Clause Set Structures and Polynomial-Time SAT-Decidable Classes

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Abstract. Proposing a fibre view on propositional clause sets, we investigate satisfiability testing for several CNF subclasses. Specifically, we show how to decide SAT in polynomial time for formulas where each pair of different clauses intersect either in all or in one variable.

Keywords: CNF satisfiability, hypergraph, fibre-transversal

1 Introduction

The intention of the present paper is to investigate certain structural properties of clause sets representing CNF formulas. Exploiting these properties we also search for subclasses of clause sets for which satisfiability is testable in polynomial time.

As the main topic, via introducing some concepts, we propose to regard clause sets from a slightly different perspective, namely as pairs of mutually set-complemented formulas with respect to the total clause set over a common intrinsic hypergraph called the *base hypergraph*. This yields no new structure from the logical point of view, because each clause set defined as usual and not behaving trivial regarding satisfiability can easily be seen to correspond to such a clause set pair. The hope is that this perspective may help to gain new structural insight into clause sets and even new algorithmic concepts. To supply such a hope we develop the basic theory around the concepts introduced and provide some new clause set classes that we prove to be polynomial time solvable via the methods presented. So e.g. satisfiability of CNF formulas, where every two distinct clauses share exactly one common variable or all (neglecting negations) can be decided in polynomial time, cf. Theorem 3. In order to establish our theory we have to introduce new concepts and notions, and we are not aware of a similar approach representing our view at satisfiability in a convenient framework.

There are several polynomial time SAT-testable classes known, as quadratic formulas, (extended and q-)Horn formulas, matching formulas etc. [1, 3, 4, 6, 7, 9, 10, 15, 16]. The classes studied in this paper, as far as we inspected, appear not to belong to one of these classes. On the other hand, mixing polynomial-time classes, in general, yields classes for which SAT becomes NP-complete, as alraedy is the case for Horn and quadratic formulas [12], cf. also [8].

To fix notation let CNF denote the set of duplicate-free conjunctive normal form formulas over propositional variables $x \in \{0,1\}$. A *positive (negative)* literal is a (negated) variable. The *negation (complement)* of a literal l is $\bar{l}$. Each formula $C \in \mathrm{CNF}$ is considered as a clause set, and each clause $c \in C$ is represented as a literal set free of $\{x, \bar{x}\}$. For formula C, clause c, literal l, by $V(C), V(c), V(l)$ we denote the variables contained (neglecting negations), correspondingly. $L(C)$ is the set of all literals in C. The length of C is denoted by $\|C\|$. For $U \subset V(C)$, let $\hat{C}(U) := \{c \in C : V(c) \cap U \neq \varnothing\}$, for $U' \subset L(C)$, set $C(U') := \{c \in C : c \cap U' \neq \varnothing\}$. For $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$, the power set of a set M is 2^M. The satisfiability problem (SAT) asks, whether input $C \in \mathrm{CNF}$ has a *model*, which is a truth assignment $t : V(C) \to \{0,1\}$ assigning at least one literal in each clause of C to 1. For, $C \in \mathrm{SAT}$, let $M(C)$ be the space of all (total) models of C, and $\mathrm{UNSAT} := \mathrm{CNF} - \mathrm{SAT}$. It turns out to be convenient to identify truth assignments with vectors in the following simple way: Let $x^0 := \bar{x}$, $x^1 := x$. Then we can identify a truth assignment $t : V \to \{0,1\}$ with the literal set $\{x^{t(x)} : x \in V\}$, and, for $b \subset V$, the *restriction* $t|_b$ is identified with the literal set $\{x^{t(x)} : x \in b\}$. We call W_V, the collection of the literal sets obtained in the described way by running through all total truth assignments $V \to \{0,1\}$, the *hypercube formula (over V)*, since its clauses correspond 1:1 to the vertices of a hypercube of dimension $|V|$. E.g., for $V = \{x, y\}$ we have $W_V = \{xy, \bar{x}y, x\bar{y}, \bar{x}\bar{y}\}$ writing clauses as literal strings. For a clause c we denote by c^γ the clause in which all its literals are complemented. Similarly, let $t^\gamma = 1 - t : V \to \{0,1\} \in W_V$, $C^\gamma := \{c^\gamma : c \in C\}$, and for $\mathcal{C} \subseteq \mathrm{CNF}$, let $\mathcal{C}^\gamma := \{C^\gamma : C \in \mathcal{C}\}$. We call C *symmetric* if $C = C^\gamma$, and *asymmetric* if for each $c \in C$ holds $c^\gamma \notin C$. Sym(Asym) $\subset$ CNF, denotes the set of all symmetric (asymmetric) formulas.

2 A Fibre-View on Clause Sets: Basic Concepts and Results

The fibre-view essentially corresponds to a projection of clauses on their underlying sets of variables via neglecting negations yielding monotone base clauses. A formula then appears to be the collection of all its fibres-subformulas, each of which is composed of all clauses projecting on the same monotone base. In order to exploit this rather natural perspective on formulas with regard to its impact on CNF satisfiability testing, let us formulate it in somewhat precise terms.

A *(variable-) base hypergraph* $\mathcal{H} = (V, B)$ is a hypergraph whose vertices $x \in V$ are regarded as Boolean variables. So the set B of (hyper)edges can be considered as a positive monotone clause set. It is required throughout that *each* vertex $x \in V$ is contained in at least one edge $b \in B$. For each $b \in B$, W_b denotes the hypercube formula over b, which therefore consists of all possible clauses over variable set b. Therfore $K_{\mathcal{H}} := \bigcup_{b \in B} W_b$ is the set of all possible clauses over $\mathcal{H}$, and is called the *total clause set over $\mathcal{H}$*. Regarding each b as a *point* in the space B, we obtain the following mapping

$$\pi : K_{\mathcal{H}} \ni c \mapsto V(c) \in B$$

recalling that $V(c)$ is the set of variables in clause c. We call $\pi^{-1}(b) = W_b$ the *fibre of* $K_{\mathcal{H}}$ over b. Obviously the fibres are mutually disjoint, w.r.t. clause-points, and π is surjective, thus is a projection.

A *formula over* $\mathcal{H}$ *(or* $\mathcal{H}$*-formula)* is any subset $C \subset K_{\mathcal{H}}$ such that $C \cap W_b \neq \varnothing$ for each $b \in B$, implying that the restriction $\pi_C := \pi|_C$ of π to C also is a projection $\pi_C : C \to B$. Let $C_b := \pi_C^{-1}(b) \subseteq W_b$ denote the *fibre(-subformula)* of C over b. Note that any $C \in \mathrm{CNF}$ can be viewed in the framework above, for it is has the base hypergraph $\mathcal{H}(C) := (V(C), B(C))$ with $B(C) := \{V(c) : c \in C\}$).

For each $C \subset K_{\mathcal{H}}$ having the property $(*)$: $W_b - C_b \neq \varnothing$, for all $b \in B$, we define its *(*$\mathcal{H}$*-based) complement formula* $\bar{C}$ via $\bar{C} := \bigcup_{b \in B}(W_b - C_b)$. By construction $\bar{C}$ has the same base hypergraph as C.[1] A *fibre-transversal (f-transversal)* of $K_{\mathcal{H}}$ is a $\mathcal{H}$-formula $F \subset K_{\mathcal{H}}$ meeting each fibre in exactly one point: $|F \cap W_b| = 1$, for each $b \in B$.[2] Let the unique clause of fibre $\pi^{-1}(b)$ contained in F be refered to as $F(b)$. Let $\mathcal{F}(K_{\mathcal{H}})$ denote the set of all f-transversals of $K_{\mathcal{H}}$. The notion of f-transversals also carries over to a $\mathcal{H}$-formula C different from the total clause set. To that end a f-transversal of C is restricted to those fractions of the fibres belonging to C. Let $\mathcal{F}(C)$ denote the set of f-transversals of C. The next definition introduces some complementary types of f-transversals, namely *compatible* and *diagonal* ones. The first are related to satisfiable formulas and the latter turn out to be always unsatisfiable.

Definition 1 *Let* $\mathcal{H} = (V, B)$ *and* $K_{\mathcal{H}}$ *as defined above.*
(1) $F \in \mathcal{F}(K_{\mathcal{H}})$ *is called* compatible *if* $\bigcup_{b \in B} F(b) \in W_V$*, meaning that* F *contains each variable of* V *as a pure literal. Let* $\mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$ *be the collection of all compatible f-transversals of* $K_{\mathcal{H}}$*.*
(2) $F \in \mathcal{F}(K_{\mathcal{H}})$ *is called* diagonal *if for each* $F' \in \mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$ *holds* $F \cap F' \neq \varnothing$*, hence meeting each compatible f-transversal in at least one (clause-)point. Let* $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}})$ *be the collection of all diagonal f-transversals of* $K_{\mathcal{H}}$*.*
(3) For any $\mathcal{H}$*-based formula* $C \subseteq K_{\mathcal{H}}$*, let* $\mathcal{F}_{\mathrm{comp}}(C) := \mathcal{F}(C) \cap \mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$ *and* $\mathcal{F}_{\mathrm{diag}}(C) := \mathcal{F}(C) \cap \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}})$*.*

As a simple example for a compatible f-transversal consider the base hypergraph $\mathcal{H}$ with variable set $V := \{x_1, x_2, x_3\}$ and base points $\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}$. Then e.g. the clauses $c_1 := \{\bar{x}_1, x_2\}$, $c_2 := \{\bar{x}_1, \bar{x}_3\}$, and $c_3 := \{x_2, \bar{x}_3\}$ form a compatible f-transversal of the corresponding $K_{\mathcal{H}}$, because $c_1 \cup c_2 \cup c_3 = \{\bar{x}_1, x_2, \bar{x}_3\} \in W_V$.

Whereas compatible f-transversals always exist as the example indicates, it is, in advance, not clear whether diagonal transversals exist at all, a question

[1] Clearly, any hypercube formula is unsatisfiable, therefore in case that C does not have property $(*)$ it is unsatisfiable trivially, which therefore can be ruled out. More precisely, it can be treated by a simple preprocessing checking in linear time whether there is $b \in B$ such that $W_b = C_b$.

[2] $K_{\mathcal{H}}$ can be viewed as a hypergraph having all literals over V as vertex set. So, a fibre-transversal should not be mixed up with a hypergraph-transversal which, as usually defined, is a subset of its vertex set meeting all its edges, thus is a hitting set.

that will be addressed below. However, if there are diagonal transversals, then each fixed compatible transversal in turn meets all diagonal transversals.

We have some simple observations regarding the f-transversals introduced.

Proposition 1 *(1)* $\mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}}) \cong W_V$ *(means isomorphism),*
(2) $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}}) = \{F \in \mathcal{F}(K_{\mathcal{H}}) : \forall t \in W_V \exists b \in B : F(b) = t|_b\}$,
(3) $\mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})^\gamma = \mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$,
(4) $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}})^\gamma = \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}})$,
(5) $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}}) \cap \mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}}) = \varnothing$.

Proof. Assertion (1) is easily obtained by observing that

$$\varphi : \mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}}) \ni F \mapsto \bigcup_{b \in B} F(b) \in W_V$$

is a bijection with $[\varphi^{-1}(t)](b) := t|_b$ for each $t \in W_V$, $b \in B$, recalling that by assumption $\bigcup_{b\in B} b = V$. (2) immediately follows from (1).

Assertion (3) is obvious, and implies $\varphi(F^\gamma) = \varphi(F)^\gamma$, for $F \in \mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$.

Let $F \in \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}})$ and assume there is $F' \in \mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$ such that $F'(b) \neq F^\gamma(b)$ for all $b \in B$ equivalent to $F'^\gamma(b) \neq F(b)$ for all $b \in B$, by (3) contradicting that F is diagonal yielding (4).

Assume $F \in \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}}) \cap \mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$, then by (3) also $F^\gamma \in \mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$ but $F^\gamma(b) \neq F(b)$ for each $b \in B$ therefore $F \notin \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}})$ yielding a contradiction implying (5). □

The next assertion essentially states that a formula C is satisfiable if and only if its $\mathcal{H}$-based complement formula admits a compatible f-transversal:

Theorem 1 *For* $\mathcal{H} = (V, B)$, *and* $K_{\mathcal{H}}$ *let* $C \subset K_{\mathcal{H}}$ *be a* $\mathcal{H}$*-formula such that* $\bar{C} \subset K_{\mathcal{H}}$ *also is a* $\mathcal{H}$*-formula (hence* $B(C) = B = B(\bar{C})$*), we have:*
(i) $C \in \mathrm{SAT}$ *if and only if* $\mathcal{F}_{\mathrm{comp}}(\bar{C}) \neq \varnothing$.
(ii) If $C \in \mathrm{SAT}$ *then* $M(C) \cong \mathcal{F}_{\mathrm{comp}}(\bar{C})$.

Proof. We claim that if $C \in \mathrm{SAT}$, hence $W_V \supseteq M(C) \neq \varnothing$, then $\mathcal{F}_{\mathrm{comp}}(\bar{C}) = \varphi^{-1}(M(C)^\gamma)$, where φ is defined as in the proof of Prop. 1 (1). From this claim (ii) follows, as obviously $M(C)^\gamma \cong M(C)$ and φ is a bijection. Further (i) follows: If $M(C)$ is empty then also $\mathcal{F}_{\mathrm{comp}}(\bar{C})$ must be empty, otherwise by the claim holds $\varphi(F)^\gamma \in M(C)$, for any $F \in \mathcal{F}_{\mathrm{comp}}(\bar{C})$, yielding a contradiction. The reverse direction of (i) is immediately implied by the claim.

So it remains to verify the claim: Let $t \in M(C)$ be chosen arbitrarily. Clearly, $F_t := \varphi^{-1}(t^\gamma) \in \mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$ by definition of φ. Suppose there is $b \in B$ such that $F_t(b) = t^\gamma|_b \in C$. Clearly, $t|_b$ is a total truth assignment of the hypercube formula W_b satisfying all of its clauses except $(t|_b)^\gamma = t^\gamma|_b \in C$ thus $t \notin M(C)$ contradicting the assumption. Therefore $F_t(b) \in \bar{C}$, for all $b \in B$, hence $F_t \in \mathcal{F}_{\mathrm{comp}}(\bar{C})$ thus $\varphi^{-1}(M(C)^\gamma) \subseteq \mathcal{F}_{\mathrm{comp}}(\bar{C})$. Conversely, let $F \in \mathcal{F}_{\mathrm{comp}}(\bar{C})$ then we claim that $t_F := \varphi(F)^\gamma \in M(C) \subseteq W_V$. Indeed, supposing the contrary, there is $b \in B$ with $t_F^\gamma|_b \in C$ equivalent to $F(b) = \varphi(F)|_b \notin \bar{C}$ contradicting the assumption and finishing the proof because $\varphi^{-1}(t_F^\gamma) = F$. □

Proposition 2 *Let $F \in \mathcal{F}(K_{\mathcal{H}})$, then holds*

$$\begin{aligned} (1) \quad & F \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \Leftrightarrow F \in \text{UNSAT} \\ (2) \quad & F \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}}) \Rightarrow F \in \text{SAT} \end{aligned}$$

Proof. By Theorem 1 (i), we have $F \in$ UNSAT iff $\mathcal{F}_{\text{comp}}(\bar{F}) = \varnothing$ iif $\forall F' \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ there is $b \in B$ such that $F'(b) = F(b) \in F$ iif $F \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$, hence (1). (2) is implied by (1) due to Prop. 1 (5); moreover for $F \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$, $\varphi(F) \in W_V$ specifically satisfies F. □

Thus, we have three types of possible f-transversals composing $\mathcal{F}(K_{\mathcal{H}})$, namely compatible f-transversals which always are satisfiable formulas, diagonal ones (which may not exist) which always are unsatisfiable, and, finally, f-transversals that are neither compatible nor diagonal but always are satisfiable.

Definition 2 *A formula $D \subseteq K_{\mathcal{H}}$ is called a* diagonal *formula if for each $F \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ holds $F \cap D \neq \varnothing$.*

Obviously each $F \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ (if existing) is a diagonal formula. Since a diagonal formula D contains a member of each compatible f-transversal the complement formula $\bar{D}$ cannot have a compatible f-transversal. Therefore $D \in$ UNSAT due to Theorem 1, and we have:

Proposition 3 *A formula is unsatisfiable iff it contains a subformula that is diagonal.* □

Consider a simple application of the concepts above: Recall that a hypergraph is called *Sperner* (or sometimes also called *simple*) if no hyperedge is contained in another hyperedge [2]. Clearly, a formula C, regarded as a hypergraph $(L(C), C)$ over its literal set, that is non-Sperner can be turned into a SAT-equivalent one having that property: For clauses c, c' with $c \subset c'$ we can remove c' from C because c already has to be satisfied implying satisfiability of c'. Let C be Sperner then its base hypergraph $\mathcal{H}(C) = (V(C), B(C))$ can either be Sperner or non-Sperner, assume $\mathcal{H}(C) = \mathcal{H}(\bar{C})$. Clearly, if $\mathcal{H}(C)$ is Sperner then so is $\bar{C}$. Specifically, all these objects are Sperner if all clauses have the same length. However, if $\mathcal{H}(C)$ is non-Sperner, $\bar{C}$ can be Sperner or non-Sperner. For the first case, i.e., C and $\bar{C}$ Sperner but $\mathcal{H}(C) = \mathcal{H}(\bar{C})$ non-Sperner, consider the following example (simply representing clauses as strings of the literals contained):

$$\begin{aligned} C &= \{xy, x\bar{y}z, \bar{x}yz, \bar{x}\bar{y}z, x\bar{y}\bar{z}, \bar{x}y\bar{z}, \bar{x}\bar{y}\bar{z}\} \\ \bar{C} &= \{x\bar{y}, \bar{x}y, \bar{x}\bar{y}, xyz, xy\bar{z}\} \\ B(C) &= \{xy, xyz\} \end{aligned}$$

Theorem 2 *Let $C \in$ CNF be Sperner such that its base hypergraph $\mathcal{H}(C)$ is non-Sperner but the complement formula $\bar{C}$ is Sperner: Then both C and $\bar{C}$ are unsatisfiable.*

Proof. According to Theorem 1 we show that $\bar{C}$ cannot have a compatible f-transversal under the assumptions stated above. Because $\mathcal{H}(C)$ non-Sperner there are $b, b' \in B(C)$ with $b \subset b'$ and $b \neq b'$. Now for each f-transversal $F \in \mathcal{F}(\bar{C})$ holds $F(b) \not\subset F(b')$ as $\bar{C}$ is assumed to be Sperner. That means there is $x \in b$ such that $x \in F(b)$, $\bar{x} \in F(b')$ or vice versa, hence $F(b) \cup F(b') \supset \{x, \bar{x}\}$ is not compatible implying that $C \in$ UNSAT. By exchanging the roles of C and $\bar{C}$ we also obtain that $\bar{C}$ cannot be satisfiable. ☐

Corollary 1 *If C is Sperner and satisfiable then either*
(i) $\mathcal{H}(C)$ and $\bar{C}$ both are Sperner or
(ii) $\mathcal{H}(C)$ and $\bar{C}$ both are non-Sperner and for each two $b_1 \subset b_2 \in B(C)$ there are $c_1 \subset c_2 \in \bar{C}$ such that $V(c_i) = b_i$, $i = 1, 2$.

Remark 1 *The criterion in (ii) of the Corollary is not sufficient for satisfiability of C: Let $b_1 \subset b \in B(C)$ such that $c_1 \subset c \in \bar{C}$ and moreover let $b'_1 \subset b' \in B(C)$ such that $c'_1 \subset c' \in \bar{C}$ where $V(c) = b$, $V(c') = b'$, $V(c_1) = b_1$, and $V(c'_1) = b'_1$. Now assume that $b \cap b' \neq \varnothing$, and that c, c' are the only clauses over b, b' in $\bar{C}$. Clearly, if $c|_{b\cap b'} \neq c'|_{b\cap b'}$ then there is no compatible f-transversal of $\bar{C}$ hence no model of C.*

Returning to the general discussion, let $\mathcal{H} = (V, B)$ be a non-empty base hypergraph, then clearly $\mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$ is not empty we even have $|\mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})| = 2^{|V|}$ due to Prop. 1 (1). However, a priori it is not clear whether also holds $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}}) \neq \varnothing$ in any case. It turns out that this depends strongly on the structure of the base hypergraph $\mathcal{H}$: To that end, let us consider an interesting and guiding example regarding satisfiability of certain formulas over *(exactly) linear* base hypergraphs. In [13, 14] linear formulas (variable sets of distinct clauses have at most one member in common) are discussed in more detail and satisfiability of exactly linear formulas is shown by simple matching techniques.

Lemma 1 *[13] Each exactly linear formula C is satisfiable.*

From the last result we immediately conclude that if the base hypergraph $\mathcal{H} = (V, B)$ is exactly linear then for the corresponding total clause set holds $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}}) = \varnothing$. Indeed, then no unsatisfiable f-transversal can exist, as each is exactly linear and we are done by Proposition 2. This answers the earlier stated question whether there are hypergraphs admitting no diagonal f-transversal. The reverse question, namely are there hypergraphs at all such that the total clause sets has diagonal f-transversals, also is answered positive: In [13, 14] it is shown that there are unsatisfiable linear formulas, these formulas must be f-transversals hence correspond to diagonal f-transversals of the total clause set over the underlying base hypergraph:

Fact 1 *The notion of (diagonal) f-transversals immediately generalizes the notion of (unsatisfiable) linear formulas.*

However formulas having an exactly linear base hypergraph in general may not be satisfiable, because they can contain a diagonal subformula (which cannot be a f-transversal):

Theorem 3 *Let $\mathcal{H} = (V, B)$ be an exactly linear base hypergraph with corresponding total clause set $K_{\mathcal{H}}$. Let $C \subseteq K_{\mathcal{H}}$ be any $\mathcal{H}$-formula. Then we can check in polynomial time whether C contains a diagonal subformula, i.e., whether $C \in$* UNSAT.

PROOF. Recall that $C_b \subset W_b$ denotes the fibre subformula of C over $b \in B$, and that $C_b(l) \subset C_b$ is the subformula of C_b of all clauses containing literal l, where $V(l) \in b$. Let l be an arbitrary literal occuring in C, and first observe that, if $b \in B$ is an edge containing the underlying variable $V(l) \in b$, then the clauses in C_b cover (i.e., have intersection with) exactly $\mu(l) := \mu(l, b) := |C_b(l)| \cdot 2^{n-|b|}$ of all 2^{n-1} truth assignments containing l, where $n := |V|$.

We intend to determine the number of truth assignments met by the clauses in the input formula C. This essentially is organized by performing two independent runs of a Procedure `ComputeCoverNumber`(l, p), one for $l = x$ and a second one for $l = \bar{x}$. Here x is the *maximum variable* that together with the determined edges $b_1, b'_1 \in B$ (smallest index if ambigous), has to be computed first according to

$$\mu(x, b_1) + \mu(\bar{x}, b'_1) = \max\{\mu(y, b) + \mu(\bar{y}, b') : y \in L(C_b), \bar{y} \in L(C_{b'}), b, b' \in B\}$$

here $\mu(l, b) = |C_b(l)| \cdot 2^{n-|b|}$ is computed for all $(l, b) \in L(V) \times B$ such that $C_b(l) \neq \varnothing$. It is possible that $b_1 = b'_1$.

Both executions of Procedure `ComputeCoverNumber`(l, p) are initiated only if $\mu(l) < 2^{n-1}$ meaning that the fibre subformula corresponding to the maximum does not cover all 2^{n-1} possible truth assignments containing l. Finally, the corresponding cover numbers returned in p are added, and the algorithm returns unsatisfiable iff the total value equals 2^n. Clearly, the runs of the procedure for x and $\bar{x}$ can be processed independently because both compute coverings in different ranges in the set of all truth assignments

Now procedure `ComputeCoverNumber`(l, p) consists of two main subprocedures. A first is entered only if there is at least one fibre subformula C_b containing l besides C_{b_1} and computes all additional truth assignments containing x covered by these fibre subformulas. The second subprocedure is entered only in case there are any remaining fibre subformulas not containing l, and the subprocedure is devoted to determine all additionally covered truth assignments containing l covered by these subformulas.

The first subprocedure proceeds as follows: W.l.o.g. (otherwise relabel the members in B) let $\{C_{b_2}, \ldots, C_{b_s}\}$, for $s \geq 1$, denote the collection of all remaining fibre subformulas with $V(l) \in b_i$, $2 \leq i \leq s$. Assume that its members are ordered due to decreasing cardinalities of its subsets $|C_{b_j}(l)|$ containing l, for $2 \leq j \leq s$.

For simplicity let $m_j := |C_{b_j}(l)|$ and $m'_j := |W_{b_j}(l) - C_{b_j}(l)| = 2^{|b_j|-1} - |C_{b_j}(l)|$, for $1 \leq j \leq s$. Then the number of truth assignments containing l covered by the subformulas in C_l is given by:

$$(*) \quad m'_1 \sum_{j=2}^{s} \left[m_j \cdot 2^{n+(j-1)-\sum_{q=1}^{j} |b_q|} \cdot \prod_{k=2}^{j-1} m'_k \right]$$

where, as usual, $\prod_{i=b}^{k} a_i := 1$, for $k < b$.

Clearly, number $(*)$ can be determined performing a simple loop recalling that by assumption $m_j > 0$, for all $1 \leq j \leq s$:

$z \leftarrow m'_1 \cdot m_2 \cdot 2^{n+1-|b_1|-|b_2|}$
$p \leftarrow z$
for $j = 2$ **to** $s-1$ **do**
  $z \leftarrow z \cdot m'_j \cdot \frac{m_{j+1}}{m_j} \cdot 2^{1-|b_{j+1}|}$
  $p \leftarrow p + z$
od

So finally, we have to check whether the resulting value $p = 2^{n-1}$. In order to avoid calculations with possibly large number 2^n it is sufficient instead to compute $p' := p/2^n$ and finally checking whether $p' = 1/2$. Observe that the second subprocedure needs to be started only if the answer is negative.

For explaining the second subprocedure, let c be any clause of a fibre subformula over $b \in B - B(x)$, then c covers a truth assignment containing l if and only if for each $b_i \in B(x)$ there are $c_i \in W_{b_i} - C_{b_i}(x)$ with $c \cap c_i \neq \emptyset$. Observe that none of these truth assignments is covered by those computed in the first subprocedure, because each of the latter ones fixes all literals of at least one complete clause in any x-fibre subformula whereas each of the newly as covered determined truth assignments are composed of missing clauses in each hypercube formula $W_b(l) - C_b(l)$, for all $e \in B(x)$. So each corresponding truth assignment is different to each detected in the first subprocedure in at least one position.

W.l.o.g. (which always can be achieved via relabeling), let $\mathcal{C}(x) := \{C_{b_{s+1}}, \ldots, C_{b_{s+r}}\}$, for $r \geq 1$, be the collection of all fibre subformulas neither containing x nor $\bar{x}$, hence it has exactly one member for each edge in $B - B(x)$. For $C_{b_{s+1}} \in C'_x$ and $c \in C_{b_{s+1}}$, let $\{y_i\} = V(c) \cap C_{b_i}$, $1 \leq i \leq s$, which are uniquely determined because of exact linearity. Assume that $l_i \in c$ is the corresponding literal with $V(l_i) = y_i \neq x$, where clearly $|c| \geq s$ and each variable in c different from y_i, $1 \leq i \leq s$, cannot occur in any member of C_l.

Let $n_l := \sum_{q=1}^{j} |b_q| - (s-1)$ be the number of variables already fixed by $b_i, 1 \leq i \leq s$. Let $\lambda_i(c) := |c \cap [W_{b_i}(l) - C_{b_i}(l)]|$ be the number of occurences of literal l_i in $W_{b_i}(l) - C_{b_i}(l)$ which is the fibre complement of $C_{b_i}(l)$. Clearly l_i occurs in exactly $2^{|b_i|-2}$ clauses in $W_{b_i}(l)$. So, if l_i occurs t_i times in $C_{b_i}(l)$, we obvioulsy have

$$\lambda_i(c) = 2^{|b_i|-2} - t_i$$

Now the clauses in $C_{b_{s+1}}$ exactly cover the following number of additional truth assignments containing l:

$$2^{n-n_l-(|b_{s+1}|-s)} \sum_{c \in C_{b_{s+1}}} \prod_{j=1}^{s} \lambda_j(c)$$

Therefore, we obtain for the number of covered truth assignments containing l by all members of C'_x,

$$\sum_{k=1}^{r}\left[2^{n-n_l-\sum_{j=1}^{k} f(j)} \sum_{c\in C_{b_{s+k}}}\left(\prod_{j=1}^{s+k-1}\lambda_j(c)\right)\right]$$

where

$$f(j) := |b_{s+j}| - \left|\bigcup_{i=1}^{s+j-1}(b_{s+j}\cap b_i)\right| \in \{0,\ldots,|b_{s+j}|-s\}$$

$1 \le j \le r$.

Having processed `ComputeCoverNumber`(l,p) for $l := x$ we again check whether $p' = 1/2$ and only in the positive case we run `ComputeCoverNumber`(l,p) for $l := \bar{x}$, because otherwise not all truth assignments containing x are covered, immediately enabling us to conclude that $C \in$ SAT. □

Obviously, the method above is not able to solve the search problem, we only obtain a decision whether C is satisfiable, but in positive case we are not aware of a model.

So, there are cases where no diagonal f-transversal of the total clause set exists, but unsatisfiable formulas $C \subset K_{\mathcal{H}}$ can exist although, so we conclude that, despite of Proposition 2, in general $C \in$ UNSAT is not equivalent to $\mathcal{F}(C) \neq \varnothing$. However, things may be different if H is structured such that $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \neq \varnothing$. So, we next pose the question whether under this assumption holds $C \in$ UNSAT iff $\mathcal{F}_{\text{diag}}(C) \neq \varnothing$. Observe that the implication $\Leftarrow$ holds because if C admits a diagonal f-transversal then $\bar{C}$ cannot have a compatible f-transversal therefore $C \in$ UNSAT due to Theorem 1 (i).

Definition 3 *Let $\mathcal{H} = (V, B)$ be a base hypergraph.*
We call $\mathcal{H}$ a diagonal base hypergraph *if $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \neq \varnothing$.*
$\mathcal{H}$ is called strictly diagonal *if it is diagonal, and additionally:*

$$(*) : \forall C \subset K_{\mathcal{H}} : B(C) = B = B(\bar{C}) : C \in \text{UNSAT} \Leftrightarrow \mathcal{F}_{\text{diag}}(C) \neq \varnothing$$

We first consider the question whether the class of strictly diagonal base hypergraphs coincides with the class of all diagonal base hypergraphs. To give an answer constructively: Start with a linear hypergraph $\mathcal{H} = (V, B)$ that admits an unsatisfiable polarization hence admits a diagonal f-transversal and therefore $\mathcal{H}$ is diagonal. Assume that $\mathcal{H}$ results by a block construction over a base block hypergraph $\mathcal{H}' = (V', B')$ with $V' \subset V, B' \subset B$ as shown in [13]. Then we claim that we can construct a (small) unsatisfiable formula $C' \subset K_{\mathcal{H}'}$ with $\pi'(C') = B' = \pi'(\bar{C}')$. Now we claim that it is possible to add to C' exactly one member of each fibre of $\pi^{-1}(b)$, forall $b \in B - B'$ such that the resulting formula C has the property that each of its f-transversals is satisfiable, hence cannot be diagonal. Thereofore the above stated question gets a negative answer. The question whether there exist strictly diagonal hypergraphs is still open.

Lemma 2 *For $\mathcal{H}$ strictly diagonal holds that each f-transversal meeting all diagonal f-transversals is compatible, formally:*

$$\mathcal{F}_{\text{comp}}(K_{\mathcal{H}}) = \{F \in \mathcal{F}(K_{\mathcal{H}}) : \forall F' \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) : F \cap F' \neq \varnothing\}$$

Proof. Let $F \in \mathcal{F}(K_{\mathcal{H}})$ meeting all members of $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ then clearly F cannot be diagonal as F^γ is diagonal. If F is compatible we are done. So assume that F is neither compatible nor diagonal, then specifically $\bar{F} \in \text{UNSAT}$ due to Theorem 1. Since $\mathcal{H}$ is strictly diagonal it is implied that $\mathcal{F}_{\text{diag}}(\bar{F}) \neq \varnothing$ meaning there is $F' \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ such that $F \cap F' = \varnothing$. □

We next provide some further considerations, besides the discussion of diagonality: For $\mathcal{H} = (V, B)$, again let $C \subset K_{\mathcal{H}}$ such that $B(C) = B = B(\bar{C})$. If $C \in \text{SAT}$ then due to Prop. 1 (1) each $t \in M(C)$ satisfies $\varphi^{-1}(t) \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$. We now address the question in which case each model t of C even satisfies $\varphi^{-1}(t) \in \mathcal{F}_{\text{comp}}(C)$, i.e., corresponds to a compatible f-transversal of the formula itself.

Lemma 3 *If $C \in \text{CNF}\cap\text{SAT}$, $B(C) = B(\bar{C})$, such that for each $t \in M(C)$ holds $\varphi^{-1}(t) \in \mathcal{F}_{\text{comp}}(C)$ then $\bar{C} \in \text{SAT}$ and $\varphi(F) \in M(\bar{C})$ for each $F \in \mathcal{F}_{\text{comp}}(\bar{C})$; and vice versa.*

Proof. Let $F \in \mathcal{F}_{\text{comp}}(\bar{C})$ be arbitrary with $t := \varphi(F) \in W_V$, then according to the proof of Theorem 1 (ii) t^γ is a model of C. By assumption there is $F' \in \mathcal{F}_{\text{comp}}(C)$ such that $t^\gamma = \varphi(F')$. Hence, again by Theorem 1 (ii), t is a model of $\bar{C}$ as claimed, specifically $\bar{C} \in \text{SAT}$. The vice versa assertion follows by exchanging the roles of C and $\bar{C}$. □

Next we provide a formula class admitting the assumption of the last lemma. Recall that a symmetric formula satisfies $C = C^\gamma$ Clearly, if $C \in \text{Sym}$ then also $\bar{C} \in \text{Sym}$, since for $c \in \bar{C}$ holds $c \notin C$ thus $c^\gamma \notin C$ implying $c^\gamma \in \bar{C}$.

Lemma 4 *Let $C \in \text{CNF}$ such that $B(C) = B(\bar{C})$ and $\bar{C} \in \text{Asym}$. Then $C \in \text{SAT}$ implies $\bar{C} \in \text{SAT}$ and each $t \in M(C)$ satisfies $\varphi^{-1}(t) \in \mathcal{F}_{\text{comp}}(C)$; and vice versa.*

Proof. Let $t \in M(C) \neq \varnothing$ then for each $b \in B(C)$ $t|_b$ satisfies all of W_b except for $(t|_b)^\gamma$ which thus must be a clause of $\bar{C}$. And $\bar{C} \in \text{Asym}$ implies that $t|_b \in C$ for each $b \in B(C)$. Hence $\{t|_b : b \in B(C)\}$ is a compatible f-transversal of C. It follows that $\bar{C} \in \text{SAT}$ and that for each $t \in M(C)$ holds $\varphi^{-1}(t) \in \mathcal{F}_{\text{comp}}(C)$. The vice versa assertion follows by exchanging the roles of C and $\bar{C}$. □

Corollary 2 *Let $C \in \text{Asym}$ such that also $\bar{C} \in \text{Asym}$ and $B(C) = B(\bar{C})$. Then $C \in \text{SAT}$ if and only if $\bar{C} \in \text{SAT}$.* □

A formula is satisfiable if and only if the complement formula admits a compatible f-transversal. Therefore the specific class of formulas C such that every f-transversal of C is compatible is of interest, because then any f-transversal gives rise to a model of $\bar{C}$, and vice versa. To provide a characterization of that very specific class, for $C \in \text{CNF}$, let $I(C) := \{r = b \cap b' : b \neq b' \in B(C)\}$, and let $C[r] = \{c \in C : r \subseteq V(c)\}$, for each $r \in I(C)$.

Lemma 5 *Let* $C \in \mathrm{CNF}$ *such that* $B(C) = B(\bar{C})$. *Then* $\mathcal{F}_{\mathrm{comp}}(C) = \mathcal{F}(C)$, *i.e., any f-transversal of* C *is compatible iff* $(*)$: *for each fixed* $r \in I(C)$ *holds* $c|_r = c_0|_r, \forall c \in C[r]$ *and arbitrary fixed* $c_0 \in C[r]$.

Proof. Consider the hypergraph $\tilde{B} := B(C) \cup I(C)$ having $V(C)$ as vertex set. Similarly consider $\tilde{C} := C \cup \{c \cap r : c \in C, r \in I(C)\}$ then it is easy to see that $(*)$ is equivalent to: For each $r \in I(C)$ holds $|\tilde{C}_r| = 1$ where $\tilde{C}_r := \{c \in \tilde{C} : V(c) = r\}$ is the fibre of $\tilde{C}$ over r, from which the assertion immediately follows. □

So we obtain a class of satisfiable formulas recognizable in polynomial time: Let $\mathrm{CNF}_{\mathrm{comp}}$ denote the class of all formulas $C \in \mathrm{CNF}$ with $B(C) = B(\bar{C})$, and such that $\mathcal{F}(\bar{C}) = \mathcal{F}_{\mathrm{comp}}(\bar{C})$. As an example for $C \in \mathrm{CNF}_{\mathrm{comp}}$, let $\mathcal{H} = (V, B)$ with $V = \{q, r, s, t, u, v, x, y\}$, $B = \{b_1 = xy, b_2 = yuv, b_3 = vxr, b_4 = rst, b_5 = txq\}$ where brackets for edges are omitted, then the following (linear) formula is maximal w.r.t. to membership in $\mathrm{CNF}_{\mathrm{comp}}$, i.e., any additional clause over any $b \in B$ disturbs that membership, of course polarity of variables in $V(I(C))$ can be chosen differently:

$$\begin{array}{rcccc} \bar{C} = x\bar{y} & \bar{y}uv & vxr & rs\bar{t} & \bar{t}xq \\ & \bar{y}\bar{u}v & & r\bar{s}\bar{t} & \bar{t}x\bar{q} \end{array}$$

arranged fibrewise. It is obvious that any f-transversal of $\bar{C}$ is compatible, hence $C = K_{\mathcal{H}} - \bar{C} \in \mathrm{CNF}_{\mathrm{comp}}$.

Theorem 4 *We can check in polynomial time whether an input formula* $C \in \mathrm{CNF}$ *belongs to* $\mathrm{CNF}_{\mathrm{comp}} \neq \emptyset$ *and in positive case, implying that* C *is satisfiable, a model can be provided in polynomial time.*

Proof. Clearly, if $C \in \mathrm{CNF}_{\mathrm{comp}}$, then we only need to select a clause $c_b \in W_b - C_b$ for each $b \in B(C)$ ensuring that $\bigcup_{b \in B(C)} c_b^{\gamma} \in M(C)$ due to Theorem 1 (ii). For fixed $b = \{b_{i_1}, \ldots, b_{i_{|b|}}\}$, the selection can be performed e.g. by ordering the members $c = \{b_{i_1}^{\varepsilon_{i_1}(c)}, \ldots, b_{i_{|b|}}^{\varepsilon_{i_{|b|}}(c)}\}$ in C_b by lexicographic order of the vectors $(\varepsilon_{i_1}(c), \ldots, \varepsilon_{i_{|b|}}(c)) \in \{0,1\}^{|b|}$.

To decide whether $C \in \mathrm{CNF}_{\mathrm{comp}}$, according to the discussion above, first compute $I(C)$ and $V(I(C))$ i.e., all variables occuring in members of $I(C)$. Then check whether each $x \in V(I(C))$ occurs in $\bar{C}$ with a fixed polarity, only in the positive case holds $C \in \mathrm{CNF}_{\mathrm{comp}}$. □

References

1. B. Aspvall, M. R. Plass, and R. E. Tarjan, A linear-time algorithm for testing the truth of certain quantified Boolean formulas, Inform. Process. Lett. 8 (1979) 121-123.
2. C. Berge, Hypergraphs, North-Holland, Amsterdam, 1989.
3. E. Boros, Y. Crama, and P. L. Hammer, Polynomial time inference of all valid implications for Horn and related formulae, Annals of Math. Artif. Intellig. 1 (1990) 21-32.

4. E. Boros, P. L. Hammer, and X. Sun, Recognition of q-Horn formulae in linear time, Discrete Appl. Math. 55 (1994) 1-13.
5. R. G. Downey and M. R. Fellows, Parameterized Complexity, Springer-Verlag, New York, 1999.
6. J. Franco, A. v. Gelder, A perspective on certain polynomial-time solvable classes of satisfiability, Discrete Appl. Math. 125 (2003) 177-214.
7. H. Kleine Büning and T. Lettman, Propositional logic, deduction and algorithms, Cambridge University Press, Cambridge, 1999.
8. D. E. Knuth, Nested satisfiability, Acta Informatica 28 (1990) 1-6.
9. H. R. Lewis, Renaming a Set of Clauses as a Horn Set, J. ACM 25 (1978) 134-135.
10. M. Minoux, LTUR: A Simplified Linear-Time Unit Resolution Algorithm for Horn Formulae and Computer Implementation, Inform. Process. Lett. 29 (1988) 1-12.
11. B. Monien, and E. Speckenmeyer, Solving satisfiability in less than 2^n steps, Discrete Appl. Math. 10 (1985) 287-295.
12. S. Porschen, and E. Speckenmeyer, Satisfiability of Mixed Horn Formulas, Discrete Appl. Math., 2007, doi:10.1016/j.dam.2007.02.010
13. S. Porschen, E. Speckenmeyer, and B. Randerath, On linear CNF formulas, in: Proceedings of the 9th International Conference on Theory and Applications of Satisfiability Testing (SAT 2006), Lecture Notes in Comp. Science, Vol. 4121, pp. 221-225, Springer-Verlag, Berlin, 2006.
14. S. Porschen, E. Speckenmeyer, X. Zhao, Linear CNF formulas and satisfiability, Techn. Report zaik2006-520, Univ. Köln, 2006.
15. J. Schlipf, F. S. Annexstein, J. Franco, R. P. Swaminathan, On finding solutions for extended Horn formulas. Inform. Process. Lett. **54** (1995) 133-137.
16. C. A. Tovey, A Simplified NP-Complete Satisfiability Problem, Discrete Appl. Math. 8 (1984) 85-89.