On graphs for which the connected domination number is at most the total domination number

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Abstract

In this note we give a finite forbidden subgraph characterization of the connected graphs for which any non-trivial connected induced subgraph has the property that the connected domination number is at most the total domination number. This question is motivated by the fact that any connected dominating set of size at least 2 is in particular a total dominating set. It turns out that in this characterization, the total domination number can equivalently be substituted by the upper total domination number, the paired-domination number and the upper paired-domination number respectively. Another equivalent condition is given in terms of structural domination.

keywords: connected domination, total domination, paired-domination, perfection of parameters.

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A *dominating set* of a graph G is a vertex subset such that every vertex of G belongs to X or has a neighbor in X. The minimal size of a dominating set of G, the *domination number*, is denoted $\gamma(G)$. A *total dominating set* X of G is a vertex subset that every vertex of G has a neighbor in. That is, X is a dominating set and the subgraph induced by X, henceforth denoted $G[X]$, does not have an isolated vertex. Note that any graph that does not have an isolated vertex has a total dominating set (and vice versa). The minimal size of a total dominating set of G is denoted $\gamma_t(G)$ and is called the *total domination number* of G. A total dominating set of minimal size is called a *minimum total dominating set*. The maximal size of an inclusionwise minimal total dominating set, the *upper total domination number*, is denoted $\Gamma_t(G)$. Total domination has been introduced by Cockayne, Dawes and Hedetniemi [3] and is well-studied now. A survey of some recent results is given by Henning [7]. A variant of (total) domination is paired-domination. A *paired-dominating set* of G is a dominating set X such that $G[X]$ has a perfect matching. In particular, any paired-dominating set is a total dominating set. Furthermore, paired-dominating sets always exist in graphs that do not have isolated vertices. The minimal size of a paired-dominating set is denoted $\gamma_p(G)$ and is called the *paired-domination number* of G. Similar to the total domination case one defines the *upper paired-domination number* $\Gamma_p(G)$. Apparently, paired-domination was first studied by Haynes and Slater [6].

Another variant of domination is connected domination. A *connected dominating set* of G is a dominating set such that $G[X]$ is connected. Clearly, a graph has a connected dominating set iff it is connected. The minimal size of a connected dominating set, the *connected domination number*, is denoted $\gamma_c(G)$.

One can say that total domination and connected domination (together with independent domination) belong to the most intensively studied variants of domination. There are a lot of sharp bounds on γ_t and γ_c and for many graph classes we know the computational complexity of the two parameters. Although a little less studied yet, similar things can be said about paired-domination. Still a good introduction into the theory of domination is given by the book of Haynes, Hedetniemi and Slater [5]. The property that two parameters are equal for all induced subgraphs is usually called *perfection* of the two parameters. Finding the forbidden induced subgraph characterization for a certain type of perfection, in particular for parameters from the context of domination, seems to be accepted as a step in the understanding of the relation of the parameters involved. A prominent example for the perfection of two domination parameters are the so-called domination perfect graphs. A graph is *domination perfect* iff for any induced subgraph the domination number equals the minimal size of an independent dominating set. After the problem was open for some time, a forbidden induced subgraph characterization of the domination perfect graphs was finally given by Zverovich and Zverovich [9]. A characterization of the connected graphs for which in any connected subgraph $\gamma = \gamma_c$ holds is given by Zverovich [10]. An extension of this result to total domination and clique-domination was given by Goddard and Henning [4]. We call a connected graph *non-trivial* if it is not an isolated vertex. It is clear that any connected dominating set of size at least 2 is also a total dominating set. Thus any connected graph with $\gamma_c \geq 2$ fulfills $\gamma_c \geq \gamma_t$. However, an open problem seems to be the characterization of the connected graphs for which we can find, in any non-trivial connected induced subgraph, a minimum total dominating set that is connected, i.e. $\gamma_c \leq \gamma_t$. These graphs then fulfill $\gamma_c = \gamma_t$, provided $\gamma_c \geq 2$.

The following Theorem gives a characterization of the connected graphs for which any non-trivial connected induced subgraph fulfills $\gamma_c \leq \gamma_t$, in terms of forbidden induced subgraphs. Somewhat surprisingly, it turns out that in this characterization γ_t can be substituted by any of the parameters Γ_t, γ_p and Γ_p. Furthermore, the set of forbidden induced subgraphs yields the equivalence of another condition in terms of structural domination.

Theorem 1. *Let G be a connected graph. The following conditions are equivalent:*

1. *Any non-trivial connected induced subgraph of G fulfills $\gamma_c \leq \gamma_t$.*
2. *Any non-trivial connected induced subgraph of G fulfills $\gamma_c \leq \Gamma_t$.*
3. *Any non-trivial connected induced subgraph of G fulfills $\gamma_c \leq \gamma_p$.*
4. *Any non-trivial connected induced subgraph of G fulfills $\gamma_c \leq \Gamma_p$.*
5. *G is $\{P_7, C_7, F_1, F_2\}$-free (see Figure 1).*

6. *Any connected induced subgraph H of G has a connected dominating set X such that $H[X]$ is $\{P_5, G_1, G_2\}$-free (see Figure 2).*

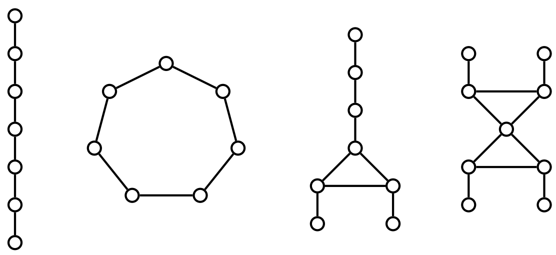

Figure 1: The graphs P_7, C_7, F_1 and F_2.

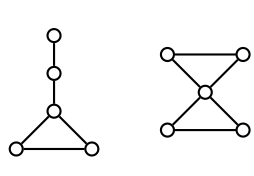

Figure 2: The graphs P_5, G_1 and G_2.

We observe that the class of connected $\{P_7, C_7, F_1, F_2\}$-free graphs properly contains the class of connected split graphs. It is well-known that the computation of the domination number γ in split graphs is NP-complete [2]. From [4] it follows that in any non-trivial connected $\{P_5, C_5\}$-free graph γ equals γ_c and γ_t, provided $\gamma \geq 2$. Thus, the computation of the parameters γ_c and γ_t remains NP-complete if the instances are restricted to split graphs. Therefore, computing the parameters γ_c and γ_t on connected $\{P_7, C_7, F_1, F_2\}$-free graphs remains NP-complete.

In view of the forbidden subgraphs of Theorem 1 (see Figures 1 and 2) we obtain the following immediate consequence:

Corollary 1. *Let G be a $\{C_3, C_7\}$-free graph. The following statements are equivalent:*

1. *Any non-trivial connected induced subgraph fulfills $\gamma_c \leq \gamma_t$ ($\gamma_c \leq \Gamma_t$, $\gamma_c \leq \gamma_p$, $\gamma_c \leq \Gamma_p$ respectively).*
2. *G is P_7-free.*
3. *Any connected induced subgraph H of G has a connected dominating set X such that $H[X]$ is P_5-free.*

Note that any biparite graph is in particular $\{C_3, C_7\}$-free. Hence, Corollary 1 applies to bipartite graphs.

The main step of the proof of Theorem 1 is formulated in the following Lemma:

Lemma 1. *If G is a non-trivial connected graph with $\gamma_c(G) > \gamma_t(G)$, then G contains P_7, C_7, F_1 or F_2 as induced subgraph (see Figure 1).*

Proof. Let G be a connected graph with $\gamma_c(G) > \gamma_t(G)$. Among the minimum total dominating sets of G let T be minimal with respect to the number of connected components of $G[T]$. We find two connected components of T, say T_1 and T_2, such that there are vertices $u \in T_1$ and $v \in T_2$ that have distance at most three. Since T is a total dominating set, T_1 and T_2 consist of at least two vertices each. By choice of T_1 and T_2, at least one of the following six cases holds:

(a) There is a vertex $x \in V \setminus T$ such that $N(x) \cap T_1 \neq \emptyset$ and $N(x) \cap T_2 \neq \emptyset$, and one of the following cases holds:

- (a.1) $T_1 \not\subseteq N(x)$ and $T_2 \not\subseteq N(x)$.
- (a.2) $T_1 \subseteq N(x)$ and $T_2 \not\subseteq N(x)$.
- (a.3) $T_1 \subseteq N(x)$ and $T_2 \subseteq N(x)$.

(b) There are two adjacent vertices $x, y \in V \setminus T$ such that $N(x) \cap T_1 \neq \emptyset$, $N(x) \cap T_2 = \emptyset$, $N(y) \cap T_1 = \emptyset$ and $N(y) \cap T_2 \neq \emptyset$, and one of the following cases holds: Further, it appears that:

- (b.1) $T_1 \not\subseteq N(x)$ and $T_2 \not\subseteq N(y)$.
- (b.2) $T_1 \subseteq N(x)$ and $T_2 \not\subseteq N(y)$.
- (b.3) $T_1 \subseteq N(x)$ and $T_2 \subseteq N(y)$.

We will show that in each of the cases (a.1) - (b.3) G contains P_7, C_7, F_1 or F_2 as induced subgraph. For symmetry, we do not need to consider the cases "$T_1 \not\subseteq N(x)$ and $T_2 \subseteq N(x)$" and "$T_1 \not\subseteq N(x)$ and $T_2 \subseteq N(y)$". For each vertex $v \in T$ we denote by $P(v)$ the set of *private neighbors* of v, i.e. the vertices for which the only neighbor among T is v. Note that $P(v)$ may also contain vertices of T. Since T is a minimum total dominating set, any member of T has at least one private neighbor.

To (a.1): Let $u, u' \in T_1$ such that $u \in N(x)$ and $u' \in N(u) \setminus N(x)$. Similar, let $v, v' \in T_2$ such that $v \in N(x)$ and $v' \in N(v) \setminus N(x)$. If the subgraph induced by the set $(T \setminus \{u'\}) \cup \{x\}$ has fewer connected components than $G[T]$, it is not a total dominating set. Thus there is a private neighbor u'' of u' that is not adjacent to x. If the subgraph induced by $(T \setminus \{u'\}) \cup \{x\}$ does not have fewer connected components than $G[T]$, u' is a cut-vertex of $G[T_1 \cup \{x\}]$. Then we can choose a vertex $u'' \in N(u') \cap T_1$ that is not adjacent to u or x, since they belong to the same component of $G[T_1 \cup \{x\}]$. For symmetry, there is neighbor v'' of v' that is not adjacent to u', u, x or v. Hence, $G[\{u'', u', u, x, v, v', v''\}]$ is isomorphic to P_7 or C_7, depending on the adjacency of u'' and v''.

To (b.1): Again let $u, u' \in T_1$ such that $u \in N(x)$ and $u' \in N(u) \setminus N(x)$ and let $v, v' \in T_2$ such that $v \in N(y)$ and $v' \in N(v) \setminus N(y)$. If $P(u') \not\subseteq N(x) \cup N(y)$, then $G[\{u'', u', u, x, y, v, v'\}] \cong P_7$ for any $u'' \in P(u') \setminus (N(x) \cup N(y))$. Hence we can assume $P(u') \subseteq N(x) \cup N(y)$ and $P(v') \subseteq N(x) \cup N(y)$ by symmetry. If the subgraph induced by the set $(T \setminus \{u', v'\}) \cup \{x, y\}$ has fewer connected components than $G[T]$, it is not a total dominating set. Thus there is a vertex $w \in N(u') \cap N(v')$ that is not adjacent to any member of $(T \setminus \{u', v'\}) \cup \{x, y\}$. Therefore $G[\{w, u', u, x, y, v, v'\}] \cong C_7$. If the subgraph induced by

$(T \setminus \{u', v'\}) \cup \{x, y\}$ does not have fewer connected components than $G[T]$, $\{u', v'\}$ is a cut-set of $G[T_1 \cup T_2 \cup \{x, y\}]$. Since the edge $\{x, y\}$ is a bridge of $G[T_1 \cup T_2 \cup \{x, y\}]$, u' is a cut-vertex of $G[T_1 \cup \{x\}]$ or v' is a cut-vertex of $G[T_2 \cup \{y\}]$. Say u' is such a cut-vertex. Then we can choose a vertex $u'' \in N(u') \cap T_1$ that is not adjacent to u or x, since they belong to the same component of $G[T_1 \cup \{x\}]$. Therefore $G[\{u'', u', u, x, y, v, v'\}] \cong P_7$.

To (a.2): We choose two adjacent vertices $u, v \in T_1$. Further, let $w, w' \in T_2$ such that $w \in N(x)$ and $w' \in N(w) \setminus N(x)$. As described in case (a.1), we find vertices $u' \in P(u) \setminus N(x)$ and $v' \in P(v) \setminus N(x)$, since neither u nor v is a cut-vertex of $G[T_1 \cup \{x\}]$. If the subgraph induced by the set $(T \setminus \{w'\}) \cup \{x\}$ has fewer connected components than $G[T]$, it is not a total dominating set. Thus there is a private neighbor w'' of w' that is not adjacent to x. If u' or v' is adjacent to w'', say u', then we have the following: If $w'' \notin T_2$, u' and w'' fulfill the condition of (b.1). If $w'' \in T_2$, u' fulfills the condition of (a.1). Since we dealt with both cases above, we can assume that u' and v' are both not adjacent to w''. If the subgraph induced by $(T \setminus \{w'\}) \cup \{x\}$ does not have fewer connected components than $G[T]$, w' is a cut-vertex of $G[T_2 \cup \{x\}]$. We can choose a vertex $w'' \in N(u') \cap T_2$ that is not adjacent to w or x, since they belong to the same component of $G[T_2 \cup \{x\}]$. If u' is not adjacent to v', $G[\{u', v', u, v, x, w, w', w''\}] \cong F_1$. Otherwise $G[\{u', v', u, x, w, w', w''\}] \cong P_7$.

To (a.3): We choose two adjacent vertices $u, v \in T_1$ and two adjacent vertices $w, z \in T_2$. As described in case (a.1), we find vertices $u' \in P(u) \setminus N(x)$, $v' \in P(v) \setminus N(x)$, $w' \in P(w) \setminus N(x)$ and $z' \in P(z) \setminus N(x)$, since none of the vertices u, v, w or z is a cut-vertex of $G[T_1 \cup \{x\}]$ (resp. $G[T_2 \cup \{x\}]$). Further, as described in case (a.2), we can assume that there is not an edge from u' or v' to w' or z'. If u' is not adjacent to v' and w' is not adjacent to z', $G[\{u', v', u, v, x, w, z, w', z'\}] \cong F_2$. If u' is adjacent to v' and w' is not adjacent to z' (or conversely), $G[\{u', v', u, x, w, z, w', z'\}] \cong F_1$ (resp. $G[\{u', v', u, v, x, w, w', z'\}] \cong F_1$). If u' is adjacent to v' and w' is adjacent to z', $G[\{u', v', u, x, w, w', z'\}] \cong P_7$.

To (b.2): We find two adjacent vertices $u, v \in T_1$ and $w, w' \in T_2$ such that $w \in N(y)$ and $w' \in N(w) \setminus N(y)$. If there is a private neighbor, say z, of u or v that is adjacent to y, then z and y fulfill the condition of (b.1), as $N(y) \cap T_1 = \emptyset$ and thus $z \notin T$. Hence, we can assume that no private neighbor of u or v is adjacent to y. Further, if $P(u) \subseteq N(x)$, then $T' = (T \setminus \{u\}) \cup \{x\}$ is a minimum total dominating set. The number of connected components of $G[T']$ equals the number of connected components of $G[T]$, as $T_1 \subseteq N(x)$. With respect to y, T' fulfills (a.1) which we already dealt with. Thus we can choose $u' \in P(u) \setminus (N(x) \cup N(y))$ and, for symmetry, $v' \in P(v) \setminus (N(x) \cup N(y))$. Since $w, w' \in T_2$, there is not an edge from u' or v' to w or w'. If u' is adjacent to v', $G[\{u', v', u, x, y, w, w'\}] \cong P_7$. Otherwise, $G[\{u', v', u, v, x, y, w, w'\}] \cong F_1$.

To (b.3): We choose two adjacent vertices $u, v \in T_1$ and $w \in T_2$. As described in (b.2), we find private neighbors $u' \in P(u) \setminus (N(x) \cup N(y))$, $v' \in P(v) \setminus (N(x) \cup N(y))$ and $w' \in P(w) \setminus (N(x) \cup N(y))$ (otherwise, case (a.2) or (b.2) holds). We observe that if u' or v' is adjacent to w', say u', then u' and w' fulfill the condition of (b.1), as $T_1 \subseteq N(x)$ and $T_2 \subseteq N(y)$ give $u', w' \notin T$. Hence, we can assume that u' and v' are both not adjacent to w'. If u' is adjacent to v', $G[\{u', v', u, x, y, w, w'\}] \cong P_7$. Otherwise, $G[\{u', v', u, v, x, y, w, w'\}] \cong F_1$. □

We are now ready to prove Theorem 1:

Proof of Theorem 1. Let G be a connected graph. We have to show the equivalence of the conditions 1 - 6 formulated in Theorem 1.

Since by definition γ_t is a lower bound for Γ_t, γ_p and Γ_p, it is clear that 1 implies 2, 3 and 4. Furthermore, we observe that

$$\gamma_t(H) = \Gamma_t(H) = \gamma_p(H) = \Gamma_p(H) = 4$$

and $\gamma_c(H) = 5$ for all $H \in \{P_7, C_7, F_1, F_2\}$. Hence, 1, 2, 3 and 4 imply 5 each. By Lemma 1, 5 implies 1.

We finish the proof by showing that condition 5 is equivalent to 6. We need a recent Theorem from Tuza [8] (that was independently proven by Bacsó [1]) about structural domination in graphs. Let $\mathcal{G}$ be any non-empty class of connected graphs closed under taking connected induced subgraphs. Their Theorem yields a forbidden subgraph characterization of the connected graphs for which any connected induced subgraph H has a connected dominating set X such that $H[X]$ is isomorphic to some member of $\mathcal{G}$. By applying their Theorem we obtain that if $\mathcal{G}$ is the class of $\{P_5, G_1, G_2\}$-free graphs, the set of forbidden induced subgraphs is $\{P_7, C_7, F_1, F_2\}$. This completes the proof. □

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