The QAP-Polytope and the Star-Transformation *

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Abstract

The quadratic assignment problem (QAP) maybe was for a long time the one among the prominent $\mathcal{NP}$-hard combinatorial optimization problems about which the fewest polyhedral results had been known. Recent work of Rijal (1995) and Padberg and Rijal (1996) has on the one hand yielded some basic facts about the associated quadratic assignment polytope, but has on the other hand shown that "naive" investigations even of the very basic questions (like the dimension, the affine hull, and the trivial facets) soon become extremely complicated. In this paper, we propose an isomorphic transformation of the "natural" realization of the quadratic assignment polytope, which simplifies the polyhedral investigations enormously. We demonstrate this by giving short proofs of the basic results on the polytope that indicate that exploiting the techniques developed in this paper deeper polyhedral investigations of the QAP now become possible. Moreover, an "inductive construction" of the QAP-Polytope is derived that might be useful in branch-and-cut algorithms.

Keywords: Quadratic Assignment Problem, Polyhedral Combinatorics, QAP-Polytope

MSC Classification: 90C09, 90C10, 90C27

1 Introduction

The methods of polyhedral combinatorics have yielded structural results and practical solvability of many combinatorial optimization problems over the past 30-40 years. The most prominent examples among the $\mathcal{NP}$-hard problems might be the traveling salesman problem, the max cut problem, the linear ordering problem, or the stable set problem. If one compares this list with the list of $\mathcal{NP}$-hard problems that are usually considered "classical" one might miss the *quadratic assignment problem*, which we consider in the formulation

$$
\begin{array}{lll}
(\mathrm{QAP})^{(n)}_{c,d} & \min & \displaystyle\sum_{\substack{i,k=1\\ i<k}}^{n} \sum_{\substack{j,l=1\\ j\neq l}}^{n} d_{ijkl} x_{ij} x_{kl} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} \\
 & \text{s.t.} & \displaystyle\sum_{j=1}^{n} x_{ij} = 1 \qquad (i \in \{1,\dots,n\}) \\
 & & \displaystyle\sum_{i=1}^{n} x_{ij} = 1 \qquad (j \in \{1,\dots,n\}) \\
 & & x_{ij} \in \{0,1\} \qquad (i,j \in \{1,\dots,n\})
\end{array}
$$

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of Lawler (1963), who slightly generalized the original formulation of Koopmans and Beckmann (1957). And in fact, while the literature concerning polyhedral investigations of the other mentioned problems is vast, one finds only a few occurences of the quadratic assignment polytope.

Basically, this polytope was investigated only twice. First, it is treated in the work of Barvinok (1992) as an example for the connection between the theory of representations of finite groups and combinatorial optimization polyhedra. Exploiting that deeply developed theory Barvinok derives the dimension and some first facets of the quadratic assignment polytope. However, this method seems to apply only to these very basic questions. The second polyhedral investigations were done by Rijal (1995) and Padberg and Rijal (1996). They derived basically the same results as Barvinok, using "classical" methods of polyhedral combinatorics. However, their treatment revealed that dealing with the quadratic assignment polytope as it is defined naturally leads to enormous technical difficulties of the following kind.

If one starts to investigate the structure of a polytope defined as the convex hull of some points, one is very soon confronted with tasks like computing the rank of a subset of these points or showing that such a subset spans a certain subspace. In both cases, one has to deal with linear combinations of the vertices of the polytope. Working with the natural realization of the quadratic assignment polytope, it turns out that such combinations with well-structured, sparse supports (i.e., nonzero components) are hard to obtain. This is mainly due to the facts that the coordinate vectors of the vertices look all the same up to certain permutations of the coordinates, and that there are no pairs among them having only slightly different supports.

In this paper, we describe how to overcome this "nastiness" by mapping the polytope isomorphically into a lower dimensional vector space, where the vertices allow some nice and simple linear combinations. This transformation seems to be crucial for the success of theoretical investigations of the quadratic assignment polytope. Without the simplifications of the proof-techniques that it yields, deeper results on the facial structure of the quadratic assignment polytope might be hard to derive. We demonstrate the power of our transformation by deriving in a relatively simple way the dimension, the affine hull, and the trivial facets of the polytope. Furthermore, our transformation gives insight into an interesting "inductive construction" of the quadratic assignment polytope, which might be exploited effectively in branching strategies of branch-and-cut algorithms.

The paper is organized in the following way. In Section 2 we introduce a new way of formulating the quadratic assignment problem in graph theoretical terms and give the definition of the quadratic assignment polytope within this notational setting. Section 3 is the central part of the paper, where we develop the "star-transformation", and finally, we give short proofs for the dimension, the affine hull, and the trivial facets of the quadratic assignment polytope in Section 4.2.

2 The Polytope $\mathcal{QAP}_n$

2.1 Formulation as a Graph Problem

The set $\{1, \dots, n\}$ will be used so frequently that it receives an own symbol. We will always denote $\mathcal{N} = \{1, \dots, n\}$. Let $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$ be the graph with node set $\mathcal{V}_n = \{(i,j) \mid i,j \in \mathcal{N}\}$ and edges

$$\mathcal{E}_n = \left\{ \{(i,j),(k,l)\} \in \binom{\mathcal{V}_n}{2} \,\middle|\, i \neq k, j \neq l \right\}$$

(where $\binom{\mathcal{V}_n}{2}$ is the set of all subsets of $\mathcal{V}_n$ having cardinality two). Figure 1 shows an example of such a graph. For ease of notation we define $[i,j,k,l] = \{(i,j),(k,l)\}$ for all edges $\{(i,j),(k,l)\} \in \mathcal{E}_n$. We call the subset $\mathrm{row}_i^{(n)} = \{(i,j) \mid j \in \mathcal{N}\}$ the *i-th row* of $\mathcal{V}_n$ (for $i \in \mathcal{N}$). The subset

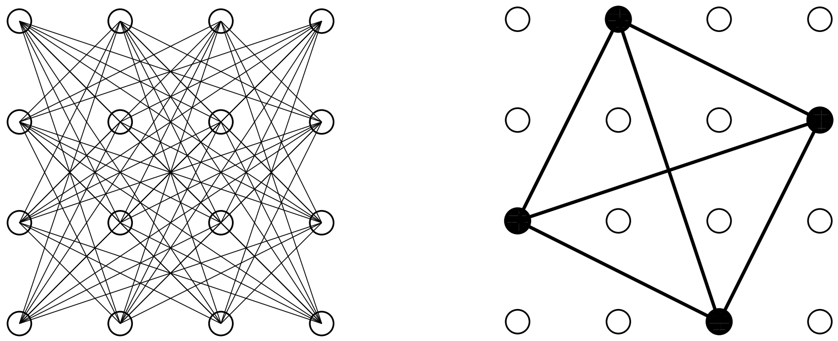


Figure 1: The graph $\mathcal{G}_n$ and an example of an n-clique in it.

$\mathrm{col}_j^{(n)} = \{(i,j) \mid i \in \mathcal{N}\}$ is called the *j-th column* of $\mathcal{V}_n$ (for $j \in \mathcal{N}$). If the context preserves from any ambiguity, then we usually omit the superscript and simply write row_i and col_j.

The connection between the graph $\mathcal{G}_n$ and the quadratic assignment problem comes from the fact that the maximum cliques of $\mathcal{G}_n$ are the n-cliques, and these correspond precisely to the $n \times n$-permutation matrices (see Figure 1). Hence, given an instance $(\mathrm{QAP})_{c,d}^{(n)}$ of the quadratic assignment problem, we weight the nodes and edges of $\mathcal{G}_n$ by $(c', d') \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$, where we set $c'_{(i,j)} = c_{ij}$ for each node $(i,j) \in \mathcal{V}_n$ and $d'_{[i,j,k,l]} = d_{ijkl}$ for each edge $[i,j,k,l] \in \mathcal{E}_n$. Then, solving $(\mathrm{QAP})_{c,d}^{(n)}$ means to find a minimally node- and edge-weighted n-clique in the graph $\mathcal{G}_n$ weighted by (c', d').

2.2 Definition and Elementary Properties of $\mathcal{QAP}_n$

Now we are ready to introduce the *quadratic assignment polytope.* We denote the characteristic vector of a subset $W \subseteq \mathcal{V}_n$ of nodes by $x^W \in \mathbb{R}^{\mathcal{V}_n}$ and the characteristic vector of a subset $F \subseteq \mathcal{E}_n$ of edges by y^F, i.e., we have

$$x_v^W = \begin{cases} 1 & \text{if } v \in W \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad x_e^F = \begin{cases} 1 & \text{if } f \in F \\ 0 & \text{otherwise.} \end{cases}$$

In particular, ommitting the brackets for singletons, x^v (for $v \in \mathcal{V}_n$) and y^e (for $e \in \mathcal{E}_n$) are the canonical unit vectors of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$.

For a subset $W \subseteq \mathcal{V}_n$ we denote by $\mathcal{E}_n(W) = \{\{v,w\} \in \mathcal{E}_n \mid v,w \in W\}$ the set of all edges having both nodes in W. The *incidence vector* of an n-clique $C \subset \mathcal{V}_n$ in $\mathcal{G}_n$ is the 0/1-vector $(x^C, y^C) = (x^C, y^{\mathcal{E}_n(C)})$. We define the *quadratic assignment polytope* to be the convex hull

$$\mathcal{QAP}_n = \mathrm{conv}\left\{(x^C, y^C) \mid C \text{ is an } n\text{-clique of } \mathcal{G}_n\right\}$$

of all incidence vectors of n-cliques of $\mathcal{G}_n$.

It can be shown that some well-known polytopes as the *traveling salesman polytope* or the *linear ordering polytope* are certain projections of the quadratic assignment polytope. Furthermore, the quadratic assignment polytope is isomorphic to a face of the *boolean quadric polytope* (introduced by Padberg, 1989), which is itself isomorphic to the *cut polytope* on the complete graph (De Simone, 1989). From this, for example, it can easily be deduced that the diameter of the quadratic assignment polytope equals one, since this holds for the cut polytope (Barahona and Mahjoub, 1986).

An important property of the quadratic assignemnt polytope is the fact that it is invariant under permuting the rows or columns of the node set $\mathcal{V}_n$ and under "transposing" $\mathcal{V}_n$. That means that these operations induce *symmetries* of $\mathcal{QAP}_n$.

2.3 An Integer Linear Programming Formulation

For a vector $x \in \mathbb{R}^{\mathcal{V}_n}$ ($y \in \mathbb{R}^{\mathcal{E}_n}$) and for any subset $W \subseteq \mathcal{V}_n$ ($F \subseteq \mathcal{E}_n$) we denote by $x(W)$ ($y(F)$) the sum $\sum_{v \in W} x_v$ ($\sum_{e \in F} y_e$). For two disjoint subsets $S, T \subseteq \mathcal{V}_n$ the set of all edges in $\mathcal{E}_n$ with one endpoint in S and the other one in T is denoted by $(S : T)$. In case of singletons $S = \{s\}$ we ommit the curly brackets.

Clearly, the equations

$$x(\mathrm{row}_i) = 1 \qquad (i \in \mathcal{N}) \tag{1}$$

$$x(\mathrm{col}_j) = 1 \qquad (j \in \mathcal{N}) \tag{2}$$

$$-x_{(i,j)} + y\left((i,j) : \mathrm{row}_k\right) = 0 \qquad (i,j,k \in \mathcal{N}, i \neq k) \tag{3}$$

$$-x_{(i,j)} + y\left((i,j) : \mathrm{col}_l\right) = 0 \qquad (i,j,l \in \mathcal{N}, j \neq l) \tag{4}$$

hold for all points in $\mathcal{QAP}_n$ (see Figure 2).

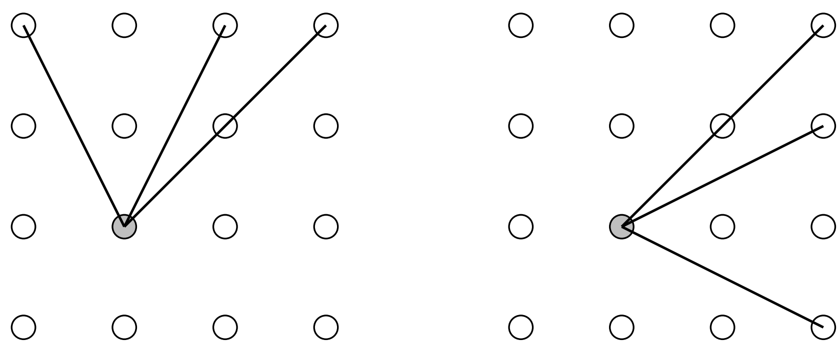


Figure 2: The left-hand-side vectors of equations (3), where the solid lines indicate coefficients $+1$ and the grey dots indicate coefficients -1. and (4).

In fact, it was observed by several authours (Johnson, 1992; Drezner, 1994; Rijal, 1995; Padberg and Rijal, 1996) that a vector $(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ is a vertex of $\mathcal{QAP}_n$ if and only if it satisfies (1), (2), (3), $y \geq 0$, and $x \in \{0,1\}^{\mathcal{V}_n}$. Moreover, Johnson (1992) has proved that the lower bound which one can compute by solving the linear program arising from (1), (2), (3), and the nonnegativity constraints $(x, y) \geq 0$ is always at least as good as the Gilmore/Lawler bound (Gilmore, 1962; Lawler, 1963). Extensive computational tests of Resende *et al.* (1995) have shown that this bound also is very tight in practice.

3 A Different Representation: $\mathcal{QAP}^{\star}_{n^{\star}}$

3.1 An Isomorphic Projection of $\mathcal{QAP}_n$

The $2n + 2n^2(n-1)$ many equations (1), ... ,(4) that are valid for the polytope $\mathcal{QAP}_n$ indicate some redundancy in the problem formulation. We will use this redundancy for finding another representation of the quadratic assignment polytope via a certain projection.

Let $\mathcal{A} \subset \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ be the affine subspace of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ defined by the equations (1), ... ,(4). We will show that the variables corresponding to vertices and edges involving the n-th row or the n-th column (the same holds for any row and any column) are redundant for $\mathcal{A}$ in the sense that the projection onto the linear subspace of the original space obtained by setting all these variables to zero produces an isomorphic image of this affine subspace. Since the polytope under consideration is contained in the affine subspace $\mathcal{A}$, this implies that the projection yields an isomorphic image of $\mathcal{QAP}_n$.

Let $W^\star = \mathrm{row}_n^{(n)} \cup \mathrm{col}_n^{(n)}$ and $F^\star = \{e \in \mathcal{E}_n \mid e \cap W^\star \neq \emptyset\}$. Define $\mathcal{U} = \{(x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid x_{W^\star} = 0, y_{F^\star} = 0\}$, and let $\pi^{(n)} : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathcal{U}$ be the orthogonal projection onto $\mathcal{U}$.

Proposition 1. *The projection $\pi^{(n)}$ restricted to the affine subspace $\mathcal{A}$ of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ induces a one-to-one map.*

Proof. We will first show that there is a way to express the components of points in $\mathcal{A}$ belonging to elements in $W^\star$ and $F^\star$ linearly by the components belonging to elements in $\mathcal{V}_n \setminus W^\star$ and $\mathcal{E}_n \setminus F^\star$.

This is possible for the elements in $W^\star$ using the equations (1) and (2). In order to show the claim for $F^\star$, it suffices to consider three possibilities for an edge $[i,j,k,l] \in F$. The first two are $i,j,k < n$, $l = n$ and $i,j,l < n$, $k = n$. Using the suitable equation from (3) (with i,j,k in the first case) and (4) (with i,j,l in the second case), these two possibilities are done. It remains the possibility that $i,j < n$, $k = n$, $l = n$. We exploit (3) for i,j,n, which allows to express $y_{[i,j,n,n]}$ since we can already express $y_{[i,j,n,l]}$ for $l < n$.

Up to now, we have shown that there is a linear function $\psi : \mathbb{R}^{\mathcal{V}_n \setminus W^\star} \times \mathbb{R}^{\mathcal{E}_n \setminus F^\star} \longrightarrow \mathbb{R}^{W^\star} \times \mathbb{R}^{F^\star}$ such that for all $(x,y) \in \mathcal{A}$ we have $(x_{W^\star}, y_{F^\star}) = \psi(x_{\mathcal{V}_n \setminus W^\star}, y_{\mathcal{E}_n \setminus F^\star})$. Hence $\phi : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ defined via $\phi(x,y) = (x',y')$ with

$$\begin{aligned} (x'_{W^\star}, y'_{F^\star}) &= (x_{W^\star}, y_{F^\star}) - \psi(x_{\mathcal{V}_n \setminus W^\star}, y_{\mathcal{E}_n \setminus F^\star}), \\ (x'_{\mathcal{V}_n \setminus W^\star}, y'_{\mathcal{E}_n \setminus F^\star}) &= (x_{\mathcal{V}_n \setminus W^\star}, y_{\mathcal{E}_n \setminus F^\star}) \end{aligned}$$

is an affine transformation (since the corresponding matrix is a triangular one having ones everywhere on the main diagonal) of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ that induces on $\mathcal{A}$ the orthogonal projection onto $\mathcal{U}$. □

We identify the linear space $\mathcal{U}$ with the space $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$. Hence, for $n^\star = n-1$

$$\mathcal{QAP}^\star_{n^\star} = \pi^{(n)}(\mathcal{QAP}_n) \subset \mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}$$

is a polytope in $\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}$ that is isomorphic to $\mathcal{QAP}_n$.

Since the vertices of this polytope arise from the projections of the vertices of the original polytope "forgetting" the last row and the last column of $\mathcal{G}_n$, one obtains that they are the characteristic vectors of the $n^\star$- and the $(n^\star-1)$-cliques of $\mathcal{G}_{n^\star}$ (see Figure 3). Thus, by adapting the notations for the incidence vectors to $(n^\star-1)$-cliques of $\mathcal{G}_{n^\star}$, we have

$$\mathcal{QAP}^\star_{n^\star} = \mathrm{conv}\left\{(x^{C^\star}, y^{C^\star}) \mid C^\star \text{ is an } n^\star\text{- or an } (n^\star-1)\text{-clique of } \mathcal{G}_{n^\star}\right\}.$$

We want to make the isomorphism that $\pi^{(n)}$ induces between $\mathcal{QAP}_n$ and $\mathcal{QAP}^\star_{n^\star}$ as well as between the corresponding face lattices a little more explicit. Denote by κ the map that assigns to every n-clique $C \subset \mathcal{V}_n$ of $\mathcal{G}_n$ the $n^\star$- or $(n^\star-1)$-clique $C^\star \subset \mathcal{V}_{n^\star}$ of $\mathcal{G}_{n^\star}$ that arises from C by removing the node(s) in the n-th row and in the n-th column. Notice that κ is one-to-one.

Remark 2. *If two faces of $\mathcal{QAP}_n$ and $\mathcal{QAP}^\star_{n^\star}$ correspond to each other with respect to the isomorphism induced by $\pi^{(n)}$, then their vertices (identified with cliques) correspond to each other by the bijection κ.*

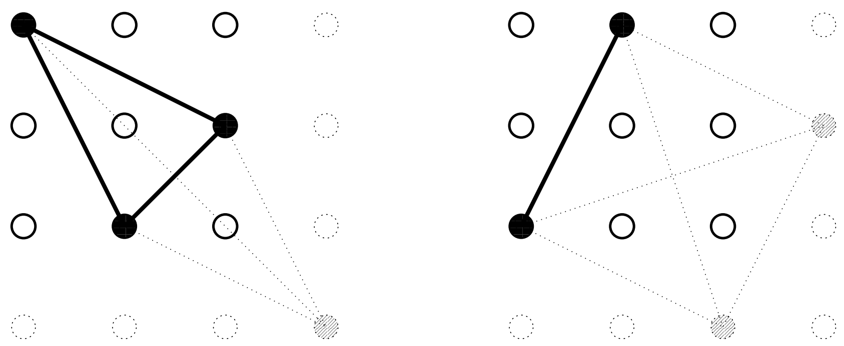


Figure 3: The effect of the projection $\pi^{(n)}$.

This remark describes the relationship between the faces from the "inner view", i.e., in terms of the vertices. Next, we want to describe the "outer relationship", i.e., the relationship between inequalities defining corresponding faces.

Remark 3.

(i) If a face of $\mathcal{QAP}_n$ is defined by an inequality that has zero-coefficients for all elements in $W^\star \cup F^\star$, then an inequality defining the corresponding face of $\mathcal{QAP}^\star_{n^\star}$ is obtained by projecting the coefficient vector of that inequality via $\pi^{(n)}$. In fact, for every face of $\mathcal{QAP}_n$ there is a defining inequality that has zero coefficients at $W^\star$ and $F^\star$, since the columns corresponding to $W^\star \cup F^\star$ of the equation system defining the affine subspace $\mathcal{A}$ are linearly independent, as shown in the proof of Proposition 1.

(ii) From every inequality defining a face of $\mathcal{QAP}^\star_{n^\star}$ one obtains an inequality defining the corresponding face of $\mathcal{QAP}_n$ by "zero-lifting", i.e., choosing zero as coefficient for every variable corresponding to $\mathcal{V}_n \setminus \mathcal{V}_{n^\star}$ or $\mathcal{E}_n \setminus \mathcal{E}_{n^\star}$.

As for $\mathcal{QAP}_n$ (see Section 2.2), permuting the rows or columns as well as transposing the node set yields symmetries of the polytope $\mathcal{QAP}^\star_{n^\star}$, i.e., it suffices also for $\mathcal{QAP}^\star_{n^\star}$ to prove all results up to permutations of the rows or the columns as well as transposition of the node set.

3.2 A System of Equations

By mapping the polytope $\mathcal{QAP}_n \subset \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ isomorphically (in particular, not changing its dimension) into the lower-dimensional space $\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}} = \mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$, we have reduced the dimensional gap between the polytope and the space it is located in. It would have been the best to make that gap even vanish, i.e., to obtain a full-dimensional representation of the quadratic assignment polytope. However, this is not reached by the projection $\pi^{(n)}$, as one sees from the equations coming up next.

Ending up with a full-dimensional polytope would be nice with respect to such goals like the uniqueness of facet-defining inequalities and clearly, for every low-dimensional polytope there is a possibility to map it isomorphically into another space where it is full-dimensional. However, we are extremely dependent from the fact that the vertices of the representation of the quadratic assignment polytope have some nice combinatorial structure, as they do in the case of $\mathcal{QAP}^\star_{n^\star}$. It seems that a full-dimensional representation of the quadratic assignment polytope satisfying this requirement is not possible.

We shall exhibit the equations that are still holding for $\mathcal{QAP}^\star_{n^\star}$ now. Since every $n^\star$- or $(n^\star - 1)$-clique of $\mathcal{G}_{n^\star}$ has an empty intersection with at most one row and with at most one column of $\mathcal{V}_{n^\star}$, the equations (where $\mathcal{N}^\star = \{1, \dots, n^\star\}$)

$$x(\mathrm{row}_i \cup \mathrm{row}_k) - y(\mathrm{row}_i : \mathrm{row}_k) = 1 \qquad (i, k \in \mathcal{N}^\star, i < k) \tag{5}$$

$$x(\mathrm{col}_j \cup \mathrm{col}_l) - y(\mathrm{col}_j : \mathrm{col}_l) = 1 \qquad (j, l \in \mathcal{N}^\star, j < l) \tag{6}$$

are valid for $\mathcal{QAP}^\star_{n^\star}$ (see Figure 4). Theorem 11 will show that (5) and (6) form a complete system of equations for $\mathcal{QAP}^\star_{n^\star}$, i.e., the solution space of these equations is the affine hull of $\mathcal{QAP}^\star_{n^\star}$.

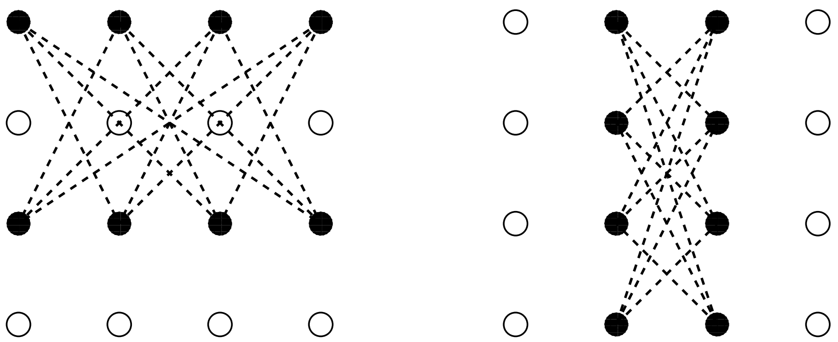


Figure 4: The left-hand-side vectors of the equations (5) and (6), where the dashed lines indicate coefficients -1, and the filled dots indicate coefficients $+1$.

Let us investigate the system $D(x, y) = d$ of the equations (5), (6) more closely. A first observation is that this system has not full row rank, since summing up all equations (5) yields the same as summing up all equations (6). Hence, the rank of these $n^\star(n^\star - 1)$ equations is at most $n^\star(n^\star - 1) - 1$.

We define a (total) ordering of the edges $\mathcal{E}_{n^\star}$ by requiring that each edge $[i, j, k, l] \in \mathcal{E}_{n^\star}$ with $i < k$ and $j < l$ has as its successor the edge $[i, l, k, j]$, and by ordering the edges $\{[i, j, k, l] \in \mathcal{E}_{n^\star} \mid i < k, j < l\}$ lexicographically according to the quadruples (i, k, j, l). After permuting the columns of D that correspond to the edges of $\mathcal{G}_{n^\star}$ with respect to this ordering of $\mathcal{E}_{n^\star}$, these columns of D form the following $n^\star(n^\star - 1) \times |\mathcal{E}_{n^\star}|$ matrix (for $n^\star = 3$):

$$\begin{pmatrix}
1&1&1&1&1&1&&&&&&&&&&&&\\
&&&&&&1&1&1&1&1&1&&&&&&\\
&&&&&&&&&&&&1&1&1&1&1&1\\
1&1&&&&&1&1&&&&&1&1&&&&\\
&&1&1&&&&&1&1&&&&&1&1&&\\
&&&&1&1&&&&&1&1&&&&&1&1
\end{pmatrix}$$

We are interested in the *bases of the matrix* D (also called the *bases of the equation system* $D(x, y) = d$), i.e., the maximal subsets of linearly independent columns of D. Since columns corresponding to edges $[i, j, k, l]$ and $[i, l, k, j]$ are identical, we can identify them for our considerations. But then, the resulting $n^\star(n^\star - 1) \times \frac{1}{2}|\mathcal{E}_{n^\star}|$ matrix is the node-edge incidence matrix of the complete bipartite graph $K_{\frac{n^\star(n^\star-1)}{2}, \frac{n^\star(n^\star-1)}{2}}$, where the left shore corresponds to the (unordered) pairs of rows, and the right shore corresponds to the (unordered) pairs of columns of $\mathcal{V}_{n^\star}$. Calling a pair $\{[i, j, k, l], [i, l, k, j]\}$ of edges of $\mathcal{G}_{n^\star}$ a pair of *mates*, we obtain a one-to-one correspondence between the edges in $K_{\frac{n^\star(n^\star-1)}{2}, \frac{n^\star(n^\star-1)}{2}}$ and the pairs of mates.

The bases of the node-edge incidence matrix of the complete bipartite graph $K_{N,N}$ are well-known to correspond to the spanning trees of $K_{N,N}$ (Balinski and Russakoff, 1974). This leads to the following characterization of all bases of $D(x,y)=d$ that do not contain columns corresponding to nodes of $\mathcal{G}_{n^\star}$.

Proposition 4. *Let $n^\star \geq 2$.*

(i) Precisely one (arbitrary) equation in the system (5), (6) is redundant, in particular, the rank of this system is $n^\star(n^\star-1)-1$.

(ii) A subset $B \subset \mathcal{E}_{n^\star}$ of edges of $\mathcal{G}_{n^\star}$ corresponds to a basis of that system if and only if

- *(a) $|B| = n^\star(n^\star-1)-1$*
- *(b) There is no pair of mates contained in B.*
- *(c) There is no sequence $(e_0, e'_0, e_1, e'_1, \dots, e_{r-1}, e'_{r-1})$ (with $r \geq 2$) of edges in B such that e_ρ and e'_ρ connect the same rows of $\mathcal{V}_{n^\star}$ and e'_ρ and $e_{(\rho+1) \bmod r}$ connect the same columns of $\mathcal{V}_{n^\star}$ for all $\rho = 0,\dots,r-1$.*

Proof. Part (ii) follows from the discussion of the connection to $K_{\frac{n(n-1)}{2},\frac{n(n-1)}{2}}$, and part (i) follows from (ii) and the observation made above that the rank of $D(x,y)=d$ is at most $n^\star(n^\star-1)-1$. □

Later, when we prove results about the dimension of $\mathcal{QAP}^\star_{n^\star}$ or of one of its faces, we will always use one special basis of the equations system (5), (6) that we exhibit now. It is illustrated in Figure 5.

Corollary 5. *The columns corresponding to the set*

$$E^{(n^\star)}_{\text{bas}} = \{[1,j,2,l] \in \mathcal{E}_{n^\star} \mid j<l\} \cup \{[i,1,k,2] \in \mathcal{E}_{n^\star} \mid i<k\}$$

form a basis of the equation system (5), (6).

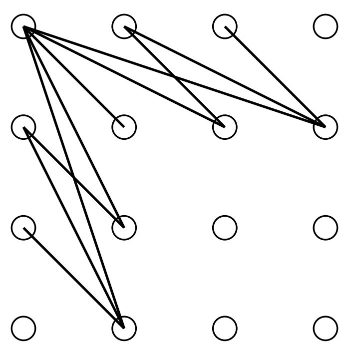


Figure 5: The edges corresponding to the basis $E^{(n^\star)}_{\text{bas}}$.

3.3 A Proof Technique

The technique we will use to prove the dimension of $\mathcal{QAP}^\star_{n^\star}$ as well as in the proofs showing that a given inequality defines a facet of $\mathcal{QAP}^\star_{n^\star}$ is a variant of the "indirect method". We give an outline of this technique here.

First, we explain the technique for the dimension proof. Let L be the set of all $n^\star$- and $(n^\star-1)$-cliques $C \subset \mathcal{V}_{n^\star}$, and let

$$\Delta_L = \{(x^{C_1}, y^{C_1}) - (x^{C_2}, y^{C_2}) \mid C_1, C_2 \in L\}$$

be the set of all difference vectors of the incidence vectors of these cliques, i.e., the set of all differences of vertices of $\mathcal{QAP}^\star_{n^\star}$. Hence, Δ_L spans the linear subspace belonging to the affine subspace $\mathrm{aff}(\mathcal{QAP}^\star_{n^\star})$. Denoting the rank of the equation system (5), (6) by $\mathrm{rank}_{\mathrm{eq}}$, we have to show that the linear dimension of Δ_L equals $\dim(\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}) - \mathrm{rank}_{\mathrm{eq}}$.

Let B be a set of edges belonging to a basis of the equation system (5), (6), in particular we have $|B| = \mathrm{rank}_{\mathrm{eq}}$. Clearly, one could also use a basis containing columns that belong to nodes, too. But we will always chose $B = E^{(n^\star)}_{\mathrm{bas}}$ as in Corollary 5, and thus restrict our notations to the case that B contains no node. Denote by $\mathcal{B} = \{y^e | \, e \in B\}$ the set of all canonical unit vectors belonging to B. Now it suffices to show

$$\mathrm{lin}(\Delta_L \cup \mathcal{B}) = \mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}},$$

what is done by successively combining all canonical unit vectors $\{x^v \mid v \in \mathcal{V}_{n^\star}\}$ and $\{y^e \mid e \in \mathcal{E}_{n^\star}\}$ of the vector space $\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}$ by using just the vectors in Δ_L and $\mathcal{B}$. In order to abbreviate the notations, we say that an edge or a node is *combined* once the corresponding unit vector is linearly combined.

If we want to prove that a *proper* face $\mathcal{F}$ is a facet of $\mathcal{QAP}^\star_{n^\star}$, then we start with the set L containing not all $n^\star$- and $(n^\star - 1)$-cliques of $\mathcal{G}_{n^\star}$ but only those ones that belong to vertices of $\mathcal{F}$. Since we do not want to prove that the dimension of $\mathcal{F}$ equals that of $\mathcal{QAP}^\star_{n^\star}$ but $\dim(\mathcal{QAP}^\star_{n^\star}) - 1$, we enlarge the set $\mathcal{B}$ by any canonical unit vector belonging either to a node v_0 or to an edge e_0, called the *extra element*, to a set $\mathcal{B}_0$. Proceeding as above with the "combination" of all canonical unit vectors in $\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}$ starting from the modified set $\Delta_L \cup \mathcal{B}_0$, it is proved that $\mathcal{F}$ is a facet of $\mathcal{QAP}^\star_{n^\star}$ as soon as all nodes and edges are combined (notice that $\mathcal{F}$ was supposed to be not the whole polytope).

Proving this way that a given proper face of $\mathcal{QAP}^\star_{n^\star}$ defines a facet even contains a proof that (5) and (6) form a complete equation system for $\mathcal{QAP}^\star_{n^\star}$. We will use this fact and give in Section 4.1 one proof for both the dimension of $\mathcal{QAP}^\star_{n^\star}$ as well as for the fact that the nonnegativity constraints on the edge variables define facets of it.

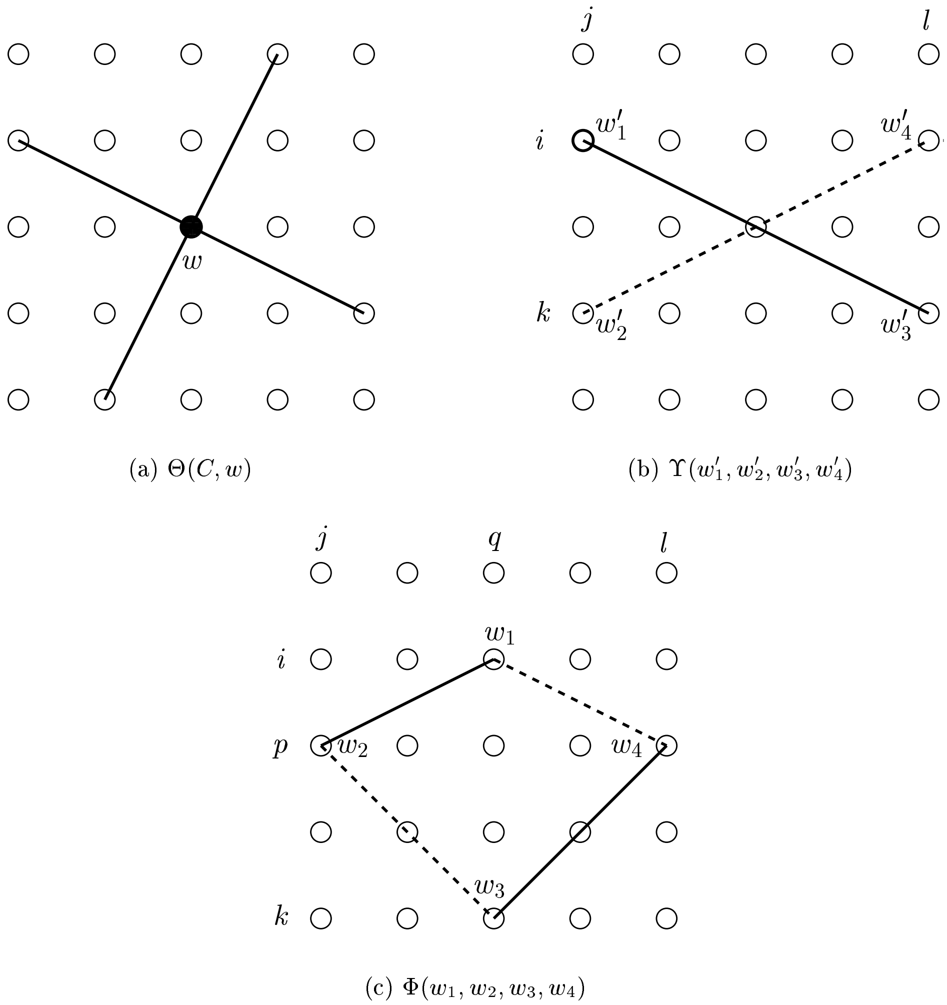
3.4 Some Useful Vectors

The first convenient gain that we took from the transition to the "star-polytope" $\mathcal{QAP}^\star_{n^\star}$ was the equation system (5), (6) (that is not yet proved to be a complete one for $\mathcal{QAP}^\star_{n^\star}$, but will be soon in Section 4.1) with its structural connection to the node-edge incidence matrix of the complete bipartite graph. Now we will show that $\mathcal{QAP}^\star_{n^\star}$ allows to combine linearly very simple vectors from its vertices.

Let $i, k, p \in \mathcal{N}^\star$ be three pairwise distinct numbers of rows of $\mathcal{V}_{n^\star}$, and let $j, l, q \in \mathcal{N}^\star$ be three pairwise distinct numbers of columns of $\mathcal{V}_{n^\star}$. The following vectors, where $w_1 = (i, q)$, $w_2 = (p, j)$, $w_3 = (k, q)$, $w_4 = (p, l)$, $w'_1 = (i, j)$, $w'_2 = (k, j)$, $w'_3 = (k, l)$, $w'_4 = (i, l)$, and $C \subset \mathcal{V}_{n^\star}$ is an $n^\star$-clique of $\mathcal{G}_{n^\star}$ containing the node $w \in C$, will be the most important auxiliaries for the combination of nodes and edges as explained in Section 3.3. They are illustrated in Figure 6.

$$\begin{aligned}
\Theta(C, w) &= x^w + \sum_{w' \in C \setminus w} y^{\{w, w'\}} \\
\Upsilon(w'_1, w'_2, w'_3, w'_4) &= y^{[i,j,k,l]} - y^{[i,l,k,j]} \\
\Phi(w_1, w_2, w_3, w_4) &= y^{[i,q,p,j]} - y^{[p,j,k,q]} + y^{[k,q,p,l]} - y^{[p,l,i,q]}
\end{aligned}$$

The following three lemmas give sufficient conditions for a set L of $n^\star$- and $(n^\star - 1)$-cliques of $\mathcal{G}_{n^\star}$ that guarantee these vectors to be members in $\mathrm{lin}(\Delta_L)$, where Δ_L is, again, the set of all difference vectors of the incidence vectors of the cliques in L. We make one more notational



(a) $\Theta(C, w)$

(b) $\Upsilon(w'_1, w'_2, w'_3, w'_4)$

(c) $\Phi(w_1, w_2, w_3, w_4)$

Figure 6: The three types of vectors provided by Lemmas 6, ... ,8

convention for stating these lemmas. Let $W \subset \mathcal{V}_{n^\star}$ be a subset of nodes. We denote by $\mathcal{G}_{n^\star}/W$ the subgraph of $\mathcal{G}_{n^\star}$ that is induced by all rows and columns that do not intersect W. If W intersects the same number of rows as of columns, then $\mathcal{G}_{n^\star}/W$ is isomorphic to some $\mathcal{G}_{\dot{n}}$ with $\dot{n} \le n^\star$.

Lemma 6. *Let L be a set of $n^\star$- and $(n^\star - 1)$-cliques of $\mathcal{G}_{n^\star}$. If for an $n^\star$-clique C of $\mathcal{G}_{n^\star}$ and a node $w \in C$ we have both $C \in L$ and $C \setminus w \in L$, then*

$$\Theta(C, w) \in \operatorname{lin}(\Delta_L)$$

holds.

Proof. The equation

$$\Theta(C, w) = \left(x^C, y^C\right) - \left(x^{C \setminus w}, y^{C \setminus w}\right)$$

shows this. □

Lemma 7. *Let L be a set of $n^\star$- and $(n^\star - 1)$-cliques of $\mathcal{G}_{n^\star}$, and let $w'_1, w'_2, w'_3, w'_4 \in \mathcal{V}_{n^\star}$ be any nodes such that $\Upsilon(w'_1, w'_2, w'_3, w'_4)$ is defined. If there is an $(n^\star - 2)$-clique C in $\mathcal{G}_{n^\star}/\{w'_1, w'_2, w'_3, w'_4\}$*

such that $C \cup \{w'_1, w'_3\} \in L$*,* $C \cup \{w'_2, w'_4\} \in L$*,* $C \cup \{w'_1\} \in L$*,* $C \cup \{w'_2\} \in L$*,* $C \cup \{w'_3\} \in L$*, and* $C \cup \{w'_4\} \in L$*, then*

$$\Upsilon(w'_1, w'_2, w'_3, w'_4) \in \mathrm{lin}(\Delta_L)$$

holds.

Proof. This is due to

$$\begin{aligned}\Upsilon(w'_1, w'_2, w'_3, w'_4) = &\left(x^{C\cup\{w'_1,w'_3\}}, y^{C\cup\{w'_1,w'_3\}}\right) - \left(x^{C\cup\{w'_1\}}, y^{C\cup\{w'_1\}}\right) \\ &- \left(x^{C\cup\{w'_3\}}, y^{C\cup\{w'_3\}}\right) - \left(x^{C\cup\{w'_2,w'_4\}}, y^{C\cup\{w'_2,w'_4\}}\right) \\ &+ \left(x^{C\cup\{w'_2\}}, y^{C\cup\{w'_2\}}\right) + \left(x^{C\cup\{w'_4\}}, y^{C\cup\{w'_4\}}\right).\end{aligned}$$

□

Lemma 8. *Let* L *be a set of* $n^\star$*- and* $(n^\star - 1)$*-cliques of* $\mathcal{G}_{n^\star}$*, and let* $w_1, w_2, w_3, w_4 \in \mathcal{V}_{n^\star}$ *be any nodes such that* $\Phi(w_1, w_2, w_3, w_4)$ *is defined. If there is an* $(n^\star - 3)$*-clique* C *in* $\mathcal{G}_{n^\star}/\{w_1, w_2, w_3, w_4\}$ *such that* $C \cup \{w_1, w_2\} \in L$*,* $C \cup \{w_2, w_3\} \in L$*,* $C \cup \{w_3, w_4\} \in L$*, and* $C \cup \{w_4, w_1\} \in L$*, then*

$$\Phi(w_1, w_2, w_3, w_4) \in \mathrm{lin}(\Delta_L)$$

holds.

Proof. This is obtained from

$$\begin{aligned}\Phi(w_1, w_2, w_3, w_4) = &\left(x^{C\cup\{w_1,w_2\}}, y^{C\cup\{w_1,w_2\}}\right) - \left(x^{C\cup\{w_2,w_3\}}, y^{C\cup\{w_2,w_3\}}\right) \\ &+ \left(x^{C\cup\{w_3,w_4\}}, y^{C\cup\{w_3,w_4\}}\right) - \left(x^{C\cup\{w_4,w_1\}}, y^{C\cup\{w_4,w_1\}}\right).\end{aligned}$$

□

3.5 An Inductive Construction of $\mathcal{QAP}_n$

Up to now, we have only considered the relationship between $\mathcal{QAP}_n$ and $\mathcal{QAP}^\star_{n-1}$ that was actually defining the latter polytope. But $\mathcal{QAP}_n$ and $\mathcal{QAP}^\star_n$ are "living" in the same vector space $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$. Hence, what is their connection? Since $x(\mathcal{V}_n) \leq n$ is a valid inequality for $\mathcal{QAP}_n$, one deduces immediately that $\mathcal{QAP}_n$ is a face of $\mathcal{QAP}^\star_n$, namely the face that has as its vertices precisely the incidence vectors of n-cliques of $\mathcal{G}_n$. However, the relationship is much stronger, and the polytope $\mathcal{QAP}^\star_n$ *decomposes* in a certain sense into $n+1$ "copies" of the polytope $\mathcal{QAP}_n$. We say that a polytope $\mathcal{P}$ *decomposes* into some faces $\mathcal{F}_1, \ldots, \mathcal{F}_r$ of itself if these faces have pairwise empty intersections and

$$\mathcal{P} = \mathrm{conv}\left(\bigcup_{\alpha=1}^{r} \mathcal{F}_\alpha\right)$$

holds (see Figure 7).

Theorem 9. *For* $n \geq 2$ *the polytope* $\mathcal{QAP}^\star_n$ *decomposes into* $n+1$ *faces that are each isomorphic to* $\mathcal{QAP}_n$*.*

Proof. We shall exhibit $n+1$ affine maps $\phi_\alpha : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ $(\alpha = 0, \ldots, n)$ such that for the $n+1$ images $\mathcal{Q}_\alpha = \phi_\alpha(\mathcal{QAP}_n)$ of $\mathcal{QAP}_n$ the following holds:

(i) Every $\mathcal{Q}_\alpha$ is isomorphic to $\mathcal{QAP}_n$.

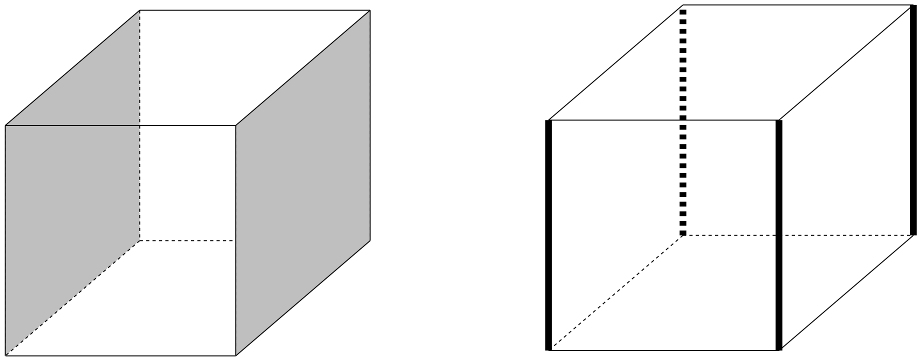


Figure 7: Two decompositions of a polytope into some faces of it.

(ii) Each $\mathcal{Q}_\alpha$ is a face of $\mathcal{QAP}_n^\star$.

(iii) The $\mathcal{Q}_\alpha$ $(\alpha = 0, \ldots, n)$ have pairwise empty intersection.

(iv) $\mathcal{QAP}_n^\star = \operatorname{conv}\left(\bigcup_{\alpha=0}^{n} \mathcal{Q}_\alpha\right)$

For any row or column $S \in \{\mathrm{row}_1, \ldots, \mathrm{row}_n, \mathrm{col}_1, \ldots, \mathrm{col}_n\}$ let

$$\sigma^S : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \{(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid x_S = 0,\, y_{\delta(S)} = 0\}$$

(where $\delta(S) \subset \mathcal{E}_{n^\star}$ is the set of all edges having one node in common with S) be the orthogonal projection. The map $\pi^{(n)}$ of Proposition 1 decomposes into

$$\pi^{(n)} = \sigma^{\mathrm{col}_n} \circ \sigma^{\mathrm{row}_n}.$$

Since Proposition 1 showed that $\pi^{(n)}$ performs an isomorphic transformation of $\mathcal{QAP}_n$, so does σ^{row_n}, too. There is nothing special about row_n, and therefore, the same holds for all choices

$$\phi_\alpha = \sigma^{\mathrm{row}_\alpha} \qquad (\alpha = 1, \ldots, n). \tag{7}$$

Finally, define ϕ_0 to be the identical map on $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$. Hence, all the

$$\mathcal{Q}_\alpha = \phi_\alpha(\mathcal{QAP}_n) \qquad (\alpha = 0, \ldots, n)$$

are isomorphic to $\mathcal{QAP}_n$, as required in (i). Claims (ii), (iii), and (iv) follow from the observation that for any $\alpha \in \{1, \ldots, n\}$ the vertices of $\mathcal{Q}_\alpha$ correspond to the $(n-1)$-cliques of $\mathcal{G}_n$ having no node in common with the α-th row of $\mathcal{V}_n$. $\square$

The projection $\pi^{(n)} : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$ maps the polytope $\mathcal{QAP}_n$ isomorphically to the polytope $\mathcal{QAP}_{n-1}^\star$. Hence $\pi^{(n)}$ induces an isomorphism between the face lattices of $\mathcal{QAP}_n$ and $\mathcal{QAP}_{n-1}^\star$, and thus, by Theorem 9, the polytope $\mathcal{QAP}_n$ decomposes into faces $\mathcal{Q}'_0, \ldots, \mathcal{Q}'_{n-1}$ of $\mathcal{QAP}_n$ with

$$\mathcal{Q}_\alpha = \pi^{(n)}(\mathcal{Q}'_\alpha) \tag{8}$$

(where the $\mathcal{Q}_\alpha$ are as in the proof of Theorem 9) for all $\alpha = 0, \ldots, n-1$. Due to (8), the polytope $\mathcal{Q}'_0$ must be the face of $\mathcal{QAP}_n$ that has as its vertices precisely the incidence vectors of n-cliques of $\mathcal{G}_n$ containing the node (n, n). For every $\alpha \in \{1, \ldots, n-1\}$ the polytope $\mathcal{Q}'_\alpha$ must be the face of $\mathcal{QAP}_n$ with vertices corresponding to those n-cliques of $\mathcal{G}_n$ containing the node (α, n).

Clearly, all the constructions can be done analogously for any other fixed row or column of $\mathcal{V}_n$ instead of $\mathrm{row}_n^{(n)}$ (this is due to the symmetries of $\mathcal{QAP}_n$ mentioned in Section 2.2). Hence, for every node $(i,j) \in \mathcal{V}_n$ the face

$$\mathcal{F}_{(i,j)} = \left\{(x,y) \in \mathcal{QAP}_n \mid x_{(i,j)} = 1\right\}$$

of $\mathcal{QAP}_n$ defined by the inequality $x_{(i,j)} \leq 1$ is isomorphic to $\mathcal{QAP}_{n-1}$, and we have the following result.

Theorem 10. *Let $n \geq 3$.*

(i) For every $i \in \mathcal{N}$ the polytope $\mathcal{QAP}_n$ decomposes into the faces $\mathcal{F}_{(i,1)}, \ldots, \mathcal{F}_{(i,n)}$ that are each isomorphic to $\mathcal{QAP}_{n-1}$.

(ii) For every $j \in \mathcal{N}$ the polytope $\mathcal{QAP}_n$ decomposes into the faces $\mathcal{F}_{(1,j)}, \ldots, \mathcal{F}_{(n,j)}$ that are each isomorphic to $\mathcal{QAP}_{n-1}$.

Theorem 10 yields an "inductive construction" of the quadratic assignment polytope that might be used for developing branching strategies in a branch-and-cut algorithm. Branching to faces of $\mathcal{QAP}_n$ that are isomorphic to $\mathcal{QAP}_{n-1}$ would have the enormous advantage that still after the branching the algorithm deals with a polytope that was investigated from the theoretical point of view instead of working on some more or less arbitrary faces of the investigated polytope. In particular, if the branching is performed in that way it is possible to work at every node of the branch-and-cut tree with inequalities that define facets of the polytope that is associated with this node.

These considerations are not only interesting for the quadratic assignment polytope. For example, the cut polytope admits similar decompositions.

4 Affine Hulls, Dimensions, and Trivial Inequalities

After doing the preparations in Section 3, we now can treat the basic polyhedral questions concerning it. We perform the corresponding investigations for $\mathcal{QAP}^{\star}_{n^\star}$ first, and carry over the results to $\mathcal{QAP}_n$ then.

4.1 Basic Facial Structures of $\mathcal{QAP}^{\star}_{n^\star}$

We have already analyzed the equation system (5), (6) holding for $\mathcal{QAP}^{\star}_{n^\star}$ in Proposition 3.2. There it turned out that precisely one (arbitrary) equation is redundant in that system. The next theorem shows in particular that we do not have to search for more valid equations for $\mathcal{QAP}^{\star}_{n^\star}$.

Theorem 11. *Let $n^\star \geq 2$.*

(i) The affine hull of $\mathcal{QAP}^{\star}_{n^\star}$ is

$$\mathrm{aff}\,(\mathcal{QAP}^{\star}_{n^\star}) = \left\{\, (x,y) \in \mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}} \,\middle|\, (x,y) \text{ satisfies } (5),\ (6)\right\}.$$

(ii) The dimension of $\mathcal{QAP}^{\star}_{n^\star}$ is

$$\dim\,(\mathcal{QAP}^{\star}_{n^\star}) = \dim\left(\mathbb{R}^{\mathcal{V}_{n^\star}} \times \mathbb{R}^{\mathcal{E}_{n^\star}}\right) - (n^\star(n^\star - 1) - 1).$$

(iii) The inequalities

$$y_e \geq 0 \qquad\qquad (e \in \mathcal{E}_{n^\star})$$

define facets of $\mathcal{QAP}^{\star}_{n^\star}$.

Proof. By Proposition 4, part (ii) is implied by part (i). We will proceed as explained in Section 3.3 and prove (i) and (iii) together. Due to the symmetries of $\mathcal{QAP}^\star_{n^\star}$, it suffices to prove (iii) for $e = [n^\star, n^\star - 1, n^\star - 1, n^\star]$.

Let L be the set of all $n^\star$- and $(n^\star - 1)$-cliques of $\mathcal{G}_{n^\star}$ that do not contain both nodes $(n^\star, n^\star - 1)$ and $(n^\star - 1, n^\star)$, i.e., L is the set of cliques belonging to the vertices of the face defined by $y_{[n^\star, n^\star-1, n^\star-1, n^\star]} \geq 0$. As in Section 3.3, we denote by Δ_L the set of differences of the incidence vectors belonging to L. We choose B to consist of E_{bas} (see Corollary 5), and take as the extra element the edge $e_0 = [n^\star, n^\star - 1, n^\star - 1, n^\star]$. Then we have to combine all nodes and edges starting from the vectors in $\mathcal{B}_0$ (the canonical unit vectors belonging to $B \cup e_0$) and Δ_L in order to prove the theorem (since $y_{[n^\star, n^\star-1, n^\star-1, n^\star]} \geq 0$ defines a proper face).

We exhibit in three lemmas some of the vectors presented in Section 3.4 that are available for our proof. From now on, we will assume $n^\star \geq 5$. This simplifies the proof and does not really leave open a gap, because one can easily check the cases $n^\star \in \{2, 3, 4\}$ by computer, for example.

Lemma 12. *Let $w'_1, w'_2, w'_3, w'_4 \in \mathcal{V}_{n^\star}$ such that $\Upsilon(w'_1, w'_2, w'_3, w'_4)$ is defined. If neither $\{w'_1, w'_3\}$ nor $\{w'_2, w'_4\}$ is the edge $[n^\star, n^\star - 1, n^\star - 1, n^\star]$, then we have*

$$\Upsilon(w'_1, w'_2, w'_3, w'_4) \in \operatorname{lin}(\Delta_L)\,.$$

Proof. Since $\mathcal{G}_{n^\star}/\{w'_1, w'_2, w'_3, w'_4\}$ has at least three rows and at least three columns, we can find an $(n^\star - 2)$-clique C of $\mathcal{G}_{n^\star}/\{w'_1, w'_2, w'_3, w'_4\}$ such that the nodes $(n^\star, n^\star - 1)$ and $(n^\star - 1, n^\star)$ are both not contained in C. Hence, Lemma 7 can be applied, yielding the claim. □

Lemma 13. *Let $w_1, w_2, w_3, w_4 \in \mathcal{V}_{n^\star}$ such that $\Phi(w_1, w_2, w_3, w_4)$ is defined. If none of $\{w_1, w_2\}$, $\{w_2, w_3\}$, $\{w_3, w_4\}$ and $\{w_4, w_1\}$ is the edge $[n^\star, n^\star - 1, n^\star - 1, n^\star]$, then we have*

$$\Phi(w_1, w_2, w_3, w_4) \in \operatorname{lin}(\Delta_L)\,.$$

Proof. There is at most one of the nodes $(n^\star, n^\star - 1)$ and $(n^\star - 1, n^\star)$ contained in $\{w_1, w_2, w_3, w_4\}$, hence we can assume (by a symmetry argument) that $(n^\star, n^\star - 1) \notin \{w_1, w_2, w_3, w_4\}$ holds. Since $\mathcal{G}_{n^\star}/\{w_1, w_2, w_3, w_4\}$ has at least two rows and at least two columns, we can find an $(n^\star - 3)$-clique C of $\mathcal{G}_{n^\star}/\{w_1, w_2, w_3, w_4\}$ with $(n^\star, n^\star - 1) \notin C$. Thus Lemma 8 yields the claim. □

Lemma 14. *Let $w \in \mathcal{V}_{n^\star}$ be any node. Then there is an $n^\star$-clique C of $\mathcal{G}_{n^\star}$, containing w, such that we have*

$$\Theta(C, w) \in \operatorname{lin}(\Delta_L)\,.$$

Proof. This is due to Lemma 6, since $\mathcal{G}_{n^\star}/w$ has at least four rows and at least four columns, and hence, it is easy to find an $(n^\star - 1)$-clique of $\mathcal{G}_{n^\star}/w$ eventually not containing a forbidden node. □

Now we combine all nodes and edges using Lemmas 12, 13, and 14. Let us partition the node set $\mathcal{V}_{n^\star}$ into four parts as indicated in the following table

$\times$	$\{1,2\}$	$\{3,\ldots,n^\star\}$
$\{1,2\}$	V_1	V_2
$\{3,\ldots,n^\star\}$	V_3	V_4

meaning that we have, e.g., $V_1 = \{1,2\} \times \{1,2\}$. Due to our supposition $n^\star \geq 5$, none of these four sets is empty, and $[n^\star, n^\star - 1, n^\star - 1, n^\star] \in V_4$. Recall that the *mate* of an edge $[i, j, k, l] \in \mathcal{E}_{n^\star}$ is the edge $[i, l, k, j]$. For any number $a \in \{1, 2\}$ we denote by $\neg a$ the number with $\{a, \neg a\} = \{1, 2\}$. We perform the necessary combinations in eight steps.

$\mathcal{E}_{\mathbf{n}^\star}(\mathbf{V_1} \cup \mathbf{V_2})$. For every edge in $\mathcal{E}_{n^\star}(V_1 \cup V_2)$ either itself or its mate is contained in B. Hence, these edges can be combined by Lemma 12.

$\mathcal{E}_{\mathbf{n}^\star}(\mathbf{V_1} \cup \mathbf{V_3})$. This is done analogously to the first step.

$(\mathbf{V_2} : \mathbf{V_3})$. Let $(i, j) \in V_2$ and $(k, l) \in V_3$, hence we have $i, l \in \{1, 2\}$ and $j, k \in \{3, \ldots, n^\star\}$. Choosing $w_1 = (\neg i, l)$, $w_2 = (i, \neg l)$, $w_3 = (k, l)$, and $w_4 = (i, j)$ (see Figure 8), we can apply Lemma 13, yielding the desired combination of $[i, j, k, l] = \{w_3, w_4\}$, since the edges $\{w_1, w_2\}$, $\{w_2, w_3\}$ and $\{w_4, w_1\}$ are already combined.

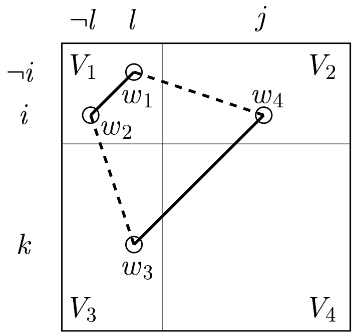


Figure 8: Combination of the edges in $(V_2 : V_3)$.

$(\mathbf{V_1} : \mathbf{V_4})$. Since all edges in $(V_2 : V_3)$ are already combined, these edges can be combined by using Lemma 12.

$(\mathbf{V_2} : \mathbf{V_4})$. Let $(i, j) \in V_2$ and $(k, l) \in V_4$, i.e., we have $i \in \{1, 2\}$ and $j, k, l \in \{3, \ldots, n^\star\}$. We choose $w_1 = (\neg i, l)$, $w_2 = (i, 1)$, $w_3 = (k, l)$, and $w_4 = (i, j)$ (see Figure 9), hence Lemma 13 applies and yields a combination of $[i, j, k, l] = \{w_3, w_4\}$, because, again, the edges $\{w_1, w_2\}$, $\{w_2, w_3\}$ and $\{w_4, w_1\}$ are already combined.

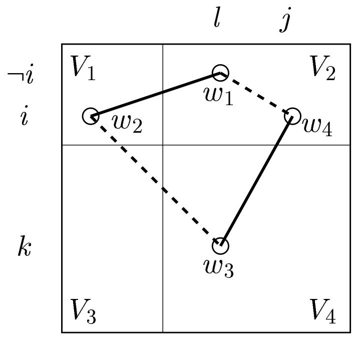


Figure 9: Combination of the edges in $(V_2 : V_4)$.

$(\mathbf{V_3} : \mathbf{V_4})$. These edges are combined analogously to the edges in $(V_1 : V_4)$.

$\mathcal{E}_{\mathbf{n}^\star}(\mathbf{V_4})$. The edge $[n^\star, n^\star - 1, n^\star - 1, n^\star]$ is already combined since it was chosen to be the extra element. Let $[i, j, k, l] \in \mathcal{E}_{n^\star}(V_4) \setminus \{[n^\star, n^\star - 1, n^\star - 1, n^\star]\}$. The nodes $w_1 = (1, j)$, $w_2 = (k, 1)$, $w_3 = (i, j)$, and $w_4 = (k, l)$ (see Figure 10) satisfy the conditions of Lemma 13, and thus, we can combine the edge $[i, j, k, l] = \{w_3, w_4\}$, because $\{w_1, w_2\}$, $\{w_2, w_3\}$ and $\{w_4, w_1\}$ have been combined in previous steps.

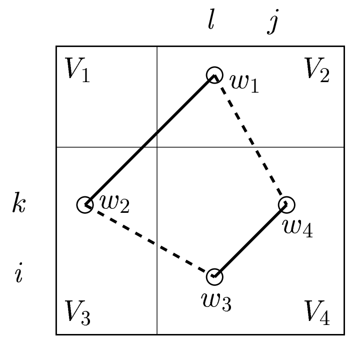


Figure 10: Combination of the edges in $\mathcal{E}_n(V_4)$.

$\mathcal{V}_{\mathbf{n}^\star}$. Now that all edges are combined, it is easy to combine also the nodes using Lemma 14.

$\square$

4.2 Basic Facial Structures of $\mathcal{QAP}_n$

The next theorem shows that we also do not have to search for other equations for $\mathcal{QAP}_n$ than for the ones given by (1), ... ,(4). Furthermore, it describes possiblities to extract from that equation system a complete and non-redundant equation system for $\mathcal{QAP}_n$.

Theorem 15. *Let $n \geq 3$.*

(i) The affine hull of $\mathcal{QAP}_n$ is described by the equations (1), ... ,(4), i.e., a point $(x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ is contained in $\operatorname{aff}(\mathcal{QAP}_n)$ if and only if it satisfies

$$x(\text{row}_i) = 1 \qquad (i \in \mathcal{N}) \tag{9}$$

$$x(\text{col}_j) = 1 \qquad (j \in \mathcal{N}) \tag{10}$$

$$-x_{(i,j)} + y((i,j) : \text{row}_k) = 0 \qquad (i,j,k \in \mathcal{N}, i \neq k) \tag{11}$$

$$-x_{(i,j)} + y((i,j) : \text{col}_l) = 0 \qquad (i,j,j \in \mathcal{N}, j \neq l). \tag{12}$$

(ii) The dimension of the quadratic assignment polytope is

$$\dim(\mathcal{QAP}_n) = \dim\left(\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}\right) - (2n^3 - 5n^2 + 5n - 2).$$

(iii) Let $r, c \in \mathcal{N}$ be two row and column indices, respectively, and let $\mathcal{R}$ be a subset of the equations (9), ... ,(12) consisting precisely of

a) one equation from (9) or (10),

b) for all $(i,j) \in \mathcal{N} \setminus r \times \mathcal{N} \setminus c$ either (11) with one arbitrary $k \neq i$ or (12) with one arbitrary $l \neq j$,

c) all equations (11), (12) with $(i,j) = (r,c)$,

d) for all $i \in \mathcal{N} \setminus r$ the equation (11) with $(k,j) = (r,c)$,

e) for all $j \in \mathcal{N} \setminus c$ the equation (12) with $(i,l) = (r,c)$,

f) for all $(k,l) \in \mathcal{N} \setminus r \times \mathcal{N} \setminus c$ either (11) with $(i,j) = (r,l)$ or (12) with $(i,j) = (k,c)$,

g) for all pairs $\{i', k'\} \in \binom{\mathcal{N} \setminus r}{2}$ either (11) with $(i,j,k) = (i',c,k')$ or (11) with $(i,j,k) = (k',c,i')$, *h) for all pairs* $\{j', l'\} \in \binom{\mathcal{N}\setminus c}{2}$ *either* (12) *with* $(i, j, l) = (r, j', l')$ *or* (12) *with* $(i, j, l) = (r, l', j')$,

i) either for one pair in g) or for one pair in h) the equation not yet chosen in g) or h), respectively

(where "either or" is always meant exclusively). Then removing $\mathcal{R}$ *from the set of equations* (9), ... , (12) *yields a complete and non-redundant equation system for* $\mathcal{QAP}_n$.

Proof. In order to prove part (i), it suffices to show that the zero-liftings of (5) and (6) from $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$ into $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ can be linearly combined from the equation system (9), ... ,(12). This is sufficient due to the fact that then the solution space $\mathcal{A}$ of (9), ... ,(12), containing $\mathcal{QAP}_n$, is mapped by the projection $\pi^{(n)}$ isomorphically (see Proposition 1) into the solution space of (5) (6), which is the affine hull of $\mathcal{QAP}^{\star}_{n-1}$ (by Theorem 11). Hence, by the isomorphism between $\mathcal{QAP}_n$ and $\mathcal{QAP}^{\star}_{n-1}$, we have

$$\dim\left(\mathcal{QAP}^{\star}_{n-1}\right) = \dim\left(\mathcal{QAP}_n\right) \leq \dim(\mathcal{A}) \leq \dim\left(\mathcal{QAP}^{\star}_{n-1}\right),$$

showing that in particular $\operatorname{aff}(\mathcal{QAP}_n) = \mathcal{A}$ must hold.

By symmetry arguments, we only need to show that the equation

$$x\left(\mathrm{row}^{(n)}_1 \setminus (1,n) \cup \mathrm{row}^{(n)}_2 \setminus (2,n)\right) - y\left(\mathrm{row}^{(n)}_1 \setminus (1,n) : \mathrm{row}^{(n)}_2 \setminus (2,n)\right) = 1 \tag{13}$$

is implied by (9), ... ,(12). We can obtain this by adding up the two equations (9) for $i = 1, 2$ as well as the two equations (11) with $j = n$ and $(i, k) \in \{(1, 2), (2, 1)\}$, subtract all equations (11) with $j \in \{1, \ldots, n-1\}$ and $(i, k) \in \{(1, 2), (2, 1)\}$, and finally divide the obtained equation by two. Figure 11 illustrates the summation by showing three of its partial sums.

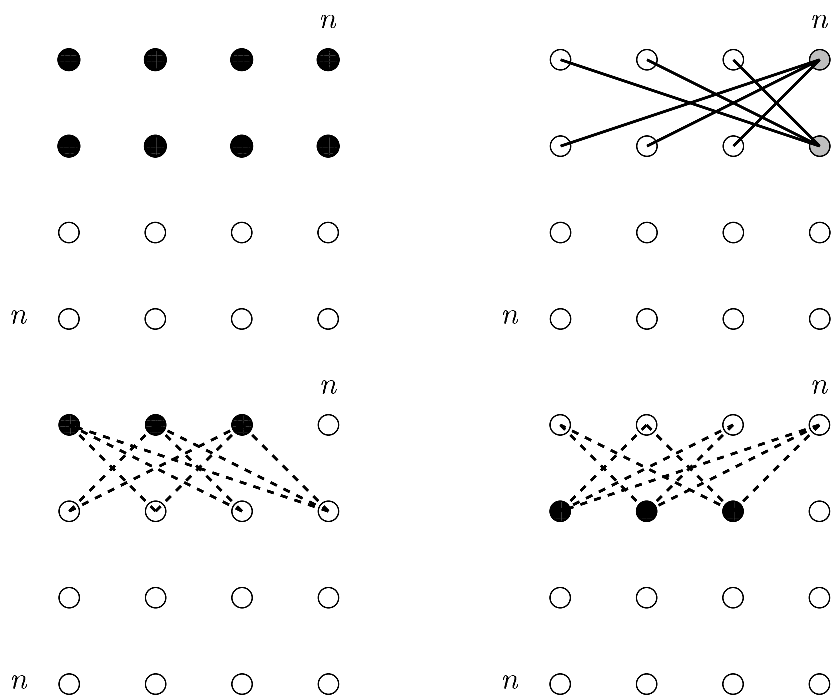


Figure 11: Combination of equation (13).

Now, we will prove part (ii). When changing from $\mathcal{G}_{n-1}$ to $\mathcal{G}_n$ one obtains $2n-1$ new nodes, $2(n-1)^2(n-2)$ new edges connecting $\text{row}_n^{(n)} \setminus (n,n)$ and $\text{col}_n^{(n)} \setminus (n,n)$ with the old nodes, $(n-1)^2$ new edges between $\text{row}_n^{(n)} \setminus (n,n)$ and $\text{col}_n^{(n)} \setminus (n,n)$, and $(n-1)^2$ new edges from (n,n) to the old nodes, summing up to

$$\begin{aligned} 2n-1+2(n-1)^2(n-2)+2(n-1)^2 &= 2n-1+2(n-1)^3 \\ &= 2n-1+2n^3-6n^2+6n-2 \\ &= 2n^3-6n^2+8n-3 \end{aligned}$$

new items. Thus, we have (using Theorem 11)

$$\begin{aligned} \dim\left(\mathcal{QAP}_n\right) &= \dim\left(\mathcal{QAP}^\star_{n-1}\right) \\ &= \dim\left(\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}\right) - ((n-1)(n-2)-1) \\ &= \dim\left(\mathbb{R}^{\mathcal{V}_{n}} \times \mathbb{R}^{\mathcal{E}_{n}}\right) - (2n^3-6n^2+8n-3) - (n^2-3n+1) \\ &= \dim\left(\mathbb{R}^{\mathcal{V}_{n}} \times \mathbb{R}^{\mathcal{E}_{n}}\right) - (2n^3-5n^2+5n-2), \end{aligned}$$

proving part (ii).

It remains to prove part (iii). The cardinality of the set $\mathcal{R}$ of equations being removed from the system (9), ... ,(12) is

$$\begin{aligned} |\mathcal{R}| &= 1+(n-1)^2+2(n-1)+(n-1)+(n-1) \\ &\quad +(n-1)^2+\frac{(n-1)(n-2)}{2}+\frac{(n-1)(n-2)}{2}+1 \\ &= 2+2(n-1)^2+4(n-1)+(n-1)(n-2) \\ &= 2+(n-1)(2(n-1)+4+(n-2)) \\ &= 2+3n(n-1) \\ &= 3n^2-3n+2. \end{aligned}$$

Hence, the remaining system consists of

$$\begin{aligned} 2n+2n^2(n-1)-(3n^2-3n+2) &= 2n+2n^3-2n^2-3n^2+3n-2 \\ &= 2n^3-5n^2+5n-2 \end{aligned}$$

equations. Due to part (ii) is suffices now to prove that this remaining system still has the same solution space as (9), ... ,(12). Hence, we will show how to combine the equations in $\mathcal{R}$ from the ones in the remaining system.

In order to simplify the notations we will denote the equations (9) by x-row(i), (10) by x-col(j), (11) by xy-row(i,j,k), and (12) by xy-col(i,j,l). Due to symmetry reasons we can restrict to $r=n$ and $c=n$.

a) We can combine the equation removed from (9), (10) from the remaining ones, since this system has not full row rank, and hence, due to symmetry reasons, every single equation is redundant.

b) For every fixed node $(i,j) \in \mathcal{V}_n$ adding up all equations xy-row(i,j,k) yields the same as adding up all equations xy-col(i,j,l). Hence, for every fixed node $(i,j)\mathcal{V}_n$ the system of equations

$$\{\text{xy-row}(i,j,k) \mid k \in \mathcal{N} \setminus i\} \cup \{\text{xy-col}(i,j,l) \mid l \in \mathcal{N} \setminus j\}$$

has not full row rank. Thus there must be at least one redundant equation among them. Due to symmetry reasons, again, this must be an arbitrary one. But for $(i,j) \in \mathcal{V}_n \setminus (\text{row}_n^{(n)} \cup \text{col}_n^{(n)})$ the set $\mathcal{R}$ contains only one of these equations that therefore can be combined from the remaining ones.

f) Suppose an equation xy-row(n,j,k) with $k,j \in \mathcal{N}\backslash n$ is contained in $\mathcal{R}$. Then xy-col(k,n,j) is not contained in $\mathcal{R}$, and furthermore, we can use all xy-row(i,j,k) and xy-col(i,j,l) with $i,j \in \{1,\ldots,n-1\}$ for the linear combination since they have already been combined in b). Adding up all xy-col(k,l,j) for $l \in \mathcal{N}\backslash j$, subtracting all xy-row$(i,j,k)$ for $i \in \mathcal{N}\backslash\{k,n\}$, and finally adding x-row(k) and subtracting x-col(j) yields a combination of xy-row(n,j,k) (see Figure 12). An equation xy-col(i,n,l) contained in $\mathcal{R}$ can be combined analogously.

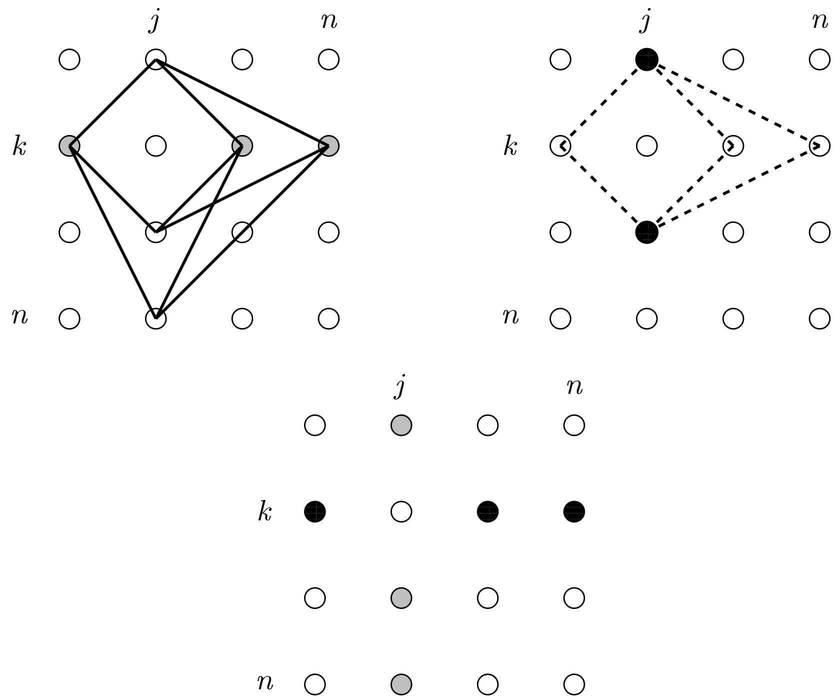


Figure 12: Combination of the equations removed in f).

g),h),i) If an equation xy-row(i,n,k) with $i,k \in \mathcal{N} \setminus n$ and $i \neq k$ is not contained in $\mathcal{R}$, then we can combine the equation

$$x\left(\text{row}_i^{(n)} \setminus (i,n) \cup \text{row}_k^{(n)} \setminus (k,n)\right) - y\left(\text{row}_i^{(n)} \setminus (i,n) : \text{row}_k^{(n)} \setminus (k,n)\right) = 1 \tag{14}$$

by just using equations from the remaining system and equations that have already been combined in a) and b) (see Figure 13). The same holds also for any equation xy-col(n,j,l) with $j,l \in \mathcal{N} \setminus n$ and $j \neq l$. We can restrict to the case, where the equation chosen in g) is xy-row$(1,n,2)$. Thus, we can combine this way all equations that are the zero-liftings of (5) and (6) but

$$x\left(\text{row}_1^{(n)} \setminus (1,n) \cup \text{row}_2^{(n)} \setminus (2,n)\right) - y\left(\text{row}_1^{(n)} \setminus (1,n) : \text{row}_2^{(n)} \setminus (2,n))\right) = 1.$$

However, this one can be combined from the other zero-lifted equations due to the fact that one arbitrary equation in (5), (6) is redundant (see Proposition 4). Proceeding "backwards" now yields also the equation xy-row$(1,n,2)$ that was put into $\mathcal{R}$ in g).

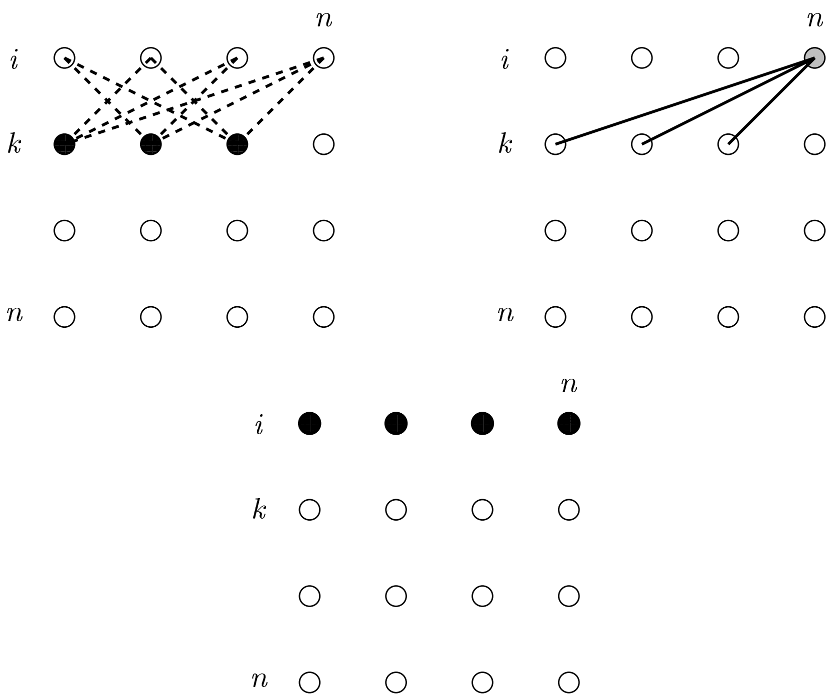


Figure 13: Combination of equation (14).

Hence, in the subsequent argumentations, we can use for every pair $\{i,k\} \in \mathcal{N}\backslash n$ with $i \neq k$ one equation of xy-row(i,n,k) and xy-row(k,n,i) as well as for every pair $\{j,l\} \in \mathcal{N} \setminus n$ with $j \neq l$ one equation of xy-col(n,j,l) and xy-col(n,l,j).

An equation xy-row(i,n,k) with $i,k \in \mathcal{N} \setminus n$ and $i \neq k$ that is contained in $\mathcal{R}$ can now be combined by adding up all xy-row(k,j,i) for all $j \in \mathcal{N}$, subtracting all xy-row(i,j,k) for $j \in \mathcal{N} \setminus n$, adding x-row$(k)$, and subtracting x-row$(i)$ (see Figure 14). Analogously, one treats an equation xy-col(n,j,l) with $j,l \in \mathcal{N}\backslash n$ that is contained in $\mathcal{R}$, where xy-col(n,l,j) is not contained in $\mathcal{R}$, and thus, all equations removed in g), h) and i) are combined.

d) Every equation xy-row(i,n,n) with $i \in \mathcal{N}\backslash n$ can be combined by adding up all xy-col(i,n,l) for all $l \in \mathcal{N} \setminus n$ and subtracting all xy-row(i,n,k) for all $k \in \mathcal{N} \setminus \{i,n\}$ (see Figure 15).

e) Here, we can proceed analogously to d).

c) We combine an equation xy-row(n,n,k) by adding up all xy-row(k,j,n) for $j \in \mathcal{N}$, subtracting all xy-row(n,j,k) for $j \in \mathcal{N}\backslash n$, adding x-row$(k)$, and finally subtracting x-row(n) (this is the same procedure as for the combination of the equations removed in g) and h), see Figure 14). Finally, the equations xy-col(n,n,l) are combined analogously.

□

We close this treatment of the basic questions concerning redundancies in the linear constraints we have considered so far by a classification of the trivial inequalities for $\mathcal{QAP}_n$.

Theorem 16. *Let $n \geq 3$.*

(i) The inequalities

$$y_e \geq 0 \qquad\qquad (e \in \mathcal{E}_n)$$

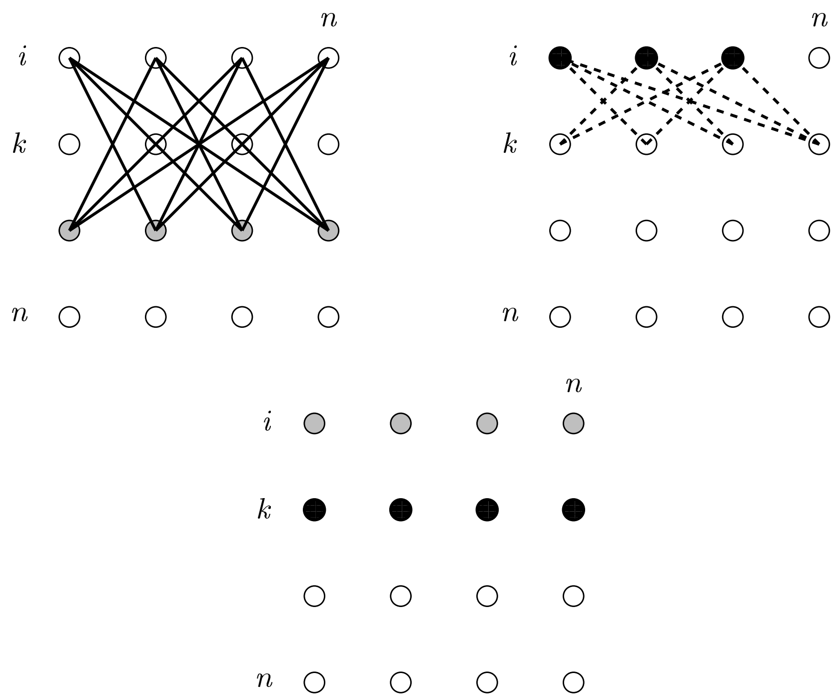


Figure 14: Combination of the equations removed in g) and h).

define facets of $\mathcal{QAP}_n$.

(ii) The inequalities

$$
\begin{aligned}
y_e &\leq 1 && (e \in \mathcal{E}_n) \\
x_v &\geq 0 && (v \in \mathcal{V}_n) \\
x_v &\leq 1 && (v \in \mathcal{V}_n)
\end{aligned}
$$

are implied by the equations (9), … ,(12) *and the nonnegativity constraints* $y \geq 0$ *on the edge variables.*

Proof. Part (i) follows immediately from part (iii) of Theorem 11. In order to prove part (ii), observe that (11), e.g., yields from $y \geq 0$ also the nonnegativity of x. From that one obtains, e.g. by (9), that $x \leq 1$ holds, and this leads, exploiting once more (11) and $y \geq 0$, to $y \leq 1$. □

5 Conclusions

With the introduction of the "star-transformation" of the quadratic assignment polytope, now a technique is available that allows to perform deeper polyhedral investigations of the quadratic assignment problem. This certainly provides the possibility of investigating large classes of inequalities with respect to the question if they define facets of the quadratic assignment polytope. Considering, e.g., the fact that polyhedral investigations of the traveling salesman problem have lead to algorithms that now can solve instances of several thousands of cities, the techniques that we have presented in this paper might give a key to utilize a large potential that polyhedral treatments of the quadratic assignment problem has for improving the practical solvability of this extremely hard one among the $\mathcal{NP}$-hard combinatorial optimization problems.

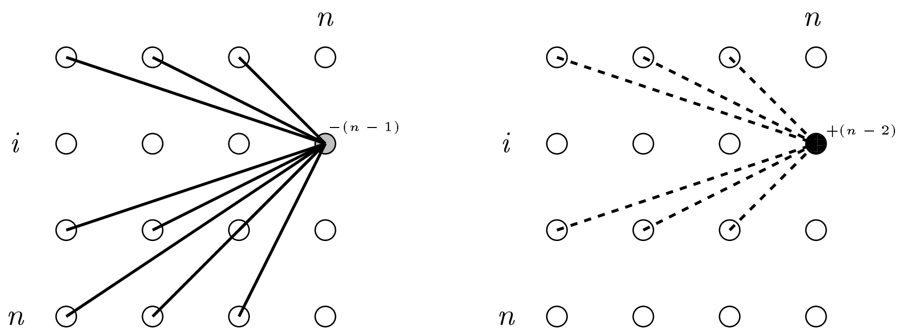


Figure 15: Combination of the equations removed in d).

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