Maximum Planar Subgraph on Graphs not Contractive to K_5 or $K_{3,3}$*

Elisabeth Gassner[1] and Merijam Percan[2]

[1] Technische Universität Graz, 5020 Institut für Mathematik B
Steyrergasse 30/II, 8010 Graz, Austria
gassner@opt.math.tu-graz.ac.at
[2] Universität zu Köln, Institut für Informatik,
Pohligstraße 1, 50969 Köln, Germany
percan@informatik.uni-koeln.de

Abstract. The maximum planar subgraph problem is well studied. Recently, it has been shown that the maximum planar subgraph problem is $\mathcal{NP}$-complete for cubic graphs [5]. In this paper we prove shortly that the maximum planar subgraph problem remains $\mathcal{NP}$-complete even for graphs without a minor isomorphic to K_5 or $K_{3,3}$, respectively.

1 Introduction

Wagner characterizes a planar graph as a graph that has no K_5 and $K_{3,3}$ minors [13]. His theorem is a significant reformulation of Kuratowski's well-known result [8]. If G has either no K_5 or no $K_{3,3}$ minor it is intuitively close to planarity.

The *maximum planar subgraph problem* (MPSP for short) is well-studied: Given a graph $G = (V, E)$ and a positive integer $k \leq |E|$, is there a subset $E' \subseteq E$ with $|E'| \geq k$ such that the graph $G' = (V, E')$ is planar?

Liu et al., Yannakatis, and Watanabe et al. all independently showed that this problem is $\mathcal{NP}$-complete [10, 14, 15].

The weighted version of MPSP is a generalization of MPSP where weights are assigned to the edges and the task is to find a planar subgraph of maximum total weight. Recently, Faria, de Figueiredo, and de Mendonça have shown that the maximum planar subgraph problem remains $\mathcal{NP}$-complete for cubic graphs [5]. For a survey on the maximum planar subgraph problem, the reader is referred to [9].

Obviously, the class of non-planar cubic graphs is not equal to the class of graphs that are either K_5-free or $K_{3,3}$-free (see Figure 1).

We strengthen the $\mathcal{NP}$-completeness result for the maximum planar subgraph problem by showing that it is $\mathcal{NP}$-complete even for graphs without a $K_{3,3}$ or K_5 minor, respectively.

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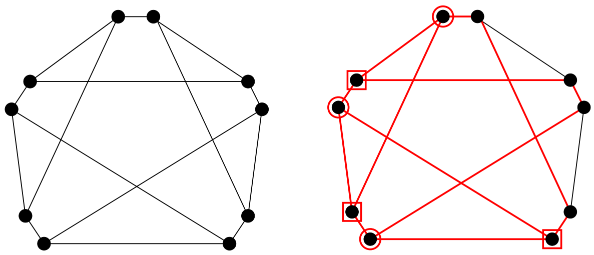

(a) Cubic graph (b) $K_{3,3}$ minor

Fig. 1. A non-planar cubic graph that has minors isomorphic to K_5 and to $K_{3,3}$

2 Preliminaries

Given a connected planar graph $G = (V, E)$ the connected vertex cover decision problem (CVC for short) asks for a vertex cover N in G of cardinality at most k, such that the subgraph induced by N is connected. CVC is known to be $\mathcal{NP}$-hard [6].

A *coloop* is an edge in a graph that does not lie in any cycle.

A minor can be defined in the following way: Let G and H be two undirected graphs. H is a minor of G if there exists a subgraph H' of G and a partition $V(H') = V_1 \uplus \cdots \uplus V_k$ of its vertex set into connected subsets such that contracting each of $V_1, \ldots, V_k$ yields a graph isomorphic to H.

Throughout this paper Tutte connectivity is used. The following definitions are given by Truemper [11]: Let $G = (V, E)$ be a connected graph. Let (E_1, E_2) be a pair of nonempty sets that partition the edge set E. Let G_1 (resp. G_2) be obtained by removal of the edges E_2 (resp. E_1). We assume G_1 and G_2 to be connected. We suppose that pairwise identification of k vertices of G_1 with k vertices of G_2 produces G. (E_1, E_2) is a (Tutte) k-separation if E_1 and E_2 have at least k edges each. G is called (Tutte) 3-connected if it has no (Tutte) 1- or 2-separation. G is called a *2-sum* (composition) of the connected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted $G = G_1 \oplus_2 G_2$, if the following process yields G: we identify an arbitrary edge e_1 of G_1 with an arbitrary edge e_2 of G_2 and delete this edge in $G_1 \cup G_2$.

3 $\mathcal{NP}$-hardness proofs

In this section we prove $\mathcal{NP}$-hardness of the maximum planar subgraph problem on K_5-free or $K_{3,3}$-free graphs, respectively.

Clearly the problem is in $\mathcal{NP}$ and we may obviously reduce the problem on connected graphs.

We use the following lemma for our $\mathcal{NP}$-hardness proofs.

Lemma 1 (Truemper [12]). *If G is a 2-sum of G_1 and G_2, then for any 3-connected minor N of G, G_1 or G_2 has a minor isomorphic to N.*

Proof. If this is not so, then N has a 2-separation induced by the 2-separation that is given by G_1 or G_2 minus their connecting edges. This leads to a contradiction to the fact that N is 3-connected. □

Let $\mathcal{K}_5$ be the following class of graphs, each constructed as follows. Let $G = (V, E)$ be a connected planar graph and E' a nonempty subset of E.

We apply an iterative processing of the edges e of E': We take the 2-sum of the current graph G with K_5, i.e. $G \oplus_2 K_5$, where $e \in E'$ is the edge of G involved in the 2-sum. Then, we redefine G to be the 2-sum.

We name $G[K_5]$ to be the final 2-sum that results from the former iterative processing.

Further, we define $\mathcal{K}_{3,3}$ and $G[K_{3,3}]$ in an analogue way, using $K_{3,3}$ instead of K_5.

Theorem 1. *The maximum planar subgraph problem is $\mathcal{NP}$-complete for the two classes $\mathcal{K}_5$ and $\mathcal{K}_{3,3}$.*

The proof of Theorem 1 uses the following result by Asano:

Theorem 2 (Asano [1]). *Let $G = (V, E)$ be a connected planar graph, and $G(2)$ obtained of G by splitting each edge of G once. The new vertices are denoted by a_i with $i = 1, 2, 3, \ldots, |E|$. Let G_2 be the graph constructed by a planar embedding of $G(2)$: for every face f add edges between two a_i, a_j, $i \neq j$, if they are belonging to the boundary of f and are adjacent to the same vertex. Let G_2^* be the dual graph of G_2 in respect to the modified planar embedding of $G(2)$. Let $N \subset V$ be a connected vertex cover of G with $|N| \leq k$, $k \in \mathbf{N}$.*

If G has a connected vertex cover of size at most k then $G(2)$ has a Steiner tree T for the terminal set $A = \{a_i \mid i = 1, 2, \ldots, |E|\}$ with $|E(T)| \leq k$ and all edges $(a_i, a_j)^* \in F^*$ with $i, j = 1, 2, 3, \ldots, |E|$, $i \neq j$ in $G_2^* - E(T)^*$ are coloops.

On the other hand, let $S \subset E(G_2^*) - F^*$ be a subset of edges in G_2^* with $|S| \leq k$ such that all edges $(a_i, a_j)^* \in F^*$ with $i, j = 1, 2, 3, \ldots, |E|$, $i \neq j$ in $G_2^* - S$ are coloops then G has a connected vertex cover of size at most k.

Asano's result [1] implies that the following multi-cut problem (MC for short) is $\mathcal{NP}$-hard. Obviously, MC is in $\mathcal{NP}$.

Corollary 1. *Given a connected planar graph $H = (V, E)$ with edge partition $E = E_1 \uplus E_2$ and an integer k, deciding whether there exist a subset $S \subset E_1$ with $|S| \leq k$ such that all edges $e \in E_2$ are coloops in $H - S$ is $\mathcal{NP}$-complete.*

Observe that MC is related to the minimum multi-cut problem [4] (MMC for short): Given a graph $G = (V, E)$, a set $S \subseteq V \times V$ of source-terminal pairs, $k \in \mathbf{N}$ and a weight function $w : E \to N$, is there a multi-cut, i.e., a set $E' \subseteq E$ such that the removal of E' from E disconnects s_i from t_i for every pair $(s_i, t_i) \in S$ such that $\sum_{e \in E'} w(e) \leq k$?

Therefore, MC seeks for a minimum multi-cut in E_1 whether MMC uses E. Hence MC is equal to MMC if and only if $E_2 = \emptyset$. Note that MMC is a generalization of the minimum multiway cut and is $\mathcal{NP}$-hard even when the graph is a tree [4, 7]. For a survey and bibliography the reader is referred to [2, 3].

Proof (Theorem 1). We show that there exists a polynomial transformation from MC (that is $\mathcal{NP}$-complete by Corollary 1) to the maximum planar subgraph problem on graphs of the classes $\mathcal{K}_5$ or $\mathcal{K}_{3,3}$, respectively.

Given an instance of MC, i.e., a planar graph $G = (V, E_1 \uplus E_2)$ and an integer k, we can construct an instance of the weighted MSPS for graphs of $\mathcal{K}_5$ or $\mathcal{K}_{3,3}$, respectively: We set $E' = E_2$ and create iteratively an instance $G[N]$ of $\mathcal{K}_5$ or $\mathcal{K}_{3,3}$, respectively, with $N = K_5$ or $N = K_{3,3}$, respectively.

Moreover, we define a weight function for the edges of $G[N]$: For each edge e of $G[N]$ that is also included in G, i.e., $e \in E_1$, we set $c(e) = 1$, otherwise $c(e) = k + 1$.

Claim: Let $G = (V, E_1 \uplus E_2)$ be a connected, planar graph, $N \in \{K_{3,3}, K_5\}$, $E' = E_2$ and let $S \subseteq E_1$. Then $G[N] - S$ does not contain any N-minor if and only if all edges $e \in E_2$ are coloops in $G - S$.

Proof of claim: First assume that there exists an edge $e \in E_2$ such that $e = (i, j)$ is no coloop in $G - S$. Then there exists a path P from i to j in $G - S$ that does not contain edge e. Since $E_2 = E'$ and hence e is involved into a 2-sum with $G - S$ and N, $G[N] - S$ contains an N-minor using path P instead of e. This leads to a contradiction to the assumption that $G[N] - S$ is N-free.

Now we assume that there exists an N-minor in $G[N] - S$. Since $G - S$ is planar and hence N-free we conclude that there is an edge $e = (i, j) \in E' = E_2$ that is involved into a 2-sum of $G - S$ and N. Furthermore, since $G[N] - S$ is not planar there is a path P in $G - S$ from i to j. This contradicts the assumption that e is a coloop in $G - S$. This concludes our claim.

Our claim implies that there exists a feasible solution $S \subset E_1$ with $|S| \leq k$ of instance $G = (V, E_1 \uplus E_2)$ for MC if and only if there exists a subset of edges S' of $G[N]$ with total weight $c(S') \leq k$ whose removal yields a planar subgraph. Observe that $c(e) = k + 1 > c(S')$ for $e \in E' = E_2$ and hence $S' \subseteq E_1$.

Moreover, if we replace every edge e in $G[N]$ with $c(e) = k + 1$ by $(k + 1)$ copies of edge e we conclude the statement of the theorem. □

Finally, we get the following result.

Corollary 2. *The maximum planar subgraph problem is $\mathcal{NP}$-complete for the following classes of graphs.*

1. *The graphs without a K_5 minor,*
2. *The graphs without a $K_{3,3}$ minor.*

Proof. Clearly the problem is in $\mathcal{NP}$ since planarity is polynomially checkable.

The $\mathcal{NP}$-hardness follows immediately by Lemma 1: The class of graphs without a K_5 or $K_{3,3}$ minor, respectively, contains the class $\mathcal{K}_{3,3}$ or $\mathcal{K}_5$, respectively, for which Theorem 1 establishes the problem to be $\mathcal{NP}$-complete. □

Conclusion

Surprisingly, the (weighted) maximum planar subgraph problem gets easy for a triconnected non-planar graph G without a minor isomorphic to $K_{3,3}$. By the major decomposition theorems [11], then G is isomorphic to K_5. Hence, the maximum planar subgraph of G is equal to K_5 minus one of the cheapest edges.

We consider a triconnected non-planar graph G without a minor isomorphic to K_5. G is called a *Δ-sum* (composition) of G_1 and G_2, denoted $G = G_1 \oplus_\Delta G_2$, if identification of an arbitrary triangle of G_1 with an arbitrary triangle in G_2 and subsequent deletion of the edges of this triangle produces G. By the major decomposition theorems [11], then G is either isomorphic to $K_{3,3}$, or to V_8, or is equal to Δ-sum compositions of planar graphs. For the first two cases, the (weighted) maximum planar subgraph problem gets easy: we delete one of the cheapest edges in $K_{3,3}$ or V_8, respectively. For the remaining case we conjecture that it is $\mathcal{NP}$-complete as well.

Furthermore, we conjecture that the crossing minimization problem on graphs without a K_5 or $K_{3,3}$ minor, respectively, is $\mathcal{NP}$-hard.

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