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by

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The critical exponents of the two-dimensional Ising spin glass revisited: Exact Ground State Calculations and Monte Carlo Simulations

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The critical exponents for $T \to 0$ of the two-dimensional Ising spin glass model with Gaussian couplings are determined with the help of exact ground states for system sizes up to $L = 50$ and by a Monte Carlo study of a pseudo-ferromagnetic order parameter. We obtain: for the stiffness exponent $y(= \theta) = -0.281 \pm 0.002$, for the magnetic exponent $\delta = 1.48 \pm 0.01$ and for the chaos exponent $\zeta = 1.05 \pm 0.05$. From Monte Carlo simulations we get the thermal exponent $\nu = 3.6 \pm 0.2$. The scaling prediction $y = -1/\nu$ is fulfilled within the error bars, whereas there is a disagreement with the relation $y = 1 - \delta$.

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I. INTRODUCTION

It is now widely believed that the bond-disordered two-dimensional Ising spin glass model with short range interactions does not have a phase transition at any non-vanishing temperature[1,2]. At zero temperature the spin glass is in its ground state (i.e. the spin configuration with the lowest possible energy), which might be degenerate or unique depending on the probability distribution of the spin interactions. This ground state is unstable with respect to thermal fluctuations and any non-vanishing temperature destroys this long range spin glass order. By decreasing the temperatures on the other hand the spatial correlations grow resulting in a divergence of the spin glass susceptibility at zero temperature. This scenario is characterized by a set of critical exponents that depend on certain features of the bond distribution. Experiments on $Rb_2Cu_{1-x}Co_xF_4$ clearly confirmed this picture[3] and reported values for the critical exponents, which are compatible with those predicted by the numerical investigations.

The latter has been pursued in four different ways: Monte Carlo simulations at finite temperatures[4,5], high temperature series expansion[6], transfer matrix calculations[7–9] and exact determination of ground states via combinatorial optimization[11–14] or replica optimization[15]. A scaling theory by Bray and Moore[16] establishes relations between exponents quantifying the stiffness of the ground state and the critical exponents characterizing the temperature dependent divergence of various thermodynamic quantities like correlation length or susceptibility.

With the most recent numerical studies a controversy has arisen on the critical exponents of the two-dimensional Ising spin glass with Gaussian couplings: A Monte Carlo study by Liang[5] (using a Swendsen-Wang-type cluster algorithm) and a numerical transfer matrix calculation by Kawashima et al.[9] yield a value for the thermal exponent ν which is significantly different from the early estimates[4,6–8]. Moreover, Kawashima et al.[15] also study the ground state magnetization of this model in an external field and report a value for the magnetic field exponent, which is, using a scaling relation[16], incompatible with the stiffness exponent found in domain wall renormalization group studies.

This observation and the progress in algorithmic developments motivated us to revisit the critical exponents of the two-dimensional Ising spin glass model. In this paper we present a synopsis of a zero-temperature (ground state) *and* a finite-temperature (Monte Carlo) approach to estimate the numerical values for these critical exponents. For the former we report results obtained from exact ground states for the largest system sizes possible to date, resulting in the most reliable estimates for the stiffness exponent y and magnetic field exponent δ reported so far. For the Monte-Carlo simulations we propose a pseudo-ferromagnetic order parameter that is defined by a projection of spin configurations onto the exactly known ground state and show that the thermal exponent ν is identical to the values that have been obtained by Bhatt and Young[4] studying the Edwards-Anderson (EA) order parameter. In the context of domain growth and non-equilibrium dynamics this concept has already been introduced and proven to be useful by direct comparison with the so called replica overlap[19].

The two-dimensional Ising spin glass model with a Gaussian distribution of couplings, which we consider throughout this paper, is defined by the Hamiltonian

$$H = \sum_{<ij>} J_{ij} S_i S_j - h \sum_i S_i \ , \qquad S_i = \pm 1 \qquad (1)$$

where $\langle ij \rangle$ denotes all nearest neighbor pairs on a $L \times L$ square lattice with periodic boundary conditions and the random interaction strengths obey a Gaussian probability distribution with mean zero and variance one. The parameter h denotes an external magnetic field strength.

The outline of the paper is as follows: In the next section we present our results from exact defect energy calculations, which provides us with an estimate for the stiffness exponent y. In section III we present a conventional finite size scaling analysis of Monte Carlo data. In contrast to earlier investigations we used exact ground state configurations instead of replica systems in order to establish an order parameter. Section IV focuses on the exact calculation of ground state magnetizations in an external field and its finite size scaling properties. Section V presents a study of the sensitivity of the ground state with respect to slight perturbations of the coupling strength. The last section is a summary plus discussion.

II. DEFECT ENERGY

A. Scaling Theory

The scaling theory by Bray and Moore[16] starts with a coarse grained picture for the spin interactions. It hypothesizes the following scaling ansatz for an effective coupling $\tilde{J}(L)$ among (block) spins on length scale L at an infinitesimal temperature:

$$\tilde{J}(L) \sim J L^y \qquad (2)$$

where J denotes the variance of the original bond distribution. For positive stiffness exponent y (sometimes θ) the coupling becomes stronger on larger length scales, which means that it is harder to flip collectively a connected set of spins of linear dimension L. Thus thermal fluctuations are irrelevant and the spin glass ordered state persists at low temperature. A negative exponent y, as we expect for $d = 2$, on the other hand indicates the instability of the spin glass ground state. In this case the spin glass transition takes place only at zero temperature and y is related to the thermal exponent ν determining the divergence of the correlation length $\xi \sim T^{-\nu}$:

The temperature dependence of the correlation length ξ near $T = 0$ can be inferred from equating the two energy scales set by the effective coupling constant and the temperature. At low temperatures where (2) holds one has then

$$\xi \sim T^{1/y} \qquad (3)$$

and therefore

$$y = -1/\nu \qquad (4)$$

In this way the exponent y, which we calculate in this section, has to be compared with ν determined in finite temperature Monte Carlo simulations discussed in the next section.

B. Algorithm and Results

The problem of finding a spin configuration with lowest energy can be transformed into the problem of finding a maximum weight cut in a special weighted graph, that represents the interaction structure of the spin glass system[12]. This is known under the name Max-Cut problem and is in general a NP-hard problem[10]. If the graph is planar, as in the two-dimensional case with *free* or *fixed* boundary conditions, the problem is solvable in polynomial time[11]. If one has periodic or anti-periodic boundary conditions or if an external field (representable as an extra node to which all other nodes are connected) is present the graph is not planar any more even in two dimensions. Hence, the situation we are studying here is indeed an NP-hard problem.

The results of this section and of sections IV and V are based on the application of a so-called branch & cut algorithm to the ground state problem[12]. This algorithm always finds an exact ground state of the given spin glass system. For further details about this algorithm and its implementation see ref.[17,13]. An important feature of this approach is, that the returned solutions are proved to be optimal. Exact ground states of grid sizes up to 100×100 can be determined in a moderate amount of computation time. The 100×100 instances take between 1.5 and 8 hours, 4 hours on average. Up to grid sizes of 50 each run takes less than 15 minutes[13].

The NP-hardness is not a serious problem as long as the system sizes L one studies are not in the region where the exponential dominates, and for really large L, where it does dominate, the exponent is very small. In the range $L \leq 32$ our empirical CPU-times can be fitted by a power law ($\tau \propto L^{3.5}$). For bigger systems there enters an exponential term $\approx 1.2^L$ inside the used size range. Note that these are only empirical observations but no rigorous bound for the complexity. For comparison Kawashima and Suzuki[15] reported about a replica optimization method which approximates ground states efficiently. They achieved an average CPU-time $\tau(L)$ for systems of linear size L which can be fitted by a power law like $\tau \propto L^{5.3}$ inside the same size range ($L \leq 32$). Although Kawashima and Suzuki only approximate the ground states while we always find optimal solutions, their CPU-times are similar to ours. On average, their biggest systems (32×32) took 260 seconds, while we needed 160 seconds for 40×40 spin glasses (1500 seconds for 60×60). Kawashima and Suzuki used a VAX 6440, our computations were carried out on a SPARC 10/612.

One can determine the defect energy by investigating the sensitivity of the ground state energy to bound-

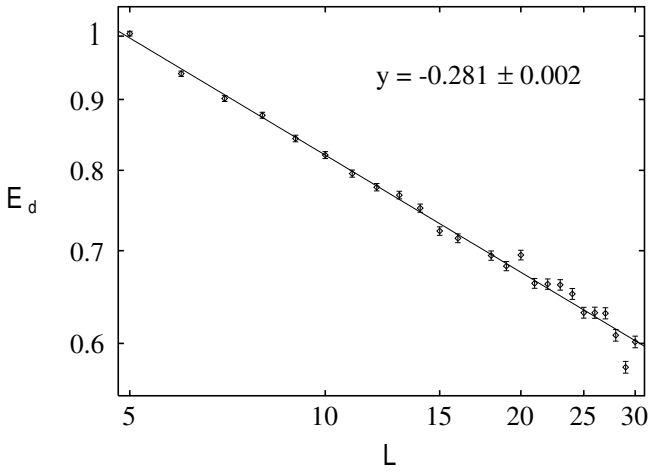

FIG. 1. Defect energy E_d as a function of the system size L in a log-log plot. The straight line is a least square fit giving the exponent $y = -0.281$.

ary conditions[16,18], which can be quantified by a stiffness exponent measuring the extra energy of a defect line through the whole sample. The block coupling J' is then given by $J' = \sqrt{(E_p - E_a)^2}$, where E_p and E_a are the ground state energies of the system under periodic and anti-periodic boundary conditions, respectively. We compute this value in the following way. First we solve the given spin glass system to optimality under periodic boundary conditions, i.e., we find an exact ground state configuration ω_p and its energy $E_p = E(\omega_p)$. Then we choose two neighboring "columns" of spins and multiply all couplings by -1 that link these two spin sets. By this modification of the couplings we impose anti-periodic boundary conditions to the original system. With this slightly changed objective function we rerun our branch & cut code to find a ground state configuration ω_a with energy $E_a = E(\omega_a)$.

Due to the very small magnitude of $E_a - E_p$ it is necessary to have a very large number of samples to obtain stable statistics. For each size $L \leq 30$ of our $L \times L$ spin glasses we ran $\lceil \frac{2 \cdot 10^5}{L} \rceil$ samples. The resulting mean values of the defect energies versus the system size L are shown in Fig. 1. A least square fit yields the value

$$y = -0.281 \pm 0.002 \quad \Rightarrow \quad \nu = 3.559 \pm 0.025 \qquad (5)$$

The errors are statistical errors only. This estimate agrees roughly with less accurate early estimates $\nu = 2.96 \pm 0.22$ and $\nu = 4.2 \pm 0.5$ from transfer matrix calculations[7] as well as $\nu = 3.56 \pm 0.06$ and $\nu = 3.4 \pm 0.1$ from domain wall renormalization calculations[8]. Note that Bray and Moore[16] report an estimate $y = -0.291 \pm 0.002$, which has an error bar that is identical to ours. However, their maximum system size is $L = 12$ and they did not calculate *exact* ground states.

Our result for y (5) implies a value for ν, if the scaling prediction (4) is correct, that differs substantially from more recent estimates[5,9]. Since in these works the thermal exponent ν has been determined directly, we shall do this, too, in the next section.

III. MONTE CARLO RESULTS

In this section we present our results from finite temperature Monte Carlo simulations. For this purpose we introduce first a number of quantities that are of interest for studying the critical properties of Ising spin glasses in zero external field h.

A. Scaling Relations and Methodology

A characteristic feature of a spin glass transition at a temperature T_c (which might be zero) is the divergence of the so called spin glass susceptibility

$$\chi = \frac{1}{N} \sum_{<ij>} [\langle S_i S_j \rangle^2]_{\rm av}, \qquad (6)$$

where $[\ldots]_{\rm av}$ denotes the average over the quenched disorder and $\langle \ldots \rangle$ a thermal average. Approaching the transition temperature from the paramagnetic phase one observes $\chi \sim (T - T_c)^{-\gamma}$, which defines the susceptibility exponent γ. As already mentioned there is a diverging length scale at the transition, the spin glass correlation length $\xi \sim (T - T_c)^{-\nu}$, which governs the scaling form of the correlation function near T_c:

$$G(r) = [\langle S_i S_{i+r} \rangle^2]_{\rm av} \sim r^{-(d-2+\eta)} \tilde{g}(r/\xi) \qquad (7)$$

Obviously $\gamma = (2 - \eta)\nu$. In the case $T_c > 0$ there is a non-vanishing Edwards-Anderson order parameter $q_{EA} = [\langle S_i \rangle^2]_{\rm av}$ below the transition and one has $q_{EA} \sim (T_c - T)^\beta$ for $T < T_c$, the order parameter exponent β obeying the hyperscaling relation $\beta = \frac{\nu}{2}(d - 2 + \eta)$.

In two dimensions $T_c = 0$ and since we are concerned with a continuous bond distribution, in which case the ground state is non-degenerate, one has

$$\begin{aligned} \eta &= 0 \\ \beta &= 0 \\ \gamma/\nu &= 2 \end{aligned} \qquad (8)$$

Thus we are left with a single unknown exponent ν, which we determined from the scaling behavior of the susceptibility and the Binder cumulant. In contrast to previous investigations we used exact ground states to define a pseudo-ferromagnetic order parameter via

$$M = [\langle q \rangle]_{\rm av} \qquad (9)$$

with

$$q = \frac{1}{N} \sum_{i=1}^{N} S_i S_i^0 \qquad (N = L^2) \qquad (10)$$

where S_i^0 denotes the value of the spin at site i in one of the two ground state configurations. Note that in contrast to the EA-order parameter here is only one fluctuating field involved, which would in principle reduce the order parameter exponent to $\beta/2$. Since we have $T_c = 0$ and thus $\beta = 0$ this is not relevant here. The corresponding order parameter susceptibility is defined via

$$\chi_L = N[\langle q^2 \rangle]_{\rm av}. \tag{11}$$

For the finite size scaling form of the susceptibility we expect according to (8)

$$\chi_L(T) = L^2 \overline{\chi}(L^{1/\nu} T). \tag{12}$$

The second quantity we studied was the disorder averaged Binder cumulant

$$g_L = \frac{1}{2}\left[3 - \frac{\langle q^4 \rangle}{\langle q^2 \rangle^2}\right]_{\rm av}. \tag{13}$$

Since this is a dimensionless combination of moments its finite size scaling form is

$$g_L(T) = \overline{g}(L^{1/\nu} T). \tag{14}$$

This quantity provides us with a second independent estimate for the scaling exponent ν.

We applied single spin flip Glauber dynamics to perform our simulations with a spin flip probability given by

$$w(S_i \to -S_i) = \frac{1}{1 + \exp(\Delta E/T)}, \tag{15}$$

and ΔE being the energy difference between the new and the old state. Time is measured in Monte Carlo sweeps (MCS) through the whole lattice.

The estimate of the critical exponents necessitates the determination of the equilibrium values of the thermodynamic quantities of interest. Due to the slow relaxation of spin glasses it is difficult to decide whether the values are stationary or not, because it is hard to discriminate between real- and quasi-stationary values of the functions. This is why we used a definite criterion analogous to the criterion introduced by Bhatt and Young[4]:

We simulated two replicas of the system, one which has been initialized with a random configuration and the other with a ground state configuration. From Fig.2 it can be clearly seen that if both estimates agree we obtained a time independent value of the susceptibility, which we took as our equilibration criterion.

B. Results

We studied the temperature dependent scaling behavior system sizes extending from $L = 6$ to $L = 12$. The number of samples is chosen that approximately

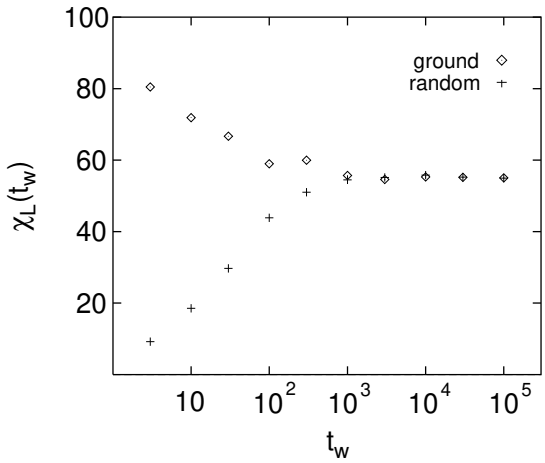

FIG. 2. Plot of the susceptibility at $T = 1.4$ for $L = 10$ averaged over 192 samples. We compare the values of $\chi_L(t_w)$ obtained from systems initialized with a ground state configuration with those obtained from systems with a random initial configuration. Obviously the results become time independent (aside from statistical fluctuations) if both estimates agree.

$N \cdot \#\text{samples} = \text{const.}$ holds. We simulated at least 128 samples for $L = 12$.

Fig.3 shows the equilibrium values of the susceptibility for various system sizes. We could reach the equilibrium value of the susceptibility in the chosen time interval for $T \geq 1.0$ ($L > 6$) and $T \geq 0.8$ ($L = 6$) respectively. With this data we got an estimate from the scaling-ansatz (12). The best data collapse we obtained (see Fig.3) for

$$\nu = 3.4 \pm 0.2. \tag{16}$$

The error bars denote the interval of exponents where we get an indistinguishable data collapse.

Fig.4 shows the equilibrium values of g_L for the same samples. The equilibrium value could be reached within the same time interval for some smaller temperature. This is why the data is more sensitive to changes of the value of the critical exponent. Thus we obtained the best data collapse for

$$\nu = 3.7 \pm 0.1. \tag{17}$$

in agreement within the error bars with the value determined above. Concluding we obtain an average value of

$$\overline{\nu} = 3.6 \pm 0.2 \tag{18}$$

for the critical exponent from our Monte Carlo simulations. This value agrees well with the estimate (5) that we obtained from the defect energy calculations in the last section. It differs substantially from the more recent estimates obtained by a cluster Monte Carlo study[5] ($\nu = 2.0 \pm 0.2$) and a numerical transfer matrix calculation[9] ($\nu = 2.08 \pm 0.01$).

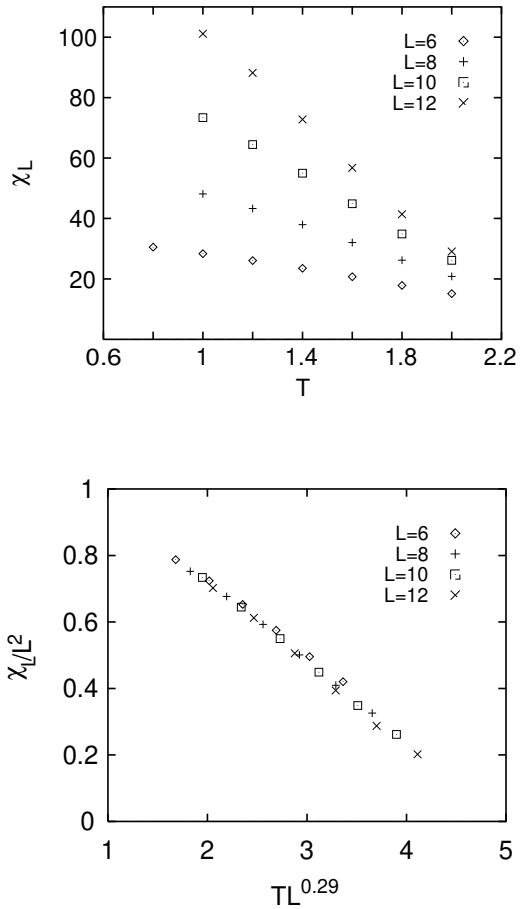

FIG. 3. Equilibrium values of the susceptibility depending on temperature and system size.

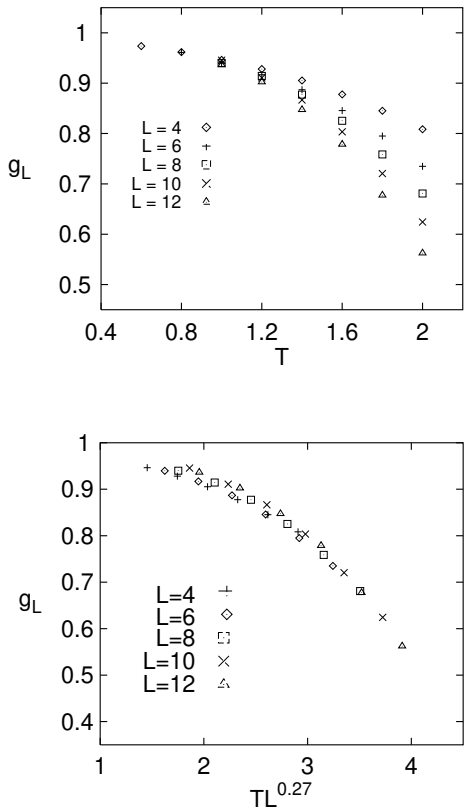

FIG. 4. Results of g_L. Only equilibrium-values are shown.

IV. GROUND STATE MAGNETIZATION

A nonzero external field h induces a non-vanishing magnetization $m = N^{-1}\sum_{i=1}^{N} S_i^0$ in a system with ground state $\{S_i^0\}$. The relation between magnetization and field strength is highly nontrivial in general and motivates the introduction of a new exponent δ characterizing this relation in the infinite system ($L \to \infty$) for small fields ($h \ll J$):

$$m_\infty(h) \sim h^{1/\delta} \tag{19}$$

The corresponding finite size scaling form and a scaling relation between δ and the already known exponent y can be obtained by the following argument[16]:

If the ground state is non-degenerate the spins are randomly oriented within an infinitesimal field at $T = 0$. Hence the magnetization m_L of a finite system in zero field is a random variable with variance $1/N$, implying $m_L(h = 0) \sim L^{-d/2}$. As a further consequence of the random orientations the total magnetic moment of a block spin of linear dimension L is of order $L^{d/2}$, thus the magnetic field on this length scale has to be rescaled according to

$$\tilde{h}(L) \sim L^{d/2} h \ . \tag{20}$$

in contrast to a ferromagnet, where we would have $\tilde{h}(L) \sim L^d h$. For nonzero field (at $T = 0$) one would expect $m_L(h) \cdot L^{d/2}$ to be a function of the dimensionless ratio of energy scales $\tilde{J}(L)$ and $\tilde{h}(L)$ only, thus

$$m_L(h) = L^{-d/2}\, \tilde{m}(L^{d/2-y}\, hJ^{-1}) \tag{21}$$

with $\tilde{m}(x \to 0) = \text{const.}$. Since for $L \to \infty$ the L-dependence of the magnetization has to drop out it is $\tilde{m}(x \to \infty) \sim x^{d/(d-2y)}$. Moreover, in this limit we have to recover (19), which implies for $d = 2$

$$\delta = 1 - y \tag{22}$$

Rewriting (21) slightly for our our purposes yields

$$m_L(h) = L^{-1}\, \overline{m}(Lh^{1/\delta}) \tag{23}$$

with $\overline{m}(x \to 0) = \text{const.}$ and $\overline{m}(x \to \infty) \propto x$. Note that (23) should hold independently of the correctness of the above derivation of the scaling relation (22): The length scale induced by the magnetic field is given by $h^{-1/\delta}$ and (23) is simply the finite size scaling form one would expect for the magnetization.

With our branch & cut algorithm we are not only able to compute $m(S, h)$ for a sample S for some specific values of h like other authors did (see Kawashima and Suzuki[15]). We can evaluate the complete piecewise

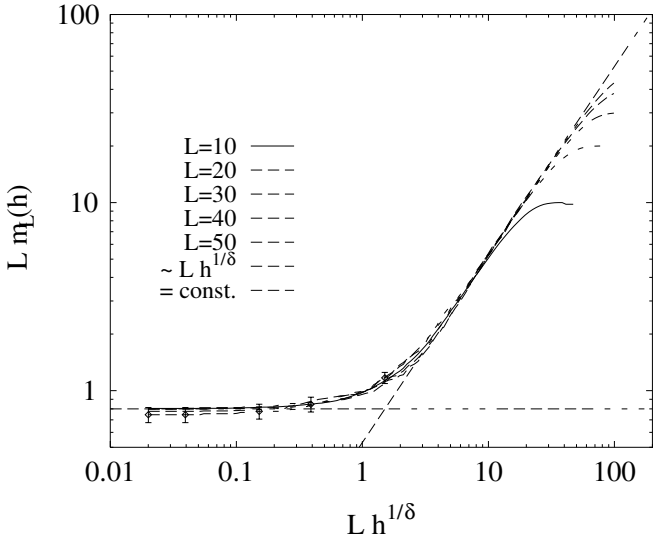

FIG. 5. Scaling plot for the ground state magnetization: $Lm_L(h)$ versus $Lh^{1/\delta}$ for various system sizes with $1/\delta = 0.675$. Note that for high fields $h \to \infty$ the curves have to saturate at $L \cdot m_L(h \to \infty) = L$.

constant function $m(S,h)$ for each sample. We do this by starting at $h = 0$, computing the ground state, and finding the next increased value of h for which the current ground state loses optimality using a sensitivity analysis technique[20]. At that point we compute the new ground state. We do this up to a given field strength or until saturation occurs.

This technique gives us the (averaged) function $m_L(h)$ for each system size L with any arbitrary resolution. We used systems of sizes $L \in \{10, 20, 30, 40, 50, 60\}$ and computed the ground states for $\lceil \frac{10^5}{L^2} \rceil$ samples for each size L. We judged the data collapse in a plot $m_L L$ versus $Lh^{1/\delta}$ as shown in Fig. 5 by visual inspection and using cubic spline interpolation. In the figure we have included some error bars for the $L = 50$ and the $L = 60$ curves to show the typical errors. The errors decrease with decreasing system size, because of the increasing number of samples.

We obtained the best data collapse at

$$1/\delta = 0.675 \pm 0.005 \quad \Rightarrow \quad \delta = 1.481 \pm 0.011 \qquad (24)$$

a value that agrees well with the result of the ground state magnetization study by Kawashima and Suzuki[15]. This value together with estimate $y = 0.281$ (5) from the defect energy calculation implies that the scaling hypothesis (22) is significantly violated. Since we estimated ν directly in a Monte Carlo simulation we conclude that it is also not legitimate to infer from the magnetic exponent δ via (22) and (4) that the thermal exponent ν should be close to 2 as found in[5,9].

V. CHAOS EXPONENT

One of the peculiar features of spin glasses is their extreme sensitivity with respect to parameter changes[21], like small temperature, field or coupling variations. For the ground state properties this means that a slight perturbation of the initial set of couplings leads to a complete reorganization of the original ground state over a length scale that depends on the strength of the perturbation. This overlap length is expected to behave as follows[21,22]:

Let us modify the interactions by replacing each coupling J_{ij} by $J'_{ij} = J_{ij} + \delta K_{ij}$. Here K_{ij} is again a Gaussian distributed random number with variance one and the parameter δ measures the strength of the perturbation. The comparison of the energy balance $\Delta E_{\rm defect}$ for turning over a connected spin cluster of linear extent L with the change of the ground state energy $\Delta E_{\rm random}$ induced by the random variation of the couplings yields an estimate for the length scale beyond which the original ground state is unstable with respect to the perturbation. $\Delta E_{\rm defect}$ is simply the defect energy, which is proportional to JL^y (see section II). The contribution to $\Delta E_{\rm random}$ coming from the L^{d_S} interface spins of the cluster (d_S being the fractal dimension of the interface) is proportional to $\delta L^{d_S/2}$. Thus for $L > L^*(\delta)$ with

$$L^*(\delta) \sim (J/\delta)^{-1/\zeta} \qquad \text{with} \quad \zeta = d_S/2 - y \qquad (25)$$

we have $\Delta E_{\rm random}(L) > \Delta E_{\rm defect}(L)$ and flipping of clusters is favored by the perturbation. Thus the ground state configurations in the original (denoted by S_i) and the perturbed sample (denoted by $S'_i(\delta)$) become uncorrelated for distances larger than L^*.

This statement can be quantified by studying the overlap correlation function

$$C_\delta(r) = \left[\frac{1}{N} \sum_{i=1}^{N} S_i \, S_{i+r} \, S'_i(\delta) \, S'_{i+r}(\delta) \right]_{\rm av} . \qquad (26)$$

According to the above mentioned argument one expects in the limit $N \to \infty$ a scaling form

$$C_\delta(r) \sim \tilde{c}(r\delta^{1/\zeta}) . \qquad (27)$$

In Fig. 6 we show the result of our calculation of the overlap correlation function $C_\delta(r)$. We fixed the system size to $L = 50$, for which reason one has to neglect the data points for $r > L/4$ (note the upwards bending due to the periodic boundary conditions). For the rest of the data we obtain the best data collapse for

$$1/\zeta = 1.05 \pm 0.05 \qquad \text{i.e.} \quad \zeta = 0.95 \pm 0.05 \qquad (28)$$

which agrees well with the estimate from Bray and Moore[21] obtained in a different way and by considering smaller system sizes. With the value for y we reported in section II the fractal dimension of the interface of an excitation is given by $d_S = 1.34 \pm 0.10$.

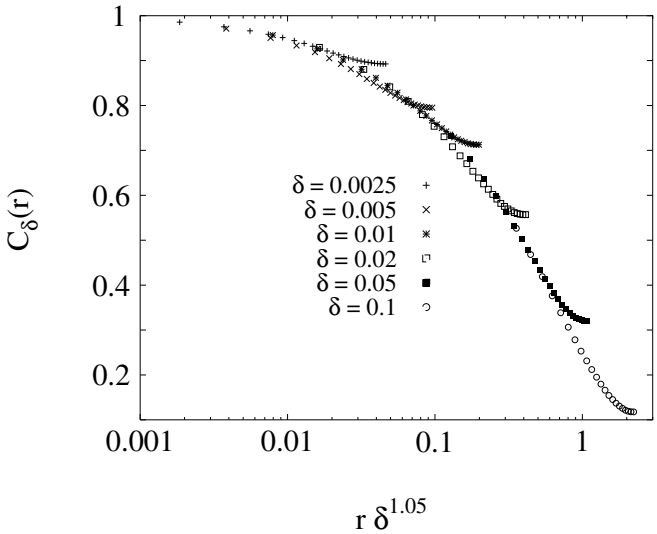

FIG. 6. Scaling plot of the overlap correlation function $C_\delta(r)$ versus r/L^* with $L^* = \delta^{-1/\zeta}$. The best data collapse (for data confined to $r < L/4$) is obtained for $1/\zeta = 1.05$. The system size is $L = 50$ and the data are averaged over 400 samples. These were obtained by creating > 80 reference instances and creating 5 random perturbations of strength δ for each.

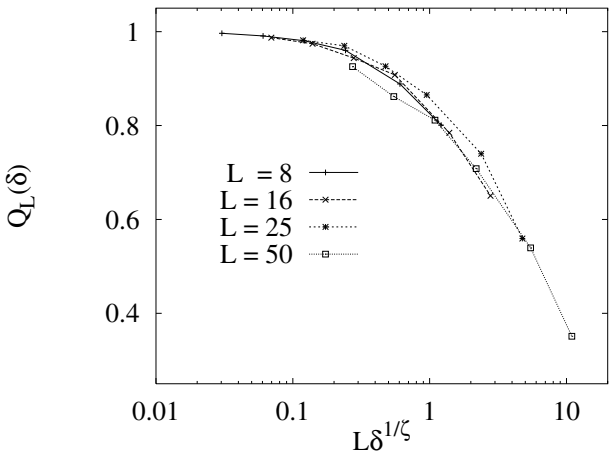

FIG. 7. Scaling plot of the ground state overlap $Q_L(\delta)$. The best data collapse is obtained with $1/\zeta = 1.2$.

In passing we mention that the dependency of $C(r)$ on distance r is neither exponential nor algebraic: it can nicely be fitted with a stretched exponential

$$C(r) \approx \exp(-r^a/b) + \exp(-(L-r)^a/b) \tag{29}$$

with fit parameters a and b. For instance $\delta = 0.1$ for $L = 50$ yields $a = 0.8$ and $b = 0.75$. Since a and b seem to depend on the perturbation strength δ we do not expect the form (29) to be universal.

Defining $\xi_L(\delta) = \sum_{r=0}^{L} C_\delta(r)$ one expects from (27)

$$\xi_L(\delta) \sim \tilde{\xi}(L\delta^{1/\zeta}) \tag{30}$$

and in a more direct way for the ground state overlap[21] $Q_L(\delta) = |\sum_{i=1}^{N} S_i\, S_i'(\delta)|$

$$Q_L(\delta) = \tilde{Q}(L\delta^{1/\zeta}). \tag{31}$$

Note that $\xi_L(\delta) = Q_L^2(\delta)$. We show a finite size scaling plot for $Q_L(\delta)$ in fig. 7, from which we estimate $1/\zeta = 1.2 \pm 0.1$. The quality of the data collapse is good (cf. Fig. 2 of ref.[21]).

VI. SUMMARY

With the help of an improved branch & cut algorithm we were able to reinvestigate the critical behavior of the two-dimensional Ising spin glass model with a continuous bond distribution with much better accuracy. We found that the stiffness exponent is given by $y = -0.281 \pm 0.002$ implying a correlation length exponent of $\nu = 3.56 \pm 0.02$, which agrees well with our independent estimate $\nu = 3.6 \pm 0.2$ from Monte Carlo simulations. For the latter we introduced a pseudo-ferromagnetic order parameter with the help of exactly known ground states and analyzed its finite size scaling behavior at non-zero temperatures.

We hope that our calculation settles the controversy regarding the thermal exponent ν initiated by the cluster Monte Carlo study of Liang[5] and the numerical transfer matrix study by Kawashima et al.[9]: Their values for ν are substantially smaller than ours indicating a violation of the scaling prediction by Bray and Moore[16]. Our results for y and ν are clearly compatible with this scaling prediction $\nu = -1/y$.

Furthermore we determined exact ground states for systems within an external field and from a finite size scaling analysis of the magnetization we obtained an independent estimate for the magnetic exponent $\delta = 1.48 \pm 0.01$. This confirms an earlier observation[15] that there seems to be a disagreement between the scaling theory[16] predicting $\delta = 1 - y$ and the numerical values obtained so far. In particular this discrepancy does not fade away for larger system sizes, which we were able to study here. Therefore our conclusion is that there must be a deeper reason for this disagreement than some finite size effect which might disappear if one only considers large enough system sizes.

Moreover, we calculated the overlap correlation function by perturbing the bonds slightly in a random manner. We found a chaos exponent $\zeta = 0.95 \pm 0.05$ in agreement with earlier estimates from the analysis of smaller system sizes.

Finally a few words concerning future perspectives: First we would like to point out that in principle it is possible to improve the system sizes *and* quality of statistics even further with the algorithm we have at hand, provided we could simply run it on a powerful parallel machine. However, our algorithm relies heavily on a commercial linear program solver for which we do not

have a license to run it on hundreds of processors of a e.g. Paragon XP/S10. On such a machine we could possibly obtain an acceptable quality of statistics for $L = 100$, for which we can presently do only a few samples in reasonable time on individual workstations.

As has been mentioned in the introduction, recently a finite temperature phase transition in the site disordered Ising spin glass has been reported[2]. Since the critical temperature is pretty small, though, Monte Carlo studies might be hampered by equilibration problems. Therefore this result could be put on a much firmer base, if the stiffness exponent y would indeed turn out to be positive in this particular two-dimensional model and so signaling the stability of the spin glass ordered phase for small, non-vanishing temperatures. We intend to answer this question with our algorithm soon.

Furthermore an obvious and highly rewarding step would be to perform the same study in three dimensions. To calculate ground states for the three-dimensional Ising spin glass model is an NP-hard problem and the two-dimensional problem we have studied here is NP-hard, too (note we have a continuous bond distribution, periodic boundary conditions and an external field). However, although both questions belong to the same class of *hard* combinatorial problems, the three-dimensional Ising spin glass is *much* harder, which means that the operation count will be much higher: either the power of the L, the system size, or the coefficient in the exponent will be larger for three dimensions than for two dimensions. Nevertheless we are currently undertaking efforts in this direction, our progress in this matter will be reported elsewhere.

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