A Graph Class related to the Structural Domination Problem

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Abstract

In the structural domination problem one is concerned with the question if a given graph has a connected dominating set whose induced subgraph has certain structural properties. For most of the common graph properties, the associated decision problem is $\mathcal{NP}$-hard.

Recently, Bacsó and Tuza gave a full characterization of the graphs whose every induced subgraph has a connected dominating set satisfying an arbitrary prescribed hereditary property. Using the Theorem of Bacsó and Tuza, we derive a finite forbidden subgraph characterization of the distance-hereditary graphs that have a dominating induced tree. Furthermore, we show that for distance-hereditary graphs minimum dominating induced trees can be found efficiently. The main part of the paper studies a new class of graphs, the *structural domination class*, which is closely related to the structural domination problem. Among other results, we give new characterizations of certain subclasses of distance-hereditary graphs (in particular for ptolemaic graphs) and analyse the structure of minimum connected dominating sets of structural domination graphs. It turns out that many of the problems associated to structural domination become tractable on the hereditary part of the structural domination class.

Keywords: connected domination, structural domination, dominating induced trees, distance-hereditary graphs

1. Introduction

1.1. Preliminaries

All graphs considered in this paper are finite, undirected, without loops and multiple edges. Let G be any graph and $v \in V(G)$. The *open neighborhood* $N(v)$ of v is defined as the set vertices that v is adjacent to. The *closed neighborhood* of v is defined as $N[v] = N(v) \cup \{v\}$. A *pendant vertex* is a vertex v that has

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exactly one neighbor, the so called *support vertex* of v. Two vertices are *adjacent (non-adjacent) twins* if they share the same closed (open) neighborhood. A vertex v *separates* G if the graph obtained from G by deleting v has more connected components than G. A *dominating set* of G is a subset $X \subseteq V(G)$ of its vertices such that $\bigcup_{x \in X} N[x] = V(G)$. A dominating set is a *connected dominating set* if the subgraph induced by X, denoted by $G[X]$, is a connected graph. A connected dominating set is *minimal* if it is minimal with respect to set inclusion and *minimum* if it is minimal with respect to cardinality. The size of any minimum connected dominating set of G is denoted by $\gamma_c(G)$. Note that any minimum connected dominating set is also minimal, but the converse is not true. In fact, determining if a given graph has a connected dominating set of a certain size is known to be a $\mathcal{NP}$-complete decision problem (see [1]). If X is a minimal connected dominating set, any vertex of X is either separating $G[X]$ or has a *private neighbor* (with respect to X). This is a neighbor in $V \setminus X$ which is not adjacent to any other vertex of X. Some graphs have special names which will be used throughout the paper: The graph P_k is the path of k vertices. The graph C_k is the cycle of k vertices. The graph W_k is obtained from C_k by attaching a vertex which is adjacent to all other vertices. A *hole* is a cycle of length at least 5. The graphs *house*, *antenna*, *mouse*, *domino* and *gem* are displayed in Figure 1.

A graph G is *distance-hereditary* if all induced paths of G are also shortest paths. For distance-hereditary graphs, several characterizations have been found. In particular, they can be decomposed iteratively in the following way.

Theorem 1 ([2]). *A graph is distance-hereditary iff any non-trivial subgraph has a pendant vertex or a pair of twins.*

This theorem can be restated as follows: A graph G with $|V(G)| = n$ is distance-hereditary iff there is an ordering $(v_1, v_2, \ldots, v_n)$ of its vertices such that v_i is a pendant vertex or a twin in the graph $G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for any $1 \leq i \leq n-1$. Note that $G_1 = G$.

There is also a characterization of distance-hereditary graphs by minimal forbidden subgraphs.

Theorem 2 ([2]). *A graph G is distance-hereditary iff it is house-free, hole-free, domino-free and gem-free.*

A graph G is $(5,2)$-chordal if any cycle of length at least 5 has two chords. That is, G is house, hole and domino free (see [7]). Hence, any distance-hereditary graph is $(5,2)$-chordal. A graph is *ptolemaic* if it is chordal and distance-hereditary. A graph is *chordal bipartite* if all induced cycles are of length 4.

1.2. The structural domination problem

From now on we only consider connected graphs. Since there is no danger of confusion, we say that a graph H is a *subgraph* of a graph G if H is connected and there is a set $X \subseteq V(G)$ with $H \cong G[X]$. For any set of graphs $\mathcal{G}$ let

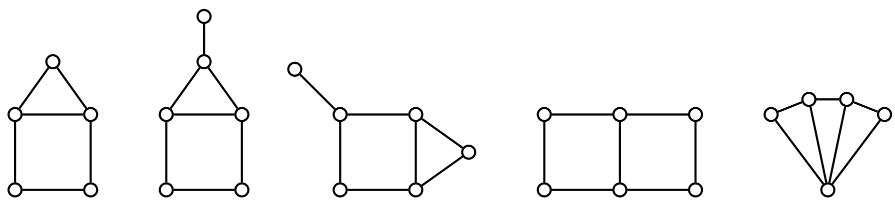

Figure 1: house, antenna, mouse, domino and gem

$\mathcal{G}^*$ denote the collection of all graphs of $\mathcal{G}$ whose every subgraph belongs to $\mathcal{G}$ and let $\overline{\mathcal{G}}$ be the set of graphs not contained in $\mathcal{G}$. A non-empty set of graphs $\mathcal{G}$ is called *concise* if $\mathcal{G} = \mathcal{G}^*$. For concise $\mathcal{G}$ let $Fb(\mathcal{G})$ be the set of minimal forbidden subgraphs of $\mathcal{G}$. Furthermore, let $Dom(\mathcal{G})$ be the collection of all graphs having a connected dominating set inducing a graph of $\mathcal{G}$. $Dom(\mathcal{G})$ is called the *domination class* of $\mathcal{G}$. Note that $Dom(\mathcal{G})$ is concise iff $\mathcal{G}$ is the set of all graphs. For any graph class $\mathcal{G}$ and any graph $G \in Dom(\mathcal{G})$ let

$$\gamma_{\mathcal{G}}(G) = \min\{|X| : X \text{ is a connected dominating set and } G[X] \in \mathcal{G}\}$$

and observe $\gamma_c(G) \leq \gamma_{\mathcal{G}}(G)$.

As is shown in [3], for most of the common graph classes $\mathcal{G}$ the recognition of $Dom(\mathcal{G})$ and the decision problem related to $\gamma_{\mathcal{G}}$ are $\mathcal{NP}$-hard. One of the aims of this paper is the development of a graph class, the structural domination class, in which $Dom(\mathcal{G})$ can be recognized and $\gamma_{\mathcal{G}}$ can be computed efficiently. This aim is fulfilled with Corollary 6 of this paper.

In contrast, for all concise $\mathcal{G}$ the class $Dom(\mathcal{G})^*$ has a characterization in terms of forbidden subgraphs which was discovered by Bacsó [4] and Tuza [5] independently. The *articulation graph* of a graph F is obtained from F by simultaneously attaching a pendant vertex to any non-separating vertex of F and is denoted by $Art(F)$.

Theorem 3 ([4], [5]). *Let $\mathcal{G}$ be a concise set of graphs.*

$$Fb(Dom(\mathcal{G})^*) = \{Art(F) : F \in Fb(\mathcal{G})\} \cup \{C_{t+2} : P_{t-1} \in \mathcal{G}, P_t \notin \mathcal{G}\}.$$

That is, a graph belongs to $Dom(\mathcal{G})^*$ iff it is $\{Art(F) : F \in Fb(\mathcal{G})\}$-free and $\{C_{t+2} : P_{t-1} \in \mathcal{G}, P_t \notin \mathcal{G}\}$-free. For example, any subgraph of a graph G has a dominating clique iff this graph is P_5- and C_5-free. This classical result was first stated in [6], according to our knowledge.

2. Dominating trees in distance-hereditary graphs

Let **tree** denote the set of all acyclic graphs. Dominating induced trees were first studied 2004 in [8] and 2007 in [9], according to our knowledge. We first proove a finite forbidden subgraph characterization of $Dom(\textbf{tree})$ in distance-hereditary graphs, based on Theorem 3. Corollary 1 shows that $Dom(\textbf{tree})$ is recognizable in polynomial time if the instances are restricted to be distance-hereditary graphs and, furthermore, a minimum dominating induced tree can be computed efficiently. In contrast, the recognition of $Dom(\mathbf{tree})$ is $\mathcal{NP}$-complete if the instances are restricted to be regular graphs, as is shown in [9]. Our result is, in particular, an answer to the question stated in [8] that asks for graph classes allowing an efficient computation of the minimal size of a dominating induced tree.

Lemma 1. *Let G be $(5,2)$-chordal graph and X be a minimal connected dominating set such that $G[X]$ is not a path. Then $Art(G[X])$ is a subgraph of G.*

Proof. Let G be a $(5,2)$-chordal graph and let X be a minimal connected dominating set such that $G[X]$ is not a path. Let $\{n_x : x \text{ not separating } G[X]\}$ be a set of private neighbors of the non-separating vertices of $G[X]$. Assume $Art(G[X])$ is not a subgraph of G. Since G is hole-free, there is an adjacent pair $x, y \in X$ of vertices which do not separate $G[X]$ such that n_x is adjacent to n_y. Since $G[X]$ is not a path, x and y belong to an induced cycle C of $G[X]$. But $G[V(C) \cup \{n_x, n_y\}]$ is not $(5,2)$-chordal, as is easily seen. □

In particular, for a $(5,2)$-chordal graph that is $Art(C_3)$-free and $Art(C_4)$-free, any minimum connected dominating set is a tree. However, there are $(5,2)$-chordal graphs which have a dominating induced tree but are not $Art(C_3)$-free and $Art(C_4)$-free, e.g. the graph obtained from $Art(C_3)$ by attaching a vertex that is adjacent to all other vertices. Necessity holds for a slightly smaller graph class, as the following result shows.

Theorem 4. *For a distance-hereditary graph G the following statements are equivalent:*

1. *$G \in Dom(\boldsymbol{tree})$.*
2. *$G \in Dom(\boldsymbol{tree})^*$.*
3. *G does not contain $Art(C_3)$ or $Art(C_4)$ as subgraph.*
4. *Any minimal connected dominating set induces a tree.*
5. *There is a minimal connected dominating set that induces a tree.*

Proof. By Theorem 3, $Fb(Dom(\mathbf{tree})^*) = \{Art(C_3), Art(C_4), Art(C_5), \ldots\}$. Hence, by Theorem 2, 2 implies 3. By Lemma 1 and since any distance-hereditary graph is $(5,2)$-chordal, 3 implies 4. 4 implies 5, and 5 implies 1 clearly.

To see that 1 implies 2, assume there is a distance-hereditary graph in $Dom(\mathbf{tree}) \setminus Dom(\mathbf{tree})^*$ and choose G minimal with respect to this property. By Theorem 3, $V(G)$ admits a partition $V(G) = U \cup W$ such that $G[U]$ is the articulation graph of C_3 or C_4 and there is a minimal dominating set X of G inducing a tree. By minimality of G, $W \subseteq X$. Let $n = |V(G)|$. We use a decomposition $(v_1, v_2, \ldots, v_n)$ of G which fulfills the following rule: If there is a twin $v \notin X$ in G_i, choose $v_i = v$. It follows that $X \cap V(G_i)$ is a connected dominating set of G_i for any $1 \le i \le n$.

Assume $G[U] \cong Art(C_3)$. By minimality, v_1 is a pendant vertex of $G[U]$ and of G_1. Thus, the neighbor c_1 of v_1 in $G[U]$ belongs to X. By minimality again, v_2 is a pendant vertex of $G[U]$ and of G_2. Thus, there is a second vertex c_2 of C_3 contained in X. Since $G[X]$ is acyclic, $c_3 \notin X$ and thus, by minimality again, v_3 is one of the adjacent twins c_1 and c_2. Furthermore, the remaining pendant vertex of $G_3[U]$ must be dominated, but $c_3 \notin X$. Hence, c_1 and c_2 have a common neighbor in X which is a contradiction to acyclicity of $G[X]$.

Assume $G[U] \cong Art(C_4)$ where the vertices of the C_4 are clockwise called c_1, c_2, c_3 and c_4. By minimality, v_1 is a pendant vertex of $G[U]$ and of G_1. We can assume that c_1 is the support vertex of v_1 and so $c_1 \in X$. By minimality again, v_2 is a pendant vertex of $G[U]$ and of G_2. We can assume that c_2 or c_3 is the support vertex of v_2.

We first assume that c_2 is the support vertex of v_2 and so $c_2 \in X$. Then, by minimality, v_3 is a pendant vertex of $G[U]$ and of G_3. We can assume that c_3 is the support vertex of v_3 and thus $c_3 \in X$. By minimality again, v_4 is either a pendant vertex of $G[U]$ and of G_4, or v_4 equals c_1 or c_3 (in the case that c_1 and c_3 are non-adjacent twins in G_4). The first case is impossible, since c_4 would belong to X, in contradiction to acyclicity of $G[X]$. We can assume $v_4 = c_1$. By acyclicity, $N_{G_4}(c_1) = N_{G_4}(c_3) = \{c_2, c_4\}$, and hence, by minimality, W contains no pendant vertex or twin in G_5. Neither the remaining pendant vertex of $G_4[U]$, nor c_2 can be pendant vertices of G_5, since X is a connected dominating set. This is a contradiction to the existence of v_5.

We now assume that c_3 is the support vertex of v_2 and so $c_3 \in X$. By minimality, v_3 is a pendant vertex of $G[U]$ and of G_3, or v_3 equals c_1 or c_3 (in the case that c_1 and c_3 are non-adjacent twins in G_4). The first case is dealt with above and we may assume the latter. By acyclicity, c_1 and c_3 have at most one neighbor $x \in W$. If they have, then by acyclicity $c_2, c_4 \notin X$ and therefore $N_{G_3}(x) \cap W \neq \emptyset$. Thus, by minimality, v_4 is a pendant vertex of $G[U]$ and of G_4, which is a contradiction to $c_2, c_4 \notin X$. The case that c_1 and c_3 have no neighbor in W leads to a situation which is dealt with above. □

Note that, for distance-hereditary graphs, minimum connected dominating set can be found efficiently, as is shown in [10]. Given a distance-hereditary graph G, one efficiently computes a minimum connected dominating set X. By Lemma 1 and Theorem 4, $G[X]$ is a tree iff $G \in Dom(\textbf{tree})$. Hence, the following holds:

Corollary 1. *$Dom(\textbf{tree})$ can be recognized in polynomial time if the instances are restricted to be distance-hereditary graphs. Any distance-hereditary graph $G \in Dom(\textbf{tree})$ fulfills $\gamma_{\textbf{tree}}(G) = \gamma_c(G)$, and such tree can be computed efficiently.*

3. The structural domination property

We say that a graph G is a *structural domination graph* if the following condition holds: For any connected dominating set X of G, each subgraph H

of G has a connected dominating set Y such that $H[Y]$ is a subgraph of $G[X]$. Note that, by definition of the term subgraph, not necessarily $Y \subseteq X$. A graph is a *hereditary* structural domination graph if every subgraph is a structural domination graph.

The *structural domination class* is the set of all structural domination graphs. Note that this class is not concise and that the concise part of the structural domination class is the set of hereditary structural domination graphs. It is easy to see that the structural domination class equals the set of all graphs which are contained in $\overline{Dom(\mathcal{G})} \cup Dom(\mathcal{G})^*$ for any concise graph class $\mathcal{G}$. That is, $Dom(\mathcal{G}) \Leftrightarrow Dom(\mathcal{G})^*$ for any concise graph class $\mathcal{G}$. In fact, we have the following slightly simpler formulation:

Observation 1. *The structural domination class equals the set of all graphs which are contained in $\overline{Dom(F\text{-free})} \cup Dom(F\text{-free})^*$ for any graph F.*

Proof. Let G be a structural domination graph, and let F be an arbitrary graph. Since the set of F-free graphs is concise, $G \in \overline{Dom(F\text{-free})} \cup Dom(F\text{-free})^*$.

Let $G \in \overline{Dom(F\text{-free})} \cup Dom(F\text{-free})^*$ for all graphs F, and let $\mathcal{G}$ be a concise class. We observe

$$Dom(Fb(\mathcal{G})\text{-free}) \subseteq \bigcap_{F \in Fb(\mathcal{G})} Dom(F\text{-free}),$$

and, by Theorem 3,

$$Dom(Fb(\mathcal{G})\text{-free})^* = \bigcap_{F \in Fb(\mathcal{G})} Dom(F\text{-free})^*.$$

By choice of G, if $G \notin \bigcup_{F \in Fb(\mathcal{G})} \overline{Dom(F\text{-free})}$, then $G \in \bigcap_{F \in Fb(\mathcal{G})} Dom(F\text{-free})^*$. Hence, $G \in \overline{Dom(Fb(\mathcal{G})\text{-free})} \cup Dom(Fb(\mathcal{G})\text{-free})^* = \overline{Dom(\mathcal{G})} \cup Dom(\mathcal{G})^*$. □

3.1. Structural domination graphs versus distance-hereditary graphs

Although there is no finite forbidden subgraph characterization of the concise part of the structural domination class, there are some small graphs that are obviously minimal forbidden subgraphs.

Observation 2. *Any hereditary structural domination graph is antenna-free, mouse-free, domino-free, gem-free and W_4-free.*

This leads us to the following connection between the concise part of the structural domination class and the distance-hereditary graphs:

Theorem 5. *A graph is house-free, hole-free and a hereditary structural domination graph iff it is W_4-free and distance-hereditary.*

Proof. To see the sufficiency, note that, by Theorem 2 and Observation 2, a house-free and hole-free hereditary structural domination graph is W_4-free and distance-hereditary.

To see the necessity, let $G = (V, E)$ be a W_4-free distance-hereditary graph. Since G is W_4-free, it is clear that $G \in \overline{Dom(\{K_1\})} \cup Dom(\{K_1\})^*$. We assume that there is a graph F such that G induces $Art(F)$ but has a connected dominating set X such that $G[X]$ is F-free. That is, by Observation 1, G is not a structural domination graph. We furthermore assume that G is minimal with respect to this property. Let $V = U \cup W$ be a partition such that $G[U] = Art(F)$. By minimality, $W \subseteq X$. Furthermore, no vertex of W separates G. Let $n = |V(G)|$. We use a decomposition $(v_1, v_2, \dots, v_n)$ of G which fulfills the following rule: If there is a twin $v \notin X$ in G_i, choose $v_i = v$. It is easy to see that $X \cap V(G_i)$ is a connected dominating set of G_i for any $1 \leq i \leq n$. We choose $1 \leq i \leq n$ minimal such that v_i is a pendant vertex with support vertex in W or a twin of a vertex of W.

Assume $w \in W$ is support vertex of v_i, but is not a twin of v_j in G_j for any $j \leq i$. Then w is separating in G_i and hence also in G, which is a contradiction. Thus, we can assume that v_i is a twin of w. The following cases are possible:

1. v_i is a non-adjacent twin of w.
2. v_i is an adjacent twin of w and $v_i \in W$.
3. v_i is an adjacent twin of w and $v_i \in U$.

In case 1 w is not a support vertex of v_j in G_j for any $j \leq i$. Thus, $N_G(w) \subseteq N_G(v_i)$. Since G is minimal, $v_i \notin X$. Thus, v_i is not a support vertex of v_j in G_j for any $j \leq i$. Furthermore, v_i is not an adjacent twin of v_j in G_j for any $j \leq i$ with $v_j \in X$. Therefore, $N_G(v_i) \cap X = N_G(w) \cap X$ and so $Y = (X \setminus \{w\}) \cup \{v_i\}$ is a connected dominating set of $G[V \setminus \{w\}]$ that does not induce F. This is a contradiction to the minimality of G.

In case 2 w and v_i are not support vertices or non-adjacent twins of v_j in G_j for any $j < i$. Thus, $N_G[w] = N_G[v_i]$, in contradiction to the minimality of G.

In case 3 we have to deal with the sub-cases $v_i \in X$ and $v_i \notin X$.

In the case $v_i \in X$ we have $N_G[w] \not\subseteq N_G[v_i]$ by minimality. Since w is not a support vertex of v_j in G_j for any $j < i$, there is a $j < i$ such that v_i is a non-adjacent twin of v_j in G_j. In the case $v_i \notin X$, v_i is not a support vertex of v_j in G_j for any $j < i$. Hence, $N_G[v_i] \subseteq N_G[w]$. By minimality, $N_G[w] \not\subseteq N_G[v_i]$ and thus there is a $j < i$ such that v_i is a non-adjacent twin of v_j in G_j. Both cases lead to the following contradiction: As G is W_4-free, $G[N_{G_j}[w] \setminus \{v_i, v_j\}]$ is isomorphic to K_n for some $n \in \mathbb{N}$. Assume $n = 1$. Then w is not separating in G_j, since it is not separating in G. Thus, there is a maximal index k with $j < k < i$ such that $v_k \in N_{G_j}(v_i)$. As $n = 1$, w is not adjacent to v_k and thus v_k is not a twin of v_i in G_k. Furthermore, v_k is not a pendant vertex in G_k, since otherwise it would be a non-adjacent twin of w in G_k. Hence, there is an index k' with $k < k' < i$ such that v_k is a twin of $v_{k'}$ in G_k. But then $v_{k'} \in N_{G_j}(v_i)$, which is a contradiction to the maximality of k.

Therefore, $n \geq 2$. For arbitrary $v \in N_{G_j}[w] \setminus \{v_i, v_j\}$ we observe that v is not a non-adjacent twin of v_l in G_l for any $l < j$, since G is W_4-free. Thus, $N_G[w] \subseteq N_G[v]$ and hence $v \notin X$ by minimality of G. Therefore, v is not a support vertex of v_l in G_l for any $l < j$ and so $N_G[w] = N_G[v]$. Hence,

$Y = (X \setminus \{w\}) \cup \{v\}$ is a connected dominating set of $G[V \setminus \{w\}]$ that does not induce F. This is a contradiction to the minimality of G. □

Corollary 2. *A graph is chordal and a hereditary structural domination graph iff it is ptolemaic.*

Note that this corollary can also be read as a characterization of the class of ptolemaic graphs which was previously unknown, by the best of our knowledge.

Corollary 3. *A graph is chordal bipartite and a hereditary structural domination graph iff it is bipartite and distance-hereditary.*

Theorem 1 and 5 imply the following decomposition theorem for the house-free, hole-free hereditary structural domination graphs:

Corollary 4. *A graph is a house-free, hole-free hereditary structural domination graph iff any non-trivial subgraph has a pendant vertex, a pair of adjacent twins or a pair of non-adjacent twins whose open neighborhood induces a disjoint union of complete graphs.*

Using results of the next section, we derive the following relationship:

Observation 3. *Any minimal connected dominating set of a hole-free hereditary structural domination graph induces a distance-hereditary subgraph.*

Proof. Note that any hole-free hereditary structural domination graph does not induce the articulation graph of a hole, a domino or a gem. Furthermore, the articulation graph of a house contains antenna as induced subgraph and is therefore not a hereditary structural domination graph. Theorem 3 and 7 complete the proof. □

3.2. Minimum connected dominating sets of structural domination graphs

For any structural domination graph G let

$$M(G) = \{F : Art(F) \text{ subgraph of } G\} \cup \{P_k : C_{k+2} \text{ subgraph of } G\}.$$

Theorem 6. *For any structural domination graph G holds*

$$\gamma_c(G) = \max\{|V(F)| : F \in M(G)\}.$$

Proof. Let G be a structural domination graph and let $k = \max\{|V(F)| : F \in M(G)\}$. By Theorem 3 and the definition of k, $G \in Dom(\{H : |V(H)| \leq k\})^*$. Thus, $\gamma_c(G) \leq k$.

Assume $Art(F)$ resp. C_{k+2} is a subgraph of G. By Theorem 3, and since G is a structural domination graph, $G \notin Dom(F\text{-free})$ resp. $G \notin Dom(P_k\text{-free})$. Thus, any connected dominating set of G induces a graph that contains F resp. P_k as subgraph. Hence, $\gamma_c(G) \geq k$. □

Corollary 5. *Let G be a structural domination graph and $\mathcal{G}$ be a concise graph class. If X is a minimum connected dominating set of G, then $G \in Dom(\mathcal{G})$ iff $G[X] \in \mathcal{G}$. In particular, if $G \in Dom(\mathcal{G})$, then $\gamma_c(G) = \gamma_{\mathcal{G}}(G)$.*

Proof. Let G be a structural domination graph and $\mathcal{G}$ be any concise graph class. As described in the proof of Theorem 6, if X is any minimum connected dominating set, then $G[X]$ contains all graphs of $M(G)$ as induced subgraphs and furthermore $G[X] \in M(G)$. Assume $G \in Dom(\mathcal{G})$. Thus, $G \in Dom(\mathcal{G})^*$, since G is a structural domination graph. By Theorem 3, $M(G)$ is $Fb(\mathcal{G})$-free and thus $M(G) \subseteq \mathcal{G}$. By Theorem 6, $G[X] \in \mathcal{G}$ and thus $\gamma_c(G) = |X| = \gamma_{\mathcal{G}}(G)$. □

Hence, a minimum connected dominating set of a structural domination graph G contains all information about the domination classes G belongs to. Furthermore, if X and Y are two minimum connected dominating sets of a structural domination graph G, then $G[X] \cong G[Y]$.

Theorem 6 can also be used to draw a connection between minimal and minimum connected dominating sets on the class of hereditary structural domination graphs:

Theorem 7. *Let G be a hereditary structural domination graph with $\gamma_c(G) \geq 3$, and let X be a minimal connected dominating set of G. If $G[X]$ is not a path, then $Art(G[X])$ is a subgraph of G. In particular, X is a minimum connected dominating set.*

Proof. Let X be a minimal connected dominating set of a hereditary structural domination graph G such that $G[X]$ is not a path. Assume that $Art(G[X])$ is not a subgraph of G. Let $\{n_x : x \in X \text{ not separating } G[X]\}$ be a set of private neighbors of the vertices of X which do not separate $G[X]$. Let $Y \subseteq X$ be a maximum set such that any $y \in Y$ does not separate $G[X]$ and furthermore $G[Y]$ and $G[\{n_y : y \in Y\}]$ are complete graphs.

Assume $|Y| = 1$. Choose x and y as a pair of non-separating vertices of $G[X]$ such that n_x is adjacent to n_y and x and y have minimal distance. Since $|Y| = 1$, x and y are not adjacent. Let P be a shortest path between x and y in $G[X]$. Since $G[X]$ is not a path, there is a vertex $z \in X \setminus V(P)$ with $N(z) \cap V(P) \neq \emptyset$.

Since P is a shortest path, only the following cases can occur. They are displayed in Figure 2.

1. z is adjacent to exactly three vertices u, v, w of P.
2. z is adjacent to exactly two non-adjacent vertices u, v of P.
3. z is adjacent to exactly two adjacent vertices u, v of P.
4. z is adjacent to exactly one vertex u of P.

In case 1, $G[\{u, v, w\}] \cong P_3$. Assume that $dist_{G[X]}(x, u) < dist_{G[X]}(x, w) < dist_{G[X]}(x, v)$.

We show, that there are two vertices w' and z' such that w' is a neighbor of w, z' is a neighbor of z and w' as well as z' are pendant vertices in $G' = G[V(P) \cup \{z, n_x, n_y, w', z'\}]$. But then G' is not a structural domination graph, as can be seen in the following way. We can assume that $dist_{G'}(x, u) \leq dist_{G'}(v, y)$. Let $S = V(P) \cup \{w, z\}$ and observe that this is

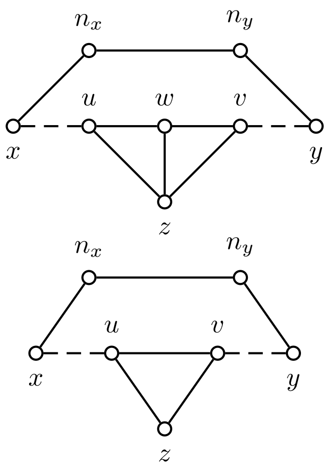

Figure 2: The cases 1, 2, 3 and 4. Dashed lines stand for paths of arbitrary length.

a connected dominating set of G'. Observe that for $G'' = G'[V(G') \setminus \{u\}]$ any connected dominating set S' necessarily contains w, z and all vertices that lie on the unique path between v and n_y. Hence, $G''[S']$ is not a subgraph of $G'[S]$ and so G is not a structural domination graph.

Assume w and z are both not separating $G[X]$. Thus, they have private neighbors n_w and n_z in G. These private neighbors are non-adjacent and not adjacent to n_x resp. n_y each, since x and y were chosen to have minimal distance.

Assume w is not separating $G[X]$, but z is separating $G[X]$. By minimality, n_w is not adjacent to n_x or n_y. Furthermore, z has a neighbor z' in X which is not adjacent to any vertex of $V(P) \cup \{n_x, n_y, n_w\}$.

The case that w is separating $G[X]$, but z is not separating $G[X]$ is dealt with in a similar way.

The case that w and z both are separating $G[X]$ is clear.

In case 2, there is a third vertex w of P such that $G[\{u, v, w\}] \cong P_3$, since P is a shortest path. Similar to case 1, w and z have neighbors w' resp. z' which are not adjacent to any vertex of $V(P) \cup \{n_x, n_y\}$. If w' and z' are not adjacent, $G[V(P) \cup \{z, n_x, n_y, w', z'\}]$ is not a structural domination graph, like in case 1. If w' and z' are adjacent, $G' = G[V(P) \cup \{n_x, n_y, w', z'\}]$ is not a structural domination graph. This can be seen in the following way. $V(P) \cup \{w'\}$ is a connected dominating set of G' such that the longest induced path of $G'[S]$ has $|V(P)|$ vertices. Since $G'' = G'[V(G') \setminus \{u\}]$ is a path containing $|V(P)| + 3$ vertices, any connected dominating set of G'' induces a path of at least $|V(P)|+1$ vertices. Hence, G' is not a structural domination graph.

To case 3: If z is separating $G[X]$, then there clearly is a neighbor $z' \in X$ which is not adjacent to any vertex of $V(P) \cup \{n_x, n_y\}$. If z is not separating, n_z is not adjacent to any vertex of $V(P)$. Since x and y are chosen to have minimal distance, n_z is not adjacent to n_x or n_y. But $G[V(P) \cup \{z, z', n_x, n_y\}]$ resp. $G[V(P) \cup \{z, n_z, n_x, n_y\}]$ has a dominating induced path but contains $Art(C_3)$ as subgraph, in contradiction to Theorem 3.

To case 4: If z is separating $G[X]$, it has a neighbor $z' \in X$ which is not adjacent to any vertex of $V(P) \cup \{n_x, n_y\}$. But $G[V(P) \cup \{n_x, n_y, z'\}]$ is not a structural domination graph, like in the second subcase of case 2.

If z is not separating $G[X]$, the case that n_z is not adjacent to any vertex of $V(P) \cup \{n_x, n_y\}$ leads to the same contradiction as above. Hence, n_z is adjacent to n_x or n_y, say n_x. Then u is different from x and adjacent to y, since x and y were chosen to have minimal distance. If n_z is not adjacent to n_y, $G' = G[V(P) \cup \{z, n_z, n_x, n_y\}]$ is not a structural domination graph. This can be seen as follows. If x is adjacent to u, G' has the dominating set $S = V(P) \cup \{z\}$, for which $G'[S]$ is a claw. But G' contains the subgraph $G'' = G'[V(G') \setminus \{x\}] \cong C_6$ for which any connected dominating set induces a path of length 3. Since the claw graph is P_4-free, G' is not a structural domination graph. If x is not adjacent to u, G' is dominated by the induced path between u and n_x that contains x, but G' induces the articulation graph of a claw. Hence, G' is not a structural domination graph.

Hence, n_z is adjacent to n_y, too, and thus P consists of three vertices only. But $G' = G[V(P) \cup \{z, n_z, n_y\}]$ is not a structural domination graph, since it has the dominating clique $\{n_x, n_y, n_z\}$ but contains the cycle $G'[\{x, n_x, n_z, z, u\}]$ of length 5.

Since all four cases lead to a contradiction we see that x is adjacent to y and hence $|Y| \geq 2$.

Assume $Y \neq X$. There is a maximal 2-connected subset Z of X that contains Y properly. Obviously there are two distinct vertices of Y, say x and x', which belong to an induced cycle C of Z that contains at least one vertex z that is not in Y. If C consists of more than three vertices, then $G[V(C) \cup \{n_x, n_{x'}\}]$ is not a structural domination graph, as is easily seen. Hence, z is a neighbor of both, x and x'.

Assume there is a vertex $y \in Y$ that is not adjacent to z. But $G[\{x, x', n_{x'}, y, n_y, z\}]$ is not a structural domination graph, which is a contradiction. Thus, z is adjacent to all vertices of Y.

Assume z separates $G[X]$. There is a vertex $z' \in X$ which is not adjacent to any vertex of Y. It is easy to see that the graph $G[\{x, n_x, x', n_{x'}, z, z'\}]$ is not a structural domination graph, which is a contradiction.

Assume z does not separate $G[X]$. As seen above, n_z necessarily has a neighbor n_x for some $x \in X$. Since Y is maximal, there is a private neighbor $n_y \notin N(n_z)$ for some $y \in Y$. It is easy to see that the graph $G[\{x, n_x, y, n_y, z, n_z\}]$ is not a structural domination graph, which is a contradiction.

Therefore, $Y = X$. We have to show $\gamma_c(G) \leq 2$ and can therefore assume $|X| \geq 3$. Let $x \in X$ be arbitrary and assume $S = \{x, n_x\}$ is not a dominating set of G. Then there is a vertex $v \notin X$ which is not dominated by S. Since X is a dominating set, v has a neighbor in X, say y. Since $|X| \geq 3$, there is a third vertex $z \in X$ with private neighbor n_z. By the choice of Y, n_z is adjacent to n_x. Thus, $G(\{v, y, x, n_x, n_z\})$ is either a path or a cycle of length 5. By Theorem 3 $G \notin Dom(\{K_n : n \in \mathbb{N}\})^*$, which is a contradiction to the fact that G has the dominating complete subgraph $G[X]$.

In the case of $\gamma_c(G) \geq 3$, for any minimal connected dominating set X that does not induce a path holds that $Art(G[X])$ is a subgraph of G. By Theorem 6, we then have $\gamma_c(G) \geq |X|$ and so X is a minimum connected dominating set. In the case that X is a minimal dominating set such that $G[X]$ is a path P_k, either $Art(G[X])$ or C_{k+2} is a subgraph of G. Hence, X is a minimum connected dominating set of G by Theorem 6. □

It is easy to see that a minimal connected dominating set can be found in polynomial time. Hence, Corollary 5 and Theorem 7 lead us to the following results about the computational complexities of the structural domination problems:

Corollary 6. *In the class of hereditary structural domination graphs, the following holds:*

1. *Minimum connected dominating sets can be computed in polynomial time.*
2. *For any concise graph class $\mathcal{G}$ which can be recognized in polynomial time, $Dom(\mathcal{G})$ can be recognized in polynomial time.*
3. *For any concise graph class $\mathcal{G}$, $\gamma_{\mathcal{G}}$ can be computed in polynomial time for the graphs in $Dom(\mathcal{G})$.*

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