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by
Michael Jünger and Volker Kaibel

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Institut für Informatik
Universität zu Köln
Pohligstr. 1
50969 Köln

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On the SQAP-Polytope [*]

Michael Jünger
Institut für Informatik
Universität zu Köln
mjuenger@informatik.uni-koeln.de

Volker Kaibel
Institut für Informatik
Universität zu Köln
kaibel@informatik.uni-koeln.de

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Abstract

The study of the *QAP-Polytope* was started by Rijal (1995), Padberg and Rijal (1996), and Jünger and Kaibel (1996), investigating the structure of the feasible points of a (Mixed) Integer Linear Programming formulation of the QAP that provides good lower bounds by its continious relaxation. Rijal (1995) and Padberg and Rijal (1996) propose an alternative (Mixed) Integer Linear Programming formulation for the case that the QAP-instance is *symmetric* in a certain sense and define analogously the *SQAP-Polytope*. They give a conjecture about the dimension of that polytope, whose proof is one part of this paper. Moreover, we investigate the trivial faces of the SQAP-Polytope and present a first class of non-trivial facets of it. The polyhedral results are used to compute lower bounds for symmetric QAPs.

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1 Introduction

For many classical $\mathcal{NP}$-hard combinatorial optimization problems like, e.g., the *Traveling Salesman Problem (TSP)*, the *Max Cut Problem*, or the *Stable Set Problem*, the methods of polyhedral combinatorics have yielded a lot of structural insight that lead to big improvements in practical problem solving via cutting plane based methods like Branch&Cut. However, the *Quadratic Assignment Problem (QAP)* – where the task is to find a permutation π that minimizes $\sum_i \sum_k a_{ik} b_{\pi(i)\pi(k)} + \sum_i c_{i\pi(i)}$ for some matrices $A = (a_{ik})$, $B = (b_{jl})$, and $C = (c_{ij})$ – was not considered from a polyhedral point of view until the work of Rijal (1995), Padberg and Rijal (1996), and Jünger and Kaibel (1996). These papers defined the QAP-Polytope via a long time known, quite natural, Mixed Integer Programming (MIP) formulation of the QAP and proved some basic properties of that polytope, in particular its dimension.

There might be two reasons, why the QAP-Polytope had not been considered before. One is the fact that this polytope looks in some sense "nasty", which can be overcome

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by mapping it in a certain way into a different space (cf. Jünger and Kaibel, 1996). The other reason is a computational one. The MIP-formulation on which the QAP-Polytope is based, has a lot of variables, such that (at least) in former times it might have seemed to be unpractical to solve the arising LP's, for instance within a Branch&Cut algorithm. However, the LP-solvers have improved a lot during the last years, especially due to the success of interior point methods. Nowadays, it seems to become promising to attack QAP-instances of size about 20 or 25 (and maybe even larger) by cutting plane based algorithms that use structural insight into the QAP-Polytope. When considering these orders of magnitudes, one has to note that existing Branch&Bound algorithms (mostly using the *Gilmore/Lawler Bound*) need a large amount of (parallel) computer power to solve instances of size about 20, since they produce Branch&Bound trees with very many nodes (cf. Clausen and Perregaard, 1994). Due to this fact, it sounds attractive to try to reduce this "tendency to implicit enumeration" by exploiting more structural information about the problem that result from the polyhedral investigations.

Actually, the kind of QAP we defined above, is a so called *Koopmans&Beckmann Problem (KB-QAP)*. Koopmans and Beckmann (1957) introduced this problem in order to model the situation of a set of n facilities that have certain amounts of "flow" between them and a set of n locations having certain distances, and the requirement is to assign the facilities to the locations in such a way that the sum of the products of flows and the respective distances is minimized. The c_{ij} model fixed costs that arise when placing facility i to location j, independently from the assignment of the other facilities. One calls matrix A the *flow matrix*, matrix B the *distance matrix*, and matrix C the matrix of the *linear costs*. Clearly, this problem is $\mathcal{NP}$-hard, since it has many $\mathcal{NP}$-hard optimization problems as special cases, e.g., the TSP.

We call instances with the property that assigning object i to location j and object k to location l always causes the same costs as assigning i to l and k to j *symmetric*. For example, all instances having a symmetric distance or flow matrix are symmetric in that sense. It turns out (first observed by Rijal, 1995; Padberg and Rijal, 1996) that for such symmetric instances one can drop nearly 50% of the variables in the MIP-formulation underlying the polyhedral approach. This yields a different polytope, the SQAP-Polytope. Rijal (1995) and Padberg and Rijal (1996) derived a set of valid equations and conjectured the dimension of the SQAP-Polytope.

In this paper, we present some basic properties of the SQAP-Polytope including a proof of that conjecture. The main tool we use is a transformation that is similar to the one that allowed us to derive basic results about the QAP-Polytope in a (relatively) simple way (cf. Jünger and Kaibel, 1996). Section 2 presents our formulation of the QAP as a minimization problem in a certain graph. Using that terminology, we give the MIP formulations for QAP and SQAP that underly the polyhedral approaches. In Section 3, we give definitions of both the QAP- and the SQAP-Polytope and describe a connection between them. Then, we map these polytopes isomorphically to other spaces, where they "look much nicer". (When saying a certain polytope P is *isomorphic* to a polytope P', we always mean that there is an affine transformation from $\mathrm{aff}(P)$ to $\mathrm{aff}(P')$ mapping P to P'. In particular, this implies that the two polytopes are combinatorially isomorphic, i.e., they have isomorphic face lattices.) Section 4 establishes the dimension of the SQAP-Polytope as well as the fact that the nonnegativity constraints on the variables define facets of it. In Section 5, we present a first class of non-trivial facets of the SQAP-Polytope. Section 6 reports on some preliminary computational results concerning a lower bound obtained by exploiting these first results about the SQAP-Polytope. It will turn out that – not surprisingly – the basic work which is presented in this paper, does not yet lead to a breakthrough in attacking (symmetric) QAP's. We discuss in Section 7, which further investigations are promising, in our opinion, and why we think that this polyhedral approach should be pursued further (although we do not want to predict that breakthrough).

2 Problem Definition

We will define the QAP as the problem of finding among certain cliques in a special graph one of minimal node- and edge-weight. The SQAP will be defined as a similar problem in a closely related hypergraph. We use the symbol $\binom{M}{k}$ for the set of all subsets of cardinality k of a set M.

Let the graph $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$ have nodes

$$\mathcal{V}_n := \big\{(i,j) \bigm| i,j \in \{1,\dots,n\}\big\}$$

and edges

$$\mathcal{E}_n := \Big\{\{(i,j),(k,l)\} \in \tbinom{\mathcal{V}_n}{2} \Bigm| i \neq k, j \neq l\Big\}.$$

We define $[i,j,k,l] := \{(i,j),(k,l)\}$ for all edges $\{(i,j),(k,l)\} \in \mathcal{E}_n$. This implies $[i,j,k,l] = [k,l,i,j]$. We usually draw $\mathcal{G}_n$ as shown in Figure 1.

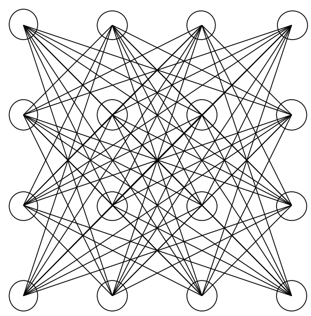


Figure 1: The graph $\mathcal{G}_n$ has all possible edges but the "horizontal" and the "vertical" ones.

The graph $\mathcal{G}_n$ has clique-number $\omega(\mathcal{G}_n) = n$, and the n-cliques of $\mathcal{G}_n$ correspond to the $n \times n$-permutation matrices. We denote the set of (node sets of) k-cliques of $\mathcal{G}_n$ by

$$\mathcal{CLQ}_k^n := \{C \subseteq \mathcal{V}_n \mid C \text{ } k\text{-clique of } \mathcal{G}_n\}.$$

For any $S \subseteq \mathcal{V}_n$, we denote by $\mathcal{E}_n(S) := \{\{v,w\} \in \mathcal{E}_n \mid v,w \in S\}$ the set of edges having both endpoints in S. As usual, for a subset $N \subseteq M$ of a finite set M and a vector $a \in \mathbb{R}^M$, we define $a(N) := \sum_{e \in N} a_e$.

The *Quadratic Assignment Problem* is to solve

$$(\mathrm{QAP}_{g,h}) \qquad \begin{array}{rl} \min & g(C) + h(\mathcal{E}_n(C)) \\ \text{s.t.} & C \in \mathcal{CLQ}_n^n. \end{array}$$

for given node weights $g \in \mathbb{R}^{\mathcal{V}_n}$ and edge weights $h \in \mathbb{R}^{\mathcal{E}_n}$. (If we have a KB-QAP defined by the matrices $A = (a_{ik})$, $B = (b_{jl})$ and $C = (c_{ij})$ we choose $g_{(i,j)} = c_{ij} + a_{ii}b_{jj}$ and $h_{[i,j,k,l]} = a_{ik}b_{jl} + a_{ki}b_{lj}$.)

The nodes and edges of $\mathcal{G}_n$ will correspond to variables in the polyhedral approach. If the instance (g, h) is *symmetric* in the sense that $h_{[i,j,k,l]} = h_{[i,l,k,j]}$ for all pairs of edges $([i, j, k, l], [i, l, k, j])$ (cf. Figure 2) then we can identify these two edges in our formulation, and hence reduce the number of variables by nearly 50%.

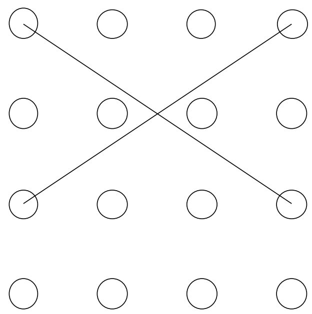


Figure 2: A pair of edges that can be identified in the symmetric case.

This observation (first made by Rijal, 1995; Padberg and Rijal, 1996) gives the motivation to study also a specific formulation for the special case of symmetric instances of the QAP, the *Symmetric Quadratic Assignment Problem (SQAP)*.

In order to derive an appropriate formulation for SQAP, we model the described identification of edges by passing from the graph $\mathcal{G}_n$ having nodes $\mathcal{V}_n$ and edges $\mathcal{E}_n$ to the hypergraph $\mathcal{H}_n$ having the same nodes $\mathcal{V}_n$, but hyperedges

$$\mathcal{F}_n := \left\{\{(i,j),(k,l),(i,l),(k,j)\} \in \tbinom{\mathcal{V}_n}{4} \middle| i \neq k, j \neq l\right\}.$$

There will be no hypergraph theory involved, we just use the notions of "hypergraph" and "hyperedges". For $i \neq k$ and $j \neq l$, we denote $\langle i, j, k, l\rangle := \{(i, j), (k, l), (i, l), (k, j)\}$. This implies $\langle i, j, k, l\rangle = \langle k, l, i, j\rangle = \langle i, l, k, j\rangle = \langle k, j, i, l\rangle$ for all $i \neq k$ and $j \neq l$. For an edge $[i, j, k, l] \in \mathcal{E}_n$ we call the edge $\tau([i, j, k, l]) := [i, l, k, j]$ the *mate* of $[i, j, k, l]$. Then we can assign to every edge $e \in \mathcal{E}_n$ a hyperedge $\mathrm{HYP}(e) := e \cup \tau(e) \in \mathcal{F}_n$. For a subset $R \subseteq \mathcal{E}_n$, we denote $\mathrm{HYP}(R) := \{\mathrm{HYP}(e) \mid e \in R\}$. For a subset $S \subseteq \mathcal{V}_n$, we define the set $\mathcal{F}_n(S) := \mathrm{HYP}(\mathcal{E}_n(S))$. We refer to a subset $C \subset \mathcal{V}_n$ as a *clique of* $\mathcal{H}_n$ if, and only if, C is a clique of the graph $\mathcal{G}_n$.

Because we need to express relationships between the asymmetric and the symmetric version of the problem, we introduce the map

$$\sigma_n \;:\; \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathbb{R}^{\mathcal{F}_n}$$

by defining $\sigma_n(y) = z$ via $z_{\langle i,j,k,l\rangle} := y_{[i,j,k,l]} + y_{[i,l,k,j]}$.

If $(g, h) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ and h is symmetric then $(\mathrm{QAP}_{g,h})$ is equivalent to solving the *Symmetric Quadratic Assignment Problem*

$$(\mathrm{SQAP}_{g,\widehat{h}}) \qquad \begin{array}{ll} \min & g(C) + \widehat{h}(\mathcal{F}_n(C)) \\ \text{s.t.} & C \in \mathcal{CLQ}_n^n \end{array}$$

for $\widehat{h} := \frac{1}{2}\sigma_n(h)$.

In the rest of this section, we will develop Mixed Integer Programming (MIP) formulations for the problems QAP and SQAP. These formulations are the starting points for the polyhedral approach. The MIP formulation for QAP was proposed independently also by Johnson (1992), Drezner (1994), and Rijal (1995). The one for SQAP is due to Rijal (1995) and Padberg and Rijal (1996). Nevertheless, we will give short proofs of the respective theorems in our notational setting.

We need the notion of a *characteristic vector* $\chi^N \in \{0,1\}^M$ for a subset $N \subseteq M$ of a (finite) set M, defined by setting $\chi_p^N := 1$ for $p \in M$ if, and only if, $p \in N$. We will denote characteristic vectors of subsets of

$$\begin{array}{lll} \mathcal{V}_n & \text{by} & x^{(\cdots)}, \\ \mathcal{E}_n & \text{by} & y^{(\cdots)}, \text{ and} \\ \mathcal{F}_n & \text{by} & z^{(\cdots)}. \end{array}$$

Define $\mathrm{VERT}_n := \{(x^C, y^{\mathcal{E}_n(C)} \mid C \in \mathcal{CLQ}_n^n\}$ and $\mathrm{SVERT}_n := \{(x^C, z^{\mathcal{F}_n(C)} \mid C \in \mathcal{CLQ}_n^n\}$, i.e., VERT_n resp. SVERT_n are the characteristic vectors of feasible solutions to QAP resp. SQAP.

We denote by $\mathrm{row}_i^{(n)} := \{(i,j) \in \mathcal{V}_n \mid j = 1, \ldots, n\}$ the i-th row and by $\mathrm{col}_j^{(n)} := \{(i,j) \in \mathcal{V}_n \mid i = 1, \ldots, n\}$ the j-th column of the nodes $\mathcal{V}_n$. The next two theorems provide the desired MIP formulations for QAP resp. SQAP. As usual, for any two disjoint subsets $S, T \subseteq \mathcal{V}_n$, $(S:T)$ is the set of all edges in $\mathcal{E}_n$ having one endpoint in S and the other one in T. For a singleton $\{v\}$, in this as well as in some other contexts, we often omit the brackets and simply write v.

Figures 3 and 4 illustrate the used equations. We draw a hyperedge from $\mathcal{F}_n$ simply by drawing both mates from $\mathcal{E}_n$ belonging to that hyperedge. In all our figures, dashed nodes or (hyper)edges indicate coefficients -1, solid ones stand for $+1$.

Theorem 1. *A vector* $(x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ *is a member of* VERT_n *if, and only if, it satisfies the following conditions:*

$$x(\mathrm{row}_i^{(n)}) = 1 \qquad (i = 1, \ldots, n) \tag{1}$$

$$x(\mathrm{col}_j^{(n)}) = 1 \qquad (j = 1, \ldots, n) \tag{2}$$

$$-x_{(i,j)} + y((i,j) : \mathrm{row}_k^{(n)}) = 0 \qquad (i,j,k = 1, \ldots, n,\ i \neq k) \tag{3}$$

$$-x_{(i,j)} + y((i,j) : \mathrm{col}_l^{(n)}) = 0 \qquad (i,j,l = 1, \ldots, n,\ j \neq l) \tag{4}$$

$$y_e \geq 0 \qquad (e \in \mathcal{E}_n) \tag{5}$$

$$x_v \in \{0,1\} \qquad (v \in \mathcal{V}_n) \tag{6}$$

We make one more notational convention in order to increase the readability of the following equations. For any pair $v, w \in \mathcal{V}_n$ of nodes belonging to the same row or column of $\mathcal{V}_n$, we denote by $\Delta_v^w := \{f \in \mathcal{F}_n \mid v, w \in f\}$ the set of all hyperedges in $\mathcal{F}_n$ containing both v and w (cf. Figure 4).

Theorem 2. *A vector* $(x,z) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ *is a member of* SVERT_n *if, and only if, it* *satisfies the following conditions:*

$$x(\mathrm{row}_i^{(n)}) = 1 \qquad (i = 1, \ldots, n) \tag{7}$$

$$x(\mathrm{col}_j^{(n)}) = 1 \qquad (j = 1, \ldots, n) \tag{8}$$

$$-x_{(i,j)} - x_{(k,j)} + z(\Delta_{(i,j)}^{(k,j)}) = 0 \qquad (i, j, k = 1, \ldots, n, i \neq k) \tag{9}$$

$$-x_{(i,j)} - x_{(i,l)} + z(\Delta_{(i,j)}^{(i,l)}) = 0 \qquad (i, j, l = 1, \ldots, n, j \neq l) \tag{10}$$

$$z_e \geq 0 \qquad (e \in \mathcal{F}_n) \tag{11}$$

$$x_v \in \{0, 1\} \qquad (v \in \mathcal{V}_n) \tag{12}$$

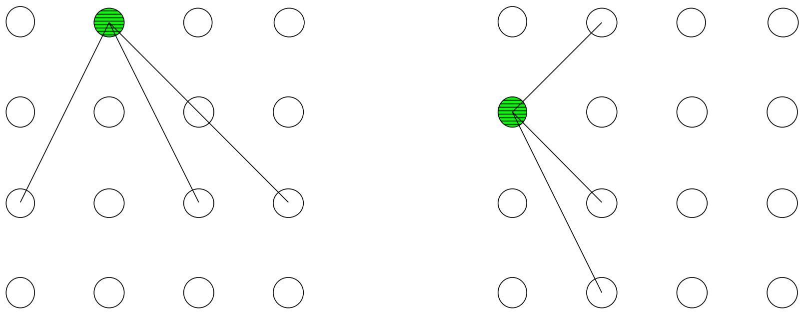


Figure 3: Equations (3) and (4).

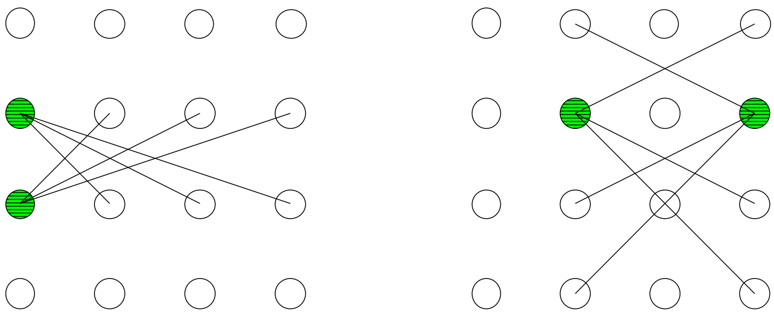


Figure 4: Equations (9) and (10).

Proof of Theorem 1. The "only if" part is clear. To see the other direction, let $(x, y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ satisfy conditions (1), ... ,(6). Obviously, x is the characteristic vector of an n-clique of $\mathcal{G}_n$, and one deduces (e.g., using two equations from (3) and the nonnegativity of y) that $y_{[i,j,k,l]} > 0$ implies $x_{(i,j)} = x_{(k,l)} = 1$. These two facts imply that it is impossible for two components of y belonging to mates to be both non-zero. Observing that $(x, \sigma_n(y))$ satisfies the conditions of Theorem 2, one obtains Theorem 1 from Theorem 2. $\square$

Proof of Theorem 2. Again, the "only if" part is obvious. Let $(x, z) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ satisfy conditions (7), ... ,(12), hence x is the characteristic vector of an n-clique $C \in \mathcal{CLQ}_n^n$.

Considering four appropriate equations from (9) and (10) (and noting the nonnegativity of z), one gets that $z_{\langle i,j,k,l\rangle} > 0$ implies $x_{(i,j)} = x_{(k,l)} = 1$ or $x_{(i,l)} = x_{(k,j)} = 1$. But then, in each of the equations (9) and (10) there is at most one hyperedge involved corresponding to a non-zero component of z. This leads to the fact that $z_{\langle i,j,k,l\rangle} > 0$ implies $z_{\langle i,j,k,l\rangle} = 1$, and that $x_{(i,j)} = x_{(k,l)} = 1$ implies $z_{\langle i,j,k,l\rangle} = 1$. Hence, z must be the characteristic vector of $\mathcal{F}_n(C)$ □

3 The SQAP-Polytope and some Relatives

Theorems 1 and 2 give us the starting points for deriving and exploiting further structural information on the problems QAP and SQAP. As with many other combinatorial optimization problems, the hope is to obtain this by investigating the convex hulls of the sets of feasible solutions to the respective MIP's.

We shall define the *Quadratic Assignment Polytope*

$$\mathcal{QAP}_n := \operatorname{conv}\left(\{(x^C, y^{\mathcal{E}_n(C)}) \mid C \in \mathcal{CLQ}_n^n\}\right)$$

and the *Symmetric Quadratic Assignment Polytope*

$$\mathcal{SQAP}_n := \operatorname{conv}\left(\{(x^C, z^{\mathcal{F}_n(C)}) \mid C \in \mathcal{CLQ}_n^n\}\right).$$

Before starting to consider the connection between these two polytopes, we want to mention the following facts.

Observation 1. *The two polytopes $\mathcal{QAP}_n$ and $\mathcal{SQAP}_n$ are invariant under permutation of the rows, permutation of the columns, and "transposition" of the node set $\mathcal{V}_n$. In particular, for each of the two polytopes, all the cones induced at the vertices are isomorphic.*

For the first one, the QAP-Polytope, investigations were started by Rijal (1995), Padberg and Rijal (1996), and Jünger and Kaibel (1996). There is not much known about the second one, the SQAP-Polytope. Basically, there is only a conjecture of Rijal (1995) and Padberg and Rijal (1996) concerning the dimension of $\mathcal{SQAP}_n$, which we will prove to be valid in Theorem 7.

This paper is concerned with the SQAP-Polytope. However, it turns out that $\mathcal{SQAP}_n$ and $\mathcal{QAP}_n$ are closely related – allthough they are not isomorphic (e.g., we will see that they have different dimensions). The situation is quite similar to the relationship between the *Asymmetric* and the *Symmetric Traveling Salesman Polytope*. While it is difficult to carry over results from the symmetric to the asymmetric case, this is – sometimes – possible for the opposite direction.

Next, we want to explain the relationship between the QAP- and the SQAP-Polytope. Formally, the two polytopes are connected by

$$\mathcal{SQAP}_n = \sigma_n(\mathcal{QAP}_n).$$

(Just consider the vertices to see this.)

We define an inequality (equation) $(u,v)^T(x,y) \leq (=)\omega$ with $(u,v) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ and $\omega \in \mathbb{R}$ to be *symmetric* if, and only if, components of v that belong to mates are equal, i.e., $v_{[i,j,k,l]} = v_{[i,l,k,j]}$ for all $[i,j,k,l] \in \mathcal{E}_n$. A face of $\mathcal{QAP}_n$ is called *symmetric* if there is a symmetric inequality defining that face. Even if a face of $\mathcal{QAP}_n$ is defined by a non-symmetric inequality, it may be symmetric. This is because in general a face is defined by many different inequalities (even in case of a facet, due to the low-dimensionality of $\mathcal{QAP}_n$), but in order to be symmetric it is only required that there exists one among these inequalities which is symmetric.

Let $(u, v)^T(x, y) \leq (=)\omega$ be a symmetric valid inequality (equation) for $\mathcal{QAP}_n$. It induces a valid inequality (equation) $(u, w)^T(x, z) \leq (=)\omega$ for $\mathcal{SQAP}_n$, where $w = \frac{1}{2}\sigma_n(v)$. Conversely, every valid inequality (equation) for $\mathcal{SQAP}_n$ induces a symmetric valid inequality (equation) for $\mathcal{QAP}_n$. From this, we obtain:

Observation 2. *There is a one-to-one correspondence between the symmetric faces of $\mathcal{QAP}_n$ and the faces of $\mathcal{SQAP}_n$. If we identify the faces of $\mathcal{QAP}_n$ and $\mathcal{SQAP}_n$ with the node sets of the cliques corresponding to their vertices then that correspondence is inclusion-preserving.*

This observation translates into the relationship between the face lattices of the QAP- and the SQAP-Polytope.

Theorem 3. *The face lattice of $\mathcal{SQAP}_n$ arises by restricting the face lattice of $\mathcal{QAP}_n$ to the symmetric faces. (Note that $\emptyset$ and $\mathcal{QAP}_n$ itself are symmetric faces of $\mathcal{QAP}_n$.)*

Corollary 1. *A symmetric proper face of $\mathcal{QAP}_n$ induces a facet of $\mathcal{SQAP}_n$ if, and only if, there are only non-symmetric faces strictly between itself and $\mathcal{QAP}_n$ in the face lattice of $\mathcal{QAP}_n$.*

In general, it will be difficult to show that strictly between a certain symmetric face and the whole polytope there are only non-symmetric faces of $\mathcal{QAP}_n$, because it is hard to prove that a set of faces is the complete set of faces containing a given face. However, in the special case that the face under consideration is a *ridge* of $\mathcal{QAP}_n$ (i.e., a face of two dimensions less than the whole polytope), the chances are better, since it is a well-known fact that any ridge is the unique intersection of two facets.

Corollary 2. *If a symmetric ridge of $\mathcal{QAP}_n$ is the intersection of two non-symmetric facets of $\mathcal{QAP}_n$ then it induces a facet of $\mathcal{SQAP}_n$.*

When investigating the structure of a polytope defined as the convex hull of some points more closely, one is very soon confrontated with tasks like computing the rank of a subset of these points or showing that such a subset spans a certain subspace. In both cases, one has to deal with linear combinations of the points, which one hopes to be sparse and to look somehow nice. Working with $\mathcal{QAP}_n$ and $\mathcal{SQAP}_n$, it turns out that such nice combinations are hard to obtain. This is mainly due to the facts that the coordinate vectors of the vertices look all the same up to certain permutations of the coordinates, and that there are no pairs among them having only slightly differing supports.

On the other hand, for both of the polytopes a lot of equations are holding, indicating some redundancy in the problem definition.

This motivated us to try to map the polytopes isomorphically into other spaces (of lower dimensions) in such a way that the coordinate vectors of the resulting vertices have nicer structures.

Let $\mathcal{A} \subset \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ be the affine subspace of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ defined by the equations (1), ... ,(4), i.e.,

$$\mathcal{A} := \left\{ (x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \;\middle|\; \begin{array}{rll} x(\mathrm{row}_i^{(n)}) & = 1 & (i = 1, \dots, n) \\ x(\mathrm{col}_j^{(n)}) & = 1 & (j = 1, \dots, n) \\ -x_{(i,j)} + y((i,j) : \mathrm{row}_k^{(n)}) & = 0 & (i,j,k = 1, \dots, n, i \neq k) \\ -x_{(i,j)} + y((i,j) : \mathrm{col}_l^{(n)}) & = 0 & (i,j,l = 1, \dots, n, j \neq j) \end{array} \right\},$$

and let $\widehat{\mathcal{A}} \subset \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ be the affine subspace of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ defined by the equations (7), ... ,(10), i.e.,

$$\widehat{\mathcal{A}} := \left\{ (x,z) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \;\middle|\; \begin{array}{rll} x(\mathrm{row}_i^{(n)}) & = 1 & (i = 1, \dots, n) \\ x(\mathrm{col}_j^{(n)}) & = 1 & (j = 1, \dots, n) \\ -x_{(i,j)} - x_{(k,j)} + z(\Delta_{(i,j)}^{(k,j)}) & = 0 & (i,j,k = 1, \dots, n, i \neq k) \\ -x_{(i,j)} - x_{(i,l)} + z(\Delta_{(i,j)}^{(i,l)}) & = 0 & (i,j,l = 1, \dots, n, j \neq j) \end{array} \right\}.$$

We will show that in both cases for the affine subspaces defined above all variables corresponding to vertices and edges resp. hyperedges involving the n-th row or the n-th column (the same holds for any row and any column) are redundant in the sense that the projections onto the linear subspaces of the original spaces obtained by setting all these variables to zero produce isomorphic images of these two affine subspaces. Since the two polytopes under consideration are contained in the respective affine subspaces, this implies that these projections yield isomorphic images of the polytopes.

Let $W := \mathrm{row}_n^{(n)} \cup \mathrm{col}_n^{(n)}$, $E := \{e \in \mathcal{E}_n \mid e \cap W \neq \emptyset\}$, and $F := \{f \in \mathcal{F}_n \mid f \cap W \neq \emptyset\}$. Define $\mathcal{U} := \{(x,y) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid x_W = 0, y_E = 0\}$ and $\widehat{\mathcal{U}} := \{(x,z) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \mid x_W = 0, z_F = 0\}$. Let $\pi : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \longrightarrow \mathcal{U}$ be the orthogonal projection onto $\mathcal{U}$, and $\widehat{\pi} : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \longrightarrow \widehat{\mathcal{U}}$ be the orthogonal projection onto $\widehat{\mathcal{U}}$.

Proposition 1. *$\pi(\mathcal{A})$ is affinely isomorphic to $\mathcal{A}$ and $\widehat{\pi}(\widehat{\mathcal{A}})$ is affinely isomorphic to $\widehat{\mathcal{A}}$.*

Proof. We only prove the symmetric part of the proposition. The non-symmetric part can be shown quite similar (cf. Jünger and Kaibel, 1996).

First, we show that there is a way to express the components of points in $\widehat{\mathcal{A}}$ belonging to elements in W and F linearly by the components belonging to elements in $\mathcal{V}_n \setminus W$ and $\mathcal{F}_n \setminus F$.

The first observation is that this is possible for the elements in W using equations of the type $x(\mathrm{row}_i^{(n)}) = 1$ and $x(\mathrm{col}_j^{(n)}) = 1$.

Now, we consider F. Here, it suffices to consider three possibilities for a hyperedge $\langle i,j,k,l \rangle \in F$. The first two are $i,j,k < n$, $l = n$ and $i,j,l < n$, $k = n$. Using $-x_{(i,j)} - x_{(k,j)} + z(\Delta_{(i,j)}^{(k,j)}) = 0$ resp. $-x_{(i,j)} - x_{(i,l)} + z(\Delta_{(i,j)}^{(i,l)}) = 0$, the first two possibilies are done. It remains the possibility that $i,j < n$, $k = n$, $l = n$. Here, we consider (e.g.) $-x_{(i,j)} - x_{(i,n)} + z(\Delta_{(i,j)}^{(i,n)}) = 0$, which allows to express $z_{\langle i,j,n,n \rangle}$ since we can already express $z_{\langle i,j,k,n \rangle}$ for $k < n$.

Up to now, we have shown that there is a linear function $\widehat{\psi} : \mathbb{R}^{\mathcal{V}_n \setminus W} \times \mathbb{R}^{\mathcal{F}_n \setminus F} \longrightarrow \mathbb{R}^W \times \mathbb{R}^F$ such that for all $(x,z) \in \widehat{\mathcal{A}}$ we have $(x_W, z_F) = \widehat{\psi}(x_{\mathcal{V}_n \setminus W}, z_{\mathcal{F}_n \setminus F})$. Hence

$\widehat{\phi} : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ defined via $\widehat{\phi}(x, z) = (x', z')$ with

$$\begin{aligned}(x'_W, z'_F) &:= (x_W, z_F) - \widehat{\psi}(x_{\mathcal{V}_n \setminus W}, z_{\mathcal{F}_n \setminus F}),\\ (x'_{\mathcal{V}_n \setminus W}, z'_{\mathcal{F}_n \setminus F}) &:= (x_{\mathcal{V}_n \setminus W}, z_{\mathcal{F}_n \setminus F})\end{aligned}$$

is an affine transformation (note that the corresponding matrix is an upper triangular one having ones everywhere on the main diagonal) of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ that induces on $\widehat{\mathcal{A}}$ the orthogonal projection onto $\widehat{\mathcal{U}}$. □

We identify the linear spaces $\mathcal{U}$ and $\widehat{\mathcal{U}}$ with the spaces $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$ resp. $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{F}_{n-1}}$. Hence,

$$\mathcal{QAP}^{\star}_{n-1} := \pi(\mathcal{QAP}_n) \subset \mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$$

is a polytope in $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{E}_{n-1}}$ that is isomorphic to $\mathcal{QAP}_n$, and

$$\mathcal{SQAP}^{\star}_{n-1} := \widehat{\pi}(\mathcal{SQAP}_n) \subset \mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{F}_{n-1}}$$

is a polytope in $\mathbb{R}^{\mathcal{V}_{n-1}} \times \mathbb{R}^{\mathcal{F}_{n-1}}$ that is isomorphic to $\mathcal{SQAP}_n$.

Since the vertices of these two polytopes arise as the projections of the vertices of the two original polytopes, one obtains that they are the respective characteristic vectors of the $(n-1)$- and the $(n-2)$-cliques of $\mathcal{G}_{n-1}$ (cf. Figure 5).

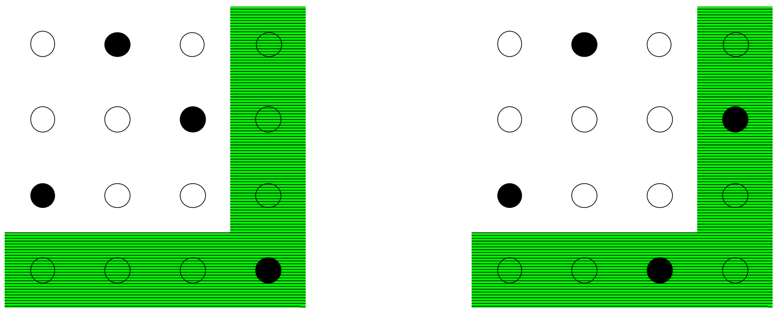


Figure 5: The effect of the projection.

We want to make the isomorphisms between $\mathcal{QAP}_n$ and $\mathcal{QAP}^{\star}_{n-1}$ as well as the one between $\mathcal{SQAP}_n$ and $\mathcal{SQAP}^{\star}_{n-1}$ a little more explicit. We denote by $\kappa : \mathcal{CLQ}^n_n \longrightarrow \mathcal{CLQ}^{n-1}_{n-1} \cup \mathcal{CLQ}^{n-1}_{n-2}$ the map defined by removing from a given n-clique in $\mathcal{G}_n$ the node(s) in the n-th row and in the n-th column. Notice that κ is one-to-one.

Remark 1. *If two faces of $\mathcal{QAP}_n$ and $\mathcal{QAP}^{\star}_{n-1}$ resp. $\mathcal{SQAP}_n$ and $\mathcal{SQAP}^{\star}_{n-1}$ correspond to each other with respect to the isomorphism induced by π resp. $\widehat{\pi}$ then their vertices (identified with cliques) correspond to each other by the bijection κ.*

This remark describes the relationship between the faces from the "inner view", i.e., in terms of the vertices. Next, we want to describe the "outer relationship", i.e., the relationship between inequalities defining corresponding faces.

Remark 2.

(i) If a face of $\mathcal{QAP}_n$ resp. $\mathcal{SQAP}_n$ is defined by an inequality that has zero-coefficients for all elements in $W \cup E$ resp. $W \cup F$ then an inequality defining the corresponding face of $\mathcal{QAP}^\star_{n-1}$ resp. $\mathcal{SQAP}^\star_{n-1}$ is obtained by projecting the coefficient vector of that inequality via π resp. $\widehat{\pi}$. (Note that for every face of $\mathcal{QAP}_n$ resp. $\mathcal{SQAP}_n$ there is a defining inequality having zero coefficients at W and E resp. F. This is due to the fact that the columns of the equation system defining the affine subspace $\mathcal{A}$ resp. $\widehat{\mathcal{A}}$ corresponding to $W \cup E$ resp. $W \cup F$ are linearly independent, as shown in the proof of Proposition 1.)

(ii) From every inequality defining a face of $\mathcal{QAP}^\star_{n-1}$ resp. $\mathcal{SQAP}^\star_{n-1}$ one obtains an inequality defining the corresponding face of $\mathcal{QAP}_n$ resp. $\mathcal{SQAP}_n$ by zero-lifting.

Up to now, we have just considered the relationship between $\mathcal{QAP}_n$ and $\mathcal{QAP}^\star_{n-1}$ as well as the one between $\mathcal{SQAP}_n$ and $\mathcal{SQAP}^\star_{n-1}$. However, $\mathcal{QAP}_n$ and $\mathcal{QAP}^\star_n$ resp. $\mathcal{SQAP}_n$ and $\mathcal{SQAP}^\star_n$ are lying in the same space $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n}$ resp. $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$. It turns out (cf. Jünger and Kaibel, 1996) that $\mathcal{QAP}^\star_n$ decomposes (with respect to taking the convex hull) into $n+1$ faces that are each isomorphic to $\mathcal{QAP}_n$. Using the isomorphism between $\mathcal{QAP}^\star_n$ and $\mathcal{QAP}_{n+1}$ one deduces that $\mathcal{QAP}_{n+1}$ decomposes into $n+1$ "isomorphic copies" of $\mathcal{QAP}_n$. The next two theorems establish corresponding results for the symmetric case.

Theorem 4. *Let $n \geq 2$. There are $n+1$ affine maps $\phi_\alpha : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ $(\alpha = 0, \ldots, n)$ such that for the $n+1$ images $\mathcal{Q}_\alpha := \phi_\alpha(\mathcal{SQAP}_n)$ $(\alpha = 0, \ldots, n)$ of $\mathcal{SQAP}_n$ the following hold:*

(i) Every $\mathcal{Q}_\alpha$ is isomorphic to $\mathcal{SQAP}_n$.

(ii) Each $\mathcal{Q}_\alpha$ is a face of $\mathcal{SQAP}^\star_n$.

(iii) The $\mathcal{Q}_\alpha$ have pairwise empty intersection.

(iv) $\mathcal{SQAP}^\star_n = \operatorname{conv}\left(\bigcup_{\alpha=0}^n \mathcal{Q}_\alpha\right)$

Proof. For any row or column $S \in \{\mathrm{row}_1^{(n)}, \ldots, \mathrm{row}_n^{(n)}, \mathrm{col}_1^{(n)}, \ldots, \mathrm{col}_n^{(n)}\}$, let $\Gamma(S) := \{f \in \mathcal{F}_n \mid f \cap S \neq \emptyset\}$, and let $\widehat{\pi}^S : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \longrightarrow \{(x, z) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \mid x_S = 0, z_{\Gamma(S)} = 0\}$ be the canonical projection. Then, the map $\widehat{\pi}$ decomposes into $\widehat{\pi} = \widehat{\pi}^{\mathrm{col}_n^{(n)}} \circ \widehat{\pi}^{\mathrm{row}_n^{(n)}}$. Since we know that $\widehat{\pi}$ performs an isomorphic transformation of $\mathcal{SQAP}_n$, so does $\widehat{\pi}^{\mathrm{row}_n^{(n)}}$, too. Clearly, there is nothing special about $\mathrm{row}_n^{(n)}$, and therefore, the same holds for all $\phi_\alpha := \widehat{\pi}^{\mathrm{row}_\alpha^{(n)}}$ $(\alpha = 1, \ldots, n)$. Finally, define ϕ_0 to be the identical map on $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$. Hence, all $\mathcal{Q}_\alpha := \phi_\alpha(\mathcal{SQAP}_n)$ $(\alpha = 0, \ldots, n)$ are isomorphic to $\mathcal{SQAP}_n$, what proves part *(i)*. Parts *(ii)*, *(iii)*, and *(iv)* follow from the observation, that for any $\alpha \in \{1, \ldots, n\}$, the vertices of $\mathcal{Q}_\alpha$ correspond to the $(n-1)$-cliques of $\mathcal{G}_n$ having no node in common with the α-th row of $\mathcal{V}_n$. $\square$

From this theorem and the isomorphism between $\mathcal{SQAP}_{n+1}$ and $\mathcal{SQAP}^\star_n$, the following "inductive construction" of $\mathcal{SQAP}_{n+1}$ follows. It establishes a kind of "self-similarity" that shows another symmetry of the SQAP-Polytope.

Theorem 5. *For $n \geq 1$ there are $n+1$ affine maps $\iota_\alpha : \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \longrightarrow \mathbb{R}^{\mathcal{V}_{n+1}} \times \mathbb{R}^{\mathcal{F}_{n+1}}$ $(\alpha = 0, \ldots, n)$ such that for the $n+1$ images $\mathcal{Q}_\alpha := \iota_\alpha(\mathcal{SQAP}_n)$ $(\alpha = 0, \ldots, n)$ of $\mathcal{SQAP}_n$ the following hold:*

(i) Every $\mathcal{Q}_\alpha$ is isomorphic to $\mathcal{SQAP}_n$.

(ii) Each $\mathcal{Q}_\alpha$ is a face of $\mathcal{SQAP}_{n+1}$.

(iii) The $\mathcal{Q}_\alpha$ have pairwise empty intersection.

(iv) $\mathcal{SQAP}_{n+1} = \operatorname{conv}\left(\bigcup_{\alpha=0}^{n} \mathcal{Q}_\alpha\right)$

We want to close this section by formally establishing "star-analogons" to some facts observed for $\mathcal{QAP}_n$ and $\mathcal{SQAP}_n$. First, as for the "non-star polytopes", also the "star-polytopes" are invariant under permutations of rows, permutations of columns, or "transposition" of the node set $\mathcal{V}_n$. Second, as in the relationship between $\mathcal{QAP}_n$ and $\mathcal{SQAP}_n$, by identifying mates any symmetric inequality (equation) for $\mathcal{QAP}_n^\star$ gives rise to an inequality (equation) for $\mathcal{SQAP}_n^\star$, and any inequality (equation) for $\mathcal{SQAP}_n^\star$ gives rise to a symmetric inequality (equation) for $\mathcal{QAP}_n^\star$.

Theorem 6. *The face lattice of $\mathcal{SQAP}_n^\star$ arises by restricting the face lattice of $\mathcal{QAP}_n^\star$ to the symmetric faces.*

Corollary 3. *A symmetric proper face of $\mathcal{QAP}_n^\star$ induces a facet of $\mathcal{SQAP}_n^\star$ if, and only if, there are only non-symmetric faces strictly between itself and $\mathcal{QAP}_n^\star$ in the face lattice of $\mathcal{QAP}_n^\star$.*

Corollary 4. *If a symmetric ridge of $\mathcal{QAP}_n^\star$ is the intersection of two non-symmetric facets of $\mathcal{QAP}_n^\star$ then it induces a facet of $\mathcal{SQAP}_n^\star$.*

4 Dimension and Trivial Facets of $\mathcal{SQAP}_n$

In this section, we will present some basic results concerning the facial structure of the SQAP-Polytope. First, we examine two sets of equations that will turn out to describe the affine hulls of $\mathcal{QAP}_n^\star$ resp. $\mathcal{SQAP}_n^\star$. For this, we make another notational convention. For two disjoint subsets $S, T \subset \mathcal{V}_n$, $S \cap T = \emptyset$, we define $\langle S : T \rangle := \{\{v, w\} \cup \tau(\{v, w\}) \mid \{v, w\} \in (S : T)\}$. Remembering that the vertices of both $\mathcal{QAP}_n^\star$ as well as $\mathcal{SQAP}_n^\star$ correspond to the n- and $(n-1)$-cliques of $\mathcal{G}_n$, one verifies that

$$x(\mathrm{row}_i^{(n)}) + x(\mathrm{row}_k^{(n)}) - y(\mathrm{row}_i^{(n)} : \mathrm{row}_k^{(n)}) = 1 \qquad (i < k) \tag{13}$$

and

$$x(\mathrm{col}_j^{(n)}) + x(\mathrm{col}_l^{(n)}) - y(\mathrm{col}_j^{(n)} : \mathrm{col}_l^{(n)}) = 1 \qquad (j < l) \tag{14}$$

are valid for $\mathcal{QAP}_n^\star$, and

$$x(\mathrm{row}_i^{(n)}) + x(\mathrm{row}_k^{(n)}) - z(\langle \mathrm{row}_i^{(n)} : \mathrm{row}_k^{(n)} \rangle) = 1 \qquad (i < k) \tag{15}$$

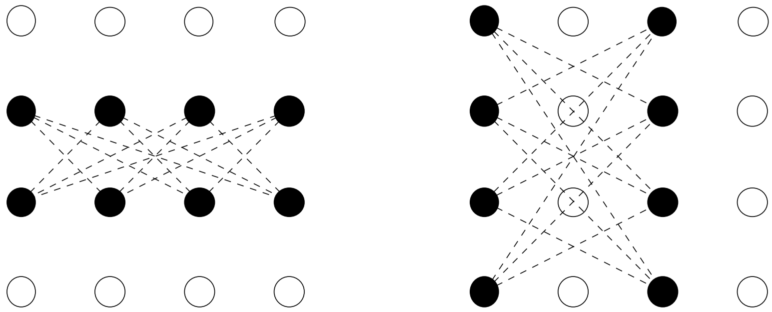


Figure 6: The equations (13), (15) and (14), (16).

and

$$x(\mathrm{col}_j^{(n)}) + x(\mathrm{col}_l^{(n)}) - z(\langle \mathrm{col}_j^{(n)} : \mathrm{col}_l^{(n)} \rangle) = 1 \qquad (j < l) \tag{16}$$

hold for $\mathcal{SQAP}_n^\star$ (cf. Figure 6).
We denote the system (13),(14) by $D(x, y) = d$ and the system (15),(16) by $\widehat{D}(x, z) = \widehat{d}$.

By saying that $\langle i, j, k, l\rangle$ ($i < k, j < l$) is smaller than $\langle i', j', k', l'\rangle$ ($i' < k', j' < l'$) if, and only if, (i, k, j, l) is lexicographically smaller than (i', k', j', l'), we introduce an ordering on the hyperedges $\mathcal{F}_n$. After permutation of the columns with respect to this order the matrix $\widehat{D}$ has the following shape ($n = 4$):

$$\begin{pmatrix}
1&1&1&1&&&&&&&&&&&&\\
&&&&1&1&1&1&&&&&&&&\\
&&&&&&&&1&1&1&1&&&&\\
&&&&&&&&&&&&1&1&1&1\\
1&&&&1&&&&1&&&&1&&&\\
&1&&&&1&&&&1&&&&1&&\\
&&1&&&&1&&&&1&&&&1&\\
&&&1&&&&1&&&&1&&&&1
\end{pmatrix}.$$

But this is the node-edge-incidence matrix of the complete bipartite graph $K_{\frac{n(n-1)}{2},\frac{n(n-1)}{2}}$, where the left shore corresponds to the (unordered) pairs of rows, and the right shore corresponds to the (unordered) pairs of columns of $\mathcal{V}_n$. The bases of the node-edge-incidence matrix of $K_{m,m}$ are well known to correspond to the spanning trees of $K_{m,m}$ (Balinski and Russakoff, 1974). This leads to the following characterization of all bases of $\widehat{D}$ that do not intersect the "x-part" of $\widehat{D}$.

Proposition 2.

(i) Precisely one (arbitrary) equation in $\widehat{D}(x, z) = \widehat{d}$ *is redundant, in particular* $\mathrm{rank}(\widehat{D}) = n(n-1) - 1$.

(ii) A subset $B \subseteq \mathcal{F}_n$ *of hyperedges corresponds to a basis of* $\widehat{D}$ *if, and only if,*

(a) $|B| = n(n-1) - 1$

(b) There is no sequence $(f_0, f'_0, f_1, f'_1, \ldots, f_{k-1}, f'_{k-1})$ $(k \geq 2)$ of hyperedges in B such that f_α and f'_α connect the same rows of $\mathcal{V}_n$ and f'_α and $f_{(\alpha+1) \bmod k}$ connect the same columns of $\mathcal{V}_n$ for all $\alpha = 0, \ldots, k-1$.

In Jünger and Kaibel (1996) we showed that $D(x,y) = d$ is a complete equation system for $\mathcal{QAP}^\star_n$. But the system $D(x,y) = d$ consists only of symmetric equations. Hence, we can deduce that $\widehat{D}(x,z) = \widehat{d}$ must be a complete system of equations for $\mathcal{SQAP}^\star_n$, since the equations for $\mathcal{SQAP}^\star_n$ correspond precisely to the symmetric equations for $\mathcal{QAP}^\star_n$. (In fact, one can deduce the "completeness" of $\widehat{D}(x,z) = \widehat{d}$ also from the proof of Theorem 9.)

Consequently, the dimension of $\mathcal{SQAP}^\star_n$ is $n^2 + \frac{n^2(n-1)^2}{4} - (n(n-1)-1)$. By the isomorphism between $\mathcal{SQAP}_n$ and $\mathcal{SQAP}^\star_{n-1}$, one obtains the following theorem.

Theorem 7.

$$\dim(\mathcal{SQAP}_n) = (n-1)^2 + \frac{(n-1)^2(n-2)^2}{4} - ((n-1)(n-2)-1)$$

Rijal (1995) and Padberg and Rijal (1996) proved that the rank of the system (7), ...,(10) equals $(n-1)^2 + \frac{n^2(n-3)^2}{4}$ (which is equal to $(n-1)^2 + \frac{(n-1)^2(n-2)^2}{4} - ((n-1)(n-2)-1)$) and conjectured that this might be the dimension of $\mathcal{SQAP}_n$. Theorem 7 proves this conjecture. Moreover, knowing that the rank of this system equals $\dim(\mathcal{SQAP}_n)$, one can even conclude that the system (7), ...,(10) describes the affine hull of $\mathcal{SQAP}_n$. In addition, we want to give another simple proof that does not compute the rank of the system explicitely.

Theorem 8.

$$\operatorname{aff}(\mathcal{SQAP}_n) = \{(x,z) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n} \mid (x,z) \textit{ satisfies } (7), \ldots, (10)\}$$

Proof. It suffices to show that one can linearly combine the zero-liftings of the equations (15) and (16) (for $n-1$) from the equations (7), ...,(10) (for n), since then it is clear that the solution space of (7), ...,(10) for n – which is $\widehat{\mathcal{A}}$ (containing $\mathcal{SQAP}_n$) – is mapped isomorphically (cf. Proposition 1) by the projection $\widehat{\pi}$ into the solution space of (15), (16) for $n-1$, which we know from our considerations to have the same dimension as $\mathcal{SQAP}_n$.

Hence, by symmetry arguments, it suffices to exhibit a linear combination of (7), ...,(10) that yields

$$x(\mathrm{row}_1^{(n)} \setminus \{(1,n)\}) + x(\mathrm{row}_2^{(n)} \setminus \{(2,n)\}) - z(\langle \mathrm{row}_1^{(n)} \setminus \{(1,n)\} : \mathrm{row}_2^{(n)} \setminus \{(2,n)\}\rangle) = 1.$$

But this is obtained by adding $x(\mathrm{row}_1^{(n)}) = 1$, $x(\mathrm{row}_2^{(n)}) = 1$, $x_{(1,j)} + x_{(2,j)} - z(\Delta^{(2,j)}_{(1,j)}) = 0$ for all $1 \leq j \leq n-1$, and $-x_{(1,n)} - x_{(2,n)} + z(\Delta^{(2,n)}_{(1,n)}) = 0$, and finally dividing the resulting equation by 2. □

We just mention that the system (1), ...,(4) describes $\operatorname{aff}(\mathcal{QAP}_n)$ (cf. Rijal, 1995; Padberg and Rijal, 1996; Jünger and Kaibel, 1996).

There is another nice gain when changing to the "star-polytopes". We pointed out in Corollary 1 that it is of interest to know that certain faces of the QAP-Polytope are non-symmetric. As mentioned above, this might be not directly seen, since a symmetric face of $\mathcal{QAP}_n$ can be defined by a non-symmetric inequality. However, this is much easier for $\mathcal{QAP}^\star_n$.

Observation 3. *Due to the fact that all equations holding for $\mathcal{QAP}_n^\star$ are symmetric, in order to show that a given face of $\mathcal{QAP}_n^\star$ is non-symmetric, it suffices to exhibit any non-symmetric inequality defining it.*

For the non-symmetric QAP-Polytope, the nonnegativity constraints on y define facets, while $0 \leq x \leq 1$ and $y \leq 1$ are already implied by $D(x, y) = d$ and $y \geq 0$ (cf. Rijal, 1995; Padberg and Rijal, 1996; Jünger and Kaibel, 1996). For the SQAP-Polytope, the situation is a little bit different:

Theorem 9. *Let $n \geq 3$.*

(i) The nonnegativity constraints $x \geq 0$ and $z \geq 0$ define facets of $\mathcal{SQAP}_n$.

(ii) The upper bounds $x \leq 1$ and $z \leq 1$ are implied by the equations (7), ... ,(10) and $x \geq 0$, $z \geq 0$.

Proof. Part *(ii)* follows from the observation that the equations (1) and (2) together with the nonnegativity of x imply that the sum of any two x-variables must be less than or equal to one.

To show part *(i)*, it suffices to prove that $x \geq 0$ and $z \geq 0$ define facets of $\mathcal{SQAP}_n^\star$ (for all $n \geq 2$). We will show this only for $n \geq 5$, since this simplifies the proof. However, the claim is also true for $n = 2, 3, 4$, as one may check by computer, for instance.

At this point, we introduce some techniques which we will refer to also in later proofs. Our usual way to prove that some inequality defines a facet of $\mathcal{SQAP}_n^\star$ is an indirect one. We denote by $L \subseteq \mathcal{CLQ}_n^n \cup \mathcal{CLQ}_{n-1}^n$ the set of cliques corresponding to the vertices of the considered face and by $\mathcal{L} := \{(x^C, z^{\mathcal{F}_n(C)}) - (x^{C'}, z^{\mathcal{F}_n(C')}) \mid C, C' \in L\}$ the set of all difference vectors of vertices of that face, i.e., $\text{lin}(\mathcal{L})$ is the subvectorspace belonging to the affine hull of the face. We choose a subset $B \subset \mathcal{F}_n$ that corresponds to a basis of the equation system $\widehat{D}(x, z) = \widehat{d}$ as well as one extra element $v_0 \in \mathcal{V}_n$ or $f_0 \in \mathcal{F}_n \setminus B$. Setting $\mathcal{B} := \{x^{v_0}\} \cup \{z^f \mid f \in B\}$ resp. $\mathcal{B} := \{z^{f_0}\} \cup \{z^f \mid f \in B\}$, and providing that the face is a proper one, it remains to show that $\text{lin}(\mathcal{L} \cup \mathcal{B}) = \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$, since this implies that the dimension of $\text{lin}(\mathcal{L})$, which equals the dimension of the face, is at least $\dim(\mathcal{SQAP}_n^\star) - 1$. We show $\text{lin}(\mathcal{L} \cup \mathcal{B}) = \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ by successively combining the canonical unit vectors of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ from elements in $\mathcal{L} \cup \mathcal{B}$.

For constructing the necessary linear combinations, the following two lemmata are useful. For a subset $S \subseteq \mathcal{V}_n$ we denote by $\mathcal{H}_n/S = (\mathcal{V}_n/S, \mathcal{F}_n/S)$ the hypergraph obtained from $\mathcal{H}_n$ by deleting all nodes lying in a common row or column with a node in S and all hyperedges involving such nodes. Note that – if S intersects the same number of rows as of column – $\mathcal{H}_n/S$ is isomorphic to an $\mathcal{H}_k$ for some $k \leq n$.

Lemma 1. *Let $C \in \mathcal{CLQ}_n^n$ be an n-clique and $v \in C$ a node in C such that $C, C \setminus \{v\} \in L$. Then we have*

$$x^v + z^{\langle v:C\setminus\{v\}\rangle} \in \text{lin}(\mathcal{L}).$$

Proof of Lemma 1. This is due to $x^v + z^{\langle v:C\setminus\{v\}\rangle} = (x^C, z^{\mathcal{F}_n(C)}) - (x^{C\setminus\{v\}}, z^{\mathcal{F}_n(C\setminus\{v\})}) \in \text{lin}(\mathcal{L})$. □

Lemma 2. *Let $1 \leq r, r_1, r_2 \leq n$ be pairwise distinct, and let $1 \leq c, c_1, c_2 \leq n$ be pairwise distinct. If there is an $(n-3)$-clique C in $\mathcal{H}_n / \{(r_1, r_2), (r, c), (r_2, c_2)\}$ such that*

$$\{(r_1, c_1), (r, c), (r_2, c_2)\} \cup C, \{(r_1, c_2), (r, c), (r_2, c_1)\} \cup C, \\ \{(r_1, c_1), (r_2, c_2)\} \cup C, \{(r_1, c_2), (r_2, c_1)\} \cup C \in L \tag{17}$$

or

$$\{(r_1, c), (r, c_2)\} \cup C, \{(r, c_2), (r_2, c)\} \cup C, \{(r_2, c), (r, c_1)\} \cup C, \{(r, c_1), (r_1, c)\} \cup C \in L \tag{18}$$

then

$$z^{\langle r_1, c_1, r, c \rangle} + z^{\langle r, c, r_2, c_2 \rangle} - z^{\langle r_1, c_2, r, c \rangle} - z^{\langle r, c, r_2, c_1 \rangle} \in \operatorname{lin}(\mathcal{L})$$

(cf. Figure 7).

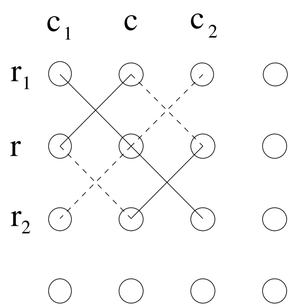


Figure 7: Notations of Lemma 2.

Proof of Lemma 2. In the first case, observe that

$$z^{\langle r_1, c_1, r, c \rangle} + z^{\langle r, c, r_2, c_2 \rangle} - z^{\langle r_1, c_2, r, c \rangle} - z^{\langle r, c, r_2, c_1 \rangle} = \\ z^{\{(r_1,c_1),(r,c),(r_2,c_2)\} \cup C} - z^{\{(r_1,c_1),(r_2,c_2)\} \cup C} - z^{\{(r_1,c_2),(r,c),(r_2,c_1)\} \cup C} + z^{\{(r_1,c_2),(r_2,c_1)\} \cup C} \in \operatorname{lin}(\mathcal{L}).$$

For the second case, we have

$$z^{\langle r_1, c_1, r, c \rangle} + z^{\langle r, c, r_2, c_2 \rangle} - z^{\langle r_1, c_2, r, c \rangle} - z^{\langle r, c, r_2, c_1 \rangle} = \\ - z^{\{(r_1,c),(r,c_2)\} \cup C} + z^{\{(r,c_2),(r_2,c)\} \cup C} - z^{\{(r_2,c),(r,c_1)\} \cup C} + z^{\{(r,c_1),(r_1,c)\} \cup C} \in \operatorname{lin}(\mathcal{L}).$$

$\square$

Now, we proceed with the proof of Theorem 9. First, note that all trivial inequalities define proper faces of $\mathcal{SQAP}_n^\star$. To show that the nonnegativity constraints on x define facets of $\mathcal{SQAP}_n^\star$, it suffices to show this for $x_{(n,n)} \geq 0$. Hence, L consists of all n- and $(n-1)$-cliques of $\mathcal{H}_n$ that do not contain (n, n). We choose $B := \langle \mathrm{row}_1^{(n)} : \mathrm{row}_2^{(n)} \rangle \cup \langle \mathrm{col}_1^{(n)} : \mathrm{col}_2^{(n)} \rangle$ (cf. Proposition 2) and the extra element as $v_0 := (n, n)$.

Since in $\mathcal{H}_k$ there is always a k-clique not involving a prescribed node as long as $k \geq 2$, we can apply Lemma 2 for every choice of r, r_1, r_2, c, c_1, c_2 (recall that we assume $n \geq 5$). We combine all canonical unit vectors in $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ successively in five steps that are illustrated in Figure 8. For a number $a \in \{1, 2\}$, we denote by $\overline{a}$ the number with $\{\overline{a}\} = \{1, 2\} \setminus \{a\}$.

Step 1: $z^{\langle i,j,k,l\rangle} \in \text{lin}(\mathcal{L} \cup \mathcal{B})$ for $i, j \in \{1, 2\}$

The case $k \in \{1, 2\}$ or $l \in \{1, 2\}$ is already clear by the choice of B. Hence, assume $k, l \notin \{1, 2\}$. Choosing $r := i$, $r_1 := \bar{i}$, $r_2 := k$, $c := j$, $c_1 := \bar{j}$, and $c_2 := l$ Lemma 2 yields $z^{\langle \bar{i},\bar{j},i,j\rangle} + z^{\langle i,j,k,l\rangle} - z^{\langle \bar{i},l,i,j\rangle} - z^{\langle i,j,k,\bar{j}\rangle} \in \text{lin}(\mathcal{L})$. Since all involved unit vectors but $z^{\langle i,j,k,l\rangle}$ are in $\mathcal{B}$, we are done.

Step 2: $z^{\langle i,j,k,l\rangle} \in \text{lin}(\mathcal{L} \cup \mathcal{B})$ for $i \in \{1, 2\}, j, k, l \geq 3$

With $r := i$, $r_1 := \bar{i}$, $r_2 := k$, $c := j$, $c_1 := 1$, $c_2 := l$ one obtains from Lemma 2 that $z^{\langle \bar{i},1,i,j\rangle} + z^{\langle i,j,k,l\rangle} - z^{\langle \bar{i},l,i,j\rangle} - z^{\langle i,j,k,1\rangle} \in \text{lin}(\mathcal{L})$. All involved unit vectors but $z^{\langle i,j,k,l\rangle}$ are either in $\mathcal{B}$ or already shown to be in $\text{lin}(\mathcal{L} \cup \mathcal{B})$ in Step 1.

Step 3: $z^{\langle i,j,k,l\rangle} \in \text{lin}(\mathcal{L} \cup \mathcal{B})$ for $j \in \{1, 2\}, i, k, l \geq 3$

This is done analogously to Step 2.

Step 4: $z^{\langle i,j,k,l\rangle} \in \text{lin}(\mathcal{L} \cup \mathcal{B})$ **for** $i, j, k, l \geq 3$

This time, we choose $r := i$, $r_1 := 1$, $r_2 := k$, $c := j$, $c_1 := 1$, and $c_2 := l$. Lemma 2 gives $z^{\langle 1,1,i,j\rangle} + z^{\langle i,j,k,l\rangle} - z^{\langle 1,l,i,j\rangle} - z^{\langle i,j,k,1\rangle} \in \text{lin}(\mathcal{L})$, which proves the claim, since all involved unit vectors but $z^{\langle i,j,k,l\rangle}$ are already shown to be in $\text{lin}(\mathcal{L} \cup \mathcal{B})$ in Step 1, 2, or 3.

Step 5: $x^v \in \text{lin}(\mathcal{L} \cup \mathcal{B})$ for all $v \in \mathcal{V}_n$

If $v = (n, n)$, we are done since $x^{(n,n)} \in \mathcal{B}$. So assume, $v \neq (n, n)$. Let $C \in \mathcal{CLQ}_n^n$ be any n-clique involving v but not (n, n). Using Lemma 1, we can combine x^v, since all unit vectors corresponding to hyperedges are already known to be in $\text{lin}(\mathcal{L} \cup \mathcal{B})$.

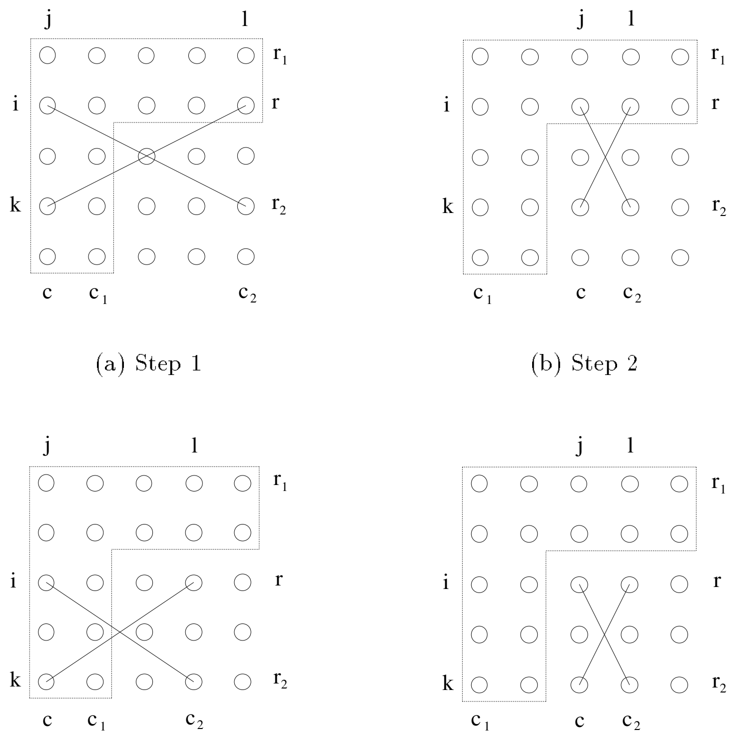
It remains to show that $z \geq 0$ define facets of $\mathcal{SQAP}_n^\star$. It suffices to show this for $z_{\langle n,n-1,n-1,n\rangle} \geq 0$. Now, L is the set of all n- and $(n-1)$-cliques of $\mathcal{H}_n$ that contain at most one node from $\{(n, n-1), (n-1, n), (n-1, n-1), (n, n)\}$. Note that it is always possible to find a k-clique in $\mathcal{H}_k$ that intersects $\{(k, k-1), (k-1, k), (k-1, k-1), (k, k)\}$ in at most one node as long as $k \geq 3$.

We choose B as above, and as the extra element, we take the hyperedge $\langle n, n-1, n-1, n\rangle$. Then, Steps 1, 2, and 3 work analogously. The only case in which Step 4 does not work is the case of the hyperedge $\langle n, n-1, n-1, n\rangle$, but this time this one is covered by the extra element. In Step 5, now we do not need an extra element anymore, since we can extend every node (also one from $\{(n, n-1), (n-1, n), (n-1, n-1), (n, n)\}$), to an n-clique not containing more than one node from $\{(n, n-1), (n-1, n), (n-1, n-1), (n, n)\}$. $\square$

There is an alternative way of proving that the nonnegativity constraints $z \geq 0$ define facets of $\mathcal{SQAP}_n^\star$. In Jünger and Kaibel (1996) we showed that $y \geq 0$ define facets of $\mathcal{QAP}_n^\star$. By a slight modification of that proof, one can show that $y_e + y_{\tau(e)} \geq 0$ defines a ridge of $\mathcal{QAP}_n^\star$ for any edge $e \in \mathcal{E}_n$. Since that symmetric ridge is the intersection of the two non-symmetric (cf. Observation 3) facets defined by $y_e \geq 0$ and $y_{\tau(e)} \geq 0$, the claim follows from Corollary 4.

5 The Curtain Facets

For any subset $S \subseteq \{1, \dots, n\}$, we define for $i \in \{1, \dots, n\}$ the *restriction of* $\text{row}_i^{(n)}$ *to* S as $\text{row}_i^{(n)}|_S := \{(i, j) \in \text{row}_i^{(n)} \mid j \in S\}$, and for $j \in \{1, \dots, n\}$, we define $\text{col}_j^{(n)}|_S :=$



(a) Step 1 (b) Step 2

(c) Step 3 (d) Step 4

Figure 8: Examples for the hyperedges considered in Steps 1–4 of the proof of Theorem 9. The hyperedges inside the "angled box" are those forming the set B.

$\{(i,j) \in \mathrm{col}_j^{(n)} \mid i \in S\}$ to be the *restriction of* $\mathrm{col}_j^{(n)}$ *to* S.

One immediately verifies that the *row curtain inequalities*

$$-x(\mathrm{row}_i^{(n)}|_S) + z(\langle \mathrm{row}_i^{(n)}|_S : \mathrm{row}_k^{(n)}|_S \rangle) \leq 0 \qquad (i \neq k, S \subseteq \{1,\dots,n\}) \tag{19}$$

and the *column curtain inequalities*

$$-x(\mathrm{col}_j^{(n)}|_S) + z(\langle \mathrm{col}_j^{(n)}|_S : \mathrm{col}_l^{(n)}|_S \rangle) \leq 0 \qquad (j \neq l, S \subseteq \{1,\dots,n\}) \tag{20}$$

are valid for $\mathcal{SQAP}_n$ (cf. Figure 9).

These inequalities dominate the inequalities

$$-x(\mathrm{row}_i^{(n)}|_S) + z(\langle (i,j) : \mathrm{row}_k^{(n)}|_S \rangle) \leq 0 \qquad (i \neq k, S \subseteq \{1,\dots,n\}, j \in S) \tag{21}$$

and

$$-x(\mathrm{col}_j^{(n)}|_S) + z(\langle (i,j) : \mathrm{col}_l^{(n)}|_S \rangle) \leq 0 \qquad (j \neq l, S \subseteq \{1,\dots,n\}, i \in S) \tag{22}$$

proposed by Rijal (1995) and Padberg and Rijal (1996).

We first address the question whether all curtain inequalities define distinct faces of $\mathcal{SQAP}_n$.

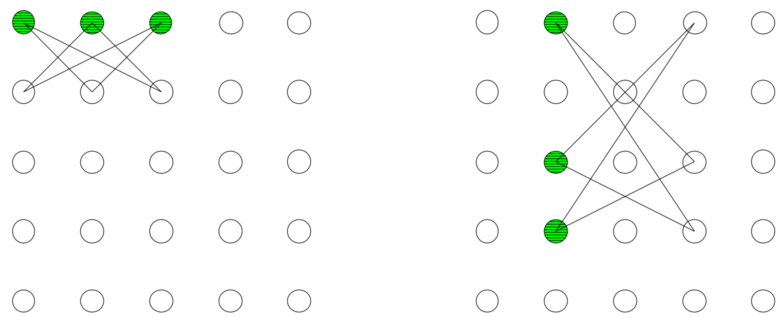


Figure 9: The curtain inequalities.

For any subset $S \subseteq \{1, \dots, n\}$, we denote by $\overline{S} := \{1, \dots, n\} \setminus S$ the complement of S. Then, the equations

(23)
$$x(\mathrm{row}_i^{(n)}|_S) - z(\langle \mathrm{row}_i^{(n)}|_S : \mathrm{row}_k^{(n)}|_S\rangle) - x(\mathrm{row}_k^{(n)}|_{\overline{S}}) + z(\langle \mathrm{row}_k^{(n)}|_{\overline{S}} : \mathrm{row}_i^{(n)}|_{\overline{S}}\rangle) = 0 \quad (i \neq k)$$

and

(24)
$$x(\mathrm{col}_j^{(n)}|_S) - z(\langle \mathrm{col}_j^{(n)}|_S : \mathrm{col}_l^{(n)}|_S\rangle) - x(\mathrm{col}_l^{(n)}|_{\overline{S}}) + z(\langle \mathrm{col}_l^{(n)}|_{\overline{S}} : \mathrm{col}_j^{(n)}|_{\overline{S}}\rangle) = 0 \qquad (j \neq l)$$

are valid for $\mathcal{SQAP}_n$ (cf. Figure 10).

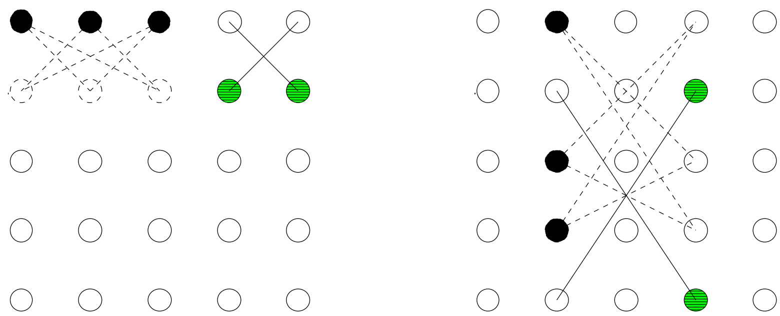


Figure 10: The equations (23) and (24).

Hence, the inequalities $-x(\mathrm{row}_i^{(n)}|_S) + z(\langle \mathrm{row}_i^{(n)}|_S : \mathrm{row}_k^{(n)}|_S\rangle) \leq 0$ and $-x(\mathrm{row}_k^{(n)}|_{\overline{S}}) + z(\langle \mathrm{row}_k^{(n)}|_{\overline{S}} : \mathrm{row}_i^{(n)}|_{\overline{S}}\rangle) \leq 0$ define the same face of $\mathcal{SQAP}_n$, and this does also hold for $-x(\mathrm{col}_j^{(n)}|_S) + z(\langle \mathrm{col}_j^{(n)}|_S : \mathrm{col}_l^{(n)}|_S\rangle) \leq 0$ and $-x(\mathrm{col}_l^{(n)}|_{\overline{S}}) + z(\langle \mathrm{col}_l^{(n)}|_{\overline{S}} : \mathrm{col}_j^{(n)}|_{\overline{S}}\rangle) \leq 0$. This means that it suffices to consider curtain inequalities with $|S| \leq \lfloor \frac{n}{2} \rfloor$.

Furthermore, if $|S| = 1$, any curtain inequality reduces to a simple nonnegativity constraint on a node variable. If $|S| = 2$, a row resp. column curtain inequality becomes a conical combination of an equation of type (9) resp. (10) and some nonnegativity constraints on the hyperedge variables. This yields the following observation:

Observation 4. *It suffices to consider curtain inequalities with* $3 \leq |S| \leq \lfloor \frac{n}{2} \rfloor$.

Theorem 10. *All curtain inequalities with $3 \leq |S| \leq \lfloor \frac{n}{2} \rfloor$ define facets of $\mathcal{SQAP}_n$. (Notice that $3 \leq \lfloor \frac{n}{2} \rfloor$ implies $n \geq 6$.)*

Proof. It suffices to show that all row curtain inequalities with $i = n$, $k = n-1$, and $S = \{a, \ldots, n\}$ with some $3 \leq a \leq n-2$ define facets of $\mathcal{SQAP}_n^\star$ for $n \geq 5$.

We will proceed as in the proof of Theorem 9. Clearly, the considered curtain inequalities define proper faces of $\mathcal{SQAP}_n^\star$. This time, the set L of n- and $(n-1)$-cliques that correspond to vertices of the considered face contains precisely all n- and $(n-1)$-cliques C satisfying

$$C \cap \mathrm{row}_n^{(n)}|_S \neq \emptyset \Longrightarrow C \cap \mathrm{row}_{n-1}^{(n)}|_S \neq \emptyset.$$

We choose $B := \langle \mathrm{row}_1^{(n)} : \mathrm{row}_2^{(n)} \rangle \cup \langle \mathrm{col}_1^{(n)} : \mathrm{col}_2^{(n)} \rangle$, as in the proof of Theorem 9. The extra element will be the hyperedge $\langle n, a+1, n-1, a \rangle$.

Observation 5. *As long as $\mathrm{row}_{n-1}^{(n)}|_S \cap (\mathcal{V}_n / \{(r_1, c_1), (r, c), (r_2, c_2)\}) \neq \emptyset$, Lemma 2 can be applied.*

Again, we denote by $\overline{a}$ the number with $\{\overline{a}\} = \{1, 2\} \setminus \{a\}$. This time, it will take us twelve steps to combine sucessively all unit vectors of $\mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ from elements in $\mathcal{L} \cup \mathcal{B}$. Figure 11 illustrates some of these steps.

Step 1: $z^{\langle i,j,k,l \rangle} \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for $i, j \in \{1, 2\}$, $k \neq n-1$
Using Observation 5, this can be done analogously to Step 1 in the proof of Theorem 9.

Step 1′: $z^{\langle i,j,n-1,l \rangle} \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for $i, j \in \{1, 2\}$, $l \geq 3$
Choosing $r := i$, $r_1 := n$, $r_2 := n-1$, $c := j$, $c_1 := \overline{j}$, and $c_2 := l$, condition (18) of Lemma 2 will be satisfied for any C. Hence, $z^{\langle n,\overline{j},i,j \rangle} + z^{\langle i,j,n-1,l \rangle} - z^{\langle n,l,i,j \rangle} - z^{\langle i,j,n-1,\overline{j} \rangle} \in \mathrm{lin}(\mathcal{L})$. Since all involved unit vectors but $z^{\langle i,j,n-1,l \rangle}$ are either in $\mathcal{B}$ or already linearly combined in Step 1, we are done.
(Note that up to now, we have linearly combined all unit vectors that we had combined after Step 1 in the proof of Theorem 9.)

Step 2: $z^{\langle i,j,k,l \rangle} \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for $i \in \{1, 2\}$, $j, k, l \geq 3$, $k \neq n-1$
As with Step 1, this can be done analogously to Step 1 in the proof of Theorem 9, using Observation 5.

Step 2′: $z^{\langle i,j,n-1,l \rangle} \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for $i \in \{1, 2\}$, $j, l \geq 3$
Choosing $r := i$, $r_1 := \overline{i}$, $r_2 := n-1$, $c := j$, $c_1 := 1$, and $c_2 := l$, condition (17) of Lemma 2 will be satisfied for any C that contains $(n, 2)$. Hence, $z^{\langle \overline{i},1,i,j \rangle} + z^{\langle i,j,n-1,l \rangle} - z^{\langle \overline{i},l,i,j \rangle} - z^{\langle i,j,n-1,1 \rangle} \in \mathrm{lin}(\mathcal{L})$. Since all involved unit vectors but $z^{\langle i,j,n-1,l \rangle}$ are either in $\mathcal{B}$ or are already combined in Steps 1', we are done.
(Now we have linearly combined all unit vectors that we had combined after Step 2 in the proof of Theorem 9.)

Step 3: $z^{\langle i,j,k,l \rangle} \in \mathrm{lin}(\mathcal{L} \cup \mathcal{B})$ for $j \in \{1, 2\}, i, k, l \geq 3$, $i, k \neq n-1$
Again, this can be done analogously to Step 3 of the proof of Theorem 9, using Observation 5.

Step 3′: $z^{\langle i,j,n-1,l\rangle} \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for $j \in \{1,2\}, i,l \geq 3$, $i \neq n$

This time, we choose $r := i$, $r_1 := n$, $r_2 := n-1$, $c := j$, $c_1 := \overline{j}$, and $c_2 := l$. Then, condition (18) of Lemma 2 is satisfied for any C. Hence, $z^{\langle n,\overline{j},i,j\rangle} + z^{\langle i,j,n-1,l\rangle} - z^{\langle n,l,i,j\rangle} - z^{\langle i,j,n-1,\overline{j}\rangle} \in \operatorname{lin}(\mathcal{L})$, and since all involved unit vectors but $z^{\langle i,j,n-1,l\rangle}$ are in $\mathcal{B}$ or are already combined in Step 3, we are done, again.

Step 3″: $z^{\langle n-1,j,n,l\rangle} \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for$j \in \{1,2\}, l \geq 3$

Choose $r := n-1$, $r_1 := 1$, $r_2 := n$, $c := j$, $c_1 := \overline{j}$, and $c_2 := l$. Then, condition (18) of Lemma 2 will be satisfied for any C, and we obtain $z^{\langle 1,\overline{j},n-1,j\rangle} + z^{\langle n-1,j,n,l\rangle} - z^{\langle 1,l,n-1,j\rangle} - z^{\langle n-1,j,n,\overline{j}\rangle} \in \operatorname{lin}(\mathcal{L})$, where all involved unit vectors but $z^{\langle n-1,j,n,l\rangle}$ are either in $\mathcal{B}$ or already combined in Step 1'.

(Now, we have combined all unit vectors that we had combined after Step 3 in the proof of Theorem 9.)

Step 4: $z^{\langle i,j,k,l\rangle} \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for $i,j,k,l \geq 3$, $i,k \neq n-1$

By Observation 5, this can be done analogously to Step 4 in the proof of Theorem 9.

Step 4′: $z^{\langle i,j,n-1,l\rangle} \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for $i,j,l \geq 3$, $(i,j) \notin \operatorname{row}_n^{(n)}|_S$

Choose $r := i$, $r_1 := n-1$, $r_2 := 1$, $c := j$, $c_1 := 1$, and $c_2 := l$. Since $2 \notin \{c, c_1, c_2\}$, we can choose a C as needed for applying Lemma 2 that avoids $\operatorname{row}_n^{(n)}|_S$. Hence, we can apply that lemma, and derive the claim as in Step 4.

Step 4″: $z^{\langle n,a+1,n-1,l\rangle} \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for $l \geq a$

The case $l = a$ is covered by the extra element. If $l \neq a$, choose $r := n$, $r_1 := 1$, $r_2 := n-1$, $c := a+1$, $c_1 := a$, and $c_2 := l$. With that choice, any C will satisfy condition (17) of Lemma 2. Hence, $z^{\langle 1,a,n,a+1\rangle} + z^{\langle n,a+1,n-1,l\rangle} - z^{\langle 1,l,n,a+1\rangle} - z^{\langle n,a+1,n-1,a\rangle} \in \operatorname{lin}(\mathcal{L})$. All involved unit vectors but $z^{\langle n,a+1,n-1,l\rangle}$ are either in $\mathcal{B}$ (the extra element) or already combined by Step 2.

Step 4‴: $z^{\langle n,j,n-1,l\rangle} \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for $j \geq a+2, l \geq a$

This can be done analogously to Step 4″. Now, the hyperedge $\langle n, a+1, n-1, j\rangle$ plays the role the extra element played in Step 4″, which is feasible, since $z^{\langle n,a+1,n-1,j\rangle}$ was combined in Step 4″.

(Now, we have combined all unit vectors corresponding to hyperedges.)

Step 5: $x^v \in \operatorname{lin}(\mathcal{L} \cup \mathcal{B})$ for all $v \in \mathcal{V}_n$

If $v \notin \operatorname{row}_n^{(n)}|_S$ extend v to an n-clique $C \in \mathcal{CLQ}_n^n$ of $\mathcal{H}_n$ such that $C \cap \operatorname{row}_n^{(n)}|_S = \emptyset$ (remember that $|S| \leq n-2$). If $v \in \operatorname{row}_n^{(n)}|_S$ extend v to an n-clique C such that $C \cap \operatorname{row}_{n-1}^{(n)}|_S \neq \emptyset$ (remember that $|S| \geq 3$). In both cases, application of Lemma 1 yields the claim, since all unit vectors corresponding to hyperedges are already combined.

□

We conclude this section with a consideration of the separation problem associated with the class of curtain inequalities. For this, let a (fractional) point $(\tilde{x}, \tilde{z}) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{F}_n}$ be given. We want to find, e.g., a row curtain inequality using rows 1 and 2 (ordered) that "cuts off" the point $(\tilde{x}, \tilde{z})$. Hence, we want to find a subset $S \subseteq \{1, \ldots, n\}$ such that $-\tilde{x}(\text{row}_1^{(n)}|_S) + \tilde{z}(\langle \text{row}_1^{(n)}|_S : \text{row}_2^{(n)}|_S \rangle) > 0$. But this is exactly the task to find a characteristic vector ξ of $\{1, \ldots, n\}$ that solves the *(Unconstrained) Boolean Quadratic 0/1 Problem (BQP)*

$$\begin{array}{ll} \max & \displaystyle\sum_{j=1}^{n} \sum_{l=j+1}^{n} \alpha_{jl} \xi_j \xi_l + \sum_{j=1}^{n} \beta_j \xi_j \\ \text{s.t.} & \xi \in \{0,1\}^n \end{array}$$

with $\alpha_{jl} := \tilde{z}_{\langle 1,j,2,l \rangle}$ and $\beta_j := -\tilde{x}_{(1,j)}$.

Hence, for each (ordered) pair of rows resp. columns, a BQP has to be solved, and this is known to be $\mathcal{NP}$-hard, in general. However, for small instances ($n \leq 20$) even very simple heuristics produce (empirically) very good solutions. This means that – although separation over this class is hard – the curtain inequalities seem to be computationally attractive. We will treat this aspect in the next section more closely.

6 Lower Bounds

For any instance of the QAP, the minimum the objective function achieves over the intersection of $\text{aff}(\mathcal{QAP}_n)$ and the nonnegative orthant is a lower bound for the optimal value of the respective QAP, called the *Equation Bound (EQB)*. This bound can be computed by solving the linear program arising from equations (1), … ,(4) and the nonnegativity constraints on the y-variables. Similarly, if the instance is symmetric, the minimum over the intersection of $\text{aff}(\mathcal{SQAP}_n)$ and the nonnegative orthant gives a lower bound, called the *Symmetric Equation Bound (SEQB)*. This may be computed by solving the linear program defined by the equations (7), … ,(10) and the nonnegativity constraints on x and z.

Let $(x, y) \in \text{aff}(\mathcal{QAP}_n) \cap (\mathbb{R}_{\geq 0}^{\mathcal{V}_n} \times \mathbb{R}_{\geq 0}^{\mathcal{E}_n})$ have value θ with respect to a symmetric objective function. Defining $z := \sigma_n(y)$, we obtain a vector $(x, z) \in \text{aff}(\mathcal{SQAP}_n) \cap (\mathbb{R}_{\geq 0}^{\mathcal{V}_n} \times \mathbb{R}_{\geq 0}^{\mathcal{F}_n})$ that has also value θ (with respect to the corresponding objective function for the symmetric formulation). Hence, SEQB can never be tighter than EQB.

It is possible to strengthen SEQB by the curtain inequalities. However, again one cannot obtain a lower bound that is tighter than EQB, since the curtain inequalities induce symmetric inequalities for the non-symmetric problem that are already implied by the equations defining $\text{aff}(\mathcal{QAP}_n)$ and by the nonnegativity of the y-variables.

Hence, do the curtain inequalities have any computational value at all? Potentially, they do. Namely, by changing (in case of a symmetric instance) from the non-symmetric problem formulation to the symmetric one, the number of variables is approximately devided by two. This leads to easier linear programs on the one hand, but to a potentially weaker bound SEQB on the other hand. So the question is, if the curtain inequalities can improve (empirically) the bound SEQB significantly towards EQB without loosing too much of the efficiency gain made by the transition.

We want to mention at this point that EQB has turned out to be a very good lower bound for the QAP. The theoretical basis for this is a result due to Johnson (1992) and Drezner (1994) showing that EQB is always at least as good as the classical *Gilmore/Lawler Bound*, proposed independently by Gilmore (1962) and Lawler (1963).

The practical indication for the quality of EQB was given most extensively in a computational study by Resende, Ramakrishnan, and Drezner (1994). They solved the linear programs that give the EQB for all instances in the QAPLIB (Burkard, Karisch, and Rendl, 1996) of size not exceeding $n = 30$ and found that EQB turned out to be the best known lower bound in most cases.

In order to investigate empirically the relative behaviour of EQB, SEQB, and the curtain inequalities, we implemented a rudimentary cutting plane procedure for symmetric QAPs. This procedure initially solves the linear program that yields SEQB and afterwards performs up to five cutting plane iterations with curtain inequalities. At each cutting plane iteration, we try to separate the current (fractional) solution by solving heuristically (i.e., repeating 100 times to guess a solution and improving it by a 2-opt procedure) a BQP for each ordered pair of rows/columns, as indicated at the end of Section 5. If such a BQB ends with value greater than zero then we add the corresponding curtain inequality to the current linear program. This way, up to $2n(n-1)$ curtain inequalities may be added per iteration.

The experiments were carried out on an SGI Power Challenge computer (Silicon Graphics) having 16 CPU's and 8 gigabytes of main memory. The code was written in C, and for solving the linear programs the package CPLEX 4.0 was used. The only parallel parts are inside the CPLEX code. The number of possible threads was restricted to 4 by setting the environment variable `MPC_NUM_THREADS`. The amount of main memory that was accessible for our runs was restricted to one gigabyte.

It turned out to be by far faster to use the Barrier than any Simplex code of CPLEX in order to solve the linear programs for both EQB as well as SEQB. Moreover, even for the linear programs arising after the addition of cutting planes to the SEQB formulation were solved much faster from the scratch by calling the Barrier solver again than by calling the Dual Simplex code with the optimal basis of the foregoing iteration. This is a bit surprising and not satisfactory at all. But for these preliminary computational experiments, we took this as a fact and simply solved any linear program from scratch by using the Barrier code of CPLEX.

We report our computational results in Tables 1 and 2. Our code was run on all QAPLIB instances of size up to 20. Table 1 reports on the lower bounds we obtained. The column labelled *ratio in %* shows the ratio of SEQB and EQB. The next five columns tell what part of the gap between SEQB and EQB has been closed after the respective cutting plane iteration. The final column, labelled *qual. in %*, gives the ratio of EQB and the best known (mostly optimal) solution as reported in Burkard, Karisch, and Rendl (1996). Table 2 tells how long it took to compute SEQB and EQB, as well as what the ratio of these times is. Furthermore, for each iteration the ratio of the totally elapsed time after that iteration and the time it needed to compute EQB is shown. The final column reports the ratio of the time spent for the linear programs and the total running time of the cutting plane procedure. All times are measured in seconds and are the sums of the CPU times of all involved (parallel) threads. Entries "—" mean that the cutting plane procedure stopped before the corresponding iteration because it could not find any violated curtain inequality anymore.

The results show that in most cases, SEQB is not significantly worse than EQB. Consequently, the curtain inequalities do not improve SEQB very much. The CPU-times that are needed to compute SEQB are about three to four times smaller than the corresponding

instance	size	SEQB	EQB	ratio in %	SEQB/EQB-gap reduced by % 1. it	2. it	3. it	4. it	5. it	qual. in %
chr12a	12	9552.00	9552.00	100.00	—	—	—	—	—	100.00
chr12b	12	9742.00	9742.00	100.00	—	—	—	—	—	100.00
chr12c	12	11156.00	11156.00	100.00	—	—	—	—	—	100.00
had12	12	1618.19	1621.54	99.79	32.24	37.01	40.00	40.30	40.30	98.16
nug12	12	520.61	522.89	99.56	12.28	12.72	12.72	12.72	12.72	90.47
rou12	12	222212.04	224302.02	99.07	20.25	29.76	33.07	33.76	33.80	95.23
scr12	12	29557.22	29827.33	99.09	37.96	43.46	43.62	43.65	43.65	94.96
tai12a	12	220018.71	222186.42	99.02	24.50	31.09	33.10	33.81	34.55	99.01
tai12b	12	30581824.50	31697152.48	96.48	64.76	77.49	80.93	82.77	83.52	80.32
had14	14	2659.86	2666.12	99.77	32.43	40.89	44.25	46.33	46.81	97.88
chr15a	15	9370.32	9513.12	98.50	66.34	73.61	77.27	77.61	77.70	96.13
chr15b	15	7894.12	7990.00	98.80	100.00	—	—	—	—	100.00
chr15c	15	9504.00	9504.00	100.00	—	—	—	—	—	100.00
nug15	15	1030.60	1040.99	99.00	22.04	26.28	31.67	34.26	35.61	90.52
rou15	15	322944.47	324901.61	99.40	18.74	24.89	26.96	27.35	27.52	91.73
scr15	15	48816.54	49264.73	99.09	37.17	43.52	46.39	47.54	47.64	96.33
tai15a	15	351289.64	352890.92	99.55	16.83	20.72	21.94	22.23	22.33	90.90
tai15b	15	51528935.02	51559404.81	99.94	37.80	49.49	54.49	55.95	56.70	99.60
esc16a	16	48.00	48.00	100.00	—	—	—	—	—	70.59
esc16b	16	278.00	278.00	100.00	—	—	—	—	—	95.21
esc16c	16	118.00	118.00	100.00	—	—	—	—	—	73.75
esc16d	16	4.00	4.00	100.00	—	—	—	—	—	25.00
esc16e	16	14.00	14.00	100.00	—	—	—	—	—	50.00
esc16f	16	0.00	0.00	100.00	—	—	—	—	—	100.00
esc16g	16	14.00	14.00	100.00	—	—	—	—	—	53.85
esc16h	16	704.00	704.00	100.00	—	—	—	—	—	70.68
esc16i	16	0.00	0.00	100.00	—	—	—	—	—	0.00
esc16j	16	2.00	2.00	100.00	—	—	—	—	—	25.00
had16	16	3548.12	3560.19	99.66	19.55	31.23	34.30	35.79	36.62	95.70
nug16a	16	1413.50	1425.64	99.15	17.30	24.96	28.67	30.89	31.71	88.55
nug16b	16	1080.05	1088.17	99.25	19.21	21.06	21.80	22.41	22.66	87.76
nug17	17	1490.79	1505.83	99.00	21.61	30.19	33.98	36.10	—	86.94
tai17a	17	440094.36	442702.77	99.41	14.61	18.73	20.78	21.48	21.75	90.01
chr18a	18	10738.55	10758.25	99.82	60.66	71.42	75.03	—	—	96.94
chr18b	18	1534.00	1534.00	100.00	0.00	—	—	—	—	100.00
had18	18	5071.09	5087.86	99.67	22.66	31.84	36.26	38.70	40.13	94.96
nug18	18	1649.70	1662.96	99.20	13.95	21.12	24.43	26.32	27.07	86.16
els19	19	16502856.83	16883302.96	97.75	29.69	47.04	53.95	57.46	—	98.09
chr20a	20	2169.67	2175.40	99.74	28.27	51.48	—	—	—	99.24
chr20b	20	2287.00	2287.00	100.00	—	—	—	—	—	99.52
chr20c	20	14006.73	14142.00	99.04	39.27	52.79	62.50	—	—	100.00
had20	20	6559.39	6578.77	99.71	25.70	35.35	40.71	—	—	95.04
lipa20a	20	3683.00	3683.00	100.00	—	—	—	—	—	100.00
lipa20b	20	27076.00	27076.00	100.00	—	—	—	—	—	100.00
nug20	20	2165.01	2181.60	99.24	28.27	32.37	35.20	36.83	37.67	84.89
rou20	20	639678.30	643363.25	99.43	13.16	18.19	20.75	21.93	22.69	88.68
scr20	20	94557.12	95117.84	99.41	17.75	28.17	31.21	33.09	34.34	86.45
tai20a	20	614849.18	618525.14	99.41	12.62	17.09	19.09	20.03	20.64	87.92
tai20b	20	84501939.93	97394937.98	86.76	14.42	39.66	63.67	—	—	79.54

Table 1: Bounds

instance	size	SEQB in sec	EQB in sec	ratio in %	ratio in % after					perc. LP
					1. it	2. it	3. it	4. it	5. it	
chr12a	12	22	86	25.58	—	—	—	—	—	72.73
chr12b	12	22	82	26.83	—	—	—	—	—	68.18
chr12c	12	28	74	37.84	—	—	—	—	—	75.00
had12	12	40	149	26.85	63.09	100.00	143.62	191.95	242.28	91.97
nug12	12	29	106	27.36	55.66	84.91	116.98	150.94	183.02	86.08
rou12	12	30	125	24.00	57.60	99.20	146.40	195.20	245.60	90.88
scr12	12	33	93	35.48	79.57	132.26	187.10	247.31	298.92	89.57
tai12a	12	32	120	26.67	67.50	118.33	172.50	224.17	283.33	91.18
tai12b	12	37	153	24.18	56.86	98.69	146.41	195.42	245.10	92.27
had14	14	122	463	26.35	64.36	115.77	179.70	256.16	358.32	96.14
chr15a	15	339	845	40.12	90.06	149.70	231.60	304.14	383.08	97.56
chr15b	15	227	612	37.09	70.75	—	—	—	—	92.61
chr15c	15	130	386	33.68	—	—	—	—	—	83.85
nug15	15	179	789	22.69	50.70	85.68	131.69	189.23	246.89	96.10
rou15	15	191	667	28.64	65.37	113.49	162.97	214.84	286.51	96.02
scr15	15	199	594	33.50	76.09	133.33	195.12	263.80	331.82	96.14
tai15a	15	189	696	27.16	57.47	100.72	142.39	189.37	233.91	95.27
tai15b	15	250	898	27.84	65.37	112.36	170.71	240.87	321.71	97.13
esc16a	16	136	447	30.43	—	—	—	—	—	78.68
esc16b	16	134	358	37.43	—	—	—	—	—	78.36
esc16c	16	134	394	34.01	—	—	—	—	—	78.36
esc16d	16	131	398	32.91	—	—	—	—	—	78.63
esc16e	16	147	394	37.31	—	—	—	—	—	80.95
esc16f	16	103	295	34.92	—	—	—	—	—	72.82
esc16g	16	142	332	42.77	—	—	—	—	—	80.28
esc16h	16	144	402	35.82	—	—	—	—	—	79.86
esc16i	16	116	321	36.14	—	—	—	—	—	75.00
esc16j	16	131	340	38.53	—	—	—	—	—	78.63
had16	16	367	2347	15.64	38.65	73.11	109.76	151.09	203.20	97.65
nug16a	16	481	3569	13.48	31.16	50.38	74.95	105.72	137.07	97.59
nug16b	16	256	960	26.67	67.92	104.38	142.81	185.52	234.79	96.10
nug17	17	622	2760	22.54	52.79	94.96	142.32	197.64	—	97.43
tai17a	17	470	1955	24.04	61.59	109.16	166.96	225.68	285.83	97.37
chr18a	18	1606	5079	31.62	76.83	186.10	284.41	—	—	99.00
chr18b	18	557	1240	44.92	90.73	—	—	—	—	94.40
had18	18	1222	4831	25.29	68.99	109.46	164.25	237.78	310.37	98.51
nug18	18	1019	4906	20.77	52.10	91.52	139.07	195.17	261.07	98.37
els19	19	2303	10449	22.04	55.14	102.71	203.46	265.54	—	99.08
chr20a	20	5124	12693	40.37	78.60	131.84	—	—	—	98.90
chr20b	20	1820	5407	33.66	—	—	—	—	—	95.33
chr20c	20	3941	9702	40.62	91.76	170.24	265.71	—	—	99.11
had20	20	3130	13675	22.89	66.98	114.01	184.32	—	—	98.88
lipa20a	20	1295	4806	26.95	—	—	—	—	—	93.51
lipa20b	20	1077	3748	28.74	—	—	—	—	—	92.20
nug20	20	2562	10081	25.41	65.13	106.34	154.00	208.57	271.73	98.74
rou20	20	2355	9402	25.05	62.69	109.44	155.88	215.65	282.22	98.63
scr20	20	2778	8354	33.25	78.74	138.21	197.09	260.59	338.21	99.02
tai20a	20	2310	8658	26.68	62.76	112.25	170.87	227.96	287.65	98.61
tai20b	20	3810	20251	18.81	43.53	84.47	127.07	—	—	98.79

Table 2: Times

ones for EQB. These two facts show that it is worth to investigate the special formulation for symmetric instances.

In those cases, where there is a gap between SEQB and EQB of more than 1% (e.g., `tai12b`, `chr15a`, `chr15b`), the curtain inequalities close a large part of that gap. However, at least with our implementation that solves each linear program from the scratch, the running time advantage SEQB has in comparison to EQB disappears in most cases already after the second cutting plane iteration.

7 Conclusion

We shortly want to discuss the context in which the work presented in this paper is located, in our opinion. Clearly, what we are finally concerned with is the exact (or at least provably good) solution of QAPs. The hope is that deeper polytopal knowledge of the problem will yield the necessary very good lower bounding procedures. Important steps that had already been performed were

- the evidence that EQB is empirically and theoretically a good lower bound,
- the basic polyhedral results on the QAP-Polytope, and
- the definition of the SQAP-Polytope.

The steps for the (quite natural) symmetric QAP that are done by the present paper are, from our point of view, the following.

- Our computational results indicate that changing the LP giving EQB in case of a symmetric instance in the natural way to a "symmetric LP" yielding SEQB does not decrease the quality of the lower bound significantly while accelerating the computations by a factor between three and four.
- It is useless to search for additional equations in order to improve the quality of SEQB, since the used equation system is already complete.
- The curtain inequalities seem to be computationally not very attractive, but they cannot be strengthened, since they already define facets.
- The methods presented in this paper, in particular the transition to the star-polytopes, provide possibilities for further investigations of the facial structure of the SQAP-Polytope.

To conclude, we present the steps that are to be done next, in our opinion. These steps might also show if the polyhedral approach can really yield progress in the attempt to solve QAPs to optimality or to a provably good solution.

- There is need for other cutting planes that (in contrast to the curtain inequalities) improve SEQB even beyond EQB at least for some instances. This improvement is necessary if one wants to reduce the tendency to implicit enumeration for larger instances, as one might see from the – with increasing size – decreasing quality of EQB.

- In order to get far beyond size 20 (i.e., towards size 30), the linear programs must be kept smaller by methods similar to the Dynamic Simplex Method (i.e., adding and removing rows as well as columns in a dynamical way, see e.g. Padberg (1995) for details).
- The fact that after each iteration the linear program has to be solved from scratch (which might be due to the structure of the curtain inequalities) should be overcome.

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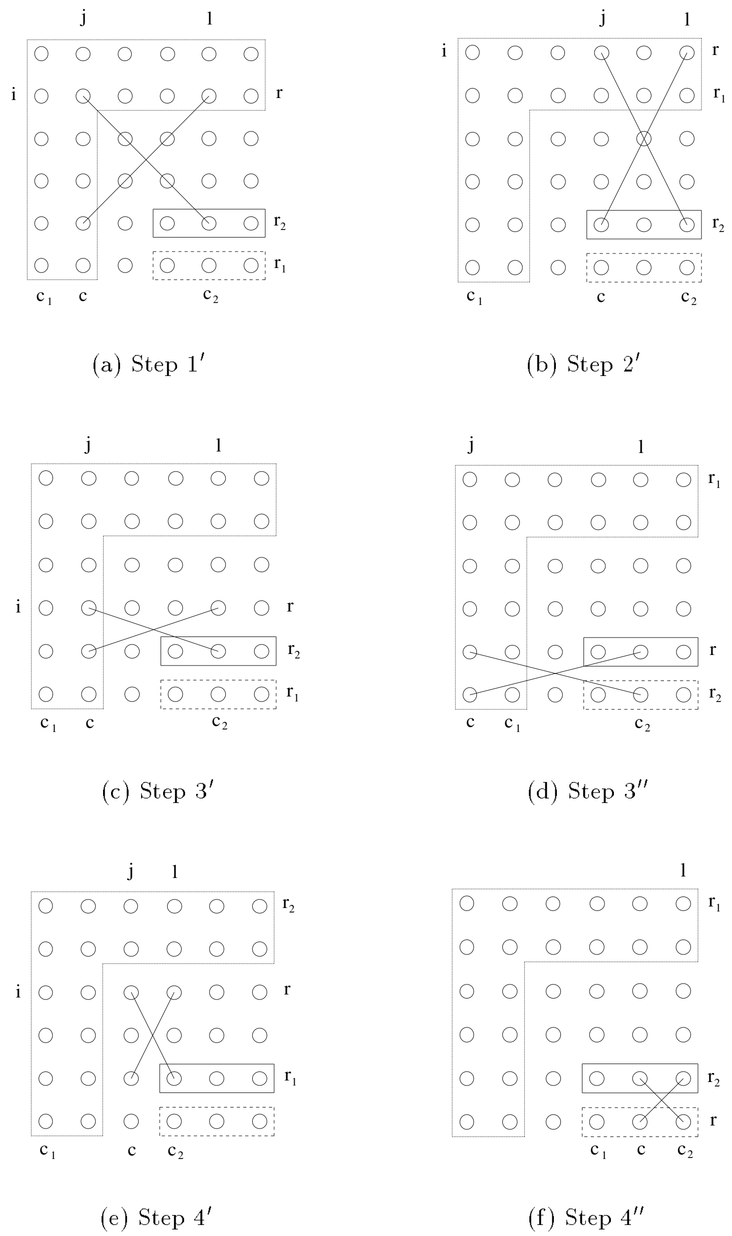

Figure 11: Some steps in the proof of Theorem 10. Again, the "angled box" contains the set B. The dashed small box indicates $\text{row}_n^{(n)}|_S$ and the solid small box is $\text{row}_{n-1}^{(n)}|_S$.