Encoding True Second-order Arithmetic in the Real-Algebraic Structure of Models of Intuitionistic Elementary Analysis

(Draft)

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Abstract. Based on the paper [1] we show that true second-order arithmetic is interpretable over the real-algebraic structure of models of intuitionistic analysis built upon a certain class of complete Heyting algebras.

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1 Introduction

Let L denote the language of ordered rings. In [2] we showed the undecidability of the L-structure of Scott's model (see [3]). Continuing the investigation, in [1], we showed that true first-order arithmetic is interpretable in the L-structure of a class of models which includes the well-known topological models as well as Scowcroft's model (defined in [4]) and its generalizations. Here we improve that result showing the interpretability of true second-order arithmetic in these structures. We shall use the notations, definitions and the results of that earlier paper.

2 Basic notions

We quote the main (standard) definitions about the models we are interested in from [1].

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Let $\mathcal{H} = (H, \bot, \top, \wedge, \vee, \rightarrow, \leq)$ be a complete Heyting algebra – the *truth value algebra* of the model to be defined. Infinite infimum and supremum will be denoted by $\bigwedge$ and $\bigvee$ respectively. We say that $A \in H$ is *complemented* if there is an element denoted by A^c of H such that $A \wedge A^c = \bot$ and $A \vee A^c = \top$. Note that if A is complemented then $A^c = A \rightarrow \bot$. $A \rightarrow \bot$ is the *pseudocomplement* of A, denoted by $\neg A$. We say that two elements $U_1, U_2 \in H$ are *disjoint* if $U_1 \wedge U_2 = \bot$. An element $U \in H$ is *dense* if $\neg\neg U = \top$.

A *choice sequence*, i.e. a sequence of natural numbers is represented in the model by a function $\xi : \omega \times \omega \rightarrow H$ such that $\bigvee_m \xi(l, m) = \top$ and $\xi(l, m) \wedge \xi(l, n) = \bot$ for all l, m, n with $m \neq n$. From this follows that the elements $\xi(l, m)$ are complemented. Here $\xi(l, m)$ is the truth value of the statement that the l-th element of the sequence represented by ξ is m. Let Ξ denote the set of choice sequences.

The language L_1 we shall use is the one used in [4]. It contains two sorts of variables — x, y, z, etc. ranging over the elements of ω, and α, β, etc. ranging over choice sequences — and constant symbols for each $m \in \omega$ and for each choice sequence ξ. Since it will not cause any confusion, we shall use the same symbol for the constant and the corresponding element of the model. The language contains symbols for certain primitive recursive functions and relations defined on the elements of ω - e.g. $|x - y|$, $\leq$ etc. — and we also have the equality symbol $=$. It will be used in atomic formulas of the form $t = t'$ or $\xi(t) = t'$ where t and t' are terms of natural-number sort and ξ is a choice sequence (constant).

Atomic sentences receive truth values as follows:

1. $\|m \leq n\| = \begin{cases} \top & \text{if } m \leq n \\ \bot & \text{if } m \not\leq n \end{cases}$

   and similarly for other primitive-recursive relations.

2. Concerning choice sequences, we have:

   $\|\xi(m) = n\| = \xi(m, n).$

Arbitrary sentences φ receive truth values $\|\varphi\| \in H$ in the usual way:

1. $\|\varphi_1 \vee \varphi_2\| = \|\varphi_1\| \vee \|\varphi_2\|$

2. $\|\varphi_1 \wedge \varphi_2\| = \|\varphi_1\| \wedge \|\varphi_2\|$

3. $\|\varphi_1 \rightarrow \varphi_2\| = \|\varphi_1\| \rightarrow \|\varphi_2\|$

4. $\|\neg\varphi\| = \|\varphi\| \to \bot$

5. $\|\exists x\varphi(x)\| = \bigvee_{n\in\omega} \|\varphi(n)\|$

6. $\|\forall x\varphi(x)\| = \bigwedge_{n\in\omega} \|\varphi(n)\|$

7. $\|\exists\alpha\varphi(\alpha)\| = \bigvee_{\xi\in\Xi} \|\varphi(\xi)\|$

8. $\|\forall\alpha\varphi(\alpha)\| = \bigwedge_{\xi\in\Xi} \|\varphi(\xi)\|$

A sentence ψ is true in the model just in case $\|\psi\| = \top$.

In [5, pages 134-135] Vesley considers a species R of *real-number generators*: $\xi \in R$ if and only if the sequence $2^{-x}\xi(x)$ $(x \in \omega)$ of diadic fractions is a Cauchy-sequence with $\forall k\exists x\forall p|2^{-x}\xi(x) - 2^{-x-p}\xi(x+p)| < 2^{-k}$, i.e. if and only if $\forall k\exists x\forall p 2^k|2^p\xi(x) - \xi(x+p)| < 2^{x+p}$.

Equality, ordering, addition and multiplication on R are defined as follows.

1. $\xi = \eta$ if and only if $\forall k\exists x\forall p 2^k|\xi(x+p) - \eta(x+p)| < 2^{x+p}$,
2. $\xi < \eta$ if and only if $\exists k\exists x\forall p 2^k(\eta(x+p)\dot{-}\xi(x+p)) \geq 2^{x+p}$,
3. $(\xi+\eta)(x) := \xi(x) + \eta(x)$ and
4. $(\xi\eta)(x) := \lfloor 2^{-x}\xi(x)\eta(x)\rfloor$.

The following facts are also proved in [5]. If $\xi, \eta \in R$ then $\xi+\eta \in R$, $=$ is a congruence relation with respect to $<$ and $+$. Similar facts are true for multiplication as well (cf. also [6, pages 20-21]).

Let us expand the language L_1 with a unary predicate symbol R, binary predicate symbols $=$ and $<$, and binary function symbols for addition and multiplication on choice sequences.

The corresponding truth values may be defined as follows:

$$\|R(\xi)\| := \|\forall k\exists x\forall p 2^k|2^p\xi(x) - \xi(x+p)| < 2^{x+p}\|$$

using obvious abbreviations.

Similarly:

$$\|\xi = \eta\| := \|\forall k\exists x\forall p 2^k|\xi(x+p) - \eta(x+p)| < 2^{x+p}\|,$$

and

$$\|\xi < \eta\| := \|\exists k\exists x\forall p 2^k(\eta(x+p)\dot{-}\xi(x+p)) \geq 2^{x+p}\|.$$

Using the facts mentioned above, these definitions can be extended readily to polynomials of choice sequences.

ξ is a *global real-number generator* just in case $\|R(\xi)\| = \top$. Let $\mathcal{G}$ denote their set. $\mathcal{G}$ is closed under addition and multiplication. From now on elements of $\mathcal{G}$ will be denoted by f, g, h etc.

For each natural number n there is a corresponding global real-number generator f_n defined as follows: $f_n(l,m) = \top$ if $m = n2^l$ and $f_n(l,m) = \bot$ otherwise. Then

$$\|f_n f = \overbrace{f + \cdots + f}^{n}\| = \top.$$

If it does not cause any confusion, we shall denote $f_n \in \mathcal{G}$ by n.
Then $\mathcal{G}$ is a model for the language of ordered rings with addition and multiplication defined above where the interpretation of 0 and 1 is f_0 and f_1 respectively. Note that $=$ has the usual properties in the model, e.g. $f = g \leftrightarrow \neg(g < f \vee f < g)$, $f = g \wedge \varphi(f) \to \varphi(g)$ etc. has truth value $\top$. $f \neq g$ is defined as $f < g \vee g < f$. Finally note that in general $\neg f = g \not\leftrightarrow f \neq g$.

In what follows a *base* of a Heyting algebra is a set of elements with the property that every element of the algebra is the supremum of base elements.

If $\mathcal{H}$ has a base of complemented elements, than it is easy to see that quantification over reals in the corresponding model can be reduced to quantification over global real-number generators. I.e. in this case

$$\|\exists\alpha(R(\alpha) \wedge \varphi(\alpha))\| = \bigvee_{f\in\mathcal{G}} \|\varphi(f)\| \text{ and } \|\forall\alpha(R(\alpha) \to \varphi(\alpha))\| = \bigwedge_{f\in\mathcal{G}} \|\varphi(f)\|.$$

Definition 1. *A Heyting algebra $\mathcal{H}$ is* nice *if for any $g \in \mathcal{G}$ there is a subalgebra $\mathcal{H}_1 \leq \mathcal{H}$ containing the elements $g(l,m)$ with the following properties.*

1. *$\mathcal{H}_1$ has a base $\mathcal{D}$ of complemented elements forming a tree of height μ. We shall call μ the height of the algebra. Each base element $D \in \mathcal{D}$ has a level, an ordinal $\kappa < \mu$ such that D has level κ ($D \in \mathcal{D}_\kappa$) if and only if the following conditions hold.*
   - *(i)* $D \notin \bigcup_{\lambda<\kappa} \mathcal{D}_\lambda$
   - *(ii)* $\forall\lambda < \kappa\ \exists!\, D_\lambda \in \mathcal{D}_\lambda$ *such that* $D < D_\lambda$
   - *(iii)* $\forall D' \in \mathcal{D}(D' \in \bigcup_{\lambda<\kappa} \mathcal{D}_\lambda$ *or* $D' \wedge D = \bot$ *or* $D' \leq D)$
2. *If an element of $\mathcal{H}_1$ is the supremum of countably many disjoint elements, then it is the supremum of countably many pairwise disjoint complemented elements.*

3. *An element $A \in \mathcal{H}_1$ is maximal if $A \neq \top$ and for any $B \in \mathcal{H}_1$ if $A \leq B \leq \top$, then $B = A$ or $B = \top$. Each $C \in \mathcal{H}_1$, $C \neq \top$ is contained in a maximal element.*

In the next lemma we collect some simple facts about $\mathcal{D}$.

Lemma 2. *1. Elements on the same level are pairwise disjoint.*

2. *Let $D_i \in \mathcal{D}_{\lambda_i}$ $(i = 1, 2)$. If $\lambda_1 < \lambda_2$ and $D_1 \wedge D_2 \neq \bot$, then $D_2 < D_1$.*

3. *Let $D \in \mathcal{D}_\nu$, $\mathcal{D}' = \{D' \in \mathcal{D}_{\nu+1} \mid D' < D\}$. If $\mathcal{D}' \neq \emptyset$, let $D_1 = \bigvee \mathcal{D}'$. Then $D = D_1$.*

4. *If $D_1 \in \mathcal{D}_\nu$ and $D_1 < D_2$ then there is some $\lambda < \nu$ such that $D_2 \in \mathcal{D}_\lambda$.*

5. *For every $n \in \omega$ there is a finest pairwise disjoint cover $\mathcal{C}_n$ (ie. $\dot{\bigvee}\mathcal{C}_n = \top$) from base elements such that $\mathcal{D}_n \subseteq \mathcal{C}_n$ and $\mathcal{C}_n \subseteq \bigcup\{\mathcal{D}_k \mid k \leq n\}$.*

Proof. 1. Immediate from *(iii)*.

2. By *(ii)* there is a unique $D_1' \in \mathcal{D}_{\lambda_1}$ such that $D_2 < D_1'$, so if $D_1 \wedge D_2 \neq \bot$, then $D_1 \wedge D_1' \neq \bot$. Then from *1.* of this lemma follows that $D_1 = D_1'$.

3. We have to show that $D \leq D_1$, the other direction is obvious. Assume that $D \not\leq D_1$, let $D' \in \mathcal{D}'$ and $C = D \wedge \neg D'$. Then there is a base element $D" \leq C$ such that $D" \not\leq D_1$. From $D" < D$ and $D" \not\leq D_1$ follows that $D" \in \mathcal{D}_\lambda$ for some $\lambda > \nu + 1$. By *(ii)* there is a unique $D_{\nu+1} \in \mathcal{D}_{\nu+1}$ with $D" < D_{\nu+1}$. $D \wedge D_{\nu+1} \neq \bot$ follows, so by *2.*, $D_{\nu+1} < D$ and $D" \leq D_1$ follows, a contradiction.

4. Follows from *1.* and *2.*

5. Follows from *(iii)* and *3.* by induction on n. Start with $\mathcal{D}_\emptyset$ and refine elements using *3.* whenever it is possible.

□

Lemma 3. *1. Let us assume that $\mathcal{H}$ is a complete Heyting algebra with a base described in the previous definition. Using the notation above assume that $B \neq \top$, $B \in \mathcal{H}_1$. B contains a set $\mathcal{B} = \{B_i : i \in I\}$ of pairwise disjoint complemented elements of $\mathcal{H}_1$ (base elements) such that $B = \bigvee \mathcal{B}$ and for every $C \in \mathcal{H}_1$, if $C \vee B > B$ then there is some $i \in I$ with $B_i \leq C$.*

2. *For every element C of $\mathcal{H}_1$ there is a countably infinite set of pairwise disjoint elements $\mathcal{U} = \{U_n \mid n \in \mathbb{N}^+\}$ such that $C = \bigvee \mathcal{U}$.*

3. *If the algebra has height ω, then requirement 2 in the definition above follows from 1. in the following stronger form: any element C of $\mathcal{H}_1$ is the supremum of countably many pairwise disjoint complemented elements.*

Proof. 1. For $\lambda < \mu$ (μ is the height of the tree as above) let $\mathcal{B}_\lambda = \{D \in \mathcal{D}_\lambda : D \leq B \text{ and } \forall \nu < \lambda,\ \forall E \in \mathcal{B}_\nu,\ E \wedge D = \bot\}$.

Let $\mathcal{B} = \bigcup_{\lambda<\mu} \mathcal{B}_\lambda$. By definition $\bigvee \mathcal{B} \leq B$. For the other direction assume that there is some base element $D \in \mathcal{D}_\lambda$ (here we are using property (h)) such that $D \not\in \bigvee \mathcal{B}$. Then there are $\nu < \lambda$ and $E \in \mathcal{B}_\nu$ such that $D \wedge E \neq \bot$. Since $D \in \mathcal{D}_\lambda$, $\nu < \lambda$ and $E \in \mathcal{D}_\nu$, $D < E$. But then $D < \bigvee \mathcal{B}$. So $B = \bigvee \mathcal{B}$.

Now let $C \in \mathcal{H}_1$, $C \vee B > B$. Again there is some $D \in \bigcup_\lambda \mathcal{D}_\lambda$ such that $D \leq C$ but $D \not\leq B$. Then $D \neq \bot$ and since $\neg B = \bot$ (B is *dense*), $D \wedge B \neq \bot$. So there is some $D' \in \mathcal{D}_j$ with $j < \kappa$ minimal such that $D' \leq D \wedge B$.

We claim that $D' \in \mathcal{B}_j$. Assume that for some $\nu < j$ and $E \in \mathcal{B}_\nu$ E and D' are not disjoint. Then by properties (a) and (f) above $D' \leq E$. Then $D' \leq E \wedge D$, so $E \wedge D \neq \bot$. $E \leq D$ would contradict the minimality of j, so by property (a) $D < E$, but then $E \not\leq B$ follows, a contradiction again. So $D' \in \mathcal{B}_j$ and $D' \leq C$ proving the claim.

2. Use the construction from *1.* inside C and partition the set $\mathcal{B}$ obtained into countably many disjoint sets $\mathcal{B} = \dot{\bigcup}\{\mathcal{B}_n \mid n \in \mathbb{N}\}$. Let $U_n = \bigvee \mathcal{B}_n$. Then the set $\mathcal{U} = \{U_n \mid n \in \mathbb{N}^+\}$ has the required property. Note that since $\mathcal{B}$ is a set of pairwise disjoint elements, $\mathcal{U}$ is a set of pairwise disjoint, but not necessarily complemented elements.

3. Here we use the results and notations of Lemma 2. Since $\mathcal{C}_n$ is a disjoint cover for every $n \in \omega$, the supremum $\dot{\bigvee}\mathcal{C}'_n$ of any subset $\mathcal{C}'_n \subseteq \mathcal{C}_n$ is complemented, if $\mathcal{C}_n" = \mathcal{C}_n \setminus \mathcal{C}'_n$ then $\dot{\bigvee}\mathcal{C}'_n \dot{\vee} \dot{\bigvee}\mathcal{C}_n" = \top$ and $\dot{\bigvee}\mathcal{C}'_n \wedge \dot{\bigvee}\mathcal{C}_n" = \bot$. For $i \in \omega$ let $\mathcal{E}_i = \{E \in \mathcal{C}_i \mid E \leq C \wedge \neg\dot{\bigvee}\{E_j \mid j < i\}$ and $E_i = \dot{\bigvee}\mathcal{E}_i$. Then, as a supremum of a subset of $\mathcal{C}_i$, E_i is complemented. $\mathcal{E} = \{E_i \mid i \in \omega\}$ is a countable set of pairwise disjoint complemented elements. We claim that $C = \dot{\bigvee}\{E_i \mid i \in \omega\}$. $\dot{\bigvee}\{E_i \mid i \in \omega\} \leq C$ is true by definition. For the other direction, if $C \not\leq \dot{\bigvee}\{E_i \mid i \in \omega\}$ then there is a base element $D_k \in \mathcal{D}_k$ such that $D_k \leq C$ but $D_k \not\leq \dot{\bigvee}\{E_i \mid i \in \omega\}$, in particular for every $i \in \omega$ and $E \in \mathcal{E}_i$, $D_k \not\leq E$. If $i < k$ then, since $\mathcal{C}_i \subseteq \bigcup\{\mathcal{D}_j \mid j \leq i\}$, $E \not\leq D_k$. Thus, for every $i < k$ and $E \in \mathcal{E}_i$, $D_k \wedge E = \bot$, so $D_k \leq C \wedge \neg\dot{\bigvee}\{E_i \mid i < k\}$. But then, since $\mathcal{D}_k \subseteq \mathcal{C}_k$, $D_k \in \mathcal{E}_k$, so $D_k \leq \dot{\bigvee}\{E_i \mid i \in \omega\}$, a contradiction.

□

Example 1. Let κ be any cardinal with the discrete topology, and $\mathcal{H}$ be the Heyting algebra of the open sets of $X = {}^{\omega}\kappa$. Then $\mathcal{H}$ with $\mathcal{H}_1 = \mathcal{H}$ fulfills the requirements of Definition 1. Note that the open set algebra gives us Scott's model in the case of $\kappa = \omega$ (see [3]).

Proof. The height of the tree of base elements required in *Definition1.1.* is ω. For $\sigma \in {}^{n}\kappa$ $(n \in \omega)$ let $B_\sigma = \{p \in {}^{\omega}\kappa \mid \sigma \subset p\}$. Let $\mathcal{D}_n = \{B_\sigma \mid \sigma \in {}^{n}\kappa\}$ and $\mathcal{D} = \bigcup_{n\in\omega} \mathcal{D}_n$. For every $p \in {}^{\omega}\kappa$, $U_p = {}^{\omega}\kappa \setminus \{p\}$ is a maximal element and $\bigwedge\{U_p \mid p \in {}^{\omega}\kappa\} = \bot$. $\square$

Example 2. Let again κ be an infinite cardinal, but now let $\mathcal{H}$ be the Heyting algebra of the open sets of $X = {}^{\kappa}({}^{\omega}2)$ with the product topology. For each $g \in \mathcal{G}$ there is a subalgebra $\mathcal{H}_1$ isomorphic to the open-set algebra of ${}^{\omega}2$ and containing the elements $g(m, n)$ – the subalgebra of elements with support equal to the support of g (see [7]). Then $\mathcal{H}$ is nice, this can be shown by using the isomorphism between $\mathcal{H}_1$ and the open-set algebra of ${}^{\omega}2$ and the previous example. Note that if $\kappa > \omega$ than these models are elementarily equivalent to Krol's model defined in [8] (cf. [7]).

Note. The results in [1] were true for a third class of Heyting algebras, the algebras of the coperfect open sets of $X = {}^{\omega}\kappa$. In particular true first order arithmetic can be interpreted in the corresponding real algebras (see below). These Heyting algebras however have no maximal elements, so the proof of the interpretability of second order arithmetic below does not go through.

Definition 4. *Let $h_1, h_2 \in \mathcal{G}$ global real number generators, $B \in \mathcal{H}_1$. A positive natural number n is an* NE-quotient *(non-excluded quotient) of h_1, h_2 with respect to B if $\|\neg n h_2^W = h_1\| \leq B$.*

We shall encode subsets of natural numbers as NE-quotients of appropriate elements of $\mathcal{G}$.

Definition 5. *Using the notation above, for each $W \subseteq \mathbb{N}$ we define the encoding real number generators in a nice Heyting algebra $\mathcal{H}$ as follows. Let us assume that $B \neq \top$, $\mathcal{B} = \{B_i : i \in I\}$ has the property of the previous lemma. For every $i \in I$ let $\{U_i^n \neq \bot : n \in \mathbb{N}^+\}$ be a set of disjoint elements with $\bigvee_{n\in\mathbb{N}^+} U_i^n = B_i$ (using property* 2. *of nice Heyting algebras) and $U_i^1 = B_i \wedge \neg \bigvee_{n>1} U_i^n$. Let $U^n = \bigvee_{i\in I} U_i^n$. Using property 3., let $\bigvee_{n\in\mathbb{N}^+} U^n = \bigvee_{i\in\omega} E_i$ where $\{E_i\}$ is a countable sequence of pairwise disjoint complemented elements. Let $W \subseteq \mathbb{N}^+$, $W \neq \emptyset$, $F : \mathbb{N}^+ \to W$ an onto function. Let*

$$h_1(l, m) = \begin{cases} (\bigvee_{i\leq l} E_i)^c & \text{if } m = 0 \\ E_i & \text{if } m = 2^{l-i},\ (0 \leq i \leq l) \\ \bot & \text{otherwise,} \end{cases}$$

$$h_2^W(l,m) = \begin{cases} (\bigvee_{i\leq l} E_i)^c & \text{if } m=0 \\ \bigvee\{E_i \wedge U^n : m = \lceil \frac{2^{l-i}}{F(n)} \rceil,\ 0\leq i\leq l\} & \text{if } m = \lceil \frac{2^{l-i}}{F(n)} \rceil \text{ for some } n \text{ and } i \\ \bot & \text{otherwise.} \end{cases}$$

Note that $m = \lceil \frac{k}{n} \rceil$ (m is the least integer greater than or equal to the rational number k/n) is definable in our language as $k \leq mn \wedge mn < k+n$.

If $W = \mathbb{N}^+$ then we shall use the notation h_2 for $h_2^{\mathbb{N}}$ with the function F being the identity function.

For $W = \emptyset$ let $h_2^{\emptyset} = 0$.

Note that every element of the form $h_2^W(m,k)$ and $h_1(m,k)$ is in $\mathcal{H}_1$.

Lemma 6. *Using the notation above here we extend and modify Lemma 3. in [1].*

1. $U^l \wedge U^m = \bot$ *if* $l \neq m$.
2. $\neg \bigvee_{n\in\mathbb{N}^+} U^n = \neg B = \bot$.
3. $h_2^W, h_1 \in \mathcal{G}$, *ie.* $\|R(h_2^W)\| = \|R(h_1)\| = \top$.
4. $\bigvee_{n\in\mathbb{N}^+} U^n = \|h_1 \neq 0\| = \|h_2^W \neq 0\|$
5. $U^n \leq \|F(n)h_2^W = h_1\|$
6. $\neg \bigvee_{n\in\mathbb{N}^+} \|F(n)h_2^W = h_1\| = \|h_1 = 0\| = \|h_2^W = 0\| = \neg \bigvee_{n\in\mathbb{N}^+} U^n = \bot$
7. *For all* $k \in \mathbb{N}^+$, $\|\neg k h_2^W = h_1\| \leq B$ *if and only if* $k = F(n)$ *for some* $n \in \mathbb{N}$, *ie.* W *is the set of NE-quotients of* h_1, h_2^W *with respect to* B. *In particular for all* $n \in \mathbb{N}^+$, $\|\neg n h_2 = h_1\| \leq B$.
8. *For all* $k, l \in \mathbb{N}^+$ *if* $k \neq l$ *then* $\|l h_2^W = h_1\| \leq \|\neg k h_2^W = h_1\|$.
9. *For all* $n \in \mathbb{N}^+$ $\|n h_2 = h_1\| \leq \|\neg\neg F(n) h_2^W = h_1\|$.
10. *For all* $n, k \in \mathbb{N}^+$ *if* $k \neq F(n)$ *then* $\|n h_2 = h_1\| \leq \|\neg k h_2^W = h_1\|$.
11. $\neg\neg \bigvee_{n\in\mathbb{N}^+} \|F(n)h_2^W = h_1\| = \top$.
12. *If for all* $n \in \mathbb{N}^+$ *if* $k \neq F(n)$ *then* $\|\neg k h_2^W = h_1\| = \top$.
13. *If* $W = \emptyset$, $\|\neg k h_2^{\emptyset} = h_1\| = \top$ *for every* $k \in \mathbb{N}^+$.
14. *For any element* $B \in \mathcal{H}_1$ *there is* $g \in \mathcal{G}$ *such that* $B = \|g \neq 0\|$

Proof. First assume that $W \neq \emptyset$.

1. $U^l \wedge U^m = \bot$ if $l \neq m$ immediately follows from the definition.

2. $\neg B = \bot$ by assumption (B is dense). For $\neg \bigvee_{n\in\mathbb{N}^+} U^n = \bot$ we have

$$\neg \bigvee_{n\in\mathbb{N}^+} U^n = \neg \bigvee_{n\in\mathbb{N}^+} \bigvee_{i\in I} U_i^n = \bigwedge_{i\in I} \bigwedge_{n\in\mathbb{N}^+} \neg U_i^n = \bigwedge_{i\in I}(\neg U_i^1 \wedge \bigwedge_{n>1} \neg U_i^n)$$

$$= \bigwedge_{i\in I}(\neg(B_i \wedge \neg \bigvee_{n>1} U_i^n) \wedge \neg \bigvee_{n>1} U_i^n) \leq \bigwedge_{i\in I} \neg B_i = \neg \bigvee_{i\in I} B_i = \neg B = \bot$$

3. $h_2^W \in \mathcal{G}$, ie. $\|R(h_2^W)\| = \top$. ($\|R(h_1)\| = \top$ is similar.)

   First of all h_2^W is a choice sequence, since $h_2^W(l,m)$ and $h_2^W(l,n)$ are obviously disjoint if $m \neq n$, and

$$\bigvee_{m\in\omega} h_2^W(l,m) = (\bigvee_{i\leq l} E_i)^c \vee \bigvee_{i\leq l} \bigvee_{n\in\mathbb{N}^+} (E_i \wedge U^n) =$$

$$(\bigvee_{i\leq l} E_i)^c \vee \bigvee_{i\leq l}(E_i \wedge (\bigvee_{n\in\mathbb{N}^+} U^n)) = (\bigvee_{i\leq l} E_i)^c \vee \bigvee_{i\leq l}(E_i \wedge (\bigvee_{j\in\omega} E_j)) = \top.$$

   Next we want to show that $\top \leq \|\forall k \exists x \forall p 2^k |2^p h_2^W(x) - h_2^W(x+p)| < 2^{x+p}\|$. Fix $k \in \omega$ and let $x > k$. Then for all $i \leq x$, $p \in \omega$ and $n \in \mathbb{N}^+$,

$$E_i \wedge U^n \leq \|h_2^W(x+p) = \left\lceil \frac{2^{x+p-i}}{F(n)} \right\rceil\|.$$

   Then

$$E_i \wedge U^n \leq \|2^{x+p-i} \leq h_2^W(x+p)F(n) \wedge h_2^W(x+p)F(n) \leq 2^{x+p-i} + F(n)\|$$

   and

$$E_i \wedge U^n \leq \|2^{x+p-i} \leq 2^p h_2^W(x)F(n) \wedge 2^p h_2^W(x)F(n) \leq 2^{x+p-i} + 2^p F(n)\|$$

   we have

$$E_i \wedge U^n \leq \|2^k|2^p h_2^W(x) - h_2^W(x+p)| < 2^k(2^p+1)\| \wedge \|2^k(2^p+1) \leq 2^{x+p}\|.$$

   Also,

$$(\bigvee_{i\leq x} E_i)^c \leq \|h_2^W(x) = 0\| \wedge \|h_2^W(x+p) \leq 2^{x+p-(x+1)}\| \wedge \|2^{x+p-(x+1)} = 2^{p-1}\|$$

for all p, so

$$(\bigvee_{i\le x} E_i)^c \le \bigwedge_{p\in\omega} \|2^k|2^p h_2^W(x) - h_2^W(x+p)| \le 2^{k+p-1}\| \wedge \|2^{k+p-1} < 2^{x+p}\|.$$

From these

$$\top \le \|\forall k \exists x \forall p 2^k|2^p h_2^W(x) - h_2^W(x+p)| < 2^{x+p}\|$$

follows.

4. $\bigvee_{n\in\mathbb{N}^+} U^n = \|h_1 \ne 0\| = \|h_2^W \ne 0\|$.

   We show only that $\bigvee_{n\in\mathbb{N}^+} U^n = \|h_2^W \ne 0\|$, $\bigvee_{n\in\mathbb{N}^+} U^n = \|h_1 \ne 0\|$ is similar.

   Since $(\bigvee_{i\le x+p} E_i)^c \le \|h_2^W(x+p) = 0\|$, $\|2^k h_2^W(x+p) \ge 2^{x+p}\| \le \bigvee_{i\le x+p} E_i$.

   From this follows that

$$\|h_2^W \ne 0\| = \bigvee_{k\in\omega}\bigvee_{x\in\omega}\bigwedge_{p\in\omega} \|2^k h_2^W(x+p) \ge 2^{x+p}\| \le \bigvee_{k\in\omega}\bigvee_{x\in\omega}\bigwedge_{p\in\omega}\bigvee_{i\le x+p} E_i =$$

$$\bigvee_{k\in\omega}\bigvee_{x\in\omega}\bigvee_{i\le x} E_i = \bigvee_{k\in\omega}\bigvee_{x\in\omega} E_x = \bigvee_{x\in\omega} E_x = \bigvee_{n\in\mathbb{N}^+} U^n.$$

   On the other hand for $n \in \mathbb{N}^+$ and $i \in \omega$ fixed, if $F(n) < 2^y$ and $x = k = y + i$, then for all $p \in \omega$,

$$E_i \wedge U^n \le \|2^k h_2^W(x+p) = 2^k\left\lceil \frac{2^{x+p-i}}{F(n)} \right\rceil\| \wedge \|2^k\left\lceil \frac{2^{x+p-i}}{F(n)} \right\rceil > 2^{k+x+p-i-y}\| \wedge$$

$$\|2^{k+x+p-i-y} = 2^{x+p}\|.$$

   From this $\bigvee_{n\in\mathbb{N}^+} U^n \le \|h_2^W \ne 0\|$ follows.

5. We claim that for all $n \in \mathbb{N}^+$ and $i \in \omega$, $E_i \wedge U^n \le \|F(n)h_2^W = h_1\|$. Since $U^n = \bigvee_{i\in\omega}(E_i \wedge U^n)$, from this follows that $U^n \le \|F(n)h_2^W = h_1\|$ as claimed.

   If $i \le l$ then

$$E_i \wedge U^n \le \|h_2^W(l) = \left\lceil \frac{2^{l-i}}{F(n)} \right\rceil\| \wedge \|h_1(l) = 2^{l-i}\|,$$

   so for all $k \in \omega$, if $x > k$, $x > i$ and $2^x > F(n)2^k$, then for all $p \in \omega$,

$$E_i \wedge U^n \le \|2^k|F(n)h_2^W(x+p) - h_1(x+p)| < F(n)2^k\| \wedge \|F(n)2^k < 2^{x+p}\|.$$

   From this the statement follows.

6. $\neg \bigvee_{n\in\mathbb{N}^+} \|F(n)h_2^W = h_1\| = \|h_1 = 0\| = \|h_2^W = 0\| = \neg \bigvee_{n\in\mathbb{N}^+} U^n = \bot$ follows from *2.*, *4.* and *5.*

7. First we show that for all $n \in \mathbb{N}^+$ $\|\neg F(n)h_2^W = h_1\| \leq B$. Otherwise $\|\neg F(n)h_2^W = h_1\| \vee B > B$, so by Lemma 3. $B_i \leq \|\neg F(n)h_2^W = h_1\|$ for some $i \in I$. Thus $U_i^n \leq \|\neg F(n)h_2^W = h_1\|$, but $U_i^n \leq \|F(n)h_2^W = h_1\|$, a contradiction.

   Now let us assume that $k \notin W$. Then $\|kh_2^W = h_1\| \wedge \bigvee_{n\in\mathbb{N}} \|F(n)h_2^W = h_1\| = \bot$, so $\|kh_2^W = h_1\| \leq \neg \bigvee_{n\in\mathbb{N}} \|F(n)h_2^W = h_1\| = \bot$ and $\|\neg kh_2^W = h_1\| = \top$ and thus k is not an NE-quotient.

8. Assume that $k \neq l \in \mathbb{N}^+$. Then $\|lh_2^W = h_1\| \wedge \|kh_2^W = h_1\| = \|lh_2^W = h_1\| \wedge \|kh_2^W = h_1\| \wedge \|k \neq l\| \leq \|h_2^W = 0\| = \bot$, so $\|lh_2^W = h_1\| \leq \|\neg kh_2^W = h_1\|$.

9. For $n \in \mathbb{N}^+$ let $A_n = \|nh_2 = h_1\| \wedge \|\neg F(n)h_2^W = h_1\|$. We claim that $A_n = \bot$, from this $\|nh_2 = h_1\| \leq \|\neg\neg F(n)h_2^W = h_1\|$ follows. Since $A_n \leq \|nh_2 = h_1\|$ they are pairwise disjoint. Also, for all $n \in \mathbb{N}^+$ $A_n \wedge U^n = \bot$, since $A_n \leq \|\neg F(n)h_2^W = h_1\|$ and $U^n \leq \|F(n)h_2^W = h_1\|$. From *3.* and the fact that $U^k \leq \|kh_2 = h_1\|$ and $A_n \leq \|nh_2 = h_1\|$ follows that if $k \neq n$ then $U^k \wedge A_n = \bot$. So $A_n \vee (\bigvee_{k\in\mathbb{N}^+} U^k)$ is a disjoint union. Then from $\neg \bigvee_{n\in\mathbb{N}^+} U^n = \bot$ (see *1.*) follows that $A_n = \bot$ as claimed.

10. Assume $k \neq F(n)$. From *8.* $\|F(n)h_2^W = h_1\| \leq \|\neg kh_2^W = h_1\|$, so $\|\neg\neg F(n)h_2^W = h_1\| \leq \|\neg\neg\neg kh_2^W = h_1\|$, ie. $\|\neg\neg F(n)h_2^W = h_1\| \leq \|\neg kh_2^W = h_1\|$. By *9.* $\|nh_2 = h_1\| \leq \|\neg\neg F(n)h_2^W = h_1\|$. From these $\|nh_2 = h_1\| \leq \|\neg kh_2^W = h_1\|$ as claimed.

11. $\neg\neg \bigvee_{n\in\mathbb{N}^+} \|F(n)h_2^W = h_1\| = \top$ follows from $\neg \bigvee_{n\in\mathbb{N}^+} \|F(n)h_2^W = h_1\| = \bot$.

12. If for all $n \in \mathbb{N}^+$ if $k \neq F(n)$ then by *10.* for all $n \in \mathbb{N}^+$ $\|nh_2 = h_1\| \leq \|\neg kh_2^W = h_1\|$, ie. $\bigvee_{n\in\mathbb{N}^+} \|nh_2 = h_1\| \leq \|\neg kh_2^W = h_1\|$. Then by *11.* $\top = \neg\neg \bigvee_{n\in\mathbb{N}^+} \|nh_2 = h_1\| \leq \neg\neg\|\neg kh_2^W = h_1\| = \|\neg kh_2^W = h_1\|$ and the statement follows.

13. $\|h_2^\emptyset = 0\| = \top$ by definition, so $\|kh_2^\emptyset = h_1\| \leq \|h_1 = 0\| = \bot$ and $\|\neg kh_2^\emptyset = h_1\| = \top$.

14. Use the construction of the previous definition.

□

4 Coding $\mathbb{N}^+$

This section is from [1]. The variables x, y, u, v, w etc. range over reals, k, l, n will range over natural numbers. From now on let $B(y)$ denote the L-formula $y = 0 \vee y \neq 0$. Let $\varphi_{\mathbb{N}^+}(x, y, u, v) \equiv$

$$\neg x < 1 \wedge (\neg v = u \vee \neg xv = u \rightarrow B(y)) \wedge \forall w[(\neg wv = u \rightarrow B(y)) \rightarrow$$

$$[(w < 1 \rightarrow B(y)) \wedge (w > 1 \rightarrow \exists w'(w \neq w' \vee \neg w'v = u + v \rightarrow B(y)))]].$$

Note that $x \neq y$ is defined as $x < y \vee y < x$ and $\neg x = y$, which is equivalent to $\neg\neg x \neq y$, is weaker intuitionistically than $x \neq y$.

The following sentences are used as axioms, they are true in the models of intuitionistic second-order arithmetic we have mentioned:

1. $\forall y \exists u \exists v(\forall n \in \mathbb{N}^+(\neg nv = u \rightarrow B(y) \wedge \neg v = 0 \wedge \neg\neg\exists n \in \mathbb{N}^+ (nv = u))$;
2. $\neg\forall y B(y)$.

Note that if $y = 0 \vee y \neq 0$ is true, then in 1. arbitrary $u = v \neq 0$ work, so this sentence is true classically.

Theorem 7. *(From [1]) Let $\psi_{\mathbb{N}^+}(x)$ denote the L-formula $\forall y \exists u \exists v \varphi_{\mathbb{N}^+}(x, y, u, v)$. Then from the axioms and the properties of real numbers mentioned above and from the usual axioms of natural numbers $\exists k \in \mathbb{N}^+(x = k) \equiv \psi_{\mathbb{N}^+}(x)$ follows in two-sorted intuitionistic predicate calculus with equality. In particular the statement holds in our models: for all $h \in \mathcal{G}$, $\bigvee_{k \in \mathbb{N}^+} \|h = k\| = \|\psi_{\mathbb{N}^+}(h)\|$.* $\square$

5 Coding Second-Order Arithmetic

Let $\varphi(\vec{x}, \vec{S})$ be a second-order formula of the language $L' = \langle 1, +, \times \rangle$. Here $\vec{x}$ is a tuple of first-order variables, $\vec{S}$ is a tuple of second-order variables. Without loss of generality, we assume that φ does not contain any implications. $\varphi_1(S/\emptyset)$ denotes the formula obtained from φ by regarding S as the empty set: replace all subformulas of φ of the form $x \in S$ where x is a numeric variable with the (false) formula $\neg x = x$. For each second-order variable S let v_S be a new variable, and let y be a new variable occuring only in the indicated places. The formula φ will be encoded by $\tilde{\varphi}(\vec{x}, \vec{v}_S, y, u)$, a formula of the language of ordered rings. Here the free variables in $\vec{x}$ correspond to free natural number variables in φ, $\vec{v}_S$ corresponds to free second order variables in φ, y corresonds to $g \in \mathcal{G}$ above, u to h_1 and v to h_2, the pair of reals in the model encoding the set of natural numbers. $\tilde{\varphi}$ is defined inductively as follows ($\tilde{\varphi}$ may contain other variables than the ones indicated):

(i) If φ is first-order atomic, then $\tilde{\varphi} \equiv \neg\varphi \to y \neq 0$;

(ii) $\widetilde{x \in S} \equiv \neg x v_S = u \to y \neq 0$;

(iii) $\widetilde{\varphi_1 \circ \varphi_2} \equiv \tilde{\varphi}_1 \circ \tilde{\varphi}_2$ where $\circ = \wedge, \vee$;

(iv) $\widetilde{\neg\varphi_1} \equiv \tilde{\varphi}_1 \to y \neq 0$;

(v) $\widetilde{\exists x \varphi_1(x)} \equiv \exists x(\psi_{\mathbb{N}^+}(x) \wedge \tilde{\varphi}_1(x))$;

(vi) $\widetilde{\forall x \varphi_1(x)} \equiv \forall x(\psi_{\mathbb{N}^+}(x) \to \tilde{\varphi}_1(x))$;

(vii) $\widetilde{\exists S \varphi_1(S)} \equiv \widetilde{\varphi_1(S/\emptyset)} \vee \exists v_S \exists u \; \tilde{\varphi}_1(v_S, u)$;

(viii) $\widetilde{\forall S \varphi_1(S)} \equiv \forall v_S \forall u \; \tilde{\varphi}_1(v_S, u)$.

In the next lemmas let g be a fixed element of $\mathcal{G}$ used in the definition of h_2^W and h_1, let $A_g = \|g \neq 0\|$ and let $C_g = \|\zeta(g)\|$. For $k \in \mathbb{N}^+$ k also denote the corresponding real-number generator as before.

Definition 8. *Let $U \in \mathcal{H}_1$. An element $C \in \mathcal{H}_1$ is maximal in U if $C < U$ and for any $C' \in \mathcal{H}_1$ if $C \leq C' \leq U$ then either $C' = C$ or $C' = U$.*

Lemma 9. *Let $A < \top$ and M maximal. If $A < M \to A$, then A is maximal in $M \to A$.*

Proof. Let $A \leq C \leq M \to A$. If $M = M \wedge C$, then $C \leq M$, so since $C \leq M \to A$, $C \leq A$ and $C = A$ follows. Otherwise $M < M \vee C$. Then, since M is maximal by assumption, $M \vee C = \top$, so $M \to A = (M \to A) \wedge (M \vee C) = ((M \to A) \wedge M) \vee ((M \to A) \wedge C) \leq A \vee C = C$, so $C = M \to A$ and we are done. □

Lemma 10. *Let $A < \top$. For each complemented $D \not\leq A$ there is an element $B \in \mathcal{H}_1$ such that $B \leq D \vee A$ and $A = B$ or A is maximal in B.*

Proof. Let E be a maximal element such that $\neg D \vee A \leq E$. By our assumption on $\mathcal{H}_1$ such an element exists. If $E = A$, then $B = \top$ fulfills the requirements. If A is maximal in E, then with $B = E$ we are done. Otherwise let $B = E \to A$. First we claim that $E \to A \leq D \vee A$. $E \to A \leq (\neg D \vee A) \to A \leq \neg D \to A = (\neg D \to A) \wedge (D \vee \neg D) = ((\neg D \to A) \wedge D) \vee A \leq D \vee A$.
Next assume that $A < B$. Then A is maximal in B by Lemma 9. □

Lemma 11. *1. If $C \in \mathcal{H}_1$ is maximal with $C = \|g \neq 0\|$ for some $g \in \mathcal{G}$ in B, then $B \leq \|\zeta(g)\|$.*

2. Let $A_g < C_g$ (notation as above) and let $\mathcal{C} = \{C \leq C_g \mid A_g$ is maximal in $C\}$. Then $\bigvee \mathcal{C} = C_g$.

3. $C = \bigwedge\{B \in \mathcal{H}_1 \mid B$ is maximal $\} = \bot$

Proof. 1. Let $h \in \mathcal{G}$ arbitrary, $A = \|h \neq 0\|$. We have to show that $B \leq (A \to C) \vee A$. Since C is maximal, either $C = (C \vee A) \wedge B$, or $(C \vee A) \wedge B = B$. In the first case $C = (C \vee A) \wedge B = (C \wedge B) \vee (A \wedge B) = C \vee (A \wedge B)$, so $A \wedge B \leq C$ and then $B \leq A \to C$, so $B \leq (A \to C) \vee A$. In the second case $B = (C \vee A) \wedge B = (C \wedge B) \vee (A \wedge B) \leq (A \to C) \vee A$ and we are done.

2. Assume that $C_g \not\leq \bigvee \mathcal{C}$. Then there is a complemented $D \in \mathcal{H}_1$ such that $D \leq C_g$ and $D \not\leq \bigvee \mathcal{C}$. By Lemma 10. there is $B \leq D \vee C_g$ such that A_g is maximal in B. $\neg D \vee \bigvee \mathcal{C} < \top$, so there is a maximal element M such that $\neg D \vee \bigvee \mathcal{C} \leq M$. There is $h \in \mathcal{G}$ such that $M = \|h \neq 0\|$. If $A_g = M \to A_g$, then $D \leq C_g \leq (M \to A_g) \vee M = A_g \vee M$, so $\top = D \vee \neg D \leq A_g \vee M$, but $A_g \leq \bigvee \mathcal{C} \leq M$, so $M = \top$ follows, a contradiction. So $A_g < M \to A_g$ and by Lemma 9 A_g is maximal in $M \to A_g$. Thus by part *1.* $M \to A_g \leq C_g$, so $M \to A_g \in \mathcal{C}$ and $M \to A_g \leq \bigvee \mathcal{C} \leq M$. Then $M \to A_g \leq A_g$, a contradiction.

3. Suppose that $C > \bot$ and let $D < C$ a complemented element. Then $\neg D$ is contained in a maximal element $M \neq \top$, so $\top = D \vee \neg D \leq C \vee \neg D \leq M$, a contradiction.

□

Lemma 12. *Using the notation above, for each second order L'-formula φ, $A_g \leq \|\tilde{\varphi}(g)\|$.*

Proof. By formula induction. □

Lemma 13. *Let $\varphi(\vec{x}, \vec{S})$ be a second-order L'-formula, $\vec{a}$ be a tuple of positive integers, and $\vec{W}$ be a tuple of subsets of $\mathbb{N}^+$. Let B be an arbitrary element such that A_g is maximal in B. Using the notation of Theorem 6 ($\tilde{\varphi}$ may contain other parameters than the ones indicated):*

1. *If* $\mathbb{N}^+ \models \varphi[\vec{a}, \vec{W}]$ *then* $\|\tilde{\varphi}[\vec{a}, \vec{h}_2^W, g, h_1]\| \geq B$

2. *If* $\mathbb{N}^+ \not\models \varphi[\vec{a}, \vec{W}]$ *then* $\|\tilde{\varphi}[\vec{a}, \vec{h}_2^W, g, h_1]\| \leq A_g$

3. *Let* h_2', h_1' *be an arbitrary pair of elements of* $\mathcal{G}$, *let* $W = \{k \in \mathbb{N}^+ \mid \|\neg k h_2' = h_1'\| \to A_g \geq B\}$, h_2^W *the element of* $\mathcal{G}$ *corresponding to* W, *then if* $W \neq \emptyset$,

$$\|\tilde{\varphi}[h_2', g, h_1']\| \geq B \Leftrightarrow \|\tilde{\varphi}[h_2^W, g, h_1]\| \geq B$$

$$\|\tilde{\varphi}[h_2', g, h_1']\| \leq A_g \Leftrightarrow \|\tilde{\varphi}[h_2^W, g, h_1]\| \leq A_g$$

Ie. each element h_2' *of* $\mathcal{G}$ θ *can be regarded as* h_2^W *for the appropriate subset* $W \subseteq \mathbb{N}^+$.

Proof. By formula induction.

(i) If φ is first-order atomic and $\mathbb{N}^+ \models \varphi$ then $\|\neg\varphi\| = \bot$ and $\|\neg\varphi\| \to A_g = \top$. If $\mathbb{N}^+ \not\models \varphi$, then $\|\neg\varphi\| = \top$ and $\|\neg\varphi\| \to A_g = A_g$. Finally *3.* holds since $\tilde{\varphi}$ does not contain the variable v_S.

(ii) $\widetilde{x \in S} \equiv \neg x v_S = u \to B(y)$.

1. follows from Theorem 6.7: $\|\neg a h_2^W = h_1\| \to A_g = \top \geq B$.

2. Assume that $\mathbb{N}^+ \not\models a \in W$. It is enough to show that $\|\neg a h_2^W = h_1\| = \top$. First assume that $W \neq \emptyset$ and let $F : \mathbb{N}^+ \to W$ be the surjection corresponding to W. Then $\forall n \in \mathbb{N}^+ F(n) \neq a$, so by Theorem 6.10. $\forall n \in \mathbb{N}^+ \; \|n h_2 = h_1\| \leq \|\neg a h_2^W = h_1\|$, so $\bigvee_{n \in \mathbb{N}^+} \|n h_2 = h_1\| \leq \|\neg a h_2^W = h_1\|$. Using Theorem *6.11*, $\top = \neg\neg \bigvee_{n \in \mathbb{N}^+} \|n h_2 = h_1\| \leq \neg\neg\|\neg a h_2^W = h_1\| = \|\neg a h_2^W = h_1\|$.

If $W = \emptyset$ then $\|\neg a h_2^W = h_1\| = \top$ by Theorem *6.13*.

3. If $a \in W$ then by *1.* $\|\neg a h_2^W = h_1\| \to A_g \geq B$. Also, by the definition of W, $\|\neg a h_2' = h_1'\| \to A_g \geq B$.

If $a \notin W$ then $\|\neg a h_2^W = h_1\| = \top$ by Theorem *6.12*, so $\|\neg a h_2^W = h_1\| \to A_g = A_g$. By the definition of W, $\|\neg a h_2' = h_1'\| \to A_g \not\geq B$ so, since A_g is maximal in B, $\|\neg a h_2' = h_1'\| \to A_g \leq A_g$ and we are done.

(iii) $\widetilde{\varphi_1 \wedge \varphi_2} \equiv \tilde{\varphi}_1 \wedge \tilde{\varphi}_2$. For *1.* if $\mathbb{N}^+ \models \varphi_1 \wedge \varphi_2$, then $\mathbb{N}^+ \models \varphi_1$ and $\mathbb{N}^+ \models \varphi_2$. By the inductive hypothesis $\|\tilde{\varphi}_1\| \geq B$ and $\|\tilde{\varphi}_2\| \geq B$, the statement follows. For *2.* if $\mathbb{N}^+ \not\models \varphi_1 \wedge \varphi_2$ then $\mathbb{N}^+ \not\models \varphi_1$ or $\mathbb{N}^+ \not\models \varphi_2$. Let us assume that $\mathbb{N}^+ \not\models \varphi_1$. Then by the inductive hypothesis $\|\tilde{\varphi}_1\| \leq A_g$. From this $\|\tilde{\varphi}\| \leq A_g$. *3.* again easily follows from the inductive hypothesis and the maximality of A_g.

(iv) $\widetilde{\varphi_1 \vee \varphi_2} \equiv \tilde{\varphi}_1 \vee \tilde{\varphi}_2$. Similar to the previous case.

(v) $\widetilde{\neg\varphi_1} \equiv \tilde{\varphi}_1 \to B(y)$. Let us assume that $\mathbb{N}^+ \models \neg\varphi_1$. By the inductive hypothesis $\|\tilde{\varphi}_1\| \leq A_g$, so $\|\tilde{\varphi}_1\| \to A_g = \top \geq B$. If $\mathbb{N}^+ \not\models \neg\varphi_1$, ie. $\mathbb{N}^+ \models \varphi_1$, then $\|\tilde{\varphi}_1\| \geq B$. If $\|\tilde{\varphi}_1\| \to A_g \geq B$, then $B \leq (\|\tilde{\varphi}_1\| \wedge (\|\tilde{\varphi}_1\| \to A_g) \leq A_g$, so $B \leq A_g$, a contradiction. From this $\|\tilde{\varphi}_1\| \to A_g \not\geq B$ and by the maximality of A_g, $\|\tilde{\varphi}_1\| \to A_g \leq A_g$.

(vi) $\widetilde{\exists x\varphi_1(x)} \equiv \exists x(\psi_{\mathbb{N}^+}(x) \wedge \tilde{\varphi}_1(x))$. Assume first that $\mathbb{N}^+ \models \exists x\varphi_1(x)$. Then $\mathbb{N}^+ \models \varphi_1(a)$ for some $a \in \mathbb{N}^+$ and $\|\psi_{\mathbb{N}^+}(a)\| \geq \|a = a\| = \top$. By the inductive hypothesis $\|\tilde{\varphi}_1(a)\| \geq B$, from these *1.* follows. If $\mathbb{N}^+ \not\models \exists x\varphi_1(x)$, then for all $a \in \mathbb{N}^+$ $\mathbb{N}^+ \not\models \varphi_1(a)$ and by the inductive hypothesis $\|\tilde{\varphi}_1(a)\| \leq A_g$. Let $h \in \mathcal{G}$ arbitrary. Then by Theorem 7. $\|\psi_{\mathbb{N}^+}(h)\| = \bigvee_{k\in\mathbb{N}^+} \|h = k\|$, so $\|\psi_{\mathbb{N}^+}(a) \wedge \tilde{\varphi}_1(a)\| = \bigvee_{k\in\mathbb{N}^+}(\|h = k\| \wedge \|\tilde{\varphi}_1(h)\|) \leq \bigvee_{k\in\mathbb{N}^+}(\|\tilde{\varphi}_1(k)\|) \leq A_g$ and *2.* follows. For *3.*, since A_g is maximal in B, it is enough to prove that

$$\|\exists x(\psi_{\mathbb{N}^+}(x) \wedge \tilde{\varphi}_1(x, h_2', g, h_1'))\| \geq B \Leftrightarrow \|\exists x(\psi_{\mathbb{N}^+}(x) \wedge \tilde{\varphi}_1(x, h_2^W, g, h_1))\| \geq B.$$

If $\|\exists x(\psi_{\mathbb{N}^+}(x) \wedge \tilde{\varphi}_1(x, h_2', g, h_1'))\| \geq B$ then there is $u \in \mathcal{G}$ such that $\|\psi_{\mathbb{N}^+}(u) \wedge \tilde{\varphi}_1(u, h_2', g, h_1'))\| \geq B$. From this $\bigvee_{k\in\mathbb{N}^+}(\|u = k\| \wedge \|\tilde{\varphi}_1(u, h_2', g, h_1')\|) \geq B$ by Theorem 7 so for some $k \in \mathbb{N}^+$ $\|u = k\| \wedge \|\tilde{\varphi}_1(u, h_2', g, h_1')\| \geq B$. From this by the inductive hypothesis $\|u = k\| \wedge \|\tilde{\varphi}_1(u, h_2^W, g, h_1)\| \geq B$ and $\|\exists x(\psi_{\mathbb{N}^+}(x) \wedge \tilde{\varphi}_1(x, h_2^W, g, h_1))\| \geq B$ follows. The other direction is similar.

(vii) $\widetilde{\forall x\varphi_1(x)} \equiv \forall x(\psi_{\mathbb{N}^+}(x) \to \tilde{\varphi}_1(x))$. Similar to the previous case.

(viii) $\widetilde{\exists S\varphi_1(S)} \equiv \widetilde{\varphi_1(S/\emptyset)} \vee \exists v_S \exists u(\tilde{\varphi}_1(v_S, u))$.

Let us assume that $\mathbb{N}^+ \models \exists S\varphi_1(S)$, so $\mathbb{N}^+ \models \varphi_1(W)$ for some $W \subseteq \mathbb{N}^+$. If $W = \emptyset$, $\|\widetilde{\varphi_1(S/\emptyset)}\| \geq B$ by the inductive hypothesis. If $W \neq \emptyset$, by the inductive hypothesis $\|\tilde{\varphi}_1(h_2^W, h_1)\| \geq B$, so $\|\exists v_S \exists u\ \tilde{\varphi}_1(v_S, u)\| \geq B$, ie. *1.* follows. If $\mathbb{N}^+ \not\models \exists S\varphi_1(S)$ then for all $W \subseteq \mathbb{N}^+$, $\mathbb{N}^+ \not\models \varphi_1(W)$, and $\|\tilde{\varphi}_1(h_2^W, h_1)\| \leq A_g$. For every pair h_2', h_1' in $\mathcal{G}$ we have to show that $\|\tilde{\varphi}_1(h_2', h_1')\| \leq A_g$. By *3.* applied to φ_1, for some $W \subseteq \mathbb{N}^+$ $\|\tilde{\varphi}_1(h_2', h_1')\| \leq A_g$ if and only if $\|\tilde{\varphi}_1(h_2^W, h_1)\| \leq A_g$ and *2.* follows. Using the inductive hypothesis *3.* is immediate, since $\widetilde{\exists S\varphi_1(S)}$ does not contain v_s and u free.

(ix) $\widetilde{\forall S\varphi_1(S)} \equiv \forall v_S \forall u\ \tilde{\varphi}_1(v_S, u)$. Let us assume that $\mathbb{N}^+ \models \forall S\varphi_1(S)$, so $\mathbb{N}^+ \models \varphi_1(W)$ for all $W \subseteq \mathbb{N}^+$. By the inductive hypothesis $\|\tilde{\varphi}_1(h_2^W, h_1)\| \geq B$. If $h_1', h_2' \in \mathcal{G}$ are arbitrary, apply *3.* to φ_1. There is some $W \subseteq \mathbb{N}^+$ such that $\|\tilde{\varphi}_1(h_2', h_1')\| \geq B$ if and only if $\|\tilde{\varphi}_1(h_2^W, h_1)\| \geq B$. From these *1.* follows. If $\mathbb{N}^+ \not\models \forall S\varphi_1(S)$ then for some $W \subseteq \mathbb{N}^+$, $\mathbb{N}^+ \not\models \varphi_1(W)$, and $\|\tilde{\varphi}_1(h_2^W, h_1)\| \leq A_g$

by the inductive hypothesis. From this *2.* follows. Again, using the inductive hypothesis, *3.* is immediate.

□

Theorem 14. *Let φ be a second order L' sentence, $\psi \equiv \forall y\,(\zeta(y) \to \tilde{\varphi}(y))$.*

1. *If $\mathbb{N}^+ \models \varphi$, then $\|\psi\| = \top$.*
2. *If $\mathbb{N}^+ \not\models \varphi$, then $\|\psi\| = \bot$.*

Proof. 1. If $\mathbb{N}^+ \models \varphi$. Let $g \in \mathcal{G}$ arbitrary, we have to show that $C_g \leq \|\tilde{\varphi}[g]\|$. If $A_g = C_g$, then the statement follows from Lemma 12. By Lemma 13.1. $\|\tilde{\varphi}[g]\| \geq B$ for every B with A_g being maximal in B. By Lemma 11.1. if A_g is maximal in B, then $B \leq C_g$, so we can apply Lemma 11.2. to get $\|\tilde{\varphi}[g]\| \geq \bigvee\{C \leq C_g \mid A_g \text{ is maximal in } C\} = C_g$.

2. If $\mathbb{N}^+ \not\models \varphi$, then by Lemma 13.2. for all $g \in \mathcal{G}$ $\|\tilde{\varphi}[g]\| \leq A_g$. Let $\mathcal{G}_0 = \{g \in \mathcal{G} \mid A_g \text{ is maximal }\}$. For any $h \in \mathcal{G}$, $g \in \mathcal{G}_0$, since A_g is maximal, $A_h \vee A_g = A_g$, or $A_h \vee A_g = \top$. In the firs case $(A_h \leq A_g)$, so $(A_h \to A_g) = \top$, so $(A_h \to A_g) \vee A_h = \top$. In the second case $\top = A_h \vee A_g \leq A_h \vee (A_h \to A_g))$ so in both cases $(A_h \to A_g) \vee A_h = \top$. Since h was an arbitrary element, if A_g is maximal, $C_g = \top$ and $C_g \to A_g = A_g$ then, since $A_g \leq C_g$. So, using Lemma 11.2 and the fact that each $B \in \mathcal{H}_1$ is of the form A_g for some $g \in \mathcal{G}$, $\|\psi\| \leq \bigwedge\{B \in \mathcal{H}_1 \mid B \text{ is maximal }\} = \bot$.

□

Theorem 15. *True second-order arithmetic can be interpreted in the real algebraic structure of models of intuitionistic analysis built on nice Heyting algebras.* □

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