{1} + \mathcal{K}_\eta) \text{ s.t. } 0 \stackrel{\mathcal{R}}{\rightarrow} x, x \stackrel{\mathcal{R}}{\rightarrow} w\}$ ;
         $\mathcal{A} := \mathcal{A} \setminus \{w \in \mathbb{Z}^d : w \notin \mathcal{R}, \exists x \in \partial^+(-k\mathbf{1} + \mathcal{K}_\eta) \text{ s.t. } 0 \stackrel{\mathcal{R}}{\rightarrow} x, x \stackrel{\mathcal{R}}{\rightarrow} w\}$ ;
    if  $0 \stackrel{\mathcal{R}}{\rightarrow} (-n\mathbf{1} + \mathcal{K}_\eta)^c$  then return I
    i := i + 1;

```

$$\sum_{k=1}^n \mathbb{1}\{k > k_v + 1\} \text{Rev}_v(T_k) \leq \sum_{k=1}^n \mathbb{1}\{k > k_v + 1\} \left(\theta_k(p) + \sum_{w \sim v} \mathbb{P}_p(w \rightarrow \partial^+(-k\mathbf{1} + \mathcal{K}_\eta)) \right).$$

Let

$$d_k^w := \sup\{l \in \mathbb{Z} : w - l\mathbf{1} \in -k\mathbf{1} + \mathcal{K}_\eta\}.$$

Note that $d_k^w \geq 0$ for all $k > k_v + 1$. Furthermore $w - d_k^w \mathbf{1} \in -k\mathbf{1} + \mathcal{K}_\eta$, and

$$w - d_k^w \mathbf{1} + \mathcal{K}_\eta \subseteq -k\mathbf{1} + \mathcal{K}_\eta,$$

since $-k\mathbf{1} + \mathcal{K}_\eta$ is a convex cone. Therefore, using translation invariance, it follows that

$$\begin{aligned} \sum_{k=1}^n \mathbb{1}\{k > k_v + 1\} \text{Rev}_v(T_k) &\leq \sum_{k=1}^n \mathbb{1}\{k > k_v + 1\} \left(\theta_k(p) + \sum_{w \sim v} \mathbb{P}_p(w \rightarrow \partial^+(w - d_k^w \mathbf{1} + \mathcal{K}_\eta)) \right) \\ &= \sum_{k=1}^n \mathbb{1}\{k > k_v + 1\} \left(\theta_k(p) + \sum_{w \sim v} \mathbb{P}_p(0 \rightarrow \partial^+(-d_k^w \mathbf{1} + \mathcal{K}_\eta)) \right). \end{aligned} \tag{3}$$

We have $d_{k_v+2}^w \geq 0$, and using the fact that $d_{k+1}^w = d_k^w + 1$, we know $d_{k_v+l}^w \geq l - 2$, for all $l \geq 2$. We can thus bound

$$\begin{aligned} \sum_{k=1}^n \mathbb{1}\{k > k_v + 1\} \text{Rev}_v(T_k) &\leq \sum_{k=1}^n \theta_k(p) + 2d \sum_{k=0}^{n-1} \mathbb{P}_p(0 \rightarrow \partial^+(-k\mathbf{1} + \mathcal{K}_\eta)) \\ &= \sum_{k=1}^n \theta_k(p) + 2d \sum_{k=0}^{n-1} \theta_k(p). \end{aligned} \tag{4}$$

Now suppose $k < k_v$, so $v \notin -k\mathbf{1} + \mathcal{K}_\eta \cup \partial^+(-k\mathbf{1} + \mathcal{K}_\eta)$. If v is revealed, it holds that $0 \xrightarrow{\mathcal{R}} v$. In particular, we have $0 \rightarrow \partial^+(-k\mathbf{1} + \mathcal{K}_\eta)$. We find

$$\sum_{k=1}^n \mathbb{1}\{k < k_v\} \text{Rev}_v(T_k) \leq \sum_{k=1}^n \theta_k(p). \tag{5}$$

Combining (4) and (5) gives

$$\sum_{k=1}^n \text{Rev}_v(T_k) \leq 2 + 2 \sum_{k=1}^n \theta_k(p) + 2d \sum_{k=0}^n \theta_k(p) = (2d + 2) \sum_{k=0}^n \theta_k(p).$$

Writing $S_n := \sum_{k=0}^n \theta_k(p)$, gives

$$\sum_{v \in \mathbb{Z}^d} \text{Inf}_v \geq \frac{1}{2d + 2} \frac{n}{S_n} \theta_n(p) (1 - \theta_n(p)). \tag{6}$$

3.3 Analysis of the Differential Inequality

We are now able to complete the proof of Theorem 1. We can obtain a differential inequality by using Russo’s formula. However, since f_n depends on infinitely many vertices, $\theta_n(p)$ is not necessarily differentiable in p . Instead we have to work with the upper-right Dini derivative:

$$D^+ \theta_n(p) := \limsup_{h \downarrow 0} \frac{\theta_n(p + h) - \theta_n(p)}{h}.$$

Using the fact that $0 \rightarrow (-n\mathbf{1} + \mathcal{K}_\eta)^c$ is a decreasing event, i.e., f_n is a decreasing function of ω , Russo’s formula gives

$$-D^+\theta_n(p) \geq \sum_{v \in \mathbb{Z}^d} \mathbb{P}_p(v \text{ is pivotal for } \{f_n = 1\}) = \frac{1}{2p(1-p)} \sum_{v \in \mathbb{Z}^d} \text{Inf}_v.$$

This version of Russo’s formula can be found in the book on Percolation by Grimmett [9]. This is the point in the proof where we use the monotonicity of the half-orthant model, as well as the coupling given at the start of Section 2. Combining the above inequality with (6) gives

$$-D^+\theta_n(p) \geq \frac{1}{2d} \frac{n}{S_n} \theta_n(p)(1 - \theta_n(p)), \tag{7}$$

where we use $(2d+2)2p(1-p) \leq 2d$ for simplicity. The rest of the proof consists of analysing the above differential inequality. This analysis follows the line of Duminil-Copin, Raoufi and Tassion [7], but since it differs on several points, we choose to include it. We have to work with Dini derivatives instead of regular derivatives, and, more importantly, in our case we cannot give a simple lower bound on $1 - \theta_n(p)$.

To analyse the differential inequality, we introduce the auxiliary critical point

$$\hat{p}_c(\eta) := \sup \left\{ p : \limsup_{n \rightarrow \infty} \frac{\log S_n(p)}{\log n} = 1 \right\}.$$

Note that by the monotonicity of the model, $\limsup_{n \rightarrow \infty} \frac{\log S_n(p)}{\log n} = 1$ for all $p < \hat{p}_c(\eta)$, and $\limsup_{n \rightarrow \infty} \frac{\log S_n(p)}{\log n} < 1$ for all $p > \hat{p}_c(\eta)$. We will first show that $\hat{p}_c(\eta) \leq \tilde{p}_c(\eta)$, for $\eta \geq 0$. To prove this, we assume the contrary, and let $p \in (\tilde{p}_c(\eta), \hat{p}_c(\eta))$. Since $p > \tilde{p}_c(\eta)$, we can fix $l \in \mathbb{N}$, such that for all $n > l$ it holds that $1 - \theta_n(p) \geq 1/2$. We define $T_n(p) := \frac{1}{\log n} \sum_{k=l}^n \frac{\theta_k(p)}{k}$. Taking the upper-right Dini derivative and using (7) gives

$$-D^+T_n \geq \frac{1}{2d} \frac{1}{\log n} \sum_{k=l}^n \frac{\theta_k(p)}{S_k} (1 - \theta_k(p)) \geq \frac{1}{4d} \frac{1}{\log n} \sum_{k=l}^n \frac{\theta_k(p)}{S_k} \geq \frac{1}{4d} \frac{\log S_{n+1} - \log S_l}{\log n}, \tag{8}$$

where in the last inequality we used

$$\frac{\theta_k(p)}{S_k} \geq \int_{S_k}^{S_{k+1}} \frac{1}{x} dx = \log S_{k+1} - \log S_k.$$

Now let $p_1 \in (p, \hat{p}_c(\eta))$. We will integrate the differential inequality between p and p_1 and use the following result regarding Dini derivatives: the Dini derivative of a decreasing function $f : [a, b] \rightarrow \mathbb{R}$ satisfies

$$f(b) - f(a) \leq \int_a^b D^+ f(x) dx. \tag{9}$$

Applying this to $T_n(p)$ and using (8) gives

$$T_n(p_1) - T_n(p) \leq \int_p^{p_1} D^+T_n(s) ds \leq -(p_1 - p) \frac{1}{4d} \frac{\log S_{n+1}(p_1) - \log S_l(p)}{\log n}.$$

Furthermore, $T_n(p)$ converges to $\theta(p) := \lim_{n \rightarrow \infty} \theta_n(p)$ for $n \rightarrow \infty$, since for all $l < m < n$:

$$\theta_n(p) \frac{\sum_{k=l}^n \frac{1}{k}}{\log n} \leq T_n(p) \leq \theta_m(p) \frac{\sum_{k=m}^n \frac{1}{k}}{\log n} + \frac{\sum_{k=l}^{m-1} \frac{1}{k}}{\log n},$$

from which the limit follows by first taking $n \rightarrow \infty$, and then $m \rightarrow \infty$. We find

$$\theta(p_1) - \theta(p) \leq -(p_1 - p) \frac{1}{4d} \limsup_{n \rightarrow \infty} \frac{\log S_{n+1}(p_1) - \log S_l(p)}{\log n}.$$

Since $p < \hat{p}_c(\eta)$, we have that $\limsup_{n \rightarrow \infty} \frac{\log S_n(p)}{\log n} = 1$ and the same holds for p_1 , so that also

$$\limsup_{n \rightarrow \infty} \frac{\log S_{n+1}(p) - \log S_l(p_1)}{\log n} = 1.$$

We conclude

$$\theta(p) \geq \theta(p) - \theta(p_1) \geq \frac{p_1 - p}{4d} > 0, \tag{10}$$

which contradicts $p > \tilde{p}_c(\eta)$, so that we have established that $\hat{p}_c(\eta) \leq \tilde{p}_c(\eta)$.

Now suppose $p > \hat{p}_c(\eta)$. Then there exists $N_1 \in \mathbb{N}$ and $\beta < 1$ such that $S_n(p) \leq n^\beta$ for all $n \geq N_1$, and there exists $N_2 \in \mathbb{N}$ such that $\theta_n(p) \leq \frac{1}{2}$ for all $n \geq N_2$. Combining this with (7) and using the chain rule for Dini derivatives gives

$$D^+ \log \theta_n(p) \leq -\frac{1}{2d} n^{1-\beta} (1 - \theta_n(p)) \leq -\frac{1}{4d} n^{1-\beta},$$

for all $n > N := N_1 \vee N_2$. Let $p_1 := (\tilde{p}_c(\eta) + p)/2$. Integrating the above inequality between p_1 and p and using (9) gives

$$\log \theta_n(p) \leq \log \theta_n(p) - \log \theta_n(p_1) \leq -\frac{1}{4d} (p - p_1) n^{1-\beta}.$$

It follows that

$$\theta_n(p) \leq \exp\left(-\frac{1}{8d} (p - \tilde{p}_c(\eta)) n^{1-\beta}\right).$$

It remains to improve the above stretched exponential decay to proper exponential decay. From the stretched exponential decay it follows that $S(p) := \lim_{n \rightarrow \infty} S_n(p) < \infty$. Combining this fact with (7), and using that $\theta_n(p) \leq \frac{1}{2}$ for $n > N$, since $p > \hat{p}_c(\eta)$, gives

$$D^+ \log \theta_n(p) \leq -\frac{1}{4dS(p)} n.$$

From here the proof is similar as for the stretched exponential decay, and we conclude

$$\theta_n(p) \leq \exp\left(-\frac{1}{8dS(p)} (p - \tilde{p}_c(\eta)) n\right).$$

It follows that $\tilde{p}_c(\eta) = \hat{p}_c(\eta)$, and that Theorem 1 holds with

$$c_p := \frac{1}{8dS(p)} (p - \tilde{p}_c(\eta)) \wedge \sup \{C > 0 : \theta_n(p) \leq \exp(-Cn) \text{ for all } n \leq N\} > 0.$$

Remark A mean-field lower bound can often be obtained from the analysis of a differential inequality such as the above one, which would be $\theta(p) \geq c(\tilde{p}_c(\eta) - p)$ in our case, for $p < \tilde{p}_c(\eta)$, and some constant $c > 0$, independent of p . This does not directly follow from the above analysis, since we have assumed $p \in (\tilde{p}_c(\eta), \hat{p}_c(\eta))$ in the first part of the analysis. If instead we take $p < \hat{p}_c(\eta)$ such that $\theta(p) < 1/2$, then we can still bound $1 - \theta_n(p) \geq 1/2$, for n large enough, and obtain the mean-field lower bound at (10). However, this bound is of little interest for the orthant model, since if $p < p_c$, $\mathcal{C}^*(0) = \mathbb{Z}^d$ almost surely, so that $\theta(p) = 1$. On the other hand, if $p \in (\tilde{p}_c, \tilde{p}_c(\eta))$ for some $\eta > 0$, Corollary 1 implies that $\theta(p) = 1$ as well. This leaves the interval (p_c, \tilde{p}_c) to be considered, but we conjecture this interval to be empty.

4 Proof of the Shape Theorem

To prove Corollary 1, we modify the proof of Holmes and Salisbury [14] in the places where they require p to be large. Their proof is structured in seven lemmas, two of which require a large p . The first of these is Lemma 1 of [14]. This lemma asserts the existence of $\theta > 1$, such that for every $\eta \in [0, 1)$, there exists $p_0 = p_0(\eta, d) < 1$, such that for $p > p_0$, there exists $c_1 > 0$ such that $\mathbb{P}_p(0 \rightarrow (-n\mathbf{1} + \mathcal{K}_\eta)^c) \leq c_1\theta^{-nd}$, for all $n \in \mathbb{N}$. In the remainder of their proof, this lemma is only used for the case $\eta = 0$. Therefore, we can replace this lemma by Theorem 1, and require $p > \tilde{p}_c$, instead of $p > p_0$.

The second lemma in the proof of Holmes and Salisbury which require large p is Lemma 5 of [14]. We will prove this lemma for $p > \tilde{p}_c$, instead of for large p , using Theorem 1. To state this lemma, we let $u \in \mathbb{Z}^d \setminus \mathbb{Z}\mathbf{1}$, and fix $v \in \mathbb{R}^d$ such that $u \cdot v > 0$ and $v \cdot \mathbf{1} = 0$. We define the slab

$$\Lambda_{u,v}(m, n) := \{z \in \mathbb{Z}^d : mu \cdot v \leq z \cdot v < nu \cdot v\}.$$

We are interested in the following three events. Let $A'_n(M)$ be the event there exists a path starting in 0 and ending in a point $k\mathbf{1} + nu$ with $k < n\gamma(u)$ that hits $\Lambda_{u,v}(-\infty, -M)$. Let $A''_n(M)$ be the event there exists a path starting in 0 and ending in a point $k\mathbf{1} + nu$ with $k < n\gamma(u)$ that hits $\Lambda_{u,v}(M + n, \infty)$. Lastly, let \hat{A}_n be the event that there is a path starting in 0 and ending in some point $k\mathbf{1}$, with $k < 0$, and reaches $\Lambda_{u,v}(n, \infty)$. We will prove the following lemma regarding these events:

Lemma 1 *Let $p > \tilde{p}_c$. There exists $c > 0$, such that $\mathbb{P}_p(A'_n(\lfloor cn \rfloor) \text{ i.o.}) = \mathbb{P}_p(A''_n(\lfloor cn \rfloor) \text{ i.o.}) = \mathbb{P}_p(\hat{A}_n(\lfloor cn \rfloor) \text{ i.o.}) = 0$.*

We will prove the above lemma for the event $A'_n(\lfloor cn \rfloor)$, the other two events can be proven similarly. The event $A'_n(\lfloor cn \rfloor)$ is shown in Fig. 3. Let $p > \tilde{p}_c$. By the definition of this critical point there exists $\eta > 0$ such that $p > \tilde{p}_c(\eta)$. We fix such an η . Let $c > 0$, and let $M := M(n) := \lfloor cn \rfloor$. We will choose the precise value of

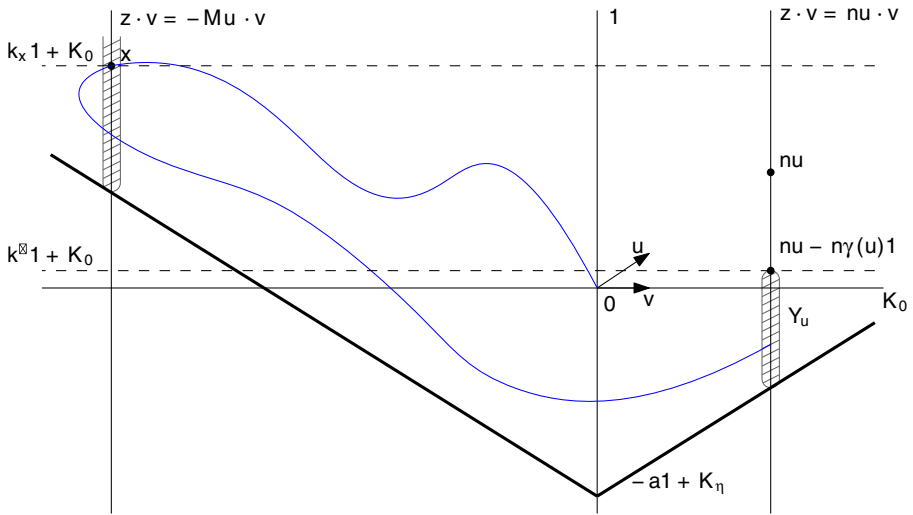


Fig. 3 When the event $A'_n(M)$ occurs, there exists a path from 0 to Y_u going through the shaded region

c later on. Let $a > 0$ and suppose $\mathcal{C}_0^* \subseteq -a\mathbf{1} + \mathcal{K}_\eta$. If $A'_n(M)$ occurs, there exists $x \in \mathbb{Z}^d$ satisfying

$$\begin{cases} (x + a\mathbf{1}) \cdot \mathbf{1} \geq \eta \|x + a\mathbf{1}\|_1, \\ x \cdot v = -Mu \cdot v, \end{cases} \implies \begin{cases} x \cdot \mathbf{1} \geq -2da + \eta \|x\|_1, \\ x \cdot v = -Mu \cdot v, \end{cases} \quad (11)$$

such that $0 \rightarrow x$, and $x \rightarrow y$, with $y = k\mathbf{1} + nu$ for some $k < n\gamma(u)$. Since the L^1 -norm is equivalent to the L^2 -norm, and since the L^2 -norm is invariant under an orthonormal basis change, it follows from the above equation that $\|x\|_1 \geq c_0M$, for some constant $c_0 = c_0(u, v) > 0$. Combining this with the above inequality gives

$$x \cdot \mathbf{1} \geq -2da + c_0\eta M.$$

Define $k_x := x \cdot \mathbf{1}/d$, so that $x \in k_x\mathbf{1} + \mathcal{K}_0$. Then the above inequality implies

$$k_x \geq -2a + \frac{c_0\eta M}{d}.$$

We define the set

$$Y_u := \{y \in \mathbb{Z}^d : y = k\mathbf{1} + nu, \text{ with } k < n\gamma(u)\}.$$

We use Theorem 1 and the union bound to obtain

$$\begin{aligned} \mathbb{P}_p(A'_n(M)) &\leq \exp(-c_p a) + \sum_{k_x \geq -2a + c_0\eta M/d}^{\infty} \mathbb{P}_p(\exists x \in \Lambda_{u,v}(-\infty, -M) : x \cdot \mathbf{1} = dk_x, 0 \rightarrow x, x \rightarrow Y_u) \\ &\leq \exp(-c_p a) + \sum_{k_x \geq -2a + c_0\eta M/d}^{\infty} \sum_{\substack{x \in \Lambda_{u,v}(-\infty, -M), \\ x \cdot \mathbf{1} = dk_x}} \mathbb{P}_p(x \rightarrow Y_u). \end{aligned}$$

We define $k^* := k^*(n) := n(\gamma(u) + u \cdot \mathbf{1}/d)$. With this choice, it follows that $y \in (k^*\mathbf{1} + \mathcal{K}_0)^c$ for all $y \in Y_u$, and all $n \in \mathbb{N}$. We can now use translation invariance to bound

$$\mathbb{P}_p(x \rightarrow Y_u) \leq \mathbb{P}_p(x \rightarrow (k^*\mathbf{1} + \mathcal{K}_0)^c) = \mathbb{P}_p(x - k_x \mathbf{1} \rightarrow (-(k_x - k^*)\mathbf{1} + \mathcal{K}_0)^c).$$

We now fix

$$c := \left(\frac{d\gamma(u) + u \cdot \mathbf{1}}{c_0\eta} + 1 \right) \vee 1.$$

Let $k' \geq 0$ such that $k_x = -2a + \frac{c_0\eta M}{d} + k'$. It holds, that

$$\begin{aligned} k_x - k^* &\geq -2a + \frac{c_0\eta}{d} \left[n \left(\frac{d\gamma(u) + u \cdot \mathbf{1}}{c_0\eta} + 1 \right) \right] + k' - n \left(\gamma(u) + \frac{u \cdot \mathbf{1}}{d} \right) \\ &\geq -2a + \frac{c_0\eta}{d} (n - 1) + k' =: f(n, k'). \end{aligned}$$

It follows, that

$$\mathbb{P}_p(A'_n(M)) \leq \exp(-c_p a) + \sum_{k'=0}^{\infty} \sum_{\substack{x \in \Lambda_{n,v}(-\infty, -M), \\ x \cdot \mathbf{1} = -2da + c_0\eta M + dk'}} \mathbb{P}_p(x - k_x \cdot \mathbf{1} \rightarrow (-f(n, k')\mathbf{1} + \mathcal{K}_0)^c).$$

Combining $x \cdot \mathbf{1} = -2da + c_0\eta M + dk'$ with (11), shows that

$$\|x\|_1 \leq c_0M + \frac{dk'}{\eta}.$$

Using another union bound, translation invariance, and Theorem 1, we find

$$\begin{aligned} \mathbb{P}_p(A'_n(M)) &\leq \exp(-c_p a) + \sum_{k'=0}^{\infty} \left| \left\{ x \in \mathbb{Z}^d : \|x\|_1 \leq c_0M + \frac{dk'}{\eta} \right\} \right| \mathbb{P}_p(0 \rightarrow (-f(n, k')\mathbf{1} + \mathcal{K}_0)^c) \\ &\leq \exp(-c_p a) + \sum_{k'=0}^{\infty} \left(2c_0cn + 2\frac{dk'}{\eta} \right)^d \exp(-c_p f(n, k')). \end{aligned} \tag{12}$$

We now take

$$a := a(n) := \frac{c_0\eta}{4}n,$$

so that

$$f(n, k') = -2a + \frac{c_0\eta}{d}(n - 1) + k' = c_0\eta \left(\frac{n}{2} - 1 \right) + k'.$$

A careful examination of (12) shows that the sum over k' converges for all $n \in \mathbb{N}$, and that the result is summable with respect to n , so that by the Borel-Cantelli lemma $\mathbb{P}_p(A'_n(\lfloor cn \rfloor) \text{ i.o.}) = 0$. The same result can be proven similarly for the events $A''_n(\lfloor cn \rfloor)$ and $\hat{A}_n(\lfloor cn \rfloor)$, and we omit the proof.

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