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by

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Higher Supergeometry and Mathematical Physics

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Introduction

This dissertation is embedded in the field of higher supermanifolds and their applications in mathematical physics. Higher supermanifolds or $\mathbb{Z}_2^n$-manifolds have been introduced in foundational papers by T. Covolo, J. Grabowski, V. Ovsienko and N. Poncin. Here $\mathbb{Z}_2^n$ stands for the Cartesian product $(\mathbb{Z}_2)^{\times n}$ of n copies of $\mathbb{Z}_2$. In the case $n = 1$, we recover standard supermanifolds. The main difference from ordinary supergeometry is that coordinates not only have a parity, but carry a $\mathbb{Z}_2^n$-degree – a degree that is an n-tuple of zeros and ones. These coordinates are $\mathbb{Z}_2^n$-commutative, i.e., the sign in their commutation rule results not from the product of their parities – which is the parity of the sum of the components of their $\mathbb{Z}_2^n$-degrees – but from the standard scalar product of their $\mathbb{Z}_2^n$-degrees – just like with differential forms on a supermanifold, if one agrees on using the Deligne sign convention. It follows that there are odd parameters that commute and non-zero degree even parameters that anticommute. Since the non-zero degree even parameters are not nilpotent, the $\mathbb{Z}_2^n$-functions are the formal power series in the non-zero degree parameters with coefficients in the smooth functions in the zero degree variables. The study of such higher degrees and the corresponding manifolds is on the one hand necessary, as natural examples in physics and mathematics show, on the other hand it is sufficient because every sign rule given by a commutative semigroup and a commutation factor in the standard sense, can be described as $\mathbb{Z}_2^n$-sign rule in our sense.

The theory of $\mathbb{Z}_2^n$-manifolds is currently well established, $\mathbb{Z}_2^n$-differential-calculus does exist and $\mathbb{Z}_2^n$-integration-theory is gradually being fully understood. In particular, the $\mathbb{Z}_2^n$-Berezinian is entirely described.

The $\mathbb{Z}_2^n$-geometry of mixed symmetry tensors

The paper 'The graded differential geometry of mixed symmetry tensors' is published in Archivum Mathematicum, 55(2) (2019).

In this work we show how the new theory of $\mathbb{Z}_2^n$-manifolds can be used in a geometric approach to mixed symmetry tensors such as the dual graviton. By mixed symmetry tensor fields we mean tensors which are neither fully antisymmetric nor symmetric. Such fields play an important role in supergravity, superstring and gauge theories. We discuss the geometric aspects of such tensor fields on both flat and curved space-times. The text is one of many examples of the potential of higher supergeometry in theoretical physics. It gives reason to hope that the new $\mathbb{Z}_2^n$-geometric lens will allow us to expand our understanding of interacting mixed-symmetric tensors.

We refer the reader to the introduction of the paper for more details – see page 10.

The Schwarz–Voronov embedding of $\mathbb{Z}_2^n$-manifolds

The paper 'The Schwarz–Voronov embedding of $\mathbb{Z}_2^n$-manifolds' is published in SIGMA, 16(002) (2020), 47 pages.

A question arises: To what extent can $\mathbb{Z}_2^n$-geometry be developed? supergeometry proposes a series of results that require special care to be generalized into the $\mathbb{Z}_2^n$-graded geometry. As indicated above, the theory of $\mathbb{Z}_2^n$-manifolds can be understood in a sheaf-theoretic framework, as supermanifolds can, but with significant differences, in particular in integration theory. In this paper, we reformulate the notion of a $\mathbb{Z}_2^n$-manifold within a categorical framework via the functor of points. We show that it is sufficient to consider $\mathbb{Z}_2^n$-points, i.e., trivial $\mathbb{Z}_2^n$-manifolds for which the reduced manifold is just a single point, as 'probes' when employing the functor of points. This allows us to construct a fully faithful restricted Yoneda embedding of the category of $\mathbb{Z}_2^n$-manifolds into a subcategory of contravariant functors from the category of $\mathbb{Z}_2^n$-points to a category of nuclear Fréchet manifolds over nuclear Fréchet algebras. We refer to this embedding as the *Schwarz–Voronov embedding*. We further prove that the category of $\mathbb{Z}_2^n$-manifolds is equivalent to the full subcategory of locally trivial functors in the preceding subcategory. We are convinced that the functor of points approach to $\mathbb{Z}_2^n$-geometry elaborated in this text will allow more physicists and mathematicians to take advantage of higher supergeometry in the future.

We refer the reader to the introduction of the paper for more details – see page 20.

$\mathbb{Z}_2^n$-Lie groups and linear actions

The paper 'Linear $\mathbb{Z}_2^n$-Manifolds and Linear Actions' is published in SIGMA 17(060) (2021), 58 pages.

In $\mathbb{Z}_2^n$-geometry, $\mathbb{Z}_2^n$-Lie groups and their actions on $\mathbb{Z}_2^n$-graded vector spaces represent a natural application of the functor of points. This study is based on the point functors of categories such as $\mathbb{Z}_2^n$-vector spaces (the zero rules functor), $\mathbb{Z}_2^n$-manifolds (the Schwarz-Voronov functor), $\mathbb{Z}_2^n$-Lie groups... The values of these functors are all functors restricted to the test category of $\mathbb{Z}_2^n$-manifolds over a single topological point. Throughout the paper, particular attention must therefore be paid to the full faithfulness and to the target category of the restricted point functors we use. Building on this, we establish the representability of the general linear $\mathbb{Z}_2^n$-group defined as a functor valued in a certain category of Fréchet Lie groups, and use the functor of points to define its smooth linear actions on $\mathbb{Z}_2^n$-graded vector spaces and linear $\mathbb{Z}_2^n$-manifolds. For this we must prove the quite unsurprising isomorphism between the category of finite-dimensional $\mathbb{Z}_2^n$-graded vector spaces and the category of linear $\mathbb{Z}_2^n$-manifolds. We do this by explicitly constructing the manifoldification functor and its inverse vectorization functor. In order to properly treat actions, need to show that the zero rules functor agrees with the Schwarz-Voronov functor up to composition with the manifoldification functor. While the mentioned isomorphism of categories is, for the Cartesian space $\mathbb{R}^{p|q}$ $(p, q \in \mathbb{N})$, fairly obvious and readily accepted, the rigorous proof is in general $\mathbb{Z}_2^n$-context unpredictably challenging – to the extent that we coined the temporary subtitle 'On the unbearable heaviness of higher supergeometry' for our paper, in reference to Milan Kundera's somewhat frivolous novel 'On the unbearable lightness of being' from 1984. The results of the work we just presented are needed to study vector bundles, in particular to prove that the geometric and sheaf-theoretical approaches to $\mathbb{Z}_2^n$-vector bundles lead to the same categories.

We refer the reader to the introduction of the paper for more details – see page 70.

$\mathbb{Z}_2^n$-Lie algebra representations

The text '$\mathbb{Z}_2^n$-Lie algebra representations by coderivations' is a work in progress and requires a follow up research project.

After $\mathbb{Z}_2^n$-Lie groups we turn to $\mathbb{Z}_2^n$-Lie algebras. More precisely, the last chapter concerns representations of a $\mathbb{Z}_2^n$-Lie algebra $\mathfrak{g}$ by derivations of the dual symmetric algebra $S(\mathfrak{g})^*$. To this end, we describe the constructions of tensor, symmetric and universal enveloping algebras associated with a $\mathbb{Z}_2^n$-Lie algebra. A weak $\mathbb{Z}_2^n$-version of the Poincaré-Birkoff-Witt theorem (PBW) allows us to build a faithful representation of $\mathfrak{g}$ into its universal enveloping algebra $U(\mathfrak{g})$, and a strong version of the PBW allows us to construct an explicit representation by coderivations of $U(\mathfrak{g})$. We examine the Hopf structures for each of the considered universal algebras and show why the adapted $\mathbb{Z}_2^n$-symmetrization map is a coalgebra isomorphism. We use this to transport the representation by coderivations to the symmetric algebra $S(\mathfrak{g})$ and finally get a representation of $\mathfrak{g}$ by derivations of $S(\mathfrak{g})^*$. We continue to study an adapted version of $\mathbb{Z}_2^n$-Weil algebras and intend to use this representation to obtain a formula for embedding a $\mathbb{Z}_2^n$-Lie algebra into a $\mathbb{Z}_2^n$-Weil algebra.

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Chapter 1

The graded differential geometry of mixed symmetry tensors

The following research paper was published in “Archivum Mathematicum”, Volume 55 (2019), No. 2 (joint work with Andrew James Bruce).

Abstract

We show how the theory of $\mathbb{Z}_2^n$-manifolds - which are a non-trivial generalisation of supermanifolds - may be useful in a geometrical approach to mixed symmetry tensors such as the dual graviton. The geometric aspects of such tensor fields on both flat and curved space-times are discussed.

1.1 Introduction

Recall that differential forms are covariant tensor fields that are completely antisymmetric in their indices. Furthermore, it is well-known that supermanifolds offer a convenient set-up in which to deal with differential forms. In particular, differential forms can be understood as functions on the supermanifold $\Pi \mathsf{T} M$ known as the antitangent bundle. This supermanifold is constructed by taking the tangent bundle of a manifold and then declaring the fibre coordinates to be Grassmann odd. Moreover, the antitangent bundle canonically comes equipped with an odd vector field which ‘squares to zero’, this vector field is identified with the de Rham differential. Mixed symmetry tensor fields are covariant tensors fields with more than one set of antisymmetrised indices. Mixed symmetry tensor fields represent a natural generalisation of differential forms in which the tensors are neither fully symmetric nor antisymmetric. From a representation theory point of view, they correspond to Young diagrams with more than one column. In physics, such tensor fields appear in the context of higher spin fields, dual gravitons, double dual gravitons etc. as found in various formulations of supergravity and string theory. In particular, the particle spectrum of string theory contains beyond the massless particles of the effective supergravity theory, an infinite tower of massive particles of ever higher spin. In the tensionless limit, these higher spin excitations become massless. Thus, if one wants to consider the effective theory beyond the effective supergravity theory, one is forced to contend with mixed symmetry tensors. Moreover, it is known that in string theory certain mixed symmetry tensors couple to exotic branes [4]. To our knowledge, the first study of mixed symmetry tensors field from a physics perspective was Curtright [9] who studied a generalised version of gauge theory. For a review of mixed symmetry tensors, including some historical remarks, the reader may consult Campoleoni [3]. Recently, Chatzistavrakidis *et al.* [5] showed how to reformulate Galileon action functionals in an index-free framework using a generalised notion of a supermanifold. The reader should also note that these results are part of Khoo’s

PhD dissertation [15]. Their theory involves two sets of Grassmann variables that mutually commute. However, assigning a degree of one to *all* the Grassmann variables does not lead to a consistent notion of a "graded supermanifold", the commutation rules of the coordinates are not defined by their degree. Thus, it is impossible to make global sense of the geometry: what is the commutation rule for two arbitrary degree one functions? These difficulties are cured by using a bi-grading and the theory of $\mathbb{Z}_2^n$-manifolds with $n = 2$. Moreover, the formalism of bi-forms (and multi-forms) as developed by Dubois-Violette & Henneaux [11], de Medeiros & Hull [10], and Bekaert & Boulanger [2], is naturally accommodated within this setting.

The locally ringed space approach to $\mathbb{Z}_2^n$-manifolds is currently work in progress initially started by Covolo *et al.* [19, 20, 21]. However, with the basic tenets in place, the time is ripe to seek applications and links with known constructions. Very loosely, $\mathbb{Z}_2^n$-manifolds are 'manifolds' in which we have $\mathbb{Z}_2^n$-graded, $\mathbb{Z}_2^n$-commutative coordinates. The sign rules are controlled by the standard scalar product on $\mathbb{Z}_2^n$. Hence, in general, we have sets of coordinates that anti-commute amongst themselves while commuting across the sets. This is exactly what we require in order to describe mixed symmetry tensors. The one complication is that, in general, there are also formal coordinates that are not nilpotent. This means that we must consider formal power series and not just polynomials in the formal coordinates. However, with the applications to mixed symmetry tensors in mind, we will not need to dwell on this subtlety. We will concentrate on mixed tensors with two 'blocks' of antisymmetric indices and so we will only employ very particular $\mathbb{Z}_2^2$-manifolds with no non-nilpotent formal coordinates.

We liken the current situation to the early days of supersymmetry and in particular the initial works on superspace methods. In particular, physicists worked rather formally with commuting and anticommuting coordinates largely unaware of that the mathematical theory of supermanifolds was concurrently being developed in the Soviet Union by Berezin and collaborators. We speculate that $\mathbb{Z}_2^n$-manifolds will shed light on various aspects of theoretical physics and here we suggest just one potentially useful facet.

1.2 Basics of $\mathbb{Z}_2^n$-geometry

The first reference to $\mathbb{Z}_2^n$-manifolds (*coloured manifolds*) is Molotkov [16] who developed a functor of points approach. The locally ringed space approach to $\mathbb{Z}_2^n$-manifolds is presented in [19]. We will draw upon this heavily and not present proofs of any formal statements. We work over the field $\mathbb{R}$ and in our notation $\mathbb{Z}_2^n := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (n-times). A $\mathbb{Z}_2^n$*-graded algebra* is an $\mathbb{R}$-algebra with a decomposition into vector spaces $\mathcal{A} := \oplus_{\gamma \in \mathbb{Z}_2^n} \mathcal{A}_\gamma$, such that the multiplication respect the $\mathbb{Z}_2^n$-grading, i.e., $\mathcal{A}_\alpha \cdot \mathcal{A}_\beta \subset \mathcal{A}_{\alpha+\beta}$. Furthermore, we will always assume the algebras to be associative and unital. If for any pair of homogeneous elements $a \in \mathcal{A}_\alpha$ and $b \in \mathcal{A}_\beta$ we have that

$$a \cdot b = (-1)^{\langle \alpha, \beta \rangle} b \cdot a, \tag{1.2.1}$$

where $\langle -, - \rangle$ is the standard scalar product on $\mathbb{Z}_2^n$, then we have a $\mathbb{Z}_2^n$*-commutative algebra*.

The basic objects we will employ are smooth $\mathbb{Z}_2^n$-manifolds. Essentially, such objects are 'manifolds' equipped with both standard commuting coordinates and formal coordinates of non-zero $\mathbb{Z}_2^n$-degree that $\mathbb{Z}_2^n$-commute according to the general sign rule (3.5.1). Note that in general - and in stark contrast to the $n = 1$ case of supermanifolds - we have formal coordinates that are *not* nilpotent.

In order to keep track of the various formal coordinates, we need to introduce a convention on how we fix the order of elements in $\mathbb{Z}_2^n$, we do this *lexicographically*. For example, with this choice of ordering

$$\mathbb{Z}_2^2 = \{(0,0),\ (0,1),\ (1,0),\ (1,1)\}\,.$$

Note that other choices of ordering have appeared in the literature. A tuple $\mathbf{q} = (q_1, q_2, \cdots, q_N)$, where $N = 2^n - 1$ provides *all* the information about the formal coordinates. We can now recall the definition of a $\mathbb{Z}_2^n$-manifold.

Definition 1.2.1. A (smooth) $\mathbb{Z}_2^n$-*manifold* of dimension $p|\mathbf{q}$ is a locally $\mathbb{Z}_2^n$-ringed space $\mathcal{M} := (M, \mathcal{O}_M)$, which is locally isomorphic to the $\mathbb{Z}_2^n$-ringed space $\mathbb{R}^{p|\mathbf{q}} := (\mathbb{R}^p, C^\infty_{\mathbb{R}^p}[[\xi]])$. Local sections of M are formal power series in the $\mathbb{Z}_2^n$-graded variables ξ with smooth coefficients,

$$\mathcal{O}_M(U) \simeq C^\infty(U)[[\xi]] := \left\{ \sum_{\hat{\alpha} \in \mathbb{N}^N}^{\infty} \xi^{\hat{\alpha}} f_{\hat{\alpha}} \mid f_{\hat{\alpha}} \in C^\infty(U) \right\},$$

for 'small enough' open domains $U \subset M$. *Morphisms* between $\mathbb{Z}_2^n$-manifolds are morphisms of $\mathbb{Z}_2^n$-ringed spaces, that is, pairs $\Phi = (\phi, \phi^*) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ consisting of a continuous map $\phi : M \to N$ and sheaf morphism $\phi^* : \mathcal{O}_N \to \mathcal{O}_M$, i.e., a family of $\mathbb{Z}_2^n$-algebra morphisms $\phi^*_V : \mathcal{O}_N(V) \to \mathcal{O}_M(\phi^{-1}(V))$, where $V \subset N$ is open. We will refer to the global sections of the structure sheaf $\mathcal{O}_M$ as *functions* on M and denote them as $C^\infty(\mathcal{M}) := \mathcal{O}_M(M)$.

Example 1.2.2 (The local model)**.** The locally $\mathbb{Z}_2^n$-ringed space $\mathcal{U}^{p|\mathbf{q}} := \big(\mathcal{U}^p, C^\infty_{\mathcal{U}^p}[[\xi]]\big)$, where $\mathcal{U}^p \subseteq \mathbb{R}^p$ is naturally a $\mathbb{Z}_2^n$-manifold – we refer to such $\mathbb{Z}_2^n$-manifolds as $\mathbb{Z}_2^n$-*superdomains* of dimension $p|\mathbf{q}$. We can employ (natural) coordinates (x^a, ξ^α) on any $\mathbb{Z}_2^n$-superdomain, where x^a form a coordinate system on $\mathcal{U}^p$ and the ξ^α are formal coordinates.

Many of the standard results from the theory of supermanifolds pass over to $\mathbb{Z}_2^n$-manifolds. For example, the topological space M comes with the structure of a smooth manifold of dimension p, hence our suggestive notation. Moreover, there exists a canonical projection $\epsilon : \mathcal{O}(M) \to C^\infty(M)$. What makes $\mathbb{Z}_2^n$-manifolds a very workable form of noncommutative geometry is the fact that we have well-defined local models. Much like the theory of manifolds, one can construct global geometric concepts via the glueing of local geometric concepts. That is, we can consider a $\mathbb{Z}_2^n$-manifold as being cover by $\mathbb{Z}_2^n$-superdomains together with specified glueing information given by coordinate transformations, composed by homomorphisms

$$\Psi_{\beta\alpha} := \Psi_\beta^{-1}\Psi_\alpha : \Psi_\alpha^{-1}(\Psi_\alpha(U_\alpha) \cap \Psi_\beta(U_\beta)) \to \Psi_\beta^{-1}(\Psi_\alpha(U_\alpha) \cap \Psi_\beta(U_\beta)),$$

which are labelled by the different local models $(U_\alpha, C^\infty(U_\alpha)[[\xi]])$, $\{\Psi_\alpha : U_\alpha \to \Psi_\alpha(U_\alpha) \subset M\}$, whenever $U_\alpha \cap U_\beta \neq \emptyset$; and a graded unital $\mathbb{R}-$algebra morphism $\Psi^*_{\beta\alpha} : C^\infty(U_\beta)[[\xi']] \longrightarrow C^\infty(U_\alpha)[[\xi]]$.

We have the *chart theorem* ([19, Theorem 7.10]) that basically says that morphisms between $\mathbb{Z}_2^n$-superdomains can be completely described by local coordinates and that these local morphisms can then be extended uniquely to morphisms of locally $\mathbb{Z}_2^n$-ringed spaces. This allows one to proceed to describe the theory much as one would on a standard smooth manifold in terms of local coordinates. Indeed, we will employ the standard abuses of notation when dealing with coordinate transformations and morphisms. In particular, the explicit way of computing change of coordinates concerning any geometrical object are well understood and work identically as in classical differential geometry. In essence, one need only take into account that $\mathbb{Z}_2^n$-degree needs to be preserved under any permissible changes of coordinates. For example, *vector fields* are defined as $\mathbb{Z}_2^n$-graded derivations of the global sections, $X \in \mathrm{Der}(C^\infty(\mathcal{M}) \subset \mathrm{End}(C^\infty(\mathcal{M}))$, that are compatible with restrictions. That is, given some open subset $U \subset M$, we can always 'localise' the vector field, i.e., $X|_U = X_U \in \mathrm{Der}(\mathcal{O}_M(U))$. Furthermore, if this open is 'small enough', we can employ local coordinates (x^a, ξ^α) and write

$$X_U = X^a(x, \xi)\frac{\partial}{\partial x^a} + X^\alpha(x, \xi)\frac{\partial}{\partial \xi^\alpha}\,.$$

Under changes of local coordinates

$$x^{a'} = x^{a'}(x,\xi), \qquad \xi^{\alpha'} = \xi^{\alpha'}(x,\xi),$$

remembering the abuses of notation and that $\mathbb{Z}_2^n$-degree is preserved, the induced transformation law on the components of the vector field follow from the chain rule and are given by

$$X^{a'} = X^b \frac{\partial x^{a'}}{\partial x^b} + X^\beta \frac{\partial x^{a'}}{\partial \xi^\beta}, \qquad X^{\alpha'} = X^b \frac{\partial \xi^{\alpha'}}{\partial x^b} + X^\beta \frac{\partial \xi^{\alpha'}}{\partial \xi^\beta}.$$

See Covolo *et al.* [21, Lemma 2.2] for details. The reader can easily verify that the $\mathbb{Z}_2^n$-graded commutator of two vector fields is again a vector field and that the obvious $\mathbb{Z}_2^n$-graded version of the Jacobi identity holds.

As is customary in classical differential geometry, we will not write out the restrictions of geometric objects explicitly and simply write objects in terms of there components in some chosen local coordinate system. In other words, one can work locally on $\mathbb{Z}_2^n$-manifolds in more-or-less the same way as one works on classical manifolds and indeed, supermanifolds. The glaring exception here is the theory of integration on $\mathbb{Z}_2^n$-manifolds which is expected to be quite involved (see Poncin [37] for work in this direction).

1.3 Mixed symmetry tensors over Minkowski space-time

Consider D-dimensional Minkowski space-time $M = (\mathbb{R}^D, \eta)$. The Poincaré transformations we write as

$$x^\mu \mapsto x^{\mu'} = x^\nu \Lambda_\nu^{\ \mu'} \ + \ a^{\mu'}.$$

We now wish to construct a $\mathbb{Z}_2^2$-manifold built from M in a canonical way. In particular, consider

$$\mathcal{M} := \mathsf{T}M[(0,1)] \times_M \mathsf{T}M[(1,0)],$$

where we have indicated the assignment of the $\mathbb{Z}_2^2$-grading to the fibre coordinates on each tangent bundle. It is straightforward to see that we do indeed obtain a $\mathbb{Z}_2^2$-manifold in this way by using coordinates (see [19, Proposition 6.1]). Specifically, we can always employ (global) coordinates of the form

$$\big(\underbrace{x^\mu}_{(0,0)}, \underbrace{\xi^\nu}_{(0,1)}, \underbrace{\theta^\rho}_{(1,0)}\big),$$

where we have signalled the assignment of $\mathbb{Z}_2^2$-grading. Note that we have the non-trivial $\mathbb{Z}_2^2$-commutation rules

$$\xi^\mu \xi^\nu = -\xi^\nu \xi^\mu, \qquad \theta^\mu \theta^\nu = -\theta^\nu \theta^\mu, \qquad \xi^\mu \theta^\nu = +\theta^\nu \xi^\mu.$$

Thus, while each 'species' of non-zero degree coordinate are themselves nilpotent, across 'species' they commute. This is, of course, very different to the case of standard supermanifolds. The Poincaré transformations induce the obvious linear coordinate transformations on the formal coordinates

$$\xi^{\nu'} = \xi^\nu \Lambda_\nu^{\ \nu'}, \qquad \theta^{\rho'} = \theta^\rho \Lambda_\rho^{\ \rho'}.$$

Clearly, these transformation laws respect the assignment of $\mathbb{Z}_2^2$-grading and satisfy (rather trivially) the cocycle condition. Thus, we do indeed obtain a $\mathbb{Z}_2^2$-manifold in this way. As the coordinate transformations respect the obvious bundle structure and do not 'mix' the non-zero degree coordinates we have an example of a so-called *split* $\mathbb{Z}_2^2$*-manifold* [20]. The fact that we do not, in this case, have non-zero degree coordinates that are not nilpotent means that we only deal with polynomials in the formal coordinates.

The space of (p,q)-forms on M we define as

$$\Omega^{(p,q)}(M) := C^\infty(\mathcal{M})_{(p,q)},$$

where we naturally have the $\mathbb{N} \times \mathbb{N}$-grading given by the polynomial order in each formal coordinate. By considering all possible degrees we obtain a unital $\mathbb{Z}_2^2$-commutative algebra

$$\Omega(M) := C^\infty(\mathcal{M}) = \bigoplus_{(p,q)\in\mathbb{N}\times\mathbb{N}}^{(D,D)} \Omega^{(p,q)}(M),$$

which we refer to as the algebra of *bi-forms*. Note that we naturally, have a $C^\infty(M) = \Omega^{(0,0)}(M)$ module structure on the space of all bi-forms.

In coordinates, any (p,q)-form can be written as

$$\omega^{(p,q)}(x,\xi,\theta) = \frac{1}{p!q!}\,\theta^{\nu_1}\cdots\theta^{\nu_p}\xi^{\mu_1}\cdots\xi^{\mu_q}\,\omega_{\mu_q\cdots\mu_1|\nu_q\cdots\nu_1}(x).$$

Due to the $\mathbb{Z}_2^2$-commutation rules, we have the relation that $\omega_{[\mu_q\cdots\mu_1]|[\nu_q\cdots\nu_1]} = \omega_{\mu_q\cdots\mu_1|\nu_q\cdots\nu_1}$ and $\omega_{[\mu_q\cdots\mu_1]|[\nu_q\cdots\nu_1]} = \omega_{[\nu_q\cdots\nu_1]|[\mu_q\cdots\mu_1]}$ Note that we will not insist on any further relations in general.

Example 1.3.1. The dual graviton in D-dimensions is a $(1, D-3)$-form and so is given in coordinates as

$$\mathrm{C}(x,\xi,\theta) = \frac{1}{(D-3)!}\,\theta^{\nu}\xi^{\mu_1}\cdots\xi^{\mu_{D-3}}\,C_{\mu_{D-3}\cdots\mu_1|\nu}(x).$$

Similarly, the double dual graviton in D-dimensions of a $(D-3, D-3)$-form and so is given in coordinates as

$$\mathrm{D}(x,\xi,\theta) = \frac{1}{(D-3)!(D-3)!}\,\theta^{\nu_1}\cdots\theta^{\nu_{D-3}}\xi^{\mu_1}\cdots\xi^{\mu_{D-3}}\,D_{\mu_{D-3}\cdots\mu_1|\nu_{D-3}\cdots\nu_1}(x).$$

See Hull [13, 14] for details of the rôle of dual gravitons and double dual gravitons in electromagnetic duality of gravitational theories.

Canonically, the algebra of bi-forms on D-dimensional Minkowski space-time comes equipped with a pair of de Rham differentials. These differentials we consider as homological vector fields on the $\mathbb{Z}_2^2$-manifold $\mathcal{M}$. That is, they 'square to zero', i.e., $2\mathrm{d}^2 = [\mathrm{d},\mathrm{d}] = 0$. In coordinate we have

$$\mathrm{d}_{(0,1)} = \xi^\mu \frac{\partial}{\partial x^\mu}, \qquad \mathrm{d}_{(1,0)} = \theta^\mu \frac{\partial}{\partial x^\mu}\,.$$

It is important to note that do indeed have a pair of vector fields in this way. In particular, the partial derivatives change under Poincaré transformations as

$$\frac{\partial}{\partial x^{\mu'}} = \Lambda_{\mu'}^{\ \mu}\frac{\partial}{\partial x^\mu}, \qquad \frac{\partial}{\partial \xi^{\nu'}} = \Lambda_{\nu'}^{\ \nu}\frac{\partial}{\partial \xi^\nu}, \qquad \frac{\partial}{\partial \theta^{\rho'}} = \Lambda_{\rho'}^{\ \rho}\frac{\partial}{\partial \theta^\rho}\,.$$

Thus, the pair of de Rham differentials are well-defined. It is also clear that they $\mathbb{Z}_2^2$-commute, i.e,

$$[\mathrm{d}_{(1,0)}, \mathrm{d}_{(0,1)}] := \mathrm{d}_{(1,0)} \circ \mathrm{d}_{(0,1)} - \mathrm{d}_{(0,1)} \circ \mathrm{d}_{(1,0)} = 0\,.$$

In this way, we obtain a *de Rham bi-complex*. Also, note that the interior product and Lie derivative can also be directly 'doubled'.

Canonically we also have a pair of vector fields of $\mathbb{Z}_2^2$-degree $(1,1)$, given by

$$\Delta_{(0,1)} = \xi^\mu \frac{\partial}{\partial \theta^\mu}, \qquad \Delta_{(1,0)} = \theta^\nu \frac{\partial}{\partial \xi^\nu}.$$

A direct calculation shows that the non-trivial $\mathbb{Z}_2^2$-commutators are

$$[\Delta_{(0,1)}, \mathrm{d}_{(1,0)}] = \mathrm{d}_{(0,1)}, \qquad [\Delta_{(1,0)}, \mathrm{d}_{(0,1)}] = \mathrm{d}_{(1,0)}.$$

Rather conveniently, we can understand the metric as a $(1,1)$-form and the inverse of the metric as a second-order differential operator given by

$$\eta := \theta^\mu \xi^\nu \eta_{\nu\mu}, \qquad \eta^{-1} := \eta^{\mu\nu} \frac{\partial^2}{\partial \xi^\nu \partial \theta^\mu},$$

respectively.

Example 1.3.2. Consider the Curtright field on $D = 5$ Minkowski space-time [9]. Such a field is understood to be the electromagnetic dual of the graviton field. In our language, the Curtright field is an example of a $(1,2)$-form and as such can be written in coordinates as

$$\mathrm{C}(x,\xi,\theta) = \frac{1}{2!}\theta^\rho \xi^\nu \xi^\mu C_{\mu\nu|\rho}(x).$$

There is a further symmetry condition on the Curtright field, i.e., $C_{\mu\nu|\rho} + C_{\rho\mu|\nu} + C_{\nu\rho|\mu} = 0$, which comes from wanting an irreducible representation of the Poincaré group. This condition can be expressed as

$$\Delta_{(0,1)}\mathrm{C} = \frac{1}{2!3}\xi^\rho \xi^\nu \xi^\mu \big(C_{\mu\nu|\rho} + C_{\rho\mu|\nu} + C_{\nu\rho|\mu}\big) = 0.$$

Furthermore, a direct calculation shows that

$$\mathrm{F} := \mathrm{d}_{(0,1)}\mathrm{C} = \frac{1}{3!}\theta^\rho \xi^\nu \xi^\mu \xi^\lambda \left(\frac{\partial C_{\mu\nu|\rho}}{\partial x^\lambda} + \frac{\partial C_{\nu\lambda|\rho}}{\partial x^\mu} + \frac{\partial C_{\lambda\nu|\rho}}{\partial x^\nu} \right) = \frac{1}{3!}\theta^\rho \xi^\nu \xi^\mu \xi^\lambda F_{\lambda\mu\nu|\rho}(x),$$

which we recognise (up to possible conventions) to be the *Curtright field strength.* Applying $\mathrm{d}_{(1,0)}$ to the Curtright field strength yields

$$\mathrm{E} := \mathrm{d}_{(1,0)}\big(\mathrm{d}_{(0,1)}\mathrm{C}\big) = \frac{1}{2!3!}\theta^\omega \theta^\rho \xi^\nu \xi^\mu \xi^\lambda \left(\frac{\partial F_{\lambda\mu\nu|\rho}}{\partial x^\omega} - \frac{\partial F_{\lambda\mu\nu|\omega}}{\partial x^\lambda} \right) = \frac{1}{2!3!}\theta^\omega \theta^\rho \xi^\nu \xi^\mu \xi^\lambda E_{\lambda\mu\nu|\rho\omega}(x),$$

which we recognise (up to possible conventions) to be the *Curtright curvature tensor*, which is fully gauge invariant, see Bekaert, Boulanger & Henneaux [**?**] for details. Similarly the *Curtright–Ricci tensor* and its trace (again, up to conventions) can be constructed by applying the inverse metric, i.e.,

$$\eta^{-1}(\mathrm{E}) = \frac{1}{2!}\theta^\rho \xi^\mu \xi^\lambda \eta^{\omega\nu} E_{\lambda\mu\nu|\rho\omega}(x) = \frac{1}{2!}\theta^\rho \xi^\mu \xi^\lambda E_{\lambda\mu|\rho}(x),$$
$$\eta^{-1}\big(\eta^{-1}(\mathrm{E})\big) = \xi^\lambda \eta^{\rho\mu} E_{\lambda\mu|\rho}(x) = \xi^\lambda E_\lambda(x).$$

Remark 1.3.3. The procedure to describe mixed symmetry tensors with more antisymmetric 'blocks' is clear. In particular, if we have n such blocks, then we should consider the $\mathbb{Z}_2^n$-manifold

$$\mathcal{M} := \mathsf{T}M[(0,\cdots,0,1)] \times_M \mathsf{T}M[(0,\cdots,0,1,0)] \times_M \cdots \times_M \mathsf{T}M[(1,\cdots,0,0)],$$

where we have signalled the $\mathbb{Z}_2^n$-degree of the fibre coordinates. Note that we have a canonical de Rham differential in each sector. Thus, the previous statements of this section can be generalised verbatim.

1.4 Mixed symmetry tensors over curved space-times

Directly extending the constructions to curved space-times (M, g) is not possible. This was for sure noticed in [5], albeit with no reference to $\mathbb{Z}_2^n$-manifolds. The two de Rham differentials cannot be naïvely be considered as vector fields on $\mathcal{M} = \mathsf{T}M[(0,1)] \times_M \mathsf{T}M[(1,0)]$. The resolution to this problem is the standard one: we use the Levi-Civita connection to lift the vector fields. The $\mathbb{Z}_2^2$-manifold $\mathcal{M}$ comes equipped with natural coordinates

$$\big(\underbrace{x^\mu}_{(0,0)}, \underbrace{\xi^\nu}_{(0,1)}, \underbrace{\theta^\rho}_{(1,0)}\big),$$

where again we have signalled the assignment of $\mathbb{Z}_2^2$-grading. The permissible changes of local coordinates are

$$x^{\mu'} = x^{\mu'}(x), \qquad \xi^{\nu'} = \xi^\nu \frac{\partial x^{\nu'}}{\partial x^\nu}, \qquad \theta^{\rho'} = \theta^\rho \frac{\partial x^{\rho'}}{\partial x^\rho}.$$

As standard, we define a covariant derivative

$$\nabla_\mu := \frac{\partial}{\partial x^\mu} - \xi^\nu \Gamma^\rho_{\nu\mu} \frac{\partial}{\partial \xi^\rho} - \theta^\nu \Gamma^\rho_{\nu\mu} \frac{\partial}{\partial \theta^\rho},$$

where $\Gamma^\rho_{\nu\mu}$ are the Christoffel symbols of the Levi-Civita connection. We then define the covariant de Rham derivatives as

$$\nabla_{(0,1)} := \xi^\mu \nabla_\mu = \xi^\mu \frac{\partial}{\partial x^\mu} - \xi^\mu \theta^\nu \Gamma^\rho_{\nu\mu} \frac{\partial}{\partial \theta^\rho}, \qquad \nabla_{(1,0)} := \theta^\mu \nabla_\mu = \theta^\mu \frac{\partial}{\partial x^\mu} - \xi^\mu \theta^\nu \Gamma^\rho_{\nu\mu} \frac{\partial}{\partial \xi^\rho},$$

remembering that the Christoffel symbols are symmetric in the lower indices, i.e., the Levi–Civita connection is torsion free. Due to the transformation rules for the Christoffel symbols both these covariant de Rham derivatives are well-defined vector fields on $\mathcal{M}$. However, in general, we lose the fact that these vector fields are homological and that they commute. This is in stark contrast to the case of standard differential forms where the covariant derivative (with respect to any torsionless connection) reduces to the de Rham differential. Direct calculation shows that

$$\begin{aligned}
[\nabla_{(0,1)}, \nabla_{(0,1)}] &= R_{(0,1)} = \theta^\mu \xi^\lambda \xi^\nu R^\rho{}_{\mu\nu\lambda}(x) \frac{\partial}{\partial \theta^\rho}, \\
[\nabla_{(1,0)}, \nabla_{(1,0)}] &= R_{(1,0)} = \xi^\mu \theta^\lambda \theta^\nu R^\rho{}_{\mu\nu\lambda}(x) \frac{\partial}{\partial \xi^\rho}, \\
[\nabla_{(1,0)}, \nabla_{(0,1)}] &= R_{(1,1)} = \xi^\mu \theta^\lambda \theta^\nu R^\rho{}_{\mu\nu\lambda} \frac{\partial}{\partial \theta^\rho}(x) - \theta^\mu \xi^\lambda \xi^\nu R^\rho{}_{\mu\nu\lambda}(x) \frac{\partial}{\partial \xi^\rho},
\end{aligned}$$

where $R^\rho{}_{\mu\nu\lambda}$ is the Riemann curvature of the Levi-Civita connection (similar expressions can be found in [12]). The vector fields $\Delta_{(0,1)}$ and $\Delta_{(1,0)}$ have exactly the same local form as on Minkowski space-time. A direct calculation shows that

$$[\Delta_{(0,1)}, \nabla_{(1,0)}] = \nabla_{(0,1)}, \qquad [\Delta_{(1,0)}, \nabla_{(0,1)}] = \nabla_{(1,0)}.$$

where one has to take care with the signs due to the $\mathbb{Z}_2^2$-grading.

Example 1.4.1. The covariant Riemann tensor is an example of a $(2,2)$-form on (M, g):

$$R(x, \xi, \theta) = \frac{1}{2!2!} \theta^\nu \theta^\mu \xi^\sigma \xi^\rho \, R_{\rho\sigma|\mu\nu}(x),$$

here $R_{\rho\sigma|\mu\nu} := g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu}$ and $R^{\lambda}{}_{\sigma\mu\nu}$ is the Riemann curvature of the Levi–Civita connection. A direct computation shows that the *first Bianchi identity* can be written as

$$\Delta_{(0,1)} R = \frac{1}{3!} \theta^{\nu} \xi^{\rho} \xi^{\mu} \xi^{\sigma} \big(R_{\nu\sigma|\mu\rho} + R_{\nu\mu|\rho\sigma} + R_{\mu\rho|\sigma\mu} \big) = 0\,.$$

Similarly, a direct computation shows that the *second Bianchi identity* can be written as

$$\begin{aligned}\nabla_{(0,1)} R = \frac{1}{2!3!} \theta^{\nu} \theta^{\rho} \xi^{\mu} \xi^{\sigma} \xi^{\lambda} \Bigg(\bigg(\frac{\partial R_{\sigma\mu|\rho\nu}}{\partial x^{\lambda}} - \Gamma^{\omega}_{\nu\lambda} R_{\sigma\mu|\rho\omega} - \Gamma^{\omega}_{\rho\lambda} R_{\sigma\mu|\nu\omega} \bigg) \ + \ \bigg(\frac{\partial R_{\mu\lambda|\rho\nu}}{\partial x^{\sigma}} - \Gamma^{\omega}_{\nu\sigma} R_{\mu\lambda|\rho\omega} - \Gamma^{\omega}_{\rho\sigma} R_{\mu\lambda|\nu\omega} \bigg) \\ + \ \bigg(\frac{\partial R_{\lambda\sigma|\rho\nu}}{\partial x^{\mu}} - \Gamma^{\omega}_{\nu\mu} R_{\lambda\sigma|\rho\omega} - \Gamma^{\omega}_{\rho\nu} R_{\lambda\sigma|\nu\omega} \bigg) \Bigg) \ = \ 0\,.\end{aligned}$$

1.5 Concluding Remarks

As remarked in the introduction, differential forms on a manifold M are naturally understood as functions of the antitangent bundle $\Pi\mathsf{T}M$, which itself canonically comes equipped with the de Rham differential, here understood as a homological vector field. Similarly, bi-forms on a (pseudo-)Riemannian manifold (M, g), are naturally understood as functions on the $\mathbb{Z}_2^2$-manifold $\mathsf{T}M[(0,1)] \times_M \mathsf{T}M[(1,0)]$, which canonically comes equipped with the odd vector fields (generally, non-homological) $\nabla_{(0,1)}$ and $\nabla_{(1,0)}$. Similar statements can be made for more general multi-forms.

While the goals of this note have been modest, we hope that the observations here will prove useful in further studies of mixed symmetry tensors. In particular, it is well-known that constructing consistent theories of interacting mixed symmetry tensors is problematic. We hope that further geometric insight can be gained via $\mathbb{Z}_2^n$-manifolds and that this will lead to a better understanding of how to build actions involving mixed symmetry tensors.

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Chapter 2

The Schwarz–Voronov embedding of $\mathbb{Z}_2^n$-manifolds

The following research paper was published in "SIGMA" 16 (2020), 002, 47 pages (joint work wth Andrew James Bruce and Norbert Poncin).

Abstract

Informally, $\mathbb{Z}_2^n$-manifolds are 'manifolds' with $\mathbb{Z}_2^n$-graded coordinates and a sign rule determined by the standard scalar product of their $\mathbb{Z}_2^n$-degrees. Such manifolds can be understood in a sheaf-theoretic framework, as supermanifolds can, but with significant differences, in particular in integration theory. In this paper, we reformulate the notion of a $\mathbb{Z}_2^n$-manifold within a categorical framework via the functor of points. We show that it is sufficient to consider $\mathbb{Z}_2^n$-points, i.e., trivial $\mathbb{Z}_2^n$-manifolds for which the reduced manifold is just a single point, as 'probes' when employing the functor of points. This allows us to construct a fully faithful restricted Yoneda embedding of the category of $\mathbb{Z}_2^n$-manifolds into a subcategory of contravariant functors from the category of $\mathbb{Z}_2^n$-points to a category of Fréchet manifolds over algebras. We refer to this embedding as the *Schwarz–Voronov embedding*. We further prove that the category of $\mathbb{Z}_2^n$-manifolds is equivalent to the full subcategory of locally trivial functors in the preceding subcategory.

2.1 Introduction

Various notions of *graded geometry* play an important rôle in mathematical physics and can often provide further insight into classical geometric constructions. For example, supermanifolds, as pioneered by Berezin and collaborators, are essential in describing quasi-classical systems with both bosonic and fermionic degrees of freedom. Very loosely, supermanifolds are 'manifolds' for which the structure sheaf is $\mathbb{Z}_2$-graded. Such geometries are of fundamental importance in perturbative string theory, supergravity, and the BV-formalism, for example. While the theory of supermanifolds is firmly rooted in theoretical physics, it has since become a respectable area of mathematical research. Indeed, supermanifolds allow for an economical description of Lie algebroids, Courant algebroids as well as various related structures, many of which are of direct interest to physics. We will not elaborate any further and urge the reader to consult the ever-expanding literature.

Interestingly, $\mathbb{Z}_2^n$*-gradings* ($\mathbb{Z}_2^n = \mathbb{Z}_2^{\times n}$, $n \geq 2$) can be found in the theory of parastatistics, see for example [22, 25, 26, 49], behind an alternative approach to supersymmetry [45], in relation to the symmetries of the Lévy-Lebond equation [2], and behind the theory of mixed symmetry tensors [11]. Generalizations of the super Schrödinger algebra (see [3]) and the super Poincaré algebra (see [10]) have also appeared in the literature. That said, it is unknown if these 'higher gradings' are of the same importance in fundamental physics as $\mathbb{Z}_2$-gradings. It must also be remarked that the quaternions and more general Clifford algebras can be understood as $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative (see below) algebras [4, 5]. Thus, one may expect $\mathbb{Z}_2^n$-gradings to be important in studying Clifford algebras and modules, though the implications for classical and quantum field theory remain as of yet unexplored. It should be further mentioned that any 'sign rule' can be understood in terms of a $\mathbb{Z}_2^n$-grading (see [15]). A natural question here is to what extent can $\mathbb{Z}_2^n$*-graded geometry* be developed.

A locally ringed space approach to $\mathbb{Z}_2^n$*-manifolds* has been constructed in a series of papers by Bruce, Covolo, Grabowski, Kwok, Ovsienko & Poncin [19, 15, 16, 17, 18, 36, 11, 13]. It includes the $\mathbb{Z}_2^n$-differential-calculus, the $\mathbb{Z}_2^n$-Berezinian, as well as a low dimensional $\mathbb{Z}_2^n$-integration-theory. Integration on $\mathbb{Z}_2^n$-manifolds turns out to be fundamentally different from integration on $\mathbb{Z}_2^1$-manifolds (i.e., supermanifolds) and is currently being constructed in full generality by authors of the present paper. The novel aspect of integration on $\mathbb{Z}_2^n$-manifolds is integration with respect to the non-zero degree even parameters (for some preliminary results see [36]).

Loosely, $\mathbb{Z}_2^n$-manifolds are 'manifolds' for which the structure sheaf has a $\mathbb{Z}_2^n$-grading and the commutation rule for the local coordinates comes from the standard scalar product of their $\mathbb{Z}_2^n$-degrees. This is not just a trivial or straightforward generalization of the notion of a supermanifold as one has to deal with formal coordinates that anticommute with other formal coordinates, but are themselves *not* nilpotent. Due to the presence of formal variables that are not nilpotent, formal power series are used rather than polynomials (for standard supermanifolds all functions are polynomial in the Grassmann odd variables). The use of formal power series is unavoidable in order to have a well-defined local theory (see [15]), and a well-defined differential calculus (see [17]). Heuristically, one can view supermanifolds as 'mild' noncommutative geometries: the noncommutativity is seen simply as anticommutativity of the odd coordinates. In a similar vein, one can view $\mathbb{Z}_2^n$-manifolds ($n > 1$) as examples of 'mild' nonsupercommutative geometries: the sign rule involved is not determined by the coordinates being even or odd, i.e., by their total degree, but by their $\mathbb{Z}_2^n$-degree.

The idea of understanding supermanifolds, i.e., $\mathbb{Z}_2^1$-manifolds, as 'Grassmann algebra valued manifolds' can be traced back to the pioneering work of Berezin [9]. An informal understanding along these lines has continuously been employed in physics, where one chooses a 'large enough' Grassmann algebra to capture the aspects to the theory needed. This informal understanding leads to the DeWitt–Rogers approach to supermanifolds which seemed to avoid the theory of locally ringed spaces altogether. However, arbitrariness in the choice of the underlying Grassmann algebra is somewhat displeasing. Furthermore, developing the mathematical consistency of DeWitt–Rogers supermanifolds takes one back to the sheaf-theoretic approach of Berezin & Leites: for a comparison of these approaches, the reader can consult Rogers [39] or Schmitt [43]. From a physics perspective, there seems no compelling reason to think that there is any physical significance to the choice of underlying Grassmann algebra. To quote Schmitt [43]: "However, no one has ever measured a Grassmann number, everyone measures real numbers". The solution here is, following Schwarz & Voronov [44, 45, 55], not to fix the underlying Grassmann algebra, but rather understand supermanifolds as functors from the category of finite-dimensional Grassmann algebras to, in the first instance, the category of sets. For a given, but arbitrary, Grassmann algebra Λ, one speaks of the set of Λ-points of a supermanifold. It is well known that the set of Λ-points of a given supermanifold comes with the further structure of a Λ_0-smooth manifold. That is we, in fact, do not only have a set, but also the structure of a finite-dimensional manifold whose tangent spaces are Λ_0-modules. Moreover, thinking of supermanifolds as functors, not all natural transformations between the Λ-points correspond to genuine supermanifold morphisms, only those that respect the Λ_0-smooth structure do. A similar approach is used by Molotkov [35], who defines Banach supermanifolds roughly speaking as specific functors from the category of finite-dimensional Grassmann algebras to the category of smooth Banach manifolds of a particular type. The classical roots of these ideas go back to Weil [57] who considered the A-points of a manifold as the set of maps from the algebra of smooth functions on the manifold to a specified finite-dimensional commutative local algebra A. Today one refers to *Weil functors* and these have long been utilised in the theory of jet structures over manifolds, see for example [29].

In this paper, we study *Grothendieck's functor of points* [27] of a $\mathbb{Z}_2^n$-manifold M, which is a contravariant functor $M(-)$ from the category of $\mathbb{Z}_2^n$-manifolds to the category of sets, and restrict it to the category of $\mathbb{Z}_2^n$*-points*, i.e., trivial $\mathbb{Z}_2^n$-manifolds $\mathbb{R}^{0|\underline{q}}$ that have no degree zero coordinates. More precisely, we consider the restricted *Yoneda functor* $M \mapsto M(-)$ from the category of $\mathbb{Z}_2^n$-manifolds to the category of contravariant functors from $\mathbb{Z}_2^n$-points to sets. Dual to $\mathbb{Z}_2^n$-points $\mathbb{R}^{0|\underline{q}}$ are what we will call $\mathbb{Z}_2^n$*-Grassmann algebras* Λ (see Definition 2.2.3). The aim of this paper is to carefully prove and generalise the main results of Schwarz & Voronov [45, 55] to the 'higher graded' setting. In particular, we show that $\mathbb{Z}_2^n$-points $\mathbb{R}^{0|\underline{q}} \simeq \Lambda$ are actually sufficient to act as 'probes' when employing the functor of points (see Theorem 2.3.8). However, not all natural transformations $\eta_\Lambda : M(\Lambda) \to N(\Lambda)$ (where Λ is a variable) between the sets $M(\Lambda)$, $N(\Lambda)$ of Λ-points correspond to morphisms $\phi : M \to N$ of the underlying $\mathbb{Z}_2^n$-manifolds. By carefully analysing the image of the functor of points, we prove that the set $M(\Lambda)$ of Λ-points of a $\mathbb{Z}_2^n$-manifold M comes with the extra structure of a Fréchet Λ_0-manifold (see Theorem 2.3.22 ; by Λ_0 we mean the subalgebra of degree zero elements of the $\mathbb{Z}_2^n$-Grassmann algebra Λ). Note that we are not trying to define infinite-dimensional $\mathbb{Z}_2^n$-manifolds, yet infinite-dimensional manifolds, specifically Fréchet manifolds, are fundamental to our paper. Moreover, we show that natural transformations η_Λ between sets of Λ-points arise from morphisms ϕ of $\mathbb{Z}_2^n$-manifolds if and only if they respect the Fréchet Λ_0-manifold structures (see Proposition 2.3.24). By restricting accordingly the natural transformations allowed, we get a *full and faithful embedding* of the category of $\mathbb{Z}_2^n$-manifolds into the category of contravariant functors from the category of $\mathbb{Z}_2^n$-points to the category of nuclear Fréchet manifolds over nuclear Fréchet algebras. This embedding we refer to as the *Schwarz–Voronov embedding* (see Definition 2.3.28). We finally study representability of such contravariant functors and prove that the category of $\mathbb{Z}_2^n$-manifolds is equivalent to the full subcategory of locally trivial functors in the just depicted subcategory of contravariant functors from $\mathbb{Z}_2^n$-points to nuclear Fréchet manifolds (see Theorem 2.3.34).

Methodology: As $\mathbb{Z}_2^n$-manifolds have well defined local models, we work with $\mathbb{Z}_2^n$-domains and then 'globalize' the results to general $\mathbb{Z}_2^n$-manifolds. We modify the approach of Schwarz & Voronov [45, 55] and draw on Balduzzi, Carmeli & Fioresi [7, 8] and Konechny & Schwarz [30, 31], making all changes necessary to encompass $\mathbb{Z}_2^n$-manifolds. Let us mention that Balduzzi, Carmeli & Fioresi study functors from the category of super Weil algebras and not that of Grassmann algebras. However, if we truly want to build a restricted Yoneda embedding, the source category of the functors of points must be a category of algebras that is opposite to some category of supermanifolds – and super Weil algebras are not the algebras of functions of some class of supermanifolds (unless they are Grassmann algebras). Moreover, the idea behind our restriction of the Yoneda embedding is 'the smaller the class of test algebras, the better' – which points again to Grassmann algebras as being the somewhat privileged objects. The most striking difference between supermanifolds and $\mathbb{Z}_2^n$-manifolds ($n > 1$) is that we are forced, due to the presence of non-zero degree even coordinates, to work with (infinite-dimensional) Fréchet spaces, algebras and manifolds. Interestingly, nuclearity of the values $M(\Lambda)$ of the functor of points of a $\mathbb{Z}_2^n$-manifold M, i.e., nuclearity of the local models of the Fréchet Λ_0-manifolds $M(\Lambda)$ or of their tangent spaces, does not play a rôle in the proofs of the statements in this paper. More precisely, the functor of points $M(-)$ has values $M(\Lambda)$ that *are* nuclear Fréchet Λ_0-manifolds. Conversely, a functor $\mathcal{F}(-)$ whose values $\mathcal{F}(\Lambda)$ are Fréchet Λ_0-manifolds *and* which is representable, has nuclear values (nuclearity is encrypted in the representability condition (see Theorem 2.3.34)). Although nuclearity of the tangent spaces of the manifolds $M(\Lambda)$ is not explicitly used throughout this work, we do *not* at all claim that nuclearity is not of importance in the theory of $\mathbb{Z}_2^n$-manifolds. For instance, the function sheaf of a $\mathbb{Z}_2^n$-manifold *is* a nuclear Fréchet sheaf of $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative algebras – a fact that is crucial for product $\mathbb{Z}_2^n$-manifolds and $\mathbb{Z}_2^n$-Lie groups [13].

Applications: The functor of points has been used informally in Physics as from the very beginning. It is actually of importance in situations where there is no good notion of point (see also Section 2.2.2), for instance in Algebraic Geometry and in Super- and $\mathbb{Z}_2^n$-Geometry. Constructing a set-valued functor and showing that it is representable as a locally ringed space, e.g., a scheme or a $\mathbb{Z}_2^n$-manifold, is often easier than building that scheme or manifold directly. Functors that are not representable can be interpreted as generalised schemes or generalised $\mathbb{Z}_2^n$-manifolds. Further, the category of functors is better behaved than the corresponding category of supermanifolds or of other types of spaces. Also Homotopical Algebraic Geometry [50, 51], as well as its generalisation that goes under the name of Homotopical Algebraic $\mathcal{D}$-Geometry (where $\mathcal{D}$ refers to differential operators) [20, 21], are fully based on the functor of points approach. Finally, the functor of points turns out to be an indispensable tool when it comes to the investigation of $\mathbb{Z}_2^n$-Lie groups and their actions on $\mathbb{Z}_2^n$-manifolds, of geometric $\mathbb{Z}_2^n$-vector bundles... These concepts are explored in upcoming texts that are currently being written down.

Arrangement: In Section 2.2, we review the basic tenets of $\mathbb{Z}_2^n$-geometry and the theory of $\mathbb{Z}_2^n$-manifolds. The bulk of this paper is to be found in Section 2.3. We rely on two appendices: in Appendix 2.4.1 we recall the notion of a generating set of a category, and in Appendix 2.4.2 we review indispensable concepts from the theory of Fréchet spaces, algebras and manifolds.

2.2 Rudiments of $\mathbb{Z}_2^n$-graded geometry

2.2.1 The category of $\mathbb{Z}_2^n$-manifolds

The locally ringed space approach to $\mathbb{Z}_2^n$-manifolds is presented in a series of papers [19, 15, 16, 17, 18, 36] by Covolo, Grabowski, Kwok, Ovsienko, and Poncin. We will draw upon these works heavily and not present proofs of any formal statements.

Definition 2.2.1. A *locally $\mathbb{Z}_2^n$-ringed space*, $n \in \mathbb{N}$, is a pair $X := (|X|, \mathcal{O}_X)$, where $|X|$ is a second-countable Hausdorff space, and $\mathcal{O}_X$ is a sheaf of $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebras, such that the stalks $\mathcal{O}_p$, $p \in |X|$, are local rings.

In this context, $\mathbb{Z}_2^n$-commutative means that any two sections a, $b \in \mathcal{O}_X(|U|)$, $|U| \subset |X|$ open, of homogeneous degrees $\deg(a) = \underline{a} \in \mathbb{Z}_2^n$ and $\deg(b) = \underline{b} \in \mathbb{Z}_2^n$ commute according to the sign rule

$$ab = (-1)^{\langle \underline{a}, \underline{b} \rangle}\, ba,$$

where $\langle -, - \rangle$ is the standard scalar product on $\mathbb{Z}_2^n$. We will say that a section a is *even* or *odd* if $\langle \underline{a}, \underline{a} \rangle \in \mathbb{Z}_2$ is 0 or 1.

Just as in standard supergeometry, which we recover for $n = 1$, a locally $\mathbb{Z}_2^n$-ringed space is a $\mathbb{Z}_2^n$-manifold if it is locally isomorphic to a specific local model. Given the central rôle of (finite dimensional) Grassmann algebras in the theory of supermanifolds, we consider here $\mathbb{Z}_2^n$-Grassmann algebras.

Remark 2.2.2. In the following, we order the elements in $\mathbb{Z}_2^n$ *lexicographically*, and refer to this ordering as the *standard ordering*. For example, we thus get

$$\mathbb{Z}_2^2 = \{(0,0),\ (0,1),\ (1,0),\ (1,1)\}.$$

Definition 2.2.3. A $\mathbb{Z}_2^n$-*Grassmann algebra* $\Lambda^{\underline{q}} := \mathbb{R}[[\xi]]$ is the $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebra of all formal power series with coefficients in $\mathbb{R}$ generated by homogeneous parameters ξ^α subject to the commutation relation

$$\xi^\alpha \xi^\beta = (-1)^{\langle \underline{\alpha}, \underline{\beta} \rangle} \xi^\beta \xi^\alpha,$$

where $\underline{\alpha} := \deg(\xi^\alpha) \in \mathbb{Z}_2^n \setminus \underline{0}$, $\underline{0} = (0, \ldots, 0)$. The tuple $\underline{q} = (q_1, q_2, \cdots, q_N)$, $N = 2^n - 1$, provides the number q_i of generators ξ^α, which have the i-th degree in $\mathbb{Z}_2^n \setminus \underline{0}$ (endowed with its standard order).

A *morphism of* $\mathbb{Z}_2^n$-*Grassmann algebras*, $\psi^* : \Lambda^{\underline{q}} \to \Lambda^{\underline{p}}$, is a map of $\mathbb{R}$-algebras that preserves the $\mathbb{Z}_2^n$-grading and the units.

We denote the category of $\mathbb{Z}_2^n$-Grassmann algebras and corresponding morphisms by $\mathbb{Z}_2^n\texttt{GrAlg}$.

Example 2.2.4. For $n = 0$, we simply get $\mathbb{R}$ considered as an algebra over itself.

Example 2.2.5. If $n = 1$, we recover the classical concept of Grassmann algebra with the standard supercommutation rule for generators. In this case, all formal power series truncate to polynomials. In particular, the Grassmann algebra generated by a single odd generator is isomorphic to the algebra of dual numbers.

Example 2.2.6. The $\mathbb{Z}_2^2$-Grassmann algebra $\Lambda^{(1,1,1)}$ is described by three generators

$$(\underbrace{\xi}_{(0,1)}, \underbrace{\theta}_{(1,0)}, \underbrace{z}_{(1,1)}),$$

where we have indicated the $\mathbb{Z}_2^2$-degree. Note that $\xi\theta = \theta\xi$, while $\xi^2 = 0$ and $\theta^2 = 0$. Moreover, $\xi z = -z\xi$ and $\theta z = -z\theta$, while z is not nilpotent. A general (inhomogeneous) element of $\Lambda^{(1,1,1)}$ is then of the form

$$f(\xi, \theta, z) = f_z(z) + \xi f_\xi(z) + \theta f_\theta(z) + \xi\theta f_{\xi\theta}(z),$$

where $f_z(z)$, $f_\xi(z)$, $f_\theta(z)$ and $f_{\xi\theta}(z)$ are formal power series in z. As a subalgebra we can consider $\Lambda^{(1,1,0)}$, whose generators are ξ and θ. A general element of this subalgebra is a polynomial in these generators.

Within any $\mathbb{Z}_2^n$-Grassmann algebra $\Lambda := \Lambda^{\underline{q}}$, we have the ideal generated by the generators of Λ, which we will denote as $\mathring{\Lambda}$. In particular we have the decomposition

$$\Lambda = \mathbb{R} \oplus \mathring{\Lambda}\ ,$$

which will be used later on. Moreover, the set of degree 0 elements, $\Lambda_0 \subset \Lambda$, is a commutative associative unital $\mathbb{R}$-algebra.

Very informally, a $\mathbb{Z}_2^n$-manifold is a smooth manifold whose structure sheaf has been 'deformed' to now include the generators of a $\mathbb{Z}_2^n$-Grassmann algebra.

Definition 2.2.7. A (smooth) $\mathbb{Z}_2^n$-*manifold* of dimension $p|\underline{q}$ is a locally $\mathbb{Z}_2^n$-ringed space $M := (|M|, \mathcal{O}_M)$, which is locally isomorphic to the locally $\mathbb{Z}_2^n$-ringed space $\mathbb{R}^{p|\underline{q}} := (\mathbb{R}^p, C^\infty_{\mathbb{R}^p}[[\xi]])$. Local sections of $\mathcal{O}_M$ are thus formal power series in the $\mathbb{Z}_2^n$-graded variables ξ with smooth coefficients,

$$\mathcal{O}_M(|U|) \simeq C^\infty_{\mathbb{R}^p}(|U|)[[\xi]] := \left\{ \sum_{\alpha \in \mathbb{N}^{\sum_i q_i}} \xi^\alpha f_\alpha : f_\alpha \in C^\infty_{\mathbb{R}^p}(|U|) \right\},$$

for 'small enough' open subsets $|U| \subset |M|$. A $\mathbb{Z}_2^n$-*morphism*, i.e., a morphism between two $\mathbb{Z}_2^n$-manifolds, say M and N, is a morphism of $\mathbb{Z}_2^n$-ringed spaces, that is, a pair $\phi = (|\phi|, \phi^*) : (|M|, \mathcal{O}_M) \to (|N|, \mathcal{O}_N)$ consisting of a continuous map $|\phi| : |M| \to |N|$ and a sheaf morphism $\phi^* : \mathcal{O}_N \to |\phi|_* \mathcal{O}_M$, i.e., a family of $\mathbb{Z}_2^n$-graded unital $\mathbb{R}$-algebra morphisms $\phi^*_{|V|} : \mathcal{O}_N(|V|) \to \mathcal{O}_M(|\phi|^{-1}(|V|))$ ($|V| \subset |N|$ open), which commute with restrictions. We will refer to the global sections of the structure sheaf $\mathcal{O}_M$ as *functions* on M and denote them as $C^\infty(M) := \mathcal{O}_M(|M|)$.

Example 2.2.8 (The local model)**.** The locally $\mathbb{Z}_2^n$-ringed space $\mathcal{U}^{p|\underline{q}} := \big(\mathcal{U}^p, C^\infty_{\mathcal{U}^p}[[\xi]]\big)$, where $\mathcal{U}^p \subset \mathbb{R}^p$ is open, is naturally a $\mathbb{Z}_2^n$-manifold – we refer to such $\mathbb{Z}_2^n$-manifolds as $\mathbb{Z}_2^n$-*domains* of dimension $p|\underline{q}$. We can employ (natural) coordinates (x^a, ξ^α) on any $\mathbb{Z}_2^n$-domain, where the x^a form a coordinate system on $\mathcal{U}^p$ and the ξ^α are formal coordinates.

Canonically associated to any $\mathbb{Z}_2^n$-graded algebra $\mathcal{A}$ is the homogeneous ideal J of $\mathcal{A}$ generated by all homogeneous elements of $\mathcal{A}$ having nonzero degree. If $f : \mathcal{A} \to \mathcal{A}'$ is a morphism of $\mathbb{Z}_2^n$-graded algebras, then $f(J_{\mathcal{A}}) \subset J_{\mathcal{A}'}$. The J-adic topology plays a fundamental rôle in the theory of $\mathbb{Z}_2^n$-manifolds. In particular, these notions can be 'sheafified'. That is, for any $\mathbb{Z}_2^n$-manifold M, there exists an ideal sheaf $\mathcal{J}_M$, defined by $\mathcal{J}(|U|) = \langle f \in \mathcal{O}_M(|U|) : \deg(f) \neq 0 \rangle$. The $\mathcal{J}_M$-adic topology on $\mathcal{O}_M$ can then be defined in the obvious way.

Many of the standard results from the theory of supermanifolds pass over to $\mathbb{Z}_2^n$-manifolds. For example, the topological space $|M|$ comes with the structure of a smooth manifold of dimension p and the continuous base map of any $\mathbb{Z}_2^n$-morphism is actually smooth. Further, for any $\mathbb{Z}_2^n$-manifold M, there exists a short exact sequence of sheaves of $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative associative $\mathbb{R}$-algebras

$$0 \longrightarrow \ker \epsilon \longrightarrow \mathcal{O}_M \stackrel{\epsilon}{\longrightarrow} C^\infty_{|M|} \longrightarrow 0 \ ,$$

such that $\ker \epsilon = \mathcal{J}_M$.

The immediate problem with $\mathbb{Z}_2^n$-manifolds is that $\mathcal{J}_M$ is *not* nilpotent – for supermanifolds the ideal sheaf is nilpotent and this is a fundamental property that makes the theory of supermanifolds so well-behaved. However, this loss of nilpotency is compensated by Hausdorff completeness of $\mathcal{O}_M$ with respect to the $\mathcal{J}_M$-adic topology.

Proposition 2.2.9. *Let M be a $\mathbb{Z}_2^n$-manifold. Then $\mathcal{O}_M$ is $\mathcal{J}_M$-adically Hausdorff complete as a sheaf of $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebras, i.e., the morphism*

$$\mathcal{O}_M \to \lim_{\leftarrow k} \mathcal{O}_M / \mathcal{J}_M^k \ ,$$

naturally induced by the filtration of $\mathcal{O}_M$ by the powers of $\mathcal{J}_M$, is an isomorphism.

The presence of formal power series in the coordinate rings of $\mathbb{Z}_2^n$-manifolds forces one to rely on the Hausdorff-completeness of the $\mathcal{J}$-adic topology. This completeness replaces the standard fact that supermanifold functions of Grassmann odd variables are always polynomials – a result that is often used in extending results from smooth manifolds to supermanifolds.

What makes $\mathbb{Z}_2^n$-manifolds a very workable form of noncommutative geometry is the fact that we have well-defined local models. Much like the theory of manifolds, one can construct global geometric concepts via the gluing of local geometric concepts. That is, we can consider a $\mathbb{Z}_2^n$-manifold as being covered by $\mathbb{Z}_2^n$-domains together with specified gluing information, i.e., coordinate transformations. Moreover, we have the *chart theorem* ([15, Theorem 7.10]) that says that $\mathbb{Z}_2^n$-morphisms from a $\mathbb{Z}_2^n$-manifold $(|M|, \mathcal{O}_M)$ to a $\mathbb{Z}_2^n$-domain $(\mathcal{U}^p, C^\infty_{\mathcal{U}^p}[[\xi]])$, are completely described by the pullbacks of the coordinates (x^a, ξ^α). In other words, to define a $\mathbb{Z}_2^n$-morphism valued in a $\mathbb{Z}_2^n$-domain, we only need to provide total sections $(s^a, s^\alpha) \in \mathcal{O}_M(|M|)$ of the source structure sheaf, whose degrees coincide with those of the target coordinates (x^a, ξ^α). Let us stress the condition $(\ldots, \epsilon s^a, \ldots)(|M|) \subset \mathcal{U}^p$, which is often understood in the literature.

A few words about the atlas definition of a $\mathbb{Z}_2^n$-manifold are necessary. Let $p|\underline{q}$ be as above. A $p|\underline{q}$-*chart* (or $p|\underline{q}$-coordinate-system) over a (second-countable Hausdorff) smooth manifold $|M|$ is a $\mathbb{Z}_2^n$-domain

$$\mathcal{U}^{p|\underline{q}} = (\mathcal{U}^p, C^\infty_{\mathcal{U}^p}[[\xi]]) ,$$

together with a diffeomorphism $|\psi| : |U| \to \mathcal{U}^p$, where $|U|$ is an open subset of $|M|$. Given two $p|\underline{q}$-charts

$$(\mathcal{U}_\alpha^{p|\underline{q}}, |\psi_\alpha|) \quad \text{and} \quad (\mathcal{U}_\beta^{p|\underline{q}}, |\psi_\beta|) \tag{2.2.1}$$

over $|M|$, we set $V_{\alpha\beta} := |\psi_\alpha|(|U_{\alpha\beta}|)$ and $V_{\beta\alpha} := |\psi_\beta|(|U_{\alpha\beta}|)$, where $|U_{\alpha\beta}| := |U_\alpha| \cap |U_\beta|$. We then denote by $|\psi_{\beta\alpha}|$ the diffeomorphism

$$|\psi_{\beta\alpha}| := |\psi_\beta| \circ |\psi_\alpha|^{-1} : V_{\alpha\beta} \to V_{\beta\alpha} . \tag{2.2.2}$$

Whereas in classical differential geometry the coordinate transformations are completely defined by the coordinate systems, in $\mathbb{Z}_2^n$-geometry, they have to be specified separately. A *coordinate transformation* between two charts, say the ones of (3.5.2), is an isomorphism of $\mathbb{Z}_2^n$-manifolds

$$\psi_{\beta\alpha} = (|\psi_{\beta\alpha}|, \psi^*_{\beta\alpha}) : \mathcal{U}_\alpha^{p|\underline{q}}|_{V_{\alpha\beta}} \to \mathcal{U}_\beta^{p|\underline{q}}|_{V_{\beta\alpha}} , \tag{2.2.3}$$

where the source and target are the open $\mathbb{Z}_2^n$-submanifolds

$$\mathcal{U}_\alpha^{p|\underline{q}}|_{V_{\alpha\beta}} = (V_{\alpha\beta}, C^\infty_{V_{\alpha\beta}}[[\xi]])$$

(note that the underlying diffeomorphism is (3.5.3)). A $p|\underline{q}$-*atlas* over $|M|$ is a covering $(\mathcal{U}_\alpha^{p|\underline{q}}, |\psi_\alpha|)_\alpha$ by charts together with a coordinate transformation (3.5.4) for each pair of charts, such that the usual cocycle condition $\psi_{\beta\gamma}\psi_{\gamma\alpha} = \psi_{\beta\alpha}$ holds (appropriate restrictions are understood).

Definition 2.2.10. A (smooth) $\mathbb{Z}_2^n$-*manifold* of dimension $p|\underline{q}$ is a (second-countable Hausdorff) smooth manifold $|M|$ together with a preferred $p|\underline{q}$-atlas over it.

As in standard supergeometry, the definitions 2.2.7 and 2.2.10 are equivalent [32]. For instance, if $M = (|M|, \mathcal{O}_M)$ is a $\mathbb{Z}_2^n$-manifold of dimension $p|\underline{q}$ in the sense of Definition 2.2.7, there are $\mathbb{Z}_2^n$-isomorphisms (isomorphisms of $\mathbb{Z}_2^n$-manifolds)

$$h_\alpha = (|h_\alpha|, h^*_\alpha) : U_\alpha = (|U_\alpha|, \mathcal{O}_M|_{|U_\alpha|}) \to \mathcal{U}_\alpha^{p|\underline{q}} = (\mathcal{U}^p_\alpha, C^\infty_{\mathbb{R}^p}|_{\mathcal{U}^p_\alpha}[[\xi]]) ,$$

such that $(|U_\alpha|)_\alpha$ is an open cover of $|M|$. For any two indices α, β, the restriction $h_\alpha|_{U_{\alpha\beta}}$ of h_α to the open $\mathbb{Z}_2^n$-submanifold $U_{\alpha\beta} = (|U_{\alpha\beta}|, \mathcal{O}_M|_{|U_{\alpha\beta}|})$, $|U_{\alpha\beta}| = |U_\alpha| \cap |U_\beta|$, is a $\mathbb{Z}_2^n$-isomorphism between $U_{\alpha\beta}$ and

$$\mathcal{U}_\alpha^{p|\underline{q}}|_{V_{\alpha\beta}} = (V_{\alpha\beta}, C^\infty_{\mathbb{R}^p}|_{V_{\alpha\beta}}[[\xi]]), \quad V_{\alpha\beta} = |h_\alpha|(|U_{\alpha\beta}|) .$$

Therefore, the composite

$$\psi_{\beta\alpha} = h_\beta|_{U_{\beta\alpha}} h_\alpha|^{-1}_{U_{\alpha\beta}} \tag{2.2.4}$$

is a $\mathbb{Z}_2^n$-isomorphism

$$\psi_{\beta\alpha} : \mathcal{U}_\alpha^{p|\underline{q}}|_{V_{\alpha\beta}} \to \mathcal{U}_\beta^{p|\underline{q}}|_{V_{\beta\alpha}} ,$$

such that the cocycle condition is satisfied. As a matter of some formality, $\mathbb{Z}_2^n$-manifolds and their morphisms form a category. The category of $\mathbb{Z}_2^n$-manifolds we will denote as $\mathbb{Z}_2^n\texttt{Man}$. We remark this category is locally small. Moreover, as shown in [13, Theorem 19], the category of $\mathbb{Z}_2^n$-manifolds admits (finite) products. More precisely, let M_i, $i \in \{1, 2\}$, be $\mathbb{Z}_2^n$-manifolds. Then there exists a $\mathbb{Z}_2^n$-manifold $M_1 \times M_2$ and $\mathbb{Z}_2^n$-morphisms $\pi_i : M_1 \times M_2 \to M_i$ (with underlying smooth manifold $|M_1 \times M_2| = |M_1| \times |M_2|$ and with underlying smooth morphisms $|\pi_i| : |M_1| \times |M_2| \to |M_i|$ given by the canonical projections), such that for any $\mathbb{Z}_2^n$-manifold N and $\mathbb{Z}_2^n$-morphisms $f_i : N \to M_i$, there exists a unique morphism $h : N \to M_1 \times M_2$ making the obvious diagram commute. It follows that, if $\phi \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(M, M')$ and $\psi \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(N, N')$, there is a unique morphism $\phi \times \psi \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(M \times N, M' \times N')$.

Remark 2.2.11. It is known that an analogue of the Batchelor–Gawędzki theorem holds in the category of (real) $\mathbb{Z}_2^n$-manifolds, see [16, Theorem 3.2]. That is, any $\mathbb{Z}_2^n$-manifold is noncanonically isomorphic to a $\mathbb{Z}_2^n \setminus \{\underline{0}\}$-graded vector bundle over a smooth manifold. While this result is quite remarkable, we will not exploit it at all in this paper.

2.2.2 The functor of points

Similar to what happens in classical supergeometry, a $\mathbb{Z}_2^n$-manifold M is not fully described by its topological points in $|M|$. To remedy this defect, we broaden the notion of 'point', as was suggested by Grothendieck in the context of algebraic geometry.

More precisely, set $V = \{z \in \mathbb{C}^n : P(z) = 0\} \in \mathtt{Aff}$, where P denotes a polynomial in n indeterminates with complex coefficients and $\mathtt{Aff}$ denotes the category of affine varieties. Grothendieck insisted on solving the equation $P(z) = 0$ not only in $\mathbb{C}^n$, but in A^n, for any algebra A in the category $\mathtt{CA}$ of commutative (associative unital) algebras (over $\mathbb{C}$). This leads to an arrow

$$\mathrm{Sol}_P : \mathtt{CA} \ni A \mapsto \mathrm{Sol}_P(A) = \{a \in A^n : P(a) = 0\} \in \mathtt{Set} \,,$$

which turns out to be a functor

$$\mathrm{Sol}_P \simeq \mathsf{Hom}_{\mathtt{CA}}(\mathbb{C}[V], -) \in [\mathtt{CA}, \mathtt{Set}] \,,$$

where $\mathbb{C}[V]$ is the algebra of polynomial functions of V. The dual of this functor, whose value $\mathrm{Sol}_P(A)$ is the set of A-points of V, is the functor

$$\mathsf{Hom}_{\mathtt{Aff}}(-, V) \in [\mathtt{Aff}^{\mathrm{op}}, \mathtt{Set}] \,,$$

whose value $\mathsf{Hom}_{\mathtt{Aff}}(W, V)$ is the set of W-points of V.

The latter functor can be considered not only in $\mathtt{Aff}$, but in any locally small category, in particular in $\mathbb{Z}_2^n\mathtt{Man}$. We thus obtain a covariant functor (functor in $\bullet$)

$$\bullet(-) = \mathsf{Hom}(-, \bullet) : \mathbb{Z}_2^n\mathtt{Man} \ni M \mapsto M(-) = \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(-, M) \in [\mathbb{Z}_2^n\mathtt{Man}^{\mathrm{op}}, \mathtt{Set}] \,. \tag{2.2.5}$$

As suggested above, the contravariant functor $\mathsf{Hom}(-, M)$ (we omit the subscript $\mathbb{Z}_2^n\mathtt{Man}$) (functor in $-$) is referred to as the *functor of points* of M. If $S \in \mathbb{Z}_2^n\mathtt{Man}$, an *S-point* of M is just a morphism $\pi_S \in \mathsf{Hom}(S, M)$. One may regard an S-point of M as a 'family of points of M parameterised by the points of S'. The functor $\bullet(-)$ is known as the *Yoneda embedding.* For any underlying locally small category $\mathtt{C}$ (here $\mathtt{C} = \mathbb{Z}_2^n\mathtt{Man}$), the functor $\bullet(-)$ is *fully faithful*, what means that, for any $M, N \in \mathbb{Z}_2^n\mathtt{Man}$, the map

$$\bullet_{M,N}(-) : \mathsf{Hom}(M, N) \ni \phi \mapsto \mathsf{Hom}(-, \phi) \in \mathrm{Nat}(\mathsf{Hom}(-, M), \mathsf{Hom}(-, N))$$

is bijective (here Nat denotes the set of natural transformations). It can be checked that the correspondence $\bullet_{M,N}(-)$ is natural in M and in N. Moreover, any fully faithful functor is automatically injective up to isomorphism on objects: $M(-) \simeq N(-)$ implies $M \simeq N$. Of course, the functor $\bullet(-)$ is not surjective up to isomorphism on objects, i.e., not every functor $X \in [\mathbb{Z}_2^n\mathtt{Man}^{\mathrm{op}}, \mathtt{Set}]$ is isomorphic to a functor of the type $M(-)$. However, if such M does exist, it is, due to the mentioned injectivity, unique up to isomorphism and it is called *'the'* *representing* $\mathbb{Z}_2^n$*-manifold* of X. Further, if $X, Y \in [\mathbb{Z}_2^n\texttt{Man}^{\rm op}, \texttt{Set}]$ are two representable functors, represented by M, N respectively, a morphism or natural transformation between them, provides, due to the mentioned bijectivity, a *unique morphism between the representing* $\mathbb{Z}_2^n$*-manifolds* M and N. It follows that, instead of studying the category $\mathbb{Z}_2^n\texttt{Man}$, we can just as well focus on the functor category $[\mathbb{Z}_2^n\texttt{Man}^{\rm op}, \texttt{Set}]$ (which has better properties, in particular it has all limits and colimits). A *generalized* $\mathbb{Z}_2^n$*-manifold* is an object in the functor category $[\mathbb{Z}_2^n\texttt{Man}^{\rm op}, \texttt{Set}]$ and morphisms of such objects are natural transformations. The category $[\mathbb{Z}_2^n\texttt{Man}^{\rm op}, \texttt{Set}]$ of generalised $\mathbb{Z}_2^n$-manifolds has finite products. Indeed, if F, G are two generalized manifolds, we define the functor $F \times G$, given on objects S, by $(F \times G)(S) = F(S) \times G(S)$, and on morphisms $\Psi : S \to T$, by

$$(F \times G)(\Psi) = F(\Psi) \times G(\Psi) : F(T) \times G(T) \to F(S) \times G(S)\ .$$

It is easily seen that $F \times G$ respects compositions and identities. Further, there are canonical natural transformations $\eta_1 : F \times G \to F$ and $\eta_2 : F \times G \to G$. If now (H, α_1, α_2) is another functor with natural transformations from it to F and G, respectively, it is straightforwardly checked that there exists a unique natural transformation $\beta : H \to F \times G$, such that $\alpha_i = \eta_i \circ \beta$.

One passes from the category of $\mathbb{Z}_2^n$-manifolds to the larger category of generalised $\mathbb{Z}_2^n$-manifolds in order to understand, for example, the *internal Hom* objects. In particular, there always exists a generalised $\mathbb{Z}_2^n$-manifold such that the so–called *adjunction formula* holds

$$\underline{\mathsf{Hom}}_{\mathbb{Z}_2^n\texttt{Man}}(M, N)(-) := \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(- \times M, N)\ .$$

This internal Hom functor is defined on $\phi \in \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(P, S)$ by

$$\begin{aligned} \underline{\mathsf{Hom}}_{\mathbb{Z}_2^n\texttt{Man}}(M, N)(\phi) : \underline{\mathsf{Hom}}_{\mathbb{Z}_2^n\texttt{Man}}(M, N)(S) &\longrightarrow \underline{\mathsf{Hom}}_{\mathbb{Z}_2^n\texttt{Man}}(M, N)(P)\,, \\ \Psi_S &\longmapsto \Psi_S \circ (\phi \times \mathbb{1}_M)\,. \end{aligned}$$

In general, a mapping $\mathbb{Z}_2^n$-manifold $\underline{\mathsf{Hom}}_{\mathbb{Z}_2^n\texttt{Man}}(M, N)$ will not be representable. We will refer to 'elements' of a mapping $\mathbb{Z}_2^n$-manifold as *maps* reserving morphisms for the categorical morphisms of $\mathbb{Z}_2^n$-manifolds.

Composition of maps between $\mathbb{Z}_2^n$-manifolds is naturally defined as a natural transformation

$$\underline{\circ} : \underline{\mathsf{Hom}}(M, N) \times \underline{\mathsf{Hom}}(N, L) \longrightarrow \underline{\mathsf{Hom}}(M, L)\,, \tag{2.2.6}$$

defined, for any $S \in \mathbb{Z}_2^n\texttt{Man}$, by

$$\begin{aligned} \mathsf{Hom}(S \times M, N) \times \mathsf{Hom}(S \times N, L) &\longrightarrow \mathsf{Hom}(S \times M, L) \\ (\Psi_S, \Phi_S) &\longmapsto (\Phi \underline{\circ} \Psi)_S := \Phi_S \circ (\mathbb{1}_S \times \Psi_S) \circ (\Delta \times \mathbb{1}_M)\,, \end{aligned} \tag{2.2.7}$$

where $\Delta : S \longrightarrow S \times S$ is the diagonal of S and $\mathbb{1}_S : S \longrightarrow S$ is its identity.

Similarly to the cases of smooth manifolds and supermanifolds, morphisms between $\mathbb{Z}_2^n$-manifolds are completely determined by the corresponding maps between the global functions. We remark that this is not, in general, true for complex (super)manifolds. More carefully, we have the following proposition that was proved in [13, Theorem 3.7.].

Proposition 2.2.12. *Let* $M = (|M|, \mathcal{O}_M)$ *and* $N = (|N|, \mathcal{O}_N)$ *be* $\mathbb{Z}_2^n$*-manifolds. Then the natural map*

$$\mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Man}}\big(M, N\big) \longrightarrow \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Alg}}\big(\mathcal{O}(|N|), \mathcal{O}(|M|)\big)\,,$$

where $\mathbb{Z}_2^n\texttt{Alg}$ *denotes the category of* $\mathbb{Z}_2^n$*-graded* $\mathbb{Z}_2^n$*-commutative associative unital* $\mathbb{R}$*-algebras, is a bijection.*

It is worth recalling how a morphism $\psi \in \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Alg}}\big(\mathcal{O}(|N|), \mathcal{O}(|M|)\big)$ defines a continuous base map $|\phi| : |M| \to |N|$. We denote by $\epsilon_m \in \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Alg}}\big(\mathcal{O}(|M|), \mathbb{R}\big)$, $m \in |M|$, the morphism

$$\epsilon_m : \mathcal{O}(|M|) \ni f \mapsto (\epsilon_{|M|} f)(m) \in \mathbb{R} \, ,$$

and by $\mathrm{Spm}(\mathcal{O}(|M|))$ the maximal spectrum of the algebra $\mathcal{O}(|M|)$. The map

$$\flat : |M| \ni m \mapsto \ker \epsilon_m \in \mathrm{Spm}(\mathcal{O}(|M|))$$

is a homeomorphism, both, when the target space is endowed with its Zariski topology and when it is endowed with its Gel'fand topology. The continuous map $|\phi| : |M| \to |N|$ that is induced by the morphism ψ is now given by

$$|\phi| : |M| \simeq \mathrm{Spm}(\mathcal{O}(|M|)) \ni m \simeq \ker \epsilon_m \mapsto \ker(\epsilon_m \circ \psi) \simeq n \in \mathrm{Spm}(\mathcal{O}(|N|)) \simeq |N| \, .$$

The fact that the functor $\mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(S, -)$ respects limits and in particular products directly implies that

$$\big(M \times N\big)(S) \simeq M(S) \times N(S) \, . \tag{2.2.8}$$

The latter result is essential in dealing with $\mathbb{Z}_2^n$-Lie groups. A (super) Lie group can be defined as a group object in the category of smooth (super)manifolds. This leads us to the following definition.

Definition 2.2.13. A $\mathbb{Z}_2^n$-*Lie group* is a group object in the category of $\mathbb{Z}_2^n$-manifolds.

A convenient fact here is that, if G is a $\mathbb{Z}_2^n$-Lie group, then the set $G(S)$ is a group (see (2.2.8)). In other words, $G(-)$ is a functor from $\mathbb{Z}_2^n\texttt{Man}^{\mathrm{op}} \to \texttt{Grp}$.

Remark 2.2.14. We leave details and examples of $\mathbb{Z}_2^n$-Lie groups for future publications. However, we will remark at this point that the idea of "colour supergroup manifolds" has already appeared in the physics literature, albeit without a proper mathematical definition (see [1, 3, 37, 38], for example). Another approach to $\mathbb{Z}_2^n$-Lie groups is via a generalisation of Harish-Chandra pairs (see [34] for work in this direction).

2.3 $\mathbb{Z}_2^n$-points and the functor of points

In view of (2.2.5), we need to 'probe' a given $\mathbb{Z}_2^n$-manifold $M \simeq M(-)$ with *all* $\mathbb{Z}_2^n$-manifolds. We will show that this is however not the case, since, much like for the category of supermanifolds, we have a rather convenient generating set that we can employ, namely the set of $\mathbb{Z}_2^n$-points.

2.3.1 The category of $\mathbb{Z}_2^n$-points

Definition 2.3.1. A $\mathbb{Z}_2^n$-*point* is a $\mathbb{Z}_2^n$-manifold $\mathbb{R}^{0|\underline{m}}$ with vanishing ordinary dimension. We denote by $\mathbb{Z}_2^n\texttt{Pts}$ the full subcategory of $\mathbb{Z}_2^n\texttt{Man}$, whose collection of objects is the (countable) set of $\mathbb{Z}_2^n$-points.

Morphisms $\phi : \mathbb{R}^{0|\underline{m}} \to \mathbb{R}^{0|\underline{n}}$ of $\mathbb{Z}_2^n$-points are exactly morphisms $\phi^* : \Lambda^{\underline{n}} \to \Lambda^{\underline{m}}$ of $\mathbb{Z}_2^n$-Grassmann algebras:

Proposition 2.3.2. *There is an isomorphism of categories*

$$\mathbb{Z}_2^n\texttt{Pts} \simeq \mathbb{Z}_2^n\texttt{GrAlg}^{\mathrm{op}} \, .$$

We can think of $\mathbb{Z}_2^n$-points as *formal thickenings* of an ordinary point by the non-zero degree generators. The simplest $\mathbb{Z}_2^n$-point is the one with trivial formal thickening, $\mathbb{R}^{0|\underline{0}} := \big(\mathbb{R}^0, \mathbb{R}\big)$:

Proposition 2.3.3. *The $\mathbb{Z}_2^n$-point $\mathbb{R}^{0|\underline{0}} = \mathbb{R}^0$ is a terminal object in both, $\mathbb{Z}_2^n\mathtt{Man}$ and $\mathbb{Z}_2^n\mathtt{Pts}$.*

Proof. The unique morphism $M \longrightarrow \mathbb{R}^{0|\underline{0}}$ corresponds to the morphism $\mathbb{R} \ni r \cdot 1 \mapsto r \cdot \mathbb{1}_M \in \mathcal{O}_M(|M|)$, where $\mathbb{1}_M$ is the unit function. □

Proposition 2.3.4. *The object set* $\mathrm{Ob}(\mathbb{Z}_2^n\mathtt{Pts}) \simeq \mathrm{Ob}(\mathbb{Z}_2^n\mathtt{GrAlg})$ *is a directed set.*

Proof. Given any $\underline{m} = (m_1, m_2, \cdots, m_N)$ and $\underline{n} = (n_1, n_2, \cdots, n_N)$, we write $\Lambda^{\underline{m}} \leq \Lambda^{\underline{n}}$ if and only if $m_i \leq n_i$, for all i . This preorder makes the non-empty set of $\mathbb{Z}_2^n$-Grassmann algebras into a directed set, since, any $\Lambda^{\underline{m}}$ and $\Lambda^{\underline{n}}$ admit $\Lambda^{\underline{p}}$, where $p_i = \sup\{m_i, n_i\}$, as upper bound. □

We will need the following functional analytic result in later sections of this paper. See Definition 2.4.4 and Definition 2.4.8 for the notion of Fréchet space and Fréchet algebra, respectively.

Proposition 2.3.5. *The algebra of functions of any $\mathbb{Z}_2^n$-point is a $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative nuclear Fréchet algebra.*

The proposition is a special case of the fact that the structure sheaf of *any* $\mathbb{Z}_2^n$-manifold is a nuclear Fréchet sheaf of $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative algebras [12, Theorem 14].

Moreover, as a direct consequence of [13, Theorem 19, Definition 13], we observe that the category of $\mathbb{Z}_2^n$-points admits all finite categorical products; in particular: $\mathbb{R}^{0|\underline{m}} \times \mathbb{R}^{0|\underline{n}} \simeq \mathbb{R}^{0|\underline{m}+\underline{n}}$. By restricting attention to elements of degree $0 \in \mathbb{Z}_2^n$, we get the following corollary. See Definition 2.4.10 for the concept of Fréchet module.

Corollary 2.3.6. *The set Λ_0 of degree 0 elements of an arbitrary $\mathbb{Z}_2^n$-Grassmann algebra Λ is a commutative nuclear Fréchet algebra. Moreover, the algebra Λ can canonically be considered as a Fréchet Λ_0-module.*

Remark 2.3.7. Specialising to the $n = 1$ case, we recover the standard and well-known facts about superpoints and their relation with Grassmann algebras.

2.3.2 A convenient generating set of $\mathbb{Z}_2^n\mathtt{Man}$

It is clear that studying just the underlying topological points of a $\mathbb{Z}_2^n$-manifold is inadequate to probe the graded structure. Much like the category of supermanifolds, where the set of superpoints forms a generating set, the set of $\mathbb{Z}_2^n$-points forms a generating set for the category of $\mathbb{Z}_2^n$-manifolds. For the classical case of standard supermanifolds, see for example [41, Theorem 3.3.3]. For the general notion of a generating set, see Definition 2.4.1.

Theorem 2.3.8. *The set* $\mathrm{Ob}\big(\mathbb{Z}_2^n\mathtt{Pts}\big)$ *constitutes a generating set for* $\mathbb{Z}_2^n\mathtt{Man}$*.*

Proof. Let $\phi = (|\phi|, \phi^*)$ and $\psi = (|\psi|, \psi^*)$ be two distinct $\mathbb{Z}_2^n$-morphisms $\phi, \psi : M \to N$ between two $\mathbb{Z}_2^n$-manifolds $M = (|M|, \mathcal{O}_M)$ and $N = (|N|, \mathcal{O}_N)$. These morphisms have distinct smooth base maps

$$|\phi|, |\psi| : |M| \to |N| \ ,$$

or, if $|\phi| = |\psi|$, they have distinct pullback morphisms of sheaves of algebras

$$\phi^*, \psi^* : \mathcal{O}_N \to |\phi|_* \mathcal{O}_M \ .$$

If $|\phi| \neq |\psi|$, there is at least one point $m \in |M|$, such that $|\phi|(m) \neq |\psi|(m)$. Let now $s : \mathbb{R}^{0|\underline{0}} \to M$ be the $\mathbb{Z}_2^n$-morphism, which corresponds to the $\mathbb{Z}_2^n\mathtt{Alg}$ morphism $s^* : \mathcal{O}_M(|M|) \ni$

$f \mapsto (\epsilon f)(m) \in \mathbb{R}$, where ϵ is the sheaf morphism $\epsilon : \mathcal{O}_M \to C^\infty_{|M|}$. It follows from the reconstruction theorem [13, Theorem 9] that the base morphism $|s| : \{\star\} \to |M|$ maps $\star$ to m. Hence, the $\mathbb{Z}_2^n$-morphisms $\phi \circ s$ and $\psi \circ s$ have distinct base maps.

Assume now that $|\phi| = |\psi|$, so that there exists $|V| \subset |N|$, such that $\phi^*_{|V|} \neq \psi^*_{|V|}$, i.e., such that $\phi^*_{|V|} f \neq \psi^*_{|V|} f$, for some function $f \in \mathcal{O}_N(|V|)$. A cover of $|V|$ by coordinate patches $(\mathcal{V}_i)_i$, induces a cover $|U_i| := |\phi|^{-1}(\mathcal{V}_i)$ of $|U| := |\phi|^{-1}(|V|)$. It follows that

$$(\phi^*_{|V|} f)|_{|U_i|} \neq (\psi^*_{|V|} f)|_{|U_i|} ,$$

for some fixed i, i.e., that

$$\phi^*_{\mathcal{V}_i}(f|_{\mathcal{V}_i}) \neq \psi^*_{\mathcal{V}_i}(f|_{\mathcal{V}_i}) ,$$

so that $\phi^*_{\mathcal{V}_i} \neq \psi^*_{\mathcal{V}_i}$.

Recall that, for any open subset $|X| \subset |M|$, there is a $\mathbb{Z}_2^n$-morphism

$$\iota_X : (|X|, \mathcal{O}_M|_{|X|}) \to (|M|, \mathcal{O}_M) ,$$

whose base map $|\iota_X|$ is the inclusion and whose pullback ι_X^* is the obvious restriction. Further, any $\mathbb{Z}_2^n$-morphism $\phi : M \to N$, whose base map $|\phi| : |M| \to |N|$ is valued in an open subset $|Y|$ of $|N|$, induces a $\mathbb{Z}_2^n$-morphism

$$\phi_Y : (|M|, \mathcal{O}_M) \to (|Y|, \mathcal{O}_N|_{|Y|}) ,$$

whose base map $|\phi_Y|$ is the map $|\phi| : |M| \to |Y|$ and whose pullback ϕ_Y^* is the pullback ϕ^* restricted to $\mathcal{O}_N|_{|Y|}$.

In view of the above, if $(\mathcal{U}_j)_j$ is a cover of $|U_i|$ by coordinate domains, we have

$$(\phi^*_{\mathcal{V}_i}(f|_{\mathcal{V}_i}))|_{\mathcal{U}_j} \neq (\psi^*_{\mathcal{V}_i}(f|_{\mathcal{V}_i}))|_{\mathcal{U}_j} , \tag{2.3.1}$$

for some fixed j. This implies that the $\mathbb{Z}_2^n$-morphisms $(\phi \circ \iota_{\mathcal{U}_j})_{\mathcal{V}_i}$ and $(\psi \circ \iota_{\mathcal{U}_j})_{\mathcal{V}_i}$ from the $\mathbb{Z}_2^n$-domain $\mathcal{U}_j = (\mathcal{U}_j, C^\infty_{\mathcal{U}_j}[[\xi]])$ to the $\mathbb{Z}_2^n$-domain $\mathcal{V}_i = (\mathcal{V}_i, C^\infty_{\mathcal{V}_i}[[\theta]])$ are different. More precisely, they have the same base map $|\phi| = |\psi| : \mathcal{U}_j \to \mathcal{V}_i$, but their pullbacks are distinct. Indeed, these sheaf morphisms' algebra maps at $\mathcal{V}_i$ are the maps $\iota^*_{\mathcal{U}_j,|U_i|} \circ \phi^*_{\mathcal{V}_i}$ and $\iota^*_{\mathcal{U}_j,|U_i|} \circ \psi^*_{\mathcal{V}_i}$ from $C^\infty_{\mathcal{V}_i}(y)[[\theta]]$ to $C^\infty_{\mathcal{U}_j}(x)[[\xi]]$, where y runs through $\mathcal{V}_i$ and x through $\mathcal{U}_j$, and the values of these algebra maps at $f|_{\mathcal{V}_i}$ are different (see Equation (2.3.1)).

In view of Lemma 2.3.9, there is a $\mathbb{Z}_2^n$-morphism $s : \mathbb{R}^{0|\underline{m}} \to \mathcal{U}_j$, such that

$$(\phi \circ \iota_{\mathcal{U}_j})_{\mathcal{V}_i} \circ s \neq (\psi \circ \iota_{\mathcal{U}_j})_{\mathcal{V}_i} \circ s .$$

However, then the $\mathbb{Z}_2^n$-morphism $\iota_{\mathcal{U}_j} \circ s : \mathbb{R}^{0|\underline{m}} \to M$ separates ϕ and ψ, since the algebra maps at $\mathcal{V}_i$ of the pullbacks $(s^* \circ \iota^*_{\mathcal{U}_j}) \circ \phi^*$ and $(s^* \circ \iota^*_{\mathcal{U}_j}) \circ \psi^*$ differ. Indeed, as the $\mathbb{Z}_2^n$-morphisms $(\phi \circ \iota_{\mathcal{U}_j})_{\mathcal{V}_i}$ and $(\psi \circ \iota_{\mathcal{U}_j})_{\mathcal{V}_i}$ are fully determined by the pullbacks of the target coordinates, their pullbacks at $\mathcal{V}_i$ differ for at least one coordinate y^b, θ^B. It follows from the proof of Lemma 2.3.9 that the pullback $s^*_{\mathcal{U}_j} \circ (\iota^*_{\mathcal{U}_j,|U_i|} \circ \phi^*_{\mathcal{V}_i})$ at $\mathcal{V}_i$ of $(\phi \circ \iota_{\mathcal{U}_j})_{\mathcal{V}_i} \circ s$ and the similar pullback for ψ differ for the same coordinate. However, the pullback at $\mathcal{V}_i$ considered is also the algebra map at $\mathcal{V}_i$ of the pullback $(s^* \circ \iota^*_{\mathcal{U}_j}) \circ \phi^*$, so that the pullbacks $(s^* \circ \iota^*_{\mathcal{U}_j}) \circ \phi^*$ and $(s^* \circ \iota^*_{\mathcal{U}_j}) \circ \psi^*$ are actually distinct. □

It remains to prove the following

Lemma 2.3.9. *The statement of Theorem 2.3.8 holds for any two distinct $\mathbb{Z}_2^n$-morphisms between $\mathbb{Z}_2^n$-domains.*

Proof. We consider two $\mathbb{Z}_2^n$-domains $\mathcal{U}^{p|\underline{q}}$ and $\mathcal{V}^{r|\underline{s}}$ together with two distinct $\mathbb{Z}_2^n$-morphisms

$$\mathcal{U}^{p|\underline{q}} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \mathcal{V}^{r|\underline{s}} .$$

As in the general case above, there are two cases to consider: either $|\phi| \neq |\psi|$, or $|\phi| = |\psi|$ and $\phi^* \neq \psi^*$. In the proof of Theorem 2.3.8, we showed that in the first case, the maps ϕ and ψ can be separated. In the second case, since a $\mathbb{Z}_2^n$-morphism valued in a $\mathbb{Z}_2^n$-domain is fully defined by the pullbacks of the coordinates, these global $\mathbb{Z}_2^n$-functions $\phi^*_{\mathcal{V}^r}(Y^i), \psi^*_{\mathcal{V}^r}(Y^i) \in C^\infty_{\mathcal{U}^p}(x)[[\xi]]$ differ for at least one coordinate $Y^i = y^b$ or $Y^i = \theta^B$. Let B be an index, such that

$$\phi^*_{\mathcal{V}^r}(\theta^B) = \sum_{|\alpha|=1}^{\infty} \phi^B_\alpha(x)\xi^\alpha ,$$

$$\psi^*_{\mathcal{V}^r}(\theta^B) = \sum_{|\alpha|=1}^{\infty} \psi^B_\alpha(x)\xi^\alpha ,$$

where we denoted the coordinates of $\mathcal{U}^{p|\underline{q}}$ by (x^a, ξ^A) and used the standard multi-index notation, differ. This means that the functions $\phi^B_\alpha(x)$ and $\psi^B_\alpha(x)$ differ for at least one α and at least one $x \in \mathcal{U}^p$, say for $\alpha = \mathfrak{a}$ and $x = \mathfrak{x} \in \mathcal{U}^p \subset \mathbb{R}^p$. From this, we can construct the separating $\mathbb{Z}_2^n$-morphism

$$\mathbb{R}^{0|\underline{q}} \xrightarrow{s} \mathcal{U}^{p|\underline{q}} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \mathcal{V}^{r|\underline{s}} .$$

Let us denote the coordinates of $\mathbb{R}^{0|\underline{q}}$ by χ^A. We then define the $\mathbb{Z}_2^n$-morphism s by setting

$$s^*_{\mathcal{U}^p} x^a = \mathfrak{x}^a \in \mathbb{R}[[\chi]],\ \deg(\mathfrak{x}^a) = \deg(x^a) ,$$
$$s^*_{\mathcal{U}^p} \xi^A = \chi^A \in \mathbb{R}[[\chi]],\ \deg(\chi^A) = \deg(\xi^A) .$$

It is clear that $\phi \circ s \neq \psi \circ s$, since

$$\sum_{|\alpha|=1}^{\infty} \phi^B_\alpha(\mathfrak{x})\chi^\alpha = s^*_{\mathcal{U}^p}(\phi^*_{\mathcal{V}^r}(\theta^B)) \neq s^*_{\mathcal{U}^p}(\psi^*_{\mathcal{V}^r}(\theta^B)) = \sum_{|\alpha|=1}^{\infty} \psi^B_\alpha(\mathfrak{x})\chi^\alpha .$$

The case where $\phi^*_{\mathcal{V}^r}(Y^i) \neq \psi^*_{\mathcal{V}^r}(Y^i)$ for $Y^i = y^b$ is almost identical. In particular, we then have

$$\phi^*_{\mathcal{V}^r}(y^b) = |\phi|^b(x) + \sum_{|\alpha|=2}^{\infty} \phi^b_\alpha(x)\xi^\alpha ,$$

$$\psi^*_{\mathcal{V}^r}(y^b) = |\psi|^b(x) + \sum_{|\alpha|=2}^{\infty} \psi^b_\alpha(x)\xi^\alpha .$$

Since we know that $|\phi| = |\psi|$, we can proceed as for $Y^i = \theta^B$. □

In view of Proposition 2.4.3, we get the

Corollary 2.3.10. *The restricted Yoneda functor*

$$\mathcal{Y}_{\mathbb{Z}_2^n\texttt{Pts}} : \mathbb{Z}_2^n\texttt{Man} \ni M \mapsto \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Man}}\big(-, M\big) \in \big[\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{Set}\big],$$

is faithful.

Above, we wrote $M(-) \in [\mathbb{Z}_2^n\mathtt{Man}^{\mathrm{op}}, \mathtt{Set}]$ for the image of $M \in \mathbb{Z}_2^n\mathtt{Man}$ by the non-restricted Yoneda functor. If no confusion arises, we will use the same notation $M(-)$ for the image $\mathcal{Y}_{\mathbb{Z}_2^n\mathtt{Pts}}(M) \in [\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{Set}]$ of M by the restricted Yoneda functor.

Definition 2.3.11. Let M be an object of $\mathbb{Z}_2^n\mathtt{Man}$ and $\Lambda \simeq \mathbb{R}^{0|\underline{m}}$ an object of $\mathbb{Z}_2^n\mathtt{GrAlg} \simeq \mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}$. We refer to the set

$$M(\Lambda) := \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}\big(\mathbb{R}^{0|\underline{m}}, M\big) \simeq \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}\big(\mathcal{O}(|M|), \Lambda\big) \tag{2.3.2}$$

as the set of Λ-*points* of M.

Proposition 2.3.12. *Let*

$$m^* \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}\big(\mathcal{O}(|M|), \Lambda\big)$$

be a Λ-point of M and let $s \in \mathcal{O}(|M|)$. The Λ-point m^* can equivalently be viewed as a $\mathbb{Z}_2^n$-morphism

$$m = (|m|, m^*) \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(\mathbb{R}^{0|\underline{m}}, M)$$

and therefore it defines a unique topological point $x := |m|(\star) \in |M|$. If $|U| \subset |M|$ is an open neighbourhood of x, such that $s|_{|U|} = 0$, then $m^*(s) = 0$.

Proof. Since $m^* : \mathcal{O}_M \to \mathcal{O}_{\mathbb{R}^{0|\underline{m}}}$ is a sheaf morphism, it commutes with restrictions, i.e., for any open subsets $|V| \subset |U| \subset |M|$ and any $s \in \mathcal{O}_M(|U|)$, we have $m^*_{|U|}(s) \in \mathcal{O}_{\mathbb{R}^{0|\underline{m}}}(|m|^{-1}(|U|))$ and

$$(m^*_{|U|}(s))|_{|m|^{-1}(|V|)} = m^*_{|V|}(s|_{|V|}) \in \mathcal{O}_{\mathbb{R}^{0|\underline{m}}}(|m|^{-1}(|V|)) \ .$$

It follows that $m^*(s) = m^*_{|M|}(s) \in \Lambda = \mathcal{O}_{\mathbb{R}^{0|\underline{m}}}(\{\star\})$ reads

$$m^*(s) = (m^*_{|M|}(s))|_{\{\star\}} = (m^*_{|M|}(s))|_{|m|^{-1}(|U|)} = m^*_{|U|}(s|_{|U|}) = 0 \ .$$

□

Lemma 2.3.13. *There is a 1:1 correspondence*

$$M(\Lambda) \simeq \bigcup_{x \in |M|} \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}\big(\mathcal{O}_{M,x}, \Lambda\big)$$

between the set of Λ-points of M and the set of morphisms from the stalks of $\mathcal{O}_M$ to Λ. The set

$$M_x(\Lambda) := \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}\big(\mathcal{O}_{M,x}, \Lambda\big)$$

is referred to as the set of Λ-points near x.

Proof. Any Λ-point m^* or $m = (|m|, m^*)$ defines a topological point $x = |m|(\star) \in |M|$, as well as a $\mathbb{Z}_2^n\mathtt{Alg}$-morphism $\phi_x \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_{M,x}, \Lambda)$ between stalks. This morphism is given, for any $t_U \in \mathcal{O}(|U|)$ defined in some neighbourhood $|U|$ of x in $|M|$, by

$$\phi_x[t_U]_x = m^*_\star[t_U]_x = [m^*_{|U|}t_U]_\star = m^*_{|U|}t_U \ .$$

Conversely, any morphism $\psi_y \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_{M,y}, \Lambda)$ $(y \in |M|)$ between stalks defines a Λ-point $\mu^* \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}\big(\mathcal{O}(|M|), \Lambda\big)$. It suffices to set

$$\mu^* t = \psi_y[t]_y \in \Lambda \ ,$$

for all $t \in \mathcal{O}(|M|)$.

It remains to check that the composites $m^* \mapsto \phi_x \mapsto \mu^*$ and $\psi_y \mapsto \mu^* \mapsto \phi_x$ are identities. In the first case, for any $t \in \mathcal{O}(|M|)$, we get $\mu^* t = \phi_x[t]_x = m^* t$, so that $\mu^* = m^*$. In the second case, we need the following reconstruction results. Let $|U| \subset |M|$ be an open subset and set

$$S_U = \{s \in \mathcal{O}^0(|M|) : (\epsilon s)|_{|U|} \text{ is invertible in } C^\infty(|U|)\} .$$

Then the localization map $\lambda_U : \mathcal{O}(|M|) \cdot S_U^{-1} \to \mathcal{O}(|U|)$ is an isomorphism in $\mathbb{Z}_2^n\mathtt{Alg}$. More precisely, for any $t_U \in \mathcal{O}(|U|)$, there is a unique $Fs^{-1} \in \mathcal{O}(|M|) \cdot S_U^{-1}$, such that $t_U = F|_{|U|} s|_{|U|}^{-1}$ (if $s \in S_U$, then $s|_{|U|}$ is invertible in $\mathcal{O}(|U|)$), and we identify Fs^{-1} with t_U. For the proof of these statements or more details on them, see [13, Proposition 3.5.]. It is further clear from the results of [13, Proposition 3.1.] that $x = |\mu|(\star)$ is the topological point y.

We now compute the second composite above. For any t_U defined in a neighborhood $|U|$ of x, we get

$$\phi_x[t_U]_x = \mu^*_{|U|}(Fs^{-1}) = \mu^*(F)\, \mu^*(s)^{-1} =$$
$$\psi_x[F]_x\, (\psi_x[s]_x)^{-1} = \psi_x[F]_x\, \psi_x([s]_x^{-1}) = \psi_x([F|_{|U|}]_x [s|_{|U|}^{-1}]_x) = \psi_x[t_U]_x \ ,$$

where the second equality is part of the reconstruction theorem of $\mathbb{Z}_2^n$-morphisms [13]. □

Let us consider an open cover $(|U_I|)_{I\in\mathcal{A}}$ of the smooth manifold $|M|$, as well as the open $\mathbb{Z}_2^n$-submanifolds $U_I := \big(|U_I|, \mathcal{O}_M|_{|U_I|}\big)$ of the $\mathbb{Z}_2^n$-manifold M (which need *not* be coordinate charts).

Proposition 2.3.14. *For any $\mathbb{Z}_2^n$-Grassmann algebra Λ and $\mathbb{Z}_2^n$-manifold $M = \big(|M|, \mathcal{O}_M\big)$, we have a natural 1:1 correspondence*

$$M(\Lambda) \simeq \bigcup_{I\in\mathcal{A}} U_I(\Lambda) \,,$$

so that the family of sets $(U_I(\Lambda))_{I\in\mathcal{A}}$ is a cover of the set $M(\Lambda)$.

Proof. Since it is clear from the definition of a stalk that $\mathcal{O}_{U_I,x} = \mathcal{O}_{M,x}$, for any $x \in |U_I|$, it follows from Lemma 2.3.13 that

$$\bigcup_{I\in\mathcal{A}} U_I(\Lambda) \simeq \bigcup_{I\in\mathcal{A}} \bigcup_{x\in|U_I|} \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}\big(\mathcal{O}_{M,x}, \Lambda\big) = \bigcup_{x\in|M|} \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}\big(\mathcal{O}_{M,x}, \Lambda\big) \simeq M(\Lambda) \ .$$

□

Recall that

$$\mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(-,-) \in [\mathbb{Z}_2^n\mathtt{Man},\ [\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{Set}]] \ ,$$

so that,

i. any $\mathbb{Z}_2^n$-morphism $\phi = (|\phi|, \phi^*) : M \to N$ is mapped (injectively) to a natural transformation

$$\phi \simeq \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(-,\phi) : \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(-,M) \to \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(-,N) \ ,$$

whose Λ-component ($\Lambda \simeq \mathbb{R}^{0|\underline{m}}$) is the $\mathtt{Set}$-map given by

$$\phi_\Lambda := \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(\Lambda,\phi) : M(\Lambda) = \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(\mathbb{R}^{0|\underline{m}}, M) \simeq \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}(|M|),\Lambda) \ni m^* \mapsto \tag{2.3.3}$$

$$m^* \circ \phi^* \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}(|N|),\Lambda) \simeq \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(\mathbb{R}^{0|\underline{m}}, N) = N(\Lambda) \ , \text{ and } ,$$

ii. for any fixed $M \in \mathbb{Z}_2^n\texttt{Man}$, given a morphism $\psi = (|\psi|, \psi^*) : \mathbb{R}^{0|\underline{m}'} \to \mathbb{R}^{0|\underline{m}}$ of $\mathbb{Z}_2^n$-points, or, equivalently, a morphism $\psi^* : \Lambda \to \Lambda'$ of $\mathbb{Z}_2^n$-Grassmann algebras, we get the induced `Set`-map

$$M(\psi^*) := \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(\psi, M) : M(\Lambda) = \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(\mathbb{R}^{0|\underline{m}}, M) \simeq \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Alg}}(\mathcal{O}(|M|), \Lambda) \ni m^* \mapsto \tag{2.3.4}$$

$$\psi^* \circ m^* \in \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Alg}}(\mathcal{O}(|M|), \Lambda') \simeq \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(\mathbb{R}^{0|\underline{m}'}, M) = M(\Lambda')\ .$$

When reading the maps ϕ_Λ and $M(\psi^*)$ through the 1:1 correspondence

$$M(\Lambda) \ni m^* \mapsto (x, m^*_\star) \in \bigcup_{y\in|M|} \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Alg}}\big(\mathcal{O}_{M,y}, \Lambda\big)\ ,$$

where $x = |m|(\star)$, we obtain

$$\begin{aligned} \phi_\Lambda : M(\Lambda) &\longrightarrow N(\Lambda) \\ (x,\ m^*_\star)\ &\mapsto (|\phi|(x),\ m^*_\star \circ \phi^*_x)\ ,\ \text{and}\ , \end{aligned} \tag{2.3.5}$$

$$\begin{aligned} M(\psi^*) : M(\Lambda) &\longrightarrow M(\Lambda') \\ (x,\ m^*_\star)\ &\mapsto (x,\ \psi^* \circ m^*_\star)\ . \end{aligned} \tag{2.3.6}$$

2.3.3 Restricted Yoneda functor and fullness

The Yoneda functor from any locally small category `C` into the category of `Set`-valued contravariant functors on `C`, is fully faithful. This holds in particular for $\texttt{C} = \mathbb{Z}_2^n\texttt{Man}$. When we restrict the contravariant functors to the generating set $\mathbb{Z}_2^n\texttt{Pts}$, the resulting restricted Yoneda functor is automatically faithful. In the following, we show that it is not full, i.e., that not all natural transformations are induced by a $\mathbb{Z}_2^n$-morphism.

Naturality of any transformation $\phi : M(-) \to N(-)$ between `Set`-valued contravariant (resp., covariant) functors on $\mathbb{Z}_2^n\texttt{Pts}$ (resp., $\mathbb{Z}_2^n\texttt{GrAlg}$), means that the diagram

$$\begin{array}{ccc} M(\Lambda) & \xrightarrow{\ \phi_\Lambda\ } & N(\Lambda) \\ {\scriptstyle M(\psi^*)}\downarrow & & \downarrow{\scriptstyle N(\psi^*)} \\ M(\Lambda') & \xrightarrow[\ \phi_{\Lambda'}\ ]{} & N(\Lambda') \end{array} \tag{2.3.7}$$

commutes, for any morphism $\psi^* : \Lambda \to \Lambda'$ of $\mathbb{Z}_2^n$-Grassmann algebras.

A Λ-point of a $\mathbb{Z}_2^n$-manifold M is denoted by m^* or $m = (|m|, m^*)$. If the manifold is a $\mathbb{Z}_2^n$-domain $\mathcal{U}^{p|\underline{q}}$, we use the notation x^* or $\mathrm{x} = (|\mathrm{x}|, \mathrm{x}^*)$. If (x^a, ξ^A) are the coordinates of $\mathcal{U}^{p|\underline{q}}$, a Λ-point x^* in $\mathcal{U}^{p|\underline{q}}$ is completely determined by the degree-respecting pullbacks

$$(x^a_\Lambda, \xi^A_\Lambda) := (\mathrm{x}^*(x^a), \mathrm{x}^*(\xi^A))\ .$$

Since $x^a_\Lambda \in \Lambda_0 = \mathbb{R} \oplus \mathring{\Lambda}_0$, we write $x^a_\Lambda = (x^a_{||}, \mathring{x}^a_\Lambda)$. Hence, any Λ-point x^* in $\mathcal{U}^{p|\underline{q}}$ can be identified with

$$\mathrm{x}^* \simeq \big(x^a_\Lambda, \xi^A_\Lambda\big) = \big(x^a_{||}, \mathring{x}^a_\Lambda, \xi^A_\Lambda\big) \in \mathbb{R}^p \times \mathring{\Lambda}_0^p \times \mathring{\Lambda}_{\gamma_1}^{q_1} \times \cdots \times \mathring{\Lambda}_{\gamma_N}^{q_N}\ , \tag{2.3.8}$$

where

$$x_{||} = (x^a_{||}) = (..., x^a_{||}, ...) \in \mathcal{U}^p \ ,$$

and where $\gamma_1, \ldots, \gamma_N$ denote the non-zero $\mathbb{Z}_2^n$-degrees in standard order. Here the $\mathring{x}^a_\Lambda$ (resp., the ξ^A_Λ) are formal power series containing at least 2 (resp., at least 1) of the generators (θ^C) of the $\mathbb{Z}_2^n$-Grassmann algebra Λ.

As mentioned above, any $\mathbb{Z}_2^n$-morphism, in particular any morphism $\phi : \mathcal{U}^{p|\underline{q}} \to \mathcal{V}^{r|\underline{s}}$ between $\mathbb{Z}_2^n$-domains, naturally induces a natural transformation, with Λ-component

$$\phi_\Lambda : \mathcal{U}^{p|\underline{q}}(\Lambda) \ni \mathrm{x}^* \mapsto \mathrm{x}^* \circ \phi^* \in \mathcal{V}^{r|\underline{s}}(\Lambda) \, .$$

If (y^b, η^B) are the coordinates of $\mathcal{V}^{r|\underline{s}}$, the morphism ϕ reads

$$\phi^*(y^b) = \sum_{|\alpha|\geq 0} \phi^b_\alpha(x)\, \xi^\alpha \ , \tag{2.3.9a}$$

$$\phi^*(\eta^B) = \sum_{|\alpha|> 0} \phi^B_\alpha(x)\, \xi^\alpha \ , \tag{2.3.9b}$$

where the right-hand sides have the appropriate degrees and where $\phi_0(\mathcal{U}^p) \subset \mathcal{V}^r$. Further, the image Λ-point $\mathrm{x}^* \circ \phi^*$ in $\mathcal{V}^{r|\underline{s}}$ by ϕ_Λ of the Λ-point $\mathrm{x}^* \simeq \big(\mathrm{x}^*(x^a); \mathrm{x}^*(\xi^A)\big) = \big(x^a_{||}, \mathring{x}^a_\Lambda; \xi^A_\Lambda\big)$ in $\mathcal{U}^{p|\underline{q}}$, is given by

$$y^b_\Lambda = \sum_{|\alpha|\geq 0} \sum_{|\beta|\geq 0} \frac{1}{\beta!} \, (\partial^\beta_x \phi^b_\alpha)(x_{||}) \, \mathring{x}^\beta_\Lambda \, \xi^\alpha_\Lambda \ , \tag{2.3.10a}$$

$$\eta^B_\Lambda = \sum_{|\alpha|> 0} \sum_{|\beta|\geq 0} \frac{1}{\beta!} \, (\partial^\beta_x \phi^B_\alpha)(x_{||}) \, \mathring{x}^\beta_\Lambda \, \xi^\alpha_\Lambda \ . \tag{2.3.10b}$$

Let us recall that there is no convergence issue with terms in $x_{||}$ [15]. Thus the components of a natural transformation implemented by a $\mathbb{Z}_2^n$-morphism between $\mathbb{Z}_2^n$-domains, are very particular formal power series in the formal variables $\mathring{x}^a_\Lambda$ and ξ^A_Λ, which are themselves formal power series in the generators (θ^C) of Λ.

We are now able to prove that not all natural transformations between the restricted functors $M(-), N(-) \in [\mathbb{Z}_2^n\texttt{Pts}, \texttt{Set}]$ associated with $M, N \in \mathbb{Z}_2^n\texttt{Man}$, arise from a $\mathbb{Z}_2^n$-morphism $M \to N$. Since it suffices to give one counter-example, we choose $M = N = \mathbb{R}^{p|\underline{0}} = \mathbb{R}^p$.

Example 2.3.15. Consider an arbitrary diffeomorphism $\phi : \mathbb{R}^p \longrightarrow \mathbb{R}^p$. The Λ-component of the associated natural transformation is

$$\begin{aligned} \phi_\Lambda : \mathbb{R}^{p|\underline{0}}(\Lambda) &\longrightarrow \mathbb{R}^{p|\underline{0}}(\Lambda) \\ (x^b_\Lambda, 0) &\mapsto \left(\phi^b(x_{||}) + \sum_{|\beta|>0} \frac{1}{\beta!} \, (\partial^\beta_x \phi^b)(x_{||}) \, \mathring{x}^\beta_\Lambda, \ 0 \right) . \end{aligned}$$

From this data we obtain another natural transformation

$$\begin{aligned} \alpha_\Lambda : \mathbb{R}^{p|\underline{0}}(\Lambda) &\longrightarrow \mathbb{R}^{p|\underline{0}}(\Lambda) \\ (x^b_\Lambda, 0) &\mapsto \big(\phi^b(x_{||}), 0\big) \ . \end{aligned}$$

The natural transformation α is not implemented by a morphism $\psi : \mathbb{R}^p \to \mathbb{R}^p$. Indeed, otherwise $\alpha_\Lambda = \psi_\Lambda$, for all Λ. This means that

$$\big(\phi^b(x_{||}), 0\big) = \left(\psi^b(x_{||}) + \sum_{|\beta|>0} \frac{1}{\beta!} \, (\partial^\beta_x \psi^b)(x_{||}) \, \mathring{x}^\beta_\Lambda, \ 0 \right) ,$$

for all Λ and all Λ-points. Since $\phi^b(x) \equiv \psi^b(x)$, we have $\partial_x^\beta \phi^b \equiv \partial_x^\beta \psi^b$. Take now any $\beta : |\beta| = 1$, so that $\beta_a = 1$, for some fixed $a \in \{1, \ldots, p\}$. As we can choose Λ and x_Λ^b, for all $b \in \{1, \ldots, p\}$, arbitrarily, we can choose $\mathring{x}_\Lambda^b = 0$, for all $b \neq a$, and $\mathring{x}_\Lambda^a = \theta^D \theta^E$, where θ^D and θ^E are two different generators of Λ that have the same degree. The coefficient of $\theta^D\theta^E$ in the sum over all β is then $(\partial_{x^a}\psi^b)(x_{||})$, hence $\partial_{x^a}\phi^b \equiv \partial_{x^a}\psi^b \equiv 0$. The latter observation is a contradiction, since the Jacobian determinant of ϕ does not vanish anywhere in $\mathbb{R}^p$.

We now generalise a technical result [55, Theorem 1] to $\mathbb{Z}_2^n$-domains $\mathcal{U}^{p|\underline{q}}$. Let

$$\mathcal{B}_{p|\underline{q}}(\mathcal{U}^p) := \mathcal{F}(\mathcal{U}^p, \mathbb{R})[[X, \Xi]],$$

be the $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebra of formal power series in p parameters X^a of $\mathbb{Z}_2^n$-degree 0 and $q_1, \ldots, q_N$ parameters Ξ^A of non-zero $\mathbb{Z}_2^n$-degree $\gamma_1, \ldots, \gamma_N$, and with coefficients in arbitrary $\mathbb{R}$-valued functions on $\mathcal{U}^p$, i.e., we do *not* ask that these functions be continuous let alone smooth. Following [44, 45, 55], we will refer to this algebra as a $\mathbb{Z}_2^n$-*Berezin algebra*. Any element of this algebra is of the form

$$F = \sum_{|\alpha|\geq 0} \sum_{|\beta|\geq 0} F_{\alpha\beta}(x) X^\beta \Xi^\alpha, \tag{2.3.11}$$

where the x^a are coordinates in $\mathcal{U}^p$.

Theorem 2.3.16. *For any $\mathbb{Z}_2^n$-domains $\mathcal{U}^{p|\underline{q}}$ and $\mathcal{V}^{r|\underline{s}}$, there is a 1:1 correspondence*

$$\mathrm{Nat}(\mathcal{U}^{p,\underline{q}}, \mathcal{V}^{r,\underline{s}}) \to (\mathcal{B}_{p|\underline{q}}(\mathcal{U}^p))^{r|\underline{s}}$$

between

- the set of natural transformations in $[\mathbb{Z}_2^n\texttt{Pts}^{\mathrm{op}}, \texttt{Set}]$ between $\mathcal{U}^{p|\underline{q}}(-)$ and $\mathcal{V}^{r|\underline{s}}(-)$, and

- the set of 'vectors' $\mathbf{F}$ with r (resp., with $s_1, \ldots, s_N$) components F^b of degree 0 (resp., components F^B of degrees $\gamma_1, \ldots, \gamma_N$) of the type (2.3.11), such that the r-tuple (F_{00}^b) made of the coefficients $F_{00}^b(x)$ of the r series F^b satisfies

$$(F_{00}^b)(\mathcal{U}^p) \subset \mathcal{V}^r \ .$$

Proof. Let $\mathbf{F}$ be such a 'vector'. For any Λ, we define the map

$$\beta_\Lambda : \mathcal{U}^{p|\underline{q}}(\Lambda) \ni (x_{||}^a, \mathring{x}_\Lambda^a, \xi_\Lambda^A) \mapsto (y_\Lambda^b, \eta_\Lambda^B) \in \mathcal{V}^{r|\underline{s}}(\Lambda) \ ,$$

where

$$y_\Lambda^b := \sum_{|\alpha|\geq 0} \sum_{|\beta|\geq 0} F_{\alpha\beta}^b(x_{||})\, \mathring{x}_\Lambda^\beta\, \xi_\Lambda^\alpha \quad \text{and} \quad \eta_\Lambda^B := \sum_{|\alpha|\geq 0} \sum_{|\beta|\geq 0} F_{\alpha\beta}^B(x_{||})\, \mathring{x}_\Lambda^\beta\, \xi_\Lambda^\alpha \ . \tag{2.3.12}$$

Since $\mathring{x}_\Lambda^a, \xi_\Lambda^A$ have the same degrees as X^a, Ξ^A, the right-hand sides of (2.3.12) have the same degrees as F^b, F^B, hence, $y_\Lambda^b, \eta_\Lambda^B$ have the degrees required to be a Λ-point in $\mathcal{V}^{r|\underline{s}}$. Moreover, we have

$$y_{||}^b = F_{00}^b(x_{||}) \ ,$$

so that $y_{||} \in \mathcal{V}^r$. The target of the map β_Λ is thus actually $\mathcal{V}^{r|\underline{s}}(\Lambda)$. The naturality of β under morphisms of $\mathbb{Z}_2^n$-Grassmann algebras is obvious: β is a natural transformation in $[\mathbb{Z}_2^n\texttt{Pts}^{\mathrm{op}}, \texttt{Set}]$ between $\mathcal{U}^{p|\underline{q}}(-)$ and $\mathcal{V}^{r|\underline{s}}(-)$. Finally, we defined a map

$$\mathcal{I} : (\mathcal{B}_{p|\underline{q}}(\mathcal{U}^p))^{r|\underline{s}} \to \mathrm{Nat}(\mathcal{U}^{p,\underline{q}}, \mathcal{V}^{r,\underline{s}}) \ .$$

We will explain now that any natural transformation $\beta : \mathcal{U}^{p|\underline{q}}(-) \longrightarrow \mathcal{V}^{r|\underline{s}}(-)$ is the image by $\mathcal{I}$ of a unique 'vector' $\mathbf{F}$. We first show that, for any $\Lambda \simeq \mathbb{R}^{0|\underline{m}}$, the image $\beta_\Lambda(\mathrm{x}^*) \in \mathcal{V}^{r|\underline{s}}(\Lambda)$ of any Λ-point

$$\mathrm{x}^* \simeq (x^a_{||}, \mathring{x}^a_\Lambda, \xi^A_\Lambda) \in \mathcal{U}^p \times \mathring{\Lambda}^p_0 \times \mathring{\Lambda}^{q_1}_{\gamma_1} \times \cdots \times \mathring{\Lambda}^{q_N}_{\gamma_N}$$

in $\mathcal{U}^{p|\underline{q}}$, has components y^b_Λ and η^B_Λ of the type (2.3.12).

Step 1. We prove that any Λ-point in $\mathcal{U}^{p|\underline{q}}$ is the image by a $\mathbb{Z}_2^n$-Grassmann algebra map $\varphi^* : \Lambda' \to \Lambda$ of a Λ'-point in $\mathcal{U}^{p|\underline{q}}$, some of whose defining series are series in formal pairings.

Let (θ^C) be the generators of Λ. The Λ-point x^* then reads

$$\mathrm{x}^* \simeq (x^a_{||}, \textstyle\sum_{\lambda\kappa} \theta^\lambda \theta^\kappa K^a_{\kappa\lambda}, \xi^A_\Lambda) \;,$$

where the degree of $K^a_{\kappa\lambda} \in \Lambda$ is the sum of the degrees of θ^λ and θ^κ. Recall that a (resp., A) runs through $\{1, \ldots, p\}$ (resp., through $\{1, \ldots, |\underline{q}|\}$), and that λ, κ run through $\{1, \ldots, |\underline{m}|\}$. Consider now the set S of generators

$$\theta' = \big(\eta^{a\lambda},\, \zeta^b_\kappa,\, \psi^A\big) \;,$$

where b has the same range as a, and define their (non-zero) $\mathbb{Z}_2^n$-degrees by

$$\deg(\eta^{a\lambda}) = \deg(\theta^\lambda), \qquad \deg(\zeta^b_\kappa) = \deg(\theta^\kappa), \qquad \deg(\psi^A) = \deg(\xi^A_\Lambda) = \deg(\xi^A) \;.$$

Let Λ' be the $\mathbb{Z}_2^n$-Grassmann algebra defined by S, and set

$$\mathrm{x}'^* \simeq \big(x^a_{||}, \textstyle\sum_\lambda \eta^{a\lambda}\zeta^a_\lambda,\, \psi^A\big) \in \mathcal{U}^p \times \mathring{\Lambda}'^p_0 \times \mathring{\Lambda}'^{q_1}_{\gamma_1} \times \cdots \times \mathring{\Lambda}'^{q_N}_{\gamma_N} \tag{2.3.13}$$

(no sum over a in the formal pairings $\sum_\lambda \eta^{a\lambda}\zeta^a_\lambda$). The degree-respecting equalities

$$\varphi^*(\eta^{a\lambda}) = \theta^\lambda, \qquad \varphi^*(\zeta^b_\kappa) = \sum_\lambda \theta^\lambda K^b_{\lambda\kappa}, \qquad \varphi^*(\psi^A) = \xi^A_\Lambda$$

define a morphism of $\mathbb{Z}_2^n$-Grassmann algebras $\varphi^* : \Lambda' \longrightarrow \Lambda$. It suffices to set

$$\varphi^*\big(\sum_\varepsilon r_\epsilon \theta'^\varepsilon\big) := \sum_\varepsilon r_\varepsilon (\varphi^*\theta')^\varepsilon \;.$$

Indeed, any term of the right-hand side is a series in θ whose terms contain at least $|\varepsilon|$ generators. Hence, for any ϵ, only the terms $|\varepsilon| \le |\epsilon|$ can contribute to θ^ϵ, and therefore there is no convergence issue with the coefficient of θ^ϵ. Since the Λ-point $\varphi^* \circ \mathrm{x}'^*$ in $\mathcal{U}^{p|\underline{q}}$ reads

$$\varphi^* \circ \mathrm{x}'^* \simeq \varphi^*\big(x^a_{||}, \textstyle\sum_\lambda \eta^{a\lambda}\zeta^a_\lambda,\, \psi^A\big) = \big(x^a_{||}, \textstyle\sum_{\lambda\kappa} \theta^\lambda\theta^\kappa K^a_{\kappa\lambda}, \xi^A_\Lambda\big) \simeq \mathrm{x}^* \;,$$

naturality of the transformation $\beta : \mathcal{U}^{p|\underline{q}}(-) \longrightarrow \mathcal{V}^{r|\underline{s}}(-)$ implies that

$$(y^b_\Lambda, \eta^B_\Lambda) :\simeq \beta_\Lambda(\mathrm{x}^*) = \beta_\Lambda(\varphi^* \circ \mathrm{x}'^*) = \beta_\Lambda\big(\mathcal{U}^{p|\underline{q}}(\varphi^*)\,(\mathrm{x}'^*)\big) =$$

$$\mathcal{V}^{r|\underline{s}}(\varphi^*)(\beta_{\Lambda'}(\mathrm{x}'^*)) = \varphi^* \circ (\beta_{\Lambda'}(\mathrm{x}'^*)) \simeq \varphi^*(y^b_{\Lambda'}, \eta^B_{\Lambda'}) \;, \tag{2.3.14}$$

where $y^b_{\Lambda'}$ and $\eta^B_{\Lambda'}$ are series in the generators of Λ'.

Step 2. We define formal rotations under which the formal pairings are invariant. Moreover, we show that any formal series that is invariant under the formal rotations is a series in the formal pairings.

The formal part of each degree 0 component of x'^* can be viewed as a formal pairing $\eta^a\cdot\zeta^a=\sum_\lambda \eta^{a\lambda}\zeta^a_\lambda$, which is stable under formal rotations R^*. More precisely, we set

$$R^*(\eta^{a\lambda})=\sum_\kappa \eta^{a\kappa}(O^a)_\kappa^{\ \lambda},\ R^*(\zeta^b_\kappa)=\sum_\lambda (O^{b\,\ddagger})_\kappa^{\ \lambda}\zeta^b_\lambda,\ R^*(\psi^A)=\psi^A\ ,$$

where O^a and $O^{b\,\ddagger}$ are any $(m_1+\ldots+m_N)\times(m_1+\ldots+m_N)$ block-diagonal matrices with entries in $\mathbb{R}$ that satisfy

$$\sum_\lambda (O^a)_\rho^{\ \lambda}(O^{a\,\ddagger})_\lambda^{\ \omega}=\delta_\rho^{\ \omega}\ . \tag{2.3.15}$$

Since, for any fixed a (resp., b), the components $\eta^{a\lambda}$ (resp., ζ^b_κ) are ordered such that the m_1 first components have degree γ_1, the next m_2 degree γ_2, and so on, these equalities are degree-preserving. Hence, they define a $\mathbb{Z}_2^n$-Grassmann algebra morphism $R^*:\Lambda'\to\Lambda'$ via

$$R^*\big(\sum_\varepsilon r_\varepsilon\theta'^\varepsilon\big)=R^*\big(\sum_{\alpha\beta\gamma} r_{\alpha\beta\gamma}\,\eta^\alpha\zeta^\beta\psi^\gamma\big):=\sum_{\alpha\beta\gamma} r_{\alpha\beta\gamma}\,(R^*\eta)^\alpha(R^*\zeta)^\beta\psi^\gamma\ .$$

Since the images $R^*(\eta^{a\lambda})$ (resp., $R^*(\zeta^b_\kappa)$) are linear in the $\eta^{a\kappa}$ (resp., ζ^b_λ) (of the same degree), the term indexed by $\alpha\beta\gamma$ is a homogeneous polynomial of order $|\alpha|+|\beta|+|\gamma|$ in the generators θ'. Hence, for any ϵ, only the terms $|\alpha|+|\beta|+|\gamma|=|\epsilon|$ can contribute to θ'^ϵ, so that no convergence problems arise. In view of (2.3.15), it is clear that, as mentioned above, the formal pairing $\eta^a\cdot\zeta^a=\sum_\lambda\eta^{a\lambda}\zeta^a_\lambda$ is invariant under R^*. As any $\mathbb{Z}_2^n$-Grassmann algebra morphism, the formal rotation R^* induces maps $\mathcal{U}^{p|\underline{q}}(R^*)$ and $\mathcal{V}^{r|\underline{s}}(R^*)$, and due to naturality of β, we find

$$\mathcal{V}^{r|\underline{s}}(R^*)(\beta_{\Lambda'}\mathrm{x}'^*)=\beta_{\Lambda'}(\mathcal{U}^{p|\underline{q}}(R^*)(\mathrm{x}'^*))=\beta_{\Lambda'}(R^*\circ\mathrm{x}'^*)\simeq\beta_{\Lambda'}\big(R^*\big(x^a_{||},\sum_\lambda\eta^{a\lambda}\zeta^a_\lambda,\ \psi^A\big)\big)\simeq\beta_{\Lambda'}\mathrm{x}'^*\ ,$$

so that $\beta_{\Lambda'}\mathrm{x}'^*$ is invariant under rotations.

We are now prepared to continue the computation (2.3.14). Since

$$\beta_{\Lambda'}(\mathrm{x}'^*)\simeq(y^b_{\Lambda'},\eta^B_{\Lambda'})=(y^b_{||},\mathring{y}^b_{\Lambda'},\eta^B_{\Lambda'}) \tag{2.3.16}$$

is invariant under the rotations R^*, the series $\mathring{y}^b_{\Lambda'},\eta^B_{\Lambda'}$ in the generators θ' are invariant. More explicitly, for each series, we have an equality of the type

$$\sum_\gamma\big(\sum_{k,\ell}\sum_{|\alpha|=k,\,|\beta|=\ell}F_{\alpha\beta\gamma}\,\eta^\alpha\zeta^\beta\big)\psi^\gamma=\sum_\gamma\big(\sum_{k,\ell}\sum_{|\alpha|=k,\,|\beta|=\ell}F_{\alpha\beta\gamma}\,(R^*\eta)^\alpha(R^*\zeta)^\beta\big)\psi^\gamma\ ,$$

which is equivalent to

$$\sum_{|\alpha|=k,\,|\beta|=\ell}F_{\alpha\beta\gamma}\ \ldots\eta^{a\lambda}\eta^{b\mu}\zeta^c_\nu\ldots=\sum_{|\alpha|=k,\,|\beta|=\ell}F_{\alpha\beta\gamma}\,\eta^\alpha\zeta^\beta=$$

$$\sum_{|\alpha|=k,\,|\beta|=\ell}F_{\alpha\beta\gamma}\,(R^*\eta)^\alpha(R^*\zeta)^\beta=\sum_{|\alpha|=k,\,|\beta|=\ell}F_{\alpha\beta\gamma}\ \ldots\ \eta^{a\delta}(O^a)_\delta^{\ \lambda}\ \eta^{b\delta'}(O^b)_{\delta'}^{\ \mu}\ (O^{c\,\ddagger})_\nu^{\ \delta''}\zeta^c_{\delta''}\ \ldots\ ,$$

and holds for all (!) formal rotations. This is only possible, if the power series considered, i.e., the series $\mathring{y}^b_{\Lambda'}$ and $\eta^B_{\Lambda'}$, are series in pairings $\eta^a\cdot\zeta^a=\sum_\lambda\eta^{a\lambda}\zeta^a_\lambda$. In the classical setting, the result is known under the name of First Fundamental Theorem of Invariant Theory for the orthogonal group [24, 58]. It has been extended to the graded situation in [7, Proposition 4.13]. In view of (2.3.14), we thus get

$$(y^b_{||},\mathring{y}^b_\Lambda,\eta^B_\Lambda)=\beta_\Lambda(\mathrm{x}^*)=\beta_\Lambda(x^a_{||},\mathring{x}^a_\Lambda,\xi^A_\Lambda)=(y^b_{||},\varphi^*(\mathring{y}^b_{\Lambda'}),\varphi^*(\eta^B_{\Lambda'}))\ ,$$

where any image by φ^* is of the type

$$\sum_{(\alpha,\beta)\neq(0,0)} F_{\alpha\beta}\,\varphi^*\big((\eta\cdot\zeta)^\beta\big)\,\varphi^*(\psi^\alpha) = \sum_{(\alpha,\beta)\neq(0,0)} F_{\alpha\beta}\,\mathring{x}_\Lambda^\beta\,\xi_\Lambda^\alpha\ .$$

It is clear from (2.3.16) and (2.3.13) that the coefficients

$$F^b_{\alpha\beta},\ F^B_{\alpha\beta}\ \ ((\alpha,\beta)\neq(0,0))\ ,\ \text{and}\ F^b_{00} := y^b_{||}$$

depend (only) on $x_{||}\in\mathcal{U}^p$. Hence, the image

$$(y^b_\Lambda,\eta^B_\Lambda) = \beta_\Lambda(\mathrm{x}^*) = (F^b(x_{||},\mathring{x}_\Lambda,\xi_\Lambda),F^B(x_{||},\mathring{x}_\Lambda,\xi_\Lambda))$$

is actually of the type (2.3.12). Since $\beta_\Lambda(\mathrm{x}^*)$ is a Λ-point in $\mathcal{V}^{r|\underline{s}}$, the r series $F^b(x_{||},\mathring{x}_\Lambda,\xi_\Lambda)$ and the s_i series $F^B(x_{||},\mathring{x}_\Lambda,\xi_\Lambda)$ are of degree 0 and degree γ_i, respectively, i.e., the r series $F^b(x,X,\Xi)$ and the s_i series $F^B(x,X,\Xi)$ are of degree 0 and degree γ_i, respectively. For the same reason, we have $F_{00}(x_{||})\in\mathcal{V}^r$, for all $x_{||}\in\mathcal{U}^p$, so that we constructed a 'vector' $\mathbf{F}\in(\mathcal{B}_{p|\underline{q}}(\mathcal{U}^p))^{r|\underline{s}}$, whose image by $\mathcal{I}$ is obviously β.

Step 3. We show that $\mathbf{F}$ *is unique (which concludes the proof).* If there is another 'vector' $\mathbf{F}'$, such that $\mathcal{I}(\mathbf{F}')=\beta$, we have

$$\sum_{|\alpha|\geq 0,|\beta|\geq 0} F^{\mathfrak{b}}_{\alpha\beta}(x_{||})\,\mathring{x}_\Lambda^\beta\,\xi_\Lambda^\alpha = \sum_{|\alpha|\geq 0,|\beta|\geq 0} F'^{\mathfrak{b}}_{\alpha\beta}(x_{||})\,\mathring{x}_\Lambda^\beta\,\xi_\Lambda^\alpha\ , \tag{2.3.17}$$

for all $\mathfrak{b}\in\{b,B\}$, all Λ, and all x^*. Notice first that any $\mathring{x}^a_\Lambda$ (resp., any ξ^A_Λ) is a series of degree 0 (resp., of degree $\deg(\xi^A)=\gamma_A$) in the θ-s that contains at least two parameters $\theta^C\theta^{C'}$ (resp., at least one parameter $\theta^{C''}$). Hence, both sides are series in θ, and the left-hand side and right-hand side coefficients of any monomial θ^ε coincide. A term $(\alpha,\beta)\neq(0,0)$ cannot contribute to the independent term θ^0. Hence $F^{\mathfrak{b}}_{00}(x_{||})=F'^{\mathfrak{b}}_{00}(x_{||})$. We now show that $F^{\mathfrak{b}}_{\alpha\beta}(x_{||})=F'^{\mathfrak{b}}_{\alpha\beta}(x_{||})$, for an arbitrarily fixed $(\alpha,\beta)\neq(0,0)$. Since Λ is arbitrary, we can choose as many different generators θ in each non-zero degree as necessary, and, since x^* is arbitrary, we can choose $x_{||}$ arbitrarily in $\mathcal{U}^p$ and we can choose the coefficients of the series $\mathring{x}^a_\Lambda$ and ξ^A_Λ arbitrarily (except that we have to observe that the coefficient of a monomial θ^ε, which does not have the required degree, *must be zero*). Let now $\alpha_1,\ldots,\alpha_\mu$ and $\beta_1,\ldots,\beta_\nu$ be the non-zero components in the fixed α and β. For each factor $\xi^{A_i}_\Lambda$ of

$$\xi^\alpha_\Lambda = (\xi^{A_1}_\Lambda)^{\alpha_1}\ldots(\xi^{A_\mu}_\Lambda)^{\alpha_\mu}\ ,$$

we choose a monomial in one generator θ^{C_i} of degree γ_{A_i}, set its coefficient r_{C_i} to 1, and all the other coefficients in the series $\xi^{A_i}_\Lambda$ to zero. Further, for different $\xi^{A_i}_\Lambda$, we choose different generators θ^{C_i}. Similarly, for each factor $\mathring{x}^{a_j}_\Lambda$ of

$$\mathring{x}^\beta_\Lambda = (\mathring{x}^{a_1}_\Lambda)^{\beta_1}\ldots(\mathring{x}^{a_\nu}_\Lambda)^{\beta_\nu}\ ,$$

we choose monomials $\theta^{D_{jk}}\theta^{E_{jk}}$ $(k\in\{1,\ldots,\beta_j\})$ in two generators of the same odd degree (for all $\mathbb{Z}_2^n$-manifolds with $n\geq 1$, there is at least one odd degree), set their coefficient $r_{D_{jk}E_{jk}}$ to 1, and all the other coefficients in the series $\mathring{x}^{a_j}_\Lambda$ to zero. Further, we choose the generators so that all generators θ^{C_i}, $\theta^{D_{jk}}$, and $\theta^{E_{jk}}$ are different. When setting

$$\theta^\omega = \prod_{j=1}^{\nu}\theta^{D_{j1}}\theta^{E_{j1}}\ldots\theta^{D_{j\beta_j}}\theta^{E_{j\beta_j}}\prod_{i=1}^{\mu}(\theta^{C_i})^{\alpha_i}\neq 0\ ,$$

the terms indexed by (the fixed) (α,β) in both sides of (2.3.17), read

$$\beta!F^{\mathfrak{b}}_{\alpha\beta}(x_{||})\theta^\omega\ \text{and}\ \beta!F'^{\mathfrak{b}}_{\alpha\beta}(x_{||})\theta^\omega\ .$$

For any term $(\alpha', \beta') \neq (\alpha, \beta)$, we either get a new series ξ^A_Λ or $\mathring{x}^a_\Lambda$ (i.e., a series that is not present in ξ^α_Λ or $\mathring{x}^\beta_\Lambda$), or we get an old series a different number of times. In the second case, the term (α', β') does not contribute to the coefficient of θ^ω; in the first, we set all the coefficients of the new series to 0, so that the term (α', β') vanishes. Finally, we obtain $F^{\mathfrak{b}}_{\alpha\beta}(x_{||}) = F'^{\mathfrak{b}}_{\alpha\beta}(x_{||})$, for any $x_{||} \in \mathcal{U}^p$. □

We now show that $\mathbb{R}^{p|\underline{q}}(\Lambda)$ is a Fréchet space and that $\mathcal{U}^{p|\underline{q}}(\Lambda)$ is an open subset of $\mathbb{R}^{p|\underline{q}}(\Lambda)$. This means that we have a notion of directional derivative, as well as a notion of smoothness of continuous maps between the Λ-points of $\mathbb{Z}_2^n$-domains. For more details on Fréchet objects, we refer the reader to Appendix 2.4.2.

Proposition 2.3.17. *For any $\Lambda \in \mathbb{Z}_2^n\texttt{GrAlg}$, the set $\mathbb{R}^{p|\underline{q}}(\Lambda)$ is a nuclear Fréchet space and a Fréchet Λ_0-module. Moreover, the set $\mathcal{U}^{p|\underline{q}}(\Lambda)$ is an open subset of $\mathbb{R}^{p|\underline{q}}(\Lambda)$.*

Proof. Let $\Lambda \in \mathbb{Z}_2^n\texttt{GrAlg}$. As explained above, there is a 1:1 correspondence between the Λ-points x^* of $\mathbb{R}^{p|\underline{q}}$ (resp., of $\mathcal{U}^{p|\underline{q}}$) and the $(p + |\underline{q}|)$-tuples

$$\mathrm{x}^* \simeq \left(x^a_\Lambda,\, \xi^A_\Lambda\right) \in \Lambda_0^p \times \Lambda_{\gamma_1}^{q_1} \times \cdots \times \Lambda_{\gamma_N}^{q_N}$$

(resp., the same $(p+|\underline{q}|)$-tuples, but with the additional requirement that the p-tuple $(x^a_{||})$ made of the independent terms of (x^a_Λ) be a point in $\mathcal{U}^p \subset \mathbb{R}^p$). Note now that Λ is the $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative nuclear Fréchet $\mathbb{R}$-algebra of global $\mathbb{Z}_2^n$-functions of some $\mathbb{R}^{0|\underline{m}}$. Hence, all its homogeneous subspaces Λ_{γ_i} $(i \in 0, \ldots, N,\ \gamma_0 = 0)$ are nuclear Fréchet vector spaces. Since any product (resp., any countable product) of nuclear (resp., Fréchet) spaces is nuclear (resp., Fréchet), the set $\mathbb{R}^{p|\underline{q}}(\Lambda)$ of Λ-points of $\mathbb{R}^{p|\underline{q}}$ is nuclear Fréchet. The latter statements can be found in [12].

As for the second claim in Proposition 2.3.17, recall that Λ_0 is a (commutative) Fréchet algebra, see Corollary 2.3.6. The Fréchet Λ_0-module structure on $\mathbb{R}^{p|\underline{q}}(\Lambda)$ is then defined by

$$m : \Lambda_0 \times \mathbb{R}^{p|\underline{q}}(\Lambda) \ni (\mathrm{a}, \mathrm{x}^*) \mapsto (\mathrm{a} \cdot x^a_\Lambda,\ \mathrm{a} \cdot \xi^A_\Lambda) \in \mathbb{R}^{p|\underline{q}}(\Lambda)\ . \tag{2.3.18}$$

Since this action is defined using the continuous associative multiplication $\cdot : \Lambda_{\gamma_i} \times \Lambda_{\gamma_j} \to \Lambda_{\gamma_i+\gamma_j}$ of the Fréchet algebra Λ, it is (jointly) continuous.

As any closed subspace of a Fréchet space is itself a Fréchet space, the space

$$\mathring{\Lambda}_0 \simeq \{0\} \times \mathring{\Lambda}_0 \subset \mathbb{R} \times \mathring{\Lambda}_0 = \Lambda_0$$

is Fréchet. We thus see that

$$\mathcal{U}^{p|\underline{q}}(\Lambda) \simeq \mathcal{U}^p \times \mathring{\Lambda}_0^p \times \prod_{i=1}^N \Lambda_{\gamma_i}^{q_i} \subset \mathbb{R}^p \times \mathring{\Lambda}_0^p \times \prod_{i=1}^N \Lambda_{\gamma_i}^{q_i} \simeq \mathbb{R}^{p|\underline{q}}(\Lambda) \tag{2.3.19}$$

is open. □

Remark 2.3.18. In the following, we will use the isomorphisms (2.3.19) (and similar ones) without further reference.

The just described Λ_0-module structure is vital in understanding the structure of the Λ-points of any $\mathbb{Z}_2^n$-manifold. In particular, morphisms between $\mathbb{Z}_2^n$-domains induce natural transformations between the associated functors that respect this module structure. The converse is also true, that is, any natural transformation between the associated functors that respects the Λ_0-module structure comes from a morphism between the underlying $\mathbb{Z}_2^n$-domains. More carefully, we have the following proposition.

Theorem 2.3.19. *Let $\mathcal{U}^{p|\underline{q}}$ and $\mathcal{V}^{r|\underline{s}}$ be $\mathbb{Z}_2^n$-domains. A natural transformation $\beta : \mathcal{U}^{p|\underline{q}}(-) \longrightarrow \mathcal{V}^{r|\underline{s}}(-)$ comes from a $\mathbb{Z}_2^n$-manifold morphism $\mathcal{U}^{p|\underline{q}} \to \mathcal{V}^{r|\underline{s}}$ if and only if $\beta_\Lambda : \mathcal{U}^{p|\underline{q}}(\Lambda) \longrightarrow \mathcal{V}^{r|\underline{s}}(\Lambda)$ is Λ_0-smooth, for all $\Lambda \in \mathbb{Z}_2^n\texttt{GrAlg}$. That is, for all Λ, the map β_Λ must be a smooth map (from the open subset $\mathcal{U}^{p|\underline{q}}(\Lambda)$ of the Fréchet space $\mathbb{R}^{p|\underline{q}}(\Lambda)$ to the Fréchet space $\mathbb{R}^{r|\underline{s}}(\Lambda)$, see Appendix 2.4.2) and its Gâteaux derivative (see Appendix 2.4.2) must be Λ_0-linear, i.e.,*

$$\mathrm{d}_{\mathrm{x}^*}\beta_\Lambda(\mathrm{a}\cdot\mathrm{v}) = \mathrm{a}\cdot\mathrm{d}_{\mathrm{x}^*}\beta_\Lambda(\mathrm{v})\ ,$$

for all $\mathrm{x}^* \in \mathcal{U}^{p|\underline{q}}(\Lambda)$, $\mathrm{a} \in \Lambda_0$, and $\mathrm{v} \in \mathbb{R}^{p|\underline{q}}(\Lambda)$.

Proof. Part I. Let $\beta : \mathcal{U}^{p|\underline{q}}(-) \longrightarrow \mathcal{V}^{r|\underline{s}}(-)$ be a natural transformation with Λ_0-smooth components β_Λ, $\Lambda \in \mathbb{Z}_2^n\texttt{GrAlg}$. From Theorem 2.3.16, we know that β_Λ is completely specified by the systems

$$y^b_\Lambda = \sum_{|\alpha|\geq 0, |\beta|\geq 0} F^b_{\alpha\beta}(x_{||})\,\mathring{x}^\beta_\Lambda \xi^\alpha_\Lambda \quad \text{and} \quad \eta^B_\Lambda = \sum_{|\alpha|>0, |\beta|\geq 0} F^B_{\alpha\beta}(x_{||})\,\mathring{x}^\beta_\Lambda \xi^\alpha_\Lambda\ , \tag{2.3.20}$$

where the coefficients $F^{\mathfrak{b}}_{\alpha\beta}$ ($\mathfrak{b} \in \{b, B\}$) are set-theoretical maps from $\mathcal{U}^p$ to $\mathbb{R}$.

Part Ia. Smoothness of β_Λ implies that these coefficients are smooth. Indeed, we will show that $F^{\mathfrak{b}}_{\alpha\beta} \in C^0(\mathcal{U}^p)$ and that, if $F^{\mathfrak{b}}_{\alpha\beta} \in C^k(\mathcal{U}^p)$ ($k \geq 0$), then $F^{\mathfrak{b}}_{\alpha\beta} \in C^{k+1}(\mathcal{U}^p)$.

Step 1. Since

$$\beta_\Lambda : \mathcal{U}^{p|\underline{q}}(\Lambda) \to \Lambda_0^r \times \prod_{i=1}^N \Lambda_{\gamma_i}^{s_i}$$

is continuous, any of its components

$$y^{\mathfrak{b}}_\Lambda : \mathcal{U}^{p|\underline{q}}(\Lambda) \to \Lambda_{\gamma_{i(\mathfrak{b})}} = \mathbb{R}[[\theta]]_{\gamma_{i(\mathfrak{b})}} \simeq \prod_{\gamma_i(\mathfrak{b})} \mathbb{R}$$

is continuous. For simplicity, we wrote y^B_Λ instead of η^B_Λ, and we will continue doing so. Moreover, the target space are the formal power series in θ with coefficients in $\mathbb{R}$, all whose terms have the degree $\gamma_{i(\mathfrak{b})}$ of $y^{\mathfrak{b}}$, and this space is identified with the corresponding space of families of reals. For any ω such that θ^ω has the degree $\gamma_{i(\mathfrak{b})}$, the corresponding real coefficient gives rise to a continuous map

$$y^{\mathfrak{b},\omega}_\Lambda : \mathcal{U}^{p|\underline{q}}(\Lambda) \to \mathbb{R}\ .$$

Since this joint continuity implies separate continuity with respect to $x_{||} \in \mathcal{U}^p$, for any fixed $(\mathring{x}_\Lambda, \xi_\Lambda)$ and any Λ, we can proceed as at the end of the proof of Theorem 2.3.16. More precisely, select any (α, β) and select (for an appropriate Λ) the pair $(\mathring{x}_\Lambda, \xi_\Lambda)$ such that $\mathring{x}^\beta_\Lambda \xi^\alpha_\Lambda = \beta!\,\theta^\omega$, where θ^ω is now the degree $\gamma_{i(\mathfrak{b})}$ monomial defined in the proof just mentioned. The real coefficient of this monomial is $\beta!\,F^{\mathfrak{b}}_{\alpha\beta}(x_{||})$, which, as said, is an $\mathbb{R}$-valued continuous map on $\mathcal{U}^p$, so that $F^{\mathfrak{b}}_{\alpha\beta} \in C^0(\mathcal{U}^p)$, for all $\mathfrak{b}$ and all (α, β).

Step 2. Since

$$\mathcal{U}^{p|\underline{q}}(\Lambda) \subset \mathbb{R} \times \big(\mathbb{R}^{p-1} \times \mathring{\Lambda}_0^p \times \prod_{i=1}^N \Lambda_{\gamma_i}^{q_i}\big)$$

is an open subset of a product of two Fréchet spaces, smoothness of β_Λ implies (via an iterated application of Proposition 2.4.7) that, for any $\mathfrak{b} \in \{b, B\}$, any $\ell \in \mathbb{N}$ and any $\gamma \in \mathbb{N}^p$ ($|\gamma| = \ell$), the partial derivative

$$\mathrm{d}^\gamma_{x_{||}}\, y^{\mathfrak{b}}_\Lambda : \mathcal{U}^{p|\underline{q}}(\Lambda) \times \mathbb{R}^{\times \ell} \to \prod_{\gamma_i(\mathfrak{b})} \mathbb{R}$$

is continuous.

Assume now that $F^{\mathfrak{b}}_{\alpha\beta} \in C^k(\mathcal{U}^p)$ $(k \geq 0)$, for any $\mathfrak{b}$ and any (α, β), as well as that, for any $\gamma \in \mathbb{N}^p$ $(|\gamma| = k)$ and any $\mathfrak{b}$, the continuous partial Gâteaux derivative

$$\mathrm{d}^{\gamma}_{x_{||}}\, y^{\mathfrak{b}}_{\Lambda}(1, \ldots, 1) : \mathcal{U}^{p|\underline{q}}(\Lambda) \to \prod_{\gamma_i(\mathfrak{b})} \mathbb{R}$$

is given by

$$\mathrm{d}^{\gamma}_{x_{||},\mathrm{x}^*}\, y^{\mathfrak{b}}_{\Lambda}(1, \ldots, 1) = \sum_{\alpha\beta} \big(\partial^{\gamma}_x F^{\mathfrak{b}}_{\alpha\beta}\big)(x_{||})\, \mathring{x}^{\beta}_{\Lambda} \xi^{\alpha}_{\Lambda}\ . \tag{2.3.21}$$

Observe that for $k = 0$, this condition is automatically satisfied. We will now show that, under these assumptions, the same statements hold at order $k+1$. In view of (2.3.21), any order $k+1$ continuous partial Gâteaux derivative

$$\mathrm{d}_{x^a_{||}}\, \mathrm{d}^{\gamma}_{x_{||}}\, y^{\mathfrak{b}}_{\Lambda}(1, \ldots, 1) : \mathcal{U}^{p|\underline{q}}(\Lambda) \to \prod_{\gamma_i(\mathfrak{b})} \mathbb{R}$$

$(a \in \{1, \ldots, p\}, |\gamma| = k)$ is given, at any $\mathrm{x}^* \simeq (x_{||}, \mathring{x}_{\Lambda}, \xi_{\Lambda}) \in \mathcal{U}^{p|\underline{q}}(\Lambda)$, by

$$\sum_{\alpha\beta} \lim_{t\to 0} \frac{1}{t} \Big(\big(\partial^{\gamma}_x F^{\mathfrak{b}}_{\alpha\beta}\big)(x^1_{||}, \ldots, x^a_{||} + t, \ldots, x^p_{||}) - \big(\partial^{\gamma}_x F^{\mathfrak{b}}_{\alpha\beta}\big)(x^1_{||}, \ldots, x^a_{||}, \ldots, x^p_{||}) \Big)\, \mathring{x}^{\beta}_{\Lambda} \xi^{\alpha}_{\Lambda}\ . \tag{2.3.22}$$

When proceeding as in Step 1, we get that the limit is an $\mathbb{R}$-valued continuous function in $\mathcal{U}^p$. In other words, the partial derivative $\partial_{x^a}\partial^{\gamma}_x F^{\mathfrak{b}}_{\alpha\beta}$ exists and is continuous in $\mathcal{U}^p$, i.e., $F^{\mathfrak{b}}_{\alpha\beta} \in C^{k+1}(\mathcal{U}^p)$. Moreover, Formula (2.3.21) pertaining to order k derivatives, extends to the order $k+1$ derivatives, see (2.3.22).

Part Ib. We examine the further consequences of Λ_0-smoothness, in particular those of Λ_0-linearity. Since β_{Λ} is of class C^1, its components $y^{\mathfrak{b}}_{\Lambda} : \mathcal{U}^{p|\underline{q}}(\Lambda) \to \prod_{\gamma_i(\mathfrak{b})} \mathbb{R}$ are of class C^1. Further, as

$$\mathcal{U}^{p|\underline{q}}(\Lambda) \subset \big(\mathbb{R} \times \mathring{\Lambda}_0\big) \times \Big(\mathbb{R}^{p-1} \times \mathring{\Lambda}^{p-1}_0 \times \prod_{i=1}^{N} \Lambda^{q_i}_{\gamma_i}\Big)$$

is an open subset of a product of two Fréchet spaces, the partial Gâteaux derivative

$$\mathrm{d}_{(x^a_{||}, \mathring{x}^a_{\Lambda})}\, y^{\mathfrak{b}}_{\Lambda} : \mathcal{U}^{p|\underline{q}}(\Lambda) \times \big(\mathbb{R} \times \mathring{\Lambda}_0\big) \to \prod_{\gamma_i(\mathfrak{b})} \mathbb{R}$$

is continuous. It is given by

$$\mathrm{d}_{(x^a_{||}, \mathring{x}^a_{\Lambda}), \mathrm{x}^*}\, y^{\mathfrak{b}}_{\Lambda}(v_{||}, \mathring{v}_{\Lambda}) = \mathrm{d}_{x^a_{||}, \mathrm{x}^*}\, y^{\mathfrak{b}}_{\Lambda}(v_{||}) + \mathrm{d}_{\mathring{x}^a_{\Lambda}, \mathrm{x}^*}\, y^{\mathfrak{b}}_{\Lambda}(\mathring{v}_{\Lambda}) =$$

$$v_{||} \sum_{\alpha\beta} \big(\partial_{x^a} F^{\mathfrak{b}}_{\alpha\beta}\big)(x_{||}) \mathring{x}^{\beta}_{\Lambda} \xi^{\alpha}_{\Lambda} + \sum_{\alpha\beta} F^{\mathfrak{b}}_{\alpha\beta}(x_{||}) \lim_{t\to 0} \frac{1}{t} \big((\mathring{x}^a_{\Lambda} + t\mathring{v}_{\Lambda})^{\beta_a} - (\mathring{x}^a_{\Lambda})^{\beta_a} \big) \prod_{b\neq a} (\mathring{x}^b_{\Lambda})^{\beta_b} \xi^{\alpha}_{\Lambda} =: v_{||} T_1 + \mathfrak{T}_2\ .$$

As $\mathring{\Lambda}_0$ is a commutative algebra, it follows from the binomial formula that

$$\mathfrak{T}_2 = \mathring{v}_{\Lambda} \sum_{\alpha\beta} \beta_a F^{\mathfrak{b}}_{\alpha\beta}(x_{||}) \mathring{x}^{\beta - e_a}_{\Lambda} \xi^{\alpha}_{\Lambda} =: \mathring{v}_{\Lambda} T_2\ ,$$

where $(e_a)_a$ is the canonical basis of $\mathbb{R}^p$. Observe now that, in view of (3.2.22), the Λ_0-linearity of the total Gâteaux derivative of $y^{\mathfrak{b}}_{\Lambda}$ with respect to x^* is equivalent to the Λ_0-linearity of all its partial Gâteaux derivatives with respect to the $x^a_\Lambda = (x^a_{||}, \mathring{x}^a_\Lambda)$ and the ξ^A_Λ. For $a = 0 + \mathring{v}_\Lambda \in \Lambda_0$ and $\mathrm{v} = 1 + 0 \in \mathbb{R} + \mathring{\Lambda}_0 = \Lambda_0$, this implies that

$$\mathring{v}_\Lambda T_2 = \mathrm{d}_{(x^a_{||}, \mathring{x}^a_\Lambda), \mathrm{x}^*}\, y^{\mathfrak{b}}_\Lambda(\mathring{v}_\Lambda \cdot 1) = \mathring{v}_\Lambda \cdot \mathrm{d}_{(x^a_{||}, \mathring{x}^a_\Lambda), \mathrm{x}^*}\, y^{\mathfrak{b}}_\Lambda(1) = \mathring{v}_\Lambda T_1 \ ,$$

i.e., that

$$\mathring{v}_\Lambda \sum_{\alpha\beta} (\beta_a + 1) F^{\mathfrak{b}}_{\alpha,\beta+e_a}(x_{||}) \mathring{x}^\beta_\Lambda \xi^\alpha_\Lambda = \mathring{v}_\Lambda \sum_{\alpha,\gamma:\gamma_a \neq 0} \gamma_a F^{\mathfrak{b}}_{\alpha\gamma}(x_{||}) \mathring{x}^{\gamma - e_a}_\Lambda \xi^\alpha_\Lambda = \mathring{v}_\Lambda \sum_{\alpha\beta} \big(\partial_{x^a} F^{\mathfrak{b}}_{\alpha\beta}\big)(x_{||}) \mathring{x}^\beta_\Lambda \xi^\alpha_\Lambda \ .$$

Since $\Lambda \in \mathbb{Z}_2^n\mathtt{GrAlg}$, $\mathring{v}_\Lambda \in \mathring{\Lambda}_0$, and $\mathrm{x}^* \in \mathcal{U}^{p|\underline{q}}(\Lambda)$ are arbitrary, we can repeat the θ^ω-argument used above. More precisely, we select (α, β), select $(\mathring{x}_\Lambda, \xi_\Lambda)$ such that $\mathring{x}^\beta_\Lambda \xi^\alpha_\Lambda = \beta!\, \theta^\omega$, and select $\mathring{v}_\Lambda = \theta^D \theta^E \in \mathring{\Lambda}_0$ such that $\theta^D \theta^E \theta^\omega \neq 0$. The coefficients of the latter monomial in the left and right hand sides do coincide, which means that

$$(\beta_a + 1) F^{\mathfrak{b}}_{\alpha,\beta+e_a}(x_{||}) = \big(\partial_{x^a} F^{\mathfrak{b}}_{\alpha\beta}\big)(x_{||}), \text{ or, equivalently, } F^{\mathfrak{b}}_{\alpha\gamma}(x_{||}) = \frac{1}{\gamma_a}\big(\partial_{x^a} F^{\mathfrak{b}}_{\alpha,\gamma-e_a}\big)(x_{||}) \ , \quad (2.3.23)$$

for all $\mathfrak{b}, \alpha, a$, all $\gamma : \gamma_a \neq 0$, and all $x_{||} \in \mathcal{U}^p$. For any $\mathfrak{b}$, α, and $x_{||}$, we now set

$$\phi^{\mathfrak{b}}_\alpha(x_{||}) := F^{\mathfrak{b}}_{\alpha 0}(x_{||}) \in C^\infty(\mathcal{U}^p) \ .$$

An iterated application of (2.3.23) shows that

$$F^{\mathfrak{b}}_{\alpha\gamma}(x_{||}) = \frac{1}{\gamma!}\big(\partial^\gamma_x \phi^{\mathfrak{b}}_\alpha\big)(x_{||}) \ .$$

Hence, the $y^{\mathfrak{b}}_\Lambda$ have the form (3.2.29a) and (3.2.29b). This means that the natural transformation β is implemented by the $\phi^{\mathfrak{b}}_\alpha$, which define actually a $\mathbb{Z}_2^n$-morphism from $\mathcal{U}^{p|\underline{q}}$ to $\mathcal{V}^{r|\underline{s}}$. Indeed, the property $(\phi^b_0)(\mathcal{U}^p) \subset \mathcal{V}^r$ follows from the similar property of (F^b_{00}). On the other hand, the pullback

$$\phi^*(y^{\mathfrak{b}}) := \sum_\alpha \phi^{\mathfrak{b}}_\alpha(x) \xi^\alpha$$

must have the same degree as $y^{\mathfrak{b}}$. However, if $\deg(\xi^\alpha) \neq \deg(y^{\mathfrak{b}})$, then $\deg(\xi^\alpha_\Lambda) \neq \deg(y^{\mathfrak{b}}_\Lambda)$, whatever ξ_Λ. It follows therefore from (2.3.20) that $\phi^{\mathfrak{b}}_\alpha = F^{\mathfrak{b}}_{\alpha 0} = 0$.

Part II. The proof of the converse implication is less demanding. Let $\beta : \mathcal{U}^{p|\underline{q}}(-) \to \mathcal{V}^{r|\underline{s}}(-)$ be a natural transformation that is induced by a $\mathbb{Z}_2^n$-morphism $\phi : \mathcal{U}^{p|\underline{q}} \to \mathcal{V}^{r|\underline{s}}$, i.e., that is of the form (3.2.29a) and (3.2.29b). For any $\Lambda \in \mathbb{Z}_2^n\mathtt{GrAlg}$, the map β_Λ is smooth and its derivative is Λ_0-linear if and only if its components $y^{\mathfrak{b}}_\Lambda$ have these properties. The total derivative of $y^{\mathfrak{b}}_\Lambda$ with respect to x^* exists, is continuous, and is Λ_0-linear if and only if its partial derivatives with respect to the x^a_Λ and the ξ^A_Λ exist, are continuous, and are Λ_0-linear. When computing the derivative $y^{\mathfrak{b}}_\Lambda$ with respect to $\xi^{A_i}_\Lambda \in \Lambda_{\gamma_i}$ at $\mathrm{x}^* \in \mathcal{U}^{p|\underline{q}}(\Lambda)$ in the direction of $w_\Lambda \in \Lambda_{\gamma_i}$, we get

$$\sum_{\alpha\beta} \frac{1}{\beta!}\big(\partial^\beta_x \phi^{\mathfrak{b}}_\alpha\big)(x_{||})\ \mathring{x}^\beta_\Lambda\ (\xi^{A_1}_\Lambda)^{\alpha_1} \ldots \lim_{t\to 0} \frac{1}{t}\Big((\xi^{A_i}_\Lambda + t w_\Lambda)^{\alpha_i} - (\xi^{A_i}_\Lambda)^{\alpha_i}\Big)\ \ldots (\xi^{A_{|\underline{q}|}}_\Lambda)^{\alpha_{|\underline{q}|}} \ .$$

If γ_i is odd, the exponent α_i is 0 or 1. In the first (resp., the second) case, the limit vanishes (resp., is w_Λ). If γ_i is even, the multiplication of vectors in Λ_{γ_i} is commutative and the binomial formula shows that the limit is $w_\Lambda \alpha_i (\xi^{A_i}_\Lambda)^{\alpha_i - 1}$. The derivative thus exists, is continuous, and is Λ_0-linear. Similarly, the derivative of $y^{\mathfrak{b}}_\Lambda$ with respect to x^a_Λ exists if and only if its derivatives with respect to $x^a_{||}$ and with respect to $\mathring{x}^a_\Lambda$ exist. The (standard) computation of the derivative with respect to x^a_Λ at x^* in the direction of

$$v_\Lambda = (v_{||}, \mathring{v}_\Lambda) \in \mathbb{R} \times \mathring{\Lambda}_0$$

thus leads to the sum of the terms

$$v_{||}\sum_{\alpha\beta}\frac{1}{\beta!}\big(\partial_x^{\beta+e_a}\phi_\alpha^{\mathfrak{b}}\big)(x_{||})\,\mathring{x}_\Lambda^\beta\,\xi_\Lambda^\alpha$$

and

$$\mathring{v}_\Lambda\sum_{\alpha,\gamma:\gamma_a\neq 0}\frac{1}{\gamma!}\big(\partial_x^{\gamma}\phi_\alpha^{\mathfrak{b}}\big)(x_{||})\,\gamma_a\,\mathring{x}_\Lambda^{\gamma-e_a}\,\xi_\Lambda^\alpha=\mathring{v}_\Lambda\sum_{\alpha\beta}\frac{1}{\beta!}\big(\partial_x^{\beta+e_a}\phi_\alpha^{\mathfrak{b}}\big)(x_{||})\,\mathring{x}_\Lambda^\beta\,\xi_\Lambda^\alpha\;.$$

The derivative considered does therefore exist, is continuous, and is Λ_0-linear (note that it is essential that the derivative is the series over $\alpha\beta$ multiplied by v_Λ – as $a\in\Lambda_0$ does not act on $v_{||}$). □

Remark 2.3.20. The Λ_0-linearity is a strong constraint that takes us from the category of generalized $\mathbb{Z}_2^n$-manifolds to the one of $\mathbb{Z}_2^n$-manifolds. A similar phenomenon exists in complex analysis. Indeed, for any real differentiable function $f=u+\mathrm{i}\,v:\Omega\subset\mathbb{C}\simeq\mathbb{R}^2\to\mathbb{C}\simeq\mathbb{R}^2$, the Jacobian is an $\mathbb{R}$-linear map $J_f:\mathbb{R}^2\to\mathbb{R}^2$. However, if we further insist that the Jacobian be $\mathbb{C}$-linear, then we see that f must be holomorphic, that is, it must satisfy the Cauchy–Riemann equations on Ω. Imposing $\mathbb{C}$-linearity thus greatly restricts class of functions and takes us from real analysis to complex analysis.

It will also be important to understand what happens to the Λ-points of a given $\mathbb{Z}_2^n$-domain under morphisms of $\mathbb{Z}_2^n$-Grassmann algebras.

Proposition 2.3.21. *Let $\mathcal{U}^{p|\underline{q}}$ be a $\mathbb{Z}_2^n$-domain and let $\psi^*:\Lambda\to\Lambda'$ be a morphism of $\mathbb{Z}_2^n$-Grassmann algebras. The induced map (see* (2.3.4) *)*

$$\Psi:=\mathcal{U}^{p|\underline{q}}(\psi^*):\mathcal{U}^{p|\underline{q}}(\Lambda)\ni\mathrm{x}^*\simeq(x_\Lambda,\xi_\Lambda)\mapsto\psi^*\circ\mathrm{x}^*\simeq\psi^*(x_\Lambda,\xi_\Lambda)\in\mathcal{U}^{p|\underline{q}}(\Lambda')$$

is a smooth map from the open subset $\mathcal{U}^{p|\underline{q}}(\Lambda)$ of the Fréchet space and Fréchet Λ_0-module $\mathbb{R}^{p|\underline{q}}(\Lambda)$ to the open subset $\mathcal{U}^{p|\underline{q}}(\Lambda')$ of the Fréchet space and Fréchet Λ_0'-module $\mathbb{R}^{p|\underline{q}}(\Lambda')$, such that

$$\mathrm{d}_{\mathrm{x}^*}\Psi(\mathrm{a}\cdot\mathrm{v})=\psi^*(\mathrm{a})\cdot\mathrm{d}_{\mathrm{x}^*}\Psi(\mathrm{v})\;,$$

for all $\mathrm{x}^*\in\mathcal{U}^{p|\underline{q}}(\Lambda)$, $\mathrm{v}\in\mathbb{R}^{p|\underline{q}}(\Lambda)$ *and* $\mathrm{a}\in\Lambda_0$.

Proof. Since $\Lambda=\mathcal{O}_{\mathbb{R}^{0|\underline{m}}}(\{\star\})$, so that

$$\psi^*\in\mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_{\mathbb{R}^{0|\underline{m}}}(\{\star\}),\mathcal{O}_{\mathbb{R}^{0|\underline{m}'}}(\{\star\}))\;,$$

there is a unique morphism

$$\Phi=(|\phi|,\phi^*)\in\mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(\mathbb{R}^{0|\underline{m}'},\mathbb{R}^{0|\underline{m}})\;,$$

such that $\psi^*=\phi^*_{\{\star\}}$. Hence, the morphism ψ^* is continuous from $\Lambda=\mathbb{R}[[\theta]]$ to $\Lambda'=\mathbb{R}[[\theta']]$ endowed with their standard locally convex topologies [12], and so are its restrictions $\psi^*|_{\Lambda_{\gamma_i}}$ from Λ_{γ_i} to Λ'_{γ_i}. We thus see that the induced map

$$\Psi=(\psi^*|_{\Lambda_0})^{\times p}\times\prod_{i=1}^{N}(\psi^*|_{\Lambda_{\gamma_i}})^{\times q_i}$$

is continuous.

At $\mathrm{x}^*\simeq(x_\Lambda,\xi_\Lambda)=:u_\Lambda\in\mathcal{U}^{p|\underline{q}}(\Lambda)$ and $\mathrm{v}\simeq v_\Lambda\in\mathbb{R}^{p|\underline{q}}(\Lambda)$, the derivative

$$\mathrm{d}\Psi:\mathcal{U}^{p|\underline{q}}(\Lambda)\times\mathbb{R}^{p|\underline{q}}(\Lambda)\longrightarrow\mathbb{R}^{p|\underline{q}}(\Lambda')$$

is defined as

$$\begin{aligned} \mathrm{d}_{\mathrm{x}^*}\Psi(\mathrm{v}) &= \lim_{t\to 0}\frac{\Psi(\mathrm{x}^*+t\mathrm{v})-\Psi(\mathrm{x}^*)}{t} \\ &= \lim_{t\to 0}\big(\cdots,\frac{\psi^*(u^{\mathfrak{a}}_\Lambda+tv^{\mathfrak{a}}_\Lambda)-\psi^*(u^{\mathfrak{a}}_\Lambda)}{t},\cdots\big) \\ &= \big(\cdots,\psi^*(v^{\mathfrak{a}}_\Lambda),\cdots\big) \\ &=: (\psi^*(v^{\mathfrak{a}}_\Lambda))\ , \end{aligned}$$

where $\mathfrak{a}$ is the label $a\in\{1,\ldots,p\}$ or $A\in\{1,\ldots,|\underline{q}|\}$ of any coordinate in $\mathbb{R}^{p|\underline{q}}(\Lambda)$, and where we used the $\mathbb{R}$-linearity of the $\mathbb{Z}_2^n$-algebra morphism $\psi^*:\Lambda\to\Lambda'$. Hence, for any $\mathrm{a}\in\Lambda_0$, we get

$$\mathrm{d}_{\mathrm{x}^*}\Psi(\mathrm{a}\cdot\mathrm{v}) = (\psi^*(\mathrm{a}\cdot v^{\mathfrak{a}}_\lambda)) = (\psi^*(\mathrm{a})\cdot\psi^*(v^{\mathfrak{a}}_\lambda)) = \psi^*(\mathrm{a})\cdot\mathrm{d}_{\mathrm{x}^*}\psi(\mathrm{v})\ .$$

Since the higher order derivatives of Ψ vanish, all its derivatives exist and are continuous, hence, the map Ψ is actually smooth. □

2.3.4 The manifold structure on the set of Λ-points

The next theorem generalizes Propositions 2.3.17 and 2.3.21. For information about Fréchet manifolds, we refer to Appendix 2.4.2. We recall that the Λ-points $M(\Lambda)$ of a $\mathbb{Z}_2^n$-manifold M can be equivalently viewed as the maps $m=(|m|,m^*)\in\mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(\mathbb{R}^{0|\underline{m}},M)$, as the global pullbacks $m^*=m^*_{|M|}\in\mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_M(|M|),\Lambda)$, or as the induced morphisms

$$m^*_\star\in\mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_{M,x},\Lambda)\ ,$$

where $x=|m|(\star)\in|M|$. If $M=\mathcal{U}^{p|\underline{q}}$ is a $\mathbb{Z}_2^n$-domain, we often write x instead of m and we can identify $\mathrm{x}\simeq\mathrm{x}^*\simeq\mathrm{x}^*_\star$ with the pullbacks

$$(u_{||},\mathring{u}_\Lambda,\rho_\Lambda)\in\mathcal{U}^p\times\mathring{\Lambda}_0^p\times\prod_i\Lambda_{\gamma_i}^{q_i}$$

by x^* of the coordinate functions (u,ρ) in $\mathcal{U}^{p|\underline{q}}$. Recall as well that $\mathbb{Z}_2^n$-morphisms $\phi:M\to N$ are mapped injectively to natural transformations $\phi:M(-)\to N(-)$ with Λ-component

$$\phi_\Lambda: M(\Lambda)\ni(x,m^*_\star)\mapsto(|\phi|(x),m^*_\star\circ\phi^*_x)\in N(\Lambda)\ , \tag{2.3.24}$$

and that, for any fixed M, a $\mathbb{Z}_2^n$-Grassmann algebra morphism $\psi^*:\Lambda\to\Lambda'$ induces a map

$$M(\psi^*): M(\Lambda)\ni(x,m^*_\star)\mapsto(x,\psi^*\circ m^*_\star)\in M(\Lambda')\ .$$

Theorem 2.3.22. *Let M be a $\mathbb{Z}_2^n$-manifold, and let Λ and Λ' be $\mathbb{Z}_2^n$-Grassmann algebras. Then*

i. $M(\Lambda)$ has the structure of a nuclear Fréchet Λ_0-manifold, and,

ii. given a morphism of $\mathbb{Z}_2^n$-Grassmann algebras $\psi^*:\Lambda\longrightarrow\Lambda'$, the induced mapping $M(\psi^*)$ is ψ^*-smooth.

Proof.

i. Let $p|\underline{q}$ be the dimension of the $\mathbb{Z}_2^n$-manifold M. The local $\mathbb{Z}_2^n$-isomorphisms

$$h_\alpha=(|h_\alpha|,h^*_\alpha):U_\alpha=(|U_\alpha|,\mathcal{O}_M|_{|U_\alpha|})\to\mathcal{U}_\alpha^{p|\underline{q}}=(\mathcal{U}^p_\alpha,C^\infty_{\mathbb{R}^p}|_{\mathcal{U}^p_\alpha}[[\rho]])\ ,$$

where α varies in some $\mathcal{A}$ and where $|U_\alpha| \subset |M|$ is open, provide an atlas on M (see paragraph below Definition 2.2.10). As recalled above, the $\mathbb{Z}_2^n$-isomorphisms

$$h_\alpha : U_\alpha \to \mathcal{U}_\alpha^{p|\underline{q}}$$

implement natural isomorphisms h_α with Λ-components

$$h_{\alpha,\Lambda} : U_\alpha(\Lambda) \ni (x, m_\star^*) \mapsto (|h_\alpha|(x), m_\star^* \circ (h_\alpha)_x^*) \in \mathcal{U}_\alpha^{p|\underline{q}}(\Lambda) , \tag{2.3.25}$$

whose inverses are the similar maps defined using

$$|h_\alpha^{-1}| = |h_\alpha|^{-1} \quad \text{and} \quad (h_\alpha^{-1})_y^* = ((h_\alpha)^*_{|h_\alpha|^{-1}(y)})^{-1} \ \ (y \in \mathcal{U}_\alpha^p) .$$

The family $(U_\alpha(\Lambda), h_{\alpha,\Lambda})$ $(\alpha \in \mathcal{A})$ is an atlas that endows $M(\Lambda)$ with a nuclear Fréchet Λ_0-manifold structure. Indeed:

(a) Any $h_{\alpha,\Lambda} : U_\alpha(\Lambda) \to \mathcal{U}_\alpha^{p|\underline{q}}(\Lambda)$ is a bijection valued in the open subset $\mathcal{U}_\alpha^{p|\underline{q}}(\Lambda)$ of the nuclear Fréchet vector space $\mathbb{R}^{p|\underline{q}}(\Lambda)$, which is also a Fréchet module over the nuclear Fréchet algebra Λ_0. Moreover, as the $|U_\alpha|$ are an open cover of $|M|$, we have

$$M(\Lambda) = \bigcup_{\alpha\in\mathcal{A}} U_\alpha(\Lambda) ,$$

in view of Proposition 2.3.14.

(b) The image $h_{\alpha,\Lambda}(U_\alpha(\Lambda) \cap U_\beta(\Lambda))$ is open in $\mathbb{R}^{p|\underline{q}}(\Lambda)$. To see this, set $|U_{\alpha\beta}| = |U_\alpha| \cap |U_\beta| \subset |U_\alpha|$ and consider the open $\mathbb{Z}_2^n$-submanifold $U_{\alpha\beta} = (|U_{\alpha\beta}|, \mathcal{O}_M|_{|U_{\alpha\beta}|})$ of U_α. The $\mathbb{Z}_2^n$-isomorphism h_α restricts to a $\mathbb{Z}_2^n$-isomorphism

$$h_\alpha : U_{\alpha\beta} \to \mathcal{U}_{\alpha\beta}^{p|\underline{q}} ,$$

where the target is the open $\mathbb{Z}_2^n$-subdomain $\mathcal{U}_{\alpha\beta}^{p|\underline{q}}$ of $\mathcal{U}_\alpha^{p|\underline{q}}$ defined over the open subset

$$\mathcal{U}_{\alpha\beta}^p := |h_\alpha|(|U_{\alpha\beta}|) \subset \mathcal{U}_\alpha^p ,$$

obtained as the image of the open subset $|U_{\alpha\beta}| \subset |U_\alpha|$ by the diffeomorphism $|h_\alpha|$. The restricted $\mathbb{Z}_2^n$-isomorphism h_α induces a natural isomorphism h_α, whose Λ-component is a bijection

$$h_{\alpha,\Lambda} : U_{\alpha\beta}(\Lambda) \to \mathcal{U}_{\alpha\beta}^{p|\underline{q}}(\Lambda) .$$

Further, we have

$$U_{\alpha\beta}(\Lambda) = \bigcup_{x\in|U_{\alpha\beta}|} \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Alg}}(\mathcal{O}_{M,x}, \Lambda) =$$

$$\bigcup_{x\in|U_\alpha|} \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Alg}}(\mathcal{O}_{M,x}, \Lambda) \ \bigcap \ \bigcup_{x\in|U_\beta|} \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Alg}}(\mathcal{O}_{M,x}, \Lambda) = U_\alpha(\Lambda) \cap U_\beta(\Lambda) .$$

Hence, the image $h_{\alpha,\Lambda}(U_\alpha(\Lambda) \cap U_\beta(\Lambda)) = \mathcal{U}_{\alpha\beta}^{p|\underline{q}}(\Lambda) \subset \mathbb{R}^{p|\underline{q}}(\Lambda)$ is open.

(c) We have still to prove that the transition bijections

$$h_{\beta,\Lambda}(h_{\alpha,\Lambda})^{-1} : \mathcal{U}_{\alpha\beta}^{p|\underline{q}}(\Lambda) \to \mathcal{U}_{\beta\alpha}^{p|\underline{q}}(\Lambda)$$

are Λ_0-smooth. In view of Theorem 2.3.19, the $\mathbb{Z}_2^n$-isomorphism

$$h_\beta h_\alpha^{-1} : \mathcal{U}_{\alpha\beta}^{p|\underline{q}} \to \mathcal{U}_{\beta\alpha}^{p|\underline{q}}$$

induces a natural isomorphism $h_\beta h_\alpha^{-1}$ with a Λ_0-smooth Λ-component

$$(h_\beta h_\alpha^{-1})_\Lambda : \mathcal{U}^{p|\underline{q}}_{\alpha\beta}(\Lambda) \to \mathcal{U}^{p|\underline{q}}_{\beta\alpha}(\Lambda) \; .$$

In view of Equations (2.3.24) and (2.3.25), we get

$$(h_\beta h_\alpha^{-1})_\Lambda(u, \mathrm{x}^*_\star) = \big(|h_\beta h_\alpha^{-1}|(u), \mathrm{x}^*_\star \circ (h_\beta \circ h_\alpha^{-1})^*_u\big) =$$

$$\big(|h_\beta|(|h_\alpha|^{-1}(u)), \mathrm{x}^*_\star \circ ((h_\alpha)^*_{|h_\alpha|^{-1}(u)})^{-1} \circ (h_\beta)^*_{|h_\alpha|^{-1}(u)}\big) = h_{\beta,\Lambda}((h_{\alpha,\Lambda})^{-1}(u, \mathrm{x}^*_\star)) \; ,$$

for any $(u, \mathrm{x}^*_\star) \in \mathcal{U}^{p|\underline{q}}_{\alpha\beta}(\Lambda)$. It follows that $h_{\beta,\Lambda}(h_{\alpha,\Lambda})^{-1} = (h_\beta h_\alpha^{-1})_\Lambda$ is Λ_0-smooth.

ii. The statement of part (ii) is purely local, see Appendix 2.4.2. Let $(x, m^*_\star) \in M(\Lambda)$, let $(U_\alpha(\Lambda), h_{\alpha,\Lambda})$ be a chart of $M(\Lambda)$ around $(x, m^*_\star)$, and let $(U_\beta(\Lambda'), h_{\beta,\Lambda'})$ be a chart of $M(\Lambda')$, such that $M(\psi^*)(U_\alpha(\Lambda)) \subset U_\beta(\Lambda')$. We must show that the local form

$$h_{\beta,\Lambda'} \circ M(\psi^*) \circ (h_{\alpha,\Lambda})^{-1}$$

of $M(\psi^*)$ is ψ^*-smooth. Actually, we can choose $(U_\alpha(\Lambda'), h_{\alpha,\Lambda'})$ as second chart, since the image by $M(\psi^*)$ of a point $(y, n^*_\star)$ in $U_\alpha(\Lambda)$, i.e., a point

$$(y, n^*_\star) \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_{M,y}, \Lambda)$$

with $y \in |U_\alpha|$, is the point

$$(y, \psi^* \circ n^*_\star) \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_{M,y}, \Lambda') \; ,$$

i.e., in $U_\alpha(\Lambda')$. From here, we omit subscript α. Since $h : U(-) \to \mathcal{U}^{p|\underline{q}}(-)$ is a natural transformation, the diagram

$$\begin{array}{ccc} U(\Lambda) & \xrightarrow{\;M(\psi^*)\;} & U(\Lambda') \\ {\scriptstyle h_\Lambda}\downarrow & & \downarrow{\scriptstyle h_{\Lambda'}} \\ \mathcal{U}^{p|\underline{q}}(\Lambda) & \xrightarrow[\;\mathcal{U}^{p|\underline{q}}(\psi^*)\;]{} & \mathcal{U}^{p|\underline{q}}(\Lambda') \end{array}$$

commutes. Since h is in fact a natural isomorphism, we get that

$$h_{\Lambda'} \circ M(\psi^*) \circ (h_\Lambda)^{-1} = \mathcal{U}^{p|\underline{q}}(\psi^*) \; .$$

From Proposition 2.3.21 we conclude that this local form is indeed ψ^*-smooth.

□

In view of (2.3.8), in general, the local model $\mathbb{R}^{p|\underline{q}}(\Lambda)$ of $M(\Lambda)$ is infinite-dimensional, due to the non-zero degree even coordinates of Λ. If the particular $\mathbb{Z}_2^n$-Grassmann algebra has no non-zero degree even coordinates, then it is a polynomial algebra and the resulting local model $\mathbb{R}^{p|\underline{q}}(\Lambda)$ will, of course, be finite-dimensional. Further, we have the

Corollary 2.3.23. *For any $\mathbb{Z}_2^n$-manifold M, the associated functor*

$$M(-) \in [\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{Set}]$$

can be considered as a functor

$$M(-) \in [\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}] \;,$$

where the target category is either the category AFMan *of Fréchet manifolds over a Fréchet algebra or the category* ANFMan *of nuclear Fréchet manifolds over a nuclear Fréchet algebra, see Appendix 2.4.2. Therefore, the faithful restricted Yoneda functor* $\mathcal{Y}_{\mathbb{Z}_2^n\mathtt{Pts}}$*, see Corollary 2.3.10, can be viewed as a faithful functor*

$$\mathcal{Y}_{\mathbb{Z}_2^n\mathtt{Pts}} : \mathbb{Z}_2^n\mathtt{Man} \to [\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}] \;.$$

The latter statement requires that the natural transformation $\phi : M(-) \to N(-)$ induced by a $\mathbb{Z}_2^n$-morphism $\phi : M \to N$ have components $\phi_\Lambda : M(\Lambda) \to N(\Lambda)$ that are morphisms in A(N)FMan between the Fréchet Λ_0-manifolds $M(\Lambda)$ and $N(\Lambda)$, i.e., that the ϕ_Λ be ρ-smooth for some morphism $\rho : \Lambda_0 \to \Lambda_0$ of Fréchet algebras. We will show in the next subsection that this condition is satisfied for $\rho = \mathrm{id}_{\Lambda_0}$, i.e., we will show that:

Proposition 2.3.24. *Any natural transformation* $\phi : M(-) \to N(-)$ *that is implemented by a* $\mathbb{Z}_2^n$*-morphism* $\phi : M \to N$ *has* Λ_0*-smooth components* $\phi_\Lambda : M(\Lambda) \to N(\Lambda)$*.*

Theorem 2.3.25. *Let* $M \in \mathbb{Z}_2^n\mathtt{Man}$ *be of dimension* $p|\underline{q}$ *and let* $\Lambda \in \mathbb{Z}_2^n\mathtt{GrAlg}$*.*

i. The nuclear Fréchet Λ_0*-manifold* $M(\Lambda)$ *is a fiber bundle in the category* ANFMan*. Its base is the nuclear Fréchet* $\mathbb{R}$*-manifold* $M(\mathbb{R})$*, i.e., the smooth manifold* $|M|$*, and its typical fiber is the nuclear Fréchet* Λ_0*-manifold*

$$\Lambda^{p|\underline{q}} := \mathring{\Lambda}_0^p \times \prod_{i=1}^N \Lambda_{\gamma_i}^{q_i} \;. \tag{2.3.26}$$

ii. The topology of $M(\Lambda)$*, which is defined, as in the case of smooth manifolds, by the atlas providing the nuclear Fréchet* Λ_0*-structure, is a Hausdorff topology, so that* $M(\Lambda)$ *is a genuine Fréchet manifold.*

Proof. (i) We think of fiber bundles in ANFMan exactly as of fiber bundles in the category of smooth manifolds. Of course, in such a fiber bundle, all objects and arrows are ANFMan-objects and ANFMan-morphisms.

Let $p^* : \Lambda \to \mathbb{R}$ be, as above, the canonical $\mathbb{Z}_2^n\mathtt{GrAlg}$-morphism. The induced map

$$\pi := M(p^*) : M(\Lambda) \ni (x, m_\star^*) \mapsto (x, p^* \circ m_\star^*) \simeq x \in M(\mathbb{R}) \simeq |M|$$

is p^*-smooth, i.e., is a morphism in the category ANFMan.

We will show that π is surjective and that the local triviality condition is satisfied.

Let $z \in |M|$. There is a $\mathbb{Z}_2^n$-chart (U, h) of M, such that $|U| \subset |M|$ is a neighborhood of z. The $\mathbb{Z}_2^n$-isomorphism $h : U \to \mathcal{U}^{p|\underline{q}}$ induces a natural isomorphism h, whose Λ-components are Λ_0-diffeomorphisms, i.e., Λ_0-smooth maps that have a Λ_0-smooth inverse. We have the following commutative diagram:

$$\begin{array}{ccc}
U(\Lambda) & \xleftrightarrow{\quad h_\Lambda \quad} & \mathcal{U}^{p|\underline{q}}(\Lambda) \simeq \mathcal{U}^p \times \Lambda^{p|\underline{q}} \\
{\scriptstyle U(p^*) = \pi|_{U(\Lambda)}} \downarrow & & \downarrow {\scriptstyle \mathcal{U}^{p|\underline{q}}(p^*) \simeq \mathrm{prj}_1} \\
U(\mathbb{R}) \simeq |U| & \xleftrightarrow[\quad h_\mathbb{R} = |h| \quad]{} & \mathcal{U}^{p|\underline{q}}(\mathbb{R}) \simeq \mathcal{U}^p
\end{array} \;,$$

where prj_1 is the canonical projection. Let us explain that $\mathcal{U}^{p|\underline{q}}(p^*) \simeq \mathrm{prj}_1$, when read through $\flat : \mathcal{U}^p \times \Lambda^{p|\underline{q}} \leftrightarrow \mathcal{U}^{p|\underline{q}}(\Lambda)$. We need a more explicit description of the equivalent views on Λ-points of a $\mathbb{Z}_2^n$-domain, see beginning of Subsection 2.3.4. As elsewhere in this text, we denote a $\mathbb{Z}_2^n$-morphism $\mathbb{R}^{0|\underline{m}} \to \mathcal{U}^{p|\underline{q}}$ by $\mathrm{x} = (|\mathrm{x}|, \mathrm{x}^*)$ and we denote the morphism it induces between the stalks $\mathcal{O}_{\mathcal{U}^{p|\underline{q}},|\mathrm{x}|(\star)} \to \Lambda$ by $\mathrm{x}^*_\star$. The morphism $\flat$ is the succession of identifications

$$\mathcal{U}^p \times \Lambda^{p|\underline{q}} \ni (x_{||}, \mathring{x}_\Lambda, \xi_\Lambda) \simeq \mathrm{x} = (|\mathrm{x}|, \mathrm{x}^*) \simeq (|\mathrm{x}|(\star), \mathrm{x}^*_\star) \in \mathcal{U}^{p|\underline{q}}(\Lambda) \;, \tag{2.3.27}$$

where the components of the base morphism $|\mathrm{x}|$ are obtained (see [15]) by applying the base projection $\varepsilon_\star : \Lambda \to \mathbb{R}$ of $\mathbb{R}^{0|\underline{m}}$, i.e., the canonical morphism p^*, to the components $x^a_\Lambda = (x^a_{||}, \mathring{x}^a_\Lambda) \in \Lambda_0$. Hence, we get

$$|\mathrm{x}|(\star) = |\mathrm{x}| = (\ldots, p^*(x^a_\Lambda), \ldots) = x_{||} \;. \tag{2.3.28}$$

Therefore, we actually obtain that

$$\mathcal{U}^{p|\underline{q}}(p^*)(\flat(x_{||}, \mathring{x}_\Lambda, \xi_\Lambda)) = (|\mathrm{x}|(\star), p^* \circ \mathrm{x}^*_\star) \simeq |\mathrm{x}|(\star) = x_{||} = \mathrm{prj}_1(x_{||}, \mathring{x}_\Lambda, \xi_\Lambda) \;.$$

Since $\pi|_{U(\Lambda)} = |h|^{-1} \circ \mathrm{prj}_1 \circ h_\Lambda$, the local projection $\pi|_{U(\Lambda)}$ is surjective, so that z is in the image of π, which is thus surjective as well.

As just mentioned, we started from $z \in |M|$ and found a neighborhood $|U|$ of z and a Λ_0-diffeomorphism h_Λ. When identifying $|U|$ with $\mathcal{U}^p$ via $|h|$ (which then becomes id), we get the Λ_0-diffeomorphism

$$h_\Lambda : \pi^{-1}(|U|) \simeq U(\Lambda) \ni (y, m^*_\star) \mapsto (y, m^*_\star \circ h^*_y) \in |U| \times \Lambda^{p|\underline{q}} \;. \tag{2.3.29}$$

Observe that in Equation (2.3.29) we used $\flat^{-1}$ defined in Equations (2.3.27) and (2.3.28), thus identifying

$$(y, m^*_\star \circ h^*_y) \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_{\mathcal{U}^{p|\underline{q}},y}, \Lambda) \subset \mathcal{U}^{p|\underline{q}}(\Lambda)$$

with $h \circ m \in \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(\mathbb{R}^{0|\underline{m}}, \mathcal{U}^{p|\underline{q}})$, and then with

$$\big(y, \mathrm{pr}_2(m^*(h^*(x))), m^*(h^*(\xi))\big) \in |U| \times \Lambda^{p|\underline{q}} \;,$$

where we denoted the projection of Λ_0 onto $\mathring{\Lambda}_0$ by pr_2. Notice also that the conclusion that $\Lambda^{p|\underline{q}}$ is a nuclear Fréchet Λ_0-manifold comes from the facts that any subspace (resp., any closed subspace) of a nuclear (resp., a Fréchet) space is a nuclear (resp., a Fréchet) space.

Hence, the trivialization condition is satisfied as well, and $M(\Lambda)$ is a fiber bundle in `ANFMan`, as announced.

(ii) Now consider two different Λ-points $m^* = (x, m^*_\star)$ and $n^* = (y, n^*_\star)$ in $M(\Lambda)$. If $x \neq y$, then, as $|M|$ is Hausdorff, there exist open neighborhoods $|U|$ of x and $|V|$ of y, such that $|U| \cap |V| = \emptyset$. When denoting the corresponding open $\mathbb{Z}_2^n$-submanifolds by U and V, respectively, we get open neighborhoods $U(\Lambda)$ and $V(\Lambda)$ of m^* and n^*, such that $U(\Lambda) \cap V(\Lambda) = \emptyset$. We have of course to check that, for any $\mathbb{Z}_2^n$-chart (U_α, h_α), the image

$$h_{\alpha,\Lambda}(U_\alpha(\Lambda) \cap U(\Lambda))$$

is open in $\mathbb{R}^{p|\underline{q}}(\Lambda)$, and similarly for $V(\Lambda)$. To see this, it suffices to proceed as in the proof of Theorem 2.3.22.

Next, consider the situation where $x = y =: z \in |M|$, use the trivialization constructed in (i), and denote the canonical projection from $\mathcal{U}^p \times \Lambda^{p|\underline{q}}$ onto $\Lambda^{p|\underline{q}}$ by prj_2. As $m^* \neq n^*$, we have $h_\Lambda(m^*) \neq h_\Lambda(n^*)$, i.e.,

$$(|h|(z), \mathrm{prj}_2(h_\Lambda(m^*))) \neq (|h|(z), \mathrm{prj}_2(h_\Lambda(n^*))) \;.$$

Since $\mathrm{prj}_2(h_\Lambda(m^*)) \neq \mathrm{prj}_2(h_\Lambda(n^*))$ are points in the Hausdorff space $\Lambda^{p|\underline{q}}$, there are open neighborhoods V_{m^*} and V_{n^*} of these projections that do not intersect. The preimages U_{m^*} and U_{n^*} of V_{m^*} and V_{n^*} by the continuous map

$$\mathrm{prj}_2 \circ h_\Lambda : U(\Lambda) \to \Lambda^{p|\underline{q}}$$

are then open neighborhoods of m^* and n^* that do not intersect.

Finally, the space $M(\Lambda)$ is indeed a Hausdorff topological space. □

2.3.5 The Schwarz–Voronov embedding

In order to get a fully faithful functor, hence, to embed the category $\mathbb{Z}_2^n\mathtt{Man}$ as full subcategory into a functor category, we need to replace the target category $[\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]$ by a subcategory that we denote by $[\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]$ and that we define as follows:

Definition 2.3.26. The category $[\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]$ is the subcategory of the category $[\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]$,

i. whose objects are the functors $\mathcal{F}$, such that, for any $\Lambda \in \mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}$, the value $\mathcal{F}(\Lambda)$ is a (nuclear) Fréchet Λ_0-manifold, and

ii. whose morphisms are natural transformations $\eta : \mathcal{F} \to \mathcal{G}$, such that, for any Λ, the component $\eta_\Lambda : \mathcal{F}(\Lambda) \to \mathcal{G}(\Lambda)$ is Λ_0-smooth.

Proposition 2.3.27. *The restricted Yoneda functor $\mathcal{Y}_{\mathbb{Z}_2^n\mathtt{Pts}}$ can be considered as a faithful functor*

$$\mathcal{S} : \mathbb{Z}_2^n\mathtt{Man} \to [\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!] \,.$$

Proof. The image $\mathcal{Y}_{\mathbb{Z}_2^n\mathtt{Pts}}(M)$ of an object $M \in \mathbb{Z}_2^n\mathtt{Man}$ is a functor $M(-) \in [\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]$, such that, for any Λ, the value $M(\Lambda)$ is a (nuclear) Fréchet Λ_0-manifold. Further, the image $\mathcal{Y}_{\mathbb{Z}_2^n\mathtt{Pts}}(\phi)$ of a $\mathbb{Z}_2^n$-morphism $\phi : M \to N$ is a natural transformation $\phi : M(-) \to N(-)$, such that, for any Λ, the component $\phi_\Lambda : M(\Lambda) \to N(\Lambda)$ is Λ_0-smooth.

The proof of Λ_0-smoothness uses the following construction, which we will also need later on. Let $M, N \in \mathbb{Z}_2^n\mathtt{Man}$ be manifolds of dimension $p|\underline{q}$ and $r|\underline{s}$, respectively, let $|\phi| \in C^\infty(|M|, |N|)$, and let $(|V_\beta|)_\beta$ be an open cover of $|N|$ by $\mathbb{Z}_2^n$-charts

$$g_\beta : V_\beta \to \mathcal{V}_\beta^{r|\underline{s}}, \text{ where } V_\beta = (|V_\beta|, \mathcal{O}_N|_{|V_\beta|}) \,.$$

The open subsets $|U_\beta| := |\phi|^{-1}(|V_\beta|) \subset |M|$ cover $|M|$, and each $|U_\beta|$ can be covered by $\mathbb{Z}_2^n$-charts

$$h_{\beta\alpha} : U_{\beta\alpha} \to \mathcal{U}_{\beta\alpha}^{p|\underline{q}}, \text{ where } U_{\beta\alpha} = (|U_{\beta\alpha}|, \mathcal{O}_M|_{|U_{\beta\alpha}|}) \,.$$

The $\mathbb{Z}_2^n$-morphism $\phi : M \to N$ restricts to a $\mathbb{Z}_2^n$-morphism $\phi|_{U_{\beta\alpha}} : U_{\beta\alpha} \to V_\beta$. In particular, the composite

$$g_\beta \circ \phi|_{U_{\beta\alpha}} \circ (h_{\beta\alpha})^{-1} : \mathcal{U}_{\beta\alpha}^{p|\underline{q}} \to \mathcal{V}_\beta^{r|\underline{s}}$$

is a $\mathbb{Z}_2^n$-morphism.

We now show that ϕ_Λ is Λ_0-smooth. Therefore, let $(x, m_\star^*) \in M(\Lambda)$. There is a $\mathbb{Z}_2^n$-chart (V_β, g_β) of N such that $|\phi|(x) \in |V_\beta|$, and there is a $\mathbb{Z}_2^n$-chart $(U_{\beta\alpha}, h_{\beta\alpha})$ of M such that $x \in |U_{\beta\alpha}|$. These charts (we omit in the following the subscripts β and α) induce charts $(U(\Lambda), h_\Lambda)$ of $M(\Lambda)$ around $(x, m_\star^*)$, and $(V(\Lambda), g_\Lambda)$ of $N(\Lambda)$ such that $\phi_\Lambda(U(\Lambda)) \subset V(\Lambda)$. It suffices to show (see Appendix 2.4.2) that the local form

$$g_\Lambda \circ \phi_\Lambda \circ (h_\Lambda)^{-1} = (g \circ \phi|_U \circ h^{-1})_\Lambda$$

is Λ_0-smooth. This is the case in view of Theorem 2.3.19. Finally, the faithfulness is established in Corollary 2.3.10. This completes the proof. □

We will prove that the functor $\mathcal{S}$ is fully faithful, hence, injective (up to isomorphism) on objects. Therefore, it embeds the category $\mathbb{Z}_2^n\mathtt{Man}$ of $\mathbb{Z}_2^n$-manifolds as full subcategory into the larger functor category $[\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]$.

Definition 2.3.28. We refer to the faithful functor

$$\mathcal{S} : \mathbb{Z}_2^n\mathtt{Man} \longrightarrow [\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]$$

as the *Schwarz–Voronov embedding.*

Theorem 2.3.29. *The Schwarz–Voronov embedding $\mathcal{S}$ is a fully faithful functor. That is, given two $\mathbb{Z}_2^n$-manifolds M and N, the injective map*

$$\mathcal{S}_{M,N} : \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}\big(M, N\big) \to \mathsf{Hom}_{[\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]}\big(M(-), N(-)\big)$$

is bijective.

Proof. Notice first that it follows from the results of [13] and Lemma 2.3.13 that there is a 1:1 correspondence

$$|M| \simeq \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_M(|M|), \mathbb{R}) \simeq \bigcup_{x\in|M|} \mathsf{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_{M,x}, \mathbb{R}) = M(\mathbb{R}) \ ,$$

which is given by

$$x \mapsto \varepsilon_x \mapsto (x, \varepsilon_x) \ ,$$

where ε_x is the evaluation map $\varepsilon_x(f) = (\varepsilon f)(x)$ $(f \in \mathcal{O}_M(|M|))$ and where ε is the base map $\varepsilon : \mathcal{O}_M \to C^\infty_{|M|}$. Hence, any $(x, m^*_\star) \in M(\mathbb{R})$ is equal to (x, ε_x) and can be identified with x. In view of (2.3.25), this 1:1-correspondence identifies the nuclear Fréchet $\mathbb{R}$-manifold structure on $M(\mathbb{R})$ with the smooth manifold structure on $|M|$.

Let now

$$\eta : M(-) \to N(-)$$

be a natural transformation in the target set of $\mathcal{S}_{M,N}$, i.e., a natural transformation such that, for any Λ, the Λ-component η_Λ is Λ_0-smooth. In particular, the map

$$|\phi| := \eta_{\mathbb{R}} : |M| \to |N|,$$

is a smooth map between the reduced manifolds. As in the proof of Proposition 2.3.27, let $(V_\beta, g_\beta)_\beta$ be an open cover of $|N|$ by $\mathbb{Z}_2^n$-charts, and, for any β, let $(U_{\beta\alpha}, h_{\beta\alpha})_\alpha$ be an open cover of $|U_\beta| := |\phi|^{-1}(|V_\beta|)$ by $\mathbb{Z}_2^n$-charts. When denoting the canonical $\mathbb{Z}_2^n$-Grassmann algebra morphism $\Lambda \to \mathbb{R}$ by p^*, we get the commutative diagram

$$\begin{array}{ccc} \bigcup_{\beta\alpha} U_{\beta\alpha}(\Lambda) & \xrightarrow{\ \eta_\Lambda\ } & \bigcup_\beta V_\beta(\Lambda) \\ {\scriptstyle M(p^*)}\downarrow & & \downarrow{\scriptstyle N(p^*)} \\ \bigcup_{\beta\alpha} |U_{\beta\alpha}| & \xrightarrow[\ |\phi|\ ]{} & \bigcup_\beta |V_\beta| \end{array} \quad , \qquad (2.3.30)$$

which shows that, for any β, α, we get the Λ_0-smooth map

$$(\eta_\Lambda)|_{U_{\beta\alpha}(\Lambda)} : U_{\beta\alpha}(\Lambda) \to V_\beta(\Lambda) \ .$$

Indeed, if, for $(x, m^*_\star) \in U_{\beta\alpha}(\Lambda)$, we set $\eta_\Lambda(x, m^*_\star) = (y, n^*_\star)$, the commutativity of the diagram implies that

$$y \simeq (y, p^* \circ n^*_\star) = (N(p^*) \circ \eta_\Lambda)(x, m^*_\star) = (\eta_{\mathbb{R}} \circ M(p^*))(x, m^*_\star) = \eta_{\mathbb{R}}(x, p^* \circ m^*_\star) \simeq |\phi|(x) \in |V_\beta| \ .$$

Therefore, the restriction

$$\eta|_{U_{\beta\alpha}(-)} : U_{\beta\alpha}(-) \to V_\beta(-)$$

is a natural transformation with Λ_0-smooth components.

Note that

$$h_{\beta\alpha} : U_{\beta\alpha} \to \mathcal{U}^{p|\underline{q}}_{\beta\alpha} \quad \text{and} \quad g_\beta : V_\beta \to \mathcal{V}^{r|\underline{s}}_\beta$$

are $\mathbb{Z}_2^n$-isomorphisms and induce natural isomorphisms, also denoted by $h_{\beta\alpha}$ and g_β, whose components are chart diffeomorphisms

$$h_{\beta\alpha,\Lambda} : U_{\beta\alpha}(\Lambda) \to \mathcal{U}^{p|\underline{q}}_{\beta\alpha}(\Lambda) \quad \text{and} \quad g_{\beta,\Lambda} : V_\beta(\Lambda) \to \mathcal{V}^{r|\underline{s}}_\beta(\Lambda)$$

of nuclear Fréchet Λ_0-manifolds. The local form

$$g_{\beta,\Lambda} \circ (\eta_\Lambda)|_{U_{\beta\alpha}(\Lambda)} \circ (h_{\beta\alpha,\Lambda})^{-1} : \mathcal{U}^{p|\underline{q}}_{\beta\alpha}(\Lambda) \to \mathcal{V}^{r|\underline{s}}_\beta(\Lambda)$$

of η_Λ is thus Λ_0-smooth. In other words, any Λ-component of the natural transformation

$$\varphi_{\beta\alpha} := g_\beta \circ \eta|_{U_{\beta\alpha}(-)} \circ h^{-1}_{\beta\alpha} : \mathcal{U}^{p|\underline{q}}_{\beta\alpha}(-) \to \mathcal{V}^{r|\underline{s}}_\beta(-) \tag{2.3.31}$$

between functors associated to $\mathbb{Z}_2^n$-domains, is Λ_0-smooth. It therefore follows from 2.3.19 that $\varphi_{\beta\alpha}$ is implemented by a $\mathbb{Z}_2^n$-morphism

$$\varphi_{\beta\alpha} : \mathcal{U}^{p|\underline{q}}_{\beta\alpha} \to \mathcal{V}^{r|\underline{s}}_\beta \ ,$$

so that the composite

$$\phi_{\beta\alpha} := g^{-1}_\beta \circ \varphi_{\beta\alpha} \circ h_{\beta\alpha} : U_{\beta\alpha} \to N \tag{2.3.32}$$

is a $\mathbb{Z}_2^n$-morphism that is defined on an open $\mathbb{Z}_2^n$-submanifold of M. The question is whether we can patch together these locally defined $\mathbb{Z}_2^n$-morphisms, which are inherited from η, and get a globally defined $\mathbb{Z}_2^n$-morphism $\phi : M \to N$ that induces η.

Let $\phi_{\beta\alpha}|_{U_{\beta\alpha,\nu\mu}}$ and $\phi_{\nu\mu}|_{U_{\beta\alpha,\nu\mu}}$ be the $\mathbb{Z}_2^n$-morphisms obtained by restriction to the open $\mathbb{Z}_2^n$-submanifold $U_{\beta\alpha,\nu\mu}$ with base manifold $|U_{\beta\alpha,\nu\mu}| := |U_{\beta\alpha}| \cap |U_{\nu\mu}|$. They coincide as $\mathbb{Z}_2^n$-morphisms, if they do as associated natural transformations, i.e., if all Λ-components of those transformations coincide. This is the case since both Λ-components are equal to $\eta_\Lambda|_{U_{\beta\alpha,\nu\mu}(\Lambda)}$. It follows that the $\mathbb{Z}_2^n$-algebra morphisms

$$\phi_{\beta\alpha}|^*_{U_{\beta\alpha,\nu\mu}}, \phi_{\nu\mu}|^*_{U_{\beta\alpha,\nu\mu}} : \mathcal{O}_N(|N|) \to \mathcal{O}_M(|U_{\beta\alpha,\nu\mu}|)$$

coincide. This implies that we can glue the $\mathbb{Z}_2^n$-algebra morphisms $\phi^*_{\beta\alpha} : \mathcal{O}_N(|N|) \to \mathcal{O}_M(|U_{\beta\alpha}|)$ and get a $\mathbb{Z}_2^n$-algebra morphism

$$\phi^* : \mathcal{O}_N(|N|) \to \mathcal{O}_M(|M|) \ .$$

Indeed, for any $f \in \mathcal{O}_N(|N|)$, the $\phi^*_{\beta\alpha}(f) \in \mathcal{O}_M(|U_{\beta\alpha}|)$ are a family of $\mathbb{Z}_2^n$-functions on an open cover of $|M|$, which do coincide on the intersections. To see this, note that

$$(\phi^*_{\beta\alpha}(f))|_{|U_{\beta\alpha,\nu\mu}|} = \phi_{\beta\alpha}|^*_{U_{\beta\alpha,\nu\mu}}(f) = \phi_{\nu\mu}|^*_{U_{\beta\alpha,\nu\mu}}(f) = (\phi^*_{\nu\mu}(f))|_{|U_{\beta\alpha,\nu\mu}|} \ .$$

Hence, there is a unique global section $F \in \mathcal{O}_M(|M|)$ of the sheaf $\mathcal{O}_M$, such that $F|_{|U_{\beta\alpha}|} = \phi^*_{\beta\alpha}(f)$. The `Set`-morphism, which is defined by

$$\phi^*_{|N|} : \mathcal{O}_N(|N|) \ni f \mapsto F \in \mathcal{O}_M(|M|) \ ,$$

is actually a morphism of $\mathbb{Z}_2^n$-algebras. Indeed, note that

$$\rho^{|M|}_{|U_{\beta\alpha}|} \circ \phi^*_{|N|} = \phi^*_{\beta\alpha}$$

(ρ is the restriction) and observe that, for any element $|U_{\beta\alpha}|$ of the open cover of $|M|$ considered, we have

$$(\phi^*_{|N|}(f \cdot g))|_{|U_{\beta\alpha}|} = \phi^*_{\beta\alpha}(f) \cdot \phi^*_{\beta\alpha}(g) = (\phi^*_{|N|}(f) \cdot \phi^*_{|N|}(g))|_{|U_{\beta\alpha}|} \ .$$

The $\mathbb{Z}_2^n$-algebra morphism $\phi^*_{|N|}$ fully characterizes a $\mathbb{Z}_2^n$-morphism $\phi = (||\phi||, \phi^*) : M \to N$. We will show that ϕ induces the natural transformation η, which then completes the proof.

Since ϕ is glued from the $\mathbb{Z}_2^n$-morphisms $\phi_{\beta\alpha}$, we get, in view of Equations (2.3.31) and (2.3.32), in particular that

$$||\phi||\,|_{|U_{\beta\alpha}|} = |\phi_{\beta\alpha}| = \eta_{\mathbb{R}}|_{U_{\beta\alpha}(\mathbb{R})} = |\phi|\,|_{|U_{\beta\alpha}|} \ , \tag{2.3.33}$$

so that $||\phi|| = |\phi|$. Further, for any $|V_\beta|$,

$$\rho^{|U_\beta|}_{|U_{\beta\alpha}|} \circ \phi^*_{|V_\beta|} = \phi^*_{\beta\alpha,|V_\beta|} : \mathcal{O}_N(|V_\beta|) \to \mathcal{O}_M(|U_{\beta\alpha}|) \ . \tag{2.3.34}$$

Let now Λ be any $\mathbb{Z}_2^n$-Grassmann algebra and let $(x, m^*_\star) \in U_{\beta\alpha}(\Lambda)$. As $x \in |U_{\beta\alpha}|$ and $|\phi|(x) \in |V_\beta|$, it follows from Equations (2.3.33), (2.3.34), (2.3.31), and (2.3.32), that the image of $(x, m^*_\star)$ by the Λ-component of the natural transformation induced by ϕ is

$$\phi_\Lambda(x, m^*_\star) = (|\phi|(x), m^*_\star \circ \phi^*_x) = (|\phi_{\beta\alpha}|(x), m^*_\star \circ \phi^*_{\beta\alpha,x}) = (\phi_{\beta\alpha})_\Lambda(x, m^*_\star) = \eta_\Lambda(x, m^*_\star) \ .$$

□

The following theorem is of importance in the study of $\mathbb{Z}_2^n$-Lie groups.

Theorem 2.3.30. *The Schwarz–Voronov embedding $\mathcal{S}$ sends $\mathbb{Z}_2^n$-Lie groups G to functors $\mathcal{S}(G) = G(-)$ from the category $\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}$ of $\mathbb{Z}_2^n$-Grassmann algebras to the category* `ANFLg` *of nuclear Fréchet Lie groups over nuclear Fréchet algebras.*

The proof is not entirely straightforward and will be given in a paper on $\mathbb{Z}_2^n$-Lie groups, which is currently being written down.

2.3.6 Representability and equivalence of categories

As the Schwarz–Voronov embedding is fully faithful, the category $\mathbb{Z}_2^n\mathtt{Man}$ can be viewed as a full subcategory of the category $[\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]$. Functor categories are known to be well-suited for geometric constructions. Hence, when trying to build a $\mathbb{Z}_2^n$-manifold M (possibly from other $\mathbb{Z}_2^n$-manifolds M_ι), it is often easier to build a functor $\mathcal{F}$ in $[\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]$ (from the given $\mathbb{Z}_2^n$-manifolds interpreted as functors $M_\iota(-)$). However, one has then to show that $\mathcal{F}$ can be represented by a $\mathbb{Z}_2^n$-manifold M, i.e., that there is a $\mathbb{Z}_2^n$-manifold M, such that $M(-) \simeq \mathcal{F}$.

Definition 2.3.31. A functor

$$\mathcal{F} \in [\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]$$

is said to be *representable*, if there exists a $\mathbb{Z}_2^n$-manifold $M \in \mathbb{Z}_2^n\mathtt{Man}$ (which is then unique up to unique isomorphism), such that

$$M(-) \simeq \mathcal{F} \quad \text{in} \quad [\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!] \ .$$

We define the restriction $\mathcal{F}|_{|U|}$ of a functor $\mathcal{F} \in [\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]$ to an open subset $|U| \subset \mathcal{F}(\mathbb{R}) \in \mathtt{(N)FMan}$.

For any $\Lambda \in \mathbb{Z}_2^n\mathtt{GrAlg}$, let

$$p_\Lambda^* : \Lambda \longrightarrow \mathbb{R}$$

be the canonical projection, let

$$\mathcal{F}(p_\Lambda^*) : \mathcal{F}(\Lambda) \longrightarrow \mathcal{F}(\mathbb{R})$$

be the corresponding smooth map. The preimage

$$\mathcal{F}|_{|U|}(\Lambda) := (\mathcal{F}(p_\Lambda^*))^{-1}(|U|) \tag{2.3.35}$$

is an open (nuclear) Fréchet Λ_0-submanifold of $\mathcal{F}(\Lambda)$.

Consider now a morphism $\varphi^* : \Lambda \longrightarrow \Lambda'$ in $\mathbb{Z}_2^n\mathtt{GrAlg}$. As $p_{\Lambda'}^* \circ \varphi^* = p_\Lambda^*$, we get the restriction

$$\mathcal{F}|_{|U|}(\varphi^*) := \mathcal{F}(\varphi^*)|_{\mathcal{F}|_{|U|}(\Lambda)} : \mathcal{F}|_{|U|}(\Lambda) \longrightarrow \mathcal{F}|_{|U|}(\Lambda')\ , \tag{2.3.36}$$

which is a morphism in $\mathtt{A(N)FMan}$.

Definition 2.3.32. For any functor

$$\mathcal{F} \in [\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]$$

and any open subset $|U| \subset \mathcal{F}(\mathbb{R})$, the *restriction of* $\mathcal{F}$ *to* $|U|$ is the functor

$$\mathcal{F}|_{|U|} \in [\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]$$

that is defined by Equations (2.3.35) and (2.3.36).

Example 2.3.33. Let $M \in \mathbb{Z}_2^n\mathtt{Man}$, let $M(-)$ be the corresponding functor, and let $|U| \subset |M| \simeq M(\mathbb{R})$ be an open subset. The restriction $M(-)|_{|U|}$ is given:

i. on objects Λ, by

$$M(-)|_{|U|}(\Lambda) := \{(x, m_\star^*) \in M(\Lambda) : (x, p_\Lambda^* \circ m_\star^*) \simeq x \in |U|\} = U(\Lambda)\ , \tag{2.3.37}$$

where $U = (|U|, \mathcal{O}_M|_{|U|})$ is the open $\mathbb{Z}_2^n$-submanifold of M over $|U|$, and

ii. on morphisms $\varphi^* : \Lambda \to \Lambda'$, by

$$M(-)|_{|U|}(\varphi^*) := M(\varphi^*)|_{U(\Lambda)} = U(\varphi^*)\ , \tag{2.3.38}$$

since both maps are given by

$$U(\Lambda) \ni (x, m_\star^*) \mapsto (x, \varphi^* \circ m_\star^*) \in U(\Lambda')\ .$$

Let $\mathcal{F}$ be representable, let M be 'its' representing $\mathbb{Z}_2^n$-manifold, and let

$$\eta : \mathcal{F} \to M(-) \tag{2.3.39}$$

be the corresponding natural isomorphism in $[\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{A(N)FMan}]\!]$. The maps η_Λ and η_Λ^{-1} are then Λ_0-smooth, i.e., η_Λ is a Λ_0-diffeomorphism, for any Λ. In particular, the map $\eta_\mathbb{R} : \mathcal{F}(\mathbb{R}) \to M(\mathbb{R})$ is a diffeomorphism of (nuclear) Fréchet manifolds. This means that the (nuclear) Fréchet structures on $\mathcal{F}(\mathbb{R}) \simeq M(\mathbb{R})$ coincide. Further, if one identifies $\mathcal{F}(\mathbb{R}) \simeq M(\mathbb{R})$ with $|M|$, the (nuclear) Fréchet structure on $\mathcal{F}(\mathbb{R}) \simeq M(\mathbb{R})$ coincides with the smooth structure on $|M|$. We can therefore view $\mathcal{F}(\mathbb{R})$ as being the smooth manifold $|M|$. Consider now a $\mathbb{Z}_2^n$-atlas $(U_\alpha, h_\alpha)_\alpha$ of M. If we denote the dimension of M by $p|\underline{q}$, the $\mathbb{Z}_2^n$-chart map h_α is a $\mathbb{Z}_2^n$-isomorphism

$$h_\alpha : U_\alpha \to \mathcal{U}_\alpha^{p|\underline{q}}$$

valued in a $\mathbb{Z}_2^n$-domain of dimension $p|\underline{q}$, which implies that

$$h_\alpha : U_\alpha(-) \to \mathcal{U}_\alpha^{p|\underline{q}}(-) \tag{2.3.40}$$

is a natural isomorphism in $[\![\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{A(N)FMan}]\!]$. In view Equations (2.3.39), (2.3.37), (2.3.38), and (2.3.40), the family $(|U_\alpha|)_\alpha$ is an open cover of $|M| \simeq \mathcal{F}(\mathbb{R})$, such that, for any α, we have

$$\mathcal{F}|_{|U_\alpha|} \simeq M(-)|_{|U_\alpha|} = U_\alpha(-) \simeq \mathcal{U}_\alpha^{p|\underline{q}}(-)$$

in $[\![\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{A(N)FMan}]\!]$.

Theorem 2.3.34. *A functor $\mathcal{F} \in [\![\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{A(N)FMan}]\!]$ is representable if and only if there exists an open cover $(|U_\alpha|)_\alpha$ of $\mathcal{F}(\mathbb{R})$, such that, for each α, we have*

$$\mathcal{F}|_{|U_\alpha|} \simeq \mathcal{U}_\alpha^{p|\underline{q}}(-) \tag{2.3.41}$$

in $[\![\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{A(N)FMan}]\!]$, where $\mathcal{U}_\alpha^{p|\underline{q}}$ is a $\mathbb{Z}_2^n$-domain in a fixed $\mathbb{R}^{p|\underline{q}}$.

Proof. We showed already that the condition is necessary. Assume now that Condition (2.3.41) is satisfied, i.e., that we have natural isomorphisms

$$k_\alpha : \mathcal{F}|_{|U_\alpha|} \to \mathcal{U}_\alpha^{p|\underline{q}}(-)$$

in $[\![\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{A(N)FMan}]\!]$. This means that the Λ-components

$$k_{\alpha,\Lambda} : \mathcal{F}|_{|U_\alpha|}(\Lambda) \to \mathcal{U}_\alpha^{p|\underline{q}}(\Lambda)$$

are Λ_0-diffeomorphisms.

In particular, we have a diffeomorphism

$$|h_\alpha| := k_{\alpha,\mathbb{R}} : \mathcal{F}|_{|U_\alpha|}(\mathbb{R}) = (\mathcal{F}(p_\mathbb{R}^*))^{-1}(|U_\alpha|) = |U_\alpha| \to \mathcal{U}_\alpha^{p|\underline{q}}(\mathbb{R}) \simeq \mathcal{U}_\alpha^p\ .$$

Notice that $(|U_\alpha|, |h_\alpha|)_\alpha$ can be interpreted as a smooth atlas on $|M| := \mathcal{F}(\mathbb{R})$. The direct image of the structure sheaf $\mathcal{O}_{\mathcal{U}_\alpha^{p|\underline{q}}}$ over $\mathcal{U}_\alpha^p$ by the continuous map $|h_\alpha|^{-1} : \mathcal{U}_\alpha^p \to |U_\alpha|$ is a sheaf over $|U_\alpha|$, which we denote by $\mathcal{O}_{U_\alpha}$:

$$\mathcal{O}_{U_\alpha} := (|h_\alpha|^{-1})_* \mathcal{O}_{\mathcal{U}_\alpha^{p|\underline{q}}}\ .$$

The $\mathbb{Z}_2^n$-ringed space

$$U_\alpha := (|U_\alpha|, \mathcal{O}_{U_\alpha})$$

is isomorphic to the $\mathbb{Z}_2^n$-domain $\mathcal{U}_\alpha^{p|\underline{q}}$. The isomorphism is $h_\alpha := (|h_\alpha|, h_\alpha^*)$, where h_α^* is the identity map (a composite of direct images is the direct image by the composite). In other words, we have an isomorphism of $\mathbb{Z}_2^n$-manifolds

$$h_\alpha : U_\alpha \to \mathcal{U}_\alpha^{p|\underline{q}}\ .$$

Consider now an overlap $|U_{\alpha\beta}| := |U_\alpha| \cap |U_\beta| \neq \emptyset$. Omitting restrictions, we get that $k_\beta k_\alpha^{-1}$ is a natural isomorphism (in $[\![\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{A(N)FMan}]\!]$)

$$k_{\beta\alpha} := k_\beta k_\alpha^{-1} : \mathcal{U}^{p|\underline{q}}_{\alpha\beta}(-) \to \mathcal{U}^{p|\underline{q}}_{\beta\alpha}(-)$$

between functors corresponding to $\mathbb{Z}_2^n$-domains (defined as usual). In view of Theorem 2.3.19, the natural isomorphism $k_{\beta\alpha}$ is implemented by a $\mathbb{Z}_2^n$-isomorphism

$$k_{\beta\alpha} : \mathcal{U}^{p|\underline{q}}_{\alpha\beta} \to \mathcal{U}^{p|\underline{q}}_{\beta\alpha} \; .$$

It follows that

$$\psi_{\beta\alpha} := h_\beta^{-1} k_{\beta\alpha} h_\alpha : U_{\alpha\beta} \to U_{\beta\alpha}$$

is an isomorphism of $\mathbb{Z}_2^n$-manifolds, where $U_{\alpha\beta} := (|U_{\alpha\beta}|, \mathcal{O}_{U_\alpha}|_{|U_{\alpha\beta}|})$. The $\mathbb{Z}_2^n$-manifolds U_α can thus be glued and provide then a $\mathbb{Z}_2^n$-manifold M over $|M| = \mathcal{F}(\mathbb{R})$, such that there are $\mathbb{Z}_2^n$-isomorphisms $(|U_\alpha|, \mathcal{O}_M|_{|U_\alpha|}) \to U_\alpha$, if the $\psi_{\beta\alpha}$ satisfy the cocycle condition.

Since the Schwarz–Voronov embedding is fully faithful, we have that $\psi_{\gamma\beta}\psi_{\beta\alpha} = \psi_{\gamma\alpha}$ as $\mathbb{Z}_2^n$-morphisms if and only if the induced natural transformations coincide. However, for any Λ, we get

$$(\psi_{\gamma\beta}\psi_{\beta\alpha})_\Lambda = (h_{\gamma,\Lambda})^{-1} k_{\gamma,\Lambda} (k_{\beta,\Lambda})^{-1} h_{\beta,\Lambda} (h_{\beta,\Lambda})^{-1} k_{\beta,\Lambda} (k_{\alpha,\Lambda})^{-1} h_{\alpha,\Lambda} = \psi_{\gamma\alpha,\Lambda} \; .$$

It remains to show that M actually represents $\mathcal{F}$, i.e., that we can find a natural isomorphism $\eta : M(-) \to \mathcal{F}$ in the category $[\![\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{A(N)FMan}]\!]$, i.e., that, for any $\Lambda \in \mathbb{Z}_2^n\texttt{GrAlg}$, there is a Λ_0-diffeomorphism $\eta_\Lambda : M(\Lambda) \to \mathcal{F}(\Lambda)$ that is natural in Λ. As $(|U_\alpha|)_\alpha$ is an open cover of $|M|$, the source decomposes as

$$M(\Lambda) = \bigcup_\alpha U_\alpha(\Lambda) \; ,$$

the $U_\alpha(\Lambda)$ being open (nuclear) Fréchet Λ_0-submanifolds. On any $U_\alpha(\Lambda)$, we define η_Λ by setting

$$\eta_\Lambda|_{U_\alpha(\Lambda)} := (k_{\alpha,\Lambda})^{-1} h_{\alpha,\Lambda} : U_\alpha(\Lambda) \to \mathcal{F}|_{|U_\alpha|}(\Lambda) \subset \mathcal{F}(\Lambda) \; .$$

These restrictions provide a well-defined map

$$\eta_\Lambda : M(\Lambda) \to \mathcal{F}(\Lambda) \; .$$

Indeed, if $(x, m_\star^*) \in U_\alpha(\Lambda) \cap U_\beta(\Lambda)$, we have

$$(k_{\alpha,\Lambda})^{-1}(h_{\alpha,\Lambda}(x, m_\star^*)) = (k_{\beta,\Lambda})^{-1}(h_{\beta,\Lambda}(x, m_\star^*)) \quad \text{if and only if} \quad \psi_{\beta\alpha,\Lambda}(x, m_\star^*) = (x, m_\star^*) \; .$$

However, since we glued M from the U_α, the gluing $\mathbb{Z}_2^n$-isomorphisms $\psi_{\beta\alpha}$ became identities and so did the induced natural isomorphisms. The definition of η_Λ^{-1} is similar. The source $\mathcal{F}(\Lambda)$ decomposes as

$$\mathcal{F}(\Lambda) = \bigcup_\alpha \mathcal{F}|_{|U_\alpha|}(\Lambda),$$

the $\mathcal{F}|_{|U_\alpha|}(\Lambda)$ being open (nuclear) Fréchet Λ_0-submanifolds. On any $\mathcal{F}|_{|U_\alpha|}(\Lambda)$, we define η_Λ^{-1} by setting

$$\eta_\Lambda^{-1}|_{\mathcal{F}|_{|U_\alpha|}(\Lambda)} := (h_{\alpha,\Lambda})^{-1} k_{\alpha,\Lambda} : \mathcal{F}|_{|U_\alpha|}(\Lambda) \to U_\alpha(\Lambda) \subset M(\Lambda) \; .$$

The condition for these restrictions to give a well-defined map

$$\eta_\Lambda^{-1} : \mathcal{F}(\Lambda) \to M(\Lambda)$$

is equivalent to the condition for η_Λ. Clearly, the maps η_Λ and η_Λ^{-1} are inverses. Naturality and Λ_0-smoothness are local questions and are therefore consequences of the naturality and the Λ_0-smoothness of $(k_{\alpha,\Lambda})^{-1} h_{\alpha,\Lambda}$ and of $(h_{\alpha,\Lambda})^{-1} k_{\alpha,\Lambda}$. □

We are now prepared to show that the category $\mathbb{Z}_2^n\mathtt{Man}$ is equivalent to a functor category.

Theorem 2.3.35. *The category $\mathbb{Z}_2^n\mathtt{Man}$ of $\mathbb{Z}_2^n$-manifolds (defined as $\mathbb{Z}_2^n$-ringed spaces that are locally isomorphic to $\mathbb{Z}_2^n$-domains) and $\mathbb{Z}_2^n$-morphisms (defined as morphisms of $\mathbb{Z}_2^n$-ringed spaces) is equivalent to the full subcategory $[\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}},\ \mathtt{A(N)FMan}]\!]_{\mathrm{rep}}$ of representable functors in $[\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}},\ \mathtt{A(N)FMan}]\!]$.*

In other words, the category $\mathbb{Z}_2^n\mathtt{Man}$ is equivalent to the category of locally trivial functors in the subcategory of the functor category $[\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}},\ \mathtt{A(N)FMan}]$, whose objects $\mathcal{F}$ have values $\mathcal{F}(\Lambda)$ in (nuclear) Fréchet Λ_0-manifolds and whose morphisms are the natural transformations with Λ_0-smooth components.

Remark 2.3.36. This result is reminiscent of the identification of schemes with those contravariant functors from affine schemes to sets that are sheaves (for the Zariski topology on affine schemes) and have a cover by open immersions of affine schemes.

Proof. The Schwarz–Voronov embedding viewed as functor valued in $[\![\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}},\ \mathtt{A(N)FMan}]\!]_{\mathrm{rep}}$ is obviously fully faithful and essentially surjective. It thus induces an equivalence of categories. □

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2.4 Appendix

2.4.1 Generating sets of categories

We will freely use Mac Lane's book [33] as our source of categorical notions and proofs of general statements. For completeness, we recall the concept of generating set of a category.

Definition 2.4.1 ([33], page 127)**.** Let $\mathtt{C}$ be a category. A set $S = \{S_i \in \mathrm{Ob}(\mathtt{C}) : i \in I\}$, where I is any index set, is said to be a *generating set* of $\mathtt{C}$, if, for any pair of distinct $\mathtt{C}$-morphisms

$$\phi, \psi : A \longrightarrow B\ ,$$

i.e., $\phi \neq \psi$, there exists some $i \in I$ and a $\mathtt{C}$-morphism

$$s : S_i \longrightarrow A\ ,$$

such that the compositions

$$S_i \xrightarrow{s} A \underset{\psi}{\overset{\phi}{\rightrightarrows}} B\ ,$$

are distinct, i.e., $\phi \circ s \neq \psi \circ s$. In this case, we say that the object S_i *separates* the morphisms ϕ and ψ, and that the set S *generates* the category $\mathtt{C}$.

Example 2.4.2. The set $\{\mathbb{R}\}$ is a generating set of the category of finite dimensional real vector spaces. This is easily seen, as, if we have two distinct linear maps $\phi, \psi : V \to W$, then there exists a vector $v \in V$ $(v \neq 0)$, such that $\phi(v) \neq \psi(v)$. Thus, the two linear maps differ on the one dimensional subspace generated by v. Now let z be a basis of $\mathbb{R}$. Then, the linear map $s : \mathbb{R} \to V$ given by $s(z) = v$, keeps ϕ and ψ separate.

Proposition 2.4.3. *For any locally small category* $\mathtt{C}$*, a set* $S \subset \mathrm{Ob}(\mathtt{C})$ *generates* $\mathtt{C}$ *if and only if the restricted Yoneda embedding*

$$\mathcal{Y}_S : \mathtt{C} \to [S^{\mathrm{op}}, \mathtt{Set}] ,$$

where S *is viewed as full subcategory of* $\mathtt{C}$*, is faithful.*

Proof. The restricted embedding is defined on objects by

$$\mathcal{Y}_S(A) = \mathsf{Hom}_{\mathtt{C}}(-, A) \in [S^{\mathrm{op}}, \mathtt{Set}]$$

and on morphisms by

$$\mathcal{Y}_S(\phi) = \mathsf{Hom}_{\mathtt{C}}(-, \phi) : \mathcal{Y}_S(A) \to \mathcal{Y}_S(B) ,$$

where

$$(\mathcal{Y}_S(\phi))_{S_i} : \mathsf{Hom}_{\mathtt{C}}(S_i, A) \ni s \mapsto \phi \circ s \in \mathsf{Hom}_{\mathtt{C}}(S_i, B) .$$

The embedding $\mathcal{Y}_S$ is faithful if and only if, for any different $\phi, \psi : A \to B$, the corresponding natural transformations are distinct, i.e., there is at least one $i \in I$ and one $s \in \mathsf{Hom}_{\mathtt{C}}(S_i, A)$, such that $\phi \circ s \neq \psi \circ s$. □

2.4.2 Fréchet spaces, modules and manifolds

Manifolds over algebras A, also known as A-manifolds, are manifolds for which the tangent spaces are endowed with a module structure over a given finite-dimensional commutative algebra. For details, the reader may consult Shurygin [47, 48, 49], and for a discussion of the specific case of (the even part of) Grassmann algebras one may consult Azarmi [6]. A comprehensive introduction to the subject can be found in the book by Vishnevskiĭ, Shirokov, and Shurygin [54] (in Russian). The concept needed in this paper is a infinite-dimensional generalisation of an A-manifold to the category of Fréchet algebras and Fréchet manifolds. For an introduction to locally convex spaces, including Fréchet vector spaces, we refer the reader to Conway [14, Chapter IV], Trèves [53, Part I], or Rudin [40, Chapter 1]. A brief introduction to Fréchet algebras can be found in Waelbroeck [56, Chapter VII]. For Fréchet manifolds, the reader can consult Saunders [42, Chapter 7] and Hamilton [28, Part I.4].

Definition 2.4.4. A *Fréchet (vector) space* is a complete Hausdorff metrizable locally convex topological vector space.

There exist a few other, equivalent, definitions of Fréchet spaces. The topology on a locally convex space is metrizable if and only if it can be derived from a countable family of semi-norms $||-||_k$, $k \in \mathbb{N}$. The topology is Hausdorff if and only if the family of semi-norms is separating, i.e., if $||\mathrm{x}||_k = 0$, for all k, implies $\mathrm{x} = 0$. Given such a family of semi-norms, one defines a translationally invariant metric that induces the topology by setting

$$d(\mathrm{x}, \mathrm{y}) = \sum_{k=0}^{\infty} 2^{-k} \frac{||\mathrm{x} - \mathrm{y}||_k}{1 + ||\mathrm{x} - \mathrm{y}||_k} ,$$

for all x and y.

Example 2.4.5. Let $M = (|M|, \mathcal{O})$ be a $\mathbb{Z}_2^n$-manifold. For any open subset $U \subset |M|$, the space $\mathcal{O}(U)$ of $\mathbb{Z}_2^n$-functions on U is a Fréchet space. An inducing family of semi-norms is given by

$$||f||_{C,D} = \sup_{x \in C} |\epsilon(D(f))(x)| ,$$

where ϵ is the projection $\varepsilon : \mathcal{O}(U) \to C^\infty(U)$ of $\mathbb{Z}_2^n$-functions to base functions, where C is any compact subset of U, and where D is any $\mathbb{Z}_2^n$-differential operator over U. Details on the construction of a countable family of semi-norms that is equivalent to $(||-||_{C,D})_{C,D}$, can be found in the proof of the last lemma in [12].

Given two Fréchet spaces $\big(F, (||-||_k^F)_{k\in\mathbb{N}}\big)$ and $\big(G, (||-||_k^G)_{k\in\mathbb{N}}\big)$, a linear map

$$\phi : F \longrightarrow G \ ,$$

is continuous if and only if, for every semi-norm $||-||_k^G$, there exists a semi-norm $||-||_l^F$ and a positive real number $C > 0$, such that

$$||\phi(\mathrm{x})||_k^G \leq C||\mathrm{x}||_l^F \ ,$$

for every $\mathrm{x} \in F$. A similar result holds for continuous bilinear maps

$$\phi : F \times G \to H \ .$$

The *morphisms of Fréchet spaces* are the continuous linear maps, so that the category of Fréchet spaces is a full subcategory of the category of topological vector spaces.

What makes Fréchet spaces interesting, is the fact that they have just enough structure to define a derivative of a mapping between such spaces. This leads to a meaningful notion of a smooth map between Fréchet spaces, and so much of finite dimensional differential geometry can be transferred to the infinite dimensional setting, using Fréchet spaces as local models. The well known *Gâteaux (directional) derivative* is defined as follows.

Definition 2.4.6. Let F and G be Fréchet spaces and $U \subset F$ be open, and let $\phi : U \to G$ be a (nonlinear) continuous map. Then the *derivative of* ϕ *in the direction of* $\mathrm{v} \in F$ *at* $\mathrm{x} \in U$ is defined as

$$\mathrm{d_x}\phi(\mathrm{v}) := \lim_{t\to 0} \frac{\phi(\mathrm{x} + t\mathrm{v}) - \phi(\mathrm{x})}{t}$$

provided the limit exists. We say that ϕ is *continuously differentiable*, if the limit exists for all $\mathrm{x} \in U$ and $\mathrm{v} \in F$, and if the mapping

$$\mathrm{d}\phi : U \times F \longrightarrow G$$

is (jointly) continuous.

Higher order derivatives are defined inductively, i.e.,

$$\mathrm{d_x^{k+1}}\phi(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_{k+1}) := \lim_{t\to 0} \frac{\mathrm{d}^k_{\mathrm{x}+t\mathrm{v}_{k+1}}\phi(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_k) - \mathrm{d}^k_{\mathrm{x}}\phi(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_k)}{t} \ .$$

A continuous map $\phi : U \to G$ is then said to be k times continuously differentiable or to be of class C^k, if

$$\mathrm{d}^k\phi : U \times F^{\times k} \longrightarrow G$$

is continuous (or, more explicitly, if all its derivatives of order $\leq k$ exist everywhere and are continuous). If ϕ is of class C^k, its derivative $\mathrm{d}^k_{\mathrm{x}}\phi(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_k)$ is multilinear and symmetric in $F^{\times k}$ [46]. Furthermore, we say that ϕ is *smooth*, if it is of class C^k, for all k.

Proposition 2.4.7. *Let F_1, F_2 be Fréchet spaces and let $U \subset F_1 \times F_2$ be an open subset. A continuous map $\phi : U \to G$ valued in a Fréchet space G is of class C^1 if and only if its (total) derivative*

$$\mathrm{d}\,\phi : U \times (F_1 \times F_2) \ni ((f_1, f_2), (\mathrm{v}_1, \mathrm{v}_2)) \mapsto \mathrm{d}_{(f_1,f_2)}\,\phi\,(\mathrm{v}_1, \mathrm{v}_2) \in G$$

is continuous, which is the case if and only if the naturally defined partial derivatives

$$\mathrm{d}_{f_1}\,\phi : U \times F_1 \ni ((f_1, f_2), \mathrm{v}_1) \mapsto \mathrm{d}_{f_1,(f_1,f_2)}\,\phi\,(\mathrm{v}_1) \in G$$

and

$$\mathrm{d}_{f_2}\,\phi : U \times F_2 \ni ((f_1, f_2), \mathrm{v}_2) \mapsto \mathrm{d}_{f_2,(f_1,f_2)}\,\phi\,(\mathrm{v}_2) \in G$$

are continuous. In this case, we have

$$\mathrm{d}_{(f_1,f_2)}\,\phi\,(\mathrm{v}_1, \mathrm{v}_2) = \mathrm{d}_{f_1,(f_1,f_2)}\,\phi\,(\mathrm{v}_1) + \mathrm{d}_{f_2,(f_1,f_2)}\,\phi\,(\mathrm{v}_2)\;.$$

The Gâteaux or Fréchet–Gâteaux derivative gives a rather weak notion of differentiation, however, most of the standard results from calculus in the finite dimensional setting remain true. Specifically, the fundamental theorem of calculus and the chain rule still hold. However, the inverse function theorem is in general lost. For a special class of Fréchet spaces, known as 'tame' Fréchet spaces, there is an analogue of the inverse function theorem known as the Nash–Moser inverse function theorem, see Hamilton [28] for details.

A *nuclear space* is a locally convex topological vector space F, such that, for any locally convex topological vector space G, the natural map $F\widehat{\otimes}_\pi G \longrightarrow F\widehat{\otimes}_\iota G$ from the projective to the injective tensor product of F and G is an isomorphism of locally convex topological vector spaces. In particular, a *nuclear Fréchet space* is a locally convex topological vector space that is a nuclear space and a Fréchet space. Loosely, if a space F is nuclear, then, for any locally convex space G, the complete topological vector space $F\widehat{\otimes}\,G$ is independent of the locally convex topology considered on $F \otimes G$. Because of this, and their nice dual properties, nuclear spaces provide a reasonable setting for infinite dimensional analysis. All the Fréchet spaces we encounter in this paper are in fact nuclear.

The following definition is standard.

Definition 2.4.8. A *Fréchet algebra* is a Fréchet vector space A, which is equipped with an associative bilinear and (jointly) continuous multiplication $\cdot : A \times A \to A$. If $(p_i)_{i\in I}$ is a family of semi-norms that induces the topology on A, (joint) continuity is equivalent to the existence, for any $i \in I$, of $j \in I$, $k \in I$, and $C > 0$, such that

$$p_i(x \cdot y) \leq C\, p_j(x)\, p_k(y),\ \forall x, y \in A\;.$$

We can always consider an equivalent increasing countable family of semi-norms $(|| - ||_n)_{n\in\mathbb{N}}$. The preceding condition then requires that, for any $n \in \mathbb{N}$, there is $r \in \mathbb{N}$ $(r \geq n)$ and $C > 0$, such that

$$||x \cdot y||_n \leq C\, ||x||_r\, ||y||_r\ \forall x, y \in A\;.$$

In particular, the topology can be induced by a countable family of submultiplicative semi-norms, i.e., by a family $(q_n)_{n\in\mathbb{N}}$ that satisfies

$$q_n(x \cdot y) \leq q_n(x)\, q_n(y)\,,\ \forall n \in \mathbb{N}, \forall x, y \in A\;.$$

Note that many authors define a Fréchet algebra simply as a Fréchet vector space, which carries an associative bilinear multiplication, and whose topology can be induced by a countable family of submultiplicative semi-norms. This latter definition is equivalent to the former.

In general, a Fréchet algebra need not be unital, and, if it is, one does not require $p_i(1_A) = 1$, in contrast to what is usually required for Banach algebras.

Example 2.4.9 (Formal power series)**.** Consider the space

$$\mathbb{R}[[z_1, z_2, \ldots, z_q]]$$

of formal power series in q parameters and coefficients in reals. We set $j := (j_1, j_2, \ldots, j_q) \in \mathbb{N}^{\times q}$ and $|j| := j_1 + j_2 + \cdots + j_q$. A general series x now reads

$$\mathrm{x} = \sum_j z^j a_j = \sum_j z_1^{j_1} z_2^{j_2} \ldots z_q^{j_q} a_{j_q \ldots j_2 j_1} \ ,$$

with no question on the convergence. The algebra structure is the standard multiplication of formal power series. The topology of coordinate-wise convergence is metrizable and given by the family of semi-norms

$$||\mathrm{x}||_k := \sum_{|j| \leq k} |a_j| \, , \qquad \forall\, k \in \mathbb{N} \ .$$

This algebra is unital with the obvious unit, and it is submultiplicative.

Let us denote the category of *Fréchet algebras* (resp., *commutative Fréchet algebras*) as `FAlg` (resp., `CFAlg`). Morphisms in this category are defined to be continuous algebra morphisms. If we restrict attention to nuclear Fréchet algebras (resp., commutative nuclear Fréchet algebras), then we work in the full subcategory `NFAlg` (resp., `CNFAlg`).

Definition 2.4.10. Fix $A \in$ `FAlg`. A *Fréchet A-module* is a Fréchet vector space F, together with a continuous action

$$\begin{aligned} A \times F \ &\stackrel{\mu}{\longrightarrow} F \\ (a, \mathrm{v}) \ &\mapsto \mu(a, \mathrm{v}) \ , \end{aligned}$$

which we will write as $\mu(a, \mathrm{v}) := a \cdot \mathrm{v}$ (and which is of course compatible with the multiplication in A).

We give a short survey on Fréchet manifolds.

Definition 2.4.11. Let $\mathcal{M}$ be a set. An F-chart of $\mathcal{M}$ is a bijective map $\phi : U \to \phi(U) \subset F$, where $U \subset \mathcal{M}$ and $\phi(U)$ is an open subset of a Fréchet space F.

A Fréchet atlas can be defined using charts valued in various Fréchet spaces. For our purposes, it is sufficient to consider a fixed Fréchet model.

Definition 2.4.12. Let $\mathcal{M}$ be a set. A *smooth F-atlas* on $\mathcal{M}$ is a collection of F-charts $((U_\alpha, \phi_\alpha))_{\alpha \in \mathcal{A}}$, such that

i. the subsets U_α cover the set $\mathcal{M}$,

ii. the subsets $\phi_\alpha(U_\alpha \cap U_\beta)$ are open in F ,

iii. the transition maps

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subset F \longrightarrow \phi_\beta(U_\beta \cap U_\alpha) \subset F$$

are smooth.

A new F-chart (U, ϕ) on $\mathcal{M}$ is *compatible* with a given smooth F-atlas, if and only if their union is again a smooth F-atlas, i.e., the subsets $\phi(U \cap U_\alpha) \subset F$ and $\phi_\alpha(U_\alpha \cap U) \subset F$ are open, and the transition maps

$$\phi_\alpha \circ \phi^{-1} : \phi(U \cap U_\alpha) \longrightarrow \phi_\alpha(U_\alpha \cap U) \quad \text{and} \quad \phi \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U) \longrightarrow \phi(U \cap U_\alpha)$$

are smooth (for every $\alpha \in \mathcal{A}$). Similarly, two smooth F-atlases are compatible provided their union is also a smooth F-atlas. Compatibility is an equivalence relation on all possible smooth F-atlases on $\mathcal{M}$.

Definition 2.4.13. A *smooth F-structure* on a set $\mathcal{M}$ is a choice of an equivalence class of smooth F-atlases on $\mathcal{M}$. We say that $\mathcal{M}$ is a *Fréchet manifold* modelled on the Fréchet space F, if $\mathcal{M}$ comes equipped with a smooth F-structure. If the model vector space F is nuclear, we speak of a *nuclear Fréchet manifold*.

A smooth F-atlas on a Fréchet manifold $\mathcal{M}$ allows us to define in the obvious way a topology on $\mathcal{M}$, which is independent of the atlas considered in the chosen equivalence class. The domain U of an F-chart (U, ϕ) is open in this topology and the bijective map $\phi : U \subset \mathcal{M} \to \phi(U) \subset F$ is a homeomorphism for the induced topologies. Most authors confine themselves to Fréchet manifolds, whose topology is Hausdorff.

Morphisms between two Fréchet manifolds are the *smooth maps* between them, where smoothness is defined, just as in the finite dimensional case of smooth manifolds, in terms of charts and smoothness of local representatives of the maps. We denote the category of Fréchet manifolds and the morphisms between them by `FMan`.

Further, the *tangent space* $\mathbb{T}_f\mathcal{M}$ to a Fréchet manifold $\mathcal{M}$ at a point $f \in \mathcal{M}$ can be defined as usual, using the tangency equivalence relation for the smooth curves of $\mathcal{M}$ that pass through f at time 0. One can easily see that $\mathbb{T}_f\mathcal{M}$ is a Fréchet space. The concept of *Fréchet vector bundle* is the natural generalization of the notion of smooth vector bundle to the category of Fréchet manifolds. The *tangent bundle* $\mathbb{T}\mathcal{M}$ of a Fréchet manifold $\mathcal{M}$ is an example of a Fréchet vector bundle.

In general, we must make a distinction between the (*kinematic*) *tangent bundle* as defined here and the *operational tangent bundle* defined in terms of derivations of the algebra of functions of a Fréchet manifold. Indeed, the two notions do not, in general, coincide, there are derivations that do not correspond to tangent vectors. However, it is known that for nuclear Fréchet manifolds the two concepts do coincide.

Let $\mathfrak{F} : \mathcal{M} \to \mathcal{N}$ be a smooth map between Fréchet manifolds modelled on Fréchet spaces F and G, respectively. There is a *tangent map* $\mathbb{T}\mathfrak{F}$ of $\mathfrak{F}$, which is a smooth map

$$\mathbb{T}\mathfrak{F} : \mathbb{T}\mathcal{M} \to \mathbb{T}\mathcal{N} \,,$$

and restricts, for any $f \in \mathcal{M}$, to a linear map

$$\mathbb{T}_f\mathfrak{F} : \mathbb{T}_f\mathcal{M} \to \mathbb{T}_{\mathfrak{F}(f)}\mathcal{N} \,.$$

As in the finite dimensional case, the local representative of $\mathbb{T}_f\mathfrak{F}$ is the derivative $\mathrm{d}_{\phi(f)}(\psi\mathfrak{F}\phi^{-1})$ of the corresponding local representative

$$\psi\mathfrak{F}\phi^{-1} : \phi(U) \subset F \to G$$

of $\mathfrak{F}$ at the point $\phi(f)$.

Fundamental to the work in this paper are Fréchet manifolds with a further module structure on their tangent bundle.

Definition 2.4.14. Let $\mathcal{M}$ be a Fréchet manifold, whose model Fréchet space F is a module over a Fréchet algebra A. We say that $\mathcal{M}$ is a *Fréchet A-manifold*, if and only if all transition maps are A-linear, i.e.,

$$\mathrm{d}_{\phi_\alpha(f)}\phi_{\beta\alpha}(a \cdot \mathrm{v}) = a \cdot \mathrm{d}_{\phi_\alpha(f)}\phi_{\beta\alpha}(\mathrm{v}) \; ,$$

for all $f \in U_\alpha \cap U_\beta$, $a \in A$, and $\mathrm{v} \in F$.

Morphisms between Fréchet A-manifolds $\mathcal{M}$ and $\mathcal{N}$ are the *A-smooth* maps between them, i.e., are the smooth maps $\mathfrak{F} : \mathcal{M} \to \mathcal{N}$ that are A-linear at every point. This means that, for any point $f \in \mathcal{M}$, there is an $\mathcal{M}$-chart (U, ϕ) around f and an $\mathcal{N}$-chart (V, ψ) around $\mathfrak{F}(f)$ that contains $\mathfrak{F}(U)$, such that the local representative

$$\mathrm{d}_{\phi(f)}(\psi\mathfrak{F}\phi^{-1})$$

of the derivative $\mathbb{T}_f\mathfrak{F}$ is an A-linear endomorphism of the A-module F. The requirement actually means that the derivative $\mathbb{T}_f\mathfrak{F}$ must be A-linear at any point $f \in \mathcal{M}$. In this way, we obtain the category of Fréchet A-manifolds, which we denote as $A\mathtt{FMan}$.

In this paper, we will use the category $\mathtt{AFMan}$, whose objects are the Fréchet A-manifolds, where A is not a fixed Fréchet algebra, but any Fréchet algebra. The definition of $\mathtt{AFMan}$-morphisms generalizes the definition of $A\mathtt{FMan}$-morphisms. Suppose that $\mathcal{M}$ is a Fréchet A-manifold modelled on an A-module F and $\mathcal{N}$ is a Fréchet B-manifold modelled on a B-module G. The $\mathtt{AFMan}$-morphisms from $\mathcal{M}$ to $\mathcal{N}$ are the $\mathtt{A}$-smooth maps between them, i.e., those smooth maps $\mathfrak{F} : \mathcal{M} \to \mathcal{N}$ that are at any point compatible with the module structures of F and G. This means that there is a Fréchet algebra morphism $\rho : A \to B$, and, for any $f \in \mathcal{M}$, there exist charts (U, ϕ) and (V, ψ) as above, such that

$$\mathrm{d}_{\phi(f)}(\psi\mathfrak{F}\phi^{-1})(a \cdot \mathrm{v}) = \rho(a) \cdot \mathrm{d}_{\phi(f)}(\psi\mathfrak{F}\phi^{-1})(\mathrm{v}) \; ,$$

for any $a \in A$ and $\mathrm{v} \in F$. This requirement actually means that, for any f, the derivative $\mathbb{T}_f\mathfrak{F}$ is compatible with the induced actions on the tangent spaces. We will refer to an $\mathtt{A}$-smooth map with associated Fréchet algebra morphism ρ, as a ρ-smooth map. If we restrict our attention to nuclear objects, i.e, the model Fréchet vector space and the Fréchet algebra are both nuclear, then we denote the corresponding category as $\mathtt{ANFMan}$.

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Chapter 3

Linear $\mathbb{Z}_2^n$-Manifolds and Linear Actions

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Abstract

We establish the representability of the general linear $\mathbb{Z}_2^n$-group and use the restricted functor of points – whose test category is the category of $\mathbb{Z}_2^n$-manifolds over a single topological point – to define its smooth linear actions on $\mathbb{Z}_2^n$-graded vector spaces and linear $\mathbb{Z}_2^n$-manifolds. Throughout the paper, particular emphasis is placed on the full faithfulness and target category of the restricted functor of points of a number of categories that we are using.

3.1 Introduction

In order to be able to deal with the technical details of vector bundles and related structures in the category of $\mathbb{Z}_2^n$-manifolds (for $n = 1$ see [6]), we need some foundational results on $\mathbb{Z}_2^n$-Lie groups and their smooth linear actions on linear $\mathbb{Z}_2^n$-manifolds. However, the proofs of some folklore results, i.e., results that we tended to accept somewhat hands-waving, are often not at all obvious in the $\mathbb{Z}_2^n$-context. The present paper, beyond its supposed applications, intrinsic interest and the beauty of some of its developments, raises the question of the scientific value of 'results' that are partially based on speculations.

Loosely speaking, $\mathbb{Z}_2^n$-manifolds ($\mathbb{Z}_2^n = \mathbb{Z}_2^{\times n}$) are 'manifolds' for which the structure sheaf has a $\mathbb{Z}_2^n$-grading and the commutation rules for the local coordinates comes from the standard scalar product (see [11, 13, 14, 15, 19, 20, 21, 23, 37] for details). This is not just a trivial or straightforward generalization of the notion of a standard supermanifold, as one has to deal with formal coordinates that anticommute with other formal coordinates, but are themselves *not* nilpotent. Due to the presence of formal variables that are not nilpotent, formal power series are used rather than polynomials. Recall that for standard supermanifolds all functions are polynomial in the Grassmann odd variables. The theory of $\mathbb{Z}_2^n$-geometry is currently being developed and many foundational questions remain. For completeness, we include Appendix 3.5.2 in which the foundations of $\mathbb{Z}_2^n$-geometry are given. In this paper, we examine the relation between $\mathbb{Z}_2^n$-graded vector spaces and linear $\mathbb{Z}_2^n$-manifolds, and then we use this to define linear actions of $\mathbb{Z}_2^n$-Lie groups.

In the literature on supergeometry, the symbol $\mathbb{R}^{p|q}$ has two distinct, but related meanings. First, we have the notion of a $\mathbb{Z}_2$-graded, or super, vector space with p even and q odd dimensions, i.e., the real vector space $\mathbf{R}^{p|q} = \mathbb{R}^p \bigoplus \mathbb{R}^q$. Secondly, we have the locally ringed space $\mathbb{R}^{p|q} = \big(\mathbb{R}^p, C^\infty_{\mathbb{R}^p}[\xi]\big)$, where ξ^i ($i \in \{1, \ldots, q\}$) are the generators of a Grassmann algebra. The difference can be highlighted by identifying the points of these objects. The $\mathbb{Z}_2$-graded vector space has as its underlying topological space $\mathbb{R}^{p+q}$ (we just forget the "superstructure"), while for the locally ringed space the topological space is $\mathbb{R}^p$. There are several ways of showing that these two notions are deeply tied. In particular, the category of finite dimensional super vector spaces is equivalent to the category of "linear supermanifolds" (see [32, 33, 38]).

In this paper, we will show that the categories of finite dimensional $\mathbb{Z}_2^n$-graded vector spaces $\mathbf{V}$ and linear $\mathbb{Z}_2^n$-manifolds V are isomorphic. We do this by explicitly constructing a 'manifoldification' functor $\mathcal{M}$, which associates a linear $\mathbb{Z}_2^n$-manifold to every finite dimensional $\mathbb{Z}_2^n$-graded vector space, and a 'vectorification' functor, which is the inverse of the previous functor. It turns out that working in a coordinate-independent way $(\mathbf{V}, V)$ is much more complex than working with canonical coordinates $(\mathbf{R}^{p|\underline{q}}, \mathbb{R}^{p|\underline{q}})$.

Throughout this article, a special focus is placed on functors of points. The functor of points has been used informally in Physics as from the very beginning. It is actually of importance in contexts where there is no good notion of point as in Super- and $\mathbb{Z}_2^n$-Geometry and in Algebraic Geometry. For instance, Homotopical Algebraic Geometry [43, 44] and its generalisation that goes under the name of Homotopical Algebraic Geometry over Differential Operators [25, 26], are completely based on the functor of points approach. In this paper, we are particularly interested in functors of Λ-points, i.e., functors of points from appropriate locally small categories $\texttt{C}$ to a functor category whose source is not the category $\texttt{C}^{\text{op}}$ but the category $\texttt{G}$ of $\mathbb{Z}_2^n$-Grassmann algebras Λ. However, functors of points that are restricted to the very simple test category $\texttt{G}$ are fully faithful only if we replace the target category of the functor category by a subcategory of the category of sets.

More precisely, closely related to the above isomorphism of supervector spaces and linear supermanifolds is the so-called 'even rules'. Loosely this means including extra odd parameters to render everything even and in doing so one removes copious sign factors (see for example [24, §1.7]). We will establish an analogue of the even rules in our higher graded setting which we will refer to as the 'zero degree rules' (see Definition 3.2.1). To address this we will make extensive use of $\mathbb{Z}_2^n$-Grassmann algebras Λ, Λ-points and the Schwarz–Voronov embedding, which is a fully faithful functor of points $\mathcal{S}$ from $\mathbb{Z}_2^n$-manifolds to a functor category with source $\texttt{G}$ and the category of Fréchet manifolds over commutative Fréchet algebras as target (see [13]). We show that the zero degree rules functor $\mathcal{F}$, understood as an assignment of a functor from $\texttt{G}$ to the category of modules over commutative [Fréchet] algebras, given a [finite dimensional] $\mathbb{Z}_2^n$-graded vector space, is fully faithful (see Theorem 3.2.2 [and Proposition 3.2.25]). The 'zero degree rules' allow one to identity a finite dimensional $\mathbb{Z}_2^n$-graded vector space, considered as a functor, with the functor of points of its 'manifoldification'. In other words, the composite $\mathcal{S} \circ \mathcal{M}$ and $\mathcal{F}$ can be viewed as functors between the same categories and are naturally isomorphic. This identification is fundamental when describing linear group actions on $\mathbb{Z}_2^n$-graded vector spaces and linear $\mathbb{Z}_2^n$-manifolds.

Another important part of this work is the category of $\mathbb{Z}_2^n$-Lie groups and its fully faithful functor of points valued in a functor category with $\texttt{G}$ as source category and Fréchet Lie groups over commutative Fréchet algebras as target category. We define the general linear $\mathbb{Z}_2^n$-group as a functor in this functor category and show that it is representable, i.e., is a genuine $\mathbb{Z}_2^n$-manifold (see Theorem 3.3.4). This leads to interesting insights into the computation of the inverse of an invertible degree zero $\mathbb{Z}_2^n$-graded square matrix of dimension $p|\underline{q}$ with entries in a $\mathbb{Z}_2^n$-commutative algebra. Furthermore, the approach using Λ-points and the zero rules allows us to construct a canonical smooth linear action of the general linear $\mathbb{Z}_2^n$-group on $\mathbb{Z}_2^n$-graded vector spaces and linear $\mathbb{Z}_2^n$-manifolds. All these notions, in particular the equivalence between the definitions of a smooth linear action as natural transformation and as $\mathbb{Z}_2^n$-morphism, are carefully and explicitly explained in the main text.

We remark that many of the statements in this paper are not surprising in themselves. However, due to the subtleties of $\mathbb{Z}_2^n$-geometry, many of the proofs are much more involved than the analogue statements in supergeometry. The main source of difficulty is that one has to deal with formal power series in non-zero degree coordinates, rather than polynomials as one does in supergeometry. This forces one to work with infinite dimensional objects and the $\mathcal{J}$-adic topology ($\mathcal{J}$ is the ideal generated by non-zero degree elements). Many of the 'categorical' proofs are significantly more involved than the proofs for supermanifolds. In general, there is a lot of work to establish the form of natural transformations as we have non-nilpotent elements of non-zero degree. While the ethos of the proofs may be standard, they are not, in general, simple or routine checks due to the aforementioned subtleties.

Motivation from physics: $\mathbb{Z}_2^n$-gradings ($n \geq 2$) can be found in the theory of parastatistics (see for example [27, 28, 29, 49]) and in relation to an alternative approach to supersymmetry [45]. 'Higher graded' generalizations of the super Schrödinger algebra (see [3]) and the super Poincaré algebra (see [10]) have appeared in the literature. Furthermore, such gradings appear in the theory of mixed symmetry tensors as found in string theory and some formulations of supergravity (see [12]). It must also be pointed out that quaternions and more general Clifford algebras can be understood as $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative algebras [4, 5]. Finally, *any* 'sign rule' can be interpreted in terms of a $\mathbb{Z}_2^n$-grading (see [19]).

Background: For various sheaf-theoretical notions we will draw upon Hartshorne [31, Chapter II] and Tennison [42]. There are several good introductory books on the theory of supermanifolds including Bartocci, Bruzzo & Hernández Ruipérez [7], Carmeli, Caston & Fioresi [16], Deligne & Morgan [24] and Varadarajan [46]. For categorical notions we will be based on Mac Lane [35]. We will make extensive use of the constructions and statements found in our earlier publications [13, 14, 15].

3.2 $\mathbb{Z}_2^n$-graded vector spaces and Linear $\mathbb{Z}_2^n$-manifolds

3.2.1 $\mathbb{Z}_2^n$-graded vector spaces and the zero degree rules

When dealing with linear superalgebra one encounters the so-called *even rules* (see [16, §1.8], [24, §1.7] and [46, pages 123-124], for example). Very informally, the idea is to remove sign factors by allowing extra parameters to render the situation completely even. The idea has been applied in physics since the early days of supersymmetry. More precisely, let

$$V(A) = (A \otimes V)_0$$

be the even part of the extension of scalars in a (real) super vector space V, from the base field $\mathbb{R}$ to a supercommutative algebra $A \in \texttt{SAlg}$ (in the even rules that we are about to describe, it actually suffices to use supercommutative Grassmann algebras $A = \mathbb{R}[\theta_1, \ldots, \theta_N]$: the θ_i are then the extra parameters mentioned before). The main result in even rules states, roughly, that defining a morphism $\phi : V \otimes V \to V$ is equivalent to defining it functorially on the even part of V after extension of scalars, i.e., is equivalent to defining a functorial family of morphisms

$$\phi(A) : V(A) \times V(A) \to V(A)$$

(indexed by $A \in \texttt{SAlg}$). More precisely, there is a 1:1 correspondence between parity respecting $\mathbb{R}$-linear maps $\phi : V_1 \otimes \ldots \otimes V_n \to V$ and functorial families

$$\phi(A) : V_1(A) \times \ldots \times V_n(A) \to V(A)$$

($A \in \texttt{SAlg}$) of A_0-multilinear maps.

We now proceed to generalise this theorem to the $\mathbb{Z}_2^n$-setting. We will work with the category $\mathbb{Z}_2^n\texttt{GrAlg}$ of $\mathbb{Z}_2^n$-Grassmann algebras rather than the category $\mathbb{Z}_2^n\texttt{Alg}$ of all $\mathbb{Z}_2^n$-commutative algebras.

Let $V = \bigoplus_{i=0}^N V_{\gamma_i}$ be a (real) $\mathbb{Z}_2^n$-graded vector space, i.e., a (real) vector space with a direct sum decomposition over $i \in \{0, \ldots, N\}$ (we say that the vectors of V_{γ_i} are of degree $\gamma_i \in \mathbb{Z}$). The category of $\mathbb{Z}_2^n$-graded vector spaces (not necessarily finite dimensional) we denote as $\mathbb{Z}_2^n\texttt{Vec}$. Morphisms in this category are degree preserving linear maps. We denote the category of modules over commutative algebras as $\texttt{AMod}$ (see Appendix 3.5.1).

To V we associate a functor

$$V(-) \in \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$$

in the category of those functors whose value on any $\mathbb{Z}_2^n$-Grassmann algebra $\Lambda \in \mathbb{Z}_2^n\texttt{Pts}^{\text{op}}$ is a Λ_0-module, and of those natural transformations that have Λ_0-linear Λ-components. The functor $V(-)$ is essentially the tensor product functor $- \otimes V$. It is built in the following way. First, for every $\mathbb{Z}_2^n$-Grassmann algebra Λ, we define

$$V(\Lambda) := \big(\Lambda \otimes V\big)_0 \in \Lambda_0\texttt{Mod}\,,$$

where the tensor product is over $\mathbb{R}$. Secondly, for any $\mathbb{Z}_2^n$-algebra morphism $\varphi^* : \Lambda \to \Lambda'$, we define

$$V(\varphi^*) := \big(\varphi^* \otimes \mathbb{1}_V\big)_0\,,$$

where the RHS is the restriction of $\varphi^* \otimes \mathbb{1}_V$ to the degree 0 part of $\Lambda \otimes V$, so that $V(\varphi^*)$ is an $\texttt{AMod}$-morphism

$$V(\varphi^*) : V(\Lambda) \to V(\Lambda')\,, \tag{3.2.1}$$

whose associated algebra morphism is the restriction $(\varphi^*)_0 : \Lambda_0 \to \Lambda'_0$. It is clear that $V(-)$ respects compositions and identities and is thus a functor, as announced.

We thus get an assignment

$$\mathcal{F} : \mathbb{Z}_2^n\texttt{Vec} \ni V \mapsto \mathcal{F}(V) := V(-) \in \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})\,.$$

The map $\mathcal{F}$ is essentially $- \otimes \bullet$ and is itself a functor. It associates to any grading respecting linear map $\phi : V \to W$ and any $\mathbb{Z}_2^n$-Grassmann algebra Λ, a Λ_0-linear map

$$\phi_\Lambda := (\mathbb{1}_\Lambda \otimes \phi)_0 : V(\Lambda) \to W(\Lambda)\,.$$

The family $\mathcal{F}(\phi) := \phi_-$ is a natural transformation from $\mathcal{F}(V)$ to $\mathcal{F}(W)$. Since $\mathcal{F}$ respects compositions and identities, it is actually a functor valued in the restricted functor category $\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$.

Definition 3.2.1. The functor

$$\mathcal{F} : \mathbb{Z}_2^n\texttt{Vec} \longrightarrow \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$$

is referred to as the *zero degree rules functor.*

Theorem 3.2.2. *The zero degree rules functor*

$$\mathcal{F} : \mathbb{Z}_2^n\texttt{Vec} \longrightarrow \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$$

is fully faithful, i.e., for any pair of $\mathbb{Z}_2^n$-graded vector spaces V and W, the map

$$\mathcal{F}_{V,W} :\ \mathsf{Hom}_{\mathbb{Z}_2^n\texttt{Vec}}(V, W) \longrightarrow \mathsf{Hom}_{\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}},\texttt{AMod})}(\mathcal{F}(V), \mathcal{F}(W))\,,$$

is a bijection.

This result is the $\mathbb{Z}_2^n$-counterpart of the 1:1 correspondence mentioned above.

Proof. We show first that the map $\mathcal{F}_{V,W}$ is injective. Let $\phi, \psi : V \to W$ be two degree preserving linear maps, and assume that $\mathcal{F}(\phi) = \phi_- = \psi_- = \mathcal{F}(\psi)$, so that, for any $\Lambda \in \mathbb{Z}_2^n\texttt{Pts}^{\rm op}$ and any $\lambda \otimes v \in V(\Lambda)$, we have

$$\lambda \otimes \phi(v) = \phi_\Lambda(\lambda \otimes v) = \psi_\Lambda(\lambda \otimes v) = \lambda \otimes \psi(v) \ . \tag{3.2.2}$$

Notice now that

$$V(\Lambda) = (\Lambda \otimes V)_0 = \bigoplus_{i=0}^{N} \Lambda_{\gamma_i} \otimes V_{\gamma_i}$$

and let Λ be the Grassmann algebra

$$\Lambda_1 := \mathbb{R}[[\theta_1, \ldots, \theta_N]] \tag{3.2.3}$$

that has exactly one generator θ_j in each non-zero degree $\gamma_j \in \mathbb{Z}_2^n$ ($N = 2^n - 1$). For any $v_j \in V_{\gamma_j}$, Equation (3.2.2) implies that $\theta_j \otimes \phi(v_j) = \theta_j \otimes \psi(v_j)$, so that ϕ and ψ coincide on V_{γ_j}, for all $j \in \{1, \ldots, N\}$. For $v_0 \in V_0 := V_{\gamma_0}$ and $\lambda = 1$, the same equation shows that ϕ and ψ coincide also on V_0.

To prove surjectivity, we consider an arbitrary natural transformation $\Phi_- : V(-) \to W(-)$ and will define a degree 0 linear map $\phi : V \to W$, such that $\mathcal{F}(\phi) = \phi_- = \Phi_-$, i.e., such that, for any $\Lambda \in \mathbb{Z}_2^n\texttt{Pts}^{\rm op}$, we have

$$\phi_\Lambda = \Phi_\Lambda \tag{3.2.4}$$

on $V(\Lambda)$. Since an element of $V(\Lambda)$ (uniquely) decomposes into a sum over $i \in \{0, \ldots, N\}$ of (not uniquely defined) finite sums of decomposable tensors $\lambda_i \otimes v_i$, with (not uniquely defined) factors λ_i and v_i of degree γ_i, it suffices to show that

$$\phi_\Lambda(\lambda_i \otimes v_i) = \Phi_\Lambda(\lambda_i \otimes v_i) \ , \tag{3.2.5}$$

for all $i \in \{0, \ldots, N\}$.

Further, it suffices to prove Condition (3.2.5) for Λ_1 (see (3.2.3)) and for the tensors $\theta_i \otimes v_i$ ($\theta_0 := 1$, $v_i \in V_{\gamma_i}, i \in \{0, \ldots, N\}$). The observation follows from naturality of Φ. Indeed, assume that (3.2.5) is satisfied for Λ_1 and the decomposable tensors just mentioned (Assumption $(\star)$). For any fixed $i \in \{1, \ldots, N\}$ (resp., $i = 0$), and for Λ, λ_i and v_i as above, let $\varphi^* : \Lambda_1 \to \Lambda$ be the $\mathbb{Z}_2^n$-algebra map defined by $\varphi^*(\theta_i) = \lambda_i$, $\varphi^*(\theta_j) = 0$ for $j \neq i, j \neq 0$, and $\varphi^*(\theta_0) = \varphi^*(1) = 1$ (resp., $\varphi^*(\theta_j) = 0$ for all $j \neq 0$, and $\varphi^*(\theta_0) = \varphi^*(1) = 1$). For $i \in \{1, \ldots, N\}$, when applying the naturality condition

$$\begin{array}{ccc} V(\Lambda_1) & \xrightarrow{\ \Phi_{\Lambda_1}\ } & W(\Lambda_1) \\ {\scriptstyle V(\varphi^*)}\downarrow & & \downarrow{\scriptstyle W(\varphi^*)} \\ V(\Lambda) & \xrightarrow[\ \Phi_\Lambda\ ]{} & W(\Lambda) \end{array}$$

to $\theta_i \otimes v_i$, we get clockwise

$$W(\varphi^*)(\Phi_{\Lambda_1}(\theta_i \otimes v_i)) = W(\varphi^*)(\theta_i \otimes \phi(v_i)) = \varphi^*(\theta_i) \otimes \phi(v_i) = \lambda_i \otimes \phi(v_i) = \phi_\Lambda(\lambda_i \otimes v_i) \ ,$$

in view of $(\star)$, whereas anticlockwise we obtain

$$\Phi_\Lambda(V(\varphi^*)(\theta_i \otimes v_i)) = \Phi_\Lambda(\lambda_i \otimes v_i) \ .$$

Hence, Condition (3.2.5) holds for $i \in \{1, \ldots, N\}$. For $i = 0$, the above naturality condition yields $1 \otimes \phi(v_0) = \Phi_\Lambda(1 \otimes v_0)$, when applied to $1 \otimes v_0$. In view of the Λ_0-linearity of the Λ-components of the natural transformations considered, we get now

$$\phi_\Lambda(\lambda_0 \otimes v_0) = \lambda_0\, \phi_\Lambda(1 \otimes v_0) = \lambda_0(1 \otimes \phi(v_0)) = \Phi_\Lambda(\lambda_0 \otimes v_0)\ .$$

Finally, Condition (3.2.5) holds for an arbitrary Λ, if it holds for Λ_1.

Surjectivity now reduces to constructing a $\mathbb{Z}_2^n\mathtt{Vec}$-morphism $\phi : V \to W$ that satisfies (3.2.5) for Λ_1 and decomposable tensors of the type $\theta_i \otimes v_i$ $(i \in \{0, \ldots, N\})$.

We first build $\phi(v_j) \in W_{\gamma_j}$ linearly in $v_j \in V_{\gamma_j}$ for an arbitrarily fixed $j \in \{0, \ldots, N\}$. We set again $\theta_0 = 1 \in \Lambda_{1,0}$. Since $\Phi_{\Lambda_1}(\theta_j \otimes v_j) \in (\Lambda_1 \otimes W)_0$, it reads

$$\Phi_{\Lambda_1}(\theta_j \otimes v_j) = \sum_{i=0}^{N} \sum_{k=1}^{M_i} \lambda_i^k \otimes w_i^k\ ,$$

where $M_i \in \mathbb{N}$, $\lambda_i^k \in \Lambda_{1,\gamma_i}$ and $w_i^k \in W_{\gamma_i}$. When setting

$$\mathcal{A}_i = \{\alpha \in \mathbb{N}^{\times N} : \sum_{\ell=1}^{N} \alpha_\ell \gamma_\ell = \gamma_i\}$$

and

$$\lambda_i^k = \sum_{\alpha \in \mathcal{A}_i} r_{\alpha,i}^k\, \theta^\alpha \quad (r_{\alpha,i}^k \in \mathbb{R})\ ,$$

where we used the standard multi-index notation, we get

$$\Phi_{\Lambda_1}(\theta_j \otimes v_j) = \sum_{i=0}^{N} \sum_{\alpha \in \mathcal{A}_i} \theta^\alpha \otimes \big(\sum_{k=1}^{M_i} r_{\alpha,i}^k w_i^k\big) =: \sum_{i=0}^{N} \sum_{\alpha \in \mathcal{A}_i} \theta^\alpha \otimes w_{\alpha,i} \quad (w_{\alpha,i} \in W_{\gamma_i})\ .$$

Denoting the canonical basis of $\mathbb{R}^N$ by $(e_\ell)_\ell$ and decomposing the RHS with respect to the values of $|\alpha| = \alpha_1 + \ldots + \alpha_N \in \mathbb{N}$, we obtain

$$\Phi_{\Lambda_1}(\theta_j \otimes v_j) = w_{0,0} + \sum_{i=1}^{N} \theta_i \otimes w_{e_i,i} + \sum_{i=0}^{N} \sum_{\alpha \in \mathcal{A}_i : |\alpha| \geq 2} \theta^\alpha \otimes w_{\alpha,i}\ . \tag{3.2.6}$$

Let now $\varphi_{r_0}^*$ $(r_0 \in \mathbb{R}, r_0 > 0$ and $r_0 \neq 1)$ be the $\mathbb{Z}_2^n$-algebra endomorphism of Λ_1 that is defined by $\varphi_{r_0}^*(\theta_k) = r_0\theta_k$ if $k \neq 0$ and by $\varphi_{r_0}^*(\theta_0) = 1$. It follows from the naturality condition

$$\begin{array}{ccc} V(\Lambda_1) & \xrightarrow{\ \Phi_{\Lambda_1}\ } & W(\Lambda_1) \\ {\scriptstyle V(\varphi_{r_0}^*)}\downarrow & & \downarrow{\scriptstyle W(\varphi_{r_0}^*)} \\ V(\Lambda_1) & \xrightarrow[\ \Phi_{\Lambda_1}\ ]{} & W(\Lambda_1) \end{array}$$

that

$$W(\varphi_{r_0}^*)(\Phi_{\Lambda_1}(\theta_j \otimes v_j)) = w_{0,0} + \sum_{i=1}^{N} \theta_i \otimes (r_0 w_{e_i,i}) + \sum_{i=0}^{N} \sum_{\alpha \in \mathcal{A}_i : |\alpha| \geq 2} \theta^\alpha \otimes (r_0^{|\alpha|} w_{\alpha,i})$$

and

$$\Phi_{\Lambda_1}(V(\varphi_{r_0}^*)(\theta_j \otimes v_j)) = r_0^{1-\delta_{j0}} w_{0,0} + \sum_{i=1}^{N} \theta_i \otimes (r_0^{1-\delta_{j0}} w_{e_i,i}) + \sum_{i=0}^{N} \sum_{\alpha \in \mathcal{A}_i : |\alpha| \geq 2} \theta^\alpha \otimes (r_0^{1-\delta_{j0}} w_{\alpha,i}),$$

where δ_{j0} is the Kronecker symbol, coincide. As all the monomials in θ in the RHS-s of the two last equations are different, we get,

i. if $j \neq 0$: $w_{0,0} = 0$ and $w_{\alpha,i} = 0$, for all $i \in \{0, \dots, N\}$ and all $\alpha \in \mathcal{A}_i : |\alpha| \geq 2$, and,

ii. if $j = 0$: $w_{e_i,i} = 0$, for all $i \in \{1, \dots, N\}$, and $w_{\alpha,i} = 0$, for all $i \in \{0, \dots, N\}$ and all $\alpha \in \mathcal{A}_i : |\alpha| \geq 2$.

Equation (3.2.6) thus yields

$$\Phi_{\Lambda_1}(\theta_j \otimes v_j) = \sum_{i=1}^{N} \theta_i \otimes w_{e_i,i} \quad (j \neq 0) \qquad \text{and} \qquad \Phi_{\Lambda_1}(1 \otimes v_0) = w_{00} \ . \tag{3.2.7}$$

If $j \neq 0$, a new application of naturality, now for the $\mathbb{Z}_2^n$-algebra endomorphism $\varphi^*_{R_0}$ ($R_0 \in \mathbb{R}$, $R_0 \neq 1$) of Λ_1 that is defined by $\varphi^*_{R_0}(\theta_i) = R_0\theta_i$ ($i \neq 0, i \neq j$), $\varphi^*_{R_0}(\theta_j) = \theta_j$ and $\varphi^*_{R_0}(\theta_0) = 1$, leads to

$$\theta_j \otimes w_{e_j,j} + \sum_{i \neq j} \theta_i \otimes (R_0 w_{e_i,i}) = \theta_j \otimes w_{e_j,j} + \sum_{i \neq j} \theta_i \otimes w_{e_i,i} \ ,$$

so that

$$\Phi_{\Lambda_1}(\theta_j \otimes v_j) = \theta_j \otimes w_{e_j,j} \quad (j \neq 0) \ . \tag{3.2.8}$$

The vectors $w_{00} \in W_0$ (see (3.2.7)) and $w_{e_j,j} \in W_{\gamma_j}$ ($j \neq 0$) (see (3.2.8)) are well-defined and depend obviously linearly on v_0 and v_j, respectively. Hence, setting $\phi(v_0) = w_{00}$ and $\phi(v_j) = w_{e_j,j}$ ($j \neq 0$), we define a degree 0 linear map from V to W. Moreover, since (3.2.5) is clearly satisfied for Λ_1 and the $\theta_i \otimes v_i$ ($i \in \{0, \dots, N\}$), it is satisfied for any Λ, which completes the proof of surjectivity. □

Since

$$\mathcal{F} : \mathbb{Z}_2^n\texttt{Vec} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$$

is fully faithful, it is essentially injective, i.e., it is injective on objects up to isomorphism. It follows that $\mathbb{Z}_2^n\texttt{Vec}$ can be viewed as a full subcategory of the target category of $\mathcal{F}$.

The above considerations lead to the following definition.

Definition 3.2.3. A functor

$$\mathcal{V} \in \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$$

is said to be *representable*, if there exists $V \in \mathbb{Z}_2^n\texttt{Vec}$, such that $\mathcal{F}(V)$ is naturally isomorphic to $\mathcal{V}$.

As $\mathcal{F}$ is essentially injective, a *representing object* V, if it exists, is unique up to isomorphism. We therefore refer sometimes to V as 'the' representing object.

3.2.2 Cartesian $\mathbb{Z}_2^n$-graded vector spaces and Cartesian $\mathbb{Z}_2^n$-manifolds

In the literature, the space $\mathbb{R}^{p|\underline{q}}$ is viewed, either as the trivial $\mathbb{Z}_2^n$-manifold

$$\mathbb{R}^{p|\underline{q}} = (\mathbb{R}^p, C^\infty_{\mathbb{R}^p}[[\xi]])$$

with canonical $\mathbb{Z}_2^n$-graded formal parameters ξ, or as the Cartesian $\mathbb{Z}_2^n$-graded vector space

$$\mathbf{R}^{p|\underline{q}} = \mathbf{R}^p \oplus \bigoplus_{j=1}^{N} \mathbf{R}^{q_j} \ ,$$

where $\mathbf{R}^p$ (resp., $\mathbf{R}^{q_j}$) is the term of degree $\gamma_0 = 0 \in \mathbb{Z}_2^n$ (resp., $\gamma_j \in \mathbb{Z}_2^n$). Observe that we use the notation $\mathbf{R}^\bullet$ (resp., $\mathbb{R}^\bullet$), when $\mathbb{R}^\bullet$ is viewed as a vector space (resp., as a manifold). It can happen that we write $\mathbb{R}^\bullet$ for both, the vector space and the manifold, however, in these cases, the meaning is clear from the context. Further, we set $q_0 = p$, $\mathbf{q} = (q_0, q_1, \ldots, q_N)$, and $|\mathbf{q}| = \sum_i q_i$. When embedding $\mathbf{R}^{q_i}$ $(i \in \{0, \ldots, N\})$ into $\mathbf{R}^{p|\underline{q}}$, we identify each vector of the canonical basis of $\mathbf{R}^{q_i}$ with the corresponding vector of the canonical basis of $\mathbf{R}^{|\mathbf{q}|}$. We denote this basis by

$$(e^i_k)_{i,k} \quad (i \in I = \{0, \ldots, N\}, k \in K_i = \{1, \ldots, q_i\})$$

and assign of course the degree γ_i to every vector e^i_k. We can now write

$$\mathbf{R}^{p|\underline{q}} = \bigoplus_{i=0}^{N} \mathbf{R}^{q_i} = \bigoplus_{(i,k)\in I\times K_i} \mathbf{R}\, e^i_k \ .$$

The dual space of $\mathbf{R}^{p|\underline{q}}$ is defined by

$$(\mathbf{R}^{p|\underline{q}})^\vee = \underline{\mathrm{Hom}}(\mathbf{R}^{p|\underline{q}}, \mathbf{R}) = \bigoplus_{i=0}^{N} \underline{\mathrm{Hom}}_{\gamma_i}(\mathbf{R}^{p|\underline{q}}, \mathbf{R}) \ ,$$

where $\underline{\mathrm{Hom}}$ is the internal Hom of $\mathbb{Z}_2^n\mathtt{Vec}$, i.e., the $\mathbb{Z}_2^n$-graded vector space of all linear maps, and where $\underline{\mathrm{Hom}}_{\gamma_i}$ is the vector space of all degree γ_i linear maps. We sometimes write $\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\mathtt{Vec}}$ instead of $\underline{\mathrm{Hom}}$. The dual basis of $(e^i_k)_{i,k}$ is defined as usual by

$$\varepsilon^k_i(e^j_\ell) = \delta^j_i \delta^k_\ell \ ,$$

so that ε^k_i is a linear map of degree γ_i and

$$(\mathbf{R}^{p|\underline{q}})^\vee = \bigoplus_{(i,k)\in I\times K_i} \mathbf{R}\, \varepsilon^k_i \ .$$

Let us finally mention that any $\mathbb{Z}_2^n$-vector $x \in \mathbf{R}^{p|\underline{q}}$ reads $x = \sum_{j,\ell} x^\ell_j e^j_\ell$ and that

$$\varepsilon^k_i(x) = x^k_i \ , \tag{3.2.9}$$

as usual.

Notice now that if M is a smooth m-dimensional real manifold and (U, φ) is a chart of M, the coordinate map φ sends any point $x \in M$ to $\varphi(x) = (x^1, \ldots, x^m) \in \mathbb{R}^m$, so that

$$\varphi^i(x) = x^i \ . \tag{3.2.10}$$

Hence, what we refer to as coordinate function $x^i \in C^\infty(U)$ is actually the function φ^i. Equations (3.2.9) and (3.2.10) suggest to associate to any $\mathbb{Z}_2^n$-graded vector space $\mathbf{R}^{p|\underline{q}}$ a $\mathbb{Z}_2^n$-manifold $\mathbb{R}^{p|\underline{q}}$ with coordinate functions ε^k_i. In other words, the associated $p|\underline{q}$-dimensional $\mathbb{Z}_2^n$-manifold will be the locally $\mathbb{Z}_2^n$-ringed space

$$\mathbb{R}^{p|\underline{q}} = (\mathbb{R}^p, \mathcal{O}_{\mathbb{R}^{p|\underline{q}}}) = (\mathbb{R}^p, C^\infty_{\mathbb{R}^p}[[\varepsilon^1_1, \ldots, \varepsilon^{q_N}_N]]) \ ,$$

where $C^\infty_{\mathbb{R}^p}$ is the standard function sheaf of $\mathbb{R}^p$, where the degree γ_j linear maps $\varepsilon^1_j, \ldots, \varepsilon^{q_j}_j$ $(j \in \{1, \ldots, N\})$ are interpreted as coordinate functions or formal parameters of degree γ_j, and where the degree 0 linear maps $\varepsilon^1_0, \ldots, \varepsilon^p_0$ are viewed as coordinates in $\mathbb{R}^p$. We often set

$$\xi^\ell_j := \varepsilon^\ell_j \quad (j \neq 0) \quad \text{and} \quad x^\ell := \varepsilon^\ell_0 \ . \tag{3.2.11}$$

Remark 3.2.4. In the following, we denote the coordinates of $\mathbb{R}^{p|\underline{q}}$ by

$$(x^\ell, \xi^\ell_j) = (x^a, \xi^A) = (u^{\mathfrak{a}}) \ ,$$

if we wish to make a distinction between the coordinates of degree $0, \gamma_1, \ldots, \gamma_N$, if we distinguish between zero degree coordinates and non-zero degree ones, or if we consider all coordinates together.

We refer to the category of $\mathbb{Z}_2^n$-graded vector spaces $\mathbf{R}^{p|\underline{q}}$ $(p, q_1, \ldots, q_N \in \mathbb{N})$ and degree 0 linear maps, as the category $\mathbb{Z}_2^n\texttt{CarVec}$ of Cartesian $\mathbb{Z}_2^n$-vector spaces. As just mentioned, the interpretation of the dual basis as coordinates leads naturally to a map

$$\mathcal{M} : \mathbb{Z}_2^n\texttt{CarVec} \ni \mathbf{R}^{p|\underline{q}} \mapsto \mathbb{R}^{p|\underline{q}} \in \mathbb{Z}_2^n\texttt{Man} ,$$

where $\mathbb{Z}_2^n\texttt{Man}$ is the category of $\mathbb{Z}_2^n$-manifolds and corresponding morphisms. This map can easily be extended to a functor. Indeed, if $\mathbf{L} : \mathbf{R}^{p|\underline{q}} \to \mathbf{R}^{r|\underline{s}}$ is a morphism in $\mathbb{Z}_2^n\texttt{CarVec}$ (it is canonically represented by a block diagonal matrix $\mathbf{L} \in \mathrm{gl}(r|\underline{s} \times p|\underline{q}, \mathbf{R})$), its dual ($\mathbb{Z}_2^n$-transpose) $\mathbf{L}^\vee : (\mathbf{R}^{r|\underline{s}})^\vee \to (\mathbf{R}^{p|\underline{q}})^\vee$ (which is represented by the standard transpose ${}^t\mathbf{L} \in \mathrm{gl}(p|\underline{q} \times r|\underline{s}, \mathbf{R})$) is also a degree 0 linear map. If we set

$$\mathbf{L} = \left(\mathbf{L}_{ik}^{\ell i}\right) ,$$

where $i \in I$, $\ell \in \{1, \ldots, s_i\}$ label the row and $i \in I$, $k \in \{1, \ldots, q_i\}$ label the column, we get

$$\mathbf{L}^\vee(\varepsilon_i'^{\ell}) = \sum_{k=1}^{q_i} \mathbf{L}_{ik}^{\ell i} \varepsilon_i^k ,$$

where $(\varepsilon_i'^{\ell})_{i,\ell}$ is the basis of $(\mathbf{R}^{r|\underline{s}})^\vee$. When using notation (3.2.11), we obtain

$$L^*(x'^{\ell}) := \mathbf{L}^\vee(x'^{\ell}) = \sum_{k=1}^{p} \mathbf{L}_{0k}^{\ell 0}\, x^k \in \mathcal{O}^0_{\mathbb{R}^{p|\underline{q}}}(\mathbb{R}^p) \quad (\ell \in \{1, \ldots, r\}) \tag{3.2.12}$$

and

$$L^*(\xi_j'^{\ell}) := \mathbf{L}^\vee(\xi_j'^{\ell}) = \sum_{k=1}^{q_j} \mathbf{L}_{jk}^{\ell j}\, \xi_j^k \in \mathcal{O}^{\gamma_j}_{\mathbb{R}^{p|\underline{q}}}(\mathbb{R}^p) \quad (j \neq 0, \ell \in \{1, \ldots, s_j\}) . \tag{3.2.13}$$

These pullbacks define a $\mathbb{Z}_2^n$-morphism $L : \mathbb{R}^{p|\underline{q}} \to \mathbb{R}^{r|\underline{s}}$. This is the searched $\mathbb{Z}_2^n$-morphism $\mathcal{M}(\mathbf{L}) : \mathcal{M}(\mathbf{R}^{p|\underline{q}}) \to \mathcal{M}(\mathbf{R}^{r|\underline{s}})$. Since $\mathcal{M}(\mathbf{L})$ is defined interpreting the standard transpose ${}^t\mathbf{L}$ as pullback $(\mathcal{M}(\mathbf{L}))^*$ of coordinates, we have

$$(\mathcal{M}(\mathbf{M} \circ \mathbf{L}))^* \simeq {}^t\mathbf{L} \circ {}^t\mathbf{M} \simeq (\mathcal{M}(\mathbf{L}))^* \circ (\mathcal{M}(\mathbf{M}))^* = (\mathcal{M}(\mathbf{M}) \circ \mathcal{M}(\mathbf{L}))^* ,$$

so that $\mathcal{M}$ respects composition. Further, it obviously respects identities. Hence, we defined a functor $\mathcal{M}$.

The pullbacks (3.2.12) and (3.2.13) are actually linear homogeneous $\mathbb{Z}_2^n$-functions, i.e., homogeneous $\mathbb{Z}_2^n$-functions in

$$\mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{p|\underline{q}}}(\mathbb{R}^p) := \{\sum_{k=1}^{p} r_k\, x^k + \sum_{j=1}^{N} \sum_{k=1}^{q_j} r^j_k\, \xi_j^k : r_k, r^j_k \in \mathbb{R}\} = (\mathbf{R}^{p|\underline{q}})^\vee \subset \mathcal{O}_{\mathbb{R}^{p|\underline{q}}}(\mathbb{R}^p) , \tag{3.2.14}$$

where the last equality is obvious because of Equation (3.2.11). Hence, the functor $\mathcal{M}$ is valued in the subcategory $\mathbb{Z}_2^n\texttt{CarMan} \subset \mathbb{Z}_2^n\texttt{Man}$ of Cartesian $\mathbb{Z}_2^n$-manifolds $\mathbb{R}^{p|\underline{q}}$ $(p, q_1, \ldots, q_N \in \mathbb{N})$ and $\mathbb{Z}_2^n$-morphisms whose coordinate pullbacks are global linear functions of the source manifold that have the appropriate degree:

$$\mathcal{M} : \mathbb{Z}_2^n\texttt{CarVec} \to \mathbb{Z}_2^n\texttt{CarMan} .$$

The inverse 'vectorification functor' $\mathcal{V}$ of this 'manifoldification functor' $\mathcal{M}$ is readily defined: to a Cartesian $\mathbb{Z}_2^n$-manifold $\mathbb{R}^{p|\underline{q}}$ we associate the Cartesian $\mathbb{Z}_2^n$-vector space $\mathbf{R}^{p|\underline{q}}$, and to a linear $\mathbb{Z}_2^n$-morphism we associate the degree 0 linear map that is defined by the transpose of the block diagonal matrix coming from the morphism's linear pullbacks. It is obvious that $\mathcal{V} \circ \mathcal{M} = \mathcal{M} \circ \mathcal{V} = \mathrm{id}$.

Proposition 3.2.5. *We have an isomorphism of categories*

$$\mathcal{M} : \mathbb{Z}_2^n\texttt{CarVec} \rightleftarrows \mathbb{Z}_2^n\texttt{CarMan} : \mathcal{V} \tag{3.2.15}$$

between the full subcategory $\mathbb{Z}_2^n\texttt{CarVec} \subset \mathbb{Z}_2^n\texttt{Vec}$ *of Cartesian* $\mathbb{Z}_2^n$*-vector spaces* $\mathbf{R}^{p|\underline{q}}$ *and the subcategory* $\mathbb{Z}_2^n\texttt{Car Man} \subset \mathbb{Z}_2^n\texttt{Man}$ *of Cartesian* $\mathbb{Z}_2^n$*-manifolds* $\mathbb{R}^{p|\underline{q}}$ *and* $\mathbb{Z}_2^n$*-morphisms with linear coordinate pullbacks.*

Remark 3.2.6. Let us stress that the $\mathbb{Z}_2^n$-vector space of linear $\mathbb{Z}_2^n$-functions

$$(\mathbf{R}^{p|\underline{q}})^\vee \simeq \mathcal{O}^{\text{lin}}_{\mathbb{R}^{p|\underline{q}}}(\mathbb{R}^p) \subset \mathcal{O}_{\mathbb{R}^{p|\underline{q}}}(\mathbb{R}^p)$$

is of course not an algebra. In the case $p = 0$, we get

$$\mathcal{O}^{\text{lin}}_{\mathbb{R}^{0|\underline{q}}}(\{\star\}) = \Lambda^{\text{lin}} \subset \mathcal{O}_{\mathbb{R}^{0|\underline{q}}}(\{\star\}) = \Lambda$$

where $\{\star\}$ denotes the 0-dimensional base manifold $\mathbb{R}^0$ of the $\mathbb{Z}_2^n$-point $\mathbb{R}^{0|\underline{q}}$, where $\Lambda = \mathbb{R}[[\theta_1^1, \ldots, \theta_N^{q_N}]]$ is the $\mathbb{Z}_2^n$-Grassmann algebra that corresponds to $\mathbb{R}^{0|\underline{q}}$, and where Λ^{lin} is the $\mathbb{Z}_2^n$-vector space of homogeneous degree 1 polynomials in the $\theta_1^1, \ldots, \theta_N^{q_N}$ (with vanishing term Λ_0^{lin} of $\mathbb{Z}_2^n$-degree zero).

We close this subsection with some observations regarding the functor of points.

The Yoneda functor of points of the category $\mathbb{Z}_2^n\texttt{Man}$ is the fully faithful embedding

$$\mathcal{Y} : \mathbb{Z}_2^n\texttt{Man} \to \texttt{Fun}(\mathbb{Z}_2^n\texttt{Man}^{\text{op}}, \texttt{Set}) ,$$

In [13], we showed that $\mathcal{Y}$ remains fully faithful for appropriate restrictions of the source and target of the functor category, as well as of the *resulting* functor category. More precisely, we proved that the functor

$$\mathcal{S} : \mathbb{Z}_2^n\texttt{Man} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{A(N)FM}) \tag{3.2.16}$$

is fully faithful. The category $\texttt{A(N)FM}$ is the category of (nuclear) Fréchet manifolds over a (nuclear) Fréchet algebra, and the functor category is the category of those functors that send a $\mathbb{Z}_2^n$-Grassmann algebra Λ to a (nuclear) Fréchet Λ_0-manifold, and of those natural transformations that have Λ_0-smooth Λ-components. For any $M \in \mathbb{Z}_2^n\texttt{Man}$ and any $\mathbb{R}^{0|\underline{m}} \simeq \Lambda \in \mathbb{Z}_2^n\texttt{Pts}^{\text{op}}$, we have

$$M(\Lambda) := \mathcal{S}(M)(\Lambda) = \mathcal{Y}(M)(\Lambda) = \text{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(\mathbb{R}^{0|\underline{m}}, M) .$$

On the other hand, the Yoneda functor of points of the category $\mathbb{Z}_2^n\texttt{CarVec}$ is the fully faithful embedding

$$\underline{\bullet} : \mathbb{Z}_2^n\texttt{CarVec} \ni \mathbf{R}^{p|\underline{q}} \mapsto \underline{\mathbf{R}}^{p|\underline{q}} := \text{Hom}_{\mathbb{Z}_2^n\texttt{Vec}}(-, \mathbf{R}^{p|\underline{q}}) \in \texttt{Fun}(\mathbb{Z}_2^n\texttt{CarVec}^{\text{op}}, \texttt{Set}) .$$

The value of this functor on $\mathbf{R}^{0|\underline{m}} \simeq \mathbb{R}^{0|\underline{m}} \simeq \Lambda$, is the *subset*

$$\underline{\mathbf{R}}^{p|\underline{q}}(\Lambda) = \text{Hom}_{\mathbb{Z}_2^n\texttt{Vec}}(\mathbf{R}^{0|\underline{m}}, \mathbf{R}^{p|\underline{q}}) \simeq \text{Hom}_{\mathbb{Z}_2^n\texttt{CarMan}}(\mathbb{R}^{0|\underline{m}}, \mathbb{R}^{p|\underline{q}}) \subset \tag{3.2.17}$$

$$\mathbb{R}^{p|\underline{q}}(\Lambda) = \mathcal{S}(\mathbb{R}^{p|\underline{q}})(\Lambda) = \text{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(\mathbb{R}^{0|\underline{m}}, \mathbb{R}^{p|\underline{q}}) \simeq$$

$$\bigoplus_{i=0}^{N}\bigoplus_{k=1}^{q_i} \mathcal{O}_{\mathbb{R}^{0|\underline{m}},\gamma_i}(\{\star\}) = \bigoplus_{i=0}^{N}\bigoplus_{k=1}^{q_i} \Lambda_{\gamma_i} = \bigoplus_{i=0}^{N}\bigoplus_{k=1}^{q_i} \Lambda_{\gamma_i} \otimes \mathbf{R}e_k^i = \bigoplus_{i=0}^{N} \Lambda_{\gamma_i} \otimes \bigoplus_{k=1}^{q_i} \mathbf{R}e_k^i =$$

$$(\Lambda \otimes \mathbf{R}^{p|\underline{q}})_0 = \mathcal{F}(\mathbf{R}^{p|\underline{q}})(\Lambda) = \mathbf{R}^{p|\underline{q}}(\Lambda) \in \Lambda_0\texttt{Mod} \;.$$

More precisely,

$$\underline{\mathbf{R}}^{p|\underline{q}}(\Lambda) = \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Vec}}(\mathbf{R}^{0|\underline{m}}, \mathbf{R}^{p|\underline{q}}) \simeq \bigoplus_{i=0}^{N}\bigoplus_{k=1}^{q_i} \mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{0|\underline{m}},\gamma_i}(\{\star\}) = (\Lambda^{\mathrm{lin}} \otimes \mathbf{R}^{p|\underline{q}})_0 \in \texttt{Set} \;. \qquad (3.2.18)$$

Remark that, if we denote the coordinates of $\mathbb{R}^{p|\underline{q}}$ compactly by $(u^{\mathfrak{a}})$, the bijection in Equation (3.2.18) sends a degree 0 linear map $\mathbf{L}$ to the linear pullbacks $L^*(u^{\mathfrak{a}})$ of the corresponding $\mathbb{Z}_2^n$-morphism $L = \mathcal{M}(\mathbf{L})$.

Remark 3.2.7. If we restrict the functor $\underline{\mathbf{R}}^{p|\underline{q}}$ (resp., the functor $\mathcal{F}(\mathbf{R}^{p|\underline{q}})$) from $\mathbb{Z}_2^n\texttt{CarVec}^{\mathrm{op}} \simeq \mathbb{Z}_2^n\texttt{CarMan}^{\mathrm{op}}$ (resp., from $\mathbb{Z}_2^n\texttt{Pts}^{\mathrm{op}}$) to the joint subcategory $\mathbb{Z}_2^n\texttt{CarPts}^{\mathrm{op}}$ of $\mathbb{Z}_2^n$-points and $\mathbb{Z}_2^n$-morphisms with linear coordinate pullbacks, the restricted Hom functor $\underline{\mathbf{R}}^{p|\underline{q}}$ is actually a subfunctor of the restricted tensor product functor $\mathcal{F}(\mathbf{R}^{p|\underline{q}})$. This observation clarifies the relationship between the fully faithful 'functor of points' $\mathcal{F}(\bullet)(-) = \bullet(-)$ of the full subcategory $\mathbb{Z}_2^n\texttt{CarVec} \subset \mathbb{Z}_2^n\texttt{Vec}$ and its standard fully faithful Yoneda functor of points $\underline{\bullet}(-)$.

Indeed, we observed already that the values of the Hom functor on $\mathbb{Z}_2^n$-points are subsets of the values of the tensor product functor. Further, on morphisms, the values of $\mathrm{Hom}(-, \mathbf{R}^{p|\underline{q}})$ are restrictions of the values of $(- \otimes \mathbf{R}^{p|\underline{q}})_0$. Indeed, if

$$\mathrm{Hom}_{\mathbb{Z}_2^n\texttt{CarPts}}(\mathbb{R}^{0|\underline{n}}, \mathbb{R}^{0|\underline{m}}) \ni L \simeq \mathcal{V}(L) = \mathbf{L} \in \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Vec}}(\mathbf{R}^{0|\underline{n}}, \mathbf{R}^{0|\underline{m}}) \;,$$

the morphisms $\underline{\mathbf{R}}^{p|\underline{q}}(\mathbf{L})$ and $\mathcal{F}(\mathbf{R}^{p|\underline{q}})(L)$ are defined on $\underline{\mathbf{R}}^{p|\underline{q}}(\Lambda)$ and its *supset* $\mathcal{F}(\mathbf{R}^{p|\underline{q}})(\Lambda)$, respectively. When interpreting an element $\mathbf{K}$ of the first as an element of the second, we use the identifications

$$\mathbf{K} \simeq K \simeq (K^*(u^{\mathfrak{a}}))_{\mathfrak{a}} \in \bigoplus_{i=0}^{N}\bigoplus_{k=1}^{q_i} \Lambda_{\gamma_i} \;.$$

Similar identifications are of course required when Λ is replaced by Λ'. We thus get

$$\underline{\mathbf{R}}^{p|\underline{q}}(\mathbf{L})(\mathbf{K}) = \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Vec}}(\mathbf{L}, \mathbf{R}^{p|\underline{q}})(\mathbf{K}) = \mathbf{K} \circ \mathbf{L} \simeq K \circ L \simeq (L^*(K^*(u^{\mathfrak{a}})))_{\mathfrak{a}} \;.$$

On the other hand, we have

$$\mathcal{F}(\mathbf{R}^{p|\underline{q}})(L^*)((K^*(u^{\mathfrak{a}}))_{\mathfrak{a}}) = (L^*(K^*(u^{\mathfrak{a}})))_{\mathfrak{a}} \;,$$

since

$$(L^* \otimes \mathbb{1}_{\mathbf{R}^{p|\underline{q}}})_0 \simeq \bigoplus_{i=0}^{N}\bigoplus_{k=1}^{q_i} L^* \;,$$

when read through the isomorphism

$$\bigoplus_{i=0}^{N} \Lambda_{\gamma_i} \otimes \bigoplus_{k=1}^{q_i} \mathbf{R}\, e^i_k \simeq \bigoplus_{i=0}^{N}\bigoplus_{k=1}^{q_i} \Lambda_{\gamma_i} \;.$$

This completes the proof of the subfunctor-statement.

3.2.3 Finite dimensional $\mathbb{Z}_2^n$-graded vector spaces and linear $\mathbb{Z}_2^n$-manifolds

In this subsection, we extend Equivalence (3.2.15) in a coordinate-free way.

Finite dimensional $\mathbb{Z}_2^n$-graded vector spaces

We focus on the full subcategory $\mathbb{Z}_2^n\texttt{FinVec} \subset \mathbb{Z}_2^n\texttt{Vec}$ of finite dimensional $\mathbb{Z}_2^n$-graded vector spaces, i.e., of $\mathbb{Z}_2^n$-vector spaces V of finite dimension

$$p|\underline{q} \quad (p \in \mathbb{N}, \underline{q} = (q_1, \ldots, q_N) \in \mathbb{N}^{\times N})\ .$$

Clearly

$$\mathbb{Z}_2^n\texttt{CarVec} \subset \mathbb{Z}_2^n\texttt{FinVec}$$

is a full subcategory.

Above, we already used the canonical basis of $\mathbf{R}^{p|\underline{q}}$, i.e., the basis

$$e_k^i = {}^t(0 \ldots 0; \ldots; 0 \ldots 1 \ldots 0; \ldots; 0 \ldots 0)\ ,$$

where 1 sits in position k of block i. If

$$(b_k^i)_{i,k} \quad (i \in I = \{0, \ldots, N\}, k \in K_i = \{1, \ldots, q_i\}, \deg(b_k^i) = \gamma_i \in \mathbb{Z}_2^n)$$

is a basis of V, the degree respecting linear map

$$b : V \ni v = \sum_{i,k} v_i^k b_k^i \mapsto \sum_{i,k} v_i^k e_k^i = {}^t(v_0^1, \ldots, v_N^{q_N}) =: v^I \in \mathbf{R}^{p|\underline{q}} \tag{3.2.19}$$

maps a basis to a basis and is thus an isomorphism of $\mathbb{Z}_2^n$-vector spaces.

We already discussed extensively the functor of points $\mathcal{F} = \mathcal{F}(\bullet)(-) = \bullet(-)$ of $\mathbb{Z}_2^n\texttt{Vec}$. Since $\mathbb{Z}_2^n\texttt{FinVec}$ is a full subcategory of $\mathbb{Z}_2^n\texttt{Vec}$, the functor $\mathcal{F}$ remains fully faithful when restricted to $\mathbb{Z}_2^n\texttt{FinVec}$:

Proposition 3.2.8. *The functor of points* $\mathcal{F} : \mathbb{Z}_2^n\texttt{FinVec} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$ *of the category* $\mathbb{Z}_2^n\texttt{FinVec}$ *is fully faithful.*

Remark 3.2.9. Later on, we consider linear $\mathbb{Z}_2^n$-manifolds and denote them sometimes using the same letter V as for $\mathbb{Z}_2^n$-vector spaces. We often disambiguate the concept considered by writing $\mathbf{V}$ in the vector space case.

Linear $\mathbb{Z}_2^n$-manifolds

In this subsection, we investigate the category of linear $\mathbb{Z}_2^n$-manifolds, linear $\mathbb{Z}_2^n$-functions of its objects, as well as its functor of points.

2.3.2.1 *Linear $\mathbb{Z}_2^n$-manifolds and their morphisms.* A $\mathbb{Z}_2^n$-manifold of dimension $p|\underline{q}$ is a locally $\mathbb{Z}_2^n$-ringed space $M := (|M|, \mathcal{O}_M)$ that is locally isomorphic to $\mathbb{R}^{p|\underline{q}}$.

Definition 3.2.10. A *linear $\mathbb{Z}_2^n$-manifold* of dimension $p|\underline{q}$ is a locally $\mathbb{Z}_2^n$-ringed space $\mathsf{L} = (|\mathsf{L}|, \mathcal{O}_{\mathsf{L}})$ that is globally isomorphic to $\mathbb{R}^{p|\underline{q}}$, i.e., it is a $\mathbb{Z}_2^n$-manifold such that there exists a $\mathbb{Z}_2^n$-diffeomorphism

$$\mathrm{h} : \mathsf{L} \longrightarrow \mathbb{R}^{p|\underline{q}}\ .$$

The diffeomorphism h is referred to as a *linear coordinate map* or a *linear one-chart-atlas.*

We now mimic Classical Differential Geometry and say that two linear one-chart-atlases are linearly compatible, if their union is a 'linear two-chart-atlas'. In other words:

Definition 3.2.11. Two linear coordinate maps $\mathrm{h}_1, \mathrm{h}_2 : \mathsf{L} \to \mathbb{R}^{p|\underline{q}}$ are said to be *linearly compatible*, if the $\mathbb{Z}_2^n$-morphisms

$$h_2 \circ h_1^{-1},\, \mathrm{h}_1 \circ \mathrm{h}_2^{-1} : \mathbb{R}^{p|\underline{q}} \longrightarrow \mathbb{R}^{p|\underline{q}}$$

have linear coordinate pullbacks, i.e., if they are $\mathbb{Z}_2^n\texttt{CarMan}$-morphisms.

Linear compatibility is an equivalence relation on linear one-chart-atlases. There is a 1:1 correspondence between equivalence classes of linear one-chart-atlases and maximal linear atlases, i.e., the unions of all linear one-chart-atlases of an equivalence class. For simplicity, we refer to a maximal linear atlas as a *linear atlas*.

Just as a classical smooth manifold is a set that admits an atlas, or, better, a set endowed with an equivalence class of atlases, a linear $\mathbb{Z}_2^n$-manifold is a locally $\mathbb{Z}_2^n$-ringed space L equipped with a linear atlas $(\mathsf{L}, \mathrm{h}_\alpha)_\alpha$.

We continue working in analogy with Differential Geometry and define a linear $\mathbb{Z}_2^n$-morphism between linear $\mathbb{Z}_2^n$-manifolds as a locally $\mathbb{Z}_2^n$-ringed space morphism, or, equivalently, a $\mathbb{Z}_2^n$-morphism, with linear coordinate form:

Definition 3.2.12. Let L and L' be two linear $\mathbb{Z}_2^n$-manifolds of dimension $p|\underline{q}$ and $r|\underline{s}$, respectively. A $\mathbb{Z}_2^n$-morphism $\phi : \mathsf{L} \to \mathsf{L}'$ is *linear*, if there exist linear coordinate maps

$$\mathrm{h} : \mathsf{L} \to \mathbb{R}^{p|\underline{q}} \quad \text{and} \quad \mathrm{k} : \mathsf{L}' \to \mathbb{R}^{r|\underline{s}}$$

in the linear atlases of L and L', such that the $\mathbb{Z}_2^n$-morphism

$$\mathrm{k} \circ \phi \circ \mathrm{h}^{-1} : \mathbb{R}^{p|\underline{q}} \to \mathbb{R}^{r|\mathbf{s}}$$

has linear coordinate pullbacks.

It follows that any linear coordinate map h of the linear atlas of a linear $\mathbb{Z}_2^n$-manifold L, *is* a *linear* $\mathbb{Z}_2^n$-morphism between the linear $\mathbb{Z}_2^n$-manifolds L and $\mathbb{R}^{p|\underline{q}}$. This justifies the name 'linear coordinate map'. Further, the inverse h^{-1} of h is a linear $\mathbb{Z}_2^n$-morphism.

Proposition 3.2.13. *If $\phi : \mathsf{L} \to \mathsf{L}'$ is a linear $\mathbb{Z}_2^n$-morphism, then, for any linear coordinate maps $(\mathsf{L}, \mathrm{h}')$ and $(\mathsf{L}', \mathrm{k}')$ of the linear atlases of L and L', respectively, the $\mathbb{Z}_2^n$-morphism $\mathrm{k}' \circ \phi \circ \mathrm{h}'^{-1}$ has linear coordinate pullbacks.*

Proof. We use the notations of Definition 3.2.12 and Proposition 3.2.13. Since

$$\mathrm{k}' \circ \phi \circ \mathrm{h}'^{-1} = (\mathrm{k}' \circ \mathrm{k}^{-1}) \circ (\mathrm{k} \circ \phi \circ \mathrm{h}^{-1}) \circ (\mathrm{h} \circ \mathrm{h}'^{-1})$$

and each parenthesis of the RHS has linear pullbacks, their composite has linear pullbacks as well. □

Proposition 3.2.14. *Linear $\mathbb{Z}_2^n$-manifolds and linear $\mathbb{Z}_2^n$-morphisms form a subcategory $\mathbb{Z}_2^n\texttt{LinMan} \subset \mathbb{Z}_2^n\texttt{Man}$ of the category of $\mathbb{Z}_2^n$-manifolds. Further, Cartesian $\mathbb{Z}_2^n$-manifolds and $\mathbb{Z}_2^n$-morphisms with linear coordinate pullbacks form a full subcategory $\mathbb{Z}_2^n\texttt{CarMan} \subset \mathbb{Z}_2^n\texttt{LinMan}$.*

Proof. If $\phi : \mathsf{L} \to \mathsf{L}'$ and $\psi : \mathsf{L}' \to \mathsf{L}''$ are linear $\mathbb{Z}_2^n$-morphisms, the composite $\mathbb{Z}_2^n$-morphism is linear as well. Indeed, if $\mathrm{k} \circ \phi \circ \mathrm{h}^{-1}$ and $\mathrm{q} \circ \psi \circ \mathrm{p}^{-1}$ have linear pullbacks, then $\mathrm{q} \circ \psi \circ \mathrm{k}^{-1}$ has linear pullbacks and so has $\mathrm{q} \circ (\psi \circ \phi) \circ \mathrm{h}^{-1}$. Further, for any linear $\mathbb{Z}_2^n$-manifold L, the $\mathbb{Z}_2^n$-identity map id_{L} is linear, as for any linear coordinate map h, we have $\mathrm{h} \circ \mathrm{id}_{\mathsf{L}} \circ \mathrm{h}^{-1} = \mathrm{id}_{\mathbb{R}^{p|\underline{q}}}$. The second statement is obvious. □

3.3.2.2 *Sheaf of linear $\mathbb{Z}_2^n$-functions.*

Definition 3.2.15. Let $\mathsf{L} \in \mathbb{Z}_2^n\texttt{LinMan}$ be of dimension $p|\underline{q}$ and let $|U| \subset |\mathsf{L}|$ be open. A $\mathbb{Z}_2^n$-function $f \in \mathcal{O}_\mathsf{L}(|U|)$ is a *linear $\mathbb{Z}_2^n$-function*, if there exists a linear coordinate map $\mathrm{h} : \mathsf{L} \to \mathbb{R}^{p|\underline{q}}$, such that

$$(\mathrm{h}^*)^{-1}(f) \in \mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{p|\underline{q}}}(|\mathrm{h}|(|U|)) \, .$$

We denote the subset of all linear $\mathbb{Z}_2^n$-functions of $\mathcal{O}_\mathsf{L}(|U|)$ by $\mathcal{O}^{\mathrm{lin}}_\mathsf{L}(|U|)$.

The subset $\mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{p|\underline{q}}}(|\mathrm{h}|(|U|))$ is defined in the obvious way. If $f \in \mathcal{O}^{\mathrm{lin}}_\mathsf{L}(|U|)$, then for any chart $(\mathsf{L}, \mathrm{h}')$ of the linear atlas of L, we have

$$(\mathrm{h}'^*)^{-1}(f) \in \mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{p|\underline{q}}}(|\mathrm{h}'|(|U|)) \, .$$

This follows from the equation

$$(\mathrm{h}'^*)^{-1}(f) = (\mathrm{h} \circ \mathrm{h}'^{-1})^*((\mathrm{h}^*)^{-1}(f))$$

and the compatibility of the two charts.

As $\mathcal{O}^{\mathrm{lin}}_\mathsf{L}(|U|) \subset \mathcal{O}_\mathsf{L}(|U|)$ is visibly closed for linear combinations, it is a vector subspace of $\mathcal{O}_\mathsf{L}(|U|)$. Hence, the intersection

$$\mathcal{O}^{\mathrm{lin}}_{\mathsf{L},\gamma_i}(|U|) := \mathcal{O}^{\mathrm{lin}}_\mathsf{L}(|U|) \cap \mathcal{O}_{\mathsf{L},\gamma_i}(|U|) \subset \mathcal{O}_\mathsf{L}(|U|)$$

is also a vector subspace. We thus get vector subspaces $\mathcal{O}^{\mathrm{lin}}_{\mathsf{L},\gamma_i}(|U|) \subset \mathcal{O}^{\mathrm{lin}}_\mathsf{L}(|U|)$, so their direct sum over i is a vector subspace as well. Since any $f \in \mathcal{O}^{\mathrm{lin}}_\mathsf{L}(|U|)$ reads uniquely as

$$f = \sum_{i=0}^{N} f_i \quad (f_i \in \mathcal{O}_{\mathsf{L},\gamma_i}(|U|)) \, ,$$

we get

$$(\mathrm{h}^*)^{-1} f_0 + \sum_j (\mathrm{h}^*)^{-1} f_j = (\mathrm{h}^*)^{-1} f = \sum_\ell r_\ell \, x^\ell + \sum_j \sum_\ell r^j_\ell \, \xi^\ell_j \, .$$

As $(\mathrm{h}^*)^{-1}$ is $\mathbb{Z}_2^n$-degree preserving, we find that $f_i \in \mathcal{O}^{\mathrm{lin}}_{\mathsf{L},\gamma_i}(|U|)$, so that

$$\mathcal{O}^{\mathrm{lin}}_\mathsf{L}(|U|) = \bigoplus_i \mathcal{O}^{\mathrm{lin}}_{\mathsf{L},\gamma_i}(|U|) \in \mathbb{Z}_2^n\texttt{Vec} \, .$$

Remark 3.2.16. Observe that:

i. For any open subset $|U| \subset |\mathsf{L}|$ and any linear coordinate map $\mathrm{h} : \mathsf{L} \to \mathbb{R}^{p|\underline{q}}$, the map

$$\mathrm{h}^* : \mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{p|\underline{q}}}(|h|(|U|)) \to \mathcal{O}^{\mathrm{lin}}_\mathsf{L}(|U|)$$

is an isomorphism of $\mathbb{Z}_2^n$-vector spaces of dimension $p|\underline{q}$.

ii. The restriction maps and the gluing property of $\mathcal{O}_\mathsf{L}$ endow $\mathcal{O}^{\mathrm{lin}}_\mathsf{L}$ with a sheaf of $\mathbb{Z}_2^n$-vector spaces structure.

iii. A $\mathbb{Z}_2^n$-morphism $\phi : \mathsf{L} \to \mathsf{L}'$ between linear $\mathbb{Z}_2^n$-manifolds is itself linear, if and only if ϕ^* is a degree respecting linear map

$$\phi^* : \mathcal{O}^{\mathrm{lin}}_{\mathsf{L}'}(|\mathsf{L}'|) \to \mathcal{O}^{\mathrm{lin}}_\mathsf{L}(|\mathsf{L}|) \, .$$

It is straightforward to check the first two statements. For the third one, let L (resp., L') be of dimension $p|\underline{q}$ (resp., $r|\underline{s}$) and denote the coordinates of the corresponding Cartesian $\mathbb{Z}_2^n$-manifold by $u^{\mathfrak{a}} = (x^a, \xi^A)$ (resp., $v^{\mathfrak{b}} = (y^b, \eta^B)$). The morphism ϕ is linear, if and only if there exist linear coordinate maps (L, h) and $(\mathsf{L}', \mathrm{k})$, such that $\mathrm{k} \circ \phi \circ \mathrm{h}^{-1}$ has linear coordinate pullbacks, i.e., such that

$$((\mathrm{h}^*)^{-1} \circ \phi^* \circ \mathrm{k}^*)(v^{\mathfrak{b}}) \in \mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{p|\underline{q}}}(\mathbb{R}^p) \ . \tag{3.2.20}$$

On the other hand, in view of the first item of the previous remark, the condition

$$\phi^*\big(\mathcal{O}^{\mathrm{lin}}_{\mathsf{L}'}(|\mathsf{L}'|)\big) \subset \mathcal{O}^{\mathrm{lin}}_{\mathsf{L}}(|\mathsf{L}|)$$

of the third item is equivalent to asking that

$$(\phi^* \circ \mathrm{k}^*)(\sum_{\mathfrak{b}} r_{\mathfrak{b}} v^{\mathfrak{b}}) \in \mathrm{h}^*(\mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{p|\underline{q}}}(\mathbb{R}^p)) \ . \tag{3.2.21}$$

The conditions (3.2.20) and (3.2.21) are visibly equivalent.

2.3.2.3 *Functor of points of* $\mathbb{Z}_2^n\texttt{LinMan}$. We start with the following

Proposition 3.2.17. *For any linear $\mathbb{Z}_2^n$-manifold L (of dimension $p|\underline{q}$) and any $\mathbb{Z}_2^n$-Grassmann algebra $\Lambda \simeq \mathbb{R}^{0|\underline{m}}$, the set*

$$\mathsf{L}(\Lambda) := \mathrm{Hom}_{\mathbb{Z}\texttt{Man}}(\mathbb{R}^{0|\underline{m}}, \mathsf{L}) \simeq \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Alg}}(\mathcal{O}_{\mathsf{L}}(|\mathsf{L}|), \Lambda)$$

of Λ-points of L admits a unique Fréchet Λ_0-module structure, such that, for any chart $\mathrm{h} : \mathsf{L} \to \mathbb{R}^{p|\underline{q}}$ of the linear atlas of L, the induced map

$$\mathrm{h}_\Lambda : \mathsf{L}(\Lambda) \ni \mathrm{x}^* \mapsto \mathrm{x}^* \circ \mathrm{h}^* \in \mathbb{R}^{p|\underline{q}}(\Lambda)$$

is a Fréchet Λ_0-module isomorphism.

The definition of the category $\texttt{FAMod}$ of Fréchet modules over Fréchet algebras can be found in Appendix 3.5.1. In the preceding proposition, it is implicit that the (unital) Fréchet algebra morphism that is associated to h_Λ is id_{Λ_0}.

Proof. Let $\Lambda \in \mathbb{Z}_2^n\texttt{GrAlg}$. In view of the fundamental theorem of $\mathbb{Z}_2^n$-morphisms, there is a 1:1 correspondence between the Λ-points x^* of $\mathbb{R}^{p|\underline{q}}$ and the $(p + |\underline{q}|)$-tuples

$$\mathrm{x}^* \simeq \big(x^a_\Lambda,\, \xi^A_\Lambda\big) \in \Lambda_0^{\times p} \times \Lambda_{\gamma_1}^{\times q_1} \times \cdots \times \Lambda_{\gamma_N}^{\times q_N}$$

(we used this correspondence already in Equation (3.2.17)). Indeed, the algebra Λ is the $\mathbb{Z}_2^n$-commutative nuclear Fréchet $\mathbb{R}$-algebra of global $\mathbb{Z}_2^n$-functions of some $\mathbb{R}^{0|\underline{m}}$ (in particular, the degree zero term Λ_0 of Λ is a commutative nuclear Fréchet algebra). Hence, all its homogeneous subspaces Λ_{γ_i} ($i \in \{0, \ldots, N\}$, $\gamma_0 = 0$) are nuclear Fréchet vector spaces. Since any product (resp., any countable product) of nuclear (resp., Fréchet) spaces is nuclear (resp., Fréchet), the set $\mathbb{R}^{p|\underline{q}}(\Lambda)$ of Λ-points of $\mathbb{R}^{p|\underline{q}}$ is a nuclear Fréchet space. The latter statements can be found in [14]. The Fréchet Λ_0-module structure on $\mathbb{R}^{p|\underline{q}}(\Lambda)$ is then defined by

$$\triangleleft : \Lambda_0 \times \mathbb{R}^{p|\underline{q}}(\Lambda) \ni (\mathrm{a}, \mathrm{x}^*) \mapsto \mathrm{a} \triangleleft \mathrm{x}^* := (\mathrm{a} \cdot x^a_\Lambda,\ \mathrm{a} \cdot \xi^A_\Lambda) \in \mathbb{R}^{p|\underline{q}}(\Lambda) \ . \tag{3.2.22}$$

Since this action (which is compatible with addition in Λ_0 and addition in $\mathbb{R}^{p|\underline{q}}(\Lambda)$) is defined using the continuous associative multiplication $\cdot : \Lambda_{\gamma_i} \times \Lambda_{\gamma_j} \to \Lambda_{\gamma_i + \gamma_j}$ of the Fréchet algebra Λ, it is (jointly) continuous.

We now define the Λ_0-module structure on $\mathsf{L}(\Lambda)$. Observe first that, for any chart map $\mathrm{h} : \mathsf{L} \rightleftarrows \mathbb{R}^{p|\underline{q}} : \mathrm{h}^{-1}$ of the linear atlas of L, the induced maps $\mathrm{h}_\Lambda : \mathsf{L}(\Lambda) \rightleftarrows \mathbb{R}^{p|\underline{q}}(\Lambda) : (\mathrm{h}^{-1})_\Lambda$ are inverse maps: $(\mathrm{h}^{-1})_\Lambda = (\mathrm{h}_\Lambda)^{-1} =: \mathrm{h}_\Lambda^{-1}$. For $K \in \mathbb{N} \setminus \{0\}$, $k \in \{1, \ldots, K\}$, $\mathrm{a}^k \in \Lambda_0$, and $\mathrm{y}^*_k \in \mathsf{L}(\Lambda)$, we set

$$\sum_k \mathrm{a}^k \star \mathrm{y}^*_k := \mathrm{h}_\Lambda^{-1}(\sum_k \mathrm{a}^k \triangleleft \mathrm{h}_\Lambda(\mathrm{y}^*_k)) \in \mathsf{L}(\Lambda) \ .$$

This defines a Λ_0-module structure on $\mathsf{L}(\Lambda)$ that makes h_Λ a Λ_0-module isomorphism. The Λ_0-module structures $\mathsf{L}(\Lambda)_\mathrm{h}$ and $\mathsf{L}(\Lambda)_\mathrm{k}$ that are implemented by h and another chart k of the linear atlas, respectively, are related by the Λ_0-module isomorphism

$$\mathrm{k}_\Lambda^{-1} \circ \mathrm{h}_\Lambda : \mathsf{L}(\Lambda)_\mathrm{h} \to \mathsf{L}(\Lambda)_\mathrm{k} \ .$$

Hence, the Λ_0-module structure on $\mathsf{L}(\Lambda)$ is well-defined.

In order to get a Fréchet structure on the real vector space $\mathsf{L}(\Lambda)$ that we just defined, we need a countable and separating family of seminorms $(p_n)_{n\in\mathbb{N}}$, such that any sequence in $\mathsf{L}(\Lambda)$ that is Cauchy for every p_n, converges for every p_n to a fixed vector (i.e., a vector that does not depend on n). We define this family (of course) by transferring to $\mathsf{L}(\Lambda)$ the analogous family $(\rho_n)_{n\in\mathbb{N}}$ of the Fréchet vector space $\mathbb{R}^{p|\underline{q}}(\Lambda)$ (see [14, Theorem 14]). In other words, for each $\mathrm{y}^* \in \mathsf{L}(\Lambda)$, we set

$$p_n(\mathrm{y}^*) := \rho_n(\mathrm{h}_\Lambda(\mathrm{y}^*)) \in \mathbb{R}_+ \ .$$

It is straightforwardly checked that $(p_n)_{n\in\mathbb{N}}$ is a countable family of seminorms that has the required properties. Moreover, the vector space isomorphism h_Λ is an isomorphism of Fréchet vector spaces, i.e., a continuous linear map with a continuous inverse. We show that h_Λ is continuous for the seminorm topologies implemented by the p_n and the ρ_n, i.e., that, for all $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $C > 0$, such that

$$\rho_n(\mathrm{h}_\Lambda(\mathrm{y}^*)) \leq C\, p_m(\mathrm{y}^*) \ ,$$

for all $\mathrm{y}^* \in \mathsf{L}(\Lambda)$. This requirement is of course satisfied. Hence, the composite $\mathrm{k}_\Lambda^{-1} \circ \mathrm{h}_\Lambda$ of isomorphisms of Fréchet spaces is an isomorphism of Fréchet spaces, so that the Fréchet space structure on $\mathsf{L}(\Lambda)$ is well-defined.

The Λ_0-module structure and the Fréchet vector space structure on $\mathsf{L}(\Lambda)$ combine into a Fréchet Λ_0-module structure, if they are compatible, i.e., if the Λ_0-action

$$\star : \Lambda_0 \times \mathsf{L}(\Lambda) \ni (\mathrm{a}, \mathrm{y}^*) \mapsto \mathrm{h}_\Lambda^{-1}(\mathrm{a} \triangleleft \mathrm{h}_\Lambda(\mathrm{y}^*)) \in \mathsf{L}(\Lambda) \tag{3.2.23}$$

is continuous. The condition is obviously satisfied as this action is the composite of the continuous maps $\mathrm{id} \times \mathrm{h}_\Lambda$, $\triangleleft$ and h_Λ^{-1}. Further, the map h_Λ is clearly a Fréchet Λ_0-module isomorphism, for any h in the linear atlas of L.

There is obviously no other Fréchet Λ_0-module structure on $\mathsf{L}(\Lambda)$ with that property. Indeed, if there were, it would be isomorphic to the Fréchet Λ_0-module structure on $\mathbb{R}^{p|\underline{q}}(\Lambda)$, hence isomorphic to the Fréchet Λ_0-module structure that we just constructed. □

In the following, we denote the Λ_0-action $\star$ on $\mathsf{L}(\Lambda)$ by simple juxtaposition, i.e., we write $\mathrm{a}\mathrm{y}^*$ instead of $\mathrm{a} \star \mathrm{y}^*$.

To proceed, we need some preparation.

Let $\mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{FAMod})$ be the category of functors F, whose values $F(\Lambda)$ are Fréchet Λ_0-modules, and of natural transformations β, whose Λ-components β_Λ are continuous Λ_0-linear maps. We already used above the category $\mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{AFM})$ of functors, whose values are Fréchet Λ_0-manifolds, and of natural transformations, whose components are Λ_0-smooth maps.

Proposition 3.2.18. *The category* $\mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{FAMod})$ *is a subcategory of the category* $\mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{AFM})$.

Proof. Observe first that composition of natural transformations (resp., identities of functors) is (resp., are) induced by composition (resp., identities) in the target category of the functors considered, which is (resp., are) in both target categories the standard set-theoretical composition (resp., identities). Hence, composition and identities are the same in both functor categories. However, we still have to show that objects (resp., morphisms) of the first functor category are objects (resp., morphisms) of the second.

Let F be a functor with target $\mathtt{FAMod}$. Since a Fréchet Λ_0-module (i.e., a Fréchet vector space with a (compatible) continuous Λ_0-action) is clearly a Fréchet Λ_0-manifold, the functor F sends $\mathbb{Z}_2^n$-Grassmann algebras Λ to Fréchet Λ_0-manifolds $F(\Lambda)$. Let now $\varphi^* : \Lambda \to \Lambda'$ be a morphism of $\mathbb{Z}_2^n$-algebras. As $F(\varphi^*) : F(\Lambda) \to F(\Lambda')$ is a morphism between Fréchet modules over the Fréchet algebras Λ_0 and Λ'_0, respectively, it is continuous and it has an associated continuous (unital, $\mathbb{R}$-) algebra morphism $\psi : \Lambda_0 \to \Lambda'_0$, such that

$$F(\varphi^*)(\mathrm{av} + \mathrm{a'v'}) = \psi(\mathrm{a})F(\varphi^*)(\mathrm{v}) + \psi(\mathrm{a'})F(\varphi^*)(\mathrm{v'}) \ , \tag{3.2.24}$$

for all $\mathrm{a}, \mathrm{a'} \in \Lambda_0$ and all $\mathrm{v}, \mathrm{v'} \in F(\Lambda)$. We must show that $F(\varphi^*)$ is a morphism between Fréchet manifolds over Λ_0 and Λ'_0, respectively, i.e., we must show that $F(\varphi^*)$ is smooth and has first order derivatives that are linear in the sense of (3.2.24) (see [13]). Since, for any $t \in \mathbb{R}$, we have $\psi(t) = t\psi(1) = t$, it follows from (3.2.24) that

$$\mathrm{d_x}F(\varphi^*)(\mathrm{v}) := \lim_{t\to 0} \frac{1}{t}(F(\varphi^*)(\mathrm{x} + t\mathrm{v}) - F(\varphi^*)(\mathrm{x})) = F(\varphi^*)(\mathrm{v})$$

and

$$\mathrm{d}_{\mathrm{x}}^{k+1}F(\varphi^*)(\mathrm{v}_1, \ldots, \mathrm{v}_{k+1}) = 0 \ ,$$

for any $\mathrm{x}, \mathrm{v}, \mathrm{v}_1, \ldots, \mathrm{v}_{k+1} \in F(\Lambda)$ and any $k \geq 1$. Hence, all derivatives exist everywhere and are (jointly) continuous. This implies that $F(\varphi^*)$ has the required properties, so that F is a functor with target $\mathtt{AFM}$.

As for morphisms, let $\eta : F \to G$ be a natural transformation between functors valued in $\mathtt{FAMod}$. Its Λ-components $\eta_\Lambda : F(\Lambda) \to G(\Lambda)$ are continuous and Λ_0-linear maps. Repeating the proof given in the preceding paragraph for $F(\varphi^*)$, we obtain that η_Λ is Λ_0-smooth, i.e., is smooth and has Λ_0-linear first order derivatives. Therefore, the morphism η of the functor category with target $\mathtt{FAMod}$ is a morphism of the functor category with target $\mathtt{AFM}$. □

Since

$$\mathbb{Z}_2^n\mathtt{LinMan} \subset \mathbb{Z}_2^n\mathtt{Man} \quad \text{and} \quad \mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{FAMod}) \subset \mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{AFM})$$

are subcategories, we expect that:

Proposition 3.2.19. *The functor*

$$\mathcal{S} : \mathbb{Z}_2^n\mathtt{Man} \to \mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{AFM}) \tag{3.2.25}$$

(*see Equation* (3.2.16)) *restricts to a functor*

$$\mathcal{S} : \mathbb{Z}_2^n\mathtt{LinMan} \to \mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{FAMod}) \ . \tag{3.2.26}$$

Proof. We have to explain why $\mathcal{S}$ sends linear $\mathbb{Z}_2^n$-manifolds and linear $\mathbb{Z}_2^n$-morphisms to objects and morphisms, respectively, of the target subcategory.

Let $\mathsf{L} \in \mathbb{Z}_2^n\mathtt{LinMan}$. The functor $\mathcal{S}(\mathsf{L})$ is an object of the functor category with target $\mathtt{AFM}$. Since composition and identities are the same in both target categories, it suffices to show that, for any $\mathbb{Z}_2^n$-Grassmann algebra Λ, the value $\mathcal{S}(\mathsf{L})(\Lambda) = \mathsf{L}(\Lambda)$ is a Fréchet Λ_0-module and that, for any $\mathbb{Z}_2^n$-algebra morphism $\varphi^* : \Lambda \to \Lambda'$, the morphism

$$\mathsf{L}(\varphi^*) : \mathsf{L}(\Lambda) \ni \mathrm{y}^* \mapsto \varphi^* \circ \mathrm{y}^* \in \mathsf{L}(\Lambda')$$

is a morphism of the category $\mathtt{FAMod}$. The first of the preceding conditions holds in view of Proposition 3.2.17. We start proving the second condition for $\mathsf{L} = \mathbb{R}^{p|\underline{q}}$. Since $\mathbb{R}^{p|\underline{q}}(\varphi^*)$ is a morphism of $\mathtt{AFM}$, it is smooth, hence, continuous. Further, omitting the summation symbols and using our standard notation, we get

$$\mathbb{R}^{p|\underline{q}}(\varphi^*)(\mathrm{a}^k \lhd \mathrm{x}_k^*) = \mathbb{R}^{p|\underline{q}}(\varphi^*)(\mathrm{a}^k \cdot x^a_{\Lambda,k}\, , \mathrm{a}^k \cdot \xi^A_{\Lambda,k}) = (\varphi^*(\mathrm{a}^k) \cdot \varphi^*(x^a_{\Lambda,k})\, , \varphi^*(\mathrm{a}^k) \cdot \varphi^*(\xi^A_{\Lambda,k}))$$
$$= \varphi^*(\mathrm{a}^k) \lhd \mathbb{R}^{p|\underline{q}}(\varphi^*)(\mathrm{x}_k^*)\ . \tag{3.2.27}$$

It now suffices to recall that the $\mathbb{Z}_2^n$-algebra morphism φ^* is the pullback $\varphi^*_\star$ over the whole base manifold $\{\star\}$ of a $\mathbb{Z}_2^n$-morphism $\varphi : \mathbb{R}^{0|\underline{m}'} \to \mathbb{R}^{0|\underline{m}}$, and that all pullbacks of $\mathbb{Z}_2^n$-morphisms are continuous, so that the restriction $\varphi^* : \Lambda_0 \to \Lambda_0'$ is a continuous algebra morphism. We are now able to prove that the second condition holds also for an arbitrary linear $\mathbb{Z}_2^n$-manifold L. Indeed, since $\varphi^* : \Lambda \to \Lambda'$ is a morphism of $\mathbb{Z}_2^n$-algebras, the map $\mathsf{L}(\varphi^*) : \mathsf{L}(\Lambda) \to \mathsf{L}(\Lambda')$ is a morphism of $\mathtt{AFM}$, hence, it is continuous. Recall now that any chart $\mathrm{h} : \mathsf{L} \to \mathbb{R}^{p|\underline{q}}$ is a $\mathbb{Z}_2^n$-morphism, so that $\mathcal{S}(\mathrm{h}) : \mathsf{L}(-) \to \mathbb{R}^{p|\underline{q}}(-)$ is a natural transformation h_- with Λ-components $\mathrm{h}_\Lambda : \mathsf{L}(\Lambda) \to \mathbb{R}^{p|\underline{q}}(\Lambda)$ that are Fréchet Λ_0-module isomorphisms in view of Proposition 3.2.17. Naturality of h_- implies that

$$\mathrm{h}_{\Lambda'} \circ \mathsf{L}(\varphi^*) = \mathbb{R}^{p|\underline{q}}(\varphi^*) \circ \mathrm{h}_\Lambda\ ,$$

and, due to invertibility, that

$$\mathsf{L}(\varphi^*) = \mathrm{h}^{-1}_{\Lambda'} \circ \mathbb{R}^{p|\underline{q}}(\varphi^*) \circ \mathrm{h}_\Lambda\ .$$

Definition (3.2.23) yields

$$\mathsf{L}(\varphi^*)(\mathrm{a}^k \star \mathrm{y}_k^*) = (\mathrm{h}^{-1}_{\Lambda'} \circ \mathbb{R}^{p|\underline{q}}(\varphi^*) \circ \mathrm{h}_\Lambda)(\mathrm{h}^{-1}_\Lambda(\mathrm{a}^k \lhd \mathrm{h}_\Lambda(\mathrm{y}_k^*))) = \varphi^*(\mathrm{a}^k) \star \mathsf{L}(\varphi^*)(\mathrm{y}_k^*)$$

(we used our standard notation). Hence, the functor $\mathcal{S}(\mathsf{L})$ is an object of the functor category with target $\mathtt{FAMod}$.

As for morphisms, we consider a linear $\mathbb{Z}_2^n$-morphism

$$\phi : \mathsf{L} \to \mathsf{L}'$$

and will prove that $\mathcal{S}(\phi)$, which is a natural transformation ϕ_- of the functor category with target $\mathtt{AFM}$, i.e., a natural transformation with Λ_0-smooth Λ-components ϕ_Λ, has actually continuous (but this results from Λ_0-smoothness) Λ_0-linear components.

Let $p|\underline{q}$ (resp., $r|\underline{s}$) be the dimension of L (resp., of L'). We first discuss the case of a linear $\mathbb{Z}_2^n$-morphism

$$\Phi : \mathbb{R}^{p|\underline{q}} \to \mathbb{R}^{r|\underline{s}}$$

between the corresponding Cartesian $\mathbb{Z}_2^n$-manifolds with canonical coordinates (x^a, ξ^A) and (y^b, η^B), respectively. We know from [13] that, if the $\mathbb{Z}_2^n$-morphism (resp., the linear $\mathbb{Z}_2^n$-morphism) Φ reads

$$\Phi^*(y^b) = \sum_{|\alpha|\geq 0} \Phi^b_\alpha(x)\,\xi^\alpha \quad (\text{resp., } = \sum_a \mathbf{L}^b_a x^a)\ , \tag{3.2.28a}$$

$$\Phi^*(\eta^B) = \sum_{|\alpha|> 0} \Phi^B_\alpha(x)\,\xi^\alpha \quad (\text{resp., } = \sum_A \mathbf{L}^B_A \xi^A)\ , \tag{3.2.28b}$$

(where the right-hand sides have the appropriate degree and where the coefficients $\mathbf{L}^*_*$ are real numbers), then the Λ-component Φ_Λ associates to the Λ-point $\mathrm{x}^* \simeq (x^a_\Lambda; \xi^A_\Lambda) = \big(x^a_{||}, \mathring{x}^a_\Lambda; \xi^A_\Lambda\big)$ of $\mathbb{R}^{p|\underline{q}}(\Lambda)$, the Λ-point $\mathrm{x}^* \circ \Phi^* \simeq (y^b_\Lambda; \eta^B_\Lambda)$ of $\mathbb{R}^{r|\underline{s}}(\Lambda)$ that is given by

$$y^b_\Lambda = \sum_{|\alpha|\geq 0}\sum_{|\beta|\geq 0} \frac{1}{\beta!}\,(\partial^\beta_x \Phi^b_\alpha)(x_{||})\,\mathring{x}^\beta_\Lambda\,\xi^\alpha_\Lambda \quad (\text{resp., } = \sum_a \mathbf{L}^b_a x^a_\Lambda)\ , \tag{3.2.29a}$$

$$\eta^B_\Lambda = \sum_{|\alpha|> 0}\sum_{|\beta|\geq 0} \frac{1}{\beta!}\,(\partial^\beta_x \Phi^B_\alpha)(x_{||})\,\mathring{x}^\beta_\Lambda\,\xi^\alpha_\Lambda \quad (\text{resp., } = \sum_A \mathbf{L}^B_A \xi^A_\Lambda)\ . \tag{3.2.29b}$$

Here, we used the obvious decomposition $\Lambda = \mathbb{R} \times \mathring{\Lambda}$ and wrote $x^a_\Lambda = (x^a_{||}, \mathring{x}^a_\Lambda)$. The particular linear versions of Equations (3.2.29a) and (3.2.29b) (in parentheses), show that the component Φ_Λ is Λ_0-linear, as needed.

In the general case of a linear $\mathbb{Z}_2^n$-morphism $\phi : \mathsf{L} \to \mathsf{L}'$, the $\mathbb{Z}_2^n$-morphism $\Phi := \mathrm{k} \circ \phi \circ \mathrm{h}^{-1} : \mathbb{R}^{p|\underline{q}} \to \mathbb{R}^{r|\underline{s}}$ has linear coordinate pullbacks $\Phi^*(y^b)$ and $\Phi^*(\eta^B)$ (and is thus a linear $\mathbb{Z}_2^n$-morphism), for any charts h and k of L and L', respectively. Since $\phi = \mathrm{k}^{-1} \circ \Phi \circ \mathrm{h}$, we have $\phi_\Lambda = \mathrm{k}^{-1}_\Lambda \circ \Phi_\Lambda \circ \mathrm{h}_\Lambda$ and, in view of Proposition 3.2.17 and the result of the preceding paragraph, all three factors of the RHS are Λ_0-linear.

Finally, the natural transformation $\mathcal{S}(\phi)$ is a natural transformation of the functor category with target $\mathtt{FAMod}$. □

Theorem 3.2.20. *The functor of points*

$$\mathcal{S} : \mathbb{Z}_2^n\mathtt{LinMan} \to \mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{FAMod})$$

of the category $\mathbb{Z}_2^n\mathtt{LinMan}$ *is fully faithful.*

Proof. We need to prove that the map

$$\mathcal{S}_{\mathsf{L},\mathsf{L}'} : \mathrm{Hom}_{\mathbb{Z}_2^n\mathtt{LinMan}}(\mathsf{L}, \mathsf{L}') \ni \phi \mapsto \phi_- \in \mathrm{Hom}_{\mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}},\mathtt{FAMod})}(\mathsf{L}(-), \mathsf{L}'(-))$$

is bijective, for all linear $\mathbb{Z}_2^n$-manifolds L, L'.

Since $\mathcal{S}$ is the restriction of the fully faithful functor $\mathcal{S} : \mathbb{Z}_2^n\mathtt{Man} \to \mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{AFM})$, the map $\mathcal{S}_{\mathsf{L},\mathsf{L}'}$ is injective.

To prove that $\mathcal{S}_{\mathsf{L},\mathsf{L}'}$ is also surjective, it actually suffices to show that the property holds for Cartesian $\mathbb{Z}_2^n$-manifolds. Indeed, in this case, if $\eta : \mathsf{L}(-) \to \mathsf{L}'(-)$ is a natural transformation of $\mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{FAMod})$, then $\mathrm{k}_- \circ \eta \circ \mathrm{h}^{-1}_-$ is a natural transformation in the same category from $\mathbb{R}^{p|\underline{q}}(-)$ to $\mathbb{R}^{r|\underline{s}}(-)$, and this transformation is implemented by a linear $\mathbb{Z}_2^n$-morphism $\varphi : \mathbb{R}^{p|\underline{q}} \to \mathbb{R}^{r|\underline{s}}$. It follows that

$$\eta = \mathrm{k}^{-1}_- \circ \varphi_- \circ \mathrm{h}_- = (\mathrm{k}^{-1} \circ \varphi \circ \mathrm{h})_-\ ,$$

where the latter composite is a linear $\mathbb{Z}_2^n$-morphism.

Let now $H : \mathbb{R}^{p|\underline{q}}(-) \to \mathbb{R}^{r|\underline{s}}(-)$ be a natural transformation of $\mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{FAMod})$, hence, a natural transformation of $\mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{AFM})$. We know from [13] that H is implemented by a $\mathbb{Z}_2^n$-morphism $\Phi : \mathbb{R}^{p|\underline{q}} \to \mathbb{R}^{r|\underline{s}}$, but we still have to prove that this morphism is linear. It follows from Equations (3.2.29a) and (3.2.29b) that $H_\Lambda = \Phi_\Lambda$ is given by

$$y^b_\Lambda = \sum_{|\alpha|\geq 0}\sum_{|\beta|\geq 0} F^b_{\alpha\beta}(x_{||})\, \mathring{x}^\beta_\Lambda\, \xi^\alpha_\Lambda \; , \tag{3.2.30a}$$

$$\eta^B_\Lambda = \sum_{|\alpha|> 0}\sum_{|\beta|\geq 0} F^B_{\alpha\beta}(x_{||})\, \mathring{x}^\beta_\Lambda\, \xi^\alpha_\Lambda \; , \tag{3.2.30b}$$

where we set

$$F^*_{\alpha\beta}(x) := \frac{1}{\beta!}\partial^\beta_x \Phi^*_\alpha \in C^\infty(\mathbb{R}^p) \tag{3.2.31}$$

(the $\Phi^*_\alpha \in C^\infty(\mathbb{R}^p)$ are the coefficients of the coordinate pullbacks by Φ, see Equations (3.2.28a) and (3.2.28b)), and where the RHS-s have of course the same $\mathbb{Z}_2^n$-degree as the corresponding coordinates of $\mathbb{R}^{r|\underline{s}}$. Since H_Λ is Λ_0-linear, we have

$$\sum_\alpha \sum_\beta F^*_{\alpha\beta}(r\, x_{||})\, r^{|\alpha|+|\beta|} \mathring{x}^\beta_\Lambda\, \xi^\alpha_\Lambda = r \sum_\alpha \sum_\beta F^*_{\alpha\beta}(x_{||})\, \mathring{x}^\beta_\Lambda\, \xi^\alpha_\Lambda \; ,$$

i.e.,

$$r^{|\alpha|+|\beta|} F^*_{\alpha\beta}(r\, x_{||}) = r\, F^*_{\alpha\beta}(x_{||}) \; ,$$

for any $r \in \mathbb{R}_{>0} \subset \Lambda_0$, any α, β and for any $x_{||} \in \mathbb{R}^p$. When deriving with respect to r, we obtain

$$r^{|\alpha|+|\beta|-1}\big((|\alpha|+|\beta|)F^*_{\alpha\beta}(rx_{||}) + r\sum_{a=1}^p x^a_{||}(\partial_{x^a_{||}} F^*_{\alpha\beta})(rx_{||})\big) = F^*_{\alpha\beta}(x_{||}) \; ,$$

so that setting $r = 1$ yields

$$\sum_{a=1}^p x^a_{||}\partial_{x^a_{||}} F^*_{\alpha\beta} = (1-n)F^*_{\alpha\beta}(x_{||}) \quad (n := |\alpha|+|\beta| \in \mathbb{N}) \; , \tag{3.2.32}$$

again for all α, β and all $x_{||} \in \mathbb{R}^p$.

Recall now that Euler's homogeneous function theorem states that, if $F \in C^1(\mathbb{R}^p \setminus \{0\})$, then, for any $\nu \in \mathbb{R}$, we have

$$\sum_{a=1}^p x^a \partial_{x^a} F = \nu F(x),\ \forall x \in \mathbb{R}^p\backslash\{0\} \quad \text{is equivalent to} \quad F(rx) = r^\nu F(x),\ \forall r > 0, \forall x \in \mathbb{R}^p\backslash\{0\} \; .$$

In view of (3.2.32), we thus get

$$F^*_{\alpha\beta}(rx_{||}) = r^{1-n} F^*_{\alpha\beta}(x_{||}),\ \forall r > 0, \forall x_{||} \in \mathbb{R}^p \; , \tag{3.2.33}$$

where we could extend the equality from $\mathbb{R}^p \setminus \{0\}$ to $\mathbb{R}^p$ due to continuity. If r tends to 0^+, the limit of the LHS is $F^*_{\alpha\beta}(0) \in \mathbb{R}$ and, for $n = 0$ (resp., $n = 1$; resp., $n \geq 2$), the limit of the RHS is 0 (resp., $F^*_{\alpha\beta}(x_{||})$; resp., $+\infty \cdot F^*_{\alpha\beta}(x_{||})$).

In the case $n \geq 2$, we conclude that

$$F^*_{\alpha\beta}(x_{||}) = 0,\ \forall x_{||} \in \mathbb{R}^p, \forall \alpha, \beta : |\alpha|+|\beta| \geq 2 \; . \tag{3.2.34}$$

For $n = 0$, we get

$$F^*_{00}(0) = 0 \ .$$

Observe that $\alpha = \beta = 0$ is only possible in Equation (3.2.30a). Differentiating (3.2.33), in the case $n = 0$, with respect to any component $x^a_{||}$ of $x_{||}$ and simplifying by r, we obtain

$$(\partial_{x^a_{||}} F^b_{00})(r x_{||}) = \partial_{x^a_{||}} F^b_{00}(x_{||}) \ ,$$

and taking the limit $r \to 0^+$, we get

$$\partial_{x^a_{||}} F^b_{00}(x_{||}) = \partial_{x^a_{||}} F^b_{00}(0) =: \mathbf{L}^b_a \in \mathbb{R} \ .$$

Integration yields

$$F^b_{00}(x_{||}) = \sum_a \mathbf{L}^b_a x^a_{||}, \ \forall x_{||} \in \mathbb{R}^p, \forall b \ , \tag{3.2.35}$$

as $F^b_{00}(0) = 0$.

In the remaining case $n = |\alpha| + |\beta| = 1$, we have necessarily $\alpha = 0$ and $\beta = e_a$, or $\alpha = e_A$ and $\beta = 0$ (the e_* are of course the vectors of the canonical basis of $\mathbb{R}^p$ and $\mathbb{R}^{|\underline{q}|}$, respectively). For $\mathbb{Z}_2^n$-degree reasons, the first (resp., second) possibility is incompatible with Equation (3.2.30b) (resp., Equation (3.2.30a)). Hence, the only terms in (3.2.30a) that still need being investigated are the terms $(\alpha, \beta) = (0, e_a)$. It follows from Equation (3.2.33) and its limit $r \to 0^+$ (see above) that $F^b_{0\,e_a}(x_{||}) = \mathbf{K}^b_a$, where we set $\mathbf{K}^b_a := F^b_{0\,e_a}(0) \in \mathbb{R}$. However, Equations (3.2.31) and (3.2.35) imply that

$$\mathbf{K}^b_a = F^b_{0\,e_a}(x_{||}) = \partial_{x^a_{||}} F^b_{00}(x_{||}) = \mathbf{L}^b_a \ ,$$

so that

$$F^b_{0\,e_a}(x_{||}) = \mathbf{L}^b_a \in \mathbb{R}, \ \forall x_{||} \in \mathbb{R}^p, \forall a, b \ . \tag{3.2.36}$$

In Equation (3.2.30b), the only terms that still need being investigated are the terms $(\alpha, \beta) = (e_A, 0)$. Using again the limit $r \to 0^+$ of Equation (3.2.33), we find

$$F^B_{e_A 0}(x_{||}) = \mathbf{L}^B_A, \ \forall x_{||} \in \mathbb{R}^p, \forall A, B \ , \tag{3.2.37}$$

where we wrote $\mathbf{L}^B_A$ instead of $F^B_{e_A 0}(0)$.

When combining now the results of Equations (3.2.34), (3.2.35), (3.2.36), and (3.2.37), we see that Equations (3.2.30a) and (3.2.30b) reduce to

$$y^b_\Lambda = \sum_a \mathbf{L}^b_a \, (x^a_{||} + \mathring{x}^a_\Lambda) \quad \text{and} \quad \eta^B_\Lambda = \sum_A \mathbf{L}^B_A \, \xi^A_\Lambda \tag{3.2.38}$$

and that the $\mathbb{Z}_2^n$-morphism Φ that induces the natural transformation H is defined by the coordinate pullbacks

$$\Phi^*(y^b) = \sum_a \mathbf{L}^b_a \, x^a \quad \text{and} \quad \Phi^*(\eta^B) = \sum_A \mathbf{L}^B_A \, \xi^A \ ,$$

i.e., that Φ is linear (see (3.2.28a), (3.2.28b), (3.2.29a), and (3.2.29b)). $\square$

Isomorphism between finite dimensional $\mathbb{Z}_2^n$-graded vector spaces and linear $\mathbb{Z}_2^n$-manifolds

In this subsection, we extend the isomorphism

$$\mathcal{M} : \mathbb{Z}_2^n\texttt{CarVec} \rightleftharpoons \mathbb{Z}_2^n\texttt{CarMan} : \mathcal{V}$$

of Proposition 3.2.5 between the full subcategories $\mathbb{Z}_2^n\texttt{CarVec} \subset \mathbb{Z}_2^n\texttt{FinVec}$ and $\mathbb{Z}_2^n\texttt{CarMan} \subset \mathbb{Z}_2^n\texttt{LinMan}$, to an isomorphism

$$\mathcal{M} : \mathbb{Z}_2^n\texttt{FinVec} \rightleftharpoons \mathbb{Z}_2^n\texttt{LinMan} : \mathcal{V} .$$

3.2.3.1 $\mathbb{Z}_2^n$-symmetric tensor algebra. We start with some remarks on tensor and $\mathbb{Z}_2^n$-symmetric tensor algebras over a (finite dimensional) $\mathbb{Z}_2^n$-vector space (see [36] and [9]).

Let

$$V = \bigoplus_{i=0}^{N} V_i := \bigoplus_{i=0}^{N} V_{\gamma_i} \in \mathbb{Z}_2^n\texttt{FinVec}$$

be of dimension $p|\underline{q}$. The $\mathbb{Z}_2^n$-symmetric tensor algebra of V is defined exactly as in the non-graded case, as the quotient of the $\mathbb{Z}_2^n$-graded associative unital tensor algebra of V by the homogeneous ideal

$$\bar{I} = (v_i \otimes v_j - (-1)^{\langle \gamma_i, \gamma_j \rangle} \; v_j \otimes v_i : v_i \in V_i, v_j \in V_j) .$$

More precisely, for $k \geq 2$, we have

$$V^{\otimes k} = \bigoplus_{i_1,\ldots,i_k=0}^{N} V_{i_1} \otimes \ldots \otimes V_{i_k} = \bigoplus_{i_1 \leq \ldots \leq i_k} V_{i_1,\ldots,i_k} := \bigoplus_{i_1 \leq \ldots \leq i_k} \Big(\bigoplus_{\sigma \in \mathrm{Perm}} V_{\sigma_{i_1}} \otimes \ldots \otimes V_{\sigma_{i_k}} \Big) , \quad (3.2.39)$$

where Perm is the set of all permutations of $i_1 \leq \ldots \leq i_k$. For instance, if $n = 1$, i.e., in the standard super case, the space $V^{\otimes 3}$ is the direct sum of the tensor products whose three factors have the subscripts $000, 001, 010, 011, 100, 101, 110, 111$. The notation we just introduced means that we write

$$V^{\otimes 3} = V_{000} \oplus V_{001} \oplus V_{011} \oplus V_{111} , \quad (3.2.40)$$

where we used the lexicographical order and where

$$V_{000} = V_0 \otimes V_0 \otimes V_0, \; V_{001} = V_0 \otimes V_0 \otimes V_1 \oplus V_0 \otimes V_1 \otimes V_0 \oplus V_1 \otimes V_0 \otimes V_0, \text{ et cetera} .$$

Further, as we are dealing with formal power series in this paper, we define the $\mathbb{Z}_2^n$-graded tensor algebra of V by

$$\overline{T}V := \Pi_k V^{\otimes k} ,$$

where Π_k means that we consider not only finite sums of tensors of different tensor degrees, but full sequences of such tensors. The vector space structure on such sequences is obvious and the algebra structure is defined exactly as in the standard case. Indeed, for $T^k \in V^{\otimes k}$ and $U^\ell \in V^{\otimes \ell}$, we have $T^k \otimes U^\ell \in V^{\otimes (k+\ell)}$ and we just extend this tensor product by linearity. In other words, if

$$T = \sum_{k=0}^{\infty} T^k \in \overline{T}V \quad \text{and} \quad U = \sum_{\ell=0}^{\infty} U^\ell \in \overline{T}V ,$$

we set

$$T \otimes U = \sum_k \sum_\ell T^k \otimes U^\ell = \sum_m \sum_{k+\ell=m} T^k \otimes U^\ell \in \overline{T}V . \quad (3.2.41)$$

It is clear that the just defined tensor multiplication endows $\overline{T}V$ with a $\mathbb{Z}_2^n$-graded algebra structure. Indeed, since

$$V^{\otimes k} = \bigoplus_{i_1 \leq \ldots \leq i_k} V_{i_1,\ldots,i_k} = \bigoplus_{p=0}^{N} \bigoplus_{\substack{i_1 \leq \ldots \leq i_k \\ \sum_j \gamma_{i_j} = \gamma_p}} V_{i_1,\ldots,i_k} =: \bigoplus_{p=0}^{N} (V^{\otimes k})_p$$

is visibly a $\mathbb{Z}_2^n$-graded vector space, the space $\overline{T}V$ is itself $\mathbb{Z}_2^n$-graded:

$$\overline{T}V = \Pi_k \bigoplus_{p=0}^{N} (V^{\otimes k})_p = \bigoplus_{p=0}^{N} \Pi_k \, (V^{\otimes k})_p =: \bigoplus_{p=0}^{N} (\overline{T}V)_p \; .$$

Now, if $T \in (\overline{T}V)_p$ and $U \in (\overline{T}V)_q$, we have $T^k \in (V^{\otimes k})_p$ and $U^\ell \in (V^{\otimes \ell})_q$, so that $T \otimes U \in (\overline{T}V)_{p+q}$ (where $p+q$ means $\gamma_p + \gamma_q$), which shows that $\overline{T}V$ is a $\mathbb{Z}_2^n$-graded (associative unital) algebra (over $\mathbb{R}$), as announced.

The ideal $\bar{I}$ is homogeneous with respect to the decomposition

$$\overline{T}V = \Pi_k \bigoplus_{i_1 \leq \ldots \leq i_k} V_{i_1,\ldots,i_k} \; , \quad \text{i.e.,} \quad \bar{I} = \Pi_{k(\geq 2)} \bigoplus_{i_1 \leq \ldots \leq i_k} (V_{i_1,\ldots,i_k} \cap \bar{I}) \; .$$

Therefore, the $\mathbb{Z}_2^n$-symmetric tensor algebra of V is given by

$$\bar{S}V = \Pi_k \bigoplus_{i_1 \leq \ldots \leq i_k} V_{i_1,\ldots,i_k} / (V_{i_1,\ldots,i_k} \cap \bar{I}) =: \Pi_k \bigoplus_{i_1 \leq \ldots \leq i_k} V_{i_1} \odot \ldots \odot V_{i_k}$$

$$= \bigoplus_{p=0}^{N} \Pi_k \bigoplus_{\substack{i_1 \leq \ldots \leq i_k \\ \sum_j \gamma_{i_j} = \gamma_p}} V_{i_1} \odot \ldots \odot V_{i_k} \; , \tag{3.2.42}$$

see [9]. We denote by $\odot$ the $\mathbb{Z}_2^n$-commutative multiplication that is induced on $\bar{S}V$ by the multiplication $\otimes$ of $\overline{T}V$. By definition, we have, for $[T] \in (\bar{S}V)_{\gamma_i}$ and $[U] \in (\bar{S}V)_{\gamma_j}$ (obvious notation),

$$[T] \odot [U] = [T \otimes U] = (-1)^{\langle \gamma_i, \gamma_j \rangle} [U] \odot [T] \; .$$

For instance, if $v_i \in V_i \subset (\bar{S}V)_{\gamma_i}$, $v_j \in V_j \subset (\bar{S}V)_{\gamma_j}$ and if $i \leq j$, we get

$$v_i \odot v_j = [v_i \otimes v_j] = [(-1)^{\langle \gamma_i, \gamma_j \rangle} v_j \otimes v_i] = (-1)^{\langle \gamma_i, \gamma_j \rangle} v_j \odot v_i \in V_i \odot V_j \; . \tag{3.2.43}$$

Notice further that, if $i < j$, the linear map

$$\iota : V_i \otimes V_j \ni T \mapsto [T] \in V_i \odot V_j \quad (\iota : V_i \otimes V_j \ni v_i \otimes v_j \mapsto v_i \odot v_j \in V_i \odot V_j) \tag{3.2.44}$$

is a vector space isomorphism. Indeed, if $[T] = 0$, the representative T is a vector in $(V_i \otimes V_j \oplus V_j \otimes V_i) \cap \bar{I}$ and is therefore a finite sum of generators of $\bar{I}$:

$$(-1)^{\langle \gamma_i, \gamma_j \rangle} \sum_k v_j^k \otimes v_i^k = \sum_k v_i^k \otimes v_j^k - T \in (V_i \otimes V_j) \cap (V_j \otimes V_i) = \{0\} \; . \tag{3.2.45}$$

It follows that the LHS of Equation (3.2.45) vanishes; hence, the first term of the RHS vanishes, due to the isomorphism $V_i \otimes V_j \simeq V_j \otimes V_i$, and thus T vanishes as well. In order to show that ι is also surjective, consider an arbitrary vector in $V_i \odot V_j$. It reads

$$[T] = [\sum_k v_i^k \otimes v_j^k + \sum_\ell w_j^\ell \otimes w_i^\ell] .$$

The image by ι of

$$\sum_k v_i^k \otimes v_j^k + (-1)^{\langle \gamma_i, \gamma_j \rangle} \sum_\ell w_i^\ell \otimes w_j^\ell \in V_i \otimes V_j$$

is the corresponding class. This class coincides with $[T]$, since the difference of the representatives is a vector of $\bar{I}$.

It follows that, for $n = 2$ for instance, we have in particular

$$V_{00} \odot V_{00} \odot V_{01} \odot V_{10} \odot V_{10} \odot V_{10} \odot V_{11} \simeq \odot^2 V_{00} \otimes V_{01} \otimes \odot^3 V_{10} \otimes V_{11} \simeq \vee^2 V_{00} \otimes V_{01} \otimes \wedge^3 V_{10} \otimes V_{11} , \tag{3.2.46}$$

where $\vee$ (resp., $\wedge$) is the symmetric (resp., antisymmetric) tensor product. Moreover, if the (finite dimensional) vector space V has dimension $q_0|q_1, q_2, q_3$, we denote the vectors of its basis (in accordance with the notation we adopted earlier in this text) by b_j^i, where $i \in \{0, 1, 2, 3\}$ refers to the degrees $00, 01, 10, 11$ and where $j \in \{1, \ldots, q_i\}$. The basis of the $\mathbb{Z}_2^n$-symmetric tensor product (3.2.46) is then made of the tensors

$$b_{j_1}^0 \vee b_{j_2}^0 \otimes b_{j_3}^1 \otimes b_{j_4}^2 \wedge b_{j_5}^2 \wedge b_{j_6}^2 \otimes b_{j_7}^3$$

$(j_1 \leq j_2$ and $j_4 < j_5 < j_6)$, which can also be written

$$b_{j_1}^0 \odot b_{j_2}^0 \odot b_{j_3}^1 \odot b_{j_4}^2 \odot b_{j_5}^2 \odot b_{j_6}^2 \odot b_{j_7}^3$$

$(j_1 \leq j_2$ and $j_4 < j_5 < j_6)$ (see (3.2.44)). More generally, the basis of $V_{i_1} \odot \ldots \odot V_{i_k}$ $(i_1 \leq \ldots \leq i_k)$ is made of the tensors

$$b_{j_1}^{i_1} \odot \ldots \odot b_{j_k}^{i_k} \tag{3.2.47}$$

$(j_\ell \leq j_{\ell+1}$ [resp., $<$], if $i_\ell = i_{\ell+1}$ and $\langle \gamma_{i_\ell}, \gamma_{i_{\ell+1}} \rangle$ even [resp., odd]$)$. To refer to the previous condition regarding the j-s, we write in the following $j_1 \lhd \ldots \lhd j_k$.

Observe also that

$$S^k V = \bigoplus_{i_1 \leq \ldots \leq i_k} V_{i_1} \odot \ldots \odot V_{i_k} = S^k \bigoplus_i V_i = \big(\bigotimes_i S V_i \big)^k ,$$

as well as that, in order to define a linear map on $V_{i_1} \odot \ldots \odot V_{i_k}$ (see (3.2.46)), it suffices to define a k-linear map on $V_{i_1} \times \ldots \times V_{i_k}$ that is $\mathbb{Z}_2^n$-commutative in the variables $i_\ell = \ldots = i_m$.

3.2.3.2. *Manifoldification functor.* If V is a $\mathbb{Z}_2^n$-graded vector space, its dual $V^\vee$ is defined by

$$V^\vee := \underline{\mathrm{Hom}}(V, \mathbb{R}) = \bigoplus_{i=0}^N \underline{\mathrm{Hom}}_{\gamma_i}(V, \mathbb{R}) = \bigoplus_{i=0}^N \mathrm{Hom}(V_i, \mathbb{R}) = \bigoplus_{i=0}^N (V_i)^\vee \in \mathbb{Z}_2^n \mathtt{Vec} .$$

More explicitly, we consider the space of $\mathbb{R}$-linear maps from V to $\mathbb{R}$ of any $\mathbb{Z}_2^n$-degree. It is clear that the linear maps of degree γ_i are the linear maps from V_i to $\mathbb{R}$ (that vanish in any other degree). Hence,

$$(V^\vee)_i = (V_i)^\vee =: V_i^\vee .$$

It follows that, if V is finite dimensional of dimension $p|\underline{q}$, its dual $V^\vee$ has the same dimension. Moreover, any basis $(b^i_k)_{i,k}$ ($i \in \{0,\dots,N\}$ and $k \in \{1,\dots,q_i\}$, where we set $q_0 := p$) of V defines a dual basis $(\beta^k_i)_{i,k}$ of $V^\vee$.

Let now $V \in \mathbb{Z}_2^n\texttt{FinVec}$ be of dimension $p|\underline{q}$. We set

$$V^\vee_* := \bigoplus_{j=1}^N V^\vee_j \in \mathbb{Z}_2^n\texttt{FinVec} \quad (\dim(V^\vee_*) = 0|\underline{q}) .$$

Proposition 3.2.21. *If V is a $\mathbb{Z}_2^n$-graded vector space of dimension $p|\underline{q}$, there is a non-canonical isomorphism of $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebras*

$$\flat : \bar{S}(V^\vee_*) \xrightarrow{\sim} \mathbb{R}[[\xi]] , \tag{3.2.48}$$

where $\mathbb{R}[[\xi]]$ is the global function algebra of $\mathbb{R}^{0|\underline{q}}$.

Proof. As usual, we ordered the $\mathbb{Z}_2^n$-degrees lexicographically, so that the ξ^ℓ_j-s are ordered unambiguously. We have

$$\mathbb{R}[[\xi]] = \Pi_\alpha \mathbb{R}\,\xi^\alpha ,$$

where the multi-index α has components $\alpha^\ell_j \in \mathbb{N}$ (resp., $\alpha^\ell_j \in \{0,1\}$), if $\langle\gamma_j,\gamma_j\rangle$ is even (resp., odd).

On the other hand, it follows from Equations (3.2.42) and (3.2.47) that, choosing a basis $(b^j_\ell)_{j,\ell}$ of V_* (defined similarly as $V^\vee_*$) and denoting its dual basis by $(\beta^\ell_j)_{j,\ell}$, leads to

$$\bar{S}(V^\vee_*) = \Pi_k \bigoplus_{j_1\le\ldots\le j_k} \bigoplus_{\ell_1\lhd\ldots\lhd\ell_k} \mathbb{R}\,\beta^{\ell_1}_{j_1} \odot \ldots \odot \beta^{\ell_k}_{j_k} = \Pi_k \bigoplus_{|\alpha|=k} \mathbb{R}\,\beta^\alpha = \Pi_\alpha \mathbb{R}\,\beta^\alpha, \tag{3.2.49}$$

where $\alpha^\ell_j \in \mathbb{N}$ (resp., $\alpha^\ell_j \in \{0,1\}$), if $\langle\gamma_j,\gamma_j\rangle$ is even (resp., odd).

In view of (3.2.41) and (3.2.43), the multiplications of $\mathbb{R}[[\xi]]$ and $\bar{S}(V^\vee_*)$ are exactly the same, so that the two $\mathbb{Z}_2^n$-commutative algebras are canonically isomorphic, once a basis of V_* has been chosen. □

Remark 3.2.22. We denoted the isomorphism by $\flat$ to remind us of its dependence on the basis $(b^j_\ell)_{j,\ell}$.

We are now prepared to define the linear $\mathbb{Z}_2^n$-manifold associated to a finite dimensional $\mathbb{Z}_2^n$-vector space. From here we denote the vector space by $\mathbf{V}$ instead of V and reserve the notation V for the manifold $V := \mathcal{M}(\mathbf{V})$.

Hence, let $\mathbf{V} \in \mathbb{Z}_2^n\texttt{FinVec}$ be of dimension $p|\underline{q}$. The p-dimensional vector space $\mathbf{V}_0$ of degree 0 is of course a smooth manifold of dimension p, as well as a linear $\mathbb{Z}_2^n$-manifold V_0 of dimension $p|\underline{0}$. On the other hand, the algebra $\bar{S}(\mathbf{V}^\vee_*)$ is a sheaf of $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebras over $\{\star\}$, i.e., it is a $\mathbb{Z}_2^n$-ringed space with underlying topological space $\{\star\}$, and, in view of Proposition 3.2.21, this space is (non-canonically) globally isomorphic to $\mathbb{R}^{0|\underline{q}} = (\{\star\}, \mathbb{R}[[\xi]])$. Hence, the space $(\{\star\}, \bar{S}(\mathbf{V}^\vee_*))$ is a linear $\mathbb{Z}_2^n$-manifold $V_>$ of dimension $0|\underline{q}$. Finally, the product $V = V_0 \times V_>$ is a $\mathbb{Z}_2^n$-manifold of dimension $p|\underline{q}$, with base manifold $V_0 \times \{\star\} \simeq V_0$ and function sheaf $\mathcal{O}_V$ that is, for any open subset $\Omega \subset V_0 \simeq \mathbb{R}^p$, given by

$$\mathcal{O}_V(\Omega) = \mathcal{O}_{V_0\times V_>}(\Omega\times\{\star\}) = C^\infty_{V_0}(\Omega)\widehat{\otimes}_{\mathbb{R}}\,\mathcal{O}_{V_>}(\{\star\}) \simeq C^\infty(\Omega)\widehat{\otimes}_{\mathbb{R}}\,\mathbb{R}[[\xi]] = C^\infty(\Omega)[[\xi]] = \mathcal{O}_{\mathbb{R}^{p|\underline{q}}}(\Omega) \tag{3.2.50}$$

(since Ω and $\{\star\}$ are $\mathbb{Z}_2^n$-chart domains; for more information about the problem with the function sheaf of product $\mathbb{Z}_2^n$-manifolds, we refer the reader to [15]). In particular, the $\mathbb{Z}_2^n$-algebras $\mathcal{O}_V(V_0)$ and $\mathcal{O}_{\mathbb{R}^{p|\underline{q}}}(\mathbb{R}^p)$ are isomorphic (see also Definition 13 of product $\mathbb{Z}_2^n$-manifolds in [15]), so that the $\mathbb{Z}_2^n$-manifolds V and $\mathbb{R}^{p|\underline{q}}$ are diffeomorphic (given what has been said above, the diffeomorphism is implemented by the choice of a basis of $\mathbf{V}$). Finally $V \in \mathbb{Z}_2^n\texttt{LinMan}$ ($\dim V = p|\underline{q}$). We define the manifoldification functor $\mathcal{M}$ on objects by

$$\mathcal{M}(\mathbf{V}) = V \; . \tag{3.2.51}$$

We now define $\mathcal{M}$ on morphisms. A degree zero linear map $\mathbf{L} : \mathbf{V} \to \mathbf{W}$ between finite dimensional vector spaces (of dimensions $p|\underline{q}$ and $r|\underline{s}$, respectively) is a family of linear maps $\mathbf{L}_i : \mathbf{V}_i \to \mathbf{W}_i$ ($i \in \{0, \ldots, N\}$). We denote the transpose maps by ${}^t\mathbf{L}_i : \mathbf{W}_i^\vee \to \mathbf{V}_i^\vee$.

The linear map $\mathbf{L}_0 : \mathbf{V}_0 \to \mathbf{W}_0$ is of course a smooth map $L_0 : V_0 \to W_0$, where V_0, W_0 are the vector spaces $\mathbf{V}_0, \mathbf{W}_0$ viewed as smooth manifolds. The map L_0 can also be interpreted as $\mathbb{Z}_2^n$-morphism $L_0 : V_0 \to W_0$ between the $\mathbb{Z}_2^n$-manifolds V_0, W_0 (which are of dimension zero in all non-zero degrees). The base morphism of L_0 is L_0 itself and, for any open subset $\Omega \subset W_0$, the pullback $(L_0)^*_\Omega$ is the (unital) algebra morphism $- \circ L_0|_\omega : C^\infty(\Omega) \to C^\infty(\omega)$ ($\omega := L_0^{-1}(\Omega)$) that extends the transpose ${}^t\mathbf{L}_0(-) = - \circ \mathbf{L}_0$.

The linear maps ${}^t\mathbf{L}_j : \mathbf{W}_j^\vee \to \mathbf{V}_j^\vee$ ($j \in \{1, \ldots, N\}$) define a linear map

$$\bar{S}({}^t\mathbf{L}) : \bar{S}(\mathbf{W}_*^\vee) \to \bar{S}(\mathbf{V}_*^\vee) \; .$$

Observe first that to define such a map, it suffices to define a linear map in each tensor degree k, hence, it suffices to define a linear map

$$({}^t\mathbf{L})^{\odot k}_{j_1 \ldots j_k} : \mathbf{W}_{j_1}^\vee \odot \ldots \odot \mathbf{W}_{j_k}^\vee \to \mathbf{V}_{j_1}^\vee \odot \ldots \odot \mathbf{V}_{j_k}^\vee \; ,$$

for any $j_1 \leq \ldots \leq j_k$ ($j_a \in \{1, \ldots, N\}$). Since the k-linear maps

$$({}^t\mathbf{L})^{\times k}_{j_1 \ldots j_k} : \mathbf{W}_{j_1}^\vee \times \ldots \times \mathbf{W}_{j_k}^\vee \ni (\omega^1_{j_1}, \ldots, \omega^k_{j_k}) \mapsto {}^t\mathbf{L}_{j_1}(\omega^1_{j_1}) \odot \ldots \odot {}^t\mathbf{L}_{j_k}(\omega^k_{j_k}) \in \mathbf{V}_{j_1}^\vee \odot \ldots \odot \mathbf{V}_{j_k}^\vee$$

are $\mathbb{Z}_2^n$-commutative in the variables $j_\ell = \ldots = j_m$, they define the degree zero linear maps $({}^t\mathbf{L})^{\odot k}_{j_1 \ldots j_k}$ (we set $({}^t\mathbf{L})^{\odot 0} = \mathrm{id}_\mathbb{R}$) and thus the degree zero linear map $\bar{S}({}^t\mathbf{L})$ that we are looking for. In view of our definitions, the latter is a (unital) $\mathbb{Z}_2^n$-algebra morphism between the global function algebras of the $\mathbb{Z}_2^n$-manifolds $W_>$ and $V_>$, and it therefore defines a unique $\mathbb{Z}_2^n$-morphism $L_> : V_> \to W_>$. The base morphism of $L_>$ is the identity $c : \{\star\} \to \{\star\}$.

We thus get a $\mathbb{Z}_2^n$-morphism

$$\mathcal{M}(\mathbf{L}) := L := L_0 \times L_> : \mathcal{M}(\mathbf{V}) = V = V_0 \times V_> \to \mathcal{M}(\mathbf{W}) = W = W_0 \times W_> \; , \tag{3.2.52}$$

with base map $L_0 \times c \simeq L_0$ and pullback (Ω open subset of W_0, $\omega := L_0^{-1}(\Omega)$)

$$L^*_\Omega : \mathcal{O}_W(\Omega) = C^\infty_{W_0}(\Omega) \widehat{\otimes}_\mathbb{R} \bar{S}(\mathbf{W}_*^\vee) \to \mathcal{O}_V(\omega) = C^\infty_{V_0}(\omega) \widehat{\otimes}_\mathbb{R} \bar{S}(\mathbf{V}_*^\vee) \; , \tag{3.2.53}$$

which is fully defined by $(- \circ L_0|_\omega) \otimes \bar{S}({}^t\mathbf{L})$.

We must now prove that the $\mathbb{Z}_2^n$-morphism $\mathcal{M}(\mathbf{L}) = L$ is a morphism of $\mathbb{Z}_2^n\texttt{LinMan}$, i.e., that in linear coordinates it has linear coordinate pullbacks. As said above, the linear coordinate map $\mathrm{k} : W \to \mathbb{R}^{r|\underline{s}}$ is the product of the linear coordinate maps $\mathrm{k}_0 : W_0 \to \mathbb{R}^{r|\underline{0}}$ and $\mathrm{k}_> : W_> \to \mathbb{R}^{0|\underline{s}}$. The first of these coordinate maps is implemented by a basis b_W of $\mathbf{W}_0$ and its global pullback $b_W^* : C^\infty(\mathbb{R}^r) \to C^\infty_{W_0}(W_0)$ sends a coordinate function $y^\ell \in C^\infty(\mathbb{R}^r)$ to

$$b_W^*(y^\ell) = y^\ell \circ b_W = \beta^\ell_W \in C^\infty_{W_0}(W_0) \; ,$$

where β_W is the dual basis (observe that b_W^* extends the transpose of b_W viewed as vector space isomorphism). Similarly, it is clear from Proposition 3.2.21 that the global pullback $\flat_W^{-1}$ of the second coordinate map sends a coordinate function $\eta^\ell_j \in \mathbb{R}[[\eta]]$ to

$$\flat_W^{-1}(\eta^\ell_j) = \beta^\ell_j \in \bar{S}(\mathbf{W}_*^\vee) \; ,$$

where $(\beta_j^\ell)_{j,\ell}$ is the dual of a basis of $\mathbf{W}_*$. Based on what we just said and on the statement (3.2.53), we get that the coordinate pullbacks in the linear coordinate expression of L are

$$(b_V^*)^{-1}((b_W^*(y^\ell))\circ L_0) = (b_V^*)^{-1}({}^t\mathbf{L}_0(\beta_W^\ell)) = (b_V^*)^{-1}(\sum_k(\mathbf{L}_0)_k^\ell\beta_V^k) = \sum_k(\mathbf{L}_0)_k^\ell x^k$$

and

$$\flat_V({}^t\mathbf{L}_j(\flat_W^{-1}(\eta_j^\ell))) = \flat_V({}^t\mathbf{L}_j(\beta_j^\ell)) = \flat_V(\sum_k(\mathbf{L}_j)_k^\ell\beta_j^k) = \sum_k(\mathbf{L}_j)_k^\ell\xi_j^k\ ,$$

where the notations are self-explanatory. Hence, $\mathcal{M}(\mathbf{L}) : \mathcal{M}(\mathbf{V}) \to \mathcal{M}(\mathbf{W})$ is a morphism of $\mathbb{Z}_2^n\mathtt{LinMan}$.

Since $\mathcal{M}(\mathbf{L})$ is essentially the transpose of $\mathbf{L}$, we have defined a functor

$$\mathcal{M} : \mathbb{Z}_2^n\mathtt{FinVec} \to \mathbb{Z}_2^n\mathtt{LinMan}$$

and this functor coincides on $\mathbb{Z}_2^n\mathtt{CarVec}$ with the functor $\mathcal{M}$ that we defined earlier.

We already mentioned that the $\mathbb{Z}_2^n$-diffeomorphism, say h, between $V = \mathcal{M}(\mathbf{V})$ and $\mathbb{R}^{p|\underline{q}}$ is implemented by a basis $(b_k^i)_{i,k}$ of $\mathbf{V}$. Now we can explain this observation in more detail. Indeed, the basis chosen provides a $\mathbb{Z}_2^n$-vector space isomorphism $\mathbf{b} : \mathbf{V} \to \mathbf{R}^{p|\underline{q}}$, hence, the image $\mathcal{M}(\mathbf{b}) : \mathcal{M}(\mathbf{V}) \to \mathcal{M}(\mathbf{R}^{p|\underline{q}})$ is a $\mathbb{Z}_2^n$-diffeomorphism (it is even an isomorphism of $\mathbb{Z}_2^n\mathtt{LinMan}$), say $b : V \to \mathbb{R}^{p|\underline{q}}$. The diffeomorphism $b = \mathcal{M}(\mathbf{b})$ is a special case of the map $L = \mathcal{M}(\mathbf{L})$ of $\mathbb{Z}_2^n\mathtt{LinMan}$, whose construction has been described above. It is almost obvious from the penultimate paragraph that the diffeomorphism h coincides with the diffeomorphism b. Indeed, the diffeomorphism h is the product of two $\mathbb{Z}_2^n$-diffeomorphisms $\mathrm{h}_0 : V_0 \to \mathbb{R}^{p|\underline{0}}$ and $\mathrm{h}_> : V_> \to \mathbb{R}^{0|\underline{q}}$ (see k in the penultimate paragraph). The same holds for b, which is defined as $b = b_0 \times b_>$, where $b_0 : V_0 \to \mathbb{R}^{p|\underline{0}}$ and $b_> : V_> \to \mathbb{R}^{0|\underline{q}}$ (see (3.2.52) and (3.2.53)). The map h_0 is canonically induced by the basis $(b_k^0)_k$ of $\mathbf{V}_0$, and so is b_0; hence $\mathrm{h}_0 = b_0$. The $\mathbb{Z}_2^n$-diffeomorphism $b_>$ is defined by the corresponding $\mathbb{Z}_2^n$-algebra isomorphism

$$\bar{S}({}^t\mathbf{b}) : \bar{S}((\mathbf{R}^{0|\underline{q}})^\vee) \to \bar{S}(\mathbf{V}_*^\vee)\ ,$$

where the source algebra is $\Pi_\alpha\mathbb{R}\,\varepsilon^\alpha = \mathbb{R}[[\xi]]$. As seen above, this algebra morphism is fully defined by the transposes ${}^t\mathbf{b}_j : (\mathbf{R}^{q_j})^\vee \to \mathbf{V}_j^\vee$ and their action on the basis $(\varepsilon_j^\ell)_\ell$. The action is

$${}^t\mathbf{b}_j(\varepsilon_j^\ell) = \varepsilon_j^\ell \circ \mathbf{b}_j = \beta_j^\ell\ ,$$

since the image of any $v_j = \sum_k v_j^k\, b_k^j \in \mathbf{V}_j$ by the two maps is v_j^ℓ. It follows that

$$\bar{S}({}^t\mathbf{b}) = \flat^{-1}\ . \tag{3.2.54}$$

This yields $b_> = \mathrm{h}_>$. Finally, we get

$$\mathrm{h} = b = \mathcal{M}(\mathbf{b})\ . \tag{3.2.55}$$

3.3.3.3. *Vectorification functor.* In this subsection, we define the vectorification functor

$$\mathcal{V} : \mathbb{Z}_2^n\mathtt{LinMan} \to \mathbb{Z}_2^n\mathtt{FinVec}\ .$$

If $\mathsf{L} \in \mathbb{Z}_2^n\mathtt{LinMan}$ has dimension $p|\underline{q}$, we set

$$\mathcal{V}(\mathsf{L}) := \mathbf{L} := \left(\mathcal{O}_{\mathsf{L}}^{\mathrm{lin}}(|\mathsf{L}|)\right)^\vee = \bigoplus_i(\mathcal{O}_{\mathsf{L},\gamma_i}^{\mathrm{lin}}(|\mathsf{L}|))^\vee =: \bigoplus_i \mathbf{L}_i \in \mathbb{Z}_2^n\mathtt{FinVec}\ , \tag{3.2.56}$$

where $\mathbf{L}_i$ has dimension q_i ($q_0 = p$). Further, in view of Item (iii) of Remark 3.2.16, if $\phi : \mathsf{L} \to \mathsf{L}'$ is a morphism of $\mathbb{Z}_2^n\texttt{LinMan}$, then ${}^t\phi^*$ is a degree preserving linear map

$$\mathcal{V}(\phi) := \mathbf{\Phi} := {}^t\phi^* : \mathcal{V}(\mathsf{L}) = \big(\mathcal{O}^{\mathrm{lin}}_{\mathsf{L}}(|\mathsf{L}|)\big)^\vee \to \big(\mathcal{O}^{\mathrm{lin}}_{\mathsf{L}'}(|\mathsf{L}'|)\big)^\vee = \mathcal{V}(\mathsf{L}') \ .$$

The definition of $\mathcal{V}(\phi)$ implies that $\mathcal{V}$ is a functor.

2.3.3.4. *Compositions of the manifoldification and the vectorification functors.*

(i) We first turn our attention to $\mathcal{V} \circ \mathcal{M}$. If

$$\mathbf{V} \in \mathbb{Z}_2^n\texttt{FinVec} \quad (\dim \mathbf{V} = p|\underline{q}) \ ,$$

its image

$$\mathcal{M}(\mathbf{V}) = V = V_0 \times V_> \in \mathbb{Z}_2^n\texttt{LinMan} \quad (\dim V = p|\underline{q})$$

is the product of the linear $\mathbb{Z}_2^n$-manifolds V_0 and $V_>$. Let $(b^i_\ell)_{i,\ell}$ be a basis of $\mathbf{V}$ with dual $(\beta^\ell_i)_{i,\ell}$ and induced $\mathbb{Z}_2^n$-vector space isomorphism $\mathbf{b} : \mathbf{V} \to \mathbf{R}^{p|\underline{q}}$ (we denote the induced diffeomorphism from V_0 to $\mathbb{R}^p$ by b_0). As explained above, it defines a linear coordinate map

$$\mathrm{h} = \mathcal{M}(\mathbf{b}) : V \to \mathbb{R}^{p|\underline{q}} \tag{3.2.57}$$

with pullback morphism

$$\mathrm{h}^* = (- \circ b_0)\widehat{\otimes}_{\mathbb{R}}\, \flat^{-1}) : C^\infty(\mathbb{R}^p)\widehat{\otimes}_{\mathbb{R}}\, \mathbb{R}[[\xi]] \to C^\infty_{V_0}(V_0)\widehat{\otimes}_{\mathbb{R}}\, \bar{S}(\mathbf{V}^\vee_*)$$

(see (3.2.55), (3.2.53) and (3.2.54)). Using Equation (3.2.14), denoting the basis of $(\mathbf{R}^{p|\underline{q}})^\vee$ as usual by $(\varepsilon^\ell_i)_{i,\ell}$, and remembering the identifications (3.2.11), we thus get

$$\mathcal{V}(\mathcal{M}(\mathbf{V})) = \big(\mathcal{O}^{\mathrm{lin}}_V(V_0)\big)^\vee = \big(\mathrm{h}^*\mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{p|\underline{q}}}(\mathbb{R}^p)\big)^\vee = \big(\mathrm{h}^*(\bigoplus_\ell \mathbb{R}\,\varepsilon^\ell_0 \oplus \bigoplus_{j,\ell} \mathbb{R}\,\varepsilon^\ell_j)\big)^\vee =$$

$$\big(\bigoplus_\ell \mathbb{R}\,(\varepsilon^\ell_0 \circ b_0) \oplus \bigoplus_{j,\ell} \mathbb{R}\,\flat^{-1}(\xi^\ell_j)\big)^\vee = \big(\bigoplus_\ell \mathbb{R}\,\beta^\ell_0 \oplus \bigoplus_{j,\ell} \mathbb{R}\,\beta^\ell_j\big)^\vee = \mathbf{V} \ .$$

(ii) Regarding $\mathcal{M} \circ \mathcal{V}$, recall that if

$$\mathsf{L} \in \mathbb{Z}_2^n\texttt{LinMan} \quad (\dim \mathsf{L} = p|\underline{q}) \ ,$$

Definition (3.2.56) yields $\mathcal{V}(\mathsf{L}) = \mathbf{L} = \big(\mathcal{O}^{\mathrm{lin}}_{\mathsf{L}}(|\mathsf{L}|)\big)^\vee$ (notice that $\mathbf{L}$ denotes a vector space here, and not a linear map) and Definition (3.2.50) leads to $\mathcal{M}(\mathbf{L}) := L := (L_0, \mathcal{O}_L)$, where L_0 is $\mathbf{L}_0 = \big(\mathcal{O}^{\mathrm{lin}}_{\mathsf{L},\gamma_0}(|\mathsf{L}|)\big)^\vee$ viewed as smooth manifold, and where $\mathcal{O}_L(\omega)$ ($\omega \subset L_0$ open) is

$$\mathcal{O}_L(\omega) = C^\infty_{L_0}(\omega)\widehat{\otimes}_{\mathbb{R}}\, \bar{S}(\mathbf{L}^\vee_*)$$

(see (3.2.50)). If we choose a basis $(\beta^\ell_j)_{j,\ell}$ of $\mathbf{L}^\vee_*$, we have

$$\bar{S}(\mathbf{L}^\vee_*) = \Pi_k \bigoplus_{j_1 \le \ldots \le j_k} \bigoplus_{\ell_1 \lhd \ldots \lhd \ell_k} \mathbb{R}\,\beta^{\ell_1}_{j_1} \odot \ldots \odot \beta^{\ell_k}_{j_k} = \Pi_\alpha \mathbb{R}\,\beta^\alpha \ ,$$

where $\alpha^k_j \in \mathbb{N}$ (resp., $\alpha^k_j \in \{0,1\}$), if $\langle \gamma_j, \gamma_j \rangle$ is even (resp., odd) (see (3.2.49)). Just as

$$C^\infty_{\mathbb{R}^p}(\Omega)\widehat{\otimes}_{\mathbb{R}}\, \Pi_\alpha\, \mathbb{R}\,\xi^\alpha = C^\infty_{\mathbb{R}^p}(\Omega)\widehat{\otimes}_{\mathbb{R}}\, \mathbb{R}[[\xi]] = C^\infty_{\mathbb{R}^p}(\Omega)[[\xi]] = \Pi_\alpha\, C^\infty_{\mathbb{R}^p}(\Omega)\,\xi^\alpha$$

($\Omega \subset \mathbb{R}^p$ open) (see [15]), we have

$$\mathcal{O}_L(\omega) = \Pi_\alpha\, C^\infty_{L_0}(\omega)\,\beta^\alpha = \Pi_k \bigoplus_{j_1 \le \ldots \le j_k} \bigoplus_{\ell_1 \lhd \ldots \lhd \ell_k} C^\infty_{L_0}(\omega)\,\beta^{\ell_1}_{j_1} \odot \ldots \odot \beta^{\ell_k}_{j_k} \ . \tag{3.2.58}$$

Remark 3.2.23. Let us mention that L and L denote a priori different linear $\mathbb{Z}_2^n$-manifolds and that our goal is to show that they do coincide.

Recall first that, for any $\mathbb{Z}_2^n$-manifold M, there is a projection

$$\epsilon_M : \mathcal{O}_M \to C^\infty_{|M|}$$

of $|M|$-sheaves of $\mathbb{Z}_2^n$-algebras and that ϵ_M commutes with pullbacks. In particular, if $\mathrm{h} : \mathsf{L} \to \mathbb{R}^{p|\underline{q}}$ is a linear coordinate map of L [a (linear) $\mathbb{Z}_2^n$-diffeomorphism], its pullback is, for any open subset $|U| \subset |\mathsf{L}|$, a $\mathbb{Z}_2^n$-algebra isomorphism

$$\mathrm{h}^* : \mathcal{O}_{\mathbb{R}^{p|\underline{q}}}(|h|(|U|)) \to \mathcal{O}_{\mathsf{L}}(|U|)$$

and it restricts to a $\mathbb{Z}_2^n$-vector space isomorphism

$$\mathrm{h}^* : \mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{p|\underline{q}}}(|h|(|U|)) \to \mathcal{O}^{\mathrm{lin}}_{\mathsf{L}}(|U|) \;.$$

Further, as just said, we have

$$\epsilon_{\mathsf{L}} \circ \mathrm{h}^* = \mathrm{h}^* \circ \epsilon_{\mathbb{R}^{p|\underline{q}}} = (- \circ |\mathrm{h}|) \circ \epsilon_{\mathbb{R}^{p|\underline{q}}}$$

on $\mathcal{O}_{\mathbb{R}^{p|\underline{q}}}(|h|(|U|))$. Taking $|U| = |\mathsf{L}|$ and restricting the equality to degree zero linear functions

$$\mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{p|\underline{q}},\gamma_0}(\mathbb{R}^p) = (\mathbf{R}^p)^\vee$$

(see (3.2.14)), we obtain

$$\epsilon_{\mathsf{L}} \circ \mathrm{h}^* = - \circ |\mathrm{h}| \;, \tag{3.2.59}$$

or, equivalently,

$$\epsilon_{\mathsf{L}} = (- \circ |\mathrm{h}|) \circ (\mathrm{h}^*)^{-1} \;, \tag{3.2.60}$$

where $(\mathrm{h}^*)^{-1}$ is a vector space isomorphism from $(\mathbf{L}_0)^\vee = \mathcal{O}^{\mathrm{lin}}_{\mathsf{L},\gamma_0}(|\mathsf{L}|)$ to $(\mathbf{R}^p)^\vee$ and where $-\circ|\mathrm{h}|$ is an algebra isomorphism from $C^\infty(\mathbb{R}^p)$ to $C^\infty(|\mathsf{L}|)$. In view of the diffeomorphism $|\mathrm{h}| : |\mathsf{L}| \to \mathbb{R}^p$, the smooth manifold $|\mathsf{L}|$ is linear. Hence, it is a finite dimensional vector space also denoted $|\mathsf{L}|$ and $|\mathrm{h}|$ is a vector space isomorphism, whose dual ${}^t|\mathrm{h}| = - \circ |\mathrm{h}|$ is a vector space isomorphism from $(\mathbf{R}^p)^\vee \subset C^\infty(\mathbb{R}^p)$ to $|\mathsf{L}|^\vee$. It follows (see also Equation (3.2.60)) that the canonical map ϵ_{L} is a vector space isomorphism from $(\mathbf{L}_0)^\vee$ to $|\mathsf{L}|^\vee$. When identifying these vector spaces, we get $\epsilon_{\mathsf{L}} = \mathrm{id}$ and $|\mathsf{L}| = \mathbf{L}_0$, hence the corresponding linear manifolds do also coincide: $|\mathsf{L}| = L_0$.

To prove that the linear $\mathbb{Z}_2^n$-manifolds L and L coincide, it now suffices to show that their function sheaves coincide. The pullback of h is an isomorphism $\mathrm{h}^* : \mathcal{O}_{\mathbb{R}^{p|\underline{q}}} \to \mathcal{O}_{\mathsf{L}}$ of sheaves of $\mathbb{Z}_2^n$-algebras. Since h^* is a $\mathbb{Z}_2^n$-vector space isomorphism

$$\mathrm{h}^* : (\mathbf{R}^{0|\underline{q}})^\vee = \mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{0|\underline{q}}}(\mathbb{R}^p) \to \mathcal{O}^{\mathrm{lin}}_{\mathsf{L},*}(|\mathsf{L}|) = \mathbf{L}_*^\vee \;,$$

the images $(\mathrm{h}^*(\varepsilon_j^\ell))_{j,\ell}$ are a basis $(\beta_j^\ell)_{j,\ell}$ of $\mathbf{L}_*^\vee$. Moreover, we know that

$$|\mathrm{h}| = (\ldots, \epsilon_{\mathsf{L}}(\mathrm{h}^*(\varepsilon_0^\ell)), \ldots) = (\ldots, \mathrm{h}^*(\varepsilon_0^\ell), \ldots) \;,$$

as $\epsilon_{\mathsf{L}} = \mathrm{id}$ on $(\mathbf{L}_0)^\vee$. Therefore, if $f(x) \in C^\infty(\mathbb{R}^p)$, we get

$$\mathrm{h}^*(f(x)) = f(\mathrm{h}^*(x)) = f \circ (\ldots, \mathrm{h}^*(\varepsilon_0^\ell), \ldots) = f \circ |\mathrm{h}| \in C^\infty(|\mathsf{L}|) \;. \tag{3.2.61}$$

Equation (3.2.61) (which generalizes Equation (3.2.59)) shows that h^* is an algebra isomorphism $\mathrm{h}^* : C^\infty(\mathbb{R}^p) \to C^\infty(|\mathsf{L}|)$. Similarly, if $\omega \subset |\mathsf{L}|$ is open, $\Omega := |\mathrm{h}|(\omega) \subset \mathbb{R}^p$ and $f(x) \in C^\infty(\Omega)$, we have

$$\mathrm{h}^*(f(x)) = f \circ |\mathrm{h}||_\omega \in C^\infty(\omega) \;,$$

so that

$$h^* : C^\infty(\Omega) \to C^\infty(\omega) \tag{3.2.62}$$

is also an algebra isomorphism. Finally, the $\mathbb{Z}_2^n$-algebra isomorphism

$$h^* : \Pi_\alpha C^\infty(\Omega)\, \xi^\alpha \to \mathcal{O}_{\mathsf{L}}(\omega) \tag{3.2.63}$$

sends any series $\sum_\alpha f_\alpha(x)\xi^\alpha$ to

$$\sum_\alpha h^*(f_\alpha(x))(\ldots, h^*(\xi_j^\ell), \ldots)^\alpha = \sum_\alpha h^*(f_\alpha(x))(\ldots, h^*(\varepsilon_j^\ell), \ldots)^\alpha = \sum_\alpha h^*(f_\alpha(x))\beta^\alpha \in \mathcal{O}_L(\omega)$$

(see (3.2.58)). The $\mathbb{Z}_2^n$-algebra morphism

$$h^* : \Pi_\alpha C^\infty(\Omega)\xi^\alpha \to \mathcal{O}_L(\omega) \tag{3.2.64}$$

we get this way (notice that the targets of the arrows (3.2.63) and (3.2.64) are different) is visibly an isomorphism. Indeed, it is obviously injective, and it is surjective due to (3.2.62). It follows from (3.2.63) and (3.2.64) that $\mathcal{O}_{\mathsf{L}}(\omega) = \mathcal{O}_L(\omega)$, for any open subset $\omega \subset |\mathsf{L}|$. Since h^* commutes with restrictions, the sheaves $\mathcal{O}_{\mathsf{L}}$ and $\mathcal{O}_L$ coincide and $\mathcal{M}(\mathcal{V}(\mathsf{L})) = \mathsf{L}$. An alternative way of saying what we just said is to observe that in view of (3.2.63) every element of $\mathcal{O}_{\mathsf{L}}(\omega)$ is the image by h^* of a unique series $\sum_\alpha f_\alpha(x)\xi^\alpha$ and therefore belongs to $\mathcal{O}_L(\omega)$. Conversely, in view of (3.2.62) every element $\sum_\alpha g_\alpha \beta^\alpha$ ($g_\alpha \in C^\infty(\omega)$) of $\mathcal{O}_L(\omega)$ uniquely reads $\sum_\alpha h^*(f_\alpha(x))\beta^\alpha$, is therefore the image by h^* of $\sum_\alpha f_\alpha(x)\xi^\alpha$ and so belongs to $\mathcal{O}_{\mathsf{L}}(\omega)$.

(iii) We leave it to the reader to check that both functors, $\mathcal{V} \circ \mathcal{M}$ and $\mathcal{M} \circ \mathcal{V}$, coincide also on morphisms with the identity functors.

Theorem 3.2.24. *The functors*

$$\mathcal{M} : \mathbb{Z}_2^n\texttt{FinVec} \rightleftarrows \mathbb{Z}_2^n\texttt{LinMan} : \mathcal{V}$$

are an isomorphism of categories.

2.3.3.5. *Comparison of the functors of points.* Since $\mathbb{Z}_2^n\texttt{FinVec} \simeq \mathbb{Z}_2^n\texttt{LinMan}$, the fully faithful functors of points $\mathcal{F}$ (see Proposition 3.2.8) and $\mathcal{S}$ (see Theorem 3.2.20) of these categories should coincide. However, up till now, the functor $\mathcal{F}$ is valued in $\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$, whereas the functor $\mathcal{S}$ is valued in $\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})$. Since FAMod is a subcategory of AMod , the latter functor category is a subcategory of the former. Hence, if we show that the image $\mathcal{F}(\mathbf{V})$ of any object $\mathbf{V}$ of $\mathbb{Z}_2^n\texttt{FinVec}$ is a functor of $\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})$ ($\star$) and that the image $\mathcal{F}(\phi)$ of any morphism $\phi : \mathbf{V} \to \mathbf{W}$ of $\mathbb{Z}_2^n\texttt{FinVec}$ is a natural transforation of $\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})$ ($*$), we can conclude that $\mathcal{F}$ is a functor

$$\mathcal{F} : \mathbb{Z}_2^n\texttt{FinVec} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod}) \ .$$

We start proving ($\star$). Since FAMod is a subcategory of AMod , we just have to show that the image

$$\mathcal{F}(\mathbf{V})(\Lambda) = \mathbf{V}(\Lambda) = (\Lambda \otimes \mathbf{V})_0$$

of any object Λ of $\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}$ is a Fréchet Λ_0-module ($\bullet$) and that the image

$$\mathcal{F}(\mathbf{V})(\varphi^*) = \mathbf{V}(\varphi^*) = (\varphi^* \otimes \mathbb{1}_{\mathbf{V}})_0$$

of any morphism $\varphi^* : \Lambda \to \Lambda'$ of $\mathbb{Z}_2^n\texttt{Alg}$ is a morphism of FAMod ($\circ$).

To prove ($\bullet$), we consider a basis of $\mathbf{V}$ ($\dim \mathbf{V} = p|\underline{q}$), i.e., an isomorphism $\mathbf{b} : \mathbf{V} \rightleftarrows \mathbf{R}^{p|\underline{q}} : \mathbf{b}^{-1}$ of $\mathbb{Z}_2^n$-vector spaces. Since $\mathcal{F}(\mathbf{b}) = \mathbf{b}_-$ is a natural isomorphism of $\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$, any of its Λ-components is an isomorphism

$$\mathbf{b}_\Lambda : \mathbf{V}(\Lambda) \rightleftarrows \mathbf{R}^{p|\underline{q}}(\Lambda) : \mathbf{b}_\Lambda^{-1}$$

of Λ_0-modules. We use this isomorphism to transfer to $\mathbf{V}(\Lambda)$ the Fréchet vector space structure of

$$\mathbf{R}^{p|\underline{q}}(\Lambda) = (\Lambda \otimes \mathbf{R}^{p|\underline{q}})_0 = \bigoplus_i \bigoplus_k \Lambda_{\gamma_i} = \Pi_i \Pi_k \Lambda_{\gamma_i} = \Lambda_0^{\times p} \times \Lambda_{\gamma_1}^{\times q_1} \times \ldots \times \Lambda_{\gamma_N}^{\times q_N} \tag{3.2.65}$$

(see Proof of Proposition 3.2.17 and Equation (3.2.17)), thus obtaining a well-defined Fréchet structure and making $\mathbf{b}_\Lambda$ a Fréchet vector space isomorphism, i.e., a continuous linear map with continuous inverse. Since $\mathbf{b}_\Lambda$ is Λ_0-linear, the action $\cdot$ of Λ_0 on $\mathbf{V}(\Lambda)$ is related to its action $\triangleleft$ on $\mathbf{R}^{p|\underline{q}}(\Lambda)$ by

$$\mathrm{a} \cdot \mathrm{v} = \mathbf{b}_\Lambda^{-1}(\mathrm{a} \triangleleft \mathbf{b}_\Lambda(\mathrm{v})) \;,$$

for any $\mathrm{a} \in \Lambda_0$ and any $\mathrm{v} \in \mathbf{V}(\Lambda)$. The action $\cdot$ is thus the composite of the continuous maps $\mathrm{id} \times \mathbf{b}_\Lambda$, $\triangleleft$, and $\mathbf{b}_\Lambda^{-1}$, hence, it is itself continuous. The Λ_0-module and the Fréchet vector space structures on $\mathbf{V}(\Lambda)$ therefore define a Fréchet Λ_0-module structure on $\mathbf{V}(\Lambda)$ and $\mathbf{b}_\Lambda$ becomes an isomorphism of Fréchet Λ_0-modules (for any basis $\mathbf{b}$ of $\mathbf{V}$).

As concerns ($\circ$), recall that $\mathbf{V}(\varphi^*)$ is a $(\varphi^*)_0$-linear map, where the algebra morphism $(\varphi^*)_0 : \Lambda_0 \to \Lambda'_0$ is the restriction of φ^*. Observe now that, in view of (3.2.65), we have

$$\mathbf{R}^{p|\underline{q}}(\varphi^*) = (\varphi^* \otimes \mathbb{1})_0 = \Pi_i \Pi_k \, \varphi^* \;,$$

so that $\mathbf{R}^{p|\underline{q}}(\varphi^*)$ is continuous as product of continuous maps (indeed, the $\mathbb{Z}_2^n\texttt{Alg}$-morphism φ^* is continuous as pullback of the associated $\mathbb{Z}_2^n$-morphism). As $\mathbf{b}_-$ is a natural transformation of $\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$, we have

$$\mathbf{V}(\varphi^*) = \mathbf{b}_{\Lambda'}^{-1} \circ \mathbf{R}^{p|\underline{q}}(\varphi^*) \circ \mathbf{b}_\Lambda \;,$$

so that $\mathbf{V}(\varphi^*)$ is continuous (and $(\varphi^*)_0$-linear), hence, is a morphism of $\texttt{FAMod}$.

It remains to show that ($*$) holds. We know that $\mathcal{F}(\phi) = \phi_-$ is a natural transformation of $\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$, i.e., its Λ-components ϕ_Λ are Λ_0-linear maps and the naturality condition is satisfied. It thus suffices to explain that $\phi_\Lambda = (\mathbb{1} \otimes \phi)_0$ is continuous. Since Λ is a Fréchet algebra, it is a locally convex topological vector space (LCTVS) and $\mathbb{1} : \Lambda \to \Lambda$ is a degree zero continuous linear map. Further, since $\mathbf{V}$ and $\mathbf{W}$ are finite dimensional $\mathbb{Z}_2^n$-vector spaces, the degree zero linear map $\phi : \mathbf{V} \to \mathbf{W}$ is automatically continuous for the canonical LCTVS structures on its source and target. It follows that $\mathbb{1} \otimes \phi$ and $(\mathbb{1} \otimes \phi)_0$ are continuous linear maps.

Proposition 3.2.25. *The functor*

$$\mathcal{F} : \mathbb{Z}_2^n\texttt{FinVec} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})$$

is fully faithful.

Proof. The result is obvious in view of Proposition 3.2.8, since $\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})$ is a subcategory of $\texttt{Fun}_0\,(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AMod})$. □

We are now ready to refine the idea expressed at the beginning of this subsection that the (fully faithful) functors of points

$$\mathcal{F} : \mathbb{Z}_2^n\texttt{FinVec} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})$$

(see Proposition 3.2.25) and

$$\mathcal{S} : \mathbb{Z}_2^n\texttt{LinMan} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})$$

(see Theorem 3.2.20) of the isomorphic categories

$$\mathcal{M} : \mathbb{Z}_2^n\texttt{FinVec} \rightleftarrows \mathbb{Z}_2^n\texttt{LinMan} : \mathcal{V}$$

should coincide.

Theorem 3.2.26. *The functors*

$$\mathcal{S} \circ \mathcal{M},\ \mathcal{F} : \mathbb{Z}_2^n\texttt{FinVec} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})$$

are naturally isomorphic.

We first prove the theorem in the Cartesian case

$$\mathcal{M} : \mathbb{Z}_2^n\texttt{CarVec} \rightleftarrows \mathbb{Z}_2^n\texttt{CarMan} : \mathcal{V}$$

(see Proposition 3.2.5). More precisely, it follows from Proposition 3.2.25 and Theorem 3.2.20 that the functors $\mathcal{F}$ and $\mathcal{S}$ are (fully faithful) functors

$$\mathcal{F} : \mathbb{Z}_2^n\texttt{CarVec} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})$$

and

$$\mathcal{S} : \mathbb{Z}_2^n\texttt{CarMan} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})\ .$$

Actually:

Proposition 3.2.27. *The functors*

$$\mathcal{S} \circ \mathcal{M},\ \mathcal{F} : \mathbb{Z}_2^n\texttt{CarVec} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})$$

are naturally isomorphic.

Proof. In order to construct a natural isomorphism $\mathsf{I} : \mathcal{S} \circ \mathcal{M} \to \mathcal{F}$, we must define, for any $\mathbf{R}^{p|\underline{q}}$, a natural isomorphism

$$\mathsf{I}_{\mathbf{R}^{p|\underline{q}}} : \mathcal{S}(\mathbb{R}^{p|\underline{q}}) \to \mathcal{F}(\mathbf{R}^{p|\underline{q}})$$

of $\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod})$ that is natural in $\mathbf{R}^{p|\underline{q}}$. To build $\mathsf{I}_{\mathbf{R}^{p|\underline{q}}}$, we have to define, for each Λ, an isomorphism

$$\mathsf{I}_{\mathbf{R}^{p|\underline{q}},\Lambda} : \mathcal{S}(\mathbb{R}^{p|\underline{q}})(\Lambda) \to \mathcal{F}(\mathbf{R}^{p|\underline{q}})(\Lambda)$$

of Fréchet Λ_0-modules that is natural in Λ. Recalling that the source and target of this arrow are

$$\mathbb{R}^{p|\underline{q}}(\Lambda) = \operatorname{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(\mathbb{R}^{0|\underline{m}}, \mathbb{R}^{p|\underline{q}}) \quad (\mathbb{R}^{0|\underline{m}} \simeq \Lambda)$$

and

$$\mathbf{R}^{p|\underline{q}}(\Lambda) = (\Lambda \otimes \mathbf{R}^{p|\underline{q}})_0 = \bigoplus_{i=0}^{N} \bigoplus_{k=1}^{q_i} \Lambda_{\gamma_i} \otimes \mathbf{R}e_k^i = \bigoplus_{i=0}^{N} \bigoplus_{k=1}^{q_i} \Lambda_{\gamma_i} = \Lambda_0^{\times p} \times \Lambda_{\gamma_1}^{\times q_1} \times \ldots \times \Lambda_{\gamma_N}^{\times q_N}\ ,$$

respectively, we set

$$\mathsf{I}_{\mathbf{R}^{p|\underline{q}},\Lambda} : \mathrm{x} \mapsto \sum_{i,k} \mathrm{x}^*(u_i^k) \otimes e_k^i = (\mathrm{x}^*(u_i^k)) = (\mathrm{x}^*(x^k), \mathrm{x}^*(\xi_j^k)) =: (x_\Lambda^k, \xi_{j,\Lambda}^k) ,$$

where $(u_i^k) = (x^k, \xi_j^k)$ are the coordinates of $\mathbb{R}^{p|\underline{q}}$ and where $(e_k^i)_{i,k}$ is the canonical basis of $\mathbf{R}^{p|\underline{q}}$. Since we actually used this 1:1 correspondence to transfer the Fréchet Λ_0-module structure from $\mathbf{R}^{p|\underline{q}}(\Lambda)$ to $\mathbb{R}^{p|\underline{q}}(\Lambda)$ (see (3.2.22)), the bijection $\mathsf{I}_{\mathbf{R}^{p|\underline{q}},\Lambda}$ is an isomorphism of Fréchet Λ_0-modules. This isomorphism is natural with respect to Λ. Indeed, if $\varphi^* : \Lambda \to \Lambda'$ is a $\mathbb{Z}_2^n$-algebra map (with corresponding $\mathbb{Z}_2^n$-morphism φ), we have

$$\mathsf{I}_{\mathbf{R}^{p|\underline{q}},\Lambda'}\big(\mathbb{R}^{p|\underline{q}}(\varphi^*)(\mathrm{x})\big) = \mathsf{I}_{\mathbf{R}^{p|\underline{q}},\Lambda'}(\mathrm{x} \circ \varphi) = (\varphi^*(\mathrm{x}^*(x^k)), \varphi^*(\mathrm{x}^*(\xi_j^k))) = (\varphi^* \otimes \mathbb{1})_0\big(\mathsf{I}_{\mathbf{R}^{p|\underline{q}},\Lambda}(\mathrm{x})\big) .$$

It now suffices to check that $\mathsf{I}_{\mathbf{R}^{p|\underline{q}}}$ is natural with respect to $\mathbf{R}^{p|\underline{q}}$. Hence, let $\mathbf{L} : \mathbf{R}^{p|\underline{q}} \to \mathbf{R}^{r|\underline{s}}$ be a degree zero linear map and let $L : \mathbb{R}^{p|\underline{q}} \to \mathbb{R}^{r|\underline{s}}$ be the corresponding linear $\mathbb{Z}_2^n$-morphism $\mathcal{M}(\mathbf{L})$. In order to prove that

$$\mathsf{I}_{\mathbf{R}^{r|\underline{s}}} \circ \mathcal{S}(L) = \mathcal{F}(\mathbf{L}) \circ \mathsf{I}_{\mathbf{R}^{p|\underline{q}}} , \tag{3.2.66}$$

we have to show that the Λ-components of these natural transformations coincide. To find that these Fréchet Λ_0-module morphisms coincide, we must explain that they associate the same image to every $\mathrm{x} \in \mathbb{R}^{p|\underline{q}}(\Lambda)$. When denoting the coordinates of $\mathbb{R}^{r|\underline{s}}$ by $(u_i'^\ell) = (x'^\ell, \xi_j'^\ell)$, we obtain

$$\mathsf{I}_{\mathbf{R}^{r|\underline{s}},\Lambda}\big(\mathcal{S}(L)_\Lambda(\mathrm{x})\big) = \mathsf{I}_{\mathbf{R}^{r|\underline{s}},\Lambda}(L \circ \mathrm{x}) = \big(\mathrm{x}^*(L^*(x'^\ell)), \mathrm{x}^*(L^*(\xi_j'^\ell))\big) = (\mathrm{x}^*(L^*(u_i'^\ell))) ,$$

where

$$L^*(u_i'^\ell) = \sum_{k=1}^{q_i} \mathbf{L}_{ik}^{\ell i}\, u_i^k ,$$

in view of (3.2.12) and (3.2.13). It follows that

$$(\mathrm{x}^*(L^*(u_i'^\ell))) = \big(\sum_{k=1}^{q_i} \mathbf{L}_{ik}^{\ell i}\, \mathrm{x}^*(u_i^k)\big) = \sum_{i,\ell} \big(\sum_k \mathbf{L}_{ik}^{\ell i}\, \mathrm{x}^*(u_i^k)\big) \otimes e_\ell'^i = \sum_{i,k} \mathrm{x}^*(u_i^k) \otimes \big(\sum_\ell \mathbf{L}_{ik}^{\ell i} e_\ell'^i\big) =$$

$$\sum_{i,k} \mathrm{x}^*(u_i^k) \otimes \mathbf{L}(e_k^i) = (\mathbb{1} \otimes \mathbf{L})_0\big(\sum_{i,k} \mathrm{x}^*(u_i^k) \otimes e_k^i\big) = \mathcal{F}(\mathbf{L})_\Lambda(\mathsf{I}_{\mathbf{R}^{p|\underline{q}},\Lambda}(\mathrm{x})) ,$$

where $(e_\ell'^i)_{i,\ell}$ is the basis of $\mathbf{R}^{r|\underline{s}}$. □

We are now able to prove Theorem 3.2.26.

Proof. For simplicity, we set

$$\mathtt{T} := \mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{FAMod}) .$$

In order to build a natural isomorphism $\mathcal{I} : \mathcal{S} \circ \mathcal{M} \to \mathcal{F}$, we must define, for any $\mathbf{V} \in \mathbb{Z}_2^n\mathtt{FinVec}$, a natural isomorphism

$$\mathcal{I}_\mathbf{V} : \mathcal{S}(V) \to \mathcal{F}(\mathbf{V})$$

of $\mathtt{T}$ that is natural in $\mathbf{V}$.

Set

$$\dim \mathbf{V} = p|\underline{q}$$

and let $\mathbf{b}$ be a basis of $\mathbf{V}$, or, equivalently, a $\mathbb{Z}_2^n$-vector space isomorphism $\mathbf{b} : \mathbf{V} \to \mathbf{R}^{p|\underline{q}}$. In view of (3.2.57), the morphism $\mathcal{M}(\mathbf{b}) : V \to \mathbb{R}^{p|\underline{q}}$ is a linear $\mathbb{Z}_2^n$-diffeomorphism. Using Proposition 3.2.25 and Theorem 3.2.20, we obtain that

$$\mathcal{F}(\mathbf{b}) : \mathcal{F}(\mathbf{V}) \to \mathcal{F}(\mathbf{R}^{p|\underline{q}})$$

and

$$\mathcal{S}(\mathcal{M}(\mathbf{b})) : \mathcal{S}(V) \to \mathcal{S}(\mathbb{R}^{p|\underline{q}})$$

are natural isomorphisms of $\mathtt{T}$. As

$$\mathsf{I}_{\mathbf{R}^{p|\underline{q}}} : \mathcal{S}(\mathbb{R}^{p|\underline{q}}) \to \mathcal{F}(\mathbf{R}^{p|\underline{q}})$$

is a natural isomorphism of $\mathtt{T}$ as well, the transformation

$$\mathcal{I}_{\mathbf{V}} := \mathcal{F}(\mathbf{b}^{-1}) \circ \mathsf{I}_{\mathbf{R}^{p|\underline{q}}} \circ \mathcal{S}(\mathcal{M}(\mathbf{b}))$$

is a natural isomorphism

$$\mathcal{I}_{\mathbf{V}} : \mathcal{S}(V) \to \mathcal{F}(\mathbf{V})$$

as requested. In view of Equation (3.2.66), the transformation $\mathcal{I}_{\mathbf{V}}$ is well-defined, i.e., is independent of the basis chosen.

It remains to show that $\mathcal{I}_{\mathbf{V}}$ is natural in $\mathbf{V}$, i.e., that, for any degree zero linear map $\phi : \mathbf{V} \to \mathbf{W}$ ($\dim \mathbf{W} = r|\underline{s}$) and for any basis $\mathbf{b}$ (resp., $\mathbf{c}$) of $\mathbf{V}$ (resp., $\mathbf{W}$), we have

$$\mathcal{F}(\phi) \circ \mathcal{F}(\mathbf{b}^{-1}) \circ \mathsf{I}_{\mathbf{R}^{p|\underline{q}}} \circ \mathcal{S}(\mathcal{M}(\mathbf{b})) = \mathcal{F}(\mathbf{c}^{-1}) \circ \mathsf{I}_{\mathbf{R}^{r|\underline{s}}} \circ \mathcal{S}(\mathcal{M}(\mathbf{c})) \circ \mathcal{S}(\mathcal{M}(\phi)) \;,$$

or, equivalently,

$$\mathsf{I}_{\mathbf{R}^{r|\underline{s}}} \circ \mathcal{S}(\mathcal{M}(\mathbf{c} \circ \phi \circ \mathbf{b}^{-1})) = \mathcal{F}(\mathbf{c} \circ \phi \circ \mathbf{b}^{-1}) \circ \mathsf{I}_{\mathbf{R}^{p|\underline{q}}} \;.$$

Since $\mathbf{L} := \mathbf{c} \circ \phi \circ \mathbf{b}^{-1}$ is a degree zero linear map $\mathbf{L} : \mathbf{R}^{p|\underline{q}} \to \mathbf{R}^{r|\underline{s}}$, Equation (3.2.66) allows once more to conclude. □

2.3.3.6. *Internal Homs.* A topological property is a property of topological spaces that is invariant under homeomorphisms (isomorphisms of topological spaces). More intuitively, a 'topological property' is a property that only depends on the topological structure, or, equivalently, that can be expressed by means of open subsets. Similarly, equivalences of categories ("isomorphisms" of categories) preserve all 'categorical properties and concepts'. Hence, an equivalence should preserve products. It turns out that this statement is actually correct. More precisely, if $\mathcal{E} : \mathtt{S} \to \mathtt{T}$ is part of an equivalence of categories, then a functor $\mathcal{D} : \mathtt{I} \to \mathtt{S}$ has limit s if and only if the functor $\mathcal{E} \circ \mathcal{D} : \mathtt{I} \to \mathtt{T}$ has limit $\mathcal{E}(s)$. Applying the statement to the discrete index category $\mathtt{I}$ with two objects $\{1, 2\}$ and setting $\mathcal{D}(i) = s_i$ ($i \in \{1, 2\}$), we get that s_1 and s_2 have product s if and only if $\mathcal{E}(s_1)$ and $\mathcal{E}(s_2)$ have product $\mathcal{E}(s)$. Now, the category $\mathbb{Z}_2^n\mathtt{FinVec}$ has the obvious binary product $\times$. It follows that, for any vector spaces $\mathbf{V}, \mathbf{W} \in \mathbb{Z}_2^n\mathtt{FinVec}$, the manifolds $\mathcal{M}(\mathbf{V}), \mathcal{M}(\mathbf{W}) \in \mathbb{Z}_2^n\mathtt{LinMan}$ have product

$$\mathcal{M}(\mathbf{V}) \times \mathcal{M}(\mathbf{W}) = \mathcal{M}(\mathbf{V} \times \mathbf{W}) \;.$$

If $\mathsf{L}, \mathsf{L}' \in \mathbb{Z}_2^n\mathtt{LinMan}$, the categorical isomorphism implies that $\mathsf{L} = \mathcal{M}(\mathcal{V}(\mathsf{L}))$ and similarly for L' , so that the product $\mathsf{L} \times \mathsf{L}'$ exists and is

$$\mathsf{L} \times \mathsf{L}' = \mathcal{M}(\mathcal{V}(\mathsf{L}) \times \mathcal{V}(\mathsf{L}')) \;. \tag{3.2.67}$$

Hence, the category $\mathbb{Z}_2^n\mathtt{LinMan}$ has finite products.

Equation (3.2.67) shows that we got the product of $\mathbb{Z}_2^n\texttt{LinMan}$ by transferring to $\texttt{T} := \mathbb{Z}_2^n\texttt{LinMan}$ the product of $\texttt{S} := \mathbb{Z}_2^n\texttt{FinVec}$. We can similarly transfer to $\texttt{T}$ the closed symmetric monoidal structure of $\texttt{S}$. Indeed, the category $\mathbb{Z}_2^n\texttt{Vec}$ is closed symmetric monoidal for the standard tensor product $-\otimes_{\mathbb{Z}_2^n\texttt{Vec}}-$ of $\mathbb{Z}_2^n$-vector spaces and the standard internal Hom $\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\texttt{Vec}}(-,-)$ of $\mathbb{Z}_2^n$-vector spaces, which is defined, on objects for instance, by

$$\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\texttt{Vec}}(\mathbf{V},\mathbf{W}) := \bigoplus_i \underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\texttt{Vec},\gamma_i}(\mathbf{V},\mathbf{W}) \in \mathbb{Z}_2^n\texttt{Vec}\;, \tag{3.2.68}$$

for any $\mathbf{V},\mathbf{W}\in\mathbb{Z}_2^n\texttt{Vec}$. Of course, if $\mathbf{V},\mathbf{W}\in\texttt{S}$, then $\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\texttt{Vec}}(\mathbf{V},\mathbf{W})\in\texttt{S}$, and the same holds for $\mathbf{V}\otimes_{\mathbb{Z}_2^n\texttt{Vec}}\mathbf{W}$. It follows that $\texttt{S} = \mathbb{Z}_2^n\texttt{FinVec}$ is also a closed symmetric monoidal category. If we set now

$$\mathsf{L}\otimes_{\texttt{T}}\mathsf{L}' := \mathcal{M}(\mathcal{V}(\mathsf{L})\otimes_{\mathbb{Z}_2^n\texttt{Vec}}\mathcal{V}(\mathsf{L}')) \quad\text{and}\quad \underline{\mathrm{Hom}}_{\texttt{T}}(\mathsf{L},\mathsf{L}') := \mathcal{M}\big(\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\texttt{Vec}}(\mathcal{V}(\mathsf{L}),\mathcal{V}(\mathsf{L}'))\big)\;, \tag{3.2.69}$$

and similarly for morphisms, we get a closed symmetric monoidal structure on $\texttt{T} = \mathbb{Z}_2^n\texttt{LinMan}$:

Proposition 3.2.28. *The category* $\mathbb{Z}_2^n\texttt{LinMan}$ *is closed symmetric monoidal for the structure* (3.2.69).

Alternatively, we could have defined $\underline{\mathrm{Hom}}_{\texttt{T}}(\mathsf{L},\mathsf{L}')\in\texttt{T}$ using the fully faithful functor of points

$$\mathcal{S}:\texttt{T}\ni\mathsf{L}\mapsto \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(-,\mathsf{L}) =: \mathsf{L}(-)\in\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\mathrm{op}},\texttt{FAMod})\;,$$

i.e., defining first a functor $F_{\mathsf{L},\mathsf{L}'}(-)$ in the target category, and then showing that this functor is representable by some $\underline{\mathrm{Hom}}_{\texttt{T}}(\mathsf{L},\mathsf{L}')\in\texttt{T}$:

$$F_{\mathsf{L},\mathsf{L}'}(-) = \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(-,\underline{\mathrm{Hom}}_{\texttt{T}}(\mathsf{L},\mathsf{L}')) = \underline{\mathrm{Hom}}_{\texttt{T}}(\mathsf{L},\mathsf{L}')(-)\;. \tag{3.2.70}$$

This ‘functor of points approach’ is often easier.

To shed some light on our more abstract definition above, we now compute $\underline{\mathrm{Hom}}_{\texttt{T}}(\mathbb{R}^{p|\underline{q}},\mathbb{R}^{r|\underline{s}})(\Lambda)$ ($\diamond$) assuming some familiarity with $\mathbb{Z}_2^n$-graded matrices $\mathrm{gl}(r|\underline{s}\times p|\underline{q}\,,\Lambda)$ with entries in $\Lambda\in\mathbb{Z}_2^n\texttt{Alg}$. Details can be found in Subsection 3.3.1 which we leave in its natural place. However, we highly recommend reading it before working though the end of this section.

We observe first that

$$\underline{\mathsf{Hom}}_{\mathbb{Z}_2^n\texttt{Vec},\gamma_k}(\mathbf{R}^{p|\underline{q}},\mathbf{R}^{r|\underline{s}}) = \mathrm{gl}_{\gamma_k}(r|\underline{s}\times p|\underline{q},\mathbb{R})\in\texttt{Vec}\;.$$

In order to understand the gist here, we consider the case $n=2$, so that a matrix $X\in\mathrm{gl}_{\gamma_k}(r|\underline{s}\times p|\underline{q},\mathbb{R})$ has the block format

$$X = \left(\begin{array}{c|c|c|c} X_{00} & X_{01} & X_{02} & X_{03}\\ \hline X_{10} & X_{11} & X_{12} & X_{13}\\ \hline X_{20} & X_{21} & X_{22} & X_{23}\\ \hline X_{30} & X_{31} & X_{32} & X_{33}\end{array}\right)\;, \tag{3.2.71}$$

where the degree x_{ij} of the block X_{ij} is

$$x_{ij} = \gamma_i+\gamma_j+\gamma_k\;. \tag{3.2.72}$$

Since the entries of the X_{ij} are real numbers and so of degree γ_0, all the blocks with non-vanishing x_{ij} do vanish. For instance, if $\gamma_k = 01\in\mathbb{Z}$ (resp., $\gamma_k = 11$) (do not confuse with the row-column index 01 in X_{01} (resp., 11 in X_{11})), the degree $x_{ij}=0$ if and only if

$ij \in \{01, 10, 23, 32\}$ (resp., $ij \in \{03, 12, 21, 30\}$) (as in most of the other cases in this text, the $\mathbb{Z}_2^n$-degrees are lexicographically ordered), so that only these X_{ij} do not vanish. It follows that

$$\underline{\mathsf{Hom}}_{\mathbb{Z}_2^n\mathtt{Vec}}(\mathbf{R}^{p|\underline{q}}, \mathbf{R}^{r|\underline{s}}) = \mathrm{gl}(r|\underline{s} \times p|\underline{q}, \mathbb{R}) \in \mathbb{Z}_2^n\mathtt{FinVec}$$

is made of the matrices (3.2.71), where no block X_{ij} vanishes a priori. The canonical basis of this $\mathbb{Z}_2^n$-vector space are the obvious matrices $E_{ik,j\ell}$ ($i, j \in \{0, \ldots, N\}$, $k \in \{1, \ldots, s_i\}$, $\ell \in \{1, \ldots, q_j\}$) with all entries equal to 0 except the entry kl in X_{ij} which is 1 . In view of Equation (3.2.72), the vectors of this basis have the degrees $\gamma_i + \gamma_j$. We can of course identify (up to renumbering) this $\mathbb{Z}_2^n$-vector space with $\mathbf{R}^{t|\underline{u}}$, where u_n ($n \in \{0, \ldots, N\}$) is equal to

$$u_n = \sum_{i,j:\gamma_i+\gamma_j=\gamma_n} s_i q_j \tag{3.2.73}$$

(we set $s_0 = r, q_0 = p, u_0 := t$). Hence:

$$\underline{\mathsf{Hom}}_{\mathbb{Z}_2^n\mathtt{Vec}}(\mathbf{R}^{p|\underline{q}}, \mathbf{R}^{r|\underline{s}}) = \mathrm{gl}(r|\underline{s} \times p|\underline{q}, \mathbb{R}) = \mathbf{R}^{t|\underline{u}} \in \mathbb{Z}_2^n\mathtt{CarVec}\ . \tag{3.2.74}$$

Combining (3.2.69) and (3.2.74), we get

$$\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\mathtt{LinMan}}(\mathbb{R}^{p|\underline{q}}, \mathbb{R}^{r|\underline{s}}) = \mathcal{M}(\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\mathtt{Vec}}(\mathbf{R}^{p|\underline{q}}, \mathbf{R}^{r|\underline{s}})) = \mathbb{R}^{t|\underline{u}} \in \mathbb{Z}_2^n\mathtt{CarMan}\ . \tag{3.2.75}$$

We now come back to ($\diamond$). Setting as usual $\mathbb{R}^{0|\underline{m}} \simeq \Lambda$, we get the isomorphism

$$\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\mathtt{LinMan}}(\mathbb{R}^{p|\underline{q}}, \mathbb{R}^{r|\underline{s}})(\Lambda) = \mathbb{R}^{t|\underline{u}}(\Lambda) \simeq \Pi_{n=0}^N \Lambda_{\gamma_n}^{\times u_n}$$

of Fréchet Λ_0-modules. On the other hand, the vector space $\mathrm{gl}_0(r|\underline{s} \times p|\underline{q}, \Lambda)$ is a Λ_0-module and this module 'coincides' obviously with

$$\mathrm{gl}_0(r|\underline{s} \times p|\underline{q}, \Lambda) = \Pi_{n=0}^N \Lambda_{\gamma_n}^{\times u_n}\ .$$

By transferring the Fréchet structure, we get an 'equality' of Fréchet Λ_0-modules. Hence, the Fréchet Λ_0-module isomorphism

$$\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\mathtt{LinMan}}(\mathbb{R}^{p|\underline{q}}, \mathbb{R}^{r|\underline{s}})(\Lambda) = \mathbb{R}^{t|\underline{u}}(\Lambda) \simeq \Pi_{n=0}^N \Lambda_{\gamma_n}^{\times u_n} = \mathrm{gl}_0(r|\underline{s} \times p|\underline{q}, \Lambda) \in \mathtt{F}\Lambda_0\mathtt{Mod}\ . \tag{3.2.76}$$

There is a natural upgrade that is independent of the internal Homs and makes $G := \mathrm{gl}_0(r|\underline{s} \times p|\underline{q}, -)$ a functor $G \in \mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{FAMod})$. Indeed, it suffices to define G on a $\mathbb{Z}_2^n\mathtt{Alg}$-morphism $\varphi^* : \Lambda \to \Lambda'$ as

$$G(\varphi^*) : G(\Lambda) \ni X \mapsto \varphi^*(X) \in G(\Lambda')\,,$$

where $\varphi^*(X)$ is defined entry-wise. The morphism $G(\varphi^*)$ is clearly $(\varphi^*)_0$-linear. It is also continuous, as it can be viewed as a product of copies of φ^* . Since G respects compositions and identities it is actually a functor of the functor category mentioned. The functors G and $\mathbb{R}^{t|\underline{u}}(-) = \mathcal{S}(\mathbb{R}^{t|\underline{u}})$ are of course naturally isomorphic. Since $\mathcal{S}$ is a fully faithful functor

$$\mathcal{S} : \mathbb{Z}_2^n\mathtt{LinMan} \to \mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{FAMod})\,,$$

the functor G can be viewed as represented by the linear $\mathbb{Z}_2^n$-manifold $\mathbb{R}^{t|\underline{u}}$.

Proposition 3.2.29. *The functor* $\mathrm{gl}_0(r|\underline{s} \times p|\underline{q}\,, -)$ *is representable and the Cartesian* $\mathbb{Z}_2^n$-*manifold*

$$\mathrm{gl}_0(r|\underline{s} \times p|\underline{q}) := \mathbb{R}^{t|\underline{u}}\,,$$

with dimension $t|\underline{u}$ *defined in Equation* (3.2.73), *is 'its' representing object.*

Example 3.2.30. For $n = 2$, we find that $\mathrm{gl}_0(1|1, 1, 1) = \mathbb{R}^{4|4,4,4}$.

3.3 $\mathbb{Z}_2^n$-Lie groups and linear actions

3.3.1 $\mathbb{Z}_2^n$-matrices

We will consider matrices that are valued in some $\mathbb{Z}_2^n$-Grassmann algebra Λ, though everything we say generalizes to arbitrary $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebras. A homogeneous matrix $X \in \mathrm{gl}_x(r|\underline{s} \times p|\underline{q}\,, \Lambda)$ of degree $x \in \mathbb{Z}$ is understood to be a block matrix

$$X = \left(\begin{array}{c|c|c} X_{00} & \dots & X_{0N} \\ \hline \vdots & \ddots & \vdots \\ \hline X_{N0} & \dots & X_{NN} \end{array}\right), \tag{3.3.1}$$

with the entries of each block X_{ij} being elements of the $\mathbb{Z}_2^n$-Grassmann algebra Λ. Here the degree $x_{ij} \in \mathbb{Z}$ of X_{ij} is

$$x_{ij} = \gamma_i + \gamma_j + x$$

and the dimension of X_{ij} is

$$\dim(X_{ij}) = s_i \times q_j$$

(setting $s_0 = r$ and $q_0 = p$ as usual). Addition of such matrices and multiplication by reals are defined in the obvious way and they endow $\mathrm{gl}_x(r|\underline{s} \times p|\underline{q}\,, \Lambda)$ with a vector space structure. We set

$$\mathrm{gl}(r|\underline{s} \times p|\underline{q}\,, \Lambda) := \bigoplus_{x \in \mathbb{Z}} \mathrm{gl}_x(r|\underline{s} \times p|\underline{q}\,, \Lambda) \in \mathbb{Z}_2^n\mathtt{Vec}\,.$$

Multiplication by an element of Λ requires an extra sign factor given by the row of the matrix, i.e., for any homogeneous $\lambda \in \Lambda_{\gamma_k}$, we have that

$$(\lambda\, X)_{ij} = (-1)^{\langle \gamma_k, \gamma_i \rangle} \lambda\, X_{ij}\,.$$

We thus obtain on $\mathrm{gl}(r|\underline{s} \times p|\underline{q}\,, \Lambda)$ a $\mathbb{Z}_2^n$-graded module structure over the $\mathbb{Z}_2^n$-commutative algebra Λ. If $r|\underline{s} = p|\underline{q}$, we write

$$\mathrm{gl}(p|\underline{q}\,, \Lambda) := \mathrm{gl}(p|\underline{q} \times p|\underline{q}\,, \Lambda)\,.$$

Multiplication of matrices in $\mathrm{gl}(p|\underline{q}\,, \Lambda)$ is via standard matrix multiplication – now taking care that the entries are from a $\mathbb{Z}_2^n$-commutative algebra. Equipped with this multiplication, the $\mathbb{Z}_2^n$-graded Λ-module $\mathrm{gl}(p|\underline{q}\,, \Lambda)$ is a $\mathbb{Z}_2^n$-graded associative unital $\mathbb{R}$-algebra. In particular, the degree zero matrices $\mathrm{gl}_0(p|\underline{q}, \Lambda)$ form an associative unital $\mathbb{R}$-algebra. Since multiplication of matrices only uses multiplication and addition in Λ, we can replace Λ not only, as said above, by any $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebra, but also by any $\mathbb{Z}_2^n$-commutative ring R and then get a ring $\mathrm{gl}_0(p|\underline{q}, R)$. We denote by $\mathrm{GL}(p|\underline{q}, R)$ the group of invertible matrices in $\mathrm{gl}_0(p|\underline{q}, R)$. For further details the reader may consult [18].

3.3.2 Invertibility of $\mathbb{Z}_2^n$-matrices

Let R be a $\mathbb{Z}_2^n$-commutative ring which is Hausdorff-complete in the J-adic topology, where J is the (proper) homogeneous ideal of R that is generated by the elements of non-zero degree $\gamma_j \in \mathbb{Z}_2^n$, $j \in \{1, \dots, N\}$. The $\mathbb{Z}_2^n$-graded ring morphism $\varepsilon : R \to R/J$, where

$$R/J = \bigoplus_i R_i/(R_i \cap J) = R_0/(R_0 \cap J)$$

vanishes in all non-zero degrees, induces a ring morphism

$$\tilde{\varepsilon} : \mathrm{gl}_0(p|\underline{q}, R) \ni X \mapsto \tilde{\varepsilon}(X) \in \mathrm{Diag}(p|\underline{q}, R/J) \ ,$$

where $\tilde{\varepsilon}(X)$ is the block-diagonal matrix with diagonal blocks $\tilde{\varepsilon}(X_{ii})$ (with commuting entries).

The following proposition appeared as Proposition 5.1. in [22]:

Proposition 3.3.1. *Let R be a J-adically Hausdorff-complete $\mathbb{Z}_2^n$-commutative ring and let $X \in \mathrm{gl}_0(p|\underline{q}, R)$ be a degree zero $p|\underline{q} \times p|\underline{q}$ matrix with entries in R, written in the standard block format*

$$X = \left(\begin{array}{c|c|c} X_{00} & \dots & X_{0N} \\ \hline \vdots & \ddots & \vdots \\ \hline X_{N0} & \dots & X_{NN} \end{array} \right).$$

We have:

$$X \in \mathrm{GL}(p|\underline{q}, R) \Leftrightarrow X_{ii} \in \mathrm{GL}(q_i, R), \forall i \Leftrightarrow \tilde{\varepsilon}(X) \in \mathrm{GL}(p|\underline{q}, R/J) \Leftrightarrow \tilde{\varepsilon}(X_{ii}) \in \mathrm{GL}(q_i, R/J), \forall i \ .$$

In this work, we are of course mainly interested in the case $R := \Lambda = \mathbb{R} \oplus \mathring{\Lambda}$ and $J = \mathring{\Lambda}$, so that $R/J = \mathbb{R}$.

3.3.3 $\mathbb{Z}_2^n$-Lie groups and their functor of points

Groups, or, better, group objects can easily be defined in any category with finite products, i.e., any category $\mathtt{C}$ with terminal object 1 and binary categorical products $c \times c'$ $(c, c' \in \mathtt{C})$.

If $\mathtt{C}$ is a concrete category, the definition of a group object is very simple. For instance, if $\mathtt{C}$ is the concrete category $A\mathtt{FM}$ of Fréchet manifolds over a Fréchet algebra A, a group object $\mathcal{G}$ in $\mathtt{C}$ is just an object $\mathcal{G} \in \mathtt{C}$ that is group whose structure maps $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and $\mathrm{inv} : \mathcal{G} \to \mathcal{G}$ are $\mathtt{C}$-morphisms, i.e., A-smooth maps. We refer of course to a group object in $A\mathtt{FM}$ as a *Fréchet A-Lie group.*

If $\mathtt{C}$ is the category $\mathbb{Z}_2^n\mathtt{Man}$ of $\mathbb{Z}_2^n$-manifolds, the definition of a group object is similar, but all the (natural) requirements (above) have to be expressed in terms of arrows (since there are no points here). More precisely, a group object G in $\mathtt{C}$ is an object $G \in \mathtt{C}$ that comes equipped with $\mathtt{C}$-morphisms

$$\mu : G \times G \to G, \quad \mathrm{inv} : G \to G \quad \text{and} \quad e : 1 \to G$$

(the terminal object 1 is here the $\mathbb{Z}_2^n$-manifold $\mathbb{R}^{0|\underline{0}} = (\{\star\}, \mathbb{R})$), which are called multiplication, inverse and unit, and satisfy the standard group properties (expressed by means of arrows): μ is associative, inv is a two-sided inverse of μ and e is a two-sided unit of μ . To understand the arrow expressions of these properties, we need the following notations. We denote by $\Delta : G \to G \times G$ the canonical diagonal $\mathtt{C}$-morphism and we denote by $e_G : G \to G$ the composite of the unique $\mathtt{C}$-morphism $1_G : G \to 1$ and the unit $\mathtt{C}$-morphism $e : 1 \to G$. The left inverse condition now reads

$$\mu \circ (\mathrm{inv} \times \mathrm{id}_G) \circ \Delta = e_G$$

and the left unit condition reads

$$\mu \circ (e_G \times \mathrm{id}_G) \circ \Delta = \mathrm{id}_G$$

(and similarly for the right conditions). The associativity of μ is of course encoded by

$$\mu \circ (\mu \times \mathrm{id}_G) = \mu \circ (\mathrm{id}_G \times \mu) \ . \tag{3.3.2}$$

We refer to a group object in $\mathbb{Z}_2^n\texttt{Man}$ as a $\mathbb{Z}_2^n$*-Lie group*.

A morphism $F : \mathcal{G} \to \mathcal{G}'$ of Fréchet A-Lie groups is of course defined as an A-smooth map that is a group morphism. Analogously, a morphism $F : \mathcal{G} \to \mathcal{G}'$ from a Fréchet A-Lie group to a Fréchet A'-Lie group is a morphism of $\texttt{AFM}$ that is also a group morphism. We denote the *category of Fréchet A-Lie groups* by $A\texttt{FLg}$ and we write $\texttt{AFLg}$ for the *category of Fréchet Lie groups over any Fréchet algebra.*

Further, a morphism $\Phi : G \to G'$ of $\mathbb{Z}_2^n$-Lie groups is a $\mathbb{Z}_2^n$-morphism that respects the multiplications, the inverses and the units (obvious arrow definitions). The *category of* $\mathbb{Z}_2^n$*-Lie groups* we denote by $\mathbb{Z}_2^n\texttt{Lg}$.

The functor of points of $\mathbb{Z}_2^n$-manifolds

$$\mathcal{S} : \mathbb{Z}_2^n\texttt{Man} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\mathrm{op}}, \texttt{AFM}) \tag{3.3.3}$$

induces a fully faithful functor of points of $\mathbb{Z}_2^n$-Lie groups:

Theorem 3.3.2. *The functor*

$$\mathcal{S} : \mathbb{Z}_2^n\texttt{Lg} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\mathrm{op}}, \texttt{AFLg}) \tag{3.3.4}$$

is fully faithful. Moreover, if $M \in \mathbb{Z}_2^n\texttt{Man}$ *and*

$$\mathcal{S}(M) = M(-) \in \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\mathrm{op}}, \texttt{AFLg}) ,$$

then $M \in \mathbb{Z}_2^n\texttt{Lg}$.

This theorem was announced as [13, Theorem 3.30.] without proper explanation or proof.

Proof. It is clear that we have subcategories

$$\texttt{AFLg} \subset \texttt{AFM}, \quad \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\mathrm{op}}, \texttt{AFLg}) \subset \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\mathrm{op}}, \texttt{AFM}) \quad \text{and} \quad \mathbb{Z}_2^n\texttt{Lg} \subset \mathbb{Z}_2^n\texttt{Man} .$$

Therefore, in order to prove that the functor (3.3.3) restricts to a functor (3.3.4), it suffices to show that $\mathcal{S}$ sends objects G and morphisms Φ of $\mathbb{Z}_2^n\texttt{Lg}$ to objects and morphisms of the functor category with target $\texttt{AFLg}$.

Observe first that, for any $M, N \in \mathbb{Z}_2^n\texttt{Man}$, we have the functor equality

$$\mathcal{S}(M \times N) = (M \times N)(-) = M(-) \times N(-) = \mathcal{S}(M) \times \mathcal{S}(N) , \tag{3.3.5}$$

in view of the universal property of $M \times N$. Further, if $\phi : M \to M'$ and $\psi : N \to N'$ are two $\mathbb{Z}_2^n$-morphisms, the natural transformation

$$\mathcal{S}(\phi \times \psi) = (\phi \times \psi)_- : (M \times N)(-) \to (M' \times N')(-)$$

becomes $\phi_- \times \psi_-$, if we read it through the identification (3.3.5).

Now, if $G \in \mathbb{Z}_2^n\texttt{Lg}$ with structure $\mathbb{Z}_2^n$-morphisms μ, inv (and e), then the $\texttt{AFM}$-valued functor $\mathcal{S}(G) = G(-)$ is actually $\texttt{AFLg}$-valued. This means that it sends any $\mathbb{Z}_2^n$-Grassmann algebra Λ and any $\mathbb{Z}_2^n\texttt{Alg}$-morphism $\varphi^* : \Lambda \to \Lambda'$ to an object $G(\Lambda)$ and a morphism $G(\varphi^*)$ of $\texttt{AFLg}$.

For $G(\Lambda) \in \Lambda_0\texttt{FM}$, notice that the natural transformations $\mathcal{S}(\mu) = \mu_-$, $\mathcal{S}(\mathrm{inv}) = \mathrm{inv}_-$ (and $\mathcal{S}(e) = e_-$) have Λ_0-smooth Λ-components

$$\mu_\Lambda : G(\Lambda) \times G(\Lambda) \to G(\Lambda), \quad \mathrm{inv}_\Lambda : G(\Lambda) \to G(\Lambda) \quad (\text{and} \quad e_\Lambda : 1(\Lambda) \to G(\Lambda))$$

(the Fréchet Λ_0-manifold $1(\Lambda)$ is the singleton that consists of the $\mathbb{Z}_2^n\texttt{Alg}$-morphism ι_Λ that sends any real number to itself viewed as an element of Λ) that define a group structure on $G(\Lambda)$ (with unit $1_\Lambda := e_\Lambda(\iota_\Lambda)$), which is therefore a Fréchet Λ_0-Lie group. The group properties of these structure maps are consequences of the group properties of the structure maps of G. For instance, when we apply $\mathcal{S}$ to the associativity equation (3.3.2) and then take the Λ-component of the resulting natural transformation, we get

$$\mu_\Lambda \circ (\mu_\Lambda \times \mathrm{id}_{G(\Lambda)}) = \mu_\Lambda \circ (\mathrm{id}_{G(\Lambda)} \times \mu_\Lambda) \ .$$

As for $G(\varphi^*) : G(\Lambda) \to G(\Lambda')$, we know that it is an $\texttt{AFM}$-morphism and have to show that it respects the multiplications μ_Λ and $\mu_{\Lambda'}$, i.e., that

$$\mu_{\Lambda'} \circ (G(\varphi^*) \times G(\varphi^*)) = G(\varphi^*) \circ \mu_\Lambda \ . \tag{3.3.6}$$

However, this equality is nothing other than the naturalness property of μ_- .

Finally, let $\Phi : G \to G'$ be a $\mathbb{Z}_2^n\texttt{Lg}$-morphism and denote the multiplications of the source and target by μ and μ' , respectively. In order to prove that the natural transformation $\mathcal{S}(\Phi) = \Phi_- : G(-) \to G'(-)$ of the functor category with target $\texttt{AFM}$ is a natural transformation of the functor category with target $\texttt{AFLg}$, it suffices to show that Φ_Λ is a morphism of $\texttt{AFLg}$, which results from the application of the functor $\mathcal{S}$ to the commutative diagram

$$\mu' \circ (\Phi \times \Phi) = \Phi \circ \mu \ . \tag{3.3.7}$$

The next task is to show that the functor (3.3.4) is fully faithful, i.e., that the map

$$\mathcal{S}_{G,G'} : \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Lg}}(G, G') \ni \Phi \mapsto \Phi_- \in \mathrm{Hom}_{\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\mathrm{op}}, \texttt{AFLg})}(G(-), G'(-)) \tag{3.3.8}$$

is a 1:1 correspondence, for any $\mathbb{Z}_2^n$-Lie groups G, G'. Since the functor (3.3.3) is fully faithful, any natural transformation in the target set of (3.3.8) is implemented by a unique $\mathbb{Z}_2^n$-morphism $\phi : G \to G'$ and it suffices to show that ϕ respects the group operations, for instance, that is satisfies Equation (3.3.7). However, Equation (3.3.7) is satisfied if and only if

$$\mu'_\Lambda \circ (\phi_\Lambda \times \phi_\Lambda) = \phi_\Lambda \circ \mu_\Lambda \ ,$$

for all Λ . The latter condition holds, since ϕ_Λ is, by assumption, a group morphism.

We must still prove the last statement of Theorem 3.3.2. The assumption implies that, for any $\mathbb{Z}_2^n$-Grassmann algebra Λ and any $\mathbb{Z}_2^n$-algebra morphism $\varphi^* : \Lambda \to \Lambda'$, we get a Fréchet Λ_0-Lie group $M(\Lambda)$ and a $(\varphi^*)_0$-smooth group morphism $M(\varphi^*) : M(\Lambda) \to M(\Lambda')$. We denote by 1_Λ (resp., $\mu_\Lambda, \mathrm{inv}_\Lambda$) the unit element (resp., the Λ_0-smooth multiplication, the Λ_0-smooth inverse) of the group structure on the Fréchet Λ_0-manifold $M(\Lambda)$. We have already observed (see (3.3.6)) that the fact that $M(\varphi^*)$ respects the multiplications μ_Λ and $\mu_{\Lambda'}$ is equivalent to that of μ_- being natural. The natural transformation $\mu_- : (M \times M)(-) \to M(-)$ is implemented by a unique $\mathbb{Z}_2^n$-morphism $\mu : M \times M \to M$. We obtain similarly a $\mathbb{Z}_2^n$-morphism $\mathrm{inv} : M \to M$. As for $e : 1 \to M$, we notice that the maps

$$e_\Lambda : 1(\Lambda) \ni \iota_\Lambda \mapsto 1_\Lambda \in M(\Lambda) \quad (\Lambda \in \mathbb{Z}_2^n\texttt{GrAlg})$$

define visibly a natural transformation with Λ_0-smooth Λ-components. Hence, it is implemented by a unique $\mathbb{Z}_2^n$-morphism $e : 1 \to M$. We leave it to the reader to check that μ, inv and e satisfy (3.3.2) and the other group properties. □

3.3.4 The general linear $\mathbb{Z}_2^n$-group

We want to define the *general linear* $\mathbb{Z}_2^n$*-group* of order $p|\underline{q}$ so that it is a $\_2^nZ$-Lie group $\mathrm{GL}(p|\underline{q})$. In view of Theorem 3.3.2, it suffices to define a functor

$$\mathrm{GL}(p|\underline{q})(-) \in \mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{AFLg})$$

that is represented by a $\mathbb{Z}_2^n$-manifold $\mathrm{GL}(p|\underline{q})$.

Definition 3.3.3. The *general linear* $\mathbb{Z}_2^n$*-group* $\mathrm{GL}(p|\underline{q})$ is defined, for any $\Lambda \in \mathbb{Z}_2^n\mathtt{GrAlg}$, by

$$\mathrm{GL}(p|\underline{q})(\Lambda) := \mathrm{GL}(p|\underline{q}\,,\Lambda) = \big\{X \in \mathrm{gl}_0(p|\underline{q}\,,\Lambda) : X \text{ is invertible}\big\}\,,$$

and, for any $\mathbb{Z}_2^n\mathtt{Alg}$-morphism $\varphi^* : \Lambda \to \Lambda'$ and any $X \in \mathrm{GL}(p|\underline{q})(\Lambda)$, by

$$\mathrm{GL}(p|\underline{q})(\varphi^*)(X) := \tilde{\varphi}^* X\,,$$

where $\tilde{\varphi}^*$ is φ^* acting on X entry-by-entry.

Theorem 3.3.4. *The maps* $\mathrm{GL}(p|\underline{q})(-)$ *of Definition 3.3.3 define a representable functor. We refer to the representing object* $\mathrm{GL}(p|\underline{q}) \in \mathbb{Z}_2^n\mathtt{Lg}$ *as the* general linear $\mathbb{Z}_2^n$-group of dimension $p|\underline{q}$.

Proof. Recall that:

i. It follows from Equation (3.2.76) that

$$\mathrm{gl}_0(p|\underline{q}\,,\Lambda) = \Pi_{n=0}^N \Lambda_{\gamma_n}^{\times u_n} = \Lambda_0^{\times t} \times \Pi_{j=1}^N \Lambda_{\gamma_j}^{\times u_j} \simeq \mathbb{R}^{t|\underline{u}}(\Lambda)\,,$$

where u_n is given by (3.2.73) ($t = u_0$).

ii. It follows from Proposition 3.3.1 that $X \in \mathrm{gl}_0(p|\underline{q}\,,\Lambda)$ is invertible if and only if $\tilde{\varepsilon}(X) \in \mathrm{GL}(p|\underline{q}\,,\mathbb{R})$, if and only if $\tilde{\varepsilon}(X_{ii}) \in \mathrm{GL}(q_i,\mathbb{R})$, for all $i \in \{0,\ldots,N\}$, if and only if $X_{ii} \in \mathrm{GL}(q_i,\Lambda)$, for all $i \in \{0,\ldots,N\}$.

In particular, a matrix

$$X \in \mathrm{gl}_0(p|\underline{q}\,,\mathbb{R}) = \mathbb{R}^t = \mathbb{R}^{p^2+\sum_j q_j^2} = \mathrm{Diag}(p|\underline{q}\,,\mathbb{R})$$

is invertible if and only if $X_{ii} \in \mathrm{GL}(q_i,\mathbb{R})$, for all i. It follows that

$$\mathcal{U}^t := \mathrm{GL}(p|\underline{q})(\mathbb{R}) = \Pi_{i=0}^N\, \mathrm{GL}(q_i,\mathbb{R}) \subset \mathbb{R}^t\,. \qquad (3.3.9)$$

As $\mathcal{U}^t \subset \mathbb{R}^t$ is open, we can consider the $\mathbb{Z}_2^n$-domain

$$\mathcal{U}^{t|\underline{u}} := (\mathcal{U}^t, \mathcal{O}_{\mathbb{R}^{t|\underline{u}}}|_{\mathcal{U}^t})\,, \qquad (3.3.10)$$

as well as its functor of points

$$\mathcal{U}^{t|\underline{u}}(-) \in \mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{AFM})\,,$$

with value on Λ

$$\mathcal{U}^{t|\underline{u}}(\Lambda) \simeq \mathcal{U}^t \times \mathring{\Lambda}_0^{\times t} \times \Pi_{j=1}^N \Lambda_{\gamma_j}^{\times u_j}$$

(see [13]).

On the other hand, we get

$$\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)=\{X\in\mathbb{R}^t\times\mathring{\Lambda}_0^{\times t}\times\Pi_{j=1}^N\Lambda_{\gamma_j}^{\times u_j}:(...,\tilde{\varepsilon}(X_{ii}),...)\in\Pi_{i=0}^N\,\mathrm{GL}(q_i,\mathbb{R})\}=\mathcal{U}^t\times\mathring{\Lambda}_0^{\times t}\times\Pi_{j=1}^N\Lambda_{\gamma_j}^{\times u_j}\ ,$$

so that $\mathcal{U}^{t|\underline{u}}(-)$ and $\mathrm{GL}(\underline{p}|\underline{q})(-)$ 'coincide' on objects Λ: if we denote the coordinates of $\mathbb{R}^{t|\underline{u}}$ as usually by $(u^{\mathfrak{a}})=(x^a,\xi^A)$, this 'equality' reads

$$\mathcal{U}^{t|\underline{u}}(\Lambda)\ni\mathrm{x}^*\simeq(\mathrm{x}^*(u^{\mathfrak{a}}))_{\mathfrak{a}}\in\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)\ .$$

Moreover, $\mathcal{U}^{t|\underline{u}}(-)$ and $\mathrm{GL}(\underline{p}|\underline{q})(-)$ coincide on morphisms $\varphi^*:\Lambda\to\Lambda'$. Indeed, the map $\mathrm{GL}(\underline{p}|\underline{q})(\varphi^*)$ acts on a matrix

$$(\mathrm{x}^*(u^{\mathfrak{a}}))_{\mathfrak{a}}\in\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)\subset\Lambda_0^{\times t}\times\Pi_{j=1}^N\Lambda_{\gamma_j}^{\times u_j}$$

by acting on all its entries $\mathrm{x}^*(u^{\mathfrak{a}})$ by φ^*, whereas the map $\mathcal{U}^{t|\underline{u}}(\varphi^*)$ acts on a $\mathbb{Z}_2^n\mathtt{Alg}$-morphism $\mathrm{x}^*\in\mathcal{U}^{t|\underline{u}}(\Lambda)$ by left composition $\varphi^*\circ\mathrm{x}^*$; if we identify x^* with the tuple $(\mathrm{x}^*(u^{\mathfrak{a}}))_{\mathfrak{a}}$, then $\mathcal{U}^{t|\underline{u}}(\varphi^*)$ acts by acting on each $\mathrm{x}^*(u^{\mathfrak{a}})$ by φ^*, which proves the claim.

It follows that $\mathrm{GL}(\underline{p}|\underline{q})(-)$ is a functor

$$\mathrm{GL}(\underline{p}|\underline{q})(-)\in\mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}},\mathtt{AFM})$$

that is represented by

$$\mathrm{GL}(\underline{p}|\underline{q}):=\mathcal{U}^{t|\underline{u}}\in\mathbb{Z}_2^n\mathtt{Man}\ ,\qquad(3.3.11)$$

so that it now suffices to prove that this functor is valued in $\mathtt{AFLg}$, i.e., it suffices to show that $\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)\in\Lambda_0\mathtt{FLg}$ and that $\mathrm{GL}(\underline{p}|\underline{q})(\varphi^*)$ is an $\mathtt{AFLg}$-morphism.

Recall that $\mathrm{gl}_0(\underline{p}|\underline{q}\,,\Lambda)$ is an associative unital $\mathbb{R}$-algebra for the standard matrix multiplication $\cdot$ (standard matrix addition, standard matrix multiplication by reals and standard unit matrix $\mathbb{I}$) (see Subsection 3.3.1). It is clear that the subset $\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)\subset\mathrm{gl}_0(\underline{p}|\underline{q}\,,\Lambda)$ is closed under $\cdot$:

$$\mu_\Lambda:\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)\times\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)\ni(X,Y)\mapsto X\cdot Y\in\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)\qquad(3.3.12)$$

is an associative unital multiplication on $\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)$. Therefore, μ_Λ and

$$\mathrm{inv}_\Lambda:\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)\ni X\mapsto X^{-1}\in\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)\qquad(3.3.13)$$

endow $\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)$ with a group structure (with unit $\mathbb{I}$). Finally, the Fréchet Λ_0-manifold $\mathrm{GL}(\underline{p}|\underline{q})(\Lambda)$ together with its group structure $\mu_\Lambda,\mathrm{inv}_\Lambda$ (and $\mathbb{I}$) is a Fréchet Λ_0-Lie group, if its structure maps μ_Λ and inv_Λ are Λ_0-smooth. This condition is actually satisfied (see below).

As for $\mathrm{GL}(\underline{p}|\underline{q})(\varphi^*)$, we know that it is an $\mathtt{AFM}$-morphism and need to show that it respects the multiplications $\mu_\Lambda,\mu_{\Lambda'}$. This condition is clearly met because $\mathrm{GL}(\underline{p}|\underline{q})(\varphi^*)$ acts entry-wise by the $\mathbb{Z}_2^n\mathtt{Alg}$-morphism φ^*.

It remains to explain why μ_Λ and inv_Λ are Λ_0-smooth.

Notice first that the source of the multiplication (3.3.12) is the open subset $\Omega(\Lambda):=\mathcal{U}^{t|\underline{u}}(\Lambda)\times\mathcal{U}^{t|\underline{u}}(\Lambda)$ of the Fréchet space $F(\Lambda):=\mathbb{R}^{t|\underline{u}}(\Lambda)\times\mathbb{R}^{t|\underline{u}}(\Lambda)$ (see [13]) and that we can choose the Fréchet vector space (and Fréchet Λ_0-module) $\mathbb{R}^{t|\underline{u}}(\Lambda)$ as its target. Since Λ is the ($\mathbb{Z}_2^n$-commutative nuclear) Fréchet $\mathbb{R}$-algebra of global $\mathbb{Z}_2^n$-functions of some $\mathbb{Z}_2^n$-point $\mathbb{R}^{0|\underline{m}}$, its addition and internal multiplication (its multiplication by reals and subtraction) are continuous maps. It follows that each component function of the standard matrix multiplication μ_Λ

is continuous, so that μ_Λ is itself continuous. We must now explain why all directional derivatives of μ_Λ exist everywhere and are continuous, and why the first derivative is Λ_0-linear. Let $(X,Y)\in\Omega(\Lambda)$ and $(V,W)\in F(\Lambda)$. We get

$$\mathrm{d}_{(X,Y)}\mu_\Lambda\,(V,W)=\lim_{t\to 0}\frac{(X+tV)\cdot(Y+tW)-X\cdot Y}{t}=X\cdot W+V\cdot Y\ .$$

Hence, the first derivative exists everywhere, is continuous and Λ_0-linear. Indeed, for any $\mathrm{a}\in\Lambda_0$, we have

$$\mathrm{d}_{(X,Y)}\mu_\Lambda\,(\mathrm{a}\cdot V,\mathrm{a}\cdot W)=\mathrm{a}\cdot\mathrm{d}_{(X,Y)}\mu_\Lambda\,(V,W)\ .$$

It is easily checked that

$$\mathrm{d}^2_{(X,Y)}\mu_\Lambda(V_1,W_1,V_2,W_2)=V_2\cdot W_1+V_1\cdot W_2\quad\text{and}\quad \mathrm{d}^{k\geq 3}_{(X,Y)}\mu_\Lambda(V_1,W_1,\ldots,V_k,W_k)=0\ ,$$

so that μ_Λ is actually Λ_0-smooth.

As for

$$\mathrm{inv}_\Lambda:\mathcal{U}^{t|\underline{u}}(\Lambda)\subset\mathbb{R}^{t|\underline{u}}(\Lambda)\to\mathbb{R}^{t|\underline{u}}(\Lambda)\ ,$$

we start computing the directional derivative of

$$\mathbb{I}_\Lambda:=\mu_\Lambda\circ(\mathrm{inv}_\Lambda\times\mathrm{id}_\Lambda)\circ\Delta_\Lambda:\mathcal{U}^{t|\underline{u}}(\Lambda)\subset\mathbb{R}^{t|\underline{u}}(\Lambda)\ni X\mapsto X^{-1}\cdot X=\mathbb{I}\in\mathbb{R}^{t|\underline{u}}(\Lambda)$$

(Δ_Λ is the diagonal map), assuming continuity of inv_Λ , for the time being. For any $V\in\mathbb{R}^{t|\underline{u}}(\Lambda)$, we have

$$\mathrm{d}_X\mathbb{I}_\Lambda(V)=\lim_{t\to 0}\frac{(X+tV)^{-1}\cdot(X+tV)-X^{-1}\cdot X}{t}=\lim_{t\to 0}\left(f_{XV}(t)\cdot X+g_{XV}(t)\cdot V\right)=0\ ,$$

where

$$f_{XV}(t)=\frac{(X+tV)^{-1}-X^{-1}}{t}\quad\text{and}\quad g_{XV}(t)=(X+tV)^{-1}\ .$$

It follows that

$$\mathrm{d}_X\,\mathrm{inv}_\Lambda(V)=\lim_{t\to 0}f_{XV}(t)=\lim_{t\to 0}\big((f_{XV}(t)\cdot X+g_{XV}(t)\cdot V)\cdot X^{-1}-g_{XV}(t)\cdot V\cdot X^{-1}\big)=-X^{-1}\cdot V\cdot X^{-1}\ ,$$

so that the first derivative is defined everywhere, is continuous, as well as Λ_0-linear. Also the higher order derivatives exist everywhere and are continuous. For instance, the second order derivative is given by

$$\begin{aligned}\mathrm{d}^2_X\,\mathrm{inv}_\Lambda(V,W)&=-\lim_{t\to 0}\big(f_{XW}(t)\cdot V\cdot X^{-1}+g_{XW}(t)\cdot V\cdot f_{XW}(t)\big)\\&=X^{-1}\cdot W\cdot X^{-1}\cdot V\cdot X^{-1}+X^{-1}\cdot V\cdot X^{-1}\cdot W\cdot X^{-1}\ .\end{aligned}\tag{3.3.14}$$

Finally, the inverse map inv_Λ is Λ_0-smooth, provided we prove its still pending continuity.

We will show that the continuity of (3.3.13) boils down to the continuity of the inverse map $\iota_\Lambda:\Lambda^\times\ni\lambda\mapsto\lambda^{-1}\in\Lambda^\times$ in Λ. Here $\Lambda^\times\subset\Lambda$ is the group of invertible elements of Λ . Since Λ is a (unital) Fréchet $\mathbb{R}$-algebra, its inverse map ι_Λ is continuous if and only if $\Lambda^\times$ is a G_δ-set, i.e., if and only if it is a countable intersection of open subsets of Λ [48]. We will show that $\Lambda^\times$ is actually open in the specific Fréchet $\mathbb{R}$-algebra Λ considered. In view of Equation (16) in [14], the topology of $\Lambda=\mathbb{R}[[\theta]]$ ($\Lambda\simeq\mathbb{R}^{0|\underline{m}}$) is induced by the countable family of seminorms

$$\rho_\beta(\lambda)=\frac{1}{\beta!}|\varepsilon(\partial_\theta^\beta\lambda)|=|\lambda_\beta|\quad(\beta\in\mathbb{N}^{\times|\underline{m}|},\lambda=\sum_\alpha\lambda_\alpha\theta^\alpha\in\Lambda)\ ,$$

where ε is the projection $\varepsilon : \Lambda \to \mathbb{R}$. This means that the topology is made of the unions of finite intersections of the open semiballs

$$B_\beta(\nu, \varepsilon) = \{\lambda \in \Lambda : \rho_\beta(\lambda - \nu) < \varepsilon\} = \{\lambda \in \Lambda : |\lambda_\beta - \nu_\beta| < \varepsilon\} = \{\lambda \in \Lambda : \lambda_\beta \in b(\nu_\beta, \varepsilon)\}$$

($\beta \in \mathbb{N}^{\times|\underline{m}|}, \nu = \sum_\alpha \nu_\alpha \theta^\alpha \in \Lambda, \varepsilon > 0$ and $b(\nu_\beta, \varepsilon)$ is the open ball in $\mathbb{R}$ with center ν_β and radius ε). Since

$$\Lambda^\times = \{\lambda \in \Lambda : \lambda_0 \in \mathbb{R} \setminus \{0\}\} \quad \text{and} \quad \mathbb{R} \setminus \{0\} = \bigcup_{r \in \mathbb{R}\setminus\{0\}} b(r, \varepsilon_r) \quad (\text{for some } \varepsilon_r > 0) ,$$

we get

$$\Lambda^\times = \bigcup_{r \in \mathbb{R}\setminus\{0\}} \{\lambda \in \Lambda : \lambda_0 \in b(r, \varepsilon_r)\} = \bigcup_{r \in \mathbb{R}\setminus\{0\}} B_0(r, \varepsilon_r) ,$$

which implies that $\Lambda^\times$ is open and that ι_Λ is continuous, as announced.

Before we are able to deduce from this that inv_Λ is continuous, we need an inversion formula for $X \in \mathrm{GL}(p|\underline{q})(\Lambda)$. Notice first that, in view of [18, Proposition 4.7], an invertible 2×2 block matrix

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.3.15}$$

with square diagonal blocks A and D and entries (of all blocks) in a ring, has a block UDL decomposition if and only if D is invertible. In this case, the UDL decomposition is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbb{I} & BD^{-1} \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ D^{-1}C & \mathbb{I} \end{pmatrix} .$$

As upper and lower unitriangular matrices are obviously invertible, it follows that the diagonal matrix is invertible, hence that $A - BD^{-1}C$ is invertible. Similarly, the invertible matrix X has a block LDU decomposition if and only if A is invertible and in this case $D - CA^{-1}B$ is invertible. Moreover, in view of Proposition 3.3.1, a matrix $X \in \mathrm{gl}_0(p|\underline{q}, \Lambda)$ is invertible if and only if all its diagonal blocks X_{ii} are invertible. Let now

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.3.16}$$

be a 2×2 block decomposition of $X \in \mathrm{gl}_0(p|\underline{q}, \Lambda)$ that respects the $(N+1) \times (N+1)$ block decomposition

$$X = \left(\begin{array}{c|c|c} X_{00} & \dots & X_{0N} \\ \hline \vdots & \ddots & \vdots \\ \hline X_{N0} & \dots & X_{NN} \end{array}\right) .$$

Since A (resp., D) is invertible if and only if

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & \mathbb{I} \end{pmatrix} \qquad \left(\text{resp., } \tilde{D} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & D \end{pmatrix}\right) \tag{3.3.17}$$

is invertible, hence, if and only if the X_{kk} on the diagonal of A (resp., D) are invertible, we get that X is invertible if and only if A and D are invertible. If we combine everything we have said so far in this paragraph, we find that if $X \in \mathrm{GL}(p|\underline{q})(\Lambda)$, then $A, D, A - BD^{-1}C, D - CA^{-1}B$ are all invertible. Therefore, we can use the formula

$$X^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} , \tag{3.3.18}$$

for any $X \in \mathrm{GL}(p|\underline{q})(\Lambda)$.

In order to simplify proper understanding, we consider for instance the case $n = 2$,

$$p|\underline{q} = p|q_1, q_2, q_3 = 1|2, 1, 1$$

and

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{c|cc|c|c} a & b & c & d & e \\ \hline f & g & h & i & j \\ k & l & m & n & p \\ \hline q & r & s & t & u \\ \hline v & w & x & y & z \end{array}\right) \in \mathrm{GL}(1|2,1,1)(\Lambda), \quad \text{where} \quad A = \left(\begin{array}{c|cc} a & b & c \\ \hline f & g & h \\ k & l & m \end{array}\right) ,$$

and so on. We focus for instance on the first of the four block matrices in X^{-1} , i.e., on $(A - BD^{-1}C)^{-1}$. The matrix D is a 2×2 invertible matrix with square diagonal blocks and entries in Λ . Since the four diagonal block matrices in X are invertible, it follows from what we have said above that the inverse D^{-1} is given by Equation (3.3.18) with $A = t \in \Lambda$, $B = u \in \Lambda$, $C = y \in \Lambda$ and $D = z \in \Lambda$. Hence all entries of D^{-1} are composites of the addition, the subtraction, the multiplication and the inverse in Λ , and so are all entries in the invertible 2×2 block matrix

$$A - BD^{-1}C = \left(\begin{array}{c|cc} \alpha & \beta & \gamma \\ \hline \delta & \varepsilon & \zeta \\ \eta & \theta & \xi \end{array}\right) \tag{3.3.19}$$

with square diagonal blocks (which are invertible) and with entries in Λ (the square diagonal blocks have entries in Λ_0). Hence, the inverse $(A - BD^{-1}C)^{-1}$ can again be computed by (3.3.18). We focus on its entry

$$\kappa := \left(\alpha - (\beta\,\gamma) \begin{pmatrix} \varepsilon & \zeta \\ \theta & \xi \end{pmatrix}^{-1} \begin{pmatrix} \delta \\ \eta \end{pmatrix} \right)^{-1} \in \Lambda\,. \tag{3.3.20}$$

Notice that here we cannot conclude that ε and ξ are invertible and apply (3.3.18) to compute the internal inverse. However, this inverse is the inverse of a square matrix with entries in the commutative ring Λ_0 , for which the standard inversion formula holds (recall that a square matrix with entries in a commutative ring is invertible if and only if its determinant is invertible):

$$\begin{pmatrix} \varepsilon & \zeta \\ \theta & \xi \end{pmatrix}^{-1} = (\varepsilon\xi - \zeta\theta)^{-1} \begin{pmatrix} \xi & -\zeta \\ -\theta & \varepsilon \end{pmatrix} . \tag{3.3.21}$$

Since all the entries of (3.3.19) are composites of the addition, subtraction, multiplication and inverse in Λ , it follows from (3.3.20) and (3.3.21) that the same is true for the entry κ of X^{-1}. More precisely the entry κ corresponds to a map $\tilde{\kappa}$ that is a composite of the inclusion of $\mathrm{GL}(p|\underline{q})(\Lambda)$ into its topological supspace $\Lambda^{\times(t+|\underline{u}|)}$ (continuous), the projection of $\Lambda^{\times(t+|\underline{u}|)}$ onto $\Lambda^{\times v}$ ($v \le t + |\underline{u}|$) (continuous) and of products of the identity map id of Λ (continuous), the diagonal map Δ of Λ (continuous), the switching map σ of $\Lambda \times \Lambda$ (continuous), the addition a of Λ (continuous), the scalar multiplication e of Λ (continuous), its subtraction s (continuous), multiplication m (continuous) and its inverse ι (continuous). Indeed, it is for instance easily seen that the map

$$\Lambda^{\times 4} \ni (t, u, y, z) \mapsto -z^{-1}y(t - uz^{-1}y)^{-1} \in \Lambda$$

is a (continuous) composite of products of these continuous maps. We thus understand that the entry κ of X^{-1} corresponds to a continuous map $\tilde{\kappa} : \mathrm{GL}(p|\underline{q})(\Lambda) \to \Lambda$. The same holds of course also for all the other entries of X^{-1}. Finally, the inverse map

$$\mathrm{inv}_\Lambda : \mathrm{GL}(p|\underline{q})(\Lambda) \ni X \mapsto X^{-1} \in \Lambda^{\times(t+|\underline{u}|)}$$

is continuous and it remains continuous when view as valued in the subspace $\mathrm{GL}(p|\underline{q})(\Lambda)$. □

Example 3.3.5. In view of Equations (3.3.11), (3.3.10) and (3.3.9), the general linear $\mathbb{Z}_2^2$-group of order $1|1,1,1$ is

$$\mathrm{GL}(1|1,1,1) \simeq \left((\mathbb{R}^\times)^4, \mathcal{O}_{\mathbb{R}^{4|4,4,4}}|_{(\mathbb{R}^\times)^4}\right) ,$$

where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$.

3.3.5 Smooth linear actions

In this section we define linear actions of $\mathbb{Z}_2^n$-Lie groups G on finite dimensional $\mathbb{Z}_2^n$-vector spaces $\mathbf{V} \simeq V$ (we identify the isomorphic categories $\mathbb{Z}_2^n\texttt{FinVec}$ and $\mathbb{Z}_2^n\texttt{LinMan}$). The definition can be given in the category of $\mathbb{Z}_2^n$-manifolds, but it is slightly more straightforward if we use the functor of points. Notice that the functors of points of $G \in \mathbb{Z}_2^n\texttt{Lg} \subset \mathbb{Z}_2^n\texttt{Man}$ and $V \in \mathbb{Z}_2^n\texttt{LinMan} \subset \mathbb{Z}_2^n\texttt{Man}$ are functors

$$\mathcal{S}(G) = G(-) \in \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AFLg}) \subset \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AFM})$$

and

$$\mathcal{S}(V) = V(-) \in \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{FAMod}) \subset \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AFM}) .$$

Definition 3.3.6. Let $G \in \mathbb{Z}_2^n\texttt{Lg}$ and $V \in \mathbb{Z}_2^n\texttt{LinMan}$. A *smooth linear action* of G on V is a natural transformation

$$\sigma_- : (G \times V)(-) = G(-) \times V(-) \to V(-)$$

in $\texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AFM})$ (natural transformation with Λ_0-smooth Λ-components) that satisfies the following conditions:

i. Identity: for all $v_\Lambda \in V(\Lambda)$, we have

$$\sigma_\Lambda(1_\Lambda, v_\Lambda) = v_\Lambda ,$$

where 1_Λ is the unit of $G(\Lambda)$.

ii. Compatibility: for all $g_\Lambda, g'_\Lambda \in G(\Lambda)$ and all $v_\Lambda \in V(\Lambda)$, we have

$$\sigma_\Lambda(g_\Lambda, \sigma_\Lambda(g'_\Lambda, v_\Lambda)) = \sigma_\Lambda(\mu_\Lambda(g_\Lambda, g'_\Lambda), v_\Lambda) ,$$

where μ_Λ is the multiplication of $G(\Lambda)$.

iii. Λ_0-linearity: for all $g_\Lambda \in G(\Lambda)$, all $v_\Lambda, v'_\Lambda \in V(\Lambda)$ and all $\mathrm{a} \in \Lambda_0$, we have

(a) $\sigma_\Lambda(g_\Lambda, v_\Lambda + v'_\Lambda) = \sigma_\Lambda(g_\Lambda, v_\Lambda) + \sigma_\Lambda(g_\Lambda, v'_\Lambda)$,
(b) $\sigma_\Lambda(g_\Lambda, \mathrm{a} \cdot v_\Lambda) = \mathrm{a} \cdot \sigma_\Lambda(g_\Lambda, v_\Lambda)$,

where $\cdot$ is the action of Λ_0 on $V(\Lambda)$.

Since

$$\mathcal{S} : \mathbb{Z}_2^n\texttt{Man} \to \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{AFM}) \tag{3.3.22}$$

is fully faithful (for more details, see [13, 14, 15]), there is a $1:1$ correspondence between natural transformations σ_- as above and $\mathbb{Z}_2^n$-morphisms

$$\sigma : G \times V \to V\ . \tag{3.3.23}$$

This correspondence implies in particular that Condition (ii) is equivalent to the equality

$$\sigma \circ (\mathrm{id}_G \times \sigma) = \sigma \circ (\mu \times \mathrm{id}_V) \tag{3.3.24}$$

of $\mathbb{Z}_2^n$-morphisms from $G \times G \times V \to V$ ($\mu : G \times G \to G$ is the multiplication of G). The same holds for Condition (i) and the equality

$$\sigma \circ (e \times \mathrm{id}_V) = \mathrm{id}_V \tag{3.3.25}$$

of $\mathbb{Z}_2^n$-morphisms from $V \simeq 1 \times V \to V$ ($e : 1 \to G$ is the two-sided unit of μ).

Canonical action of the general linear group.

We will now define the canonical action of the general linear $\mathbb{Z}_2^n$-group $\mathrm{GL}(p|\underline{q}) = \mathcal{U}^{t|\underline{u}} \in \mathbb{Z}_2^n\texttt{Lg}$ on the Cartesian $\mathbb{Z}_2^n$-manifold $\mathbb{R}^{p|\underline{q}} \in \mathbb{Z}_2^n\texttt{LinMan}$. To do this, we use both, the fully faithful functor (3.3.22) and the fully faithful functor

$$\mathcal{Y} : \mathbb{Z}_2^n\texttt{Man} \ni M \mapsto \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(-, M) \in \texttt{Fun}(\mathbb{Z}_2^n\texttt{Man}^{\text{op}}, \texttt{Set})\ . \tag{3.3.26}$$

We start defining a natural transformation σ_- of $\texttt{Fun}(\mathbb{Z}_2^n\texttt{Man}^{\text{op}}, \texttt{Set})$ from $\mathcal{U}^{t|\underline{u}}(-) \times \mathbb{R}^{p|\underline{q}}(-)$ to $\mathbb{R}^{p|\underline{q}}(-)$. We will denote the coordinates of $\mathcal{U}^{t|\underline{u}}$ (resp., $\mathbb{R}^{p|\underline{q}}$) here by $\mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}}$ (resp., $\S^{\mathfrak{c}}$), where $\mathfrak{a}, \mathfrak{b} \in \{1, \ldots, p + |\underline{q}|\}$ (resp., where $\mathfrak{c} \in \{1, \ldots, p + |\underline{q}|\}$). For this, we must associate to any $S \in \mathbb{Z}_2^n\texttt{Man}$, a set-theoretical map σ_S that assigns to any

$$(X, \phi) \in \mathcal{U}^{t|\underline{u}}(S) \times \mathbb{R}^{p|\underline{q}}(S) = \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(S, \mathcal{U}^{t|\underline{u}}) \times \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(S, \mathbb{R}^{p|\underline{q}})\ ,$$

i.e., to any (appropriate) coordinate pullbacks

$$(\mathcal{X}^{\mathfrak{a}}_{S,\mathfrak{b}}, \S^{\mathfrak{c}}_S) := (X^*(\mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}}), \phi^*(\S^{\mathfrak{c}})) \in \mathcal{O}(S)^{\times (p+|\underline{q}|)^2} \times \mathcal{O}(S)^{\times (p+|\underline{q}|)}\ , \tag{3.3.27}$$

a unique element $\sigma_S(X, \phi) \in \mathbb{R}^{p|\underline{q}}(S)$, i.e., unique (appropriate) coordinate pullbacks

$$\sigma_S(\mathcal{X}^{\mathfrak{a}}_{S,\mathfrak{b}}, \S^{\mathfrak{c}}_S) \in \mathcal{O}(S)^{\times (p+|\underline{q}|)}\ .$$

Since $(\S^{\mathfrak{c}}_S)_{\mathfrak{c}}$ is viewed as a tuple (horizontal row), the natural definition of this image (horizontal row) is

$$\sigma_S(\mathcal{X}^{\mathfrak{a}}_{S,\mathfrak{b}}, \S^{\mathfrak{c}}_S) = (\S^{\mathfrak{b}}_S\ \mathcal{X}^{\mathfrak{a}}_{S,\mathfrak{b}})_{\mathfrak{a}}\ , \tag{3.3.28}$$

where the sum and products are taken in the global $\mathbb{Z}_2^n$-function algebra $\mathcal{O}(S)$ of S. It is clear that the elements of this target-tuple have the required degrees, as the same holds for the elements of the source-tuple. The transformation σ_- we just defined is clearly natural. Indeed, for any $\mathbb{Z}_2^n$-morphism $\psi : S' \to S$, the induced set-theoretical mapping between the Hom-sets with source S and the corresponding ones with source S' is $- \circ \psi$, so that the induced set-theoretical mapping between the tuples of global $\mathbb{Z}_2^n$-functions of S and S' is ψ^*. The naturalness of σ_- follows now from the fact that ψ^* is a $\mathbb{Z}_2^n\texttt{Alg}$-morphism.

Since (3.3.26) is fully faithful, the natural transformation σ_- is implemented by a unique $\mathbb{Z}_2^n$-morphism

$$\sigma : \mathrm{GL}(p|\underline{q}) \times \mathbb{R}^{p|\underline{q}} \to \mathbb{R}^{p|\underline{q}} , \tag{3.3.29}$$

which in turn implements, via (3.3.22), a unique natural transformation in $\mathtt{Fun}_0(\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}, \mathtt{AFM})$ between the same functors, but restricted to $\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}$. Since this transformation is the restriction of σ_- to $\mathbb{Z}_2^n\mathtt{Pts}^{\mathrm{op}}$, we use this symbol for both transformations (provided that any confusion can be excluded). It is easily seen that

$$\sigma_\Lambda(\mathcal{X}^{\mathfrak{a}}_{\Lambda,\mathfrak{b}}, \S^{\mathfrak{c}}_\Lambda) = (\S^{\mathfrak{b}}_\Lambda\, \mathcal{X}^{\mathfrak{a}}_{\Lambda,\mathfrak{b}})_{\mathfrak{a}} ,$$

with sum and products in Λ, has the properties (i), (ii) and (iii) of Definition 3.3.6, so that we defined a smooth linear action of $\mathrm{GL}(p|\underline{q})$ on $\mathbb{R}^{p|\underline{q}}$.

The interesting aspect here is that we are able to compute the $\mathbb{Z}_2^n$-morphism (3.3.29). Indeed, in view of the proof of the full faithfulness of the standard Yoneda embedding $c \mapsto \mathrm{Hom}_{\mathtt{C}}(-, c)$ of an arbitrary locally small category $\mathtt{C}$ into the functor category $\mathtt{Fun}(\mathtt{C}^{\mathrm{op}}, \mathtt{Set})$, the morphism $\sigma \in \mathrm{Hom}_{\mathtt{C}}(c, c')$ that implements a natural transformation

$$\sigma_- : \mathrm{Hom}_{\mathtt{C}}(-, c) \to \mathrm{Hom}_{\mathtt{C}}(-, c')$$

is

$$\sigma = \sigma_c(\mathrm{id}_c) \in \mathrm{Hom}_{\mathtt{C}}(c, c') .$$

In our case of interest $\mathtt{C} = \mathbb{Z}_2^n\mathtt{Man}$, the previous Yoneda embedding is the functor (3.3.26) and the morphism

$$\sigma \in \mathrm{Hom}_{\mathbb{Z}_2^n\mathtt{Man}}(\mathrm{GL}(p|\underline{q}) \times \mathbb{R}^{p|\underline{q}}, \mathbb{R}^{p|\underline{q}})$$

is

$$\sigma = \sigma_c(\mathrm{id}_c), \quad \text{with} \quad c = \mathrm{GL}(p|\underline{q}) \times \mathbb{R}^{p|\underline{q}} .$$

Since the pullback of the identity $\mathbb{Z}_2^n$-morphism id_c is identity and the coordinate pullbacks (3.3.27) are

$$(\mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}}, \S^{\mathfrak{c}}) \in \mathcal{O}(c)^{\times(p+|\underline{q}|)(p+|\underline{q}|+1)} ,$$

Equation (3.3.28) yields

$$\sigma = \sigma_c(\mathrm{id}_c) \simeq \sigma_c(\mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}}, \S^{\mathfrak{c}}) = (\S^{\mathfrak{b}}\, \mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}})_{\mathfrak{a}} ,$$

with sum and products in $\mathcal{O}(c)$. In other words:

Proposition 3.3.7. *The canonical action σ of the general linear $\mathbb{Z}_2^n$-group $\mathrm{GL}(p|\underline{q})$ on the linear $\mathbb{Z}_2^n$-manifold or $\mathbb{Z}_2^n$-graded vector space $\mathbb{R}^{p|\underline{q}}$, is the $\mathbb{Z}_2^n$-morphism that is defined by the coordinate pullbacks*

$$\sigma^*(\S^{\mathfrak{a}}) = \S^{\mathfrak{b}}\, \mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}} , \tag{3.3.30}$$

where we denoted the coordinates of $\mathrm{GL}(p|\underline{q})$ (resp., $\mathbb{R}^{p|\underline{q}}$) by $\mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}}$ (resp., $\S^{\mathfrak{c}}$).

Example 3.3.8. We know that the general linear $\mathbb{Z}_2^2$-group $\mathrm{GL}(1|1,1,1)$ can be identified with the open $\mathbb{Z}_2^2$-submanifold $\mathcal{U}^{4|4,4,4}$ of $\mathbb{R}^{4|4,4,4}$. We denote the global coordinates of this Cartesian $\mathbb{Z}_2^2$-manifold by $(x^\alpha, \xi^\beta, \theta^\gamma,\ z^\delta)$. The indices run over $\{1,2,3,4\}$ and the $\mathbb{Z}_2^2$-degrees of these coordinates are $(0,0), (0,1), (1,0)$ and $(1,1)$, respectively. We already mentioned that if we view $\mathcal{U}^{4|4,4,4}$ as $\mathrm{GL}(1|1,1,1)$ we must rearrange the coordinates:

$$\mathcal{X} = (\mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}})_{\mathfrak{a},\mathfrak{b}} = \begin{pmatrix} x^1 & \xi^1 & \theta^1 & z^1 \\ \xi^2 & x^2 & z^2 & \theta^2 \\ \theta^3 & z^3 & x^3 & \xi^3 \\ z^4 & \theta^4 & \xi^4 & x^4 \end{pmatrix}$$

In view of (3.3.30), the action σ of $\mathrm{GL}(1|1,1,1)$ on $\mathbb{R}^{1|1,1,1}$ with global coordinates

$$\S = (\S^{\mathfrak{a}})_{\mathfrak{a}} = (x^0, \xi^0, \theta^0, z^0)\,,$$

is given as

$$\begin{aligned}
\sigma^*(x^0) &= x^0x^1 + \xi^0\xi^1 + \theta^0\theta^1 + z^0z^1\ ,\\
\sigma^*(\xi^0) &= x^0\xi^2 + \xi^0x^2 + \theta^0z^2 + z^0\theta^2\ ,\\
\sigma^*(\theta^0) &= x^0\theta^3 + \xi^0z^3 + \theta^0x^3 + z^0\xi^3\ ,\\
\sigma^*(z^0) &= x^0z^4 + \xi^0\theta^4 + \theta^0\xi^4 + z^0x^4\ .
\end{aligned}$$

Connection between the canonical action and the internal Hom.

Since

$$\mathrm{GL}(p|\underline{q}) = \mathcal{U}^{t|\underline{u}}$$

(see Equation (3.3.11)) is an open $\mathbb{Z}_2^n$-submanifold (see Equation (3.3.10)) of

$$\mathrm{gl}_0(p|\underline{q}) = \mathbb{R}^{t|\underline{u}} = \underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\texttt{LinMan}}(\mathbb{R}^{p|\underline{q}}, \mathbb{R}^{p|\underline{q}})$$

(see Proposition 3.2.29 and Equation (3.2.75)), we can expect a connection between the canonical action of $\mathrm{GL}(p|\underline{q})$ on $\mathbb{R}^{p|\underline{q}}$ and $\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\texttt{LinMan}}(\mathbb{R}^{p|\underline{q}}, \mathbb{R}^{p|\underline{q}})$. It turns out that this link becomes apparent as soon as we understand the connection between the internal Hom of linear $\mathbb{Z}_2^n$-manifolds and the internal Hom of arbitrary $\mathbb{Z}_2^n$-manifolds. Indeed, for any $\Lambda \simeq \mathbb{R}^{0|\underline{m}}$, we have

$$\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\texttt{Man}}(\mathbb{R}^{p|\underline{q}}, \mathbb{R}^{p|\underline{q}})(\Lambda) := \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(\mathbb{R}^{p|\underline{m}+\underline{q}}, \mathbb{R}^{p|\underline{q}})$$

(see [13]). If we denote the coordinates of $\mathbb{R}^{0|\underline{m}}$ by $\theta = (\theta^{\mathfrak{d}})$ and those of $\mathbb{R}^{p|\underline{q}}$ by $\S = (\S^{\mathfrak{a}}) = (x^a, \xi^A)$, the RHS Hom-set can be identified with the set of (degree respecting) coordinate pullbacks:

$$\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\texttt{Man}}(\mathbb{R}^{p|\underline{q}}, \mathbb{R}^{p|\underline{q}})(\Lambda) = \{\S^{\mathfrak{a}} = \S^{\mathfrak{a}}(x,\xi,\theta) = \sum_{\alpha\beta} f^{\mathfrak{a}}_{\alpha\beta}(x)\xi^\alpha\theta^\beta\}\ .$$

On the other hand, when denoting the coordinates of $\mathbb{R}^{t|\underline{u}}$ as above by $\mathcal{X} = (\mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}})$, we get similarly

$$\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\texttt{LinMan}}(\mathbb{R}^{p|\underline{q}}, \mathbb{R}^{p|\underline{q}})(\Lambda) = \mathbb{R}^{t|\underline{u}}(\Lambda) = \mathrm{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(\Lambda, \mathbb{R}^{t|\underline{u}}) = \{\mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}} = \mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}}(\theta) = \sum_{\delta} r^{\mathfrak{a}}_{\mathfrak{b},\delta}\,\theta^\delta\}$$

$$= \mathrm{gl}_0(p|\underline{q}, \Lambda)\ . \tag{3.3.31}$$

An obvious identification leads now to

$$\underline{\mathrm{Hom}}_{\mathbb{Z}_2^n\texttt{LinMan}}(\mathbb{R}^{p|\underline{q}}, \mathbb{R}^{p|\underline{q}})(\Lambda) = \{\S^{\mathfrak{a}} = \S^{\mathfrak{a}}(x,\xi,\theta) = \sum_{\mathfrak{b}} \S^{\mathfrak{b}}\,\mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}}(\theta) = \sum_b x^b\,\mathcal{X}^{\mathfrak{a}}_b(\theta) + \sum_B \xi^B\,\mathcal{X}^{\mathfrak{a}}_B(\theta)\}\ . \tag{3.3.32}$$

When comparing (3.3.32) and (3.3.31), we see that the internal Hom of linear $\mathbb{Z}_2^n$-manifolds consists of the pullbacks of the internal Hom of arbitrary $\mathbb{Z}_2^n$-manifolds which are defined by the canonical action of $\mathrm{gl}_0(p|\underline{q})(\Lambda)$ on $\mathbb{R}^{p|\underline{q}}$, in the sense of (3.3.30).

Equivalent definitions of a smooth linear action.

Subsection 3.3.5 already implicitly contained the idea that a smooth linear action of a $\mathbb{Z}_2^n$-Lie group G on a linear $\mathbb{Z}_2^n$-manifold V in the sense of Definition 3.3.6, is equivalent to a $\mathbb{Z}_2^n$-morphism $\sigma : G \times V \to V$ that satisfies the conditions (3.3.24) and (3.3.25) and additionally has a certain linearity property with respect to V. A natural idea is that σ^* should send linear $\mathbb{Z}_2^n$-functions of V to $\mathbb{Z}_2^n$-functions of $G \times V$ that are linear along the fibers. The meaning of this concept becomes clear when we think of the classical differential geometric case in which the functions of a trivial vector bundle $E = M \times \mathbb{R}^r$ are

$$C^\infty(E) = \Gamma(\vee E^*) = C^\infty(M) \otimes \vee (\mathbf{R}^r)^*$$

($\vee$ is the symmetric tensor product), i.e., are the functions that are smooth in the base and polynomial along the fiber. Hence, linear functions of E are the functions that are smooth in the base and linear along the fiber, i.e.,

$$C^\infty_{\text{lin}}(E) = C^\infty(M) \otimes (\mathbf{R}^r)^* = C^\infty(M) \otimes C^\infty_{\text{lin}}(\mathbb{R}^r) \; .$$

We can choose the same definition in the case of the trivial $\mathbb{Z}_2^n$-vector bundle $E = G \times V$:

$$\mathcal{O}_E^{\text{lin}}(|G| \times |V|) := \mathcal{O}_G(|G|) \otimes \mathcal{O}_V^{\text{lin}}(|V|) \; .$$

This definition is of course in particular valid for $G = \text{GL}(p|\underline{q}) \in \mathbb{Z}_2^n\texttt{Lg}$. However, let us mention that the linear functions ('linear along the fibers') of the trivial $\mathbb{Z}_2^n$-vector bundle $E = \text{GL}(p|\underline{q}) \times V$ that are defined on $|\,\text{GL}(p|\underline{q})| \times |V|$ do not coincide with the linear functions ('globally linear') of the linear $\mathbb{Z}_2^n$-manifold $M = \mathbb{R}^{t|\underline{u}} \times V$ (see (3.2.67)) that are defined on the open subset $|\,\text{GL}(p|\underline{q})| \times |V|$ of its base $\mathbb{R}^t \times |V|$:

$$\mathcal{O}_E^{\text{lin}}(|\,\text{GL}(p|\underline{q})| \times |V|) \neq \mathcal{O}_M^{\text{lin}}(|\,\text{GL}(p|\underline{q})| \times |V|) \; .$$

Given what we have just said, we expect the following proposition to hold:

Proposition 3.3.9. *A smooth linear action σ_- of the $\mathbb{Z}_2^n$-Lie group $G = \text{GL}(p|\underline{q})$ on a linear $\mathbb{Z}_2^n$-manifold V in the sense of Definition 3.3.6, is equivalent to a $\mathbb{Z}_2^n$-morphism $\sigma : G \times V \to V$ that satisfies the conditions* (3.3.24) *and* (3.3.25) *and has the linearity property*

$$\sigma^*(\mathcal{O}_V^{\text{lin}}(|V|)) \subset \mathcal{O}_G(|G|) \otimes \mathcal{O}_V^{\text{lin}}(|V|) \; . \tag{3.3.33}$$

Notice first that the pullback σ^* is a morphism of $\mathbb{Z}_2^n$-algebras

$$\sigma^* : \mathcal{O}_V(|V|) \to \mathcal{O}_{G\times V}(|G| \times |V|) \; .$$

Since G and V have global coordinates, it follows from [15] that the target of σ^* is given by

$$\mathcal{O}_{G\times V}(|G| \times |V|) = \mathcal{O}_G(|G|) \widehat{\otimes}\, \mathcal{O}_V(|V|) \; ,$$

which shows that it contains

$$\mathcal{O}_G(|G|) \otimes \, \mathcal{O}_V^{\text{lin}}(|V|)$$

and that the requirement (3.3.33) actually makes sense.

Another fact is also worth noting. We know from standard supergeometry that the classical Berezinian defines a super-Lie group morphism

$$\text{Ber} : \text{GL}(p|q) \to \text{GL}(1|0) \, ,$$

so that we get a linear action of $\text{GL}(p|q)$ on $\mathbb{R}^{1|0}$. The point here is that linear actions of $\text{GL}(p|q)$ are not limited to actions on $\mathbb{R}^{p|q}$.

Proof. In the light of the observations that follow Definition 3.3.6, it suffices to prove that the Λ_0-linearity requirement (iii) in Definition 3.3.6 is equivalent to the linearity condition (3.3.33) in Proposition 3.3.9. Hence, let σ_- be a smooth action of G on V and let σ be the corresponding $\mathbb{Z}_2^n$-morphism. If $\mathrm{h} : V \to \mathbb{R}^{r|\underline{s}}$ is a linear coordinate map of V , the $\mathbb{Z}_2^n$-morphism

$$\mathfrak{S} := \mathrm{h} \circ \sigma \circ (\mathrm{id}_G \times \mathrm{h}^{-1}) : G \times \mathbb{R}^{r|\underline{s}} \to \mathbb{R}^{r|\underline{s}}$$

satisfies (3.3.33) if and only if the $\mathbb{Z}_2^n$-morphism σ does. Indeed, if σ has the property (3.3.33), then

$$\mathfrak{S}^* = (\mathrm{id}_G \times \mathrm{h}^{-1})^* \circ \sigma^* \circ \mathrm{h}^* = \left(\mathrm{id}_G^* \widehat{\otimes} (\mathrm{h}^{-1})^*\right) \circ \sigma^* \circ \mathrm{h}^*$$

has obviously the same property. We similarly find that the converse implication holds. On the other hand, if we denote the coordinates of $\mathbb{R}^{r|\underline{s}}$ by $\dagger = (\dagger^{\mathfrak{c}}) = (y^c, \eta^C)$, the Λ-components of the natural transformations σ_- and $\mathfrak{S}_-$ satisfy

$$\mathfrak{S}_\Lambda = \mathrm{h}_\Lambda \circ \sigma_\Lambda \circ (\mathrm{id}_{G(\Lambda)} \times \mathrm{h}_\Lambda^{-1})$$

and

$$\mathfrak{S}_\Lambda(g_\Lambda, \sum_k \lambda^k \dagger_{\Lambda,k}) = \mathrm{h}_\Lambda(\sigma_\Lambda(g_\Lambda, \mathrm{h}_\Lambda^{-1}(\sum_k \lambda^k \dagger_{\Lambda,k}))) \ ,$$

for any $g_\Lambda \in G(\Lambda)$, any $\dagger_{\Lambda,k} \in \mathbb{R}^{r|\underline{s}}(\Lambda)$ and any $\lambda^k \in \Lambda_0$ (where k runs through a finite set). Since

$$\mathrm{h}_\Lambda : V(\Lambda) \to \mathbb{R}^{r|\underline{s}}(\Lambda)$$

is an isomorphism of Fréchet Λ_0-modules, the Λ_0-smooth map $\mathfrak{S}_\Lambda$ is Λ_0-linear in $\dagger_\Lambda$ if and only if the Λ_0-smooth map σ_Λ is Λ_0-linear in $v_\Lambda \in V(\Lambda)$. It is therefore sufficient to prove the equivalence "(iii) if and only if (3.3.33)" for $V = \mathbb{R}^{r|\underline{s}}$.

We refrain from writing down the proof of the implication "if (iii) then (3.3.33)". It is technical and partially reminiscent of a part of the proof of Theorem 3.2.20 (for the super-case, see [17] and the references it contains).

We now prove the converse implication from scratch. Assume that

$$\mathfrak{S}^*(\mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{r|\underline{s}}}(\mathbb{R}^r)) \subset \mathcal{O}_G(|G|) \otimes \mathcal{O}^{\mathrm{lin}}_{\mathbb{R}^{r|\underline{s}}}(\mathbb{R}^r) \ . \tag{3.3.34}$$

In view of the universal property of the product of $\mathbb{Z}_2^n$-manifolds, we have

$$G(\Lambda) \times \mathbb{R}^{r|\underline{s}}(\Lambda) \ni (g_\Lambda, \dagger_\Lambda) \simeq u_\Lambda \in (G \times \mathbb{R}^{r|\underline{s}})(\Lambda) \ .$$

If we identify the $\mathbb{Z}_2^n$-morphisms $g_\Lambda, \dagger_\Lambda, u_\Lambda$ with the corresponding continuous $\mathbb{Z}_2^n$-algebra morphisms

$$g_\Lambda^* \in \mathrm{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_G(|G|), \Lambda), \dagger_\Lambda^* \in \mathrm{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_{\mathbb{R}^{r|\underline{s}}}(\mathbb{R}^r), \Lambda), u_\Lambda^* \in \mathrm{Hom}_{\mathbb{Z}_2^n\mathtt{Alg}}(\mathcal{O}_G(|G|)\widehat{\otimes}\, \mathcal{O}_{\mathbb{R}^{r|\underline{s}}}(\mathbb{R}^r), \Lambda) \ ,$$

we get

$$u_\Lambda^* = \widehat{m}_\Lambda \circ (g_\Lambda^* \widehat{\otimes}\, \dagger_\Lambda^*) \ ,$$

where $\widehat{m}_\Lambda : \Lambda \widehat{\otimes}\, \Lambda \to \Lambda$ is continuous $\mathbb{Z}_2^n$-algebra morphism that extends the multiplication $m_\Lambda = \cdot_\Lambda$ of Λ (see [15]). We denote the coordinates of $G \times \mathbb{R}^{r|\underline{s}}$ as $\ddagger = (\ddagger^{\mathfrak{d}}) = (\mathcal{X}^{\mathfrak{a}}_{\mathfrak{b}}, \dagger^{\mathfrak{c}})$. Since

$$\mathfrak{S}_\Lambda : (G \times \mathbb{R}^{r|\underline{s}})(\Lambda) \ni u_\Lambda \simeq (g_\Lambda, \dagger_\Lambda) \simeq u_\Lambda^* \simeq u_\Lambda^*(\ddagger^{\mathfrak{d}}) \mapsto \mathfrak{S} \circ u_\Lambda \simeq u_\Lambda^* \circ \mathfrak{S}^* \simeq u_\Lambda^*(\mathfrak{S}^*(\dagger^{\mathfrak{c}})) \in \mathbb{R}^{r|\underline{s}}(\Lambda) \ ,$$

we find that

$$\mathfrak{S}_\Lambda(g_\Lambda, \sum_k \lambda^k \dagger_{\Lambda,k}) \simeq \widehat{m}_\Lambda\big((g_\Lambda^* \widehat{\otimes} (\sum_k \lambda^k \dagger_{\Lambda,k})^*)(\mathfrak{S}^*(\dagger^{\mathfrak{c}}))\big) \ .$$

In view of the definition of the Λ_0-module structure on $\mathbb{R}^{r|\underline{s}}(\Lambda)$, we have

$$\big(\sum_k \lambda^k \dagger_{\Lambda,k}\big)^* = \sum_k \lambda^k \dagger^*_{\Lambda,k}$$

and in view of the assumption (3.3.34), we get for any *fixed* $\mathfrak{c}$ that

$$\mathfrak{S}^*(\dagger^{\mathfrak{c}}) = \sum_{n=1}^{M} s^n_G \otimes_{\mathbb{R}} \big(\sum_{\mathfrak{c}'} r^n_{\mathfrak{c}'} \dagger^{\mathfrak{c}'}\big) ,$$

where $M \in \mathbb{N}$, where $s^n_G \in \mathcal{O}_G(|G|)$ and where $r^n_{\mathfrak{c}'} \in \mathbb{R}$. Since $\lambda^k \in \Lambda_0$, what we just said yields

$$\mathfrak{S}_\Lambda(g_\Lambda, \sum_k \lambda^k \dagger_{\Lambda,k}) \simeq \sum_{n=1}^{M} g^*_\Lambda(s^n_G) \cdot_\Lambda \sum_k \lambda^k \cdot_\Lambda \big(\sum_{\mathfrak{c}'} r^n_{\mathfrak{c}'} \, \dagger^*_{\Lambda,k}(\dagger^{\mathfrak{c}'})\big) \simeq \sum_k \lambda^k \, \mathfrak{S}_\Lambda(g_\Lambda, \dagger_{\Lambda,k}) .$$

□

3.4 Future directions

We view the current paper as the first steps towards understanding actions of $\mathbb{Z}_2^n$-Lie groups on $\mathbb{Z}_2^n$-manifolds and we claim that it will be vital in carefully constructing the total spaces of $\mathbb{Z}_2^n$-vector bundles, for example. In both these settings, the functor of points, and in particular Λ-points, are expected to be of fundamental importance. In particular, the typical fibres of $\mathbb{Z}_2^n$-vector bundles cannot be $\mathbb{Z}_2^n$-graded vector spaces, but rather they are linear $\mathbb{Z}_2^n$-manifolds. Moreover, the transition functions will correspond to an action of the general linear $\mathbb{Z}_2^n$-group and as such a careful understanding of linear actions is needed. This paper provides some of this technical background. We plan to explore the algebraic and geometric definitions of vector bundles in the category of $\mathbb{Z}_2^n$-manifolds in a future publication.

3.5 Appendix

3.5.1 The category of modules over a variable algebra

We define the category `AMod` (resp., `FAMod`) of modules (resp., Fréchet modules) over any (unital) algebra (resp., any (unital) Fréchet algebra) A. The algebra A can vary from object to object. The objects are the modules over some A (resp., the Fréchet vector spaces that come equipped with a (compatible) continuous A-action). We denote such modules by M_A. Morphisms consist of pairs (φ, Φ), where

$$\varphi : A \longrightarrow B$$

is an algebra morphism (resp., a continuous algebra morphism), and

$$\Phi : M_A \longrightarrow M_B$$

is a map (resp., a continuous map) that satisfies

$$\Phi(am + a'm') = \varphi(a)\Phi(m) + \varphi(a')\Phi(m') ,$$

for all $a, a' \in A$ and $m, m' \in M_A$. It is evident that we do indeed obtain a category in this way.

The preceding categories `AMod` and `FAMod` are similar to the category `AFMan` that we used in [13]. They naturally appear when considering the zero degree rules functor or the functor of points. See for instance Equations (3.2.1), (3.2.27) and (3.2.24).

3.5.2 Basics of $\mathbb{Z}_2^n$-geometry,

$\mathbb{Z}_2^n$-manifolds and their morphisms: The locally ringed space approach to $\mathbb{Z}_2^n$-manifolds was pioneered in [19]. We work over the field $\mathbb{R}$ of real numbers and set $\mathbb{Z}_2^n := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ (n-times). A $\mathbb{Z}_2^n$-*graded algebra* is an $\mathbb{R}$-algebra $\mathcal{A}$ with a decomposition into vector spaces $\mathcal{A} := \oplus_{\gamma \in \mathbb{Z}_2^n} \mathcal{A}_\gamma$, such that the multiplication, say $\cdot$, respects the $\mathbb{Z}_2^n$-grading, i.e., $\mathcal{A}_\alpha \cdot \mathcal{A}_\beta \subset \mathcal{A}_{\alpha+\beta}$. We will always assume the algebras to be associative and unital. If for any pair of homogeneous elements $a \in \mathcal{A}_\alpha$ and $b \in \mathcal{A}_\beta$ we have that

$$a \cdot b = (-1)^{\langle \alpha, \beta \rangle} b \cdot a, \tag{3.5.1}$$

where $\langle -, - \rangle$ is the standard scalar product on $\mathbb{Z}_2^n$, then $\mathcal{A}$ is a $\mathbb{Z}_2^n$-*commutative algebra*.

Essentially, $\mathbb{Z}_2^n$-manifolds are 'manifolds' equipped with both, standard commuting coordinates and formal coordinates of non-zero $\mathbb{Z}_2^n$-degree that $\mathbb{Z}_2^n$-commute according to the general sign rule (3.5.1). Note that in general we need to deal with formal coordinates that are *not* nilpotent.

In order to keep track of the various formal coordinates, we need to introduce a convention on how we fix the order of elements in $\mathbb{Z}_2^n$ and we choose the *lexicographical order*. For example, with this choice of ordering

$$\mathbb{Z}_2^2 = \{(0,0),\ (0,1),\ (1,0),\ (1,1)\}\,.$$

Note that other choices of ordering have appeared in the literature. A tuple $\underline{q} = (q_1, q_2, \cdots, q_N) \in \mathbb{N}^{\times N}$ ($N = 2^n - 1$) provides the number of formal coordinates in each $\mathbb{Z}_2^n$-degree. We can now recall the definition of a $\mathbb{Z}_2^n$-manifold.

Definition 3.5.1. A (smooth) $\mathbb{Z}_2^n$-*manifold* of dimension $p|\underline{q}$ is a locally $\mathbb{Z}_2^n$-ringed space $M := (|M|, \mathcal{O}_M)$, which is locally isomorphic to the $\mathbb{Z}_2^n$-ringed space $\mathbb{R}^{p|\underline{q}} := (\mathbb{R}^p, C^\infty_{\mathbb{R}^p}[[\xi]])$. Local sections of M are formal power series in the $\mathbb{Z}_2^n$-graded variables ξ with smooth coefficients,

$$\mathcal{O}_M(|U|) \simeq C^\infty(|U|)[[\xi]] := \left\{ \sum_{\alpha \in \mathbb{N}^{\times N}}^{\infty} f_\alpha\, \xi^\alpha \,:\, f_\alpha \in C^\infty(|U|) \right\},$$

for 'small enough' opens $|U| \subset |M|$. *Morphisms* between $\mathbb{Z}_2^n$-manifolds are morphisms of $\mathbb{Z}_2^n$-ringed spaces, that is, pairs $\Phi = (\phi, \phi^*) : (|M|, \mathcal{O}_M) \to (|N|, \mathcal{O}_N)$ consisting of a continuous map $\phi : |M| \to |N|$ and a sheaf morphism $\phi^* : \mathcal{O}_N(-) \to \mathcal{O}_M(\phi^{-1}(-))$, i.e., a family of $\mathbb{Z}_2^n$-algebra morphisms $\phi^*_{|V|} : \mathcal{O}_N(|V|) \to \mathcal{O}_M(\phi^{-1}(|V|))$ ($|V| \subset |N|$ open) that commute with restrictions. We sometimes denote $\mathbb{Z}_2^n$-manifolds by $\mathcal{M} = (M, \mathcal{O}_M)$ instead of $M = (|M|, \mathcal{O}_M)$ and we sometimes denote $\mathbb{Z}_2^n$-morphisms by $\phi = (|\phi|, \phi^*)$ instead of $\Phi = (\phi, \phi^*)$.

Example 3.5.2 (The local model)**.** The locally $\mathbb{Z}_2^n$-ringed space $\mathcal{U}^{p|\underline{q}} := \big(\mathcal{U}^p, C^\infty_{\mathcal{U}^p}[[\xi]]\big)$ ($\mathcal{U}^p \subset \mathbb{R}^p$ open) is naturally a $\mathbb{Z}_2^n$-manifold – we refer to such $\mathbb{Z}_2^n$-manifolds as $\mathbb{Z}_2^n$-*domains* of dimension $p|\underline{q}$. We can employ (natural) coordinates (x^a, ξ^A) on any $\mathbb{Z}_2^n$-domain, where the x^a form a coordinate system on $\mathcal{U}^p$ and the ξ^A are formal coordinates.

Many of the standard results from the theory of supermanifolds pass over to $\mathbb{Z}_2^n$-manifolds. For example, the topological space $|M|$ comes with the structure of a smooth manifold of dimension p, hence our suggestive notation. Moreover, there exists a canonical projection $\varepsilon : \mathcal{O}_M \to C^\infty_{|M|}$. What makes the category of $\mathbb{Z}_2^n$-manifolds a very tractable form of noncommutative geometry is the fact that we have local models. Much like in the theory of smooth manifolds, one can construct global geometric concepts via the gluing of local geometric concepts. That is, we can consider a $\mathbb{Z}_2^n$-manifold as being covered by $\mathbb{Z}_2^n$-domains together with specified gluing data. More precisely, a $p|\underline{q}$-*chart* (or $p|\underline{q}$-coordinate-system) over a (second-countable Hausdorff) smooth manifold $|M|$ is a $\mathbb{Z}_2^n$-domain

$$\mathcal{U}^{p|\underline{q}} = (\mathcal{U}^p, C^\infty_{\mathcal{U}^p}[[\xi]])\ ,$$

together with a diffeomorphism $|\psi| : |U| \to \mathcal{U}^p$, where $|U|$ is an open subset of $|M|$. Given two $p|\underline{q}$-charts

$$(\mathcal{U}_\alpha^{p|\underline{q}}, |\psi_\alpha|) \quad \text{and} \quad (\mathcal{U}_\beta^{p|\underline{q}}, |\psi_\beta|) \tag{3.5.2}$$

over $|M|$, we set $V_{\alpha\beta} := |\psi_\alpha|(|U_{\alpha\beta}|)$ and $V_{\beta\alpha} := |\psi_\beta|(|U_{\alpha\beta}|)$, where $|U_{\alpha\beta}| := |U_\alpha| \cap |U_\beta|$. We then denote by $|\psi_{\beta\alpha}|$ the diffeomorphism

$$|\psi_{\beta\alpha}| := |\psi_\beta| \circ |\psi_\alpha|^{-1} : V_{\alpha\beta} \to V_{\beta\alpha}\,. \tag{3.5.3}$$

Whereas in classical differential geometry the coordinate transformations are completely defined by the coordinate systems, in $\mathbb{Z}_2^n$-geometry, they have to be specified separately. A *coordinate transformation* between two charts, say the ones of (3.5.2), is an isomorphism of $\mathbb{Z}_2^n$-manifolds

$$\psi_{\beta\alpha} = (|\psi_{\beta\alpha}|, \psi^*_{\beta\alpha}) : \mathcal{U}_\alpha^{p|\underline{q}}|_{V_{\alpha\beta}} \to \mathcal{U}_\beta^{p|\underline{q}}|_{V_{\beta\alpha}}\ , \tag{3.5.4}$$

where the source and target are the open $\mathbb{Z}_2^n$-submanifolds

$$\mathcal{U}_\alpha^{p|\underline{q}}|_{V_{\alpha\beta}} = (V_{\alpha\beta}, C^\infty_{V_{\alpha\beta}}[[\xi]])$$

(note that the underlying diffeomorphism is (3.5.3)). A $p|\underline{q}$-*atlas* over $|M|$ is a covering $(\mathcal{U}_\alpha^{p|\underline{q}}, |\psi_\alpha|)_\alpha$ by charts together with a coordinate transformation (3.5.4) for each pair of charts, such that the usual cocycle condition $\psi_{\beta\gamma}\psi_{\gamma\alpha} = \psi_{\beta\alpha}$ holds (appropriate restrictions are understood).

Moreover, we have the *chart theorem* ([19, Theorem 7.10]) that says that $\mathbb{Z}_2^n$-morphisms from a $\mathbb{Z}_2^n$-manifold $(|M|, \mathcal{O}_M)$ to a $\mathbb{Z}_2^n$-domain $(\mathcal{U}^p, C^\infty_{\mathcal{U}^p}[[\xi]])$ are completely described by the pullbacks of the coordinates (x^a, ξ^A). In other words, to define a $\mathbb{Z}_2^n$-morphism valued in a $\mathbb{Z}_2^n$-domain, we only need to provide total sections $(s^a, s^A) \in \mathcal{O}_M(|M|)$ of the source structure sheaf, whose degrees coincide with those of the target coordinates (x^a, ξ^A). Let us stress the condition

$$(\ldots, \varepsilon s^a, \ldots)(|M|) \subset \mathcal{U}^p\ ,$$

where ε is the canonical projection, is often understood in the literature.

$\mathbb{Z}_2^n$-Grassmann algebras, $\mathbb{Z}_2^n$-points and the Schwarz–Voronov embedding: It is clear that $\mathbb{Z}_2^n$-manifolds, as they are locally ringed spaces, are not fully determined by their topological points. To 'claw back' a fully useful notion of a point, one can employ Grothendieck's functor of points. This is, of course, an application of the Yoneda embedding (see [35, Chapter III, Section 2]). For the case of supermanifolds, it is well-known, via the seminal works of Schwarz & Voronov [40, 41, 47], that superpoints are sufficient to act as 'probes' for the functor of points. That is, we only need to consider supermanifolds that have a single point as their underlying topological space. Dual to this, we may consider finite dimensional Grassmann algebras $\Lambda = \Lambda_0 \oplus \Lambda_1$ as parameterizing the 'points' of a supermanifold. One can thus view supermanifolds as functors from the category of finite dimensional Grassmann algebras to sets. However, it turns out that the target category is not just sets, but (finite dimensional) Λ_0-smooth manifolds. That is, the target category consists of smooth manifolds that have a Λ_0-module structure on their tangent spaces. Morphisms in this category respect the module structure and are said to be Λ_0-smooth (we will explain this further later on). In [13], it was shown how the above considerations generalize to the setting of $\mathbb{Z}_2^n$-manifolds. We will use the notations and results of [13] rather freely. We encourage the reader to consult this reference for the subtleties compared to the standard case of supermanifolds.

A $\mathbb{Z}_2^n$-*Grassmann algebra* we define to be a formal power series algebra $\mathbb{R}[[\theta]]$ in $\mathbb{Z}_2^n$-graded, $\mathbb{Z}_2^n$-commutative parameters θ_j^ℓ. All the information about the number of generators is specified by the tuple $\underline{q}$ as before. We will denote a $\mathbb{Z}_2^n$-Grassmann algebra by Λ, as usually we do not need to specify the number of generators. A $\mathbb{Z}_2^n$-*point* is a $\mathbb{Z}_2^n$-manifold (that is isomorphic to) $\mathbb{R}^{0|\underline{q}}$. It is clear, from Definition 3.5.1, that the algebra of global sections of a $\mathbb{Z}_2^n$-point is precisely a $\mathbb{Z}_2^n$-Grassmann algebra. There is an equivalence between $\mathbb{Z}_2^n$-Grassmann algebras and $\mathbb{Z}_2^n$-points:

$$\mathbb{Z}_2^n\texttt{GrAlg} \cong \mathbb{Z}_2^n\texttt{Pts}^{\text{op}} .$$

The Yoneda functor of points of the category $\mathbb{Z}_2^n\texttt{Man}$ of $\mathbb{Z}_2^n$-manifolds is the fully faithful embedding

$$\mathcal{Y} : \mathbb{Z}_2^n\texttt{Man} \ni M \mapsto \operatorname{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(-, M) \in \texttt{Fun}(\mathbb{Z}_2^n\texttt{Man}^{\text{op}}, \texttt{Set}) .$$

In [13], we showed that $\mathcal{Y}$ remains fully faithful for appropriate restrictions of the source and target of the functor category, as well as of the *resulting* functor category. More precisely, we proved that the functor

$$\mathcal{S} : \mathbb{Z}_2^n\texttt{Man} \ni M \mapsto \operatorname{Hom}_{\mathbb{Z}_2^n\texttt{Man}}(-, M) \in \texttt{Fun}_0(\mathbb{Z}_2^n\texttt{Pts}^{\text{op}}, \texttt{A(N)FM})$$

is fully faithful. The category $\texttt{A(N)FM}$ is the category of (nuclear) Fréchet manifolds over a (nuclear) Fréchet algebra, and the functor category is the category of those functors that send a $\mathbb{Z}_2^n$-Grassmann algebra Λ to a (nuclear) Fréchet Λ_0-manifold, and of those natural transformations that have Λ_0-smooth Λ-components.

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Chapter 4

$\mathbb{Z}_2^n$-Lie algebra representations by coderivation

This chapter is a joint work in progress with Zoran Škoda.

Abstract

In this chapter we develop the required tools in the category of $\mathbb{Z}_2^n$-manifolds required to study representations by coderivations of the symmetric algebra of a Lie algebra, (see chapter 3.2.3.1), $S(\mathfrak{g})$. For this, we recall properties of the tensor and symmetric algebras of $\mathbb{Z}_2^n$-vector spaces, construct the *universal enveloping algebra* and prove the $\mathbb{Z}_2^n$ version of the Poincaré-Birkhof-Witt theorem. This allows to construct a faithfull representation of $\mathfrak{g}$ by acting over the universal enveloping algebra $U(\mathfrak{g})$, and by use of the strong Poincaré-Birkhof-Witt-theorem, we can extend this action to $S(\mathfrak{g})$. We endow this algebras with their classical Hopf algebra structure. We prove that this action is by coderivations. We set also the basis for the study of the embedding of the coderivations of $S(\mathfrak{g})$ into a $\mathbb{Z}_2^n$-Weil algebra.

Definition 4.0.1. A $\mathbb{Z}_2^n$-graded Lie algebra

$$\mathfrak{g} = \bigoplus_{\gamma \in Z_2^n} \mathfrak{g}^\gamma$$

Is a $\mathbb{Z}_2^n$-graded vector space, endowed with a graded bracket that satisfies Lie algebra and Koszul rules:

i. $[x_{\gamma_j}, y_{\gamma_k}] = -(-1)^{<\gamma_j,\gamma_k>}[y_{\gamma_k}, x_{\gamma_j}]$

ii. $(-1)^{<\gamma_k,\gamma_i>}[x_{\gamma_i}, [x_{\gamma_j}, x_{\gamma_k}]] + (-1)^{<\gamma_i,\gamma_j>}[x_{\gamma_j}, [x_{\gamma_k}, x_{\gamma_i}]] + (-1)^{<\gamma_j,\gamma_k>}[x_{\gamma_k}, [x_{\gamma_i}, x_{\gamma_j}]] = 0$

where, $< \gamma_j, \gamma_k >$, denotes the inner product of $\mathbb{Z}_2^n$.

For fixed number of $\mathbb{Z}_2^n$-vector spaces,

$$\mathfrak{g}_1, \cdots, \mathfrak{g}_p$$

the tensor product is the Universal (initial) object in the category of p-multilinear maps from the cartesian product (See [6]).

$$f : \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_p \to \mathfrak{h}$$

And morphisms,

$$(f : \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_p \to \mathfrak{h}) \Rightarrow (g : \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_p \to \mathfrak{k})$$

Given by a linear map $h : \mathfrak{h} \to \mathfrak{k}$ making the following diagram commute:

$$\begin{array}{ccc} & & \mathfrak{h} \\ & \nearrow & \\ \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_p & & \downarrow \\ & \searrow & \\ & & \mathfrak{k} \end{array}$$

Universal objects are unique up to isomorphism. One object of this isomorphism class can be obtained by looking at the free module generated by the set $\mathfrak{g}_1, \cdots, \mathfrak{g}_p$ quotiented by the relations (the submodule generated by elements of the form):

$$(x_1, \cdots, x_i + x_{i\prime}, \cdots, x_p) - (x_1, \cdots, x_i, \cdots, x_p) - (x_1, \cdots, x_{i\prime}, \cdots, x_p)$$
$$(x_1, \cdots, ax_i, \cdots, x_p) - a(x_1, \cdots, x_i, \cdots, x_p)$$

We denote such vector space as,

$$\mathfrak{g}_1 \otimes \cdots \otimes \mathfrak{g}_p$$

Its elements can always be written as a sum of decomposable terms, meaning:

$$\mathfrak{g}_1 \otimes \cdots \otimes \mathfrak{g}_p = \Big\{ \sum x_1 \otimes \cdots \otimes x_p \mid x_i \in \mathfrak{g}_i \Big\}$$

and inherits a $\mathbb{Z}_2^n$-grading

$$\mathfrak{g}_1 \otimes \cdots \otimes \mathfrak{g}_p = \bigoplus_{\gamma \in \mathbb{Z}_2^n} \bigoplus_{\sum_{i=1}^p \gamma_{\alpha_i} = \gamma} \left\{ x_1^{\gamma_{\alpha_1}} \otimes x_2^{\gamma_{\alpha_2}} \otimes \cdots \otimes x_p^{\gamma_{\alpha_p}} \;\middle|\; x_i \in \mathfrak{g}_i^{\gamma_{\alpha_i}} \right\}$$

Given a family of linear maps, $f_i : \mathfrak{g}_i \longrightarrow \mathfrak{h}_\mathfrak{i}$, there is a unique linear map,

$$\begin{aligned} T(f_1, \cdots, f_p) : \mathfrak{g}_1 \otimes \cdots \otimes \mathfrak{g}_p &\longrightarrow \mathfrak{h}_1 \otimes \cdots \otimes \mathfrak{h}_p \\ x_1 \otimes \cdots \otimes x_p &\mapsto f_1(x_1) \otimes \cdots \otimes f(x_p) \end{aligned} \tag{4.0.1}$$

Which respects composition: If $f_i \circ g_i$ is a family of composite maps, then

$$T(f_1 \circ g_1, \cdots, f_p \circ g_p) = T(f_1, \cdots, f_p) \circ T(g_1, \cdots, g_p)$$

and

$$T(id, \cdots, id) = id.$$

From $\mathfrak{g}$, we proceed to build a unital associative algebra: Start with the monoid of natural numbers $\mathbb{N}$; and for each $n \in \mathbb{N}$ define,

$$T^n(\mathfrak{g}) = \mathfrak{g} \otimes \overset{n-times}{\cdots} \otimes \mathfrak{g}$$

Then, consider the bilinear associative map,

$$T^p(\mathfrak{g}) \times T^q(\mathfrak{g}) \to T^{(p+q)}(\mathfrak{g})$$
$$((x_1 \otimes \cdots \otimes x_p, y_1 \otimes \cdots \otimes y_q) \mapsto x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q)$$

This implies that,

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} (T^n(\mathfrak{g}))$$

is a ring. It is bi-graded, such being the classic $\mathbb{N}$ grading, and the $\mathbb{Z}_2^n$ degree induced from $\mathfrak{g}$ (See Chapter II, 3.2.3.1). Using the association in morphisms (4.0.1), we get a functor, from graded vector spaces, to associative unital algebras,

$$(\mathbf{VEC} \ni \mathfrak{g} \mapsto T(\mathfrak{g}) \in \mathbf{Alg}) .$$

It is universal in the following way: For any linear map $\phi : \mathfrak{g} \to \mathcal{A}$ from the Lie algebra into an associative unital associative algebra, in particular zero degree maps, $\mathcal{A}$, there exists a unique unital algebra morphism, $h : T(\mathfrak{g}) \to \mathcal{A}$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{i} & T(\mathfrak{g}) \\ \phi \searrow & & \downarrow h \\ & & \mathcal{A} \end{array}$$

Where, $i : \mathfrak{g} \to T(\mathfrak{g})$, is the cannonical inclusion ($\otimes\mathfrak{g}^0 = \mathbb{K}$; $\otimes\mathfrak{g}^1 = \mathfrak{g}$). The map is the obvious one, given by:

$$h(x_1 \otimes x_2 \cdots \otimes x_n) = \phi(x_1) \bullet_{\mathcal{A}} \phi(x_2) \bullet_{\mathcal{A}} \cdots \bullet_{\mathcal{A}} \phi(x_n)$$

N.B. *This characterizes $T(\mathfrak{g})$ as the unique unital associative algebra constructed from $\mathfrak{g}$, up to isomorphism. The previous is equivalent to a statement about T being a left adjunction to the forgetful functor from algebras into vector spaces; $Hom_{\mathbb{Z}_2^n \boldsymbol{Vec}}(\mathfrak{g}, \bullet) \cong Hom_{\boldsymbol{Alg}}(T(\mathfrak{g}), \bullet)$. (See [8], [11])*

We now look at the two sided ideals generated by elements of the form:

$$J_s = (x \otimes y - (-1)^{<d(x),d(y)>} y \otimes x \quad | \quad x, y \text{ are homogeneous})$$

This ideals are distributed along all the degrees, (both $\mathbb{N}$ and $\mathbb{Z}_2^n$-degrees). For each $n \in \mathbb{N}$, we have the module,

$$S^n = T^n / J_n$$

and we obtain a canonical map

$$\mathfrak{g} \times \cdots \times \mathfrak{g} \to T^n(\mathfrak{g}) \to T^n / J_n = S^n.$$

The direct sum (see 3.2.42)

$$S(\mathfrak{g}) = \bigoplus_r S^r$$

is a $\mathbb{N}$-graded Z_2^n-graded-commutative algebra.

Observe how $J_1 \cong 0$, so that the inclusion $\mathfrak{g} \hookrightarrow S(\mathfrak{g})$ remains injective, and the universal property can be written as:

For any linear map, $\phi : \mathfrak{g} \to \mathcal{CA}$, from the vector space into a $\mathbb{Z}_2^n$-commutative algebra, (with $\phi(u)\phi(v) - (-1)^{<d(u),d(v)>}\phi(v)\phi(u) = 0$), then there exist a unique map of $\mathbb{Z}_2^n$-commutative algebras, $h : S(\mathfrak{g}) \to \mathcal{CA}$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xhookrightarrow{i_s} & S(\mathfrak{g}) \\ \phi \searrow & & \downarrow h \\ & & \mathcal{CA} \end{array}$$

Again, $h(u_1 \cdots u_n) = \phi(u_1) \cdots \phi(u_n)$.

As in the classical case, $S(\mathfrak{g})$ may be identified with the algebra of $\mathbb{Z}_2^n$-graded polynomials with the basis of $\mathfrak{g}$ as generators (see 3.2.47).

Definition 4.0.2. Let $\mathfrak{g}$ be a $\mathbb{Z}_2^n$-graded Lie algebra. The universal enveloping algebra of $\mathfrak{g}$ is a pair $(U(\mathfrak{g}), i_u)$ formed by a unital associative algebra, $U(\mathfrak{g})$, and a linear morphism $i_u : \mathfrak{g} \to U(\mathfrak{g})$, such that,

$$i_u(x)i_u(y) - (-1)^{<deg(x),deg(y)>} i_u(y)i_u(x) = i_u([x,y]) \tag{4.0.2}$$

for all $x, y \in \mathfrak{g}$; and with the following universal property: If $\mathfrak{A}$ is any other associative unital algebra and there exists a map $j_u : \mathfrak{g} \to \mathfrak{A}$ with the property (4.0.2), there exists a unique morphism of algebras, $h : \mathfrak{A} \to U(\mathfrak{g})$, sending 1 to 1, such that $h \circ i_u = j_u$.

We proceed to construct such a pair. We start from the tensor algebra, $T(V)$ and consider, the graded two sided ideal, J_u, generated by the elements

$$(x \otimes y - (-1)^{<deg(x),deg(y)>} y \otimes x - [x,y])$$

Again, the quotient is a unital associative algebra, with canonical map

$$\pi : T(\mathfrak{g}) \to T(\mathfrak{g})/J_u$$

There exists a map coming from the restriction of π to $T^1(\mathfrak{g}) \cong \mathfrak{g}$, which we will call $i_u : \mathfrak{g} \to U(\mathfrak{g})$, and inherits the universal property from the tensor algebra in the following way: Let $(\mathfrak{A}, j_u)$ be as before. Since $\mathfrak{A}$ is a unital associative algebra, the universal property of the tensor algebra points out at a unique algebra morphism, $h' : T(\mathfrak{g}) \longrightarrow \mathfrak{A}$, which sends 1 to 1, extends the morphism j_u and therefore, $J_u \subset Ker(h')$. This allows us to identify a unique morphism of algebras $h : T(\mathfrak{g})/J_u \to \mathfrak{A}$ such that $h \circ i_u = j_u$.

4.1 Poincaré-Birkhoff-Witt theorem

N.B. *When proving the Poincaré-Birkhoff-Witt theorem, the degree issues that might have appeared are under control by the Koszul sign rule properties and its appearance in the axioms of the $\mathbb{Z}_2^n$-Lie bracket. There are parity issues that we solve in the same fashion as the proofs to the super PBW theorem.*

Let G^n be the $\mathbb{K}$-module U_n/U_{n-1}, with $U_n = \pi(T^n)$. Let $\mathcal{G}$ be the direct sum over $n \in \mathbb{N}$. $\mathcal{G}$ inherits a multiplication map induced by the one in $U(\mathfrak{g})$ with, $G^m G^n \subset G^{n+m}$. Consider the maps

$$\phi_n : T^n \longrightarrow U_n \xrightarrow{q} G^n$$

This maps are all surjective and assemble into a linear map

$$\phi : T(\mathfrak{g}) \longrightarrow \mathcal{G}$$

which is an algebra epimorphism. If we take $x_\mu \in \mathfrak{g}_\mu$ and $x_\nu \in \mathfrak{g}_\nu$, the element in T^2, $x_\mu \otimes x_\nu - (-1)^{<\gamma_\mu,\gamma_\nu>} x_\nu \otimes x_\mu$ projects into U_1, and eventually to zero in $\mathcal{G}$. This means that there is a unique epimorphism,

$$\beta^* : S(\mathfrak{g}) \to \mathcal{G}$$

that makes the diagram

$$\begin{array}{ccc} T^n & \longrightarrow & U_n \\ \downarrow & & \downarrow \\ S^n & \longrightarrow & G^n \end{array} \tag{4.1.1}$$

commute (See [9], [11]).

Theorem 4.1.1. *Let $\mathfrak{g}$ be a $\mathbb{Z}_2^n$-Lie algebra over $\mathbb{K}$. The epimorphism, $\beta^* : S(\mathfrak{g}) \to \mathcal{G}$, is an algebra isomorphism.*

Proof. Let $\mathfrak{g}$ be a $\mathbb{Z}_2^n$-Lie algebra, with an ordered homogeneous base $\mathcal{X}$. In the context of the diamond lemma, consider then the space of words made by using the base elements as letters, $< \mathcal{X} >$ and the free group generated by $\mathcal{X}$ which is just he tensor algebra $T(\mathfrak{g}) \cong \mathbb{R} < \mathcal{X} >$ (see [1], [13]).

The order in $\mathcal{X} = \{X_\mu\}$ is given by parity in first instance, and lexicographical in each parity. At every degree we suppose to have chosen an order.

Notation. *Suppose X_a has degree γ_j, and X_b γ_k. We write,*

$$(-1)^{<\gamma_j,\gamma_k>} := (-1)^{a\cdot b}$$

The strategy is to reduce words at $< \mathcal{X} >$ into a standard form in $T(\mathfrak{g})$ which is unique in $\mathcal{U}(\mathfrak{g})$ (unique mod $J_u = (Y_a \cdot X_b - (-1)^{a\cdot b} X_b \cdot Y_a - [Y_a, X_b])$). Standard form means to have only ordered words. If $a < b < c$, then

$$X_a X_b X_c \text{ is standard, while}$$

$$X_a X_c X_b \ ; \ X_c X_b X_a \text{ are not standard and require reduction}$$

As usual when one proceeds into a proof by diamond lemma, we substitute non standard words by means of the algebra relations. The easiest example is a non standard quadratic term.

$$X_b X_a = (-1)^{a\cdot b} X_a X_b + [X_b, X_a]$$

The LHS is clearly non standard, and the RHS consists of a standard quadratic word, followed by linear terms appearing in

$$[X_a, X_b] = C_{a\ b}^{k} X_k.$$

- For $a < b$ a word $X_b X_a$ is changed by $(-1)^{a\cdot b} X_a X_b + [X_b, X_a]$.
- For a is odd. We change the word $X_a X_a = \frac{1}{2}[X_a, X_a]$

Given this rules of interchange, the only ambiguity that might occur is to have an overlap. This means that a word can be reduced in different ways, and the reduction is not unique mod J_u. We have to study two different cases:

The more difficult case is when $a < b < c$: The word

$$X_c X_b X_a$$

can be reduced starting with $X_c X_b$ or with $X_b X_a$.

In the first case,

$$X_c X_b X_a = (-1)^{c\cdot b + c\cdot a + a\cdot b} X_a X_b X_c + (-1)^{c\cdot b + c\cdot a} [X_b, X_a] X_c + (-1)^{c\cdot b} X_b [X_c, X_a] + [X_c, X_b] X_a$$

And starting by $X_b X_a$:

$$X_c X_b X_a = (-1)^{c\cdot b + c\cdot a + a\cdot b} X_a X_b X_c + (-1)^{b\cdot a + a\cdot c} X_a [X_c, X_b] + (-1)^{b\cdot a} [X_c, X_a] X_b + X_c [X_b, X_a]$$

We will compare by subtracting the two reductions of $X_c X_b X_a$; The cubic terms cancel, and the quadratic terms reduce as,

$$(-1)^{c\cdot a+c\cdot b}[X_b,X_a]X_c - X_c[X_b,X_a] = (-1)^{c\cdot a+c\cdot b}[X_b,X_a]X_c - (-1)^{c\cdot a+c\cdot b}[X_b,X_a]X_c - [X_c[X_b,X_a]]$$
$$= -[X_c[X_b,X_a]].$$

The whole difference assemble as follows:

$$-[X_a,[X_c,X_b]] - [X_c,[X_b,X_a]] - [[X_c,X_a],X_b] =$$

$$(-1)^{c\cdot a}[X_a,[X_b,X_c]] + (-1)^{a\cdot b}[X_b,[X_c,X_a]] + (-1)^{b\cdot c}[X_c,[X_a,X_b]]$$

Which is zero by the Jacobi identity.

The second case concerns the fact that odd vectors do not commute. We study now reductions of words involving odd vector fields, say X_a, of the form $X_bX_aX_a \;\; a < b$.

Starting with X_bX_a

$$X_bX_aX_a = X_aX_aX_b + (-1)^{b\cdot a}X_a[X_b,X_a] + [X_b,X_a]X_a$$

$$= \frac{1}{2}[X_a,X_a]X_b + [[X_b,X_a],X_a]$$

And starting by X_aX_a

$$X_bX_aX_a = \frac{1}{2}X_b[X_a,X_a]$$

If we subtract both expressions and, we can arrange them in the following way,

$$\frac{1}{2}[X_a,X_a]X_b + [[X_b,X_a],X_a] - \frac{1}{2}X_b[X_a,X_a] = \frac{1}{2}[[X_a,X_a],X_b] + [[X_b,X_a],X_a]$$

Which is again zero because of the Jacobi identity.

□

Example 4.1.2. Consider the $\mathbb{Z}_2^2$-Lie algebra, with one generator per degree, and the ordered base,

$$\{X_{(0,0)}, X_{(1,1)}, X_{(0,1)}, X_{(1,0)}\}$$

relabeled as before

$$\{X_1, X_2, X_3, X_4\}.$$

Quadratic terms have trivial reductions,

$$X_3X_2 = (-1)^{<(1,1),(0,1)>}X_2X_3 + [X_3,X_2] = -X_2X_3 + [X_3,X_2]$$

or,

$$X_4X_3 = (-1)^{<(1,0),(0,1)>}X_3X_4 + [X_3,X_4] = X_3X_4 + [X_3,X_4].$$

Another example with cubic term:

$$X_3X_4X_3 = X_3X_3X_4 + X_3[X_4,X_3]$$

Now In this particular case, the result of the bracket must have degree (1,1), and for this, we know that $[X_4X_3] = C_{43}^2X_2$. In this case we still need to reduce,

$$X_3X_2 = -X_2X_3 + [X_3,X_2]$$

The final form, standard word, is,

$$X_3X_4X_3 = X_3X_3X_4 - C_{43}^2X_2X_3 + C_{43}^2[X_3,X_2].$$

We follow now the classical case [3].

Corollary 1. *Let W be a submodule of $T(\mathfrak{g})$. If the restriction of π_s to W is an isomorphism. Then the restriction of π_u to W is isomorphic, as $\mathbb{Z}_2^n$-vector space, to a complement of U_{n-1} in U_n.*

Proof. This follows from the fact that the composition $\beta^* \circ \pi_s|_W$ is an isomorphism of vector spaces, so, $q \circ \pi_u|_W$ is as well. □

We have then three important results:

Corollary 2. *The map $i_u : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is injective. $U_0 \cong \mathbb{K}$ and $U_1 \cong \mathfrak{g}$.*

Proof. We use the corollary 1, with $n = 1$ and $W = T^1$. This entails that, since the composition $(T^1 \cong \mathfrak{g} \ni \pi_s|_{T^1} \to S^1 \to G^1)$ is an isomorphism, then $T^1 \cong U^1$ □

For any homogeneous $n - tuple$ in $\mathfrak{g}^n$, and for every, $\sigma \in \Sigma_n$, we define the map

$$(\mathfrak{g}^n \ni (x_1, \cdots, x_n) \overset{\sigma\bullet}{\longmapsto} \alpha(\sigma) x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \in T^n)$$

A homogeneous $\mathbb{Z}_2^n$-symmetric tensor of degree $n \in \mathbb{N}$, $z \in T^n$, is one such that, for any permutation, $\sigma \in \Sigma_n$, $\sigma \bullet z = z$. The set of all symmetric tensors form a sub$-\mathbb{K}-$module, Sym_n of T^n. If z is symmetric,

$$\sum_{\sigma \in \Sigma_n} \sigma \bullet z = n! z$$

and then, the map,

$$(T^n \ni t \overset{s}{\longmapsto} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma \bullet t \in Sym_n)$$

is a linear map and a projector of T^n into Sym_n. This means that $s \circ s = id$. This allows us to identify the image of the map, $Im(s)$, and Sym_n (See [2] II, 1, no 8). We also obtain an orthogonal projector, $(1 - s)$, which has Sym_n as kernel, and its image is $I_s \bigcap T^n$. We get, then, the $\mathbb{Z}_2^n$-vector spaces isomorphisms:

$$T^n \cong Sym_n \oplus (T^n \bigcap J_s)$$

$$\pi_s|_{Sym_n} : Sym_n \longrightarrow S^n(\mathfrak{g})$$

Restricting the diagram 4.1.1 to the submodule Sym_n, we get

$$\begin{array}{ccc} Sym_n & \to & U_n \\ \| & & \downarrow \\ S^n & \to & G^n \end{array}.$$

by corollary (1), we get a morphism,

$$S^n \overset{\sim}{\longrightarrow} Sym_n \to U_n$$

which is a $\mathbb{Z}_2^n$-vector space isomorphism between S^n and a supplement to U_{n-1} in U_n.

And, the commutative diagram of $\mathbb{Z}_2^n$-vector space isomorphisms

$$\begin{array}{ccc} Sym_n & \to & U_n \\ \downarrow & \nearrow & \downarrow \\ S^n & \to & G^n \end{array}. \tag{4.1.2}$$

Corollary 3. *Let $\mathfrak{g}$ be a $\mathbb{Z}_2^n$-Lie algebra over $\mathbb{K}$ and let $\mathcal{X}$ be an ordered homogeneous basis of $\mathfrak{g}$. Let $X_a = i_u(x_a) \in U(\mathfrak{g})$. Then 1 and the standard monomials*

$$X_{i_1}^{e_{i_1}} \cdots X_{i_r}^{e_{i_r}} \tag{4.1.3}$$

where $1 \leq i_1 < \cdots < i_r \leq p+Q$, $e_i \in \mathbb{N}$ for X_i even, and $e_j \in \{0,1\}$ for X_j odd; $Q = \sum_{\gamma \in \mathbb{Z}_2^n/0} q_\gamma$, is the number of non-zero degree dimensions, and p classic zero degree dimensions. form a free $\mathbb{K}-$basis of $U(\mathfrak{g})$.

Proof. Consider the submodule of T^n generated by the ordered base $\mathcal{X}$ of $\mathfrak{g}$ and the elements,

$$y = x_{i_1}^{e_{i_1}} \otimes \cdots \otimes x_{i_n}^{e_{i_n}}.$$

The elements, y, span the vector space S^n, and so (1) gives us an isomorphism between it and a supplement of U_{n-1} in U_n. And since

$$\pi_u(y) = i_u(x_{i_1})^{e_{i_1}} \cdots i_u(x_{i_n})^{e_{i_n}} = X_{i_1}^{e_{i_1}} \cdots X_{i_r}^{e_{i_r}}$$

We obtain a base for $U(\mathfrak{g})$. □

If $f : \mathfrak{g} \to \mathfrak{h}$ is a $\mathbb{Z}_2^n$-Lie algebra homomorphism, the composition,

$$i_{u_h} \circ f : \mathfrak{g} \to U(\mathfrak{h})$$

induces an algebra homomorphism,

$$U(f) : U(\mathfrak{g}) \to U(\mathfrak{h})$$

N.B. *The universal enveloping can be here viewed in two ways: one as a usual associative algebra and another as a $\mathbb{Z}_2^n-$graded associative algebra. The latter interpretation is clear as this holds for the tensor algebra and the ideal is $\mathbb{Z}_2^n-$homogeneous. Then there is a functor from $\mathbb{Z}_2^n-$graded associative algebras to $\mathbb{Z}_2^n-$graded Lie algebras with identity.*

Consequences of the the Poincaré-Birkhoff-Witt theorem

- The isomorphism between S^n and U_n obtained in (4.1.2), we will call it *symmetrization map*, inherits explicit formulas (See [11], [3], [9]):

$$\beta(i_s(x)^n) = i_u(x)^n \tag{4.1.4}$$

$$\beta(i_s(x_1) \cdots i_s(x_n)) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \alpha(\sigma) i_u(x_{\sigma(1)}) \cdots i_u(x_{\sigma(n)}) \tag{4.1.5}$$

  where, in this case, $\alpha(\sigma) \in \{-1, 1\}$ is the sign factor resulting from each particular permutation of $n-$elements, $\sigma \in \Sigma_n$. The formula is obtained by $\mathbb{Z}_2^n$-symmetric permutations, (multilinear Σ_n equivariant linear morphisms from the cartesian product), linearly extended to the map $T^n \to Sym_n$. One is able to invert the linear isomorphism $Sym_n \to S^n$, which composed with the canonical quotient, $\pi_u|_{Sym_n} : Sym_n \to U_n$, gives a well defined linear map.

- The faithful representation of $\mathfrak{g}$ by linear transformations, by left multiplication in $U(\mathfrak{g})$,

$$\mathfrak{g} \times U(\mathfrak{g}) \ni (x, u) \longmapsto i_u(x)u \in U(\mathfrak{g})$$

4.1.1 Hopf algebras

The diagonal map,

$$\mathfrak{g} \ni x \overset{D}{\longmapsto} (x, x) \in \mathfrak{g} \times \mathfrak{g}, \tag{4.1.6}$$

defines a linear morphism,

$$(\mathfrak{g} \ni x \overset{D^*}{\longmapsto} 1 \otimes x + x \otimes 1 \in T(\mathfrak{g}) \otimes T(\mathfrak{g})),$$

which lifts into an algebra map,

$$\Lambda : T(\mathfrak{g}) \to T(\mathfrak{g}) \otimes T(\mathfrak{g})$$

that, along with the morphisms, $(\mathfrak{g} \ni x \longmapsto \epsilon(x) = 0 \in \mathbb{R}, \epsilon(1) = 1)$, and $\forall x \in \mathfrak{g}$, $\delta(x) = -x$, turn $T(\mathfrak{g})$ into a Hopf algebra. This means that this maps verify the following properties:

$$\begin{gathered}(id \otimes \Lambda) \circ \Lambda = (\Lambda \otimes id) \circ \Lambda \\ m \circ (id \otimes \delta) \circ \Lambda = m \circ (\delta \otimes id) \circ \Lambda = \epsilon \\ m \circ (id \otimes \epsilon) \circ \Lambda = m \circ (\epsilon \otimes id) \circ \Lambda = id\end{gathered} \tag{4.1.7}$$

$m : T(\mathfrak{g}) \bigotimes T(\mathfrak{g}) \longrightarrow T(\mathfrak{g})$, $u \bigotimes v \mapsto u \otimes v$, is the product in $T(\mathfrak{g})$. We proceed to induce this operations into quotients of the tensor algebra (see [4], [8], [3]).

We use $m_s : S(\mathfrak{g}) \otimes S(\mathfrak{g}) \to S(\mathfrak{g})$, the multiplication map obtained in $S(\mathfrak{g})$ from the multiplication in the tensor algebra and the canonical map $T(\mathfrak{g}) \to S(\mathfrak{g})$. Following (Petracci [9]) σ is the $\mathbb{Z}_2^n$-cocommutation operator in $S(\mathfrak{g}) \otimes S(\mathfrak{g})$,

$$x \otimes y \overset{\sigma}{\mapsto} (-1)^{<d(x),d(y)>} y \otimes x$$

And $\alpha(\vec{p}) \in \{-1, 1\}$ is the sign factor such that,

$$\alpha(\vec{p}) x_{p_1} \cdots x_{p_j} x_1 \cdots \widehat{x_{p_1}} \cdots \widehat{x_{p_j}} \cdots x_n = x_1 \cdots x_n,$$

This maps are explicitly defined by:

For $\{x_i\}_{i=1}^n \in \mathfrak{g}$,

$$\Lambda(x_1 \cdots x_n) = \sum_{j=0}^{n} \sum_{1 \leq p1 \leq \cdots \leq pj \leq n} \alpha(\vec{p}) x_{p_1} \cdots x_{p_j} \otimes u_1 \cdots \widehat{x_{p_1}} \cdots \widehat{x_{p_j}} \cdots x_n \tag{4.1.8}$$

$$\delta(x) = -x \quad and \quad \epsilon(x) = 0 \tag{4.1.9}$$

A the special case is,

$$x_0 \in \mathfrak{g}_{\mathbf{0}} \quad \Lambda(x^n) = \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}$$

They satisfy relations (4.1.7) and

$$\Lambda = \sigma \circ \Lambda \quad (cocommutativity)$$

by construction.

For the case of the universal enveloping algebra, a short computation shows that the morphism, $x \overset{D^*}{\longmapsto} 1 \otimes x + x \otimes 1$, verifies

$$D^*(x_\mu) D^*(x_\nu) - (-1)^{<\mu,\nu>} D^*(x_\nu) D^*(x_\mu) = D^*([x_\mu, x_\nu])$$

By the universal property of $U(\mathfrak{g})$, as a morphism into $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, it lifts to a unique algebra morphism,

$$\Lambda : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

which, along with $(\mathfrak{g} \ni x \longmapsto \epsilon(x) = 0 \in \mathbb{R}, \epsilon(1) = 1)$ endows $U(\mathfrak{g})$ with a coalgebra structure, and the addition of the map $\delta(x) = -x \ \forall_{x\in\mathfrak{g}}$, makes of $U(\mathfrak{g})$ a Hopf algebra. The formulas of coproduct, counit and antipode can also be written like (4.1.8).

N.B. *One can see that both of this algebras are Hopf algebras by looking at Hopf ideals, $(\delta(J) \subset J)$ of $T(\mathfrak{g})$ such as the ideals J_s and J_u defining the universal symmetric and enveloping algebras, (see [4]).*

Given a morphism of $\mathbb{Z}_2^n$-graded Lie algebras, $f : \mathfrak{g} \longrightarrow \mathfrak{h}$, we obtain a commutative diagram

$$\begin{array}{ccc} S(\mathfrak{g}) & \overset{S(f)}{\longrightarrow} & S(\mathfrak{h}) \\ \beta\downarrow & & \beta\downarrow \\ U(\mathfrak{g}) & \overset{U(f)}{\longrightarrow} & U(\mathfrak{h}) \end{array} \tag{4.1.10}$$

since both compositions yield

$$\frac{1}{n!}\sum_{\sigma\in\Sigma_n} \alpha(\sigma) i_{u_\mathfrak{h}}(f(x_{\sigma(1)}))\cdots i_{u_\mathfrak{h}}(f(x_{\sigma(n)})).$$

We finalize this section proving that the symmetrization map, (4.1.5), is a coalgebra isomorphism.

The diagonal map, 4.1.6 $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$, lifts naturally into the coproduct, for the case of both, symmetric and enveloping coalgebras as an algebra homomorphism, $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U\mathfrak{g} \cong U(\mathfrak{g} \times \mathfrak{g})$.

Commutativity of the diagram (4.1.10) applied to $f : \mathfrak{g} \to \mathfrak{h} = \mathfrak{g} \times \mathfrak{g}$, is exactly the property required for the symmetrization map to be a coalgebra homomorphism, and ultimately, coalgebra isomorphism (see [9] Lemma 2.5.3).

4.2 Coderivations

Given a coalgebra C, with coproduct Λ_C.

Definition 4. *A linear endomorphism, $D \in End(C)$ is called a coderivation of C, if it verifies,*

$$(1 \otimes D + D \otimes 1) \circ \Lambda_C = \Lambda_C \circ D$$

The action by left multiplication of the Hopf algebra $U(\mathfrak{g})$ restricts to a Lie algebra action (faithful representation),

$$\mathfrak{g} \ni h \mapsto L_h \in End(U(\mathfrak{g})),$$

by coderivations of $U(\mathfrak{g})$. This is obvious since $\Lambda_C(h) = (1 \otimes h + h \otimes 1)$, and

$$(1 \otimes L_h + L_h \otimes 1) \circ \Lambda(h') = \Lambda(hh') = (\Lambda \circ L_h)(h')$$

This allowed us to define, by means of the symmetrization map, an action by coderivations of $\mathfrak{g}$ in $S(\mathfrak{g})$ (see [5]),

$$\beta^{-1} \circ L_h \circ \beta : S(\mathfrak{g}) \longrightarrow S(\mathfrak{g}).$$

The non degenerate pairing in $\mathfrak{g}$, allows to define a dual structure $S^{n*} = S^n(\mathfrak{g}^*)$ to S^n, for each $n \in \mathbb{N}$, bt means of the extended inner product,

$$< \delta^1 \cdots \delta^n, x_1 \cdots x_n >= \prod_{i=1}^{n} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \alpha(\sigma) < \delta^i, x_{\sigma(i)} >$$

allowes to define the derivation ${}^tD_h : S(\mathfrak{g})^* \to S(\mathfrak{g})^*$. This is the same as a $\mathbb{Z}_2^n$-Lie algebra homomorphism

$$\theta : \mathfrak{g} \to \mathrm{Der}(S(\mathfrak{g})^*).$$

4.3 Weyl algebras

There are several ways to consider the $\mathbb{Z}_2^n$-graded Weyl algebra and its completion by the degree of the differential operator. Sometimes and invariant form is useful, while below we shall also use a concrete formulation in a distinguished basis. The simplest abstract approach is as an inner object of derivations in the symmetric monoidal category of $\mathbb{Z}_2^n$-graded vector spaces. While a more internal definition is in place in effect one should consider the subspace of inner endomorphisms of graded symmetric algebra spanned by graded derivations. This subspace has a structure of a $\mathbb{Z}_2^n$-graded algebra. Then one needs classification of derivations to present this algebra in more explicit terms.

A bilinear form B on a $\mathbb{Z}_2^n$-vector space V is $\mathbb{Z}_2^n$-skew-symmetric if $B(a,b) = -(-1)^{\langle \tilde{a}, \tilde{b} \rangle} B(b,a)$ and symmetric if $B(a,b) = -(-1)^{\langle \tilde{a}, \tilde{b} \rangle} B(b,a)$ for all $a, b \in V$. Define $\epsilon = 1$ in symmetric and $\epsilon = -1$ in skew-symmetric case. Consider the category $\mathcal{C}^B$, whose objects are pairs (A, λ_A) where A is a $\mathbb{Z}_2^n$-graded associative algebra and $\lambda_A : V \to A$ a linear map such that for all $\mathbb{Z}_2^n$-homogeneous $x, y \in A$

$$\lambda(x)\lambda(y) + (-1)^{\langle \tilde{x}, \tilde{y} \rangle} \epsilon \lambda(y)\lambda(x) = B(x,y)1,$$

and where the morphisms $(A, \lambda_A) \to (B, \lambda_B)$ are the morphisms of $\mathbb{Z}_2^n$-graded associative algebras $f : A \to B$ such that $f \circ \lambda_A = \lambda_B$. Then this category has an initial object, which is in the skew-symmetric case called the $\mathbb{Z}_2^n$-graded (symplectic) Weyl algebra of (V, B) and in the nondegenerate symmetric case the $\mathbb{Z}_2^n$-graded Clifford algebra. (The terminology may be a bit confusing regarding that the usual Clifford algebra appears to be $\mathbb{Z}_2^n$-graded, but here we talk about the notion attached to already graded vector space.) A special case is obtained as follows: take a $\mathbb{Z}_2^n$-graded vector space V and its dual V^* and introduce the $\mathbb{Z}_2^n$-graded skew-symmetric form $B(v + v', w + w') = v'(w) - (-1)^{\langle \tilde{v}, \tilde{w}' \rangle} w'(v)$ on $V \oplus V^*$, where the primed elements are in the dual. It is clear that if we choose a homogeneous basis then the corresponding Weyl algebra is in the form in physics known as a $\mathbb{Z}_2^n$-superHeisenberg algebra

$$\partial^i x_j - (-1)^{\langle i,j \rangle} x_j \partial^i = \delta_i^j,$$

Like in the case of the usual Weyl algebra, there is a $\mathbb{Z}_2^n$-graded vector space isomorphism of this Weyl algebra with the tensor product $S(V) \otimes S(V^*)$ where the polynomials of the form $x_I \partial^J$ where I, J are multiindices (x_I is a generic monomial in generators of $S(V)$ and alike for ∂^J i $S(V^*)$).

We are now interested in a concrete formulas for the embeddings of $U(\mathfrak{g})$ into the (completion of) the Weyl algebra corresponding to the skew-symmetric form on the space $\mathfrak{g} \oplus \mathfrak{g}^*$ and in particular for the embedding induced by the symmetrization map.

We look for the embeddings in a form

$$\hat{x}_i \mapsto \sum_j x_j \phi_i^j \tag{4.3.1}$$

where $\hat{x}_j$ is some choice of basis of $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$ (consisting of $\mathbb{Z}_2^n$-homogeneous elements), x_j are the corresponding generators of $\mathbb{Z}_2^n$-graded symmetric algebra $S(\mathfrak{g}) \hookrightarrow W(\mathfrak{g})$ and ϕ_i^j are the $\mathbb{Z}_2^n$-homogeneous elements of (the completion of) the subalgebra generated by the elementary derivations (coming from the dual space elements) $\partial^j = \partial_{x^j}$. This form has a reason: this is a dual description of a formal $\mathbb{Z}_2^n$-graded vector field on a neighborhood of unit element of a $\mathbb{Z}_2^n$-graded Lie group. The $\mathbb{Z}_2^n$-degree of ϕ_i^j is $\tilde{x}_i + \tilde{x}_j$. It is an easy exercise that, as in the classical case, one can write uniquely every element in $\mathbb{Z}_2^n$-superHeisenberg algebra as the sum of products where elements of the $S(\mathfrak{g})$ are on the left hand side (a version of normal ordering). Notice that these are the derivations of the same $\mathbb{Z}_2^n$-homogeneity as x_j and the graded commutator is

$$[\partial^j, x_l] = \delta_l^j,$$

while the graded commutators among x-s vanish, and graded commutators among ∂-s also vanish. In the purely even case, ϕ_i^j is a usual formal function of partial derivatives and $[\phi_i^j, x_i]$ can be written as the derivative of ϕ_i^j with respect to ∂_i. In general case, before taking the derivative one may need to anticommute with some other generators. In any case, the commutation drops the polynomial degree by 1 and this generalizes the derivatives with respect to ∂_i. Thus, $[\phi_i^j, x_l] = \frac{\delta\phi_i^j}{\delta\partial^l}$. Let also the structure constants of $\mathfrak{g}$ be defined by the graded commutators

$$[\hat{x}_i, \hat{x}_j] = \sum_s C_{ij}^s \hat{x}_s. \tag{4.3.2}$$

The condition that the map (4.3.1) defines (on generators) a morphism of $\mathbb{Z}_2^n$-graded algebras is that the $\mathbb{Z}_2^n$-graded commutator

$$[\sum_k x_k \phi_i^k, \sum_l x_l \phi_j^l] = \sum_{sr} C_{ij}^s x_r \phi_s^r$$

To that point all looked like in non-graded case. The difference comes from the expansion of the left-hand side. For the simplicity of the notation we will use the Einstein summation convention. Thus, after some calculations and using that the $\mathbb{Z}_2^n$-graded commutators $[x_i, x_j] = 0$ and $[\partial^i, \partial^j] = 0$, we obtain

$$x_k \frac{\delta\phi_i^k}{\delta\partial^l} \phi_j^l - (-1)^{\langle\tilde{i},\tilde{j}\rangle} x_l \frac{\delta\phi_j^l}{\delta\partial^k} \phi_i^k = C_{ij}^s x_r \phi_s^r$$

Consequently, by equating the expressions on the right of x_r, we get a system of $\mathbb{Z}_2^n$-graded formal differential equations

$$\frac{\delta\phi_i^r}{\delta\partial^l} \phi_j^l - (-1)^{\langle\tilde{i},\tilde{j}\rangle} \frac{\delta\phi_j^r}{\delta\partial^l} \phi_i^l = C_{ij}^s \phi_s^r \tag{4.3.3}$$

Theorem 4.3.1. *Every solution of the above system for ϕ gives an inclusion of $U(\mathfrak{g})$ as a $\mathbb{Z}_2^n$-graded associative algebras into the appropriate $\mathbb{Z}_2^n$-superHeisenberg algebra.*

Now we want to discuss the form of ϕ which corresponds to the symmetrization map. For the usual supersymmetry Petracci [9] studied the corresponding representation by coderivations and she called it universal as it is related to certain scalar function. It is known that the symmetrization map is functorial in the choice of underlying Lie algebra. In particular if one takes an identity as a map and considers the formula in terms of different bases, then one gets covariance with respect to basis. This implies that the form has to be covariant. An extension of the analysis in [5] of the covariance suggests that in the coordinates the form of the embedding has to be of the form

$$\phi_j^i(x) = \sum_{N=0}^{\infty} A_N (\mathbb{C}^N)_j^i \tag{4.3.4}$$

where A_N are some numbers, $A_0 = 1$ and $\mathbb{C} = (\mathbb{C}^i_j)$ is a matrix whose upper index is a column index (non-standard convention which has a reason coming from a deformation theory interpretation) and where $\mathbb{C}^i_j = C^i_{jk}\partial^k$ and summation over k is understood. This form can be obtained by considering the exponential map in the graded case and expressing the graded Lie algebra of left invariant vector fields on the $\mathbb{Z}_2^n$-graded Lie group in the chart given by the exponential map. This calculation in the non-graded case is given in [7], chapter 3. The calculation uses the formula for the differential of the exponential map; the analogue of the formula for the differential in graded case is known and like in the classical case it involves the formal function whose coefficients are given by Bernoulli numbers and consequently $A_N = (-1)^N B_N/N!$ where B_N is the N-th Bernoulli number.

Definition 4.3.2. Bernoulli numbers B_N for $N = 0, 1, 2, \ldots$ are the rational numbers defined by the generating series

$$\sum_{N=0}^{\infty}(-1)^N \frac{B_N}{N!}t^n = \frac{t}{1-\exp(t)} \tag{4.3.5}$$

In particular, it holds $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$ and $B_{2K+1} = 0$ for all integer $K \geq 1$.

The coefficients A_N and the proof that 4.3.4 for $A_N = B_N$ is indeed a representation of the universal enveloping algebra by elements of the $\mathbb{Z}_2^n$-superHeisenberg algebra can alternatively be inferred by a direct calculation modifying the proof which is in the classical case exhibited in chapters 2-7 of [5]. There are however some nontrivial differences with respect to their proof of the nongraded case. Regarding that the derivations $\delta_s = \frac{\partial}{\partial(\partial^s)}$ are also having parity, they satisfy the $\mathbb{Z}_2^n$-graded Leibniz rule. In particular, if $f = f(\partial)$ then

$$\delta_r(\mathbb{C}^i_j f) = \delta_r(C^i_{jk}\partial_k \cdot f) = C^i_{jr} f + (-1)^{r\cdot k} C^i_{jk}(\delta_r f)$$

The cumbersome problem in many calculations is that the sign $(-1)^{r\cdot k}$ is not an overall sign in front of the second term, but it falls under the summation over k. Another problem is that while the scalars C^i_{jk} are in the ground field, the entries of the matrix $\mathbb{C}$ involve odd and even summands and hence when we contract the copies of the matrix we need to be careful with the order and some manipulations from the earlier proof are illegal and have to be replaced by more subtle calculations.

We first substitute the Ansatz with tensorial form (4.3.4) into (4.3.3). We obtain

$$\sum_{N=0}^{\infty}\sum_{I=1}^{N} A_I A_{N-I}[(\delta_r(\mathbb{C}^I)^c_i)(\mathbb{C}^{N-I})^r_j - (-1)^{i\cdot j}(\delta_r(\mathbb{C}^I)^c_j)(\mathbb{C}^{N-I})^r_i] = \sum_{N=0}^{\infty} A_{N-1} C^s_{ij}(\mathbb{C}^{N-I})^c_s$$

In the formal topology with respect to the degree of the differential operators it is sufficient to prove the identity in every degree N with respect to ∂-s. Thus we need to check the identities

$$\sum_{I=1}^{N} A_I A_{N-I}[(\delta_r(\mathbb{C}^I)^c_i)(\mathbb{C}^{N-I})^r_j - (-1)^{i\cdot j}(\delta_r(\mathbb{C}^I)^c_j)(\mathbb{C}^{N-I})^r_i] = A_{N-1} C^s_{ij}(\mathbb{C}^{N-I})^c_s \tag{4.3.6}$$

for all $N \geq 1$.

We shall extensively use the $\mathbb{Z}_2^n$-graded antisymmetry of the bracket expressed in terms of the structure constants as $C_{ij} = (-1)^{ij} C_{ji}$ and of the $\mathbb{Z}_2^n$-graded Jacobi identity,

$$(-1)^{k\cdot i} C^s_{jk} C^l_{is} + (-1)^{i\cdot j} C^s_{ki} C^l_{js} + (-1)^{j\cdot k} C^s_{ij} C^l_{ks} = 0, \tag{4.3.7}$$

and their combinations like the left and right $\mathbb{Z}_2^n$-graded Leibniz identity, e.g.

$$C^c_{ir} C^r_{jk} - (-1)^{j\cdot i} C^c_{jr} C^r_{ik} = C^s_{ij} C^c_{sk}. \tag{4.3.8}$$

Equation (4.3.6) for $N = 1$ reads simply as $A_1(C^c_{ij} - (-1)^{ji}C^c_{ji}) = C^c_{ij}$, hence it is satisfied iff $A_1 = \frac{1}{2}$.

Equation (4.3.6) for $N = 1$ reads

$$A_1^2[\delta_r(C^c_{ik}\partial^k)\mathbb{C}^r_j - (-1)^{i\cdot j}\delta_r(C^c_{jk}\partial^k)\mathbb{C}^r_i] + A_2[\delta_r(\mathbb{C}^2)^c_i\delta^r_j - (-1)^{i\cdot j}\delta_r(\mathbb{C}^2)^c_i\delta^r_j] = A_1 C^s_{ij}\mathbb{C}^c_s$$

The first summand on the left hand side is

$$A_1^2[C^c_{ir}C^r_{jk}\partial^k - C^c_{ir}C^r_{jk}\partial^k] \overset{(4.3.8)}{=} A_1^2 C^s_{ij}\mathbb{C}^c_s,$$

while the second summand is $A_1(\delta_j(\mathbb{C}^2)^c_i - (-1)^{i\cdot j}\delta_j(\mathbb{C}^2)^c_i)$. Using the $\mathbb{Z}_2^n$-graded rule for the graded derivative on the matrix square the expression in brackets expands to

$$C^s_{ij}C^c_{sk}\partial^k - (-1)^{i\cdot r}C^s_{ik}\partial^k C_{sj} - (-1)^{i\cdot j}C^s_{ji}C^c_{sk}\partial^k + (-1)^{i\cdot j + j\cdot r}C^c_{jk}\partial^k C^k_{si} \overset{(4.3.8)}{=} 3C^s_{ij}\mathbb{C}^s_s.$$

This follows by observing that the third summand equals first and the second and fourth add to the same expression using the Leibniz identity.

Thus the $N = 2$ equation holds iff $A_1^2 + A_2 = A_1$. Using $A_1 = 1/2$ we obtain $A_2 = 1/12$.

We are now going to prove that the $N = 3$ identity holds if $A_3 = 0$. This means that only $I = 1$ and $I = 2$ terms (involving $A_1 A_2$ and $A_2 A_1$) contribute to the left hand side. After dividing by A_2 and substituting $A_1 = 1/2$, we obtain the $N = 3$ condition

$$C^c_{ir}(\mathbb{C}^2)^r_j - (-1)^{i\cdot j}C^c_{jr}(\mathbb{C}^2)^r_i + \delta_r(\mathbb{C}^2)^c_i\mathbb{C}^r_j - (-1)^{i\cdot j}\delta_r(\mathbb{C}^2)^c_j\mathbb{C}^r_i = 2C^s_{ij}(\mathbb{C}^2)^c_s \tag{4.3.9}$$

Applying the graded derivative, the third summand is

$$\delta_r(\mathbb{C}^2)^c_i\mathbb{C}^r_j = C^s_{ir}\mathbb{C}^c_s\mathbb{C}^r_j - (-1)^{r\cdot k}C^s_{ik}\partial^k C^c_{sr}\mathbb{C}^r_j \tag{4.3.10}$$

and the fourth summand is

$$-(-1)^{i\cdot j}\delta_r(\mathbb{C}^2)^c_j\mathbb{C}^r_i = -(-1)^{i\cdot j}C^s_{jr}\mathbb{C}^c_s\mathbb{C}^r_i + (-1)^{i\cdot j + r\cdot k}C^s_{jk}\partial^k C^c_{sr}\mathbb{C}^r_i. \tag{4.3.11}$$

The first summand in (4.3.10) and the first summand in (4.3.11), by the Leibniz identity for the structure constants, add up to $C^s_{ij}(\mathbb{C}^2)^c_s$ what is exactly one half of the right hand side in (4.3.9). It remains to show that the remaining terms agree, that is

$$C^c_{ir}(\mathbb{C}^2)^r_j - (-1)^{r\cdot k}C^s_{ik}\partial^k\cdot C^c_{sr}\mathbb{C}^r_j - (-1)^{i\cdot j}C^c_{jr}(\mathbb{C}^2)^r_i + (-1)^{i\cdot j + r\cdot k}C^s_{jk}\partial^k\cdot C^c_{sr}\mathbb{C}^r_i = C^s_{ij}(\mathbb{C}^2)^c_s. \tag{4.3.12}$$

For the second summand on the left hand side (and similarly for the fourth) we use the Leibniz identity for the contraction of the first and third C-factor:

$$-(-1)^{r\cdot k}C^s_{ik}\partial^k\cdot C^c_{sr}\mathbb{C}^r_j = (-1)^{i\cdot j}C^c_{js}(\mathbb{C}^2)^s_i + C^s_{ir}\mathbb{C}^c_s\mathbb{C}^r_j \tag{4.3.13}$$

$$(-1)^{i\cdot j + r\cdot k}C^s_{ik}\partial^k\cdot C^c_{sr}\mathbb{C}^r_j = -(-1)^{i\cdot j}C^c_{is}(\mathbb{C}^2)^s_j - C^s_{jr}\mathbb{C}^c_s\mathbb{C}^r_i \tag{4.3.14}$$

Substituting (4.3.13) and (4.3.14) into (4.3.12) we obtain the condition $C^s_{ir}\mathbb{C}^c_s\mathbb{C}^r_j - C^s_{jr}\mathbb{C}^c_s\mathbb{C}^r_i = C^t_{ij}(\mathbb{C}^2)^c_t$, which follows by contracting the Leibniz identity with one copy of $\mathbb{C}^c_s$ multiplied from the left. This proves the identity (4.3.12) and hence (4.3.9).

Now we claim that not only $A_3 = 0$ implies that the differential equation holds for $N = 3$ (for previously inferred values for A_0, A_1, A_2), but we shall also prove inductively that if $A_{2K+1} = 0$ for all $K \in \mathbb{N}$, and any even coefficients $A_{2K} \neq 0$, then the remaining identities for odd powers $N = 2K + 1 > 3$ also hold. The basis of induction is the $N = 3$ case. If it holds then for $N + 2$ only the summands for $I = 1$ and $I = N - 1$. We need to prove for $L = 2K \geq 2$ that

$$A_1 A_L[C^c_{is}(\mathbb{C}^L)^s_j - \delta_r(\mathbb{C}^L)^c_j\mathbb{C}^r_j - (-1)^{i\cdot j}C^c_{js}(\mathbb{C}^L)^s_i + (-1)^{i\cdot j}\delta_r(\mathbb{C}^L)^c_j\mathbb{C}^r_i] = A_L C^s_{ij}(\mathbb{C}^L)^c_s$$

what is for $A_L = A_{2K} \neq 0$ equivalent to

$$[C^c_{is}(\mathbb{C}^L)^s_j + \delta_r(\mathbb{C}^L)^c_i\mathbb{C}^r_j - (-1)^{i\cdot j}C^c_{js}(\mathbb{C}^L)^s_i - (-1)^{i\cdot j}\delta_r(\mathbb{C}^L)^c_j\mathbb{C}^r_i] = 2C^s_{ij}(\mathbb{C}^L)^c_s \tag{4.3.15}$$

We shall in fact prove more, namely this identity hold for all $L \geq 2$, both odd and even.

Theorem 4.3.3. *Equation (4.3.15) and equation*

$$C^c_{ir}(\mathbb{C}^L)^r_j+(-1)^{r\cdot k}C^s_{ik}\partial^k\cdot\delta_r(\mathbb{C}^{L-1})^c_s\mathbb{C}^r_j-(-1)^{i\cdot j}C^c_{jr}(\mathbb{C}^L)^r_i-(-1)^{i\cdot j+r\cdot k}C^s_{jk}\partial^k\cdot\delta_r(\mathbb{C}^{L-1})^c_s\mathbb{C}^r_i = C^s_{ij}(\mathbb{C}^L)^c_s \tag{4.3.16}$$

hold for all natural numbers $L \geq 2$.

Proof. For any L we first show that (4.3.15) is equivalent to (4.3.16). Say, if we start with (4.3.15) and expand by the graded Leibniz rule, $\delta_r(\mathbb{C}^L)^c_i = C^p_{ir}(\mathbb{C}^{L-1})^c_i+(-1)^{r\cdot k}(C^p_{ik}\partial^k)\delta_r(\mathbb{C}^{L-1})^c_p$, and analogously for $\delta_r(\mathbb{C}^L)^c_j$, obtaining at the left-hand side

$$\begin{array}{l} C^c_{ir}(\mathbb{C}^L)^r_j + C^p_{ir}(\mathbb{C}^{L-1})^c_p\mathbb{C}^r_j + (-1)^{r\cdot k}C^s_{ik}\partial^k\delta_r(\mathbb{C}^{L-1})^c_s\mathbb{C}^r_j \\ -(-1)^{i\cdot j}C^c_{jr}(\mathbb{C}^L)^r_i - (-1)^{i\cdot j}C_{jr}(\mathbb{C}^{L-1})^c_p\mathbb{C}^r_i + (-1)^{i\cdot j+r\cdot k}C^s_{jk}\partial^k\cdot\delta_r(\mathbb{C}^{L-1})^c_s\mathbb{C}^r_i. \end{array}$$

The second and the fourth summand equal exactly a half of the right-hand side by the identity

$$C^p_{ir}(\mathbb{C}^{L-1})^c_p\mathbb{C}^r_j - (-1)^{i\cdot j}C^p_{jr}(\mathbb{C}^{L-1})^c_p\mathbb{C}^r_i = C^s_{ij}(\mathbb{C}^L)^c_s \tag{4.3.17}$$

which is just the Leibniz identity in the form

$$C^p_{ir}C^r_{jk} - (-1)^{i\cdot j}C^p_{jr}C^r_{ik} = C^s_{ij}C^p_{sk} \tag{4.3.18}$$

contracted with ∂^k and then contracting with $(\mathbb{C}^{L-1})^c_p$ from the left. Thus the remaining four summands equate the other half of the right-hand side in (4.3.15) Thus, by subtracting (4.3.17) we obtain (4.3.16). To conclude the proof, it remains to show (4.3.16) by induction on L. For the basis of induction, just notice that for $L = 2$, equation (4.3.15) coincides with equation (4.3.9) and equation (4.3.16) coincides with equation (4.3.12). The step of induction reduces to routine tensorial manipulations and usage of Jacobi identities.

Corollary 4.3.4. *If $A_N = (-1)^N\frac{B_N}{N!}$ then identities (4.3.6) hold for $N = 0, 2,$ and all odd N.*

Let us now expand how the required identities look for even $N \geq 4$. As products $A_I A_{N-I}$ vanish when I is odd, we are left with

$$\sum_{k=1}^{N/2} A_{2k}A_{N-2k}\left[\delta_r(\mathbb{C}^I)^c_i(\mathbb{C}^{N-2k})^r_j - (-1)^{i\cdot j}\delta_r(\mathbb{C}^I)^c_j(\mathbb{C}^{N-2k})^r_i\right] = -A_N\delta_j(\mathbb{C}^N)^c_i + (-1)^{i\cdot j}A_N\delta_i(\mathbb{C}^N)^c_j. \tag{4.3.19}$$

The right-hand side (coming from $I = 2k = N$ summand) can also be written as $-A_N(M_0 + \ldots + M_N)^c_{ij}$ where

$$\begin{array}{rcl} (M_0)^c_{ij} &=& (1-(-1)^{i\cdot j})C^{s_1}_{ij}\mathbb{C}^{s_2}_{s_1}\cdots\mathbb{C}^c_{s_{N-1}}, \\ (M_L)^c_{ij} &=& (\mathbb{C}^{L-1}_{(r)})^s_i C^t_{sj}(\mathbb{C}^{N-I-1})^c_t - (-1)^{i\cdot j}(\mathbb{C}^{L-1}_{(r)})^s_j C^t_{si}(\mathbb{C}^{N-I-1})^c_t,\ \ 1 \leq L < N-1, \\ (M_{N-1})^c_{ij} &=& (\mathbb{C}^{N-1}_{(r)})^s_i C^c_{sj} - (-1)^{i\cdot j}(\mathbb{C}^{N-1}_{(r)})^s_j C^c_{si}, \end{array} \tag{4.3.20}$$

where we introduce the auxiliary matrix $(\mathbb{C}_{(r)})^s_t = (-1)^{r\cdot k}C^s_{tk}\partial^k$.

A direct proof of the identities (4.3.19) for N even and bigger than 4 stays an open question in $\mathbb{Z}_2^n$-graded case. This would extend from the non-graded case the explicit formula for the representation of $\mathbb{Z}_2^n$-graded Lie algebras by derivations of the (completion) of $\mathbb{Z}_2^n$-graded symmetric algebra, hence by elements in the corresponding Weyl algebra (completed $\mathbb{Z}_2^n$-superHeisenberg algebra). We know that this representation exists by transporting the derivations of the universal enveloping algebra along the symmetrization map which is here introduced by studying the $\mathbb{Z}_2^n$-graded PBW theorem.

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