A topological Paley-Wiener-Schwartz Theorem for sections of homogeneous vector bundles on G/K

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Abstract

We study the Fourier transform for compactly supported distributional sections of complex homogeneous vector bundles on symmetric spaces of non-compact type $X = G/K$. We prove a characterization of their range. In fact, from Delorme's Paley-Wiener theorem for compactly supported smooth functions on a real redutive group of Harish-Chandra class, we deduce topological Paley-Wiener and Paley-Wiener-Schwartz theorems for sections.

Contents

1 Introduction

One of the central theorems of harmonic analysis on $\mathbb{R}^n$, is the so-called Paley-Wiener theorem, named after the two mathematicians Raymond Paley and Norbert Wiener. It describes the image of the Fourier transform of the space $C_c^\infty(\mathbb{R}^n)$ of smooth functions with compact support as the space of entire functions on $\mathbb{C}^n$ satisfying some growth condition. The theorem has a counterpart, known as Paley-Wiener-Schwartz theorem. Here, the smooth functions are replaced by distributions $T \in C_c^{-\infty}(\mathbb{R}^n)$ and the growth condition by a weaker growth condition (e.g. [Hör83], Thm. 7.3.1).

Both theorems have been generalized to more general Lie groups G and furthermore to some smooth manifolds carrying symmetries. For example, the case of Riemannian symmetric spaces of non-compact type $X = G/K$ was considered by Helgason [Hel66] and Gangolli [Gan71]. They proved a Paley-Wiener theorem for compactly supported K-invariant smooth functions and Helgason [Hel73] even showed it for general compactly supported smooth functions on X. There is also a Paley-Wiener theorem for $K \times K$-finite compactly supported smooth functions on a real reductive Lie group G of Harish-Chandra class due to Arthur [Art83] and Delorme [Del05], formulated in terms of the so-called Arthur-Campolli and Delorme conditions, respectively. Delorme even proved a version without the $K \times K$-finess. A generalization to K-finite functions on reductive symmetric spaces was presented by van den Ban and Schlichtkrull [vdBS06]. Furthermore, later van den Ban and Souaifi [vdBS14] proved, without using the proof or validity of any associated Paley-Wiener theorems of Arthur or Delorme, that the two compatibility conditions are equivalent. Concerning the Paley-Wiener-Schwartz theorem for distributions on symmetric spaces, we mention Helgason [Hel73] and Eguchi, Hashizume, Okamato [EHO73]. Moreover, van den Ban and Schlichtkrull [vdBS06] also proved a topological Paley-Wiener-Schwartz theorem for K-finite distributions on reductive symmetric spaces.

Our aim is to establish a topological Paley-Wiener theorem for (distributional) sections of homogeneous

vector bundles on X using Delorme's intertwining conditions. Thus, starting, in Section 2 with Delorme's Paley-Wiener theorem ([Del05], Thm. 2) in the setting of van den Ban and Souaifi [vdBS14], we will adjust it, in Sections 3 and 4, for our proposes. More precisely, we describe the intertwining conditions for sections and show that there are equivalent with Delorme's one by using Frobenius-reciprocity (Prop. 7 & Thm. 2). We consider three levels, (Level 1) refers to Delorme's Paley-Wiener theorem (Thm. 1), (Level 2) corresponds to the desired Paley-Wiener theorem for sections (Thm. 3) and (Level 3) stands for the Paley-Wiener theorem for 'spherical functions' (Thm. 3). For the last, we fixed an irreducible K-representations on the left while a right, not necessary irreducible, K-type $*$ is fixed by the bundle $\mathbb{E}_* \to X$. In this way, it will be much easier to manage the intertwining conditions.
Finally in Section 6, we present, a topological Paley-Wiener-Schwartz theorem for distributional sections (Thm. 4) in both levels (Level 2) and (Level 3). We used van den Ban and Schlichtkrull's technique [vdBS06] as well as Camporesi's Plancherel theorem for sections ([Ca97], Thm. 3.4 & Thm. 4.3).
This paper ends, in Section 7, by analysing consequences of this theorem for linear invariant differential operators between sections of homogeneous vector bundles (Prop. 10).

The motivation behind this work lies in solvability questions of systems of invariant differential equations on symmetric spaces G/K. In fact, the results of the present paper as well as applications to solvability questions are part of the doctoral dissertation [Pal21] of the second author. For further details, we refer to [Pal21] and the upcoming papers ([OlPa22-2], [OlPa22-3]).

2 On Delorme's Paley-Wiener Theorem

Let G be a real connected semi-simple Lie group with finite center of non-compact type with Lie algebra $\mathfrak{g}$ and $K \subset G$ its maximal compact subgroup with Lie algebra $\mathfrak{k}$. The quotient $X = G/K$, then is a Riemannian symmetric space of non-compact type.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Fix a corresponding minimal parabolic subgroup $P = MAN$ of G with split component $A = \exp(\mathfrak{a})$, nilpotent Lie group N and $M = Z_K(\mathfrak{a})$ being the centralizer of A in K. Let $(\sigma, E_\sigma) \in \widehat{M}$ be a finite-dimensional irreducible representation of M and $\lambda \in \mathfrak{a}^*_{\mathbb{C}} \cong \mathbb{C}^n$. For fixed $(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}^*_{\mathbb{C}}$, let $(\sigma_\lambda, E_{\sigma,\lambda})$ be the representation of P on the vector space $E_{\sigma,\lambda} = E_\sigma$ such that $\sigma_\lambda(man) = a^{\lambda+\rho}\sigma(m) \in \mathrm{End}(E_{\sigma,\lambda})$ for $m \in M$, $a \in A, n \in N$ and where $\rho \in \mathfrak{a}^+$ is the half sum of the positive roots of $(\mathfrak{g}, \mathfrak{a})$, counted with multiplicities. We use the notation a^λ for $e^{\lambda \log(a)}$. Then, the space

$$H^{\sigma,\lambda}_\infty := \{f : G \overset{C^\infty}{\to} E_{\sigma,\lambda} \mid f(gman) = a^{-(\lambda+\rho)}\sigma(m)^{-1}(f(g))\} \cong C^\infty(G/P, \mathbb{E}_{\sigma,\lambda})$$

together with the left regular action $(\pi_{\sigma,\lambda}(g)f)(x) := f(g^{-1}x) = (l_g f)(x)$ for $g, x \in G$ and $f \in H^{\sigma,\lambda}_\infty$, is the space of smooth vectors of the principal series representations of G induced from the P-representation σ_λ on $E_{\sigma,\lambda}$ (e.g. [Kna86], p. 168). The restriction map from $H^{\sigma,\lambda}_\infty$ to functions on K is injective by the Iwasawa decomposition $g = \kappa(g)e^{a(g)}n(g) \in KAN$ of G. In particular, for $f \in H^{\sigma,\lambda}_\infty$ we have $f(g) = f(\kappa(g)e^{a(g)}n(g)) = a(g)^{-(\lambda+\rho)}(f(\kappa(g)))$. This yields, the so-called *compact picture* of $H^{\sigma,\lambda}_\infty$ (e.g. [Kna86], p. 168). It has the advantage that the representation space

$$H^\sigma_\infty := \{\varphi : K \overset{C^\infty}{\to} E_\sigma \mid \varphi(km) = \sigma(m)^{-1}\varphi(k),\ k \in K, m \in M\} \cong C^\infty(K/M, \mathbb{E}_\sigma) \tag{2.1}$$

does not depend on λ. Here, H^σ_∞ is equipped with the usual Fréchet topology. From time to time, we need the L^2-norm. In the compact picture, the action of all elements $g \in G$, which are not in K, is slightly more involved, since we need to commute them with the argument $k \in K$, i.e.

$$(\pi_{\sigma,\lambda}(g)\varphi)(k) = a(g^{-1}k)^{-(\lambda+\rho)}\varphi(\kappa(g^{-1}k)), \quad \varphi \in H^\sigma_\infty. \tag{2.2}$$

Fourier transform for G in (Level 1)

Let

$$C^\infty_c(G) = \bigcup_{r>0} C^\infty_r(G) := \bigcup_{r>0}\{f \in C^\infty(G) \mid supp(f) \in \overline{B}_r(o)\}$$

be the space of compactly supported smooth complex functions on G, where

$$\overline{B}_r(o) := \{g \in G \mid dist_X(gK, o) \le r\} \subset G$$

denotes the preimage of the closed ball of radius r and center $o = eK$ in X under the projection $G \to X$. Here, $dist_X$ means a fixed G-invariant Riemannian distance on X and e is the neutral element of G. We equip $C_r^\infty(G)$ with the usual Fréchet topology, thus $C_c^\infty(G)$ is a LF-space. Given $\sigma \in \widehat{M}$, let us consider the map

$$\pi_{\sigma,\cdot} : G \to (\mathfrak{a}_\mathbb{C}^* \to \mathrm{End}(H_\infty^\sigma)), g \mapsto (\lambda \mapsto \pi_{\sigma,\lambda}(g)).$$

Definition 1 (Fourier transform for G in (Level 1))**.** *Fix $(\sigma,\lambda) \in \widehat{M} \times \mathfrak{a}_\mathbb{C}^*$, we define the Fourier transform of $f \in C_c^\infty(G)$ by the operator*

$$\mathcal{F}_{\sigma,\lambda}(f) := \pi_{\sigma,\lambda}(f) = \int_G f(g)\pi_{\sigma,\lambda}(g)\, dg \in \mathrm{End}(H_\infty^\sigma).$$

We denote by $\mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*)$ the space of holomorphic functions in $\mathfrak{a}_\mathbb{C}^*$ and by $\mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, \mathrm{End}(H_\infty^\sigma))$ the space of maps $\mathfrak{a}_\mathbb{C}^* \ni \lambda \mapsto \phi(\lambda) \in \mathrm{End}(H_\infty^\sigma)$ such that

(1.*i*) for $\varphi \in H_\infty^\sigma$, the function $\lambda \mapsto \phi(\lambda)\varphi \in H_\infty^\sigma$ is holomorphic.

From ([Del05], Lem. 10 (ii)), we deduce the following statement.

Proposition 1. *The family of applications $f \mapsto \mathcal{F}_{\sigma,\lambda}(f)$ is a linear map from $C_c^\infty(G)$ into $\prod_{\sigma\in\widehat{M}} \mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, \mathrm{End}(H_\infty^\sigma))$.* □

Delorme's Paley-Wiener theorem and intertwining conditions in (Level 1)

We now proceed with the definition of Delorme's Paley-Wiener space ([Del05], Def. 3). It induced Delorme's intertwining conditions for *derived* versions of H_∞^σ ([Del05], Sect. 1.5 & Déf. 3 (4.4)). Van den Ban and Souaifi present a more elegant reformulation of them ([vdBS14], Sect. 4.5, in particular Lem. 4.4. and Prop. 4.5.). In the same spirit, we present a very similar definition of derived G-representations.

Definition 2 (m-th derived representation)**.** *For $\lambda \in \mathfrak{a}_\mathbb{C}^*$, let Hol_λ be the set of germs at λ of $\mathbb{C}$-valued holomorphic functions $\mu \mapsto f_\mu$ and $m_\lambda \subset \mathrm{Hol}_\lambda$ the maximal ideal of germs vanishing at λ.*
Denote by $H_{[\lambda]}^\sigma$ the set of germs at λ of H_∞^σ-valued holomorphic functions $\mu \mapsto \phi_\mu \in H_\infty^\sigma$ with G-action

$$(g\phi)_\mu = \pi_{\sigma,\mu}(g)\phi_\mu, \quad g \in G.$$

For $m \in \mathbb{N}_0$, it induces a representation $\pi_{\sigma,\lambda}^{(m)}$ on the space

$$H_{\infty,(m)}^{\sigma,\lambda} := H_{[\lambda]}^\sigma / m_\lambda^{m+1} H_{[\lambda]}^\sigma, \tag{2.3}$$

which is equipped with the natural Fréchet topology. We call this representation the m-th derived principal series representation of G.

Here, Hol_λ acts on $H_{[\lambda]}^\sigma$ by pointwise multiplication. Note that the m =0-th derived representation $H_{\infty,(0)}^{\sigma,\lambda} \cong H_\infty^\sigma$ is the space of smooth vectors of the principal series G-representation in the compact picture. Intuitively, we can say that $H_{\infty,(m)}^{\sigma,\lambda}$ contains all Taylor polynomials of order m at λ of holomorphic families ϕ_μ. Moreover, $\phi \in \prod_{\sigma\in\widehat{M}} \mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, \mathrm{End}(H_\infty^\sigma))$ induces an operator on each $H_{\infty,(m)}^{\sigma,\lambda}$.
The following definition turns out to be equivalent to Delorme's intertwining condition ([Del05], Déf. 3 (4.4)).

Definition 3 (Delorme's intertwining condition in (Level 1))**.** *Let Ξ be the set of all 3-tuples (σ, λ, m) with $\sigma \in \widehat{M}$, $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $m \in \mathbb{N}_0$. Consider the m-th derived G-representation $H_{\infty,(m)}^{\sigma,\lambda}$ defined in (2.3). For every finite sequence $\xi = (\xi_1, \xi_2, \ldots, \xi_s) \in \Xi^s, s \in \mathbb{N}$, we define the G-representation*

$$H_\xi := \bigoplus_{i=1}^s H_{\infty,(m_i)}^{\sigma_i,\lambda_i}.$$

We consider proper closed G-subrepresentations $W \subseteq H_\xi$.
Such a pair (ξ, W) with $\xi \in \Xi^s$ and $W \subset H_\xi$ as above, is called an intertwining datum. Every function $\phi \in \prod_{\sigma\in\widehat{M}} \mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, \mathrm{End}(H_\infty^\sigma))$ induces an element

$$\phi_\xi \in \bigoplus_{i=1}^s \mathrm{End}(H_{\infty,(m_i)}^{\sigma_i,\lambda_i}) \subset \mathrm{End}(H_\xi).$$

(D.a) We say that ϕ satisfies Delorme's intertwining condition, if $\phi_\xi(W) \subseteq W$ for every intertwining datum (ξ, W).

Next, we define Delorme's Paley-Wiener space ([Del05], Déf. 3). We denote by $\mathcal{U}(\mathfrak{k})$ the universal enveloping algebra of complexification of $\mathfrak{k}$ (e.g. [Jac62] Chap. V). Note that our fixed Riemannian metric corresponds to a Ad-invariant bilinear form on $\mathfrak{g}$, which is definit on $\mathfrak{k}$ and $\mathfrak{p}$. Therefore, we get a norm $|\cdot|$ on $\mathfrak{b}_{\mathbb{C}}^*$ for each subspace $\mathfrak{b} \subset \mathfrak{k}$ or $\mathfrak{b} \subset \mathfrak{p}$.

Definition 4 (Paley-Wiener space in (Level 1))**.** *For $r > 0$, Delorme's Paley-Wiener space is the vector space*

$$PW_r(G) := \Big\{\phi \in \prod_{\sigma \in \widehat{M}} \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}(H_\infty^\sigma)) \mid \phi \textit{ satisfies the growth condition } (1.ii)_r \textit{ below and } (D.a)\Big\}. \tag{2.4}$$

Here,

$(1.ii)_r$ for all $Y_1, Y_2 \in \mathcal{U}(\mathfrak{k}), (\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_{\mathbb{C}}^*$ and $N \in \mathbb{N}_0$, there exists a constant $C_{r,N,Y_1,Y_2} > 0$ such that

$$||\pi_{\sigma,\lambda}(Y_1)\phi(\sigma,\lambda)\pi_{\sigma,\lambda}(Y_2)|| \le C_{r,N,Y_1,Y_2}(1 + |\Lambda_\sigma|^2 + |\lambda|^2)^{-N} e^{r|\mathrm{Re}(\lambda)|}$$

for $\phi \in \mathrm{End}(H_\infty^\sigma)$ and where Λ_σ is the highest weight of σ, $||\cdot||$ is the operator norms on H_∞^σ with respect to the L^2-norm of H_∞^σ.

Notice that, due to Lem. 10 (i) in [Del05], the space $PW_r(G)$ equipped with semi-norms:

$$||\phi||_{r,N,Y_1,Y_2} := \sup_{(\sigma,\lambda)\in\widehat{M}\times\mathfrak{a}_{\mathbb{C}}^*} (1 + |\Lambda_\sigma|^2 + |\lambda|^2)^N e^{-r|\mathrm{Re}(\lambda)|} ||\pi_{\sigma,\lambda}(Y_1)\phi(\sigma,\lambda)\pi_{\sigma,\lambda}(Y_2)||_{H_\infty^\sigma}, \quad \phi \in PW_r(G)$$

is a Fréchet space. Furthermore, the intertwining condition $(D.a)$ in Def. 4 is a special case of van den Ban and Souaifi's one ([vdBS14], Cor. 4.7 and Prop. 4.10.). The small difference is, that instead of the defined m-th derived representations $H_{\infty,(m)}^{\sigma,\lambda}$ (2.3), they consider

$$H_{[\lambda],E}^\sigma := H_{[\lambda]}^\sigma \otimes_{\mathrm{Hol}_\lambda} E,$$

where E is a finite-dimensional Hol_λ-module. By the following proposition, this leads to equivalent intertwining conditions.

Proposition 2. *With the previous notations, let $(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_{\mathbb{C}}^*$. Then, for $E = \mathrm{Hol}_\lambda/m_\lambda^{m+1}$, we have that $H_{[\lambda],E}^\sigma \cong H_{\infty,(m)}^{\sigma,\lambda}$.*
Moreover, for any finite-dimensional Hol_λ-module E, there exists $m_1, \dots, m_s \in \mathbb{N}_0$ such that $H_{[\lambda],E}^\sigma$ is a quotient of $H_{\infty,(m_1)}^{\sigma,\lambda} \oplus \cdots \oplus H_{\infty,(m_s)}^{\sigma,\lambda}$.

Proof. Consider a (commutative) ring R with neutral element 1, a R-module M and $I \subset R$ an ideal. Then, we have the following isomorphism

$$M \otimes_R R/I \cong M/IM.$$

In fact, by an algebraic computation, one can easily show that the two maps

$$\begin{aligned} \alpha : M \otimes_R R/I &\to M/IM & \text{and} \quad \beta : M/IM &\to M \otimes_R R/I \\ \alpha(m \otimes [r]) &:= [rm] & \beta([m]) &:= m \otimes [1] \end{aligned}$$

are well-defined and inverse to each other. Here $[\cdot]$ denotes the class in the corresponding quotient. For $m \in \mathbb{N}_0$ and $R = \mathrm{Hol}_\lambda$, consider its maximal ideal $m_\lambda^{m+1} \subset \mathrm{Hol}_\lambda$. Take $E = \mathrm{Hol}_\lambda/m_\lambda^{m+1} = R/I$ and $M = H_{[\lambda]}^\sigma$, then

$$H_{[\lambda]}^\sigma \otimes_{\mathrm{Hol}_\lambda} E \cong H_{[\lambda]}^\sigma / m_\lambda^{m+1} H_{[\lambda]}^\sigma =: H_{\infty,(m)}^{\sigma,\lambda}.$$

Moreover, by their Lem. 2.1 in [vdBS14], an ideal $\mathcal{I}$ in Hol_λ is cofinite, if and only, if there exists $m \in \mathbb{N}_0$ such that $m_\lambda^{m+1} \subset \mathcal{I}$.
Thus, for some $s \in \mathbb{N}$ and finitely many cofinite ideals $m_\lambda^{m_1+1}, \dots, m_\lambda^{m_s+1}$ of Hol_λ, we have that E is a quotient of the direct sum

$$\mathrm{Hol}_\lambda/m_\lambda^{m_1+1} \oplus \mathrm{Hol}_\lambda/m_\lambda^{m_2+1} \oplus \cdots \oplus \mathrm{Hol}_\lambda/m_\lambda^{m_s+1}.$$

Hence, the map

$$H_{\infty,(m_1)}^{\sigma,\lambda} \oplus \cdots \oplus H_{\infty,(m_s)}^{\sigma,\lambda} \longrightarrow E$$

is surjective and the result follows. ∎

Now, we can formulate Delorme's Paley-Wiener theorem.

Theorem 1 (Paley-Wiener Theorem, [Del05], Thm. 2)**.** *For $r > 0$, the Fourier transform*

$$C_r^\infty(G) \ni f \mapsto \mathcal{F}_{\sigma,\lambda}(f) \in PW_r(G), \quad (\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_{\mathbb{C}}^*$$

is a topological isomorphism between the two Fréchet spaces $C_r^\infty(G)$ and $PW_r(G)$. □

Remark 1. Delorme formulated the Paley-Wiener Thm. 1 in terms of all cuspidal parabolic subgroups. By Casselman's subrepresentation theorem (e.g. [Wal88], Thm. 3.8.3.), it is clear that it remains true if we restrict to the minimal parabolic subgroup P (compare [vdBS14], Lem. 4.4).)

3 Fourier transforms for (distributional) sections and its properties

Let (τ, E_τ) be a finite dimensional, not necessary irreducible, representation of K. We obtain a homogeneous vector bundle $\mathbb{E}_\tau$ over X, whose space $C^\infty(X, \mathbb{E}_\tau)$ of smooth sections is identified with the following space:

$$C^\infty(X, \mathbb{E}_\tau) \cong \{f : G \xrightarrow{C^\infty} E_\tau \mid f(gk) = \tau^{-1}(k)(f(g)), \forall g \in, k \in K\}.$$

The group G acts on $C^\infty(X, \mathbb{E}_\tau)$ by left translations, $(g \cdot f)(g') = f(g^{-1}g'), \forall g, g' \in G$. It is not difficult to see that we have the following G-isomorphisms:

$$C^\infty(X, \mathbb{E}_\tau) \cong C^\infty(G, E_\tau)^K \cong [C^\infty(G) \otimes E_\tau]^K.$$

Moreover, by taking the topological linear dual of $C^\infty(X, \mathbb{E}_\tau)$, we obtain the space of compactly supported distributional sections:

$$C_c^{-\infty}(X, \mathbb{E}_{\tilde{\tau}}) = \bigcup_{r \geq 0} C_r^{-\infty}(X, \mathbb{E}_\tau) := \bigcup_{r \geq 0} \{T \in C^{-\infty}(X, \mathbb{E}_\tau) \mid supp(T) \in \overline{B}_r(o)\} = C^\infty(X, \mathbb{E}_\tau))', \tag{3.1}$$

where $(\tilde{\tau}, E_{\tilde{\tau}})$ is the dual of the representation (τ, E_τ).

Fourier transform in (Level 2)

We want to study the reduced Fourier transform $\mathcal{F}$ on the space $[C_c^{\pm\infty}(G) \otimes E_\tau]^K \cong C_c^{\pm}(X, \mathbb{E}_\tau)$ by

$$\sum_{i=1}^{d_\tau} f_i \otimes v_i \mapsto \sum_{i=1}^{d_\tau} \mathcal{F}(f_i) \otimes v_i, \quad f \in C_c^{\pm\infty}(G),$$

where d_τ denotes the dimension of E_τ and $v_i, i \in \{1, \cdots, d_\tau\}$, is a basis of E_τ. Roughly, for $r > 0$, one can deduce from Thm. 1, that

$$C_r^\infty(X, \mathbb{E}_\tau) \cong [C_r^\infty(G) \otimes E_\tau]^K \overset{\text{Thm. 1}}{\cong} [PW_r(G) \otimes E_\tau]^K,$$

where $PW_r(G)$ is Delorme's Paley-Wiener space defined in (2.4). The goal is to make $[PW_r(G) \otimes E_\tau]^K$ more explicit and then do the same study for distributions. For this, let us study the map

$$\begin{aligned} C_r^\infty(X, \mathbb{E}_\tau) \ni f \quad &\mapsto \quad \sum_{i=1}^{d_\tau} f_i \otimes v_i \in [C_r^\infty(G) \otimes E_\tau]^K \\ &\overset{\text{Thm. 1}}{\mapsto} \quad \sum_{i=1}^{d_\tau} \mathcal{F}_{\sigma,\lambda}(f_i) \otimes v_i \in [\mathrm{End}(H_\infty^\sigma) \otimes E_\tau]^K \cong H_\infty^\sigma \otimes \mathrm{Hom}_K(H_\infty^\sigma, E_\tau). \end{aligned}$$

Bringing the Frobenius reciprocity into play, it gives us a better description of the space $\mathrm{Hom}_K(H_\infty^\sigma, E_\tau)$. Namely, we have

$$\begin{aligned} \mathrm{Hom}_K(H_\infty^\sigma, E_\tau) \quad &\overset{Frob}{\cong} \quad \mathrm{Hom}_M(E_\sigma, E_\tau) \text{ defined by} \\ \langle Frob(S)w, \tilde{v}\rangle \quad &= \quad \langle w, S^*\tilde{v}(e)\rangle, \quad w \in E_\sigma, \ \tilde{v} \in E_{\tilde{\tau}}, \ S^* : E_{\tilde{\tau}} \to H_\infty^{\tilde{\sigma}}. \end{aligned} \tag{3.2}$$

Let us next compute the inverse of $Frob$.

Lemma 1 ([Olb95], Lem. 2.12)**.** *Let $s \in \mathrm{Hom}_M(E_\sigma, E_\tau)$ and $f \in H^\sigma_\infty$. Then, we have*

$$Frob^{-1}(s)(f) = \int_K \tau(k) s f(k)\, dk.\square$$

The dual of $Frob$ is given by

$$\begin{aligned} \mathrm{Hom}_K(E_\tau, H^\sigma_\infty) &\overset{\widetilde{Frob}}{\cong} \mathrm{Hom}_M(E_\tau, E_\sigma) \\ \widetilde{Frob}(T)(v) &= T(v)(e), \quad v \in E_\tau \end{aligned} \tag{3.3}$$

and for $t \in \mathrm{Hom}_M(E_\tau, E_\sigma)$ and $v \in E_\tau$, the inverse of $\widetilde{Frob}$ will be

$$\widetilde{Frob}^{-1}(t)(v)(k) = t\tau(k^{-1})v. \tag{3.4}$$

Coming back to our previous computation, we get

$$\begin{aligned} [\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K \cong H^\sigma_\infty \otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau) &\overset{Frob}{\cong} H^\sigma_\infty \otimes \mathrm{Hom}_M(E_\sigma, E_\tau) \\ &\overset{(2.1)}{\cong} C^\infty(K/M, E_\sigma \otimes \mathrm{Hom}_M(E_\sigma, E_\tau)) \\ &\cong C^\infty(K/M, \mathbb{E}_{\tau|_M}(\sigma)) \\ &\cong H^{\tau|_M(\sigma)}_\infty, \end{aligned} \tag{3.5}$$

where $\mathbb{E}_{\tau|_M}(\sigma)$ is the σ-isotypic component of $\mathbb{E}_{\tau|_M}$. Here, τ is restricted to M, it is generally no more irreducible and splits into a finite direct sum $\tau|_M = \bigoplus_{\sigma\in\hat{M}} m(\sigma,\tau)\sigma$, where $m(\sigma,\tau) = \dim(\mathrm{Hom}_M(E_\sigma, E_\tau)) \geq 0$ is the multiplicity of σ in $\tau|_M$. Now by taking the algebraic direct sum over all $\sigma \in \widehat{M}$, where only finitely many of them appears, we obtain

$$\bigoplus_{\sigma\in\widehat{M}} [\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K \overset{(3.5)}{\cong} \bigoplus_{\sigma\in\widehat{M}} H^{\tau|_M(\sigma)}_\infty \cong H^{\tau|_M}_\infty = \{f : K \overset{C^\infty}{\to} E_\tau \mid f(km) = \tau(m)^{-1} f(k)\},$$

which can be viewed as the principal series representations corresponding to $\tau|_M$.

Definition 5 (Fourier transform for sections over homogeneous vector bundles in (Level 2))**.** *Let $g = \kappa(g)a(g)n(g) \in KAN = G$ be the Iwasawa decomposition. For fixed $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $k \in K$, we define the function $e^\tau_{\lambda,k}$ by*

$$\begin{aligned} e^\tau_{\lambda,k} : G &\to \mathrm{End}(E_\tau) \cong E_{\tilde{\tau}} \otimes E_\tau \\ g &\mapsto e^\tau_{\lambda,k}(g) := \tau(\kappa(g^{-1}k))^{-1} a(g^{-1}k)^{-(\lambda+\rho)}. \end{aligned} \tag{3.6}$$

(a) For $f \in C^\infty_c(X, \mathbb{E}_\tau)$, the Fourier transformation is given by

$$\mathcal{F}_\tau f(\lambda, k) = \int_G e^\tau_{\lambda,k}(g) f(g)\, dg = \int_{G/K} e^\tau_{\lambda,k}(g) f(g)\, dg, \tag{3.7}$$

where the last equality makes sense, since the integrand is right K-invariant.

(b) The Fourier transform for distributional section $T \in C^{-\infty}_c(X, \mathbb{E}_\tau)$ is defined by

$$\mathcal{F}_\tau T(\lambda, k) := \langle T, e^\tau_{\lambda,k}\rangle = T(e^\tau_{\lambda,k}) \in E_\tau, \quad (\lambda, k) \in \mathfrak{a}^*_{\mathbb{C}} \times K/M.$$

Note that the Fourier transform for sections has already been introduced and studied by Camporesi ([Ca97], (3.18)). It is a direct generalization of Helgason's Fourier transform for $E_\tau = \mathbb{C}$. It is not difficult to see that $\mathcal{F}_\tau f(\lambda, \cdot)$ and $\mathcal{F}_\tau T(\lambda, \cdot)$ are in $\mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, H^{\tau|_M}_\infty)$. Observe that, for $k \in K$ and $g \in G$, we have, by definition

$$e^\tau_{\lambda,k}(g) = l_k(e^\tau_{\lambda,1}(g)) = e^\tau_{\lambda,1}(k^{-1}g). \tag{3.8}$$

This function $e^\tau_{\lambda,k}$ in Def. 5 can be seen as the analogous of the 'exponential' function in the definition of Fourier transform in the Euclidean case $\mathbb{R}^n$. It has some interesting properties. Note that for fixed $k \in K$, $e^\tau_{\lambda,k}(g)$ is an entire function on $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, since $a(g^{-1}k)^{-(\lambda+\rho)}$ is an entire function on $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

Proposition 3. *Let $\tau \in \widehat{K}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $k \in K$. Then, we have*

$$e^\tau_{\lambda,k}(hg) = e^\tau_{\lambda,\kappa(h^{-1}k)}(g) a(h^{-1}k)^{-(\lambda+\rho)}, \quad g, h \in G. \tag{3.9}$$

Proof. Let $h, g \in G = KAN$, then by Iwasawa decomposition, we have

$$\begin{aligned} hg = h\kappa(g)a(g)n(g) &= \kappa(h(\kappa(g))\, a(h\kappa(g))\, n(h\kappa(g))\, a(g)\, n(g) \\ &= \underbrace{\kappa(h\kappa(g))}_{\in K}\, \underbrace{a(h\kappa(g))\, a(g)}_{\in A}\, \underbrace{n(h\kappa(g))\, n(g)}_{\in N}. \end{aligned}$$

In other words, we have $\kappa(hg) = \kappa(h\kappa(g)a(g)n(g)) = \kappa(h(\kappa(g))$, and
$a(hg) = a(h\kappa(g)a(g)n(g)) = a(h\kappa(g))\, a(g)$. Hence,

$$\begin{aligned} e^{\tau}_{\lambda,k}(hg) &\overset{(3.6)}{=} \tau(\kappa(g^{-1}h^{-1}k))^{-1}a(g^{-1}h^{-1}k)^{-(\lambda+\rho)} \\ &= \tau(\kappa(g^{-1}\kappa(h^{-1}k))^{-1}a(g^{-1}\kappa(h^{-1}k))^{-(\lambda+\rho)}a(h^{-1}k)^{-(\lambda+\rho)} \\ &\overset{(3.6)}{=} e^{\tau}_{\lambda,\kappa(h^{-1}k)}(g)a(h^{-1}k)^{-(\lambda+\rho)}. \end{aligned}$$ □

Fourier transform in (Level 3) and its properties

Now consider an additional finite-dimensional K-representation $\gamma : K \to GL(E_\gamma)$ with its associated homogeneous vector bundle $\mathbb{E}_\gamma$ over X. It induces a mapping

$$\mathrm{Hom}_K(E_\gamma, C_c^\infty(X, \mathbb{E}_\tau)) \longrightarrow \mathrm{Hom}_K(E_\gamma, \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, H_\infty^{\tau|_M})). \tag{3.10}$$

The LHS of (3.10) can be identified with a space of functions with values in $\mathrm{Hom}(E_\gamma, E_\tau)$, the (γ,τ)-spherical functions:

$$\begin{aligned} \mathrm{Hom}_K(E_\gamma, C_c^\infty(X, \mathbb{E}_\tau)) &\cong C_c^\infty(G, \gamma, \tau) \\ &:= \{f : G \to \mathrm{Hom}(E_\gamma, E_\tau) \mid f(k_1 g k_2) = \tau(k_2)^{-1} f(g)\gamma(k_1)^{-1}, \forall k_1, k_2 \in K\}. \end{aligned}$$

For the RHS of (3.10), we use the Frobenius reciprocity between K and M, by evaluating at $k = 1$, and we obtain the space of functions $\{\phi : \mathfrak{a}^*_{\mathbb{C}} \to \mathrm{Hom}_M(E_\gamma, E_\tau)\}$. Now we define the Fourier transformation ${}_\gamma\mathcal{F}_\tau$ of $f \in C_c^\infty(G, \gamma, \tau)$.

Definition 6 (Fourier transform in (Level 3))**.** *With the previous notations, the Fourier transformation for $f \in C_c^\infty(G, \gamma, \tau)$ is given by*

$${}_\gamma\mathcal{F}_\tau f(\lambda) := \int_G e^{\tau}_{\lambda,1}(g) f(g)\, dg, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}. \tag{3.11}$$

Similar, the Fourier transformation for distributional function $T \in C_c^{-\infty}(G, \gamma, \tau)$ is defined by

$${}_\gamma\mathcal{F}_\tau T(\lambda) := \langle T, e^{\tau}_{\lambda,1}\rangle.$$

Observe that

$$\tau(m)\mathcal{F}_\gamma f(\lambda) = \int_G e^{\tau}_{\lambda,1}(mg) f(g)\, dg = \int_G e^{\tau}_{\lambda,1}(g) f(m^{-1}g)\, dg = {}_\gamma\mathcal{F}_\tau f(\lambda)\gamma(m) \in \mathrm{Hom}_M(E_\gamma, E_\tau),$$

same for the distributions. Let us consider now the convolution G of $f \in C_c^\infty(X, \mathbb{E}_\gamma)$ to a (γ,τ)-spherical function $\varphi \in C_c^\infty(G, \gamma, \tau)$, which is defined by

$$(f * \varphi)(g) := \int_G \varphi(x^{-1}g) f(x)\, dx = \int_G \varphi(xg) f(x^{-1})\, dx, \quad g \in G. \tag{3.12}$$

By considering the corresponding Fourier transform, we obtain the following result, which is analogous as Lem. 1.4. in ([Hel89], Chap. 3).

Proposition 4. *With the notations above, we then have that*

$$\mathcal{F}_\tau(f * \varphi)(\lambda, k) = {}_\gamma\mathcal{F}_\tau\varphi(\lambda)\mathcal{F}_\gamma f(\lambda, k), \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}, k \in K.$$

Proof. For $(\lambda, k) \in \mathfrak{a}_{\mathbb{C}}^* \times K$, we compute

$$\begin{aligned}
\mathcal{F}_\tau(f*\varphi)(\lambda,k) &\overset{(3.12)}{=} \int_{G\times G} e^\tau_{\lambda,k}(g)\varphi(\underbrace{x^{-1}g}_{=:h})f(x)\,dx\,dg \\
&\overset{\text{Fubini's thm.}}{=} \int_G \Big(\int_G e^\tau_{\lambda,k}(xh)\varphi(h)\,dh\Big) f(x)\,dx \\
&\overset{(3.6)}{=} \int_G \Big(\int_G e^\tau_{\lambda,\kappa(x^{-1}k)}(h)a(x^{-1}k)^{-(\lambda+\rho)}\varphi(h)\,dh\Big) f(x)\,dx \\
&\overset{(3.8)}{=} \int_G \Big(\int_G e^\tau_{\lambda,1}(\underbrace{\kappa(x^{-1}k)^{-1}h}_{=:g})\varphi(h)\,dh\Big) a(x^{-1}k)^{-(\lambda+\rho)} f(x)\,dx \\
&= \int_G \Big(\int_G e^\tau_{\lambda,1}(g)\varphi(\kappa(g^{-1}k)g)\,dg\Big) a(x^{-1}k)^{-(\lambda+\rho)} f(x)\,dx \\
&= \int_G \Big(\int_G e^\tau_{\lambda,1}(g)\varphi(g)\,dg\Big)\gamma(\kappa(x^{-1}k))^{-1} a(x^{-1}k)^{-(\lambda+\rho)} f(x)\,dx \\
&= {}_\gamma\mathcal{F}_\tau\varphi(\lambda)\mathcal{F}_\gamma f(\lambda,k).
\end{aligned}$$

□

Remark 2. (a) If $\gamma = \tau$, then we have $\mathcal{F}_\tau(f*\varphi)(\lambda,k) = {}_\tau\mathcal{F}_\tau\varphi(\lambda)\mathcal{F}_\tau f(\lambda,k)$, for $f \in C_c^\infty(X,\mathbb{E}_\tau)$ and a spherical function $\varphi \in C_c^\infty(G,\tau,\tau)$.

(b) In a smiliar way, one can define the left convolution for scalar valued-function $\varphi \in C_c^\infty(G)$. In fact, we know that, for $f \in C_c^\infty(X,\mathbb{E}_\tau)$ and $g \in G$, we have

$$\begin{aligned}
\mathcal{F}_\tau(l_g f)(\lambda,k) = \int_G e^\tau_{\lambda,k}(x) l_g f(x)\,dx &= \int_G e^\tau_{\lambda,k}(gh)f(h)\,dh \\
&\overset{(3.9)}{=} a(g^{-1}k)^{-(\lambda+\rho)}\int_G e^\tau_{\lambda,\kappa(g^{-1}k)}(h)f(h)\,dh \\
&\overset{(2.2)}{=} (\pi_{\tau,\lambda}(g)\mathcal{F}_\tau f(\lambda,\cdot))(k).
\end{aligned}$$

Hence, we can deduce for $\varphi \in C_c^\infty(G)$:

$$\mathcal{F}_\tau(\varphi * f)(\lambda,k) = (\pi_{\tau,\lambda}(\varphi)\mathcal{F}_\tau f(\lambda,\cdot))(k). \tag{3.13}$$

(c) Analogously as for smooth compactly functions (3.12), we define the convolution for distributions $T \in C_c^{-\infty}(X,\mathbb{E}_\tau)$ by

$$(T*\varphi)(g) := T(l_g\varphi^\vee) = \langle T, l_g\varphi^\vee\rangle, \quad g \in G, \varphi \in C_c^\infty(G,\tau,\tau),$$

where $\varphi^\vee \in C_c^\infty(X,\mathbb{E}_{\tilde\tau}) \otimes E_\tau$ is given by $\varphi^\vee(g) := \varphi(g^{-1}), g \in G$. Then, the obtained results can be applied for distributions as well.

Now, for positive $\epsilon > 0$, take a K-conjugation invariant open neighbourhood $U_\epsilon \subset B_\epsilon(0)$ so that $\bigcap_{\epsilon>0} U_\epsilon = \{0\}$, and for $\epsilon_1 < \epsilon_2$, we have $U_{\epsilon_1} \subset U_{\epsilon_2}$. Consider a scalar-valued positive function $\tilde\eta_\epsilon \in C_c^\infty(U_\epsilon) \subset C_c^\infty(G)$ in G satisfying

$$\int_{U_\epsilon} \tilde\eta_\epsilon(g)\,dg = 1. \tag{3.14}$$

Note that $\tilde\eta_\epsilon$ cannot be $K\times K$-invariant. Let us construct from this an endomorphism function $\eta_\epsilon \in C^\infty(G,\tau,\tau)$ by

$$\eta_\epsilon(g) := \int_{K\times K} \tilde\eta_\epsilon(k_1 g k_2)\tau(k_1k_2)\,dk_1\,dk_2, \quad g \in G. \tag{3.15}$$

Then, we get the following observation.

Corollary 1. *For each $\epsilon > 0$, let $\eta_\epsilon \in C_c^\infty(G,\tau,\tau)$ be the $K\times K$-invariant endomorphism function (3.15). Then, its Fourier transform ${}_\tau\mathcal{F}_\tau\eta_\epsilon$ converges uniformly on compact sets C on $\mathfrak{a}_{\mathbb{C}}^*$ to the identity map:*

$${}_\tau\mathcal{F}_\tau\eta_\epsilon(\lambda) \to \mathrm{Id}, \quad \lambda \in C$$

when $\epsilon \to 0$.

Proof. Consider $\eta_\epsilon \in C^\infty(G,\tau,\tau)$, then for $g \in G$:

$$\eta_\epsilon(g) = \int_K \int_K \tilde{\eta}_\epsilon(k_1 g k_2)\tau(k_1 k_2)\, dk_1\, dk_2 = \int_K \int_K \tilde{\eta}_\epsilon(k_1 g l k_1^{-1})\tau(l)\, dk_1\, dl = \int_K \overline{\eta}_\epsilon(gl)\tau(l)\, dl,$$

where we did a change of variable and set $\overline{\eta}_\epsilon(g) := \int_K \tilde{\eta}_\epsilon(k_1 g k_1^{-1})\, dk_1$. Here, $\tilde{\eta}_\epsilon \in C_c^\infty(U_\epsilon)$ as above (3.14). By computing its Fourier transform, we obtain, for $\lambda \in \mathfrak{a}_\mathbb{C}^*$

$$\begin{aligned} {}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\lambda) \overset{(3.11)}{=} \int_G e^\tau_{\lambda,1}(g)\eta_\epsilon(g)\, dg &= \int_G \Big(\int_K e^\tau_{\lambda,1}(g)\overline{\eta}_\epsilon(gl)\tau(l)\, dl \Big)\, dg \\ &\overset{(3.18)}{=} \int_G \Big(\int_K e^\tau_{\lambda,1}(gl)\overline{\eta}_\epsilon(gl)\, dl \Big)\, dg \\ &= \int_G e^\tau_{\lambda,1}(g)\overline{\eta}_\epsilon(g)\, dg \\ &= \int_{U_\epsilon} \overline{\eta}_\epsilon(g)(e^\tau_{\lambda,1}(g) - \mathrm{Id})dg + \mathrm{Id}. \end{aligned}$$

Now, consider a compact set C on $\mathfrak{a}_\mathbb{C}^*$ and $\delta > 0$, then there exists $\epsilon > 0$ such that

$$|e^\tau_{\lambda,1}(g) - \mathrm{Id}| < \delta \text{ for } g \in U_\epsilon, \lambda \in C.$$

Thus, this implies that ${}_\tau\mathcal{F}_\tau\eta_\epsilon$ converges uniformly on compact sets to Id, when ϵ converges to 0. □

Furthermore, consider an non-zero linear G-invariant differential operator between sections over homogeneous vector bundles

$$D : C^\infty(X, \mathbb{E}_\tau) \longrightarrow C^\infty(X, \mathbb{E}_\gamma) \tag{3.16}$$

such that $D(g \cdot f) = g \cdot (Df)$, for all $g \in G, f \in C^\infty(X, \mathbb{E}_\tau)$. Denote by $\mathcal{D}_G(\mathbb{E}_\tau, \mathbb{E}_\gamma)$ the vector space of all these G-invariant differential operators on sections. We get the following relation.

Proposition 5. *Let $Q \in \mathcal{D}_G(\mathbb{E}_{\tilde{\tau}}, \mathbb{E}_{\tilde{\gamma}})$ be an invariant linear differential operator. Then, we have*

$$Qe^\tau_{\lambda,k} = (Qe^\tau_{\lambda,1}(1)) \circ e^\gamma_{\lambda,k}, \quad \lambda \in \mathfrak{a}_\mathbb{C}^*, k \in K. \tag{3.17}$$

Proof. Let us first consider the case $k = 1$. We then have for $g \in G = NAK$:

$$e^\tau_{\lambda,1}(g) = e^\tau_{\lambda,1}(nak_1) = a^{\lambda+\rho}\tau(k_1) = a^{\lambda+\rho}e^\tau_{\lambda,1}(k_1), \quad n \in N, a \in A, k_1 \in K. \tag{3.18}$$

In particular, for $n_1 a_1 \in NA$

$$\begin{aligned} l_{(n_1a_1)^{-1}}e^\tau_{\lambda,1}(nak_1) = e^\tau_{\lambda,1}(n_1a_1nak_1) = e^\tau_{\lambda,1}(n_1(a_1na_1^{-1})a_1ak_1) &\overset{(3.18)}{=} a_1^{\lambda+\rho}a^{\lambda+\rho}\tau(k_1) \\ &= a_1^{\lambda+\rho}e^\tau_{\lambda,1}(g). \end{aligned}$$

Hence, since Q is linear and G-invariant, we obtain that

$$l_{(n_1a_1)^{-1}}(Qe^\tau_{\lambda,1}(g)) = Q(l_{(n_1a_1)^{-1}}e^\tau_{\lambda,1}(g)) = Q(a_1^{\lambda+\rho}e^\tau_{\lambda,1}(g)) = a_1^{\lambda+\rho}Q(e^\tau_{\lambda,1}(g)) \tag{3.19}$$

and by setting $g = k_1 = 1$, we have

$$Qe^\tau_{\lambda,1}(n_1a_1) \overset{(3.19)}{=} a_1^{\lambda+\rho}Qe^\tau_{\lambda,1}(1). \tag{3.20}$$

Therefore, since $e^\tau_{\lambda,1} \in C^\infty(X, \mathbb{E}_{\tilde{\tau}}) \otimes E_\tau \subset C^\infty(G, \mathrm{End}(E_\tau))$, we have that $Qe^\tau_{\lambda,1} \in C^\infty(X, \mathbb{E}_{\tilde{\gamma}}) \otimes E_\tau \subset C^\infty(G, \mathrm{Hom}(E_\gamma, E_\tau))$. Therefore, for $g = n_1a_1k_2 \in G$, we can conclude that

$$Qe^\tau_{\lambda,1}(n_1a_1k_2) = Qe^\tau_{\lambda,1}(n_1a_1)\gamma(k_2) \overset{(3.20)}{=} a_1^{\lambda+\rho}(Qe^\tau_{\lambda,1}(1))\gamma(k_2) \overset{(3.18)}{=} (Qe^\tau_{\lambda,1}(1))e^\gamma_{\lambda,1}(n_1a_1k_2). \tag{3.21}$$

Now for general $k \in K$, we observe that $e^\tau_{\lambda,k} = l_k e^\tau_{\lambda,1}$. Hence

$$Qe^\tau_{\lambda,k} = Q(l_k e^\tau_{\lambda,1}) \overset{(3.21)}{=} l_k(Qe^\tau_{\lambda,1}(1))e^\gamma_{\lambda,1} = (Qe^\tau_{\lambda,1}(1)) \circ e^\gamma_{\lambda,k}.$$

Thus, we get the desired result. □

4 Delorme's intertwining conditions and some examples

We study Delorme's intertwining conditions $(D.a)$ in Def. 3 and determine the intertwining conditions in (Level 2) and (Level 3) induced by them. To do this, we firstly need some preparations. In the previous Section 3, we have seen that the identification (3.5). Let us now take a closer look. Consider the Frobenius-reciprocity (3.2) with its dual (3.3) and define the map

$$I : \bigoplus_{\sigma\in\hat{M}} H_\infty^\sigma \otimes \mathrm{Hom}_K(H_\infty^\sigma, E_\tau) \longrightarrow H_\infty^{\tau|_M}$$

by $I(\alpha) = d_\sigma \sum_{i=1}^{m(\tau,\sigma)} s_i\alpha_i$, for $\alpha = \sum_{i=1}^{m(\tau,\sigma)} \alpha_i \otimes S_i \in H_\infty^\sigma \otimes \mathrm{Hom}_K(H_\infty^\sigma, E_\tau)$, where $s_i = Frob(S_i)$ runs a basis through $\mathrm{Hom}_M(E_\sigma, E_\tau)$, for all i. Here, $m(\tau,\sigma)$ stands for the dimension of the multiplicity space $\mathrm{Hom}_K(H_\infty^\sigma, E_\tau)$. For $T \in \mathrm{Hom}_K(E_\tau, H_\infty^\sigma)$, let

$$\langle \alpha, T\rangle := \sum_{i=1}^{m(\tau,\sigma)} \alpha_i \cdot \mathrm{Tr}_\tau(S_i \circ T).$$

Now, by using the identification $[\mathrm{End}(H_\infty^\sigma)\otimes E_\tau]^K \overset{j}{\cong} H_\infty^\sigma \otimes \mathrm{Hom}_K(H_\infty^\sigma, E_\tau)$, we can define the map

$$J : \bigoplus_{\sigma\in\hat{M}} [\mathrm{End}(H_\infty^\sigma)\otimes E_\tau]^K \longrightarrow H_\infty^{\tau|_M} \tag{4.1}$$

by $J = I\circ j$. In addition, for $\beta = \sum_{i=1}^{d_\tau} \beta_i \otimes v_i \in \bigoplus_{\sigma\in\hat{M}} [\mathrm{End}(H_\infty^\sigma)\otimes E_\tau]^K$ and $T \in \mathrm{Hom}_K(E_\tau, H_\infty^\sigma)$, let

$$\langle \beta, T\rangle := \sum_{i=1}^{d_\tau} \beta_i \circ T(v_i) \in H_\infty^\sigma, \tag{4.2}$$

where $\{v_i, i = 1,\dots,d_\tau\}$ runs a vector basis of E_τ. One checks that $\langle\beta,T\rangle = \langle j(\beta), T\rangle$.

Proposition 6. *With the previous notations, let $f := \sum_i^{d_\tau} f_i \otimes v_i \in C_c^\infty(X, \mathbb{E}_\tau)$. Denote by $\mathcal{F}_\tau(f)$ its Fourier transform in $H_\infty^{\tau|_M}$ given in (3.7).*
Then, for $T \in \mathrm{Hom}_K(E_\tau, H_\infty^\sigma)$ and $t = \widetilde{Frob}^{-1}(T) \in \mathrm{Hom}_M(E_\tau, E_\sigma)$, we obtain

(1) $\langle\alpha, T\rangle = t\circ I(\alpha)$,

(2) $\langle \mathcal{F}_{\sigma,\lambda}(f), T\rangle = t\circ \mathcal{F}_\tau f(\lambda,\cdot) \in H_\infty^{\sigma,\lambda}$, for $\lambda\in\mathfrak{a}_\mathbb{C}^*$,

(3) $\mathcal{F}_\tau f(\lambda,\cdot) = J(\bigoplus_{\sigma\in\widehat{M}} \mathcal{F}_{\sigma,\lambda}(f))$, for $\lambda\in\mathfrak{a}_\mathbb{C}^*$.

Proof. (1) It is sufficient to prove it for only one summand in α, hence let $\alpha = \alpha_1\otimes S$. For $T = \widetilde{Frob}(t) \in \mathrm{Hom}_K(E_\tau, H_\infty^\sigma)$ and $S = Frob(s) \in \mathrm{Hom}_K(H_\infty^\sigma, E_\tau)$, we thus obtain

$$\begin{aligned}
\langle\alpha, T\rangle = \alpha_1 \mathrm{Tr}_\tau(S\circ T) &\overset{\text{Lem. 1+(3.4)}}{=} \alpha_1\mathrm{Tr}_\tau\Big(v\mapsto \int_K \tau(k)s\circ t(\tau(k^{-1}))v\,dk\Big),\ v\in E_\tau \\
&= \alpha_1\mathrm{Tr}_\tau\Big(\int_K \tau(k)s\circ t\tau(k^{-1})\,dk\Big) \\
&= \alpha_1\mathrm{Tr}_\tau(s\circ t) \\
&= \mathrm{Tr}_\sigma(t\circ s)\alpha_1.
\end{aligned}$$

Since $\sigma\in\widehat{M}$ is irreducible and $t\circ s\in\mathrm{End}_M(E_\sigma)$, by Schur's lemma, we have that $t\circ s = \lambda\cdot\mathrm{Id}$, for some $\lambda\in\mathbb{C}$ and thus $\mathrm{Tr}_\sigma(t\circ s) = \lambda$. Hence $\langle\alpha,T\rangle = (t\circ s)(\alpha_1) = t(I(\alpha))$.

(2) By computation, we obtain

$$\begin{aligned}
\langle\mathcal{F}_{\sigma,\lambda}(f), T\rangle = \sum_{i=1}^{d_\tau} \mathcal{F}_{\sigma,\lambda}(f_i)\circ T(v_i) &\overset{(3.4)}{=} \sum_{i=1}^{d_\tau} \mathcal{F}_{\sigma,\lambda}(f_i)(t\tau(\cdot)(v_i)) \\
&\overset{\text{Def. 1}}{=} \sum_{i=1}^{d_\tau} \int_G f_i(g)\pi_{\sigma,\lambda}(g)(t\tau(\cdot)(v_i))\,dg \\
&= \sum_{i=1}^{d_\tau} \int_G f_i(g)(\pi_{\sigma,\lambda}(g)\varphi_i)(\cdot)\,dg.
\end{aligned}$$

In the last line, we set $\varphi_i(k) := t\tau(k^{-1})(v_i)$, for $k \in K$. Fix $k \in K$, by applying (2.2), we have $(\pi_{\sigma,\lambda}(g)\varphi_i)(k) = a(g^{-1}k)^{-(\lambda+\rho)}\varphi(\kappa(g^{-1}k))^{-1}$.
Thus,

$$\begin{aligned}\sum_{i=1}^{d_\tau}\int_G f_i(g)t\tau(\kappa(g^{-1}k))^{-1}a(g^{-1}k)^{-(\lambda+\rho)}v_i\,dg &= \sum_{i=1}^{d_\tau}\int_G f_i(g)te^{\tau}_{\lambda,k}(g)v_i\,dg\\ &= t\circ\int_G\sum_{i=1}^{d_\tau}e^{\tau}_{\lambda,k}(g)f_i(g)v_i\,dg\\ &= t\circ\int_G e^{\tau}_{\lambda,k}(g)f(g)\,dg = t\circ\mathcal{F}_\tau f(\lambda,k).\end{aligned}$$

(3) By rewritting (1) and (2) in the following way:
(1') $\mathrm{Tr}_\tau(I^{-1}(\alpha)\circ T) = t\circ\alpha$,
(2') $\mathrm{Tr}_\tau(\mathcal{F}_{\sigma,\lambda}(f)\circ T) = t\circ\mathcal{F}_\tau f(\lambda,\cdot)$,
we get that

$$\mathrm{Tr}_\tau(J^{-1}(\mathcal{F}_\tau f(\lambda,\cdot))\circ T) = \mathrm{Tr}_\tau(I^{-1}(\mathcal{F}_\tau f(\lambda,\cdot))\circ T) \overset{(1')}{=} t\circ\mathcal{F}_\tau f(\lambda,\cdot) \overset{(2')}{=} \mathrm{Tr}_\tau(\mathcal{F}_{\sigma,\lambda}(f)\circ T).$$

By taking only the σ-component of $\bigoplus_{\sigma\in\widehat{M}}[\mathrm{End}(H^\sigma_\infty)\otimes E_\tau]^K$, we have that the parining in Tr_σ is non-degenerate, thus $J^{-1}(\mathcal{F}_\tau f(\lambda,\cdot)) = \bigoplus_{\sigma\in\widehat{M}}\mathcal{F}_{\sigma,\lambda}(f)$. □

We first study what happens to Delorme's intertwining condition $(D.a)$ if we tensor it with E_τ and take K-invariants.

Definition 7. *Consider $\tau\in\widehat{K}$.*

(1) We say that a function

$$\phi\in\prod_{\sigma\in\widehat{M}}[\mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}},\mathrm{End}(H^\sigma_\infty))\otimes E_\tau]^K \cong \bigoplus_{\sigma\subset\tau|_M}\mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}},[\mathrm{End}(H^\sigma_\infty)\otimes E_\tau]^K)$$

satisfies the intertwining condition, if for each $\tilde{v}\in E_{\tilde{\tau}}$:

$$\langle\phi,\tilde{v}\rangle_\tau\in\prod_{\sigma\in\widehat{M}}\mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}},\mathrm{End}(H^\sigma_\infty))$$

satisfies the intertwining condition in Def. 3.

Proposition 7. *Let $\phi\in\prod_{\sigma\in\widehat{M}}[\mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}},\mathrm{End}(H^\sigma_\infty))\otimes E_\tau]^K$ as in Def. 7 and (ξ,W) the intertwining data defined in Def. 3.*

$(D.1)$ *Then, ϕ satisfies the intertwining condition (1) of Def. 7 if, and only if, for each intertwining datum (ξ,W) and $T\in\mathrm{Hom}_K(E_\tau,W)\subset\mathrm{Hom}_K(E_\tau,H_\xi)$, the induced element $\phi_\xi\in[\mathrm{End}(H_\xi)\otimes E_\tau]^K$ satisfies*

$$\langle\phi_\xi,T\rangle\in W.$$

Proof. For each $i\in\{1,\dots,d_\tau\}$, consider $f_i\in\mathrm{End}(H_\xi)$ so that for each intertwining datum (ξ,W), we have $f_i(W)\subseteq W$. Consider

$$\phi_\xi = \sum_{i=1}^{d_\tau}f_i\otimes v_i\in[\mathrm{End}(H_\xi)\otimes E_\tau]^K$$

as in Thm. 2. It is sufficient to show that for each i and $T\in\mathrm{Hom}_K(E_\tau,W)$, we have $f_i\circ T\in W$ if, and only if, $\langle\phi_\xi,T\rangle\in W$, $\forall T\in\mathrm{Hom}_K(E_\tau,W)$.

The right implication is obvious. By using the definition of the brackets $\langle\cdot,\cdot\rangle$ as in (4.2), we have

$$\langle\phi_\xi,T\rangle = \sum_{i=1}^{d_\tau}f_i\circ T(v_i)\in W$$

since for $v_i\in E_\tau$, $T(v_i)\in W\subset H_\xi$.

For the left implication, write $f_i = \langle \phi_\xi, \tilde{v}_i \rangle_\tau$, for all $i \in \{1, \dots, d_\tau\}$, where $\tilde{v}_i$ runs a dual basis of $\mathbb{E}_{\tilde{\tau}}$. Consider the mapping $A_{ij} \in \mathrm{End}(E_\tau)$ such that $v_i \mapsto v_j$ and $v_k \mapsto 0$, $k \neq i$. Then, for all $i, j \in \{1, \dots, d_\tau\}$, we have

$$f_i \circ T(v_j) = \langle \phi_\xi, T(v_j) \cdot \tilde{v}_i \rangle = \langle \phi_\xi, T \circ A_{ij} \rangle = \langle \phi_\xi, p_K(T \circ A_{ij}) \rangle,$$

where $p_K : \mathrm{Hom}(E_\tau, W) \to \mathrm{Hom}_K(E_\tau, W)$ is the orthogonal projection. Note that $T(v_j)\cdot\tilde{v}_i \in \mathrm{Hom}(E_\tau, W)$, for all i, j. By setting, now in the last line $T'_{ij} := p_K(T \circ A_{ij})$, we get that $\langle \phi_\xi, T'_{ij} \rangle \in W$. Thus, for all $i \in \{1, \dots, d_\tau\}$, we have $f_i \circ T \in W$. □

Next, we state the intertwining condition in (Level 2) and (Level 3) induced from Delorme's intertwining condition (D.a), more presciely (1) in Def. 7.

Definition 8 (Intertwining conditions in (Level 2) and (Level 3))**.** *Let $\tau, \gamma \in \hat{K}$ and consider the map J defined in (4.1).*

(2) We say that a function $\psi \in \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, H^{\tau|_M}_\infty)$ satisfies the intertwining condition, if

$$J^{-1}\psi \in \bigoplus_{\sigma \subset \tau|_M} \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, [\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K)$$

satisfies the intertwining condition (1) *in Def. 7.*

(3) We say that a function $\varphi \in \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{Hom}_M(E_\gamma, E_\tau))$ satisfies the intertwining condition, if for all $w \in E_\gamma$:

$$(\lambda, k) \mapsto \varphi(w)(\lambda, k) := \varphi(\lambda)\gamma(k^{-1})w \in \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, H^{\tau|_M}_\infty)$$

satisfies the above intertwining condition (2)*.*

We now want to make the intertwining conditions more explicit. Let us first introduce some notations. We define

$$\mathrm{Hom}_M(E_\tau, E_\sigma)^\lambda_{(m)} := \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{Hom}_M(E_\tau, E_\sigma))/m^{m+1}_\lambda \, \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{Hom}_M(E_\tau, E_\sigma))$$

as in (2.3), similarly for $H^{\tau|_M,\lambda}_{\infty,(m)}$. For $\tau \in \widehat{K}$ and each intertwining datum (ξ, W), consider

$$\begin{aligned} D^\tau_W &:= \{t \in \bigoplus_{i=1}^s \mathrm{Hom}_M(E_\tau, E_{\sigma_i})^{\lambda_i}_{(m_i)} \mid T = \widetilde{Frob}^{-1}(t) \in \mathrm{Hom}_K(E_\tau, W) \subset \mathrm{Hom}_K(E_\tau, H_\xi)\} \\ &\subset \bigoplus_{i=1}^s \mathrm{Hom}_M(E_\tau, E_{\sigma_i})^{\lambda_i}_{(m_i)}. \end{aligned} \tag{4.3}$$

Write by $\overline{\Xi}$ the set of all 2-tuples (λ, m) with $\lambda \in \mathfrak{a}^*_\mathbb{C}$ and $m \in \mathbb{N}_0$ and we define the map

$$\Xi \longrightarrow \overline{\Xi}, \quad \xi = (\sigma, \lambda, m) \mapsto \overline{\xi} = (\lambda, m).$$

For $s \in \mathbb{N}$ and $\xi \in \Xi^s$, we have the corresponding element $\overline{\xi} \in \overline{\Xi}^s$.

Theorem 2 (Intertwining conditions in the three levels)**.** *With the notations above, we then have:*

(D.2) (Level 2) Then, $\psi \in \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, H^{\tau|_M}_\infty)$ satisfies the intertwining condition (2) of Def. 8 if, and only if, for each intertwining datum (ξ, W) and each non-zero $t = (t_1, t_2, \dots, t_s) \in D^\tau_W$, the induced element $\psi_{\overline{\xi}} \in \bigoplus_{i=1}^s H^{\tau|_M,\lambda_i}_{\infty,(m_i)} =: H^{\tau|_M}_{\overline{\xi}}$ satisfies

$$t \circ \psi_{\overline{\xi}} = (t_1 \circ \psi_1, \dots, t_2 \circ \psi_s) \in W.$$

(D.3) (Level 3) Then, $\varphi \in \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{Hom}_M(E_\gamma, E_\tau))$ satisfies the intertwining condition (3) of Def. 8 if, and only if, for each intertwining datum (ξ, W) and each non-zero $t = (t_1, t_2, \dots, t_s) \in D^\tau_W$, the induced element $\varphi_{\overline{\xi}} \in \bigoplus_{i=1}^s \mathrm{Hom}_M(E_\gamma, E_\tau)^{\lambda_i}_{(m_i)} =: H^{\gamma,\tau}_{\overline{\xi}}$ satisfies

$$t \circ \varphi_{\overline{\xi}} = (t_1 \circ \varphi_1, \dots, t_2 \circ \varphi_s) \in D^\gamma_W.$$

Proof. We obtain directly the equivalence between $(D.1)$ for $J^{-1}\psi$ and $(D.2)$ for ψ by applying the Frobenius reciprocity, Prop. 7 and Prop. 6 (2).

Concerning $(D.2)$ for $\varphi(w) \iff (D.3)$ for φ, one implication is trivial. For the other one, we have, by the inverse dual Frobenius reciprocity, that

$$W \ni t \circ \psi_{\overline{\xi}} = t \circ \widetilde{Frob}^{-1}(\varphi_{\overline{\xi}})(w)(k) \overset{(3.3)}{=} t \circ \varphi_{\overline{\xi}} \circ \gamma(k^{-1})w, \ \ \forall t \in D_W^\tau,$$

for $w \in E_\gamma$ and $k \in K$. This means that $\widetilde{Frob}^{-1}(t \circ \varphi_{\overline{\xi}})(w) \in \mathrm{Hom}_K(E_\gamma, W)$ and hence by applying the dual Frobenius-reciprocity $\mathrm{Hom}_K(E_\gamma, W) \overset{\widetilde{Frob}}{\cong} D_W^\gamma$, this implies that $t \circ \varphi_{\overline{\xi}} \in D_W^\gamma$. □

Example 1. (a) Consider $s = 1$ and $m = 0$. Let $\xi := (\sigma, \lambda, 0) \in \Xi$ and $W \subset H_\infty^{\sigma,\lambda}$. Consider $D_W^\tau \subset \mathrm{Hom}_M(E_\tau, E_\sigma)$ as in Thm. 2. Then, we have the following intertwining conditions in the corresponding levels:

(D.2a) (Level 2) For each intertwining datum (ξ, W) and $0 \neq t \in D_W^\tau$, we have

$$t \circ \psi(\lambda, \cdot) \in W.$$

Note that for each $\overline{\xi} \in \overline{\Xi}$, the induced element $\psi_{\overline{\xi}} = \psi(\lambda, \cdot)$.

(D.3a) (Level 3) For each intertwining datum (ξ, W) and $0 \neq t \in D_W^\tau$, we have

$$t \circ \varphi(\lambda) \in D_W^\gamma.$$

Note that for each $\overline{\xi} \in \overline{\Xi}$, the induced element $\varphi_{\overline{\xi}} = \varphi(\lambda)$.

(b) Consider now $s = 2$ and $m_1 = m_2 = 0$. Let $L : H_\infty^{\sigma_1,\lambda_1} \longrightarrow H_\infty^{\sigma_2,\lambda_2}$ be an intertwining operator between the two principal series representations. Let $\xi := ((\sigma_1, \lambda_1, 0), (\sigma_2, \lambda_2, 0)) \in \Xi^2$ and $W = graph(L) \subset H_\infty^{\sigma_1,\lambda_1} \oplus H_\infty^{\sigma_2,\lambda_2}$. Moreover, define $l^\tau : \mathrm{Hom}_M(E_\tau, E_{\sigma_1}) \longrightarrow \mathrm{Hom}_M(E_\tau, E_{\sigma_2})$ by

$$l^\tau(t)(v) = L(t\tau(\cdot)^{-1}v)(e)$$

for $v \in E_\tau$ and $t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})$. Then

$$\begin{aligned} D_W^\tau = \{(t_1, t_2) \mid t_2 = l^\tau(t_1)\} &= \{(t, l^\tau(t)) \mid t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})\} \\ &\subset \mathrm{Hom}_M(E_\tau, E_{\sigma_1}) \oplus \mathrm{Hom}_M(E_\tau, E_{\sigma_2}). \end{aligned}$$

In this situation, we have the following intertwining conditions.

(D2.b) (Level 2) For each intertwining datum (ξ, W) and $t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})$, we have for $\psi(\lambda_i, \cdot) \in H_\infty^{\tau|_M, \cdot}, i = 1, 2$

$$L(t \circ \psi(\lambda_1, \cdot)) = l^\tau(t) \circ \psi(\lambda_2, \cdot). \tag{4.4}$$

(D3.b) (Level 3) For each intertwining datum (ξ, W) and $t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})$, we have for $\varphi(\lambda_i) \in \mathrm{Hom}_M(E_\gamma, E_\tau), i = 1, 2$

$$l^\gamma(t \circ \varphi(\lambda_1)) = l^\tau(t) \circ \varphi(\lambda_2). \tag{4.5}$$

5 Topological Paley-Wiener theorem for sections

The Paley-Wiener space for sections over homogeneous vector bundles is defined as follows.

Definition 9 (Paley-Wiener space for sections in (Level 2) and (Level 3))**.**

(a) For $r > 0$, let $PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$ be the space of sections $\psi \in C^\infty(\mathfrak{a}_\mathbb{C}^* \times K/M, \mathbb{E}_{\tau|_M})$ be such that

(2.i) *the section ψ is holomorphic in $\lambda \in \mathfrak{a}_\mathbb{C}^*$, i.e. $\psi \in \mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, H_\infty^{\tau|_M})$.*

(2.ii)$_r$ *(growth condition) for all $Y \in \mathcal{U}(\mathfrak{k})$ and $N \in \mathbb{N}_0$, there exists a constant $C_{r,N,Y} > 0$ such that*

$$||l_Y \psi(\lambda, k)||_{E_\tau} \leq C_{r,N,Y}(1 + |\lambda|^2)^{-N} e^{r|\mathrm{Re}(\lambda)|}, \quad k \in K,$$

where $||\cdot||_{E_\tau}$ denotes the norm on finite-dimensional vector space E_τ (for convenience, we often denotes it by $|\cdot|$).

(2.iii) *(intertwining condition) (D.2) from Thm. 2.*

(b) By considering an additional K-type, let ${}_\gamma PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^*)$ be the space of functions

$$\mathfrak{a}_\mathbb{C}^* \ni \lambda \mapsto \varphi(\lambda) \in \mathrm{Hom}_M(E_\gamma, E_\tau)$$

be such that

(3.i) *the function* φ *is holomorphic in* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

(3.ii)$_r$ *(growth condition) for all* $N \in \mathbb{N}_0$*, there exists a constant* $C_{r,N} > 0$ *such that*

$$||\varphi(\lambda)||_{op} \leq C_{r,N}(1+|\lambda|^2)^{-N} e^{r|\mathrm{Re}(\lambda)|},$$

where $||\cdot||_{op}$ *denotes the operator norm on* $\mathrm{Hom}_M(E_\gamma, E_\tau)$.

(3.iii) *(intertwining condition) (D.3) from Thm. 2.*

The inequalities provide semi-norms $||\cdot||_{r,N,Y}$ (resp. $||\cdot||_{r,N}$) on $PW_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ (resp. ${}_\gamma PW_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^*)$) and made the vector space $PW_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ (resp. ${}_\gamma PW_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^*)$) to Fréchet space, e.g. one can compare Lem. 10 of Delorme [Del05].

Combining Delorme's Paley-Wiener Thm. 1 with the above identifications and observations, we obtain a Paley-Wiener theorem in (Level 2) and (Level 3).

Theorem 3 (Topological Paley-Wiener theorem for sections in (Level 2) and (Level 3))**.** *Let* (τ, E_τ) *be a* K*-representation with associated homogeneous vector bundle* $\mathbb{E}_\tau$*. For* $r > 0$*, then the Fourier transform*

$$C_r^\infty(X, \mathbb{E}_\tau) \ni \psi \mapsto \mathcal{F}_\tau(\psi)(\lambda, k) \in PW_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^* \times K/M), \quad (\lambda, k) \in \mathfrak{a}_{\mathbb{C}}^* \times K$$

is a topological isomorphism between $C_r^\infty(X, \mathbb{E}_\tau)$ *and* $PW_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$.
Moreover, by considering an additional K*-representation* (γ, E_γ) *with associated homogeneous vector bundle* $\mathbb{E}_\gamma$*, then the Fourier transform*

$$C_r^\infty(G, \gamma, \tau) \ni \varphi \mapsto {}_\gamma\mathcal{F}_\tau(\varphi)(\lambda) \in {}_\gamma PW_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^*), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*$$

is a topological isomorphism between $C_r^\infty(G, \gamma, \tau)$ *and* ${}_\gamma PW_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^*)$. □

Furthermore, by taking the union of all $r > 0$, the Paley-Wiener space $PW_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ is defined as

$$PW_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M) := \bigcup_{r>0} PW_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$$

similar for ${}_\gamma PW_\tau(\mathfrak{a}_{\mathbb{C}}^*)$. Equip $PW_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ and ${}_\gamma PW_\tau(\mathfrak{a}_{\mathbb{C}}^*)$ with the inductive limit topology (compare the next Sect. 6). Hence, by the above result (Thm. 3), we also have a linear topological Fourier transform isomorphism from $C_c^\infty(X, \mathbb{E}_\tau)$ (resp. $C_c^\infty(G, \gamma, \tau)$) onto $PW_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ (resp. ${}_\gamma PW_\tau(\mathfrak{a}_{\mathbb{C}}^*)$).

6 On topological Paley-Wiener-Schwartz theorem for sections and its proof

Distributional sections and their corresponding topology

In (3.1), we already introduced the vector space $C_c^{-\infty}(X, \mathbb{E}_\tau)$ by taking the taking the topological linear dual of $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$. We provide $C_c^{-\infty}(X, \mathbb{E}_\tau)$ with the *strong dual topology.* Actually, we know that $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$ is a Fréchet space with semi-norm

$$||h||_{\Omega,Y} := \sup_{g\in\Omega} |l_Y h(g)|, \quad h \in C^\infty(X, \mathbb{E}_{\tilde{\tau}}), \tag{6.1}$$

where $Y \in \mathcal{U}(\mathfrak{g})$ and Ω is a compact subset of G. Furthermore, a subset $B \subset C^\infty(X, \mathbb{E}_{\tilde{\tau}})$ is called bounded, if for each compact $\Omega \subset G$ and $Y \in \mathcal{U}(\mathfrak{g})$ there exists a constant $C_{\Omega,Y} > 0$ such that $\sup_{\varphi\in B} ||\varphi||_{\Omega,Y} \leq C_{\Omega,Y}$. Shortly, every semi-norm is bounded on B.

The strong dual topology on $C_c^{-\infty}(X, \mathbb{E}_\tau)$ is a locally convex topology vector space given by the semi-norm system

$$p_B(T) := ||T||_B = \sup_{\varphi\in B} |T(\varphi)| = \sup_{\varphi\in B} |\langle T, \varphi\rangle|, \quad T \in C_c^{-\infty}(X, \mathbb{E}_\tau), \tag{6.2}$$

where B belongs to the family of all bounded subsets of $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$. Similarly, we equip $C^{-\infty}(X, \mathbb{E}_\tau) = (C_c^\infty(X, \mathbb{E}_{\tilde{\tau}}))'$ with the strong dual topology. As an immediate consequence of theses dualities, the topologies on $C_c^{-\infty}(X, \mathbb{E}_\tau)$ and $C^{-\infty}(X, \mathbb{E}_\tau)$ induce the same topology on the space of distributions supported in a fixed compact subset Ω of G ([vdBS06], Sect. 14). For example, one can take $\Omega = \overline{B}_r(o)$.

A subset $B' \subset C_c^{-\infty}(X, \mathbb{E}_\tau)$ is bounded in the strong dual topology, if for each bounded $B \subset C^\infty(X, \mathbb{E}_{\tilde{\tau}})$, we have

$$\sup_{T\in B'} p_B(T) = \sup_{T\in B', \varphi\in B} |T(\varphi)| < \infty. \tag{6.3}$$

Since, by Schaefer ([Sch71], Cor. 1.6, p. 127), we know that all such sets B' are equicontinuous, this means that there exist a continuous semi-norm p on $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$ and a constant $C > 0$ such that

$$B' \subset \{T \in C_c^{-\infty}(X, \mathbb{E}_\tau) \mid |T(\varphi)| \leq Cp(\varphi), \forall \varphi \in C^\infty(X, \mathbb{E}_{\tilde{\tau}})\}.$$

Let $Y_1, \dots, Y_n$ be a basis of $\mathfrak{g}$, then for a multi-index $\alpha \in \mathbb{N}_0^n$, we set $Y_\alpha := Y_1^{\alpha_1} \cdots Y_n^{\alpha_n} \in \mathcal{U}(\mathfrak{g})$. We may assume that the semi-norm p has the form

$$p(\varphi) = \sum_{|\alpha| \leq m} ||\varphi||_{\Omega,\alpha} \overset{(6.1)}{=} \sum_{|\alpha| \leq m} \sup_{g \in \Omega} |l_{Y_\alpha} \varphi(g)|, \qquad \varphi \in C^\infty(X, \mathbb{E}_{\tilde{\tau}}), \forall \alpha \tag{6.4}$$

for some $m \in \mathbb{N}_0$ and compact $\Omega \subset G$.

It is interesting to notice that $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$ is a reflexive Fréchet space, even a Montel space, that is, it is reflexive and a subset is bounded if, and only if, it is relatively compact ([Sch71], p. 147).
Thus, since $C_c^{-\infty}(X, \mathbb{E}_\tau)$ is the strong dual space of a Montel space $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$, we can deduce by Cor. 1 in ([Sch71], p.154) that $C_c^{-\infty}(X, \mathbb{E}_\tau)$ is a bornological space, that is a locally convex space on which each semi-norm p_B, which is bounded on bounded subsets, is continuous ([Sch71], Chap.2.8, p. 61).
This observation leads us to the following general result, which will play an imporant role in the proof of the Paley-Wiener-Schwartz theorem. For bornological spaces, bounded linear maps are continuous ([Sch71], Thm. 8.3., p. 62), hence, we obtain the following.

Lemma 2. *Let W be any locally convex topological vector space and consider a linear map*

$$A : C_c^{-\infty}(X, \mathbb{E}_\tau) \to W.$$

Then A is continuous if, and only if, $A(B')$ is bounded in W, for every bounded subset $B' \subset C_c^{-\infty}(X, \mathbb{E}_\tau)$. □

Let $Y_1, \dots, Y_k$ be a basis of $\mathcal{U}(\mathfrak{k})$, then for a multi-index $\alpha \in \mathbb{N}_0^k$, we have $Y_\alpha := Y_1^{\alpha_1} \cdots Y_k^{\alpha_k} \in \mathcal{U}(\mathfrak{k})$. Now we are in the position to define Paley-Wiener-Schwartz space for sections.

Definition 10 (Paley-Wiener-Schwartz space for sections in (Level 2) and (Level 3))**.**

(a) For $r > 0$, let $PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$ be the space of sections $\psi \in C^\infty(\mathfrak{a}_\mathbb{C}^* \times K/M, \mathbb{E}_{\tau|_M})$ be such that

(2.i) the section ψ is holomorphic in $\lambda \in \mathfrak{a}_\mathbb{C}^*$, i.e. $\psi \in \mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, H_\infty^{\tau|_M})$.

$(2.iis)_r$ (growth condition) for all multi-indices α, there exist $N \in \mathbb{N}_0$ and a positive constant $C_{r,N,\alpha}$ such that

$$||l_{Y_\alpha} \psi(\lambda, k)||_{E_\tau} \leq C_{r,N,\alpha}(1 + |\lambda|^2)^{N + \frac{|\alpha|}{2}} e^{r|\mathrm{Re}(\lambda)|}, \quad k \in K.$$

(2.iii) (intertwining condition) (D.2) from Thm. 2.

(b) By considering an additional K-type, let ${}_\gamma PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^*)$ be the space of functions

$$\mathfrak{a}_\mathbb{C}^* \ni \lambda \mapsto \varphi(\lambda) \in \mathrm{Hom}_M(E_\gamma, E_\tau)$$

be such that

(3.i) the function φ is holomorphic in $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

$(3.iis)_r$ (growth condition) there exist $N \in \mathbb{N}_0$ and a positive constant $C_{r,N}$ such that

$$||\varphi(\lambda)||_{op} \leq C_{r,N}(1 + |\lambda|^2)^N e^{r|\mathrm{Re}(\lambda)|}.$$

(3.iii) (intertwining condition) (D.3) from Thm. 2.

For all $r \geq 0$ and $N \in \mathbb{N}_0$, we consider

$$PWS_{\tau,r,N} := \{\psi \in PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M) \mid ||\psi||_{r,N,\alpha} < \infty, \forall \alpha\},$$

with semi-norms

$$||\psi||_{r,N,\alpha} := \sup_{\lambda \in \mathfrak{a}_\mathbb{C}^*,\, k \in K/M} (1 + |\lambda|^2)^{-(N + \frac{|\alpha|}{2})} e^{-r|\mathrm{Re}(\lambda)|} ||l_{Y_\alpha} \psi(\lambda, k)||_{E_\tau}, \quad \forall \alpha, k \in K.$$

This gives $PWS_{\tau,r,N}$ the structure of a Fréchet space. We set $PWS_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M) := \bigcup_{r \geq 0} \bigcup_{N \in \mathbb{N}_0} PWS_{\tau,r,N}(\mathfrak{a}_\mathbb{C}^* \times K/M)$ and equip it with the locally convex inductive limit topology. It is the finest locally convex topology on $PWS_\tau(\mathfrak{a}^*_\mathbb{C} \times K/M)$ such that all the embeddings $PWS_{\tau,r,N} \overset{i_{r,N}}{\hookrightarrow} PWS_\tau(\mathfrak{a}^*_\mathbb{C} \times K/M)$ are continuous. Furthermore, this topology is characterized by the following property. A linear map

$$A : PWS_\tau \to W,$$

where W is any locally convex space, is continuous if, and only if, all the maps

$$PWS_{\tau,r,N} \overset{i_{r,N}}{\hookrightarrow} PWS_\tau \overset{A}{\longrightarrow} W$$

are continuous, i.e., $A \circ i_{r,N}$ are continuous. The exactly same procedure, can be done for ${}_\gamma PWS_\tau(\mathfrak{a}^*_\mathbb{C})$. We are now in the position to state the main theorem.

Theorem 4 (Topological Paley-Wiener-Schwartz theorem for sections)**.**

(a) Let (τ, E_τ) be a K-representation with associated homogeneous vector bundle $\mathbb{E}_\tau$. Then, for each $r \geq 0$, the Fourier transform $\mathcal{F}_\tau$ is a linear bijection between the two spaces $C^{-\infty}_r(X, \mathbb{E}_\tau)$ and the Paley-Wiener-Schwartz space $PWS_{\tau,r}(\mathfrak{a}^*_\mathbb{C} \times K/M)$. Moreover, it is a linear topological isomorphism from $C^{-\infty}_c(X, \mathbb{E}_\tau)$ onto $PWS_\tau(\mathfrak{a}^*_\mathbb{C} \times K/M)$.

(b) Similarly, if we consider an additional K-representation (γ, E_γ) with associated homogeneous vector bundle $\mathbb{E}_\gamma$. Then, the Fourier transform ${}_\gamma\mathcal{F}_\tau$ is a linear bijection between the two spaces $C^{-\infty}_r(G, \gamma, \tau)$ and ${}_\gamma PWS_{\tau,r}(\mathfrak{a}^*_\mathbb{C})$, for each $r \geq 0$, and a linear topological isomorphism from $C^{-\infty}_c(G, \gamma, \tau)$ onto ${}_\gamma PWS_\tau(\mathfrak{a}^*_\mathbb{C})$.

Remark 3. Delorme proved in his paper ([Del05]), the Paley-Wiener theorem in (Level 1) for Hecke algebra

$$\mathcal{H}(G, K) := C^{-\infty}_{r=0}(G)_K \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} C^\infty(K)_K, \tag{6.5}$$

which consists of all $K \times K$-finite distributions on G supported by $K \subset G$.

Harish-Chandra inversion and Plancherel Theorem for sections

In order to prove Thm. 4, we need the Harish-Chandra Plancherel inversion formula for sections over homogeneous vector bundles.

Theorem 5 (Plancherel Theorem for sections, [Ca97], Thm. 3.4 & Thm. 4.3)**.** *Let $\mathcal{Q}$ be a complete set of representatives of association classes of cuspidal parabolic subgroups $Q = M_Q A_Q N_Q$ with $Q \supset P = MAN$ and $A_Q \subset A$. We have $\mathfrak{a}^* = \mathfrak{a}^*_Q \oplus \mathfrak{a}^*_{M_Q}$.*
Then, there exists a finite set $A^\tau_Q \subset \mathfrak{a}^*_{M_Q} \subset \mathfrak{a}^*$ and for $\nu \in A^\tau_Q$, there exists an analytic function of at most polynomial growth

$$\mu^Q_\nu : i\mathfrak{a}^*_Q \longrightarrow \mathrm{End}_M(E_\tau)$$

such that for each $f \in C^\infty_c(X, \mathbb{E}_\tau)$, we have

$$f(e) \;=\; \sum_{Q\in\mathcal{Q}} \sum_{\nu\in A^\tau_Q} \int_{i\mathfrak{a}^*_Q} \int_K \tau(k)\mu^Q_\nu(\lambda)\mathcal{F}_\tau(f)(\nu+\lambda, k)\, dk\, d\lambda. \square$$

Note that $A^\tau_P = \{0\}$.

Corollary 2. *With the notations above, let $f \in C^\infty_c(X, \mathbb{E}_\tau)$ and $\varphi \in C^\infty_c(X, \mathbb{E}_{\tilde\tau})$. Then*

$$\int_G \langle\varphi(g), f(g)\rangle_\tau\, dg = \sum_{Q\in\mathcal{Q}} \sum_{\nu\in A^\tau_Q} \int_{i\mathfrak{a}^*_Q} \int_K \langle\mathcal{F}_{\tilde\tau}(\varphi)(-\nu-\lambda, k), \mu^Q_\nu(\lambda)\mathcal{F}_\tau(f)(\nu+\lambda, k)\rangle_\tau\, dk\, d\lambda. \tag{6.6}$$

Proof. Let $\{\tilde v_i, i = 1, \ldots, d_\tau\}$ be a vector basis of $E_{\tilde\tau}$. We write $\varphi = \sum_{i=1}^{d_\tau} \varphi_i \cdot \tilde v_i$ with $\varphi_i \in C^\infty_c(G)$. For $h \in C^\infty_c(G)$, we set $h^\vee(g) := h(g^{-1})$. Then

$$\int_G \langle\varphi(g), f(g)\rangle\, dg = \sum_{i=1}^{d_\tau} \langle(\varphi_i^\vee * f)(e), \tilde v_i\rangle,$$

where we used the usual convolution defined in (3.12). Note that $h * f = l(h)f$, where l is the (left) regular representation of G on $C^\infty_c(X, \mathbb{E}_\tau)$. By the G-equivariance of the Fourier transform, we have by (3.13): $\mathcal{F}_\tau(h * f)(\lambda, k) = \pi_{\tau,\lambda}(h)(\mathcal{F}_\tau(f)(\lambda, \cdot))(k)$. By applying Thm. 5, we obtain for all $i \in \{1, \ldots, d_\tau\}$

$$\langle\tilde v_i, (\varphi_i^\vee * f)(e)\rangle = \sum_{Q,\nu} \int_{i\mathfrak{a}^*_Q} \int_K \langle\tilde v_i, \tau(k)\mu^Q_\nu(\lambda)\pi_{\tau,\nu+\lambda}(\varphi_i^\vee)(\mathcal{F}_\tau(f)(\nu+\lambda, \cdot))(k)\rangle\, dk\, d\lambda.$$

Using that μ_ν^Q commutes with $\pi_{\tau,\nu+\lambda}$ and that integration over K gives a G-equivariant pairing between $H_\infty^{\tau,\nu+\lambda}$ and $H_\infty^{\tilde{\tau},-(\nu+\lambda)}$, we obtain that the K-integral equals

$$\begin{aligned}\int_K \langle\tilde{\tau}(k^{-1})\tilde{v}_i,\quad \pi_{\tau,\nu+\lambda}(\varphi_i^\vee)\mu_\nu^Q(\lambda)(\mathcal{F}_\tau(f)(\nu+\lambda,\cdot))(k)\rangle\, dk \\ = \int_K \langle(\pi_{\tilde{\tau},-(\nu+\lambda)}(\varphi_i)\tilde{\tau}(\cdot)^{-1}\tilde{v}_i)(k),\ \ \mu_\nu^Q(\lambda)\mathcal{F}_\tau(f)(\nu+\lambda,k)\rangle\, dk.\end{aligned}$$

Now

$$\begin{aligned}(\pi_{\tilde{\tau},-(\nu+\lambda)}(\varphi_i)\tilde{\tau}(\cdot)^{-1}\tilde{v}_i)(k) &= \int_G \varphi_i(g)\tilde{\tau}(\kappa(g^{-1}k))^{-1}a(g^{-1}k)^{\nu+\lambda-\rho}\tilde{v}_i\, dg \\ &= \int_G \varphi_i(g)e^{\tilde{\tau}}_{-(\nu+\lambda),k}(g)\tilde{v}_i\, dg.\end{aligned}$$

The sum over all i equals to $\mathcal{F}_{\tilde{\tau}}(\varphi)(-(\nu+\lambda),k)$. Combining all the previous formulas, we obtain the corollary. □

Proof of the topological Paley-Wiener-Schwartz Thm. 4

For $r \geq 0$, let us frist provide the bijection between the vector spaces $C_r^{-\infty}(X,\mathbb{E}_\tau)$ and $PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^*\times K/M)$.

Proposition 8. *Consider a K-representation (τ, E_τ).*

(a) Let $T \in C_c^{-\infty}(X,\mathbb{E}_\tau)$ such that its Fourier transform $\mathcal{F}_\tau(T) = 0$, then $T = 0$.

(b) For $r \geq 0$ and $\tilde{T} \in PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$, there exists $T \in C_r^{-\infty}(X,\mathbb{E}_\tau)$ such that $\tilde{T} = \mathcal{F}_\tau(T)$.

(c) For $r \geq 0$, let $T \in C_r^{-\infty}(X,\mathbb{E}_\tau)$, then $\mathcal{F}_\tau(T) \in PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$.

Proof. For each $\epsilon > 0$, consider $\eta_\epsilon \in C^\infty(G,\tau,\tau)$ with compact support in the closed ball $\overline{B}_\epsilon(o)$ as in Cor. 1. Let $T \in C_c^{-\infty}(X,\mathbb{E}_\tau)$ be a distribution, then

$$T_\epsilon := T * \eta_\epsilon \in C_c^\infty(X,\mathbb{E}_\tau).$$

Moreover, by using the same arguments as in the proof of Cor. 1, we have that $T_\epsilon \overset{\epsilon\to 0}{\longrightarrow} T$ (weakly). Hence, by the Paley-Wiener Thm. 3, this implies that $\mathcal{F}_\tau(T_\epsilon) \in PW_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$. Note that $\mathcal{F}_\tau(T_\epsilon)$ is holomorphic on $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and it satisfies the conditions $(2.i)$ and $(2.ii)_r$ of Def. 9. Furthermore, by Prop. 4, we have

$$\mathcal{F}_\tau(T_\epsilon)(\lambda,k) = {}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\lambda)\mathcal{F}_\tau(T)(\lambda,k), \quad (\lambda,k) \in \mathfrak{a}_\mathbb{C}^* \times K/M. \tag{6.7}$$

Due to Cor. 1, ${}_\tau\mathcal{F}_\tau(\eta_\epsilon)$ converges uniformly on compact subsets of $\mathfrak{a}_\mathbb{C}^*$ to the identity map, whenever ϵ tends to 0. Hence, $\lim_{\epsilon\to 0}\mathcal{F}_\tau(T_\epsilon) = \mathcal{F}_\tau(T)$ uniformly on compact sets on $\mathfrak{a}_\mathbb{C}^*$.

(a) Now assume that $\mathcal{F}_\tau(T) = 0$. By (6.7), we have that $\mathcal{F}_\tau(T_\epsilon) = 0$. By applying the Paley-Wiener Thm. 3, this implies that $T_\epsilon = 0$. Hence, since $T_\epsilon \overset{\epsilon\to 0}{\longrightarrow} T$ weakly, we have that $T = 0$.

(b) Consider $\psi \in PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$. For each $\epsilon > 0$ and $h \in C_c^\infty(X,\mathbb{E}_{\tilde{\tau}})$, let T_ϵ be the functional given by

$$\begin{aligned}T_\epsilon(h) \ := \ \sum_{Q\in\mathcal{Q}}\sum_{\nu\in A_Q^\tau}\int_{i\mathfrak{a}_Q^*}\int_K \langle\mathcal{F}_{\tilde{\tau}}(h)(-\nu-\lambda,k)\,, \\ \mu_\nu^Q(\lambda)\,{}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\nu+\lambda)\psi(\nu+\lambda,k)\rangle\, dk\, d\lambda\end{aligned} \tag{6.8}$$

under the same notations introduced in Thm. 5. Notice that, since $\operatorname{supp}(\eta_\epsilon) \subset \overline{B}_\epsilon(o)$ and ψ satisfies the 'slow' growth condition $(2.iis)_r$ of Def. 10, for all $r \geq 0$, this implies that for each multi-index $\alpha \in \mathbb{N}_0$ and $N \in \mathbb{N}_0$, there exists a constant $C_{r,N,\alpha} > 0$ such that

$$|l_{Y_\alpha}\,{}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\lambda)\psi(\lambda,k)| \leq C_{r,N,\alpha}(1+|\lambda|^2)^{-N}e^{(r+\epsilon)|\mathrm{Re}(\lambda)|}, \quad (\lambda,k) \in \mathfrak{a}_\mathbb{C}^* \times K. \tag{6.9}$$

In addition, for each intertwining datum (ξ, W), the induced operator $({}_\tau\mathcal{F}_\tau(\eta_\epsilon)\psi)_{\overline{\xi}} = {}_\tau\mathcal{F}_\tau(\eta_\epsilon)_{\overline{\xi}}\psi_{\overline{\xi}} \in H_\infty^{\tau|_M}$ satisfies the intertwining condition $(3.iii)$ of Def. 9. In fact, for $t \in D_W^\tau$, we have $t \circ {}_\tau\mathcal{F}_\tau(\eta_\epsilon)_{\overline{\xi}} \in D_W^\tau$ and since $\psi \in PWS_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$, this implies that

$$(t \circ {}_\tau\mathcal{F}_\tau(\eta_\epsilon)_{\overline{\xi}}) \circ \psi_{\overline{\xi}} \in W.$$

Therefore, by the Paley-Wiener Thm. 3, we have that ${}_\tau\mathcal{F}_\tau(\eta_\epsilon)\psi$ is the Fourier transform of a unique function $f_\epsilon \in C_c^\infty(X,\mathbb{E}_\tau)$, i.e.,

$$\mathcal{F}_\tau(f_\epsilon) := {}_\tau\mathcal{F}_\tau(\eta_\epsilon)\psi.$$

On the other side, by (6.8) and Cor. 2, we have $T_\epsilon = f_\epsilon$. By (6.9), we have that $\mathrm{supp}(T_\epsilon) \subset \overline{B}_{r+\epsilon}(o)$. Thus, by Cor. 1, this implies that

$$T_\epsilon(h) \overset{\epsilon\to 0}{\longrightarrow} T(h) := \sum_{Q\in\mathcal{Q}}\sum_{\nu\in A_Q^\tau}\int_{i\mathfrak{a}_Q^*}\int_K \langle \mathcal{F}_{\tilde{\tau}}(h)(-\nu-\lambda,k),\ \mu_\nu^Q(\lambda)\psi(\nu+\lambda,k)\rangle\, dk\, d\lambda \tag{6.10}$$

and thus $\mathrm{supp}(T) \subset \overline{B}_r(o)$. Note that μ_ν^Q has at most polynomial growth, thus T is well-defined and continuous. Since T is compactly supported, we can set $h := e_{\lambda,k}^\tau$. In conclusion, we have found a distribution $T \in C_r^{-\infty}(X,\mathbb{E}_\tau)$ such that

$$\begin{aligned}\mathcal{F}_\tau(T)(\lambda,k) = T(e_{\lambda,k}^\tau) \overset{(6.10)}{=} \lim_{\epsilon\to 0} T_\epsilon(e_{\lambda,k}^\tau) = \lim_{\epsilon\to 0}\mathcal{F}_\tau(f_\epsilon)(\lambda,k) &= \lim_{\epsilon\to 0}{}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\lambda)\psi(\lambda,k)\\ &= \psi(\lambda,k).\end{aligned}$$

(c) Let us check that for $r \geq 0$, $\mathcal{F}_\tau(T) \in PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$. This means that we need to verify that the Fourier transform of $T \in C_r^{-\infty}(X,\mathbb{E}_\tau)$ satisfies the conditions $(2.i) - (2.iii)$ of Def. 9.
The condition $(2.i)$ is immediate. Concerning the intertwining condition $(2.iii)$, in order to show that for each intertwining datum (ξ, W) and $t \in D_W^\tau$, we have

$$t \circ (\mathcal{F}_\tau(T))_{\overline{\xi}} \in W \subseteq H_\xi,$$

we will use a similar convolution argument as above, except that now we are interested to the convolution on the left instead on the right. For each $\epsilon > 0$, let $\delta_\epsilon \in C_c^\infty(G)$ be a delta-sequence such that $\lim_{\epsilon\to 0}\delta_\epsilon = \delta_0$. Hence, $\lim_{\epsilon\to 0}\delta_\epsilon * T = T$, for $T \in C_r^{-\infty}(X,\mathbb{E}_\tau)$. Moreover, for all representations $(\pi_{\tau,\lambda}, H)$ with Fréchet space H and $v \in H$, we have $\pi_{\tau,\lambda}(\delta_\epsilon)v \overset{\epsilon\to 0}{\longrightarrow} v$. By taking the Fourier transform on $\delta_\epsilon * T \in C_r^\infty(X,\mathbb{E}_\tau)$, we first prove that for each intertwining datum (ξ, W) and $t \in D_W^\tau$:

$$\lim_{\epsilon\to 0}(t \circ \mathcal{F}_\tau(\delta_\epsilon * T)_{\overline{\xi}}) \in W.$$

In fact, we have

$$\begin{aligned} t\circ\mathcal{F}_\tau(\delta_\epsilon * T)_{\overline{\xi}} &\overset{\text{Remark 2}}{=} t\circ(\pi_{\tau,\cdot}(\delta_\epsilon)\mathcal{F}_\tau(T))_{\overline{\xi}}\\ &= (\ldots, t_i\circ(\pi_{\tau,\lambda_i}(\delta_\epsilon)\mathcal{F}_\tau(T)(\lambda_i,\cdot))_{(m_i)},\ldots)\\ &= (\ldots, t_i\circ\pi_{\xi_i}(\delta_\epsilon)\mathcal{F}_\tau(T)_{\overline{\xi}_i},\ldots)\\ &= (\ldots, \pi_{\xi_i}(\delta_\epsilon)(t_i\circ\mathcal{F}_\tau(T)_{\overline{\xi}_i}),\ldots)\\ &= \pi_\xi(\delta_\epsilon)(t\circ\mathcal{F}_\tau(T)_{\overline{\xi}}) \in W,\end{aligned}$$

where $(\pi_{\sigma_1,\lambda_1}^{(m_1)}(\delta_\epsilon),\ldots,\pi_{\sigma_s,\lambda_s}^{(m_s)}(\delta_\epsilon)) = \pi_\xi(\delta_\epsilon) \in W \subset H_\xi$. Hence, by taking $\epsilon \to 0$ and since W is closed, we obtain that $t \circ (\mathcal{F}_\tau(T))_{\overline{\xi}} \in W$.
It remains to check that $\mathcal{F}_\tau(T)$ statisfies the slow growth condition $(2.ii s)_r$. Fix $r \geq 0$. We need to show that for each multi-index α, there exist $N \in \mathbb{N}_0$ and a constant $C_{r,N,\alpha} > 0$ such that

$$|l_{Y_\alpha}\mathcal{F}_\tau(T)(\lambda,k)| \leq C_{r,N,\alpha}(1+|\lambda|^2)^{N+\frac{|\alpha|}{2}}e^{r|\mathrm{Re}(\lambda)|}.$$

Note that $l_{Y_\alpha}\mathcal{F}_\tau(T) = \mathcal{F}_\tau(l_{Y_\alpha}T)$. Let $T \in C_r^{-\infty}(X,\mathbb{E}_\tau)$ be a distribution of order $m \in \mathbb{N}_0$. Write $X_\beta \in \mathcal{U}(\mathfrak{n})$ and $H_\gamma \in \mathcal{U}(\mathfrak{a})$ for all multi-indices β, γ. Since $G/K \cong NA$ and $\mathcal{U}(\mathfrak{n}\oplus\mathfrak{a}) \cong \mathcal{U}(\mathfrak{n})\mathcal{U}(\mathfrak{a})$, then, there exists a constant $C > 0$ such that

$$|T(h)| \leq C\sum_{|\beta|+|\gamma|\leq m}\sup_{g\in\overline{B}_r(o)}|(l_{X_\beta}(l_{H_\gamma}h))(g)|, \quad \forall h \in C^\infty(X,\mathbb{E}_{\tilde{\tau}}). \tag{6.11}$$

Next, we want to apply it to $h = e_{\lambda,1}^\tau$. We observe that

$$l_{Y_\alpha}\mathcal{F}_\tau(T)(\lambda,k) = \mathcal{F}_\tau(l_{Y_\alpha}T)(\lambda,k) = l_{Y_\alpha}T(e_{\lambda,k}^\tau) \overset{(3.8)}{=} (l_{Y_\alpha}T)(l_k e_{\lambda,1}^\tau) = (l_{k^{-1}}l_{Y_\alpha}T)(h).$$

Thus, $l_{k^{-1}} l_{Y_\alpha} T$ is a distribution of order $m + |\alpha|$. Applying (6.11) to $(l_{k^{-1}} l_{Y_\alpha} T)(h)$ instead of $T(h)$, we obtain

$$\sup_{k\in K} |(l_{k^{-1}} l_{Y_\alpha} T)(h)| \leq C' \sum_{|\beta|+|\gamma|\leq m+|\alpha|} \sup_{g\in \overline{B}_r(o)} |(l_{X_\beta}(l_{H_\gamma} h))(g)|, \quad \forall h \in C^\infty(X, \mathbb{E}_{\tilde{\tau}}).$$

In fact, since K is compact and operates continuously on $C_c^{-\infty}(X, \mathbb{E}_\tau)$, the constant $C' > 0$ can be chosen to be independently of K. Moreover, h is annihilated by each l_{X_β} for $\beta \neq 0$ and it is an eigenfunction of each l_{H_γ} with eigenvalue a polynomial in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ of degree $\leq |\gamma|$, i.e.

$$|l_{Y_\alpha} \mathcal{F}_\tau(T)(\lambda, k)| = |(l_k l_{Y_\alpha} T)(e^\tau_{\lambda,1})| \leq C_{r,N,\alpha}(1 + |\lambda|^2)^{N+\frac{|\alpha|}{2}} e^{r|\mathrm{Re}(\lambda)|},$$

for $N \geq \frac{m}{2}$. This complete the proof. □

Consequently, by (6.10), the *inverse Fourier transform* of $\psi \in PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ for a test function $h \in C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$ is given by

$$\langle \mathcal{F}_\tau^{-1}(\psi), h\rangle := \sum_{Q\in\mathcal{Q}} \sum_{\nu\in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \langle \mathcal{F}_{\tilde{\tau}}(h)(-\nu - \lambda, k),\ \mu_\nu^Q(\lambda)\psi(\nu + \lambda, k)\rangle\, dk\, d\lambda.$$

Finally, we discuss the topology on the image space by which the Fourier transform becomes a topological isomorphism.

Lemma 3. *(a) The Fourier transform $\mathcal{F}_\tau : C_c^{-\infty}(X, \mathbb{E}_\tau) \longrightarrow PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ is continuous.*

(b) The inverse Fourier transform

$$\mathcal{F}_\tau^{-1} : PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M) \longrightarrow C_c^{-\infty}(X, \mathbb{E}_\tau) \tag{6.12}$$

is continuous.

Proof. (a) We will show that for each bounded $B' \subset C_c^{-\infty}(X, \mathbb{E}_\tau)$, there exist $r \geq 0$ and $N \in \mathbb{N}_0$ such that $\mathcal{F}_\tau(B')$ is contained as a bounded set in $PWS_{\tau,r,N}$. Since $PWS_{\tau,r,N} \hookrightarrow PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ is continuous, by definition of inductive limit, then $\mathcal{F}_\tau(B')$ is also bounded in $PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$. By Lem. 2, we will have that $\mathcal{F}_\tau$ is continuous.

Now let $B' \subset C_c^{-\infty}(X, \mathbb{E}_\tau)$ be bounded. Since B' is equicontinuous and because of (6.4), there exist $r \geq 0, m \in \mathbb{N}_0$ and a constant $C > 0$ such that (6.11) holds uniformly for all $T \in B'$:

$$\sup_{T\in B'} p_B(T) = \sup_{T\in B',\ \varphi\in B} |T(\varphi)| \leq C \sum_{|\alpha|\leq m} \sup_{g\in\overline{B}_r(o)} |l_{Y_\alpha}\varphi(g)|.$$

Now by arguing as in the proof of Prop. 8 (c), we obtain, for $N = [\frac{m}{2}]$ that

$$||\mathcal{F}_\tau(T)||_{r,N,\alpha} \leq \infty, \quad \forall T \in B'$$

i.e., $\mathcal{F}_\tau(B') \subset PWS_{\tau,r,N}$ is bounded. Hence the Fourier transform is continuous.

(b) It suffices to show that if, for all $r \geq 0$ and $N \in \mathbb{N}_0$

$$\mathcal{F}_\tau^{-1} : PWS_{\tau,r,N}(\mathfrak{a}_{\mathbb{C}}^* \times K/M) \longrightarrow C^{-\infty}(X, \mathbb{E}_\tau) \tag{6.13}$$

is continuous. Indeed, by construction of the inductive limit topology and the remark between (6.2) & (6.3), as well as using $\mathcal{F}_\tau^{-1}(PWS_{\tau,r,N}) \subset C_r^{-\infty}(X, \mathbb{E}_\tau)$, we have that (6.12) is continuous.

Fix $r \geq 0$ and $N \in \mathbb{N}_0$. We want to show that (6.13) is continuous. For that, it suffices to show that for every bounded $\tilde{B} \subset C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$, we have

$$p_{\tilde{B}}(\mathcal{F}_\tau^{-1}(\psi)) \leq C||\psi||_{r,N,0}(< \infty), \quad \psi \in PWS_{\tau,r,N},$$

where $p_{\tilde{B}}(\cdot)$ is the seminorm as in (6.2) and C is a positive constant. Since $\tilde{B}$ is bounded subset in $C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$, there exsits $R \geq 0$ so that the support of all $\varphi \in \tilde{B}$ are in $\overline{B}_R(o)$. Thus, for $\psi \in PWS_{\tau,r,N}$, we have that

$$\begin{aligned}
&p_{\tilde{B}}(\mathcal{F}_\tau^{-1}(\psi)) \\
\overset{(6.2)}{=}\ &\sup_{\varphi\in\tilde{B}} |\langle \mathcal{F}_\tau^{-1}(\psi), \varphi\rangle| \\
\overset{(6.6)}{=}\ &\sup_{\varphi\in\tilde{B}} \left| \sum_{Q\in\mathcal{Q}} \sum_{\nu\in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \langle \mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu - \lambda, k)\ ,\ \mu_\nu^Q(\lambda)\psi(\nu + \lambda, k)\rangle\, dkd\lambda \right| \\
\leq\ &\sup_{\varphi\in\tilde{B}} \sum_{Q\in\mathcal{Q}} \sum_{\nu\in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \left| \langle \mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu - \lambda, k)\ ,\ \mu_\nu^Q(\lambda)\psi(\nu + \lambda, k)\rangle \right| dkd\lambda.
\end{aligned}$$

Fix now $Q \in \mathcal{Q}$ and $\nu \in A_Q^\tau$. Set

$$d_{Q,\nu} := \sup_{\varphi\in\tilde{B}} \int_{i\mathfrak{a}_Q^*} \int_K \left| \langle \mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k) \ , \ \mu_\nu^Q(\lambda)\psi(\nu+\lambda,k)\rangle \right| dkd\lambda.$$

It suffices to show that $d_{Q,\nu} \leq C||\psi||_{r,N,0}$. We have

$$\begin{aligned} d_{Q,\nu} &\leq \sup_{\varphi\in\tilde{B}} \int_{i\mathfrak{a}_Q^*} \int_K (1+|\nu+\lambda|^2)^{-d_Q}(1+|\nu+\lambda|^2)^{d_Q}|\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \\ &\qquad\qquad\qquad |\mu_\nu^Q(\lambda)\psi(\nu+\lambda,k)|\ dkd\lambda \\ &\leq C \sup_{\substack{\varphi\in\tilde{B} \\ k\in K,\lambda\in i\mathfrak{a}_Q^*}} (1+|\nu+\lambda|^2)^{d_Q}|\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)|\ |\mu_\nu^Q(\lambda)\psi(\nu+\lambda,k)| \end{aligned}$$

where $C := \int_{i\mathfrak{a}_Q^*} (1+|\nu+\lambda|^2)^{-d_Q}\ d\lambda < \infty$ and $(1+|\nu+\lambda|^2)^{d_Q}$ is a weight factor with some $d_Q \in \mathbb{N}_0$ depending on the dimension of $i\mathfrak{a}_Q^*$. For some positive constant N and growth constant $m \in \mathbb{N}_0$, we get

$$\begin{aligned} d_{Q,\nu} &\leq C \sup_{\substack{\varphi\in\tilde{B} \\ k\in K,\lambda\in i\mathfrak{a}_Q^*}} (1+|\nu+\lambda|^2)^{d_Q+N+m}|\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \\ &\qquad \cdot \sup_{k\in K,\lambda\in i\mathfrak{a}_Q^*} (1+|\nu+\lambda|^2)^{-(N+m)}|\mu_\nu^Q(\lambda)\psi(\nu+\lambda,k)| \\ &\leq C' \sup_{\substack{\varphi\in\tilde{B} \\ k\in K,\lambda\in i\mathfrak{a}_Q^*}} (1+|\nu+\lambda|^2)^{d_Q+N+m}|\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \\ &\qquad \cdot \sup_{k\in K,\lambda\in i\mathfrak{a}_{\mathbb{C}}^*} (1+|\nu+\lambda|^2)^{-N}|\psi(\nu+\lambda,k)|, \end{aligned}$$

where $||\mu_\nu^Q(\lambda)||_{\mathrm{op}} \leq C'(1+|\nu+\lambda|^2)^m$ of at most polynomial growth of $m \in \mathbb{N}_0$. Thus

$$\begin{aligned} d_{Q,\nu} &\leq C'' \sup_{\substack{\varphi\in\tilde{B} \\ k\in K,\lambda\in i\mathfrak{a}_Q^*}} e^{R|\nu|}(1+|\nu+\lambda|^2)^{d_Q+N+m}|\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \\ &\qquad \cdot \sup_{k\in K,\lambda\in i\mathfrak{a}_Q^*} e^{r|\nu|}(1+|\nu+\lambda|^2)^{-N}|\psi(\nu+\lambda,k)| \\ &= C'' \sup_{\varphi\in\tilde{B}} ||\mathcal{F}_{\tilde{\tau}}(\varphi)||_{R,d_Q+N+m}||\psi||_{r,N,0}, \end{aligned}$$

where we set $\xi := \nu+\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. By the Paley-Wiener Thm. 3, $\mathcal{F}_{\tilde{\tau}}$ is continuous, thus

$$\sup_{\varphi\in\tilde{B}} ||\mathcal{F}_{\tilde{\tau}}(\varphi)||_{R,d_Q+N+m} < \widetilde{C} < \infty.$$

Therefore, $d_{Q,\nu} \leq C'''|\psi||_{r,N,0}$ and hence the inverse Fourier transform is continuous. □

End of the proof of Thm. 4. The isomorphism of the Fourier transform map outcomes from Prop. 8 and the continuity and topology statement results from Lem. 3, hence this completes the proof.
Analogously, we obtain the topological Fourier isomorphism in (Level 3) by taking $C_c^{-\infty}(G,\gamma,\tau)$ instead of $C_c^{-\infty}(X,\mathbb{E}_\tau)$. □

7 Invariant differential operators on the Fourier range

We consider the vector space of distributional sections $C_{\{o\}}^{-\infty}(X,\mathbb{E}_\tau)$ supported at the origin $o = eK \in X$. Since $g\cdot o \neq o$, G does not act on $C_{\{o\}}^{-\infty}(X,\mathbb{E}_\tau)$, but K as well as $\mathfrak{g}$ do, thus $C_{\{o\}}^{-\infty}(X,\mathbb{E}_\tau)$ is a $(\mathfrak{g},K)$-module (e.g. [Wal88], 3.3.1). Moreover, it is generated by the so-called *vector-valued Dirac delta-distributions* δ_v at $v \in E_\tau$:

$$\delta_v(f) = \langle v, f(e)\rangle_\tau, \quad \text{with test function } f \in C_{(c)}^\infty(X,\mathbb{E}_{\tilde{\tau}}),$$

where $\langle\cdot,\cdot\rangle_\tau$ denotes the pairing in E_τ. In particular, we have the following identification:

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\tau \overset{\beta}{\cong} C_{\{o\}}^{-\infty}(X,\mathbb{E}_\tau)$$

given by $\beta(Z\otimes v)(f) := \langle r_Z f(e), v\rangle_\tau$, for $Z \in \mathcal{U}(\mathfrak{g}), v \in E_\tau, f \in C^\infty(X, \mathbb{E}_{\tilde\tau})$, with actions $Y(Z\otimes v) = YZ\otimes v$, and $k(Z\otimes v) = \mathrm{Ad}(k)Z \otimes \tau(k)v$, for $Y \in \mathfrak{k}$ (or $\mathcal{U}(\mathfrak{k})$), $k \in K$.
In addition, every invariant differential operator $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ may be viewed as a linear map between these spaces $D : C^{-\infty}_{\{o\}}(X, \mathbb{E}_\gamma) \longrightarrow C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau)$. This map defines an element

$$H_D \in \mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau)) \cong [C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau) \otimes E_{\tilde\gamma}]^K$$

given by

$$H_D(v) := D(\delta_v) \in C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau), \quad v \in E_\gamma, \delta_v \in C^{-\infty}_{\{o\}}(X, \mathbb{E}_\gamma). \tag{7.1}$$

In other words

$$\langle H_D(v), f\rangle_\tau \overset{(7.1)}{=} \langle \delta_v, D^t(f)\rangle_\gamma = \langle v, D^t(f)(1)\rangle_\gamma, \tag{7.2}$$

where $D^t \in \mathcal{D}_G(\mathbb{E}_{\tilde\tau}, \mathbb{E}_{\tilde\gamma})$ is the adjoint invariant differential operator of D defined by the corresponding pairing. Since the graded space of both Hilbert spaces $\mathcal{D}_G(\mathbb{E}_{\tilde\tau}, \mathbb{E}_{\tilde\gamma})$ and $\mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau))$ is isomorphic to $[S(\mathfrak{p}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)]^K$, we have the following isomorphism:

$$\begin{array}{rcl} \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau) & \overset{\sim}{\longrightarrow} & \mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau)) \\ D & \mapsto & H_D. \end{array}$$

Here, $S(\mathfrak{p})$ denotes the symmetric algebra of $\mathfrak{p} \subset \mathfrak{g}$.
Consequently, we have $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau) \cong \mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{0\}}(X, \mathbb{E}_\tau)) \cong C^{-\infty}_{\{0\}}(G, \gamma, \tau)$. Hence, by applying the Fourier transform in (Level 3) and the Paley-Wiener-Schwartz Thm. 4 (b), we can deduce the following result.

Proposition 9. *With the notations above, we then have*

$$\begin{array}{rcl} {}_\gamma\mathcal{F}_\tau(\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)) & \cong & {}_\gamma PWS_{\tau,0}(\mathfrak{a}^*_\mathbb{C}) \\ & = & \{P \in \mathrm{Pol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{Hom}_M(E_\gamma, E_\tau)) \mid P \text{ satisfies } (3.iii) \text{ of Def. } 10\}.\square \end{array}$$

Thus, provided one has a good understanding of the intertwining condition $(3.iii)$, one can determine $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$. The converse holds by van den Ban's and Souaifi's Lem. 5.3 and Cor. 5.4 in [vdBS14]. Strictly speaking these results are in terms of the Hecke algebra (6.5). But the $(\gamma, \tilde\tau)$-isotypic component $\mathcal{H}(G, K)(\gamma \otimes \tilde\tau)$ of the Hecke algebra is exactly $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau) \otimes \mathrm{Hom}(E_\tau, E_\gamma)$. In other words, given all invariant differential operators $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$, one can determine explicitly the intertwining condition $(3.iii)$ and the corresponding Paley-Wiener space.

Moreover, we remark that the isomorphism in Prop. 9 can also be described more algebraically as a Harish-Chandra type homomorphism, we refer to ([Olb95], p. 4) or ([Pal21], Sect. 2.1) for more details.

In addition, we also have the following result.

Proposition 10. *Let $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ be an invariant linear differential operator. For $f \in C^{\pm\infty}_c(X, \mathbb{E}_\gamma)$, we then have that*

$$\mathcal{F}_\tau(Df)(\lambda, k) = {}_\gamma\mathcal{F}_\tau(H_D)(\lambda)\mathcal{F}_\gamma(f)(\lambda, k), \quad \lambda \in \mathfrak{a}^*_\mathbb{C}, k \in K, \tag{7.3}$$

where ${}_\gamma\mathcal{F}_\tau(H_D) \in \mathrm{Pol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{Hom}_M(E_\gamma, E_\tau))$ is a polynomial in $\lambda \in \mathfrak{a}^*_\mathbb{C}$ with values in $\mathrm{Hom}_M(E_\gamma, E_\tau)$.

Proof. We know that the Fourier transform of a distribution $H_D \in \mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau))$ is defined by ${}_\gamma\mathcal{F}_\tau(H_D)(\lambda)(v) = \langle H_D(v), e^\tau_{\lambda,1}\rangle$, for $v \in E_\gamma$ and where $e^\tau_{\lambda,1} \in C^\infty(G, \tau, \tilde\tau)$. Hence by (7.2), we obtain

$${}_\gamma\mathcal{F}_\tau(H_D)(\lambda)(v) = \langle H_D(v), e^\tau_{\lambda,1}\rangle_\tau \overset{(7.2)}{=} \langle v, D^t(e^\tau_{\lambda,1})(1)\rangle_\gamma = (D^t(e^\tau_{\lambda,1})(1))v, \quad \lambda \in \mathfrak{a}^*_\mathbb{C}. \tag{7.4}$$

Now, by considering a function $f \in C^\infty_c(X, \mathbb{E}_\gamma)$, we conclude, via 'partial integration', that (7.3) holds. In fact

$$\begin{array}{rcl} \mathcal{F}_\tau(Df)(\lambda, k) = \displaystyle\int_G e^\tau_{\lambda,k}(g) D(f(g))\, dg & \overset{\text{def. of } D^t}{=} & \displaystyle\int_G D^t(e^\tau_{\lambda,k}(g)) f(g)\, dg \\ & \overset{(3.17)}{=} & \displaystyle\int_G D^t(e^\tau_{\lambda,1}(1)) \circ e^\gamma_{\lambda,k}(g) f(g)\, dg \\ & = & D^t(e^\tau_{\lambda,1}(1)) \circ \mathcal{F}_\gamma(f)(\lambda, k) \\ & \overset{(7.4)}{=} & {}_\gamma\mathcal{F}_\tau(H_D)(\lambda) \circ \mathcal{F}_\gamma(f)(\lambda, k). \end{array}$$

The same computation remains true for $f \in C^{-\infty}_c(X, \mathbb{E}_\gamma)$, by using the pairing $\langle\cdot,\cdot\rangle$ instead of the integration. $\square$

Remark 4. Consider an additional not necessarily irreducible K-representation (δ, E_δ). Then, for $D_1 \in \mathcal{D}_G(\mathbb{E}_\tau, \mathbb{E}_\delta)$ and $D_2 \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$, Prop. 10 implies that

$$_\gamma\mathcal{F}_\delta(H_{D_1} \circ H_{D_2}) = {}_\tau\mathcal{F}_\delta(H_{D_1}) \circ {}_\gamma\mathcal{F}_\tau(H_{D_2}).$$

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