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Solvability of systems of invariant differential equations on symmetric spaces G/K

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Abstract

We study the Fourier transform for distributional sections of vector bundles over symmetric spaces of non-compact type. We show how this can be used for questions of solvability of systems of invariant differential equations in analogy to Hörmander's proof of the Ehrenpreis-Malgrange theorem. We get complete solvability for the hyperbolic plane $\mathbb{H}^2$ and partial results for finite products $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2$ and the hyperbolic 3-space $\mathbb{H}^3$.

Mir studéieren Fourier Transformatioun fir Distributiounal Sektiounen vu Vektorbündelen u symmetresch Réim vun engem net-kompakten Typ. Mir beweisen wéi et fir d'Léisbarkeet vu Systémer vun invarianten Differentialequatiounen an Analogie zu Hörmander's Schätzungen, ugewand ka ginn. Mir kréien komplett Léisbarkeet fir hyperbolesch Pléng $\mathbb{H}^2$ a partial Résultater fir Produkter $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2$, wéi och fir hyperbolesch 3-Réim $\mathbb{H}^3$.

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Introduction

Since the introduction of differential calculus by Newton and Leibniz, differential equations have played an essential role in the development of mathematics as well as in applied sciences. One of the most important basic theoretical questions is the one of solvability.

In the Euclidean case $\mathbb{R}^n$ (for some positive integer n), it is well-known by Malgrange [Mal55] and Ehrenpreis [Ehr54] that all linear differential operators with constant coefficients

$$D = \sum_{|\alpha|\leq m} c_\alpha D^\alpha, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}},\ \alpha = (\alpha_1,\ldots,\alpha_n), \alpha_j \in \mathbb{N}_0,\ |\alpha| = \sum_{j=1}^{n}\alpha_j$$

are solvable. Even more, there exists a fundamental solution E for D, which is a distribution on $\mathbb{R}^n$ such that $DE = \delta$ and consequently, by convolution, there exists a solution $f := E * g$ of the equation $Df = g$ at least for g with compact support. Here, δ denotes the Dirac measure or delta-distribution at the origin on $\mathbb{R}^n$. One may wonder whether an invariant differential operator with non-constant coefficients, but smooth coefficients, is solvable. This is indeed not always the case, as the well-known example of Lewy $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} - 2i(x+iy)\frac{\partial}{\partial z}$ on $\mathbb{R}^3$ or even Mizohata's example $\frac{\partial}{\partial x} + ix\frac{\partial}{\partial y}$ on $\mathbb{R}^2$ show.
Constant coefficients operators on $\mathbb{R}^n$ are exactly invariant differential operators, if we consider $\mathbb{R}^n$ as a Lie group of translations.

Speaking of Lie groups G, one may ask the analog question for invariant differential operators, which are left invariant under G by translations:

$$D(f\circ l_g) = (Df)\circ l_g, \quad f \in C^\infty(G), \forall g \in G, \tag{0.1}$$

where l_g is the left translation on G and $C^\infty(G)$ denotes the space of smooth functions on G. Concerning the above-mentioned example, the Lewy operator can be considered as a left invariant differential operator on the Heisenberg group $\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$. Such differential operators are, in general, not even locally solvable.
On the other hand, the situation is much better for non-zero linear differential operators D, which are bi-invariant under G, this means that (0.1) holds not only for the left but also for the right translation on G. In fact, D is locally solvable for simply connected nilpotent Lie groups by Raïs [Ra71], for solvable Lie groups by Raïs-Duflo [DuRa76] and Rouvière [Ro76], and for semi-simple Lie groups by Helgason [Hel75]. Of course, other interesting results have been found on groups (e.g. [Ce75] & [Hel75]). But the

key tool of most of these works is the theory of harmonic analysis on the corresponding Lie group, in particular the application of the Fourier transform.

Now, if we consider the quotient G/K, with G a non-compact connected semi-simple Lie group with finite center and K is its maximal compact subgroup, then Helgason ([Hel89], Chap. V) proved that all G-invariant differential operators are solvable on Riemannian symmetric spaces X of non-compact type. His proof is essential based on a characterisation of the image of $C_c^\infty(X)$ under the Fourier transform on X, in particular the Radon transform.

However what happens if we genuinely extend this situation and consider systems of linear invariant differential operators, e.g. D as a $q \times p$ -matrix $(q, p \in \mathbb{N})$ of such linear invariant differential operators? Is it still (locally) solvable?
In case of $\mathbb{R}^n$, the questions have been answered completely by Malgrange ([Mal61] & [Mal64]), Ehrenpreis ([Ehr61] & [Ehr70], Chap. 6) and Palamodov ([Pal63] & [Pal70]). Already, here, the proof is much more complicated as for a single operator.
Let us now express the setting for symmetric space X of non-compact type. Suitably interpreted, it also applies to the case $X = \mathbb{R}^n$ with $G = \mathbb{R}^n$ and $K = \{0\}$. A system of invariant differential equations is an invariant differential operator between homogeneous vector bundles. Let (τ, E_τ) and (γ, E_γ) be finite dimensional, not necessarily irreducible, representations of K. They determine homogeneous vector bundles $\mathbb{E}_\gamma, \mathbb{E}_\tau \to X$. We identify the spaces of their smooth sections with the following vector spaces:

$$C^\infty(X, \mathbb{E}_*) \cong \{\varphi : G \overset{C^\infty}{\to} E_* \mid \varphi(gk) = *(k^{-1})(\varphi(g)),\ \forall g \in G, k \in K\}, \quad * = \tau, \gamma.$$

The group G acts on $C^\infty(X, \mathbb{E}_*)$ by left translation $(g \cdot \varphi)(x) = \varphi(g^{-1}x)$, for $x, g \in G$. Consider a linear G-invariant differential operator

$$D : C^\infty(X, \mathbb{E}_\gamma) \longrightarrow C^\infty(X, \mathbb{E}_\tau). \tag{0.2}$$

Given $g \in C^\infty(X, \mathbb{E}_\tau)$, does there exists a solution $f \in C^\infty(X, \mathbb{E}_\gamma)$ such that $Df = g$? For many D's, there is (another) operator

$$C^\infty(X, \mathbb{E}_\tau) \xrightarrow{D_0} C^\infty(X, \mathbb{E}_\delta),$$

where (δ, E_δ) is another K-representation such that $D_0 \circ D = 0$. Therefore such a D can not be solvable for g, if $D_0 g \neq 0$. For algebraic reasons there is, in some sense, a maximal operator, call it $\widetilde{D}$, of this kind.
Question. Is $Df = g$ solvable in $C^\infty(X, \mathbb{E}_\gamma)$ for a given $g \in C^\infty(X, \mathbb{E}_\tau)$ if, and only if, $\tilde{D}g = 0$ in $C^\infty(X, \mathbb{E}_\tau)$? In other words, is the sequence

$$C^\infty(X, \mathbb{E}_\gamma) \xrightarrow{D} C^\infty(X, \mathbb{E}_\tau) \xrightarrow{\tilde{D}} C^\infty(X, \mathbb{E}_\delta)$$

exact in the middle, i.e. $\mathrm{Im}(D) = \mathrm{Ker}(\tilde{D})$?

Conjecture. *The answer is yes.*

Maybe as an example, consider $\mathbb{E}_i = \wedge^{p+(i-1)} T^* X$ $(i = 1, 2, 3)$ as exterior powers of the cotangent bundles T^*X for $p \in \mathbb{N}$. Then, by Poincaré-Lemma (e.g. in [BoT82], Sect. 4), in this case, the conjecture is true for the exterior differention $D = d_p$ and $\tilde{D} = d_{p+1}$ on X.

Notice that if $\mathbb{E}_\gamma = \mathbb{E}_\tau = \mathbb{C}$ are trivial one-dimensional vector bundles on X, then the transposed invariant differential operator D^t is injective on $C_c^\infty(X,\mathbb{C})$. This means that $\widetilde{D} = 0$, thus we have no 'integrability' condition. This leads us back to Helgason's result ([Hel89], Chap. V). Also for single equations and elliptic operators, we have $\widetilde{D} = 0$. In the last case, the conjecture was proved by Malgrange ([Mal55], p. 341).

Furthermore, in the Euclidean analogue, Hörmander ([Hör73], Thm. 7.6.13 & Thm. 7.6.14) was one of the first to give a written proof of Ehrenpreis's results ([Ehr61] & [Ehr70], Chap. 6), also known as the *Ehrenpreis Fundamental Principle.* He even went beyond by inventing new methods for bounds for L^2 estimates and related them with the theory of analytic sheaves on a Stein manifold.

Not only Hörmander extended Ehrenpreis's fundamental principle but also Oshima, Saburi and Wakayama for symmetric spaces X. They announced in their paper ([OSW91], Sect. 8, Thm. 4) that the conjecture is true if $\mathbb{E}_\gamma = \mathbb{C}^p, \mathbb{E}_\tau = \mathbb{C}^q$ and $\mathbb{E}_\delta = \mathbb{C}^r$, $p,q,r \in \mathbb{N}$, with trivial K-representations.

Another interesting result is that of Kashiwara-Schmid [KS95]. By taking the dualization of the sequence (0.2), i.e., $D^t : C^{-\infty}_{\{o\}}(X,\mathbb{E}_{\tilde\gamma}) \longrightarrow C^{-\infty}_{\{o\}}(X,\mathbb{E}_{\tilde\tau})$ and restricting to distributions supported at the origin $o \in X$, we obtain an operator, also denoted by D^t:

$$D^t : \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_{\tilde\gamma} \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_{\tilde\tau}.$$

If D^t occurs in a projective resolution of a Harish-Chandra-module W, e.g. an irreducible $(\mathfrak{g},K)$-module

$$0 \to \cdots \stackrel{P_{s+3}}{\to} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_{s+2} \stackrel{P_{s+2}}{\to} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_{s+1} \stackrel{P_{s+1}}{\to} \cdots \stackrel{P_0}{\to} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_0 \to W \to 0$$

with $D^t = P_s, s \in \mathbb{N}_0$, then the conjecture is true for D. Here $\mathcal{U}(\mathfrak{g})$ (resp. $\mathcal{U}(\mathfrak{k})$) is the universal enveloping algebra of the complexification of the Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{k}$) of G (resp. K). The full proof appeared in [Ka08] and is based on $\mathcal{D}$-modules.

We see that for many special cases, the conjecture is true, but it is not proved in general. The aim of this thesis is to present a possible strategy to solve the conjecture and apply it for some instructive examples. The following diagram, which we will try to explain in the sequel, pictures the strategy and also the 'jungle' into which the reader is about to adventure.

$$\begin{tikzcd}
C^\infty(X,\mathbb{E}_\gamma) \arrow[r, "D"] \arrow[d, dashed] & \underset{\text{exact}}{C^\infty(X,\mathbb{E}_\tau)} \arrow[r, "\widetilde{D}"] \arrow[d, dashed, "dualization"] & C^\infty(X,\mathbb{E}_\delta) \arrow[d, dashed] & \\
C_c^{-\infty}(X,\mathbb{E}_{\tilde\delta}) \arrow[r, "\widetilde{D}^t"] \arrow[d, "\mathcal{F}_{\tilde\delta}"] & \underset{\text{exact}}{C_c^{-\infty}(X,\mathbb{E}_{\tilde\tau})} \arrow[r, "D^t"] \arrow[d, "\mathcal{F}_{\tilde\tau}"] & \underset{\text{closed range}}{C_c^{-\infty}(X,\mathbb{E}_{\tilde\gamma})} \arrow[d, "\mathcal{F}_{\tilde\gamma}"] & \\
PWS_{\tilde\delta}(\mathfrak{a}^*_{\mathbb{C}} \times K/M) \arrow[r, "Q"] \arrow[d, dashed, leftrightarrow] & \underset{\text{exact}}{PWS_{\tilde\tau}(\mathfrak{a}^*_{\mathbb{C}} \times K/M)} \arrow[r, "P"] \arrow[d, dashed, leftrightarrow] & \underset{\text{closed range}}{PWS_{\tilde\gamma}(\mathfrak{a}^*_{\mathbb{C}} \times K/M)} \arrow[d, dashed, leftrightarrow] & (\textit{Level } 2) \arrow[d, dashed, leftrightarrow] \\
{}_\mu PWS_{\tilde\delta}(\mathfrak{a}^*_{\mathbb{C}}) \arrow[r, "{}_\mu Q"] & \underset{\text{exact}}{{}_\mu PWS_{\tilde\tau}(\mathfrak{a}^*_{\mathbb{C}})} \arrow[r, "{}_\mu P"] & \underset{\text{closed range}}{{}_\mu PWS_{\tilde\gamma}(\mathfrak{a}^*_{\mathbb{C}})} & (\textit{Level } 3)
\end{tikzcd}$$

Roughly speaking, to prove that $\mathrm{Im}(D) = \mathrm{Ker}(\widetilde{D})$, the idea is to prove that D has a closed and dense range in $\mathrm{Ker}(\widetilde{D})$. In terms of duals, we wish to prove that D^t is injective modulo $\mathrm{Ker}(\widetilde{D})$ with closed range in the strong dual topology. Write $(\tilde{\tau}, E_{\tilde{\tau}})$ for the contragredient K-representation of (τ, E_τ) and

$$C_c^{-\infty}(X, \mathbb{E}_{\tilde{\tau}}) := C^\infty(X, \mathbb{E}_\tau)'$$

for the space of compactly supported distributions. The next step will be to apply the Fourier transform for sections and describe the image of the Fourier transform of $C_c^{-\infty}(X, \mathbb{E}_{\tilde{\tau}})$ as the space of holomorphic and smooth functions with some slow growth and intertwining conditions. This will lead us to the Paley-Wiener-Schwartz theorem for distributional sections.

Theorem A (Thm. 2.40)**.** *Let $PWS_{\tilde{\tau}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ be the Paley-Wiener-Schwartz space for sections of homogeneous vector bundles $\mathbb{E}_{\tilde{\tau}}$. Then, there is a topological isomorphism through the Fourier transform*

$$C_c^{-\infty}(X, \mathbb{E}_{\tilde{\tau}}) \xrightarrow{\mathcal{F}_{\tilde{\tau}}} PWS_{\tilde{\tau}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M).$$

In particular, our starting point will be Delorme's Paley-Wiener theorem for $C_c^\infty(G)$ ([Del05], Thm. 2), which we refer to as (Level 1) and which we would like to apply and make more concerete for our proposes. The most difficult, but at the same time exciting task, will be to control his intertwining conditions ([Del05] & [vdBS14]). For three examples, $G = SL(2,\mathbb{R}), G = SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R})$ (finite copies) and $G = SL(2,\mathbb{C})$, we will completely determine the intertwining conditions for the Paley-Wiener-Schwartz space. Now by using the impact of Thm. A, the Fourier transform of $\widetilde{D}^t$ (resp. D^t) will be a matrix of polynomials Q (resp. P) which also satisfies these intertwining conditions. Therefore, the conjecture can be reformulated in terms of action of Q on $PWS_{\tilde{\tau}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ (resp. P on $PWS_{\tilde{\gamma}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$). Nevertheless, the modified problem is still too difficult to be solved in general. It is appropriate to quote here Hörmander's doctoral adviser Lars Gårding:
„When a problem of partial differential operators has been fitted into the abstract theory, all that remains is usually to prove a suitable inequality and much of our new knowledge is, in fact, essentially contained in such inequalitites.“
Hörmander repeatedly followed this principle in his work. In particular, we can relate the above with Hörmander's results ([Hör73], Thm. 7.6.11 and Cor. 7.6.12) on the '$\mathfrak{a}$'-part under some conditions on the K-type. More precisely, we will fix an irreducible K-representation (μ, E_μ) on the left while a right K-type is fixed by the bundle $\mathbb{E}_{\tilde{*}} \to X$, $* = \delta, \gamma, \tau$, which we will refer to as (Level 3). Of course, (Level 2) will correspond to the desired situation. In this framework, we can immediately solve the conjecture in (Level 3).

Hypothesis B (Hyp. 2 & Hyp. 3)**.** *There exist $M \in \mathbb{N}_0$, for all $r \geq 0$ and $N \in \mathbb{N}_0$, as well as a constant $C_{r,N} \in \mathbb{N}_0$ so that for each function $u \in {}_\mu PWS_{\tilde{\tau},H}(\mathfrak{a}_{\mathbb{C}}^*)$ such that $||Pu||_{r,N} < \infty$, one can find a function $v \in {}_\mu PWS_{\tilde{\tau}}(\mathfrak{a}_{\mathbb{C}}^*)$ with*

(i) $Pu = Pv$ *and*

(ii) $||v||_{r,N+M} \leq C_{r,N}||Pu||_{r,N}$.

The constant M can be chosen to be independent of the K-type μ and $C_{r,N}$ to be a constant of at most of polynomial growth in the length of μ.

Here, $||\varphi||_{r,N} := \sup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}}(1+|\lambda|^2)^{-N}e^{-r|\mathrm{Re}(\lambda)|}||\varphi(\lambda)||_{\mathrm{op}}$ denotes the semi-norm for $\varphi \in {}_{\mu}PWS_{\tilde{\tau}}(\mathfrak{a}^*_{\mathbb{C}})$, where $||\cdot||_{\mathrm{op}}$ is the operator norm in the corresponding space and ${}_{\mu}PWS_{\tilde{\tau},H}(\mathfrak{a}^*_{\mathbb{C}})$ is the Paley-Wiener-Schwartz without the growth condition.

The final step would be to move back to (Level 2) and take the inverse Fourier transform to get to the initial problem.

Theorem C (Thm. 4.10 & Cor. 4.11)**.** *Assume that Hyp. B is satisfied. Then, under a mild additional assumption (Hyp. 1), we have solvability on the symmetric space* $X = G/K$ *and in the ball* $B_r(x) \subset X, \forall r \geq 0, x \in X$.

In particular, we will establish Hyp. 1, Hyp. B and Hyp. B in (Level 2) for $G = SL(2,\mathbb{R})$.

Theorem D (Thm. 5.1 & Thm. 5.2)**.** *The conjecture is true on the hyperbolic plane* $\mathbb{H}^2$.

Also partial results will be obtained for $G = SL(2,\mathbb{R})^d$ $(d \geq 2)$ and $G = SL(2,\mathbb{C})$.

Roadmap of the thesis. The thesis consists of five chapters. The first three provide key preparations and tools for the conjecture, which will be then merged in the fourth chapter and afterwards, in the last one, applied to several illustrative examples.

In Chapter 1, we will start with some preliminary results on harmonic analysis on semi-simple Lie groups. In particular, we will introduce the notions of invariant differential operators on Riemannian symmetric spaces in order to give the precise statement of the conjecture (Sect. 1.3 & Conj. 1).

Chapter 2 will contain crucial and important tools for the solvability questions, namely the Fourier transform and the Paley-Wiener theorems for sections over homogeneous vector bundles. Starting with Delorme's Paley-Wiener theorem, we will adjust it for our proposes. We will consider three levels, (Level 1) refers to Delorme's Paley-Wiener theorem (Thm. 2.7), (Level 2) corresponds to the desired Paley-Wiener theorem for sections (Thm. 2.31) and (Level 3) stands for the Paley-Wiener theorem for sections for 'spherical functions' (Thm. 2.31). Using Frobenius-reciprocity, we will transfer Delorme's intertwining condition to the other levels (Prop. 2.26 & Thm. 2.28). Furthermore, we will present a topological Paley-Wiener-Schwartz theorem for distributional sections (Thm. 2.40) by using a version of Plancherel theorem for sections (Thm. 2.34 & Cor. 2.35). This chapter will end by analysing the consequence of Delorme's theorem on the invariant differential operators (Thm. 2.46).

Chapter 3 will be devoted to the description of a family of examples of general interest for Delorme's intertwining conditions exposed in the previous chapter. More precisely, we will introduce the Knapp-Stein and Želobenko intertwining operators. We will develop a criterion to check when a subset of Delorme's intertwining condition is already sufficient to describe the Paley-Wiener space completely (Thm. 3.13). This criterion is already contained implicitely in Delorme's proof of his Paley-Wiener theorem. To apply it, one has to know rather the complete composition series of reducible principle series representations.
For example, for the case of $G = SL(2,\mathbb{R})$, we will describe this composition series and the corresponding intertwining conditions for each levels (Thms. 3.17, 3.18 & 3.20), in particular in (Level 2), with the help of '*box-pictures*' (Fig. 3.1). After that, we will observe that if we consider the product of $G = SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R})$, the intertwining conditions remain the 'same' (Thm. 3.25). As third example, we will consider

$G = SL(2,\mathbb{C})$. Here, the interpretation of Delorme's intertwining conditions will be more complicated than for the previous two examples (Thms. 3.28, 3.29 & 3.35). In these three cases, we will have complete explicit Paley-Wiener(-Schwartz) theorems.

In Chapter 4, we will gather all our tools from the previous chapters and explain a possible strategy to attack the conjecture, which we will refer to as our four hypotheses (Hyps. 1 - 4). In fact, the plan will be to prove the conjecture for K-finite elements, this means in (Level 3) and then go up to the desired situation (Level 2) by applying some convergence arguments and the Fourier decomposition (Thm. 4.1). We will complete the proof of the conjecture by employing abstract function analytic criteria for closedness and density of images of operators (Thm. 4.10 & Cor. 4.11).

Finally, in Chapter 5, the hypotheses stated in the previous chapter, will then be proved (partially) for three specific examples. Namely, for the hyperbolic plane $\mathbb{H}^2$ (Thms. 5.1 & 5.2), in particular we obtain Thm. D, and partial results for $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2$ (Thms. 5.4 & 5.5) and $\mathbb{H}^3$ (Thm. 5.7).

Keywords. Symmetric spaces, Fourier transform, Fourier series, Paley-Wiener theorem, Linear invariant differential operators, Intertwining operators, Homogeneous vector bundles,...

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„The analysis of PDE is a beautiful subject, combining the rigour and technique of modern analysis and geometry with the very concrete real-world of physics and other sciences.“

– Terence Tao (2006)

Chapter 1

Invariant differential operators on symmetric spaces and statement of the conjecture

In this chapter, we introduce the main objects of study, namely the invariant differential operators on Riemannian symmetric spaces X, as well as the notions of harmonic analysis on X that will be used in the rest of the thesis. The chapter ends with the precise statement of Conjecture 1.

In Section 1.1, we start to review briefly the basic terminology and important results of sections over complex homogeneous vector bundles over X. Then, we describe the invariant differential operators and give their algebraic interpretation in terms of universal enveloping algebra. We adopt the standard notation and refer to ([Wal88], [Jac62] & [KoRe00]) for more details.

Next, in Section 1.2, we move to their interpretation in terms of symmetric algebra and universal enveloping algebra. This gives rise to the discussion of their image under the so-called Harish-Chandra homomorphism, which will be useful for our exposition in Chapter 2, in particular in Section 2.4. For further details we refer to ([Olb95] & [Wal88]).

Finally, in Section 1.3, in view to state rigorously the main problem we are interested in, we construct, by dualization, an invariant differential operator $\widetilde{D}$ so that its kernel is equal to the image of a given invariant differential operator D. This turns out to attend the solvability of such invariant differential operators.

1.1 Invariant differential operators on sections of homogeneous vector bundles

Let G be a real connected semi-simple Lie group with finite center of non-compact type and $K \subset G$ a maximal compact subgroup. The quotient $X = G/K$, then is a Riemannian symmetric space of non-compact type.

Consider E_τ a vector space over $\mathbb{C}$ and denote by $GL(E_\tau)$ the group of all invertible elements of $\mathrm{End}(E_\tau)$, which is the space of continuous endomorphism of E_τ. Let $\hat{K}$ be the set of all isomorphism classes of, not necessary irreducible, unitary representations

$$\tau : K \to GL(E_\tau)$$

of K on E_τ. Recall that a representation (τ, E_τ) is irreducible, if the only invariant subspace of E_τ are $\{0\}$ and itself ([Wal88], 1.1.1). Since K is compact, every $\tau \in \hat{K}$ is finite-dimensional.

With the data (G, K, E_τ), we can construct a homogeneous vector bundle over X as follows. Let $\mathbb{E}_\tau := (G \times E_\tau)/K = G \times_K E_\tau$ be the quotient space of $G \times E_\tau$ under the equivalence relation

$$[g, v]k \sim [gk, \tau(k^{-1})v], \quad g \in G, v \in E_\tau, k \in K$$

and consider the canonical projection $p : \mathbb{E}_\tau \longrightarrow X$ mapping the equivalence class of $[g, v]$ to gK, for $g \in G$ and $v \in E_\tau$. The pair (p, E_τ) is a vector bundle over X with fiber $p^{-1}(gK) = [g, E_\tau] = \{[g, v] \mid v \in E_\tau\}$. In particular, $\mathbb{E}_\tau$ is a complex (G-)homogeneous vector bundle over X induced by E_τ since it carries the smooth left G-action given by $g[x, v] := [gx, v]$, for every $x, g \in G$ and $v \in E_\tau$. It is compatible with the map p

$$p(g[x, v]) = gp([x, v]), \quad g, x \in G, v \in E_\tau.$$

Let $s : X \to \mathbb{E}_\tau$ be a section of the (G-)homogeneous vector bundle $\mathbb{E}_\tau$, i.e. $p \circ s = \mathrm{Id}_X$. The space of its smooth sections $C^\infty(X, \mathbb{E}_\tau)$ is equipped with a smooth left G-action given by

$$(g \cdot s)(x) := g(s(g^{-1}x)), \quad g \in G, x \in X, s \in C^\infty(X, \mathbb{E}_\tau).$$

Note that the space $C^\infty(X, \mathbb{E}_\tau)$ carries a natural Fréchet topology, this means it is a locally convex topological vector space where the family of semi-norms is countable.

The next isomorphism of G-modules will be useful. Here, we denote by W^K the space of the K-invariants of a K-representation W. The complete proof can be found for example in [KoRe00].

Proposition 1.1. *In the previous notation, let*

$$C^\infty(G, E_\tau)^K := \{f : G \overset{C^\infty}{\to} E_\tau \mid f(gk) = \tau(k)^{-1}(f(g)), \forall g \in G, k \in K\}$$

denotes the space of all infinitely often differentiable maps from G to E_τ, where the group G acts on the space by translations from the left. Then, one has the G-isomorphism

$$C^\infty(G, E_\tau)^K \;\cong\; C^\infty(X, \mathbb{E}_\tau)$$

given by $f \mapsto s_f(gK) = [g, f(g)]$ with $g \cdot s_f = s_{l_g f}$, where

$$(l_g f)(x) := f(g^{-1}x), \quad g, x \in G$$

denotes the left translation on the vector space $C^\infty(G)$ of all smooth function of G. □

Moreover, let $[C^\infty(G) \otimes E_\tau]^K$ be the set of all K-invariant vectors of $C^\infty(G) \otimes E_\tau$, where K acts on $C^\infty(G)$ by right translation via

$$(r_g f)(x) := f(xg), \quad g, x \in G$$

and on E_τ by τ. While G acts by left translation and trivial on E_τ. Then, there is an isomorphism between

$$[C^\infty(G) \otimes E_\tau]^K \cong C^\infty(G, E_\tau)^K$$

given by $\sum_i f_i \otimes v_i \mapsto \sum_i f_i \cdot v_i$, for $f_i \in C^\infty(G)$ and v_i runs a vector basis of E_τ, for all i. Hence we can identify the space of sections over homogeneous vector bundles over X by

$$C^\infty(X, \mathbb{E}_\tau) \cong C^\infty(G, E_\tau)^K \cong [C^\infty(G) \otimes E_\tau]^K.$$

Consider now an additional, not necessary irreducible, K-representation (γ, E_γ) and its associated complex homogeneous vector bundle $\mathbb{E}_\gamma := G \times_K E_\gamma$ over X.

Definition 1.2. *A linear non-zero differential operator*

$$D : C^\infty(X, \mathbb{E}_\gamma) \to C^\infty(X, \mathbb{E}_\tau)$$

between sections over homogeneous vector bundles is said to be G-invariant if

$$D(g \cdot f) = g \cdot (Df), \qquad \forall g \in G, f \in C^\infty(X, \mathbb{E}_\gamma).$$

This means D is equivariant with respect to the left regular translations l_g by G. We denote by $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$, the vector space of all these G-invariant differential operators on sections.

Enveloping algebras and invariant differential operators

Here and in the following, we use the convention to denote Lie groups by Roman capitals and their Lie algebras by the corresponding lower case Gothic letters.
Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of the Lie group of G (resp. K) and $\mathcal{U}(\mathfrak{g})$ (resp. $\mathcal{U}(\mathfrak{k})$) be the universal enveloping algebra of complexification of $\mathfrak{g}$ (resp. $\mathfrak{k}$) (e.g. [Jac62] Chap. V).

Let $Z \in \mathfrak{g}$, then, we have a left invariant (first order) differential operator on G

$$Zf(g) = r_Z f(g) = \frac{d}{dt}\Big|_{t=0} f(g \exp(tZ)), \quad g \in G, f \in C^\infty(G)$$

so that $l_g(Zf) = Z(l_g f)$. Here the suffix $|_{t=0}$ means the evaluation in real variables $t = 0$ after differentation. In a similar way, one can define a right invariant differential operator on G by

$$l_Z f(g) = \frac{d}{dt}\Big|_{t=0} f((-\exp(tZ))g).$$

The left l_Z and right r_Z invariant differential operator can be extended for $Z \in \mathcal{U}(\mathfrak{g})$.

Various ways of looking at space of invariant differential operators $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ are known from the literature, e.g. in [KoRe00]. We will present an interpretation which suits our purpose. To do this, let us analyse an important construction related to $\mathcal{U}(\mathfrak{k})$-modules. Write $\mathrm{Hom}(E_\gamma, E_\tau)$ for the vector space of complex homomorphism from E_γ to E_τ. We turn $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)$ into $\mathcal{U}(\mathfrak{g})$-module by

$$Z(Y \otimes a) := ZY \otimes a,$$

where $Z \in \mathcal{U}(\mathfrak{g}), Y \in \mathcal{U}(\mathfrak{k})$ and $a \in \mathrm{Hom}(E_\gamma, E_\tau)$. Let

$$\mathrm{opp} : \mathcal{U}(\mathfrak{k}) \to \mathcal{U}(\mathfrak{k})$$

be the anti-automorphism algebra map of $\mathcal{U}(\mathfrak{k})$ given by $\mathrm{opp}(Y) := -Y$, for $Y \in \mathfrak{k}$ ([Wal88] Chap.0 p.10). Then, $\mathrm{Hom}(E_\gamma, E_\tau)$ turns into a left $\mathcal{U}(\mathfrak{k})$-module by

$Ya := a\gamma(\mathrm{opp}(Y))$. Hence, the tensor product of the modules $\mathcal{U}(\mathfrak{g})$ and $\mathrm{Hom}(E_\gamma, E_\tau)$ over $\mathcal{U}(\mathfrak{k})$ is the tensor product over $\mathbb{C}$ modulo the equivalence relation determined by the linear span J of the elements

$$ZY \otimes a - Z \otimes a\gamma(\mathrm{opp}(Y)).$$

The standard notation of this tensor product of a left and a right $\mathcal{U}(\mathfrak{k})$-module is written by

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(E_\gamma, E_\tau) := (\mathcal{U}(\mathfrak{g}) \otimes \mathrm{Hom}(E_\gamma, E_\tau))/J$$

(e.g. [KoRe00], p. 373 or [Wal88], 0.6.5.). Note that, by convention, we will not distinguish between the notation of the induced representation of $\mathfrak{k}$ resp. of $\mathcal{U}(\mathfrak{k})$ on a vector space E_γ, with the notion of a representation γ of K on E_γ. Elements in $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(E_\gamma, E_\tau)$ act on function $f \in C^\infty(X, \mathbb{E}_\gamma)$ by

$$D(f(g)) = (Z \otimes a)(f(g)) := a(Zf)(g), \; g \in G.$$

It is clear that it is left invariant and thus it commutes with l_g. Note also that, since the elements in J act trivially, the above action is well-defined. Indeed, for $Y \in \mathfrak{k}$

$$\begin{aligned}(ZY \otimes a)(f(g)) = a(ZY)(f(g)) &= a\frac{d}{dt}\Big|_{t=0}(Zf)(g\exp(tY)) \\ &= a\frac{d}{dt}\Big|_{t=0}\gamma(\exp(-tY))(Zf)(g) \\ &= a \circ \gamma(\mathrm{opp}(Y))(Zf)(g) \\ &= (Z \otimes a \circ \gamma(\mathrm{opp}(Y)))(f)(g).\end{aligned}$$

To guarantee that the elements in $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(E_\gamma, E_\tau)$, which describe all left invariant linear differential operators, operate from $C^\infty(X, \mathbb{E}_\gamma)$ to $C^\infty(X, \mathbb{E}_\tau)$, a further condition is required. The subgroup K acts on $\mathcal{U}(\mathfrak{g})$, by the adjoint representation and naturally on $\mathrm{Hom}(E_\gamma, E_\tau)$ by $k \in K$, sending any element a to $\tau(k) \circ a \circ \gamma(k)^{-1}$. Thus, we set

$$k(Z \otimes a) = \mathrm{Ad}(k)Z \otimes \tau(k)a\gamma(k)^{-1}.$$

Since the tensor product action leaves the subspace J invariant, K acts on the tensor product over $\mathcal{U}(\mathfrak{k})$. We will denote by $[\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(E_\gamma, E_\tau)]^K$ the subset of K-invariants elements. Hence, we have the following isomorphism, which is quite explicit.

Proposition 1.3 ([KoRe00], Prop. 1.2.)**.** *Given two finite-dimensional K-representations (γ, E_γ) and (τ, E_τ) with their associated complex homogeneous vector bundles $\mathbb{E}_\gamma$ resp. $\mathbb{E}_\tau$ over X. The space of G-equivariant differential operators acting from $C^\infty(X, \mathbb{E}_\gamma)$ to $C^\infty(X, \mathbb{E}_\tau)$ is isomorphic to $[\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(E_\gamma, E_\tau)]^K$:*

$$\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau) \cong [\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(E_\gamma, E_\tau)]^K$$

action induced by $(Z \otimes a)(f(g)) = a(Zf)(g)$, for $Z \in \mathcal{U}(\mathfrak{g}), a \in \mathrm{Hom}(E_\gamma, E_\tau)$ and $f \in C^\infty(X, \mathbb{E}_\gamma)$. □

Remark 1.4. (a) Notice that the above isomorphism respects the multiplication. In fact, consider an additional K-representation (δ, E_δ) and let $\{Z_i, i = 1, \ldots, n\}$ and $\{W_j, j = 1, \ldots, m\}$ in $\mathcal{U}(\mathfrak{g})$. Take two invariant differential operators of the form

$$D_1 = \sum_{i=1}^{n} Z_i \otimes a_i, \quad \text{and} \quad D_2 = \sum_{j=1}^{m} W_j \otimes b_j$$

with $a_i \in \mathrm{Hom}(E_\gamma, E_\tau)$ and $b_j \in \mathrm{Hom}(E_\tau, E_\delta)$ two constant matrices. Then for $g \in G$ and $f \in C^\infty(X, \mathbb{E}_\gamma)$:

$$\begin{aligned}(D_2 \circ D_1)(f(g)) = \sum_{j=1}^{m} b_j(W_j(D_1 f))(g) &= \sum_{j=1}^{m} b_j\Big(W_j \sum_{i=1}^{n} a_i(Z_i f)\Big)(g) \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} b_j \circ a_i(W_j Z_i f)(g).\end{aligned}$$

(b) When $(\gamma, E_\gamma) = (\tau, E_\tau) =: (*, E_*)$, then the vector space $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{End}(E_*)$ is actually an algebra $\mathcal{D}_G(\mathbb{E}_*, \mathbb{E}_*)$.

(c) When $(*, E_*)$ is the trivial one-dimensional representation, i.e. $E_* = \mathbb{C}$, one has an isomorphism of algebras

$$\mathcal{D}_G(\mathbb{C}, \mathbb{C}) \cong \mathcal{U}(\mathfrak{g})^K / \mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k},$$

where $\mathcal{D}_G(\mathbb{C}, \mathbb{C})$ coincides with the commutative algebra $\mathcal{D}(G/K)$ of left-invariant differential operators acting on smooth functions on $X = G/K$.
More generally, if $(*, E_*)$ is an irreducible representation, not necessarily trivial, one can prove (see [Olb95], Satz 2.4) that

$$\mathcal{D}_G(\mathbb{E}_*, \mathbb{E}_*) \cong \mathcal{U}(\mathfrak{g})^K / [\mathcal{U}(\mathfrak{g})\mathrm{Ker}(\tilde{*})]^K,$$

where $\mathrm{Ker}(\tilde{*})$ is the kernel of the dual of a K-representation of $*$.

Next, let us delve into another decomposition so that $\mathrm{Hom}(E_\gamma, E_\tau)$ turns into a right $\mathcal{U}(\mathfrak{k})$-module by $Ya := \tau(Y)a$, for $Y \in \mathcal{U}(\mathfrak{k})$ and $a \in \mathrm{Hom}(E_\gamma, E_\tau)$. In this case, we operate on the second term. Set I as the $\mathcal{U}(\mathfrak{g})$-submodule of $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)$ generated by the elements

$$WY \otimes a - W \otimes \tau(Y)a, \quad W \in \mathcal{U}(\mathfrak{g}).$$

In order to distinguish, between the $\mathcal{U}(\mathfrak{g})$-action on $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)$ by left and right translation in the first factor, for convenience, we rewrite the tensor product of the modules $\mathcal{U}(\mathfrak{g})$ and $\mathrm{Hom}(E_\gamma, E_\tau)$ over $\mathcal{U}(\mathfrak{k})$ by

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}),\tau} \mathrm{Hom}(E_\gamma, E_\tau) := (\mathcal{U}(\mathfrak{g}) \otimes \mathrm{Hom}(E_\gamma, E_\tau))/I$$

and

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}),\gamma} \mathrm{Hom}(E_\gamma, E_\tau) := (\mathcal{U}(\mathfrak{g}) \otimes \mathrm{Hom}(E_\gamma, E_\tau))/J.$$

Let $D = \sum_{i=1}^{n} Z_i \otimes a_i$ be an element in $[\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}),\gamma} \mathrm{Hom}(E_\gamma, E_\tau)]^K$ and consider the map

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\gamma \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\tau$$

so that D operates on $W \otimes u \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\gamma$ by

$$D(W \otimes u) = \Big(\sum_{i=1}^{n} Z_i \otimes a_i\Big)(W \otimes u) := \sum_{i=1}^{n} W Z_i \otimes a_i(u).$$

This leads us to prove the following statement.

Proposition 1.5. *The mapping*

$$[\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}),\gamma} \mathrm{Hom}(E_\gamma, E_\tau)]^K \xrightarrow{\iota} [\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}),\tau} \mathrm{Hom}(E_\gamma, E_\tau)]^K$$

defined by $\iota(\sum_{i=1}^n Z_i \otimes a_i) = \sum_{i=1}^n \mathrm{opp}(Z_i) \otimes a_i$ *is an isomorphism.*
In particular, $D \in [\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}),\gamma} \mathrm{Hom}(E_\gamma, E_\tau)]^K$ *operates on* $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\tau$ *by*

$$D(W \otimes v) = \Big(\sum_{i=1}^n Z_i \otimes a_i\Big)(W \otimes v) := \sum_{i=1}^n W \mathrm{opp}(Z_i) \otimes a_i(v)$$

for $W \in \mathcal{U}(\mathfrak{g})$ *and* $v \in E_\tau$*.*

Proof. The main point of this proof lies on the well-definedness of the isomorphism map ι. To do this, we will proceed by a directly algebraic approach. Let

$$\sum_{i=1}^n (Z_i Y_i \otimes a_i + Z_i \otimes a_i \gamma(Y_i)) \in [\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathrm{Hom}(E_\gamma, E_\tau)]^K$$

with $Z_i \in \mathcal{U}(\mathfrak{g}), Y_i \in \mathfrak{k}$ (or $\mathcal{U}(\mathfrak{k})$) and $a_i \in \mathrm{Hom}(E_\gamma, E_\tau), \forall i$. Note, that, here, the tensor product is over $\mathbb{C}$ and not $\{\mathcal{U}(\mathfrak{k}), \gamma\}$, otherwise the LHS would be equal to 0. Since $\sum_{i=1}^n Z_i \otimes a_i \mapsto \sum_{i=1}^n \mathrm{opp}(Z_i) \otimes a_i$ is K-equinvariant (also for the tensor product over $\mathbb{C}$), it is sufficient to prove that

$$\sum_{i=1}^n (\mathrm{opp}(Z_i Y_i) \otimes a_i + \mathrm{opp}(Z_i) \otimes a_i \gamma(Y_i)) = 0 \text{ in } [\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}),\tau} \mathrm{Hom}(E_\gamma, E_\tau)]^K.$$

By computation, we obtain

$$\begin{aligned}
& \sum_{i=1}^n (\mathrm{opp}(Z_i Y_i) \otimes a_i + \mathrm{opp}(Z_i) \otimes a_i \gamma(Y_i)) \\
= & \sum_{i=1}^n (-Y_i \mathrm{opp}(Z_i) \otimes a_i + \mathrm{opp}(Z_i) \otimes a_i \gamma(Y_i)) \\
= & \sum_{i=1}^n \Big(-p_K\big\{Y_i \mathrm{opp}(Z_i) \otimes a_i - \mathrm{opp}(Z_i) Y_i \otimes a_i + \mathrm{opp}(Z_i) \otimes \tau(Y_i) a_i \\
& -\mathrm{opp}(Z_i) \otimes a_i \gamma(Y_i)\big\} + p_K\{-\mathrm{opp}(Z_i) Y_i \otimes a_i + \mathrm{opp}(Z_i) \otimes \tau(Y_i) a_i\}\Big) \\
= & \; 0 + p_K\Big\{\sum_{i=1}^n (-\mathrm{opp}(Z_i) Y_i \otimes a_i + \mathrm{opp}(Z_i) \otimes \tau(Y_i) a_i)\Big\}, \qquad (1.1)
\end{aligned}$$

where p_K denotes the projection on the K-invariants. Now consider the space $\mathcal{C}_\tau$ defined by the element of the linear span I. $\mathcal{C}_\tau$ is an K-invariant subspace of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathrm{Hom}(E_\gamma, E_\tau)$. Thus, we have that $p_K(\mathcal{C}_\tau) \subset \mathcal{C}_\tau$ and therefore (1.1) $\in \mathcal{C}_\tau$. In conclusion, we proved that (1.1) $= 0$ in $[\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}),\tau} \mathrm{Hom}(E_\gamma, E_\tau)]^K$. □

Thus, by Prop. 1.5, we can regard $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ as a linear map

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\gamma \xrightarrow{D} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\tau. \qquad (1.2)$$

Both $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\gamma$ and $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\tau$ are finitely generated $\mathcal{U}(\mathfrak{g})$-modules.

1.2 Chevalley restriction and Harish-Chandra type homomorphism

In this section, we want to describe the vector space of G-invariant differential operators $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ in terms of symmetric algebra over $\mathfrak{p}$ and the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. This leads us to the definition of a general Harish-Chandra-homomorphism $\Phi_{\gamma,\tau}$, which embeeds $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ in the space of polynomials with values in a finite-dimensional space of $\mathrm{Hom}(E_\gamma, E_\tau)$.

Let us fix some notations related to the structure of the complex Lie algebra $\mathfrak{g}$ of G. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition of $\mathfrak{g}$ into an orthogonal (under the Cartan-Killing form B of $\mathfrak{g}$) direct product of the ± 1 eigenspaces of the Cartan involution.

The symmetric algebra $S(\mathfrak{g})$ generated by the vector space $\mathfrak{g}$ is build as follows. Let $T(\mathfrak{g}) := \bigoplus_{n=0}^{\infty} T^n(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (with convention that $T^0(\mathfrak{g}) = \mathbb{C}$) be the tensor algebra over the vector space $\mathfrak{g}$ and denote by $I(\mathfrak{g})$ the two-sided ideal generated by the commutators $Z \otimes Y - Y \otimes Z$, for $Z, Y \in \mathfrak{g}$. Set

$$S(\mathfrak{g}) := T(\mathfrak{g})/I(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} S^n(\mathfrak{g}),$$

where $S^n(\mathfrak{g})$ are the n-th symmetric power of $\mathfrak{g}$ ([Jac62], Chap.V.). There is a natural homomorphism between $S(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$. We define a linear (bijective) map *symm* : $S(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})$ by

$$symm(Z_1 \cdots Z_p) = \frac{1}{p!} \sum_{s \in S_p} Z_{s(1)} \times \cdots \times Z_{s(p)}, \quad \{Z_i, i = 1, \ldots, n\} \in \mathfrak{g},$$

which is equivariant under all automorphism of $\mathfrak{g}$, extended to $S(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$. In particular, under adjoint Lie algebra $\mathrm{Ad}(g), g \in G$. Here, S_p is the permuations group of the p letters. By Poincaré-Birkhoff-Witt's (short PBW) theorem ([Jac62], Thm. 3, p.159), this map *symm* is even an isomorphism and is called *symmetrization* map, ([Wal88] 0.4.2).

Note that by Wallach ([Wal88], 0.4.3), $\mathcal{U}(\mathfrak{k})$ can be identified with the associative subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by 1 and $\mathfrak{k}$. Thus, we have an injective canonical map of $\mathcal{U}(\mathfrak{k})$ into $\mathcal{U}(\mathfrak{g})$. Hence, due to the Cartan decomposition of $\mathfrak{g}$, we have $symm(S(\mathfrak{p})) \subset \mathcal{U}(\mathfrak{g})$ and a linear isomorphism

$$symm(S(\mathfrak{p})) \otimes \mathcal{U}(\mathfrak{k}) \cong \mathcal{U}(\mathfrak{g})$$

defined by $Z_1 \cdots Z_p \otimes Y \mapsto Z_1 \cdots Z_p Y$, for $Y \in \mathcal{U}(\mathfrak{k})$ and $Z_1 \cdots Z_p \in symm(S(\mathfrak{p}))$. By Wallach ([Wal88], 0.4.3) again, this means that $\mathcal{U}(\mathfrak{g})$ is the free module (resp. free $\mathcal{U}(\mathfrak{k})$-module) on the generators $symm(S(\mathfrak{p}))$ as a $\mathcal{U}(\mathfrak{k})$ module under left multiplication (resp. generated by $symm(S(\mathfrak{p}))$ under the right multiplication by $\mathcal{U}(\mathfrak{k})$). This observation, together with Prop. 1.3, breed to the following well-defined isomorphism.

Proposition 1.6. *The vector space of all invariant differential operators $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ is isomorphic to $[S(\mathfrak{p}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)]^K$.* □

Next, notice that the vector space $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ is a filtered space by degree and with $Gr(\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau))$ we denote its corresponding graded space. Similarly for $\mathcal{U}(\mathfrak{g})$, if $\mathcal{U}^n(\mathfrak{g}) \subset \mathcal{U}^{n+1}(\mathfrak{g})$ is canonical filtration of $\mathcal{U}(\mathfrak{g})$, then its associated graded algebra is defined by

$$Gr(\mathcal{U}(\mathfrak{g})) = \bigoplus_{n=0}^{\infty} \mathcal{U}^n(\mathfrak{g})/\mathcal{U}^{n-1}(\mathfrak{g})$$

([Wal88], 0.4.2). By PBW's theorem, we know that $Gr(\mathcal{U}(\mathfrak{g})) \cong S(\mathfrak{g})$, thus by Prop. 1.6 and Olbrich's result ([Olb95], Folgerung 2.5), we can easily deduce the following algebra isomorphism

$$Gr(\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)) \cong [S(\mathfrak{p}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)]^K. \tag{1.3}$$

Now, set $\mathrm{Pol}(\mathfrak{g})$ as the space of complex valued polynomial functions on the real vector space $\mathfrak{g}$. By Wallach ([Wal88], 3.2.2), we know that

$$S(\mathfrak{g}) \cong \mathrm{Pol}(\mathfrak{g}^*) \tag{1.4}$$

is isomorphic and we can identify $S(\mathfrak{g})$ and $\mathrm{Pol}(\mathfrak{g}^*)$ as $\mathfrak{g}$-modules. Thus, by replacing $\mathfrak{g}$ by $\mathfrak{p}$, we have that $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ is isomorphic to the space of all polynomials in $\mathfrak{p}^*$ with values in $\mathrm{Hom}(E_\gamma, E_\tau)$:

$$\begin{aligned}
\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau) \overset{\text{Prop.1.3}}{\cong} [\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(E_\gamma, E_\tau)]^K &\overset{\text{Prop.1.6}}{\cong} [S(\mathfrak{p}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)]^K \\
&\overset{(1.4)}{\cong} [\mathrm{Pol}(\mathfrak{p}^*) \otimes \mathrm{Hom}(E_\gamma, E_\tau)]^K \\
&\cong \mathrm{Pol}(\mathfrak{p}^*, \mathrm{Hom}(E_\gamma, E_\tau))^K.
\end{aligned}$$

Note that K acts on $\mathrm{Pol}(\mathfrak{p}^*)$ by $kp(Y) = p(\mathrm{Ad}^*(k^{-1})Y)$, for $k \in K, Y \in \mathfrak{p}^*$ and $p \in \mathrm{Pol}(\mathfrak{p}^*)$, ([Wal88], 3.1.1.). Here Ad^* denotes the dual or coadjoint adjoint representation.

Choose now a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$, called Cartan subspace, and write by $\mathfrak{a}^*$ its (real) dual. The dimension of any $\mathfrak{a}$ is a constant called the (real) rank of G. Any two Cartan subspaces are conjugate under the adjoint representation $\mathrm{Ad}(K)$ of K.

Consider $W_A = M'/M$ the analytic Weyl group, where

$$M' := N_K(\mathfrak{a}) = \{k \in K \mid \mathrm{Ad}(k) \subset \mathfrak{a}\} \subset K$$

is the normalizer of $\mathfrak{a}$ in K, and $M := Z_K(\mathfrak{a}) = \{k \in K \mid \mathrm{Ad}(k)H = H, \forall H \in \mathfrak{a}\}$ is the centralizer of $\mathfrak{a}$ in K.

The restriction of the exponential map of G to $\mathfrak{a}$ is an analytic diffeomorphism onto the abelian subgroup $A := \exp(\mathfrak{a})$. The inverse diffeomorphism is denoted by log. The action on $\mathfrak{a}$ of the Weyl group W_A induces actions of W_A on $\mathfrak{a}^*$ by duality, on A via the exponential map, and on $\mathfrak{a}_\mathbb{C}$ by complex linearity. We have

$$\mathrm{Ad}(K)\mathfrak{a} = \mathfrak{p}, \tag{1.5}$$

i.e. every point in $\mathfrak{p}$ is conjugate to some element H of $\mathfrak{a}$. Thus, since $\mathfrak{p}$ carries an $\mathrm{Ad}(K)$-invariant Euclidean scalar product, we therefore can view $\mathfrak{a}^*$ as a subspace of $\mathfrak{p}^*$:

$$\mathfrak{a}^* \cong \mathfrak{a} \subset \mathfrak{p} \cong \mathfrak{p}^*.$$

We then obtain a restriction map

$$res : \mathrm{Pol}(\mathfrak{p}^*, \mathrm{Hom}(E_\gamma, E_\tau))^K \longrightarrow \mathrm{Pol}(\mathfrak{a}^*, \mathrm{Hom}(E_\gamma, E_\tau))^{M'}.$$

By (1.5) this map is injective and we get

$$\mathrm{Pol}(\mathfrak{a}^*, \mathrm{Hom}(E_\gamma, E_\tau))^{M'} \cong \mathrm{Pol}(\mathfrak{a}^*, \mathrm{Hom}_M(E_\gamma, E_\tau))^{W_A},$$

where W_A acts on $\mathrm{Pol}(\mathfrak{a}^*)$ by $wp(H) = p(w^{-1}H)$ for $w \in W_A, H \in \mathfrak{a}^*$ and $p \in \mathrm{Pol}(\mathfrak{a}^*)$ and on $\mathrm{Hom}_M(E_\gamma, E_\tau)$ by $wa := \tau(m_w)a\gamma(m_w^{-1})$ for $a \in \mathrm{Hom}_M(E_\gamma, E_\tau), m_w \in M'$. So we get an embedding

$$res : \mathcal{D}_G(\mathbb{E}_\gamma, E_\tau) \cong \mathrm{Pol}(\mathfrak{p}^*, \mathrm{Hom}(E_\gamma, E_\tau))^K \hookrightarrow \mathrm{Pol}(\mathfrak{a}^*, \mathrm{Hom}_M(E_\gamma, E_\tau))^{W_A} \tag{1.6}$$

with respect to a graded multiplication. In some cases, the image of res is known, but it is difficult to compute in general. The complete proof of the followng proposition, can be found for example in ([Hel20], Thm. 6.10, p.430) or in ([Wal88], 3.1.2).

Proposition 1.7 (Chevalley restriction Theorem). *Let $\mathbb{E}_\gamma = \mathbb{E}_\tau = \mathbb{C}$ be trivial one dimensional vector bundles. Then, we have an algebra isomorphism between* $\mathrm{Pol}(\mathfrak{p}^*)^K$ *and* $\mathrm{Pol}(\mathfrak{a}^*)^{W_A}$. □

Remark 1.8. (i) This theorem is due to Chevally (unpublished, [HC58], I, Sect. 3).

(ii) If for instance $E_\gamma = E_\tau =: E$ is not irreducible under M', then the analog of Prop. 1.7 can not be true even on the level of constant polynomials. In fact, we have that $\mathrm{End}_K(E)$ is one-dimensional and $\mathrm{End}_{M'}(E)$ is higher dimensional.

However (1.6) gives a lot of information on $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$. To see this, we need to modify (1.6) for an embedding of algebras and decompose the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Note, that by considering two complex finite-dimensional vector spaces E_γ and E_τ, the map

$$\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau) \longrightarrow \mathrm{Pol}(\mathfrak{a}^*, \mathrm{Hom}_M(E_\gamma, E_\tau))$$

is no longer an homomorphism of algebra but of associated algebroids, which respect the multiplication.

Consider $\Delta_\mathfrak{a}^+$ as the set of non-vanishing restricted simple positive roots of the pair $\mathfrak{g}$ with respect to $\mathfrak{a}$. Then, the Lie algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta_\mathfrak{a}^+} \mathfrak{g}_\alpha$$

with $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, \forall H \in \mathfrak{a}\}$ the root subspace of a root α with multiplicity $m_\alpha := \dim(\mathfrak{g}_\alpha)$. The subspace $\mathfrak{n} := \bigoplus_{\alpha \in \Delta_\mathfrak{a}^+} \mathfrak{g}_\alpha$ is a nilpotent subalgebra of $\mathfrak{g}$. Write $\rho \in \mathfrak{a}^+$ the half sum of all positive roots $\alpha \in \Delta_\mathfrak{a}^+$ in $\mathfrak{a}^+$, counted with their multiplicities m_α:

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta_\mathfrak{a}^+} \alpha m_\alpha. \tag{1.7}$$

The Iwasawa decomposition ([Kna02], Chap. VI.4):

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k} \in NAK = G$$

provides, together with PBW's theorem, the decomposition of the universal enveloping algebra on linear subspaces ([Wal88], 3.2.1):

$$\mathcal{U}(\mathfrak{g}) = \mathfrak{n}\mathcal{U}(\mathfrak{g}) \oplus \mathcal{U}(\mathfrak{a}) \cdot \mathcal{U}(\mathfrak{k}).$$

Let p be the projection on the second summands, then we identify

$$\mathcal{U}(\mathfrak{a}) \cdot \mathcal{U}(\mathfrak{k}) \cong \mathcal{U}(\mathfrak{a}) \otimes \mathcal{U}(\mathfrak{k})$$

as linear space. Consider the corresponding algebra of the RHS of the above tensor product, then

$$\begin{aligned} p_\gamma : \mathcal{U}(\mathfrak{g}) \longrightarrow \quad & \mathcal{U}(\mathfrak{a}) \otimes \mathcal{U}(\mathfrak{k}) & \longrightarrow \mathcal{U}(\mathfrak{a}) \otimes \mathrm{End}(E_\gamma) \\ Z \overset{p}{\mapsto} \quad & \textstyle\sum_{i=1}^n H_i \otimes Y_i & \mapsto \sum_{i=1}^n H_i \otimes \gamma(\mathrm{opp}(Y_i)). \end{aligned}$$

Now, if we consider two K-representations, we obtain the linear map:

$$\begin{aligned} p_{\gamma,\tau} : \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(E_\gamma, E_\tau) \longrightarrow \quad & \mathcal{U}(\mathfrak{a}) \otimes \mathrm{Hom}(E_\gamma, E_\tau) & \cong \mathrm{Pol}(\mathfrak{a}^*_{\mathbb{C}}, \mathrm{Hom}(E_\gamma, E_\tau)) \\ Z \otimes a \mapsto \quad & (\textstyle\sum_{i=1}^n H_i \otimes Y_i) \otimes a & \mapsto \sum_{i=1}^n H_i \otimes a \circ \gamma(\mathrm{opp}(Y_i)), \end{aligned}$$

where $H \in \mathcal{U}(\mathfrak{a})$, $Y \in \mathcal{U}(\mathfrak{k})$ and $a \in \mathrm{Hom}(E_\gamma, E_\tau)$. Thus, we have

$$p_{\gamma,\tau}(Z \otimes a) = (1 \otimes a) \circ p_\gamma(Z).$$

By restricting it to $[\mathcal{U}(\mathfrak{a}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)]^K$, we obtain the desired homomorphism.

Proposition 1.9. *The restriction map* $\widetilde{\Phi}_{\gamma,\tau} := p_{\gamma,\tau}|_{[\mathcal{U}(\mathfrak{a}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)]^K}$ *is a homomorphism of algebroids and it is injective.*

Proof. For the injectivity, one checks that on the 'graded level', it coincides with res (1.6). In fact, if $D \in [\mathcal{U}^n(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(E_\gamma, E_\tau)]^K$ then, this implies that

$$p_{\gamma,\tau}(D) - res(D) \in \mathcal{U}^{n-1}(\mathfrak{a}) \otimes \mathrm{Hom}_M(E_\gamma, E_\tau).$$

Hence, form the injectivity of the map res, the assertion follows.

For the homomorphism of algebroids, since $\mathfrak{n} \subset \mathfrak{n} \oplus \mathfrak{a}$ is an ideal and $\mathfrak{n} \oplus \mathfrak{a} \to \mathfrak{a}$ is a Lie algebra homomorphism, we have that

$$p_0 : \mathcal{U}(\mathfrak{n} + \mathfrak{a}) \longrightarrow \mathcal{U}(\mathfrak{a})$$

is a linear algebra homomorphism. Note that $\mathcal{U}(\mathfrak{n}+\mathfrak{a}) \cong \mathcal{U}(\mathfrak{n})\mathcal{U}(\mathfrak{a}) = \mathcal{U}(\mathfrak{a}) \oplus \mathfrak{n}\mathcal{U}(\mathfrak{n}+\mathfrak{a})$ and $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(E_\gamma, E_\tau) \cong \mathcal{U}(\mathfrak{n} + \mathfrak{a}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)$ as vector space. Thus

$$p_{\gamma,\tau} : \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(E_\gamma, E_\tau) \hookrightarrow \mathcal{U}(\mathfrak{n} + \mathfrak{a}) \otimes \mathrm{Hom}(E_\gamma, E_\tau) \overset{p_0}{\to} \mathcal{U}(\mathfrak{a}) \otimes \mathrm{Hom}(E_\gamma, E_\tau).$$

Consider

$$\begin{aligned} D_1 &= \sum_i Y_i \otimes a_i \in \mathcal{U}(\mathfrak{n}+\mathfrak{a}) \otimes \mathrm{Hom}(E_\gamma, E_\tau) \\ D_2 &= \sum_j Z_j \otimes b_j \in \mathcal{U}(\mathfrak{n}+\mathfrak{a}) \otimes \mathrm{Hom}(E_\tau, E_\delta) \end{aligned}$$

with $Y_i, Z_j \in \mathcal{U}(\mathfrak{n}+\mathfrak{a})$ and $a_i \in \mathrm{Hom}(E_\gamma, E_\tau)$ respectively $b_j \in \mathrm{Hom}(E_\tau, E_\delta)$. Here (δ, E_δ) is a, not necessary irreducible, K-representation. Then, $p_{\gamma,\tau}(D_1) = \sum_i p_0(Y_i) \otimes a_i$ and $p_{\tau,\delta}(D_2) = \sum_j p_0(Z_j) \otimes b_j$. Hence

$$p_{\gamma,\delta}(D_2 \circ D_1) = \sum_{i,j} p_0(Z_j Y_i) \otimes b_j a_i = \sum_{i,j} p_0(Z_j) p_0(Y_i) \otimes b_j a_i = p_{\tau,\delta}(D_2) \circ p_{\gamma,\tau}(D_1). \ \square$$

Definition 1.10. *Let $p_{\gamma,\tau}$ as above and $v_\rho : \mathcal{U}(\mathfrak{a}) \to \mathcal{U}(\mathfrak{a}), H \mapsto H + \rho(H), H \in \mathfrak{a}$ be an automorphism. The Harish-Chandra homomorphism is an injective algebroid-homomorphism*

$$\Phi_{\gamma,\tau} : \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau) \longrightarrow \mathcal{U}(\mathfrak{a}) \otimes \mathrm{Hom}_M(E_\gamma, E_\tau)$$

defined by $\Phi_{\gamma,\tau} = v_\rho \circ p_{\gamma,\tau}$.

Remark 1.11. (i) If $E_\gamma = E_\tau$ is irreducible, then by Olbrich ([Olb95], Satz 2.8), we even get an isomorphism.

(iii) Furthermore, with the help of Φ_γ, it will be easy to show the necessity and sufficiently criterion for the commutativity of the algebra $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\gamma)$. For more details, we refer to Sect. 2.2. in [Olb95].

1.3 Integrability conditions and the conjecture

Given a linear invariant differential operator $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$. We wish that the equation $Df = g$ is solvable in $C^\infty(X, \mathbb{E}_\gamma)$, if and only, if $\widetilde{D} g = 0$, for all $g \in C^\infty(X, \mathbb{E}_\tau)$ given and for a certain differential operator $\widetilde{D}$. To do so, we need to construct, first, a candidate for $\widetilde{D}$. Ideally, we want to find a homogeneous vector bundle $\mathbb{F} \to X$ and a certain $\widetilde{D} \in \mathcal{D}_G(\mathbb{E}_\tau, \mathbb{F})$, such that, in particular, $\widetilde{D} \circ D = 0$.

Since D is G-invariant and by (1.2), we have that $\mathrm{Ker}(D)$ is a $\mathcal{U}(\mathfrak{g})$-submodule and a $(\mathfrak{g}, K)$-submodule of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\gamma$. Since $\mathcal{U}(\mathfrak{g})$ is Noetherian ([Wal88], 0.6.1.) and $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\gamma$ is finitely generated, there exists a finite-dimensional K-invariant generating subspace E_0 of $\mathrm{Ker}(D)$. It carries a representation $\tau_0 : K \to GL(E_0)$. Consider the natural embedding

$$i : E_0 \hookrightarrow \mathrm{Ker}(D) \hookrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\gamma$$

and recall the following isomoprhism

$$\mathrm{Hom}(E_\gamma, E_\tau) \cong E_\tau \otimes E_{\tilde{\gamma}}. \tag{1.8}$$

Note that

$$\begin{aligned} i \in \mathrm{Hom}_K(E_0, \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\gamma) &\overset{(1.8)}{\cong} [\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}),\gamma} E_\gamma \otimes E_{\tilde{0}}]^K \\ &\overset{(1.8)}{\cong} [\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}),\gamma} \mathrm{Hom}(E_0, E_\gamma)]^K \\ &\overset{\text{Prop. } 1.3}{\cong} \mathcal{D}_G(\mathbb{E}_0, \mathbb{E}_\gamma). \end{aligned} \tag{1.9}$$

Next, we consider the induced $\mathcal{U}(\mathfrak{g})$-module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_0$ and the linear map

$$D_0 : \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_0 \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\gamma$$

defined by $D_0(Z \otimes w) = Z(w)$, for $w \in E_0 \subset \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_\gamma$ and $Z \in \mathcal{U}(\mathfrak{g})$. Moreover, D_0 is surjective on $\mathrm{Ker}(D)$ and D_0 can be viewed as an element of $\mathcal{D}_G(\mathbb{E}_0, \mathbb{E}_\gamma)$, via (1.9). In particular, we have that

$$D \circ D_0 = 0. \tag{1.10}$$

However, we want the opposite direction, thus, we need to dualize. Write by $(\tilde{\gamma}, E_{\tilde{\gamma}})$ (resp. $(\tilde{\tau}, E_{\tilde{\tau}})$) the dual of the representation (γ, E_γ) (resp. (τ, E_τ)). The dual or 'adjoint' of D is the transposed invariant differential operator

$$D^t : C_c^{-\infty}(X, \mathbb{E}_{\tilde{\tau}}) \longrightarrow C_c^{-\infty}(X, \mathbb{E}_{\tilde{\gamma}})$$

defined by the natural pairing

$$\int_{G/K} \langle D^t \varphi(g), h(g) \rangle_\gamma \, dg := \int_{G/K} \langle \varphi(g), Dh(g) \rangle_\tau \, dg \tag{1.11}$$

with $\varphi \in C_c^{-\infty}(X, \mathbb{E}_{\tilde{\tau}}), h \in C^\infty(X, \mathbb{E}_\gamma)$ and where the pairing $\langle \cdot, \cdot \rangle_*$ denotes the pairing in $E_*, * = \gamma, \tau$. Note that the space of compactly supported distributions

$$C_c^{-\infty}(X, \mathbb{E}_{\tilde{*}}) := (C^\infty(X, \mathbb{E}_*))'$$

is the topological linear dual of $C^\infty(X, \mathbb{E}_*)$. So, we have that

$$D^t : \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_{\tilde{\tau}} \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_{\tilde{\gamma}}.$$

Choose $F_0 \subset \mathrm{Ker}(D^t) \subset \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_{\tilde{\tau}}$ a finite-dimensional K-invariant generating subspace of $\mathrm{Ker}(D^t)$. By the above, we obtain that $D_0 \in \mathcal{D}_G(\mathbb{F}_0, \mathbb{E}_{\tilde{\tau}})$ such that $D^t \circ D_0 = 0$. Now, set

$$\mathbb{F} := \mathbb{F}_0^* \text{ (dual of } \mathbb{F}_0) \qquad \text{and} \qquad \widetilde{D} := D_0^t \in \mathcal{D}_G(\mathbb{E}_\tau, \mathbb{F}).$$

Then, by (1.10), this implies that $(\widetilde{D} \circ D)^t = D^t \circ \widetilde{D}^t = D^t \circ D_0 = 0$. Thus, $\widetilde{D} \circ D = 0$ on $C^\infty(X, \mathbb{E}_\gamma)$, i.e. $\mathrm{Im}(D) \subset \mathrm{Ker}(\widetilde{D})$. This leads us to state the following problem.

Conjecture 1. *Let $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ and $\widetilde{D} \in \mathcal{D}_G(\mathbb{E}_\tau, \mathbb{F})$ as above.*
Then, the differential equation $Df = g$ is solvable in $C^\infty(X, \mathbb{E}_\gamma)$ for given $g \in C^\infty(X, \mathbb{E}_\tau)$, if and only, if $\widetilde{D}g = 0$ in $C^\infty(X, \mathbb{E}_\tau)$. In other words, the sequence

$$C^\infty(X, \mathbb{E}_\gamma) \xrightarrow{D} \underset{exact}{C^\infty(X, \mathbb{E}_\tau)} \xrightarrow{\widetilde{D}} C^\infty(X, \mathbb{F}) \tag{1.12}$$

is exact in the middle, i.e. $\mathrm{Im}(D) = \mathrm{Ker}(\widetilde{D})$.

Note that one implication is obvious, since $\widetilde{D} \circ D = 0$ implies that $\widetilde{D}g = \widetilde{D} \circ Df = 0$. In the coming chapter, we will state and prove an important tool for the conjecture, namely the Fourier transform and the Paley-Wiener-(Schwartz) theorems for sections over homogeneous vector bundles.

Chapter 2

Fourier transforms and the Paley-Wiener theorems

In the Euclidean case $\mathbb{R}^n$, it is well-known that the Fourier transform of an integrable measurable function on $\mathbb{R}^n$ is defined by

$$\hat{f}(\lambda) = \int_{\mathbb{R}^n} f(x) e^{i\langle \lambda, x\rangle}\, dx \in \mathbb{C}, \quad \text{for } \lambda \in \mathbb{R}^n \tag{2.1}$$

whenever the integral converges. Note that $\hat{f}(\lambda)$ is defined for $\lambda \in \mathbb{C}^n$, not only for $\mathbb{R}^n$, and $\hat{f}$ is holomorphic on $\mathbb{C}^n$. We will, however, restrict to those functions which satisfy certain properties. This leads us to one of the central theorems of harmonic analysis on $\mathbb{R}^n$, the so-called Paley-Wiener theorem, named after the two mathematicians Raymond Paley and Norbert Wiener. It describes the image of the Fourier transform of the space $C_c^\infty(\mathbb{R}^n)$ of smooth functions with compact support as the space of entire functions on $\mathbb{C}^n$ with the following growth condition. There exists $r > 0$, for each $N \in \mathbb{N}$ and one can find a positive constant $C_{N,r}$, depending on N and r, such that

$$|\hat{f}(\lambda)| \leq C_{N,r}(1 + |\lambda|^2)^{-N} e^{r|\mathrm{Im}(\lambda)|},$$

where $\mathrm{Im}(\lambda)$ is the imaginary part of $\lambda \in \mathbb{C}^n$. The theorem has a counterpart, known as Paley-Wiener-Schwartz theorem. Here, the smooth functions are replaced by distributions and the growth condition above by a 'slow' growth ([Hör83], Thm. 7.3.1). Regarding the inverse Euclidean Fourier transform of a smooth Schwartz-function $f \in \mathcal{S}(\mathbb{R}^n)$, it is given by

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\lambda) e^{-i\langle \lambda, x\rangle} \frac{d\lambda}{(2\pi)^n}, \quad \text{for } x \in \mathbb{R}^n,$$

where $\frac{d\lambda}{(2\pi)^n}$ is known as the Plancherel measure.

Now to generalize this theorem for general Lie groups G and furthermore for smooth manifold carrying some symmetries, we need first to recall some basic and abstract results from representation theory as well as harmonic analysis. Note that one can reformulate (2.1) in terms of *irreducible unitary representations* of $(\mathbb{R}^n, +)$:

$$\pi_\lambda : \mathbb{R}^n \longrightarrow U(1)$$

given by $\pi_\lambda(y) := e^{i\langle \lambda, y\rangle} \in U(1)$, the unitary Lie group isomorphic to the circle group. Thus

$$\hat{f}(\lambda) = \int_{\mathbb{R}^n} f(x) \pi_\lambda(x)\, dx \in \mathrm{End}(\mathbb{C}) \cong \mathbb{C}.$$

For a unimodular Lie group G, with bi-invariant Haar measure dg, we expect:

$$\hat{f}(\pi) = \pi(f) := \int_G f(g)\pi(g)\, dg \in \mathrm{End}(E_\pi),$$

where f is an integrable measurable function on G and (π, E_π) belongs to the set $\hat{G}$ of all equivalence classes of irreducible unitary representations of G. There is an *equivariance or intertwining* property:

$$(\widehat{l_x r_y f})(\pi) = \int_G f(x^{-1}gy)\pi(g)\, dg = \int_G f(g)\pi(xgy^{-1})\, dg = \pi(x)\hat{f}(\pi)\pi(y)^{-1}, \quad x, y \in G.$$

It will be important to find a nice family of representations of $\hat{G}$, in terms of a good parametrization, to invert the Fourier transform as well as define the Fourier transform for $C_c^\infty(G)$ and then establish a Paley-Wiener theorem.

The Paley-Wiener theorem is known in many interesting cases. For example, the case of Riemannian symmetric spaces of non-compact type $X = G/K$ was considered by Helgason [Hel66] and Gangolli [Gan71]. They proved a Paley-Wiener theorem for compactly supported K-invariant smooth functions and Helgason [Hel73] even showed it for general compactly supported smooth functions on X.
There is also a Paley-Wiener theorem for K-finite compactly supported smooth functions on a real reductive Lie group G of Harish-Chandra class due to Arthur [Art83] and Delorme [Del05], reformulated in terms of the so-called Arthur-Campolli and Delorme conditions respectively.
Furthermore, later van den Ban and Souaifi [vdBS14] proved, without using the proof nor validity of any associated Paley-Wiener theorems of Arthur or Delorme, that the two compatibility conditions are equivalent.

The main task in this chapter is to define the Fourier transform for sections over homogeneous vector bundles and use Delorme's Paley-Wiener theorem to inspect the image of the Fourier transform as well as of invariant differential operators introduced in the previous chapter. This part of the exposition will later, in Chapter 3, applied on specific examples.

In order to be more specific on the contents of this chapter, we start in Section 2.1, in particular Subsection 2.1.1, by briefly reminding Delorme's Paley-Wiener theorem (Thm. 2.7) in the setting of van den Ban and Souaifi [vdBS14]. Then, in Subsection 2.1.2, we prepare our analysis for our purpose by describing explicitly Delorme's intertwining conditions for special cases and state the Paley-Wiener theorem (Thm. 2.31) for sections.

Moreover, in Section 2.2, we also prove a version of Plancherel theorem (Thm. 2.33) for sections over homogeneous vector bundles for smooth compactly supported functions on X which behave finitely under both left and right translations by K.

Later, in Section 2.3, we do the same study for (Schwartz) distributions and establish a topological Paley-Wiener-Schwartz theorem for sections (Thm. 2.40). Here, the growth properties are replaced by similar exponential conditions of 'slow' growth, which is alike to its Euclidean analogue, whereas the compatibility conditions remain the same. We used some recent results of van den Ban and Schlichtkrull [vdBS06] as well as the Plancherel theorem exposed in the above section.

Finally, in Section 2.4, we use the Paley-Wiener-Schwartz theorem to compute the Fourier transform of invariant differential operators acting on sections (Thm. 2.46).

2.1 Fourier transform and Delorme's Paley-Wiener theorem in three different levels

In this part, we state Delorme's Paley-Wiener theorem ([Del05], Thm. 2) under certain compatibility conditions, in three levels, as follows:

(Level 1): Refers to the original Delorme's Paley-Wiener theorem reformulated in the framework of van den Ban and Souaifi [vdBS14], (Subsection 2.1.1).

(Level 2): Corresponds to the desired Delorme's Paley-Wiener theorem for our situation, that is for sections over complex homogeneous vector bundles, where the right K-type comes from the vector bundle (Subsection 2.1.2).

(Level 3): By considering an additional K-type (coming from the left) in contrast to (Level 2), we deal with a Paley-Wiener theorem with two irreducible K-representations (Subsection 2.1.2).

In this way, it will be much easier to manage the intertwining conditions. We will show, in Subsection 2.1.3, that the corresponding conditions at the three levels are equivalent. Roughly speaking, it will allow us to *easily* switch and move from one level to the other (Thm. 2.28).

2.1.1 On Delorme's Paley-Wiener Theorem

Under the previous notation, fix a minimal parabolic subgroup P of G with split component A ([Kna02], Chap. VII, Sect. 7). It has a Langlands decomposition of the form $P = MAN$, where $N := \exp(\mathfrak{n})$ is a nilpotent Lie group. Let $(\sigma, E_\sigma) \in \hat{M}$ be a finite-dimensional irreducible representation of M and $\lambda \in \mathfrak{a}^*_{\mathbb{C}} \cong \mathbb{C}^n$.

For fixed $(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}^*_{\mathbb{C}}$, let $(\sigma_\lambda, E_{\sigma,\lambda})$ be the representation of P on the vector space $E_{\sigma,\lambda} = E_\sigma$ such that

$$\sigma_\lambda(man) = a^{\lambda+\rho}\sigma(m) \in \mathrm{End}(E_{\sigma,\lambda})$$

for $m \in M$, $a \in A$ and $n \in N$. We use the notation a^λ for $e^{\lambda \log(a)}$. Then, the space

$$H^{\sigma,\lambda}_\infty := \{f : G \overset{C^\infty}{\to} E_{\sigma,\lambda} \mid f(gman) = a^{-(\lambda+\rho)}\sigma(m)^{-1}(f(g))\} \cong C^\infty(G/P, \mathbb{E}_{\sigma,\lambda})$$

together with the left regular action $\pi_{\sigma,\lambda}$ of G

$$(\pi_{\sigma,\lambda}(g)f)(x) := f(g^{-1}x) = (l_g f)(x), \quad g, x \in G, f \in H^{\sigma,\lambda}_\infty$$

is the space of smooth vectors of the principal series representations of G induced from the P-representation σ_λ on $E_{\sigma,\lambda}$ ([Kna86], p. 168).
The restriction map from continuous functions in $H^{\sigma,\lambda}_\infty$ to functions on K is injective by the Iwasawa decomposition KAN of G

$$g = \kappa(g)e^{a(g)}n(g) \in G.$$

In particular, for $f \in H_\infty^{\sigma,\lambda}$ we have

$$f(g) = f(\kappa(g)e^{a(g)}n(g)) = a(g)^{-(\lambda+\rho)}(f(\kappa(g))).$$

This yields, the so-called *compact picture* of $H_\infty^{\sigma,\lambda}$ ([Kna86], p.168). It has the advantage that the representation space

$$H_\infty^\sigma := \{\varphi : K \overset{C^\infty}{\to} E_\sigma \mid \varphi(km) = \sigma(m)^{-1}\varphi(k),\ k \in K, m \in M\} \cong C^\infty(K/M, \mathbb{E}_\sigma) \tag{2.2}$$

does not depend on λ. Here, H_∞^σ is equipped with the usual Fréchet topology. From time to time, we need the L^2-norm. In the compact picture, the action of all elements $g \in G$, which are not in K, is slightly more involved, since we need to commute them with the argument $k \in K$:

$$(\pi_{\sigma,\lambda}(g)\varphi)(k) = a(g^{-1}k)^{-(\lambda+\rho)}\varphi(\kappa(g^{-1}k)), \quad \varphi \in H_\infty^\sigma. \tag{2.3}$$

Fourier transform for G in (Level 1)

Let

$$C_c^\infty(G) = \bigcup_{r>0} C_r^\infty(G) := \bigcup_{r>0}\{f \in C^\infty(G) \mid supp(f) \in \overline{B}_r(o)\}$$

be the space of compactly supported smooth complex functions on G, where

$$\overline{B}_r(o) := \{g \in G \mid dist_X(gK, o) \leq r\} \subset G$$

denotes the preimage of the closed ball of radius r and center $o = eK$ in X under the projection $G \to X$. Here, $dist_X$ means the Riemaniann distance on X and e is the neutral element of G. We equip $C_r^\infty(G)$ with the usual Fréchet topology, thus $C_c^\infty(G)$ is a LF-space [Tre67].

Given $\sigma \in \hat{M}$, let us consider the map

$$\pi_{\sigma,\cdot} : G \to (\mathfrak{a}_\mathbb{C}^* \to \mathrm{End}(H_\infty^\sigma)), g \mapsto (\lambda \mapsto \pi_{\sigma,\lambda}(g)).$$

For $g \in \overline{B}_r(o)$ and $r > 0$, we know by Delorme ([Del05], (1.25)), the relation

$$||\pi_{\sigma,\lambda}(g)||_{\mathrm{op}} \leq e^{r|\mathrm{Re}(\lambda)|},$$

where $||\cdot||_{\mathrm{op}}$ is the operator norm with respect to the L^2-norm on H_∞^σ and $\mathrm{Re}(\lambda)$ denotes the real part of $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

Definition 2.1 (Fourier transform for G in (Level 1))**.** *Fix $(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}_\mathbb{C}^*$, we define the Fourier transform of $f \in C_c^\infty(G)$ by the operator*

$$\mathcal{F}_{\sigma,\lambda}(f) := \pi_{\sigma,\lambda}(f) = \int_G f(g)\pi_{\sigma,\lambda}(g)\, dg \in \mathrm{End}(H_\infty^\sigma).$$

We denote by $\mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*)$ the space of holomorphic functions in $\mathfrak{a}_\mathbb{C}^*$ and by $\mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, \mathrm{End}(H_\infty^\sigma))$ the space of maps $\hat{M} \times \mathfrak{a}_\mathbb{C}^* \ni (\sigma, \lambda) \mapsto \phi(\sigma, \lambda) \in \mathrm{End}(H_\infty^\sigma)$ such that

$(1.i)$ for $\varphi \in H_\infty^\sigma$, the function $\lambda \mapsto \phi(\sigma, \lambda)\varphi \in H_\infty^\sigma$ is holomorphic in $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

From ([Del05], Lem. 10 (ii)), we deduce the following statement.

Proposition 2.2. *The application $f \mapsto \mathcal{F}_{\sigma,\lambda}(f)$ is a linear map from $C_c^\infty(G)$ into $\prod_{\sigma\in\hat{M}} \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}(H_\infty^\sigma))$.*

Proof. Let us fix $\sigma \in \hat{M}$ and $\varphi \in H_\infty^\sigma$. To show that the condition (1.i) for $f \in C_c^\infty(G)$ is holomorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, it is sufficient to prove that the map in (1.i) is weakly holomorphic, due Osgood's theorem (e.g. [FrGr02], Thm. 4.3.). This means that for a vector distribution $T \in H_{-\infty}^{\tilde{\sigma}} := (H_\infty^\sigma)'$ on K, we need to show that

$$\lambda \mapsto \langle T, \pi_{\sigma,\lambda}(f)\varphi\rangle_{H_\infty^\sigma} \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*).$$

Since $a(g^{-1}k)^{-(\lambda+\rho)}$ is an entire function on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we have that

$$(\pi_{\sigma,\lambda}(g)\varphi)(k) \overset{(2.3)}{=} a(g^{-1}k)^{-(\lambda+\rho)}\varphi(\kappa(g^{-1}k)) \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*)$$

for fixed $k \in K$ and $g \in G$. By taking the integral on K, it still remains holomorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Furthermore by Wallach ([Wal88], Lem. 3.8.1), if U is a compact set of $\mathfrak{a}_{\mathbb{C}}^*$, then there exists a positive constant C_U depending on U such that

$$||\pi_{\sigma,\lambda}(f)\varphi||_U \leq C_U \sup_{k\in K} |\varphi(k)|, \quad \lambda \in U, f \in C_c^\infty(G). \tag{2.4}$$

Now, for $\epsilon > 0$, take a smooth function T_ϵ on K given by elements in $H_{-\infty}^{\tilde{\sigma}}$, such that T can be approximated by T_ϵ, i.e., $T_\epsilon \overset{\epsilon\to 0}{\longrightarrow} T$ with strong topology. Then, by (2.4) and Vitali's convergence theorem (e.g. [El18], Chap. 3), the vector-valued function

$$\int_K \langle T_\epsilon(k), \pi_{\sigma,\lambda}(f)\varphi(k)\rangle \, dk$$

is holomorphic and converges uniformly on compact sets U of λ to $\langle T, \pi_{\sigma,\lambda}(f)\varphi\rangle$. □

Delorme's Paley-Wiener theorem and intertwining conditions in (Level 1)

We now proceed with the definition of Delorme's Paley-Wiener space ([Del05], Def. 3) for a fixed P. To handle some conditions introduced by Arthur ([vdBS14], Sect. 3), Delorme considers *derived* versions of H_∞^σ ([Del05], Sect. 1.5). Unfortunately, his intertwining conditions ([Del05], Déf. 3 (4.4)) are not easy to understand and check, even in particular cases. Van den Ban and Souaifi present a more elegant reformulation of them ([vdBS14], Sect. 4.5, in particular Lem. 4.4. and Prop. 4.5.). In the same spirit, we present another definition of derived G-representations.

Definition 2.3 (m-th derived representation)**.** *For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, let Hol_λ be the set of germs at λ of $\mathbb{C}$-valued holomorphic functions $\mu \mapsto f_\mu$ and $m_\lambda \subset \mathrm{Hol}_\lambda$ the maximal ideal of germs vanishing at λ.*
Denote by $H_{[\lambda]}^\sigma$ the set of germs at λ of H_∞^σ-valued holomorphic functions $\mu \mapsto \phi_\mu \in H_\infty^\sigma$ with G-action

$$(g\phi)_\mu = \pi_{\sigma,\mu}(g)\phi_\mu, \quad g \in G.$$

We define by

$$H_{\infty,(m)}^{\sigma,\lambda} := H_{[\lambda]}^\sigma / m_\lambda^{m+1} H_{[\lambda]}^\sigma, \quad m \in \mathbb{N}_0 \tag{2.5}$$

the m-th derived G-representation, which is equipped with the natural Fréchet topology.

Here, Hol_λ acts on $H^\sigma_{[\lambda]}$ by pointwise multiplication. Note that the m =0-th derived representation $H^{\sigma,\lambda}_{\infty,(0)} \cong H^\sigma_\infty$ is the space of smooth vectors of the principal series G-representation in the compact picture. Intuitively, we can say that $H^{\sigma,\lambda}_{\infty,(m)}$ contains all Taylor polynomials of order m at λ of holomorphic families ϕ_μ. Moreover, $\phi \in \prod_{\sigma\in\hat{M}} \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, \mathrm{End}(H^\sigma_\infty))$ induces an operator on each $H^{\sigma,\lambda}_{\infty,(m)}$.

The following reminder will serve nicely. Consider a closed subspace W of $H^{\sigma,\lambda}_{\infty,(m)}$. It is a G-invariant subrepresentation (or submodule) of $H^{\sigma,\lambda}_{\infty,(m)}$, if W is stable by the action of G, i.e., $\pi_{\sigma,\lambda}(g)W \subseteq W$, for $g \in G$ [Kna02].

The following definition is equivalent to Delorme's intertwining condition ([Del05], Déf. 3 (4.4)).

Definition 2.4 (Delorme's intertwining condition in (Level 1))**.** *Let Ξ be the set of all 3-tuples (σ, λ, m) with $\sigma \in \hat{M}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $m \in \mathbb{N}_0$. Consider the m-th derived G-representation $H^{\sigma,\lambda}_{\infty,(m)}$ defined in (2.5). For every finite sequence $\xi = (\xi_1, \xi_2, \dots, \xi_s) \in \Xi^s$, $s \in \mathbb{N}$, we define the G-representations*

$$H_\xi := \bigoplus_{i=1}^{s} H^{\sigma_i,\lambda_i}_{\infty,(m_i)}.$$

We consider proper closed G-subrepresentations $W \subseteq H_\xi$.
We call (ξ, W) the intertwining data. Every function $\phi \in \prod_{\sigma\in\hat{M}} \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, \mathrm{End}(H^\sigma_\infty))$ induces an element

$$\phi_\xi \in \bigoplus_{i=1}^{s} \mathrm{End}(H^{\sigma_i,\lambda_i}_{\infty,(m_i)}) \subset \mathrm{End}(H_\xi).$$

(D.a) We say that ϕ satisfies Delorme's intertwining condition, if $\phi_\xi(W) \subseteq W$ for every intertwining datum (ξ, W).

Next, we define Delorme's Paley-Wiener space ([Del05], Déf. 3).

Definition 2.5 (Paley-Wiener space in (Level 1))**.** *For $r > 0$, Delorme's Paley-Wiener space is the vector space*

$$PW_r(G) := \Big\{\phi \in \prod_{\sigma\in\hat{M}} \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, \mathrm{End}(H^\sigma_\infty)) \mid \phi \text{ satisfies conditions } (1.ii)_r \text{ and } (D.a)\Big\} \tag{2.6}$$

with growth condition:

$(1.ii)_r$ for all $Y_1, Y_2 \in \mathcal{U}(\mathfrak{k}), (\sigma, \lambda) \in \hat{M} \times \mathfrak{a}^*_{\mathbb{C}}$ and $N \in \mathbb{N}_0$, there exists a constant $C_{r,N,Y_1,Y_2} > 0$ such that

$$||\pi_{\sigma,\lambda}(Y_1)\phi(\sigma,\lambda)\pi_{\sigma,\lambda}(Y_2)||_{H^\sigma_\infty} \leq C_{r,N,Y_1,Y_2}(1 + |\Lambda_\sigma|^2 + |\lambda|^2)^{-N} e^{r|\mathrm{Re}(\lambda)|}$$

for $\phi \in \mathrm{End}(H^\sigma_\infty)$ and where Λ_σ is the highest weight of G, $||\cdot||_{H^\sigma_\infty}$ is the operator norms on H^σ_∞ with respect to the L^2-norm.

Notice that, due Lem. 10 (i) in [Del05], the space $PW_r(G)$ equipped with semi-norms:

$$||\phi||_{r,N,Y_1,Y_2} := \sup_{(\sigma,\lambda)\in\hat{M}\times\mathfrak{a}^*_{\mathbb{C}}} (1+|\Lambda_\sigma|^2+|\lambda|^2)^N e^{-r|\mathrm{Re}(\lambda)|} ||\pi_{\sigma,\lambda}(Y_1)\phi(\sigma,\lambda)\pi_{\sigma,\lambda}(Y_2)||_{H^\sigma_\infty}, \quad \phi \in PW_r(G)$$

is a Fréchet space.

Furthermore, the intertwining condition $(D.a)$ in Def. 2.5 is a special case of van den Ban and Souaifi's one ([vdBS14], Cor. 4.7 and Prop. 4.10.). The small difference is, that instead of the defined m-th derived representations $H^{\sigma,\lambda}_{\infty,(m)}$ (2.5), they consider

$$H^{\sigma}_{[\lambda],E} := H^{\sigma}_{[\lambda]} \otimes_{\mathrm{Hol}_\lambda} E,$$

where E is a finite-dimensional Hol_λ-module.

Proposition 2.6. *With the previous notations, let $(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}^*_{\mathbb{C}}$. Then, for $E = \mathrm{Hol}_\lambda / m_\lambda^{m+1}$, we have that $H^{\sigma}_{[\lambda],E} \cong H^{\sigma,\lambda}_{\infty,(m)}$.*
Moreover, for any finite-dimensional Hol_λ-module E, there exists $m_1, \dots, m_s \in \mathbb{N}_0$ such that $H^{\sigma}_{[\lambda],E}$ is a quotient of $H^{\sigma,\lambda}_{\infty,(m_1)} \oplus \cdots \oplus H^{\sigma,\lambda}_{\infty,(m_s)}$.

Proof. Consider a (commutative) ring R with neutral element 1, a R-module M and $I \subset R$ an ideal. Then, we have the following isomorphism

$$M \otimes_R R/I \cong M/IM.$$

In fact, by an algebraic computation, one can easily show that the two maps

$$\begin{aligned} \alpha : M \otimes_R R/I &\to M/IM \quad \text{and} \quad & \beta : M/IM &\to M \otimes_R R/I \\ \alpha(m \otimes [r]) &:= [rm] & \beta([m]) &:= m \otimes [1] \end{aligned}$$

are well-defined and inverse to each other. Here, $[\cdot]$ denotes the class in the corresponding quotient. For $m \in \mathbb{N}_0$ and $R = \mathrm{Hol}_\lambda$, consider its maximal ideal $m_\lambda^{m+1} \subset \mathrm{Hol}_\lambda$. Take $E = \mathrm{Hol}_\lambda / m_\lambda^{m+1} = R/I$ and $M = H^{\sigma}_{[\lambda]}$, then

$$H^{\sigma}_{[\lambda]} \otimes_{\mathrm{Hol}_\lambda} E \cong H^{\sigma}_{[\lambda]} / m_\lambda^{m+1} H^{\sigma}_{[\lambda]} =: H^{\sigma,\lambda}_{\infty,(m)}.$$

Van den Ban and Souaifi ([vdBS14], Sect. 2) consider a cofinite ideal $\mathcal{I}$ in Hol_λ, that is, if the quotient $\mathrm{Hol}_\lambda/\mathcal{I}$ is finite-dimensional as a vector space over $\mathbb{C}$. Moreover, by their Lem. 2.1 in [vdBS14], an ideal $\mathcal{I}$ in Hol_λ is cofinite, if and only, if there exists $m \in \mathbb{N}_0$ such that $m_\lambda^{m+1} \subset \mathcal{I}$.
Thus, for some $s \in \mathbb{N}$ and finitely many cofinite ideals $m_\lambda^{m_1+1}, \dots, m_\lambda^{m_s+1}$ of Hol_λ, we have that E is a quotient of the direct sum

$$\mathrm{Hol}_\lambda / m_\lambda^{m_1+1} \oplus \mathrm{Hol}_\lambda / m_\lambda^{m_2+1} \oplus \cdots \oplus \mathrm{Hol}_\lambda / m_\lambda^{m_s+1}.$$

Hence, the map

$$H^{\sigma,\lambda}_{\infty,(m_1)} \oplus \cdots \oplus H^{\sigma,\lambda}_{\infty,(m_s)} \longrightarrow E$$

is surjective and the result follows. □

Finally, we are in the position to state a Paley-Wiener theorem for G, the complete proof can be found in [Del05].

Theorem 2.7 (Paley-Wiener Theorem, [Del05], Thm. 2)**.** *For $r > 0$, the Fourier transform*

$$C_r^\infty(G) \ni f \mapsto \mathcal{F}_{\sigma,\lambda}(f) \in PW_r(G), \quad (\sigma, \lambda) \in \hat{M} \times \mathfrak{a}^*_{\mathbb{C}}$$

is a topological isomorphism between the two Fréchet spaces $C_r^\infty(G)$ and $PW_r(G)$. □

2.1.2 Application to sections over homogeneous vector bundles over G/K

Before we proceed with the investigation to adapt Delorme's Paley-Wiener Thm. 2.7 for our purposes, we first of all need to introduce some preparations. We will start by defining the so-called isotypic components ([Wal88], Sect. 1.4.7.) and then establish the Fourier transform for smooth compactly sections over homogeneous vector bundles over X.

Isotypic compositions and multiplicities

Consider, for the moment, K as a compact topological group. Let (π, V) be fixed a K-representation on a locally convex topological vector space V. If $\tau \in \hat{K}$, then we define the τ-isotypic component of V, denoted by $V(\tau)$, as the closure of the sum of all the closed, invariant, irreducible, subspaces E of V that are in the class τ, i.e., such that $\pi|_E \cong \tau$:

$$V(\tau) = \overline{\sum} E$$

([Wal88], Sect. 1.4.7.). Note that the closure of the sum above means that $V(\tau) \subset V$ is closed.

By combining two important results from harmonic analysis, Schur's lemma ([Wal88], Lem. 12.1) and Peter-Weyl's theorem ([Wal73], Thm. 2.8.2.), we obtain the following isotypic composition. In particular, the analytic definition concides with the algebraic one

$$V(\tau) = \sum E. \tag{2.7}$$

Proposition 2.8. *Let V be a locally convex topological vector space and consider the algebraic definition (2.7). If $\tau \in \hat{K}$, then*

$$V(\tau) \cong \mathrm{Hom}_K(E_\tau, V) \otimes E_\tau, \tag{2.8}$$

where $\mathrm{Hom}_K(E_\tau, V)$ is the so-called multiplicity space.
Furthermore, by taking the algebraic direct sum over all τ, we obtain the subspace of K-finite vectors of V and

$$V = \overline{\bigoplus}_{\tau \in \widehat{K}} V(\tau) \cong \overline{\bigoplus}_{\tau \in \widehat{K}} \mathrm{Hom}_K(E_\tau, V) \otimes E_\tau. \tag{2.9}$$

Proof. Consider the natural map

$$\alpha : \mathrm{Hom}_K(E_\tau, V) \otimes E_\tau \longrightarrow V(\tau)$$

and let us show in the following that it is an isomorphism. With the algebraic definition (2.7), we have directly that α is surjective. The reason is that, by definition, every element in $V(\tau)$ is of the form

$$v := w_1 + \cdots + w_r = \sum_{i=1}^{r} w_i$$

for $w_i \in E_i \cong E_\tau, \forall i \in \{1, \ldots, r\}, r \in \mathbb{N}$, i.e., there exist $\phi_i \in \mathrm{Hom}_K(E_\tau, V)$ and $v_i \in E_\tau$ with $w_i := \phi_i(v_i) \in E_\tau$. This implies that

$$u := \alpha(\phi_1 \otimes v_1 + \cdots + \phi_r \otimes v_r) = \alpha\Big(\sum_{i=1}^{r} \phi_i \otimes v_i\Big) \in V(\tau).$$

To show that the map α is injective, we know that every element of $\mathrm{Hom}_K(E_\tau, V)\otimes E_\tau$ can be described by $T = \phi_1 \otimes v_1 + \cdots + \phi_r \otimes v_r =: \sum_{i=1}^r \phi_i \otimes v_i$, where $\{\phi_1, \ldots, \phi_r\}$ is linear independent. Now, if we can prove that

$$\sum_{i=1}^{r} \mathrm{Im}(\phi_i) \subset V(\tau) \tag{2.10}$$

is a direct sum of vector space, we can conclude that α is injective. In fact, by Schur's lemma, we have that the $\phi_i \neq 0, \forall i$. Hence

$$u = 0 \iff \alpha\Big(\sum_{i=1}^{r} \phi_i \otimes v_i\Big) = 0 \iff \sum_{i=1}^{r} \phi_i(v_i) = 0 \iff (v_1, \ldots, v_r) = 0 \iff T = 0.$$

By contradiction, assume that the sum (2.10) is not direct, this means that a vector space intersects with another one. By induction on the integer r, assume that $\sum_{i=1}^{r-1} \mathrm{Im}(\phi_i)$ is direct, i.e., $\mathrm{Im}(\phi_r) \cap \bigoplus_{i=1}^{r-1} \mathrm{Im}(\phi_i) \neq \{0\}$ by irreducibility of $\mathrm{Im}(\phi_r)$. Consider K-projections p_i on the components $\mathrm{Im}(\phi_i) \subset \bigoplus_{i=1}^{r-1} \mathrm{Im}(\phi_i)$. Then, for $i \neq 0$, we have the intertwining operator $p_i \circ \phi_r : E_\tau \longrightarrow \mathrm{Im}(\phi_i)$. By Schur's lemma again, this implies that there exists $\lambda_i \in \mathbb{C}$ such that

$$p_i \circ \phi_r = \lambda_i \phi_i, \quad \forall i$$

i.e., $\phi_r = \sum_{i=1}^{r-1} \lambda_i \phi_i$. But this contradicts the fact that ϕ_i are linear independent. Hence, we proved that the map α is injective and thus the isomorphism (2.8).

It remains to show (2.9). Denote by V' the topological dual of the vector space V and consider the matrix coefficients ([Wal88], 1.3.2):

$$c_{v',v}(k) := \langle v', \pi(k)v\rangle, \quad v \in V, v' \in V', k \in K.$$

Fix $v' \in V'$ and consider the map $c_{v',\cdot} : V \longrightarrow C(K)$ defined by $c_{v',\cdot}(v) := c_{v',v}$, where $C(K)$ denotes the space of all continuous functions on K with the usual supremum norm. Notice that for a continuous V-representations, we have that

$$K \times K \to V, (k, v) \mapsto \pi(k)v$$

is continuous, which implies that the map $c_{v',\cdot}$ is continuous as well.
Let $C(K)_\tau \subset C(K)$ be the space of all matrix coefficients defined as above. By a result of Wallach ([Wal73], Cor. 1.4.6.) and by Peter-Weyl's theorem, we have that $\dim(C(K)_\tau) \leq d_\tau^2$, where d_τ denotes the dimension of the vector space E_τ. In particular, we have that $C(K)_\tau$ is closed. Furthermore, for $k \in K, v' \in V'$ and $w = \sum_{i=1}^r \phi_i(v_i)$:

$$c_{v',w}(k) = \sum_{i=1}^{r} \langle v', \pi(k)\phi_i(v_i)\rangle = \sum_{i=1}^{r} \langle v', \phi_i(\tau(k)v_i)\rangle = \sum_{i=1}^{r} \langle \underbrace{\phi_i^t(v')}_{\in E_{\tilde\tau}}, \underbrace{\tau(k)v_i}_{\in E_\tau}\rangle \in C(K)_\tau$$

are the matrix coefficients of E_τ, where $\phi_i^t : V' \to E_{\tilde\tau}$ is the transpose of $\phi_i, \forall i$. Thus, we have that $c_{v',\cdot}\Big|_{V(\tau)}$ has range in this space $C(K)_\tau$.
Now consider the closed space

$$\mathcal{A} := \{v \in V \mid c_{v',w} \in C(K)_\tau, \forall v' \in V', w \in E_\tau\} \subset V.$$

Since $V(\tau) \subset V$, we have that $V(\tau) \subset \mathcal{A}$. Concerning the inverse inclusion, it suffices to prove that $\dim(B) := \dim(\mathrm{span}\{\pi(k)w \mid k \in K\}) \leq d_\tau^2 < \infty$ with $B \subset \mathcal{A}$.
For this, fix $w \in E_\tau$, we define the map $c_{\cdot,w} : V' \longrightarrow C(K)_\tau$ by $c_{\cdot,w}(v') := c_{v',w}$. Then, since an element $v' \in V'$ is in $\mathrm{Ker}(c_{\cdot,w})$ if, and only if, $v'\Big|_B = 0$, we have

$$d_\tau^2 \geq \dim(C(K)_\tau) = \dim(V'/\mathrm{Ker}(c_{\cdot,w})) = \dim(V'/\mathrm{ann}_{V'}(B)) = \dim(B),$$

where $\mathrm{ann}_{V'}(B)$ is the annihilator of B in V', i.e., the kernel of the map $c_{\cdot,w}$. Hence, B splits into a finite direct sum of irreducible invariant subspaces $\tilde{E}_i \cong E_\gamma \subset B \subset \mathcal{A}$ each in the class of $\gamma \in \hat{K}$, i.e., $B = \bigoplus_{i=1}^r \tilde{E}_i$.
By the above, we have that the matrix coefficients map $\tilde{E}_i$ by $C(K)_\gamma$. More precisely, if $w \in E_\tau$ and $\tilde{w} \in E_\gamma$, with τ and γ distinct, then

$$c_{\tilde{w},w} \in C(K)_\gamma \cap C(K)_\tau = \{0\}.$$

This implies that $\langle E_\gamma, E_\tau \rangle = 0$ and for $\tilde{E}_i \subset V, \forall i$, we get $B \subset V(\tau)$. Hence, each element of $\mathcal{A}$ is in $V(\tau)$, thus $\mathcal{A} \subset V(\tau)$. Finally, we proved that $\mathcal{A} = V(\tau)$ and thus we obtain the desired result (2.8). □

We now focus on the multiplicity space

$$\mathrm{Hom}_K(E_\tau, V) \overset{(1.8)}{\cong} [V \otimes E_{\tilde{\tau}}]^K. \tag{2.11}$$

Example 2.9. 1) Let $K \subset G$ be compact subgroup and consider $V = C^\infty(G)$ with right regular representation r. If $\tau \in \hat{K}$, then the multiplicity space is given by

$$\mathrm{Hom}_K(E_\tau, C^\infty(G)) \overset{(2.11)}{\cong} [C^\infty(G) \otimes E_{\tilde{\tau}}]^K \cong C^\infty(X, \mathbb{E}_{\tilde{\tau}}),$$

which corresponds to the space of sections over homogeneous vector bundles.

2) Consider now $V = C^\infty(G)$ with $K \times K$-action. For $\gamma, \tau \in \hat{K}$, let $E_\gamma \otimes E_\tau$ be an irreducible $K \times K$ representation. Then, as multiplicity space we obtain

$$\mathrm{Hom}_{K\times K}(E_\gamma \otimes E_\tau, C^\infty(G)) \overset{(2.11)}{\cong} \mathrm{Hom}_K(E_\gamma, C^\infty(X, \mathbb{E}_{\tilde{\tau}})) \overset{\beta}{\cong} C^\infty(G, \gamma, \tilde{\tau}),$$

where

$$\begin{aligned} C^\infty(G,\gamma,\tilde{\tau}) \;:=\; & \{f : G \to \mathrm{Hom}(E_\gamma, E_{\tilde{\tau}}) \mid \\ & f(k_1 g k_2) = \tilde{\tau}(k_2)^{-1} f(g) \gamma(k_1)^{-1}, \forall k_1, k_2 \in K\} \end{aligned} \tag{2.12}$$

is known as the $(\gamma, \tilde{\tau})$-spherical functions on G. More precisely, if $f \in \mathrm{Hom}_K(E_\gamma, C^\infty(X, \mathbb{E}_{\tilde{\tau}}))$ and $F \in C^\infty(G, \gamma, \tilde{\tau})$, then the isomorphism β is given by $f_F(g)v := F(v)(g)$ and $F_f(v)(g) := f(g)v, \forall v \in E_\gamma, g \in G$.

Remark 2.10. (i) (Special case of 2)) If $\gamma = \tau = 1$ are trivial representations, then the multiplicity space is the smooth space of K-bi-invariant functions on G:

$$C^\infty(K \backslash G / K) := \{f : G \to \mathbb{C} \mid f(k_1 g k_2) = f(g), k_1, k_2 \in K\}, \tag{2.13}$$

which are left K-invariant function on X.

(ii) The (γ, τ)-spherical functions on G can be viewed (by convolution from the right) as G-invariant integral operators between sections over homogeneous vector bundles:

$$C^\infty_{(c)}(X, \mathbb{E}_\gamma) \longrightarrow C^\infty_{(c)}(X, \mathbb{E}_\tau).$$

We refer to Prop. 2.21 for a more precise statement and proof.

Fourier transform for sections over homogeneous vector bundles

In this part, we want to define the Fourier transform, as well as, establish Delorme's Paley-Wiener theorem for sections over homogeneous vector bundles:

$$C_c^\infty(X, \mathbb{E}_\tau) = \bigcup_{r>0} C_r^\infty(X, \mathbb{E}_\tau) \cong \bigcup_{r>0} [C_r^\infty(G) \otimes E_\tau]^K.$$

More precisely, we want to study the reduced Fourier transform $\mathcal{F}$ on the above space by

$$\sum_{i=1}^{d_\tau} f_i \otimes v_i \mapsto \sum_{i=1}^{d_\tau} \mathcal{F}(f_i) \otimes v_i, \quad f \in C_c^\infty(G),$$

where d_τ denotes the dimension of E_τ and $v_i, i \in \{1, \cdots, d_\tau\}$, is a basis of E_τ. Roughly, for $r > 0$, one can deduce from Thm. 2.7, that

$$C_r^\infty(X, \mathbb{E}_\tau) \cong [C_r^\infty(G) \otimes E_\tau]^K \overset{\text{Thm. 2.7}}{\cong} [PW_r(G) \otimes E_\tau]^K,$$

where $PW_r(G)$ is Delorme's Paley-Wiener space defined in (2.6). The goal is to make $[PW_r(G) \otimes E_\tau]^K$ more explicit. For this, let us study the map

$$\begin{array}{rcl} C_r^\infty(X, \mathbb{E}_\tau) \ni f & \mapsto & \displaystyle\sum_{i=1}^{d_\tau} f_i \otimes v_i \in [C_r^\infty(G) \otimes E_\tau]^K \\ & \overset{\text{Thm. 2.7}}{\mapsto} & \displaystyle\sum_{i=1}^{d_\tau} \mathcal{F}_{\sigma,\lambda}(f_i) \otimes v_i \in [\mathrm{End}(H_\infty^\sigma) \otimes E_\tau]^K \cong H_\infty^\sigma \otimes \mathrm{Hom}_K(H_\infty^\sigma, E_\tau). \end{array}$$

Bringing the Frobenius-reciprocity into play, it gives us a better description of the space $\mathrm{Hom}_K(H_\infty^\sigma, E_\tau)$. Namely, we have

$$\begin{array}{rcl} \mathrm{Hom}_K(H_\infty^\sigma, E_\tau) & \overset{Frob}{\cong} & \mathrm{Hom}_M(E_\sigma, E_\tau) \text{ defined by} \\ \langle Frob(S)w, \tilde{v} \rangle & = & \langle w, S^*\tilde{v}(e) \rangle, \qquad w \in E_\sigma,\ \tilde{v} \in E_{\tilde{\tau}},\ S^* : E_{\tilde{\tau}} \to H_\infty^{\tilde{\sigma}}. \end{array} \tag{2.14}$$

Let us next compute the inverse of $Frob$.

Lemma 2.11 ([Olb95], Lem. 2.12)**.** *Let $s \in \mathrm{Hom}_M(E_\sigma, E_\tau)$ and $f \in H_\infty^\sigma$. Then, we have*

$$Frob^{-1}(s)(f) = \int_K \tau(k) s f(k)\, dk.$$

Proof. Let $S = Frob^{-1}(s)$, $f \in H_\infty^\sigma$ and $\tilde{v} \in E_{\tilde{\tau}}$. We compute, then

$$\begin{aligned} \langle Sf, \tilde{v} \rangle_\tau = \langle f, S^*\tilde{v} \rangle_{H_\infty^{\sigma,\lambda}} = \int_K \langle f(k), S^*\tilde{v}(k) \rangle_\sigma\, dk &= \int_K \langle f(k), S^*(\tilde{\tau}(k^{-1}\tilde{v}))(e) \rangle_\sigma\, dk \\ &= \int_K \langle \tau(k) s f(k), \tilde{v} \rangle_\tau\, dk. \qquad \square \end{aligned}$$

The dual of $Frob$ is given by

$$\begin{array}{rcl} \mathrm{Hom}_K(E_\tau, H_\infty^\sigma) & \overset{\widetilde{Frob}}{\cong} & \mathrm{Hom}_M(E_\tau, E_\sigma) \\ \widetilde{Frob}(T)(v) & = & T(v)(e), \qquad v \in E_\tau \end{array} \tag{2.15}$$

and for $t \in \mathrm{Hom}_M(E_\tau, E_\sigma)$ and $v \in E_\tau$, the inverse of $\widetilde{Frob}$ will be

$$\widetilde{Frob}^{-1}(t)(v)(k) = t\tau(k^{-1})v. \tag{2.16}$$

Coming back to our previous computation, we get

$$\begin{aligned}[]
[\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K \overset{(1.8)}{\cong} H^\sigma_\infty \otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau) &\overset{Frob}{\cong} H^\sigma_\infty \otimes \mathrm{Hom}_M(E_\sigma, E_\tau) \\
&\overset{(2.2)}{\cong} C^\infty(K/M, E_\sigma \otimes \mathrm{Hom}_M(E_\sigma, E_\tau)) \\
&\overset{(2.8)}{\cong} C^\infty(K/M, \mathbb{E}_{\tau|_M}(\sigma)) \\
&\cong H^{\tau|_M(\sigma)}_\infty,
\end{aligned} \tag{2.17}$$

where $\mathbb{E}_{\tau|_M}(\sigma)$ is the σ-isotypic component of $\mathbb{E}_{\tau|_M}$. Here, τ is restricted to M, it is generally no more irreducible and splits into a finite direct sum

$$\tau|_M = \bigoplus_{\sigma \in \hat{M}} m(\sigma, \tau)\sigma,$$

where $m(\sigma, \tau) = \dim(\mathrm{Hom}_M(E_\sigma, E_\tau)) \geq 0$ is the multiplicity of σ in $\tau|_M$. Now by taking the algebraic direct sum over all $\sigma \in \hat{M}$, where only finitely many of them appears, we obtain

$$\bigoplus_{\sigma \in \hat{M}} [\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K \overset{(2.17)}{\cong} \bigoplus_{\sigma \in \hat{M}} H^{\tau|_M(\sigma)}_\infty \overset{(2.9)}{\cong} H^{\tau|_M}_\infty = \{f : K \overset{C^\infty}{\to} E_\tau \mid f(km) = \tau(m)^{-1} f(k)\},$$

which can be viewed as the principal series representations corresponding to $\tau|_M$.

Let us now define the Fourier transform on this space by considering the integral kernel. Namely

$$\mathrm{End}(H^\sigma_\infty) \cong C^\infty(K \times K, \mathrm{End}(E_\sigma))^{M \times M} \tag{2.18}$$

can be characterised by $A \mapsto \phi_A$ with

$$(A\varphi)(k) = \int_{K/M} \phi_A(k, y)\varphi(y)\, dy, \quad k \in K, \varphi \in H^\sigma_\infty.$$

Therefore by using the identification (2.18) to the space $[\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K$, we obtain the same isomorphism as above (2.17) but without using the Frobenius-reciprocity:

$$\begin{aligned}[]
[\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K &\overset{(2.2)}{\cong} [C^\infty(K \times K, \mathrm{End}(E_\sigma))^{M \times M} \otimes E_\tau]^K \\
&\overset{\text{evaluate in } (k,1)}{\cong} [C^\infty(K, \mathrm{End}(E_\sigma))^M \otimes E_\tau]^{M \times K} \\
&\cong C^\infty(K, \mathbb{E}_\sigma)^M \otimes \mathrm{Hom}_M(E_\sigma, E_\tau) \\
&\cong C^\infty(K, \mathbb{E}_\tau(\sigma))^M \cong H^{\tau|_M(\sigma)}_\infty.
\end{aligned}$$

Proposition 2.12. *Let $f_0 \in C^\infty_c(G)$ be a scalar-valued function and $\varphi \in H^\sigma_\infty$. Then, for $k \in K$, we have that*

$$(\pi_{\sigma,\lambda}(f_0)\varphi)(k) = \int_{K/M} \phi_{f_0}(k, y)\varphi(y)\, dy,$$

where the integral kernel ϕ_{f_0} of $\pi_{\sigma,\lambda}(f_0)$ is given by

$$\phi_{f_0}(k, y) = \int_M \Big(\int_G f_0(g\kappa(g^{-1}k)my^{-1}) a(g^{-1}k)^{-(\lambda+\rho)}\, dg \Big) \sigma(m)\, dm.$$

Proof. By computation, we obtain

$$\begin{aligned}
(\pi_{\sigma,\lambda}(f_0)\varphi)(k) &= \int_G f_0(g)(\pi_{\sigma,\lambda}(g)\varphi)(k)\, dg \\
&\overset{G\text{-action}}{=} \int_G f_0(g)\varphi(g^{-1}k)\, dg \\
&= \int_{G=KAN} f_0(g)\varphi\Big(\kappa(g^{-1}k)e^{a(g^{-1}k)}n(g^{-1}k)\Big)\, dg \\
&= \int_G f_0(g)\varphi(\kappa(g^{-1}k))a(g^{-1}k)^{-(\lambda+\rho)}\, dg \\
&= \int_{G/K}\Big(\int_K f_0(gl)\varphi(l^{-1}\kappa(g^{-1}k))\, dl\Big)a(g^{-1}k)^{-(\lambda+\rho)}\, dg =: I(f_0,\varphi).
\end{aligned}$$

Now, if we set $y := l^{-1}\kappa(g^{-1}k)$ in the above integral, then

$$\begin{aligned}
I(f_0,\varphi) &= \int_{G/K}\Big(\int_K f_0(g\kappa(g^{-1}k)y^{-1})\varphi(y)\, dy\Big)a(g^{-1}k)^{-(\lambda+\rho)}\, dg \\
&= \int_{G/K}\Big(\int_{K/M}\Big(\int_M f_0(g\kappa(g^{-1}k)my^{-1})\sigma(m)\, dm\Big)\varphi(y)\, dy\Big)a(g^{-1}k)^{-(\lambda+\rho)}\, dg \\
&= \int_{K/M}\Big(\int_M\Big(\int_G f_0(g\kappa(g^{-1}k)my^{-1})a(g^{-1}k)^{-(\lambda+\rho)}\, dg\Big)\sigma(m)\, dm\Big)\varphi(y)\, dy \\
&= \int_{K/M}\phi_{f_0}(k,y)\varphi(y)\, dy,
\end{aligned}$$

where $\phi_{f_0}(k,y) = \int_M\Big(\int_G f_0(g\kappa(g^{-1}k)my^{-1})a(g^{-1}k)^{-(\lambda+\rho)}\, dg\Big)\sigma(m)\, dm.$ ☐

Note that by evaluating the integral kernel at $y = 1$, we get

$$\phi_{f_0}(k,1) = \int_M\Big(\int_G f_0(g\kappa(g^{-1}k)m)a(g^{-1}k)^{-(\lambda+\rho)}\, dg\Big)\sigma(m)\, dm.$$

By a slight abuse of notation, we denote the map

$$C_c^\infty(X,\mathbb{E}_\tau) \cong [C_c^\infty(G)\otimes E_\tau]^K \overset{\mathcal{F}_{\sigma,\lambda}\otimes\mathrm{Id}}{\longrightarrow} [\mathrm{End}(H_\infty^\sigma)\otimes E_\tau]^K$$

also by $\mathcal{F}_{\sigma,\lambda}$.

Corollary 2.13. *Consider a vector-valued function $f \in [C_c^\infty(G)\otimes E_\tau]^K$ given by $f = \sum_i f_i \otimes v_i$, where v_i runs a vector basis of E_τ. Then, the Fourier transform of f is given by*

$$\mathcal{F}_{\sigma,\lambda}(f) = \frac{1}{d_\sigma} pr_\sigma\Big(\int_G \tau(\kappa(g^{-1}k))^{-1}a(g^{-1}k)^{-(\lambda+\rho)}f(g)\, dg\Big), \tag{2.19}$$

where $pr_\sigma : E_\tau \to E_\tau(\sigma)$ the projection on the σ-isotypic component.

Proof. For $f \in [C_c^\infty(G)\otimes E_\tau]^K$, we have that $\phi_f(k,1) \in \mathrm{End}(E_\sigma)\otimes E_\tau$:

$$\phi_f(k,1) = \int_M\Big(\sigma(m)\otimes\int_G f(g\kappa(g^{-1}k)m)a(g^{-1}k)^{-(\lambda+\rho)}\, dg\Big)dm.$$

Denote by Tr_σ the trace with respect to E_σ. By applying $\mathrm{Tr}_\sigma \otimes 1$ and using the fact that $\mathrm{Tr}_\sigma(\sigma(m)) := \chi_\sigma(m)$ is equal to the character of M with respect to σ, we then have

$$\begin{aligned}(\mathrm{Tr}_\sigma \otimes 1)\phi_f(k,1) &= \int_M \chi_\sigma(m)\Big(\int_G \tau(m)^{-1}\tau(\kappa(g^{-1}k))^{-1}a(g^{-1}k)^{-(\lambda+\rho)}f(g)\,dg\Big)dm \\ &\overset{m\mapsto m^{-1}}{=} \int_M \overline{\chi_\sigma(m)}\tau(m)\Big(\int_G \tau(\kappa(g^{-1}k))^{-1}a(g^{-1}k)^{-(\lambda+\rho)}f(g)\,dg\Big)dm \\ &= \frac{1}{d_\sigma}pr_\sigma\Big(\int_G \tau(\kappa(g^{-1}k))^{-1}a(g^{-1}k)^{-(\lambda+\rho)}f(g)\,dg\Big).\end{aligned}$$

In the last line, we used the properties of characters in representation theory, namely that $\chi_\sigma(1_{\mathbb{E}_\sigma}) = \dim(E_\sigma) =: d_\sigma$ and that

$$\chi_\sigma(m^{-1}) = \mathrm{Tr}_\sigma(\sigma(m)^{-1}) = \sum_{j=1}^{d_\sigma}\lambda_j^{-1} = \sum_{j=1}^{d_\sigma}\overline{\lambda_j} = \overline{\chi_\sigma(m)},$$

where $\{\lambda_j, j = 1,\dots,d_\sigma\}$ are the eigenvalues of $\sigma(m)$, which are unitary, therefore $\lambda^{-1} = \overline{\lambda}$. □

More precisely, the outcome of this identification variant leads us to define the Fourier transform for a function $f \in C_c^\infty(X,\mathbb{E}_\tau)$ in (Level 2).

Definition 2.14 (Fourier transform for sections over homogeneous vector bundles in (Level 2))**.** *Let $g = \kappa(g)a(g)n(g) \in KAN = G$ be the Iwasawa decomposition. For fixed $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $k \in K$, we define the function $e_{\lambda,k}^\tau$ by*

$$\begin{aligned} e_{\lambda,k}^\tau : G &\to \mathrm{End}(E_\tau) \cong E_{\tilde\tau}\otimes E_\tau \\ g &\mapsto e_{\lambda,k}^\tau(g) := \tau(\kappa(g^{-1}k))^{-1}a(g^{-1}k)^{-(\lambda+\rho)}, \end{aligned} \tag{2.20}$$

where ρ is the half sum of the positive roots of $(\mathfrak{g},\mathfrak{a})$, (1.7). For $f \in C_c^\infty(X,\mathbb{E}_\tau)$, the Fourier transformation is given by

$$\mathcal{F}_\tau f(\lambda,k) = \int_G e_{\lambda,k}^\tau(g)f(g)\,dg = \int_{G/K} e_{\lambda,k}^\tau(g)f(g)\,dg, \tag{2.21}$$

where the last equality makes sense, since the integrand is right K-invariant.

Note that the Fourier transform for sections has already been introduced and studied by Camporesi ([Ca97], (3.18)). It is a direct generalization of Helgason's Fourier transform for $E_\tau = \mathbb{C}$.

Observe that, for $k \in K$ and $g \in G$, we have, by definition

$$e_{\lambda,k}^\tau(g) = l_k(e_{\lambda,1}^\tau(g)) = e_{\lambda,1}^\tau(k^{-1}g). \tag{2.22}$$

This function $e_{\lambda,k}^\tau$ in Def. 2.14 can be interpreted like the 'exponential' function in the definition of Fourier transform in the Euclidean case $\mathbb{R}^n$. It has some interesting properties.

Proposition 2.15. *Let $\tau \in \hat{K}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $k \in K$. Then, we have*

$$e_{\lambda,k}^\tau(hg) = e_{\lambda,\kappa(h^{-1}k)}^\tau(g)a(h^{-1}k)^{-(\lambda+\rho)}, \quad g,h \in G. \tag{2.23}$$

Proof. Let $h, g \in G = KAN$, then by Iwasawa decomposition, we have

$$\begin{aligned} hg = h\kappa(g)a(g)n(g) &= \kappa(h(\kappa(g))\, a(h\kappa(g))\, n(h\kappa(g))\, a(g)\, n(g) \\ &= \underbrace{\kappa(h\kappa(g))}_{\in K}\ \underbrace{a(h\kappa(g))\, a(g)}_{\in A}\ \underbrace{n(h\kappa(g))\, n(g)}_{\in N}. \end{aligned}$$

In other words, we have $\kappa(hg) = \kappa(h\kappa(g)a(g)n(g)) = \kappa(h(\kappa(g))$ and $a(hg) = a(h\kappa(g)a(g)n(g)) = a(h\kappa(g))\, a(g)$. Hence

$$\begin{aligned} e^\tau_{\lambda,k}(hg) &\overset{(2.20)}{=} \tau(\kappa(g^{-1}h^{-1}k))^{-1}a(g^{-1}h^{-1}k)^{-(\lambda+\rho)} \\ &= \tau(\kappa(g^{-1}\kappa(h^{-1}k))^{-1}a(g^{-1}\kappa(h^{-1}k))^{-(\lambda+\rho)}a(h^{-1}k)^{-(\lambda+\rho)} \\ &\overset{(2.20)}{=} e^\tau_{\lambda,\kappa(h^{-1}k)}(g)a(h^{-1}k)^{-(\lambda+\rho)}. \end{aligned}$$

□

Proposition 2.16. *For fixed $g \in G$, the function $e^\tau_{\lambda,k}(g)$, defined in (2.20), is an entire function on $\lambda \in \mathfrak{a}^*_\mathbb{C}$.*

Proof. It is obvious, since $a(g^{-1}k)^{-(\lambda+\rho)}$ is an entire function on $\lambda \in \mathfrak{a}^*_\mathbb{C}$. □

Furthermore, by considering an invariant differential operator on sections, we get the following relation.

Proposition 2.17. *Let $Q \in \mathcal{D}_G(\mathbb{E}_{\tilde{\tau}}, \mathbb{E}_{\tilde{\gamma}})$ be an invariant linear differential operator. Then, we have*

$$Qe^\tau_{\lambda,k} = (Qe^\tau_{\lambda,1}(1)) \circ e^\gamma_{\lambda,k}, \quad \lambda \in \mathfrak{a}^*_\mathbb{C}, k \in K. \tag{2.24}$$

Proof. Let us first consider the case $k = 1$. We then have for $g \in G = NAK$:

$$e^\tau_{\lambda,1}(g) = e^\tau_{\lambda,1}(nak_1) = a^{\lambda+\rho}\tau(k_1) = a^{\lambda+\rho}e^\tau_{\lambda,1}(k_1), \quad n \in N, a \in A, k_1 \in K. \tag{2.25}$$

In particular, for $n_1a_1 \in NA$

$$\begin{aligned} l_{(n_1a_1)^{-1}}e^\tau_{\lambda,1}(nak_1) = e^\tau_{\lambda,1}(n_1a_1nak_1) = e^\tau_{\lambda,1}(n_1(a_1na_1^{-1})a_1ak_1) &\overset{(2.25)}{=} a_1^{\lambda+\rho}a^{\lambda+\rho}\tau(k_1) \\ &= a_1^{\lambda+\rho}e^\tau_{\lambda,1}(g). \end{aligned}$$

Hence, since Q is linear and G-invariant, we obtain that

$$l_{(n_1a_1)^{-1}}(Qe^\tau_{\lambda,1}(g)) = Q(l_{(n_1a_1)^{-1}}e^\tau_{\lambda,1}(g)) = Q(a_1^{\lambda+\rho}e^\tau_{\lambda,1}(g)) = a_1^{\lambda+\rho}Q(e^\tau_{\lambda,1}(g)) \tag{2.26}$$

and by setting $g = k_1 = 1$, we have

$$Qe^\tau_{\lambda,1}(n_1a_1) \overset{(2.26)}{=} a_1^{\lambda+\rho}Qe^\tau_{\lambda,1}(1). \tag{2.27}$$

Therefore, since $e^\tau_{\lambda,1} \in C^\infty(X, \mathbb{E}_{\tilde{\tau}}) \otimes E_\tau \subset C^\infty(G, \mathrm{End}(E_\tau))$, we have that $Qe^\tau_{\lambda,1} \in C^\infty(X, \mathbb{E}_{\tilde{\gamma}}) \otimes E_\tau \subset C^\infty(G, \mathrm{Hom}(E_\gamma, E_\tau))$. Therefore, for $g = n_1a_1k_2 \in G$, we can conclude that

$$Qe^\tau_{\lambda,1}(n_1a_1k_2) = Qe^\tau_{\lambda,1}(n_1a_1)\gamma(k_2) \overset{(2.27)}{=} a_1^{\lambda+\rho}(Qe^\tau_{\lambda,1}(1))\gamma(k_2) \overset{(2.25)}{=} (Qe^\tau_{\lambda,1}(1))e^\gamma_{\lambda,1}(n_1a_1k_2). \tag{2.28}$$

Now for general $k \in K$, we observe that $e^\tau_{\lambda,k} = l_ke^\tau_{\lambda,1}$. Hence

$$Qe^\tau_{\lambda,k} = Q(l_ke^\tau_{\lambda,1}) \overset{(2.28)}{=} l_k(Qe^\tau_{\lambda,1}(1))e^\gamma_{\lambda,1} = (Qe^\tau_{\lambda,1}(1)) \circ e^\gamma_{\lambda,k}.$$

Thus, we get the desired result. □

Back to our Fourier transform $\mathcal{F}_\tau$, we thus see that it has values in the functions space

$$\{f : \mathfrak{a}_{\mathbb{C}}^* \times K/M \to E_\tau |\ f(\lambda, km) = \tau(m)^{-1} f(\lambda, k), \forall m \in M\} = \mathcal{F}_\tau(C_c^\infty(X, \mathbb{E}_\tau)),$$

which is in the range of the Fourier transform $\mathcal{F}_\tau$. In other words, we proved the following result merging Prop. 2.8.

Lemma 2.18. *Let (τ, E_γ) be a K-representation and H_∞^σ the space of smooth vectors of the principal series representations defined in (2.2). Then, we have*

$$H_\infty^{\tau|_M} \cong C^\infty(K/M, \mathbb{E}_{\tau|_M}) \cong \bigoplus_{\sigma \in \hat{M}} [\mathrm{End}(H_\infty^\sigma) \otimes E_\tau]^K \cong \bigoplus_{\sigma \in \hat{M}} C^\infty(K/M, \mathbb{E}_{\tau|_M}(\sigma)). \quad (2.29)$$

In this identification, the Fourier transform is given by (2.19). □

Fourier transform in (Level 3) and its properties

Now consider an additional finite-dimensional K-representation $\gamma : K \to GL(E_\gamma)$ with its associated homogeneous vector bundle $\mathbb{E}_\gamma$ over X. It induces a mapping

$$\mathrm{Hom}_K(E_\gamma, C_c^\infty(X, \mathbb{E}_\tau)) \longrightarrow \mathrm{Hom}_K(E_\gamma, \mathcal{F}_\tau(C_c^\infty(X, \mathbb{E}_\tau))). \quad (2.30)$$

The LHS of (2.30) can be identify with a space of functions with values in $\mathrm{Hom}(E_\gamma, E_\tau)$, the (γ, τ)-spherical functions (2.12):

$$\mathrm{Hom}_K(E_\gamma, C_c^\infty(X, \mathbb{E}_\tau)) \cong C_c^\infty(G, \gamma, \tau).$$

For the RHS of (2.30), we use the Frobenius-reciprocity between K and M, by evaluating at $k = 1$, and we obtain the space of functions

$$\{\phi : \mathfrak{a}_{\mathbb{C}}^* \to \mathrm{Hom}_M(E_\gamma, E_\tau)\} \supset \mathrm{Im}({}_\gamma\mathcal{F}_\tau), \quad (2.31)$$

which is the image of the Fourier transformation ${}_\gamma\mathcal{F}_\tau$, where, here, the double indices on the left and right, specifies the K-types. In particular, we define the Fourier transformation ${}_\gamma\mathcal{F}_\tau$ of $f \in C_c^\infty(G, \gamma, \tau)$.

Definition 2.19 (Fourier transform in (Level 3))**.** *With the previous notations, the Fourier transformation for $f \in C_c^\infty(G, \gamma, \tau)$ is given by*

$$ {}_\gamma\mathcal{F}_\tau f(\lambda) := \int_G e_{\lambda,1}^\tau(g) f(g)\, dg. \quad (2.32)$$

Moreover, for irreducible γ, it describes ${}_\gamma\mathcal{F}_\tau$ as the behaviour on the γ-isotypic component

$$C_c^\infty(X, \mathbb{E}_\tau)(\gamma) \cong C_c^\infty(G, \gamma, \tau) \otimes E_\tau.$$

Thus, with $F(g) = f(g)v$, for $v \in E_\tau$, we have

$$ {}_\gamma\mathcal{F}_\tau f(\lambda)(v) = \mathcal{F}_\tau F(\lambda, 1) = \int_G e_{\lambda,1}^\tau(g) F(g) dg = \int_{NA} F(na) a^{\lambda-\rho}\, dn\, da,$$

which is the Fourier transformation for $F \in C_c^\infty(X, \mathbb{E}_\tau)(\gamma)$ in (Level 3). The last equality makes sense since the integrand is right K-invariant.

Corollary 2.20. *Let (γ, E_γ) and (τ, E_τ) be two K-representations. Then, we have*

$$\mathrm{Hom}_M(E_\gamma, E_\tau) \cong \bigoplus_{\sigma\in\hat{M}} \mathrm{Hom}_M(E_\gamma, E_\tau(\sigma)) \cong \bigoplus_{\sigma\in\hat{M}} [\mathrm{End}(H_\infty^\sigma)\otimes \mathrm{Hom}(E_\gamma, E_\tau)]^{K\times K}. \tag{2.33}$$

Proof. By using Lem. 2.18 and the dual Frobenius-reciprocity (2.15) for $\phi \mapsto \phi(\cdot)(1)$, we obtain

$$\begin{aligned}
[\mathrm{End}(H_\infty^\sigma)\otimes \mathrm{Hom}(E_\gamma, E_\tau)]^{K\times K} &\overset{(1.8)}{\cong} [\mathrm{End}(H_\infty^\sigma)\otimes (E_\tau \otimes E_{\tilde{\gamma}})]^{K\times K}\\
&\cong \mathrm{Hom}_K(E_\gamma, [\mathrm{End}(H_\infty^\sigma)\otimes E_\tau]^K)\\
&\overset{\text{Lem. 2.18}}{=} \mathrm{Hom}_K(E_\gamma, C^\infty(K/M, \mathbb{E}_{\tau|_M}(\sigma)))\\
&\overset{\widetilde{Frob}}{\cong} \mathrm{Hom}_M(E_\gamma, E_{\tau|_M}(\sigma)).
\end{aligned}$$
□

Let us consider now the convolution G of a smooth compactly supported function f to a right $K\times K$-invariant endomorphism function φ, which is defined by

$$(f*\varphi)(g) := \int_G \varphi(x^{-1}g)f(x)\,dx = \int_G \varphi(xg)f(x^{-1})\,dx, \quad g\in G. \tag{2.34}$$

By considering the corresponding Fourier transform, we obtain the following result, which is analogous as Lem. 1.4. in ([Hel89], Chap. 3).

Proposition 2.21. *Consider a (γ,τ)-spherical function $\varphi \in C_c^\infty(G,\gamma,\tau)$ and $f \in C_c^\infty(X, \mathbb{E}_\gamma)$. Then, we have that*

$$\mathcal{F}_\tau(f*\varphi)(\lambda,k) = {}_\gamma\mathcal{F}_\tau\varphi(\lambda)\mathcal{F}_\gamma f(\lambda,k), \quad \lambda\in\mathfrak{a}_\mathbb{C}^*, k\in K.$$

Proof. For $(\lambda,k)\in\mathfrak{a}_\mathbb{C}^*\times K$, we compute

$$\begin{aligned}
\mathcal{F}_\tau(f*\varphi)(\lambda,k) &= \int_G e^\tau_{\lambda,k}(g)(f*\varphi)(g)\,dg\\
&\overset{(2.34)}{=} \int_{G\times G} e^\tau_{\lambda,k}(g)\varphi(\underbrace{x^{-1}g}_{=:h})f(x)\,dx\,dg\\
&\overset{\text{Fubini's thm.}}{=} \int_G\Big(\int_G e^\tau_{\lambda,k}(xh)\varphi(h)\,dh\Big)f(x)\,dx\\
&\overset{(2.20)}{=} \int_G\Big(\int_G e^\tau_{\lambda,\kappa(x^{-1}k)}(h)a(x^{-1}k)^{-(\lambda+\rho)}\varphi(h)\,dh\Big)f(x)\,dx\\
&\overset{(2.22)}{=} \int_G\Big(\int_G e^\tau_{\lambda,1}(\underbrace{\kappa(x^{-1}k)^{-1}h}_{=:g})\varphi(h)\,dh\Big)a(x^{-1}k)^{-(\lambda+\rho)}f(x)\,dx\\
&= \int_G\Big(\int_G e^\tau_{\lambda,1}(g)\varphi(\kappa(g^{-1}k)g)\,dg\Big)a(x^{-1}k)^{-(\lambda+\rho)}f(x)\,dx\\
&= \int_G\Big(\int_G e^\tau_{\lambda,1}(g)\varphi(g)\,dg\Big)\gamma(\kappa(x^{-1}k))^{-1}a(x^{-1}k)^{-(\lambda+\rho)}f(x)\,dx\\
&= {}_\gamma\mathcal{F}_\tau\varphi(\lambda)\int_G e^\gamma_{\lambda,k}(x)f(x)\,dx = {}_\gamma\mathcal{F}_\tau\varphi(\lambda)\mathcal{F}_\gamma f(\lambda,k),
\end{aligned}$$

thus the result follows. □

Remark 2.22. (a) If $\gamma = \tau$, then we have

$$\mathcal{F}_\tau(f * \varphi)(\lambda, k) = {}_\tau\mathcal{F}_\tau\varphi(\lambda)\mathcal{F}_\tau f(\lambda, k), \quad \lambda \in \mathfrak{a}_\mathbb{C}^*, k \in K,$$

for $f \in C_c^\infty(X, \mathbb{E}_\tau)$ and a spherical function $\varphi \in C_c^\infty(G, \tau, \tau)$.

(b) In a smiliar way, one can define the left convolution for scalar valued-function $\varphi \in C_c^\infty(G)$. In fact, we know that, for $f \in C_c^\infty(X, \mathbb{E}_\tau)$ and $g \in G$, we have

$$\begin{aligned}
\mathcal{F}_\tau(l_g f)(\lambda, k) = \int_G e_{\lambda,k}^\tau(x) l_g f(x)\, dx \quad &= \quad \int_G e_{\lambda,k}^\tau(x) f(\underbrace{g^{-1}x}_{=:h})\, dx \\
&= \quad \int_G e_{\lambda,k}^\tau(gh) f(h)\, dh \\
&\overset{(2.23)}{=} \quad a(g^{-1}k)^{-(\lambda+\rho)} \int_G e_{\lambda,\kappa(g^{-1}k)}^\tau(h) f(h)\, dh \\
&= \quad a(g^{-1}k)^{-(\lambda+\rho)} \mathcal{F}_\tau(f)(\lambda, \kappa(g^{-1}k)). \\
&\overset{(2.3)}{=} \quad (\pi_{\tau,\lambda}(g)\mathcal{F}_\tau f(\lambda, \cdot))(k).
\end{aligned}$$

Hence, we can deduce for $\varphi \in C_c^\infty(G)$:

$$\mathcal{F}_\tau(\varphi * f)(\lambda, k) = (\pi_{\tau,\lambda}(\varphi)\mathcal{F}_\tau f(\lambda, \cdot))(k). \tag{2.35}$$

Now, for positive $\epsilon > 0$, take a K-conjugation invariant open neighbourhood $U_\epsilon \subset \overline{B}_\epsilon(0)$ so that $\bigcap_{\epsilon>0} U_\epsilon = \{0\}$, and for $\epsilon_1 < \epsilon_2$, we have $U_{\epsilon_1} \subset U_{\epsilon_2}$. Consider a scalar-valued function $\tilde{\eta}_\epsilon \in C_c^\infty(U_\epsilon) \subset C_c^\infty(G)$ in G by

$$\int_{U_\epsilon} \tilde{\eta}_\epsilon(g)\, dg = 1 \tag{2.36}$$

so that $\tilde{\eta}_\epsilon(g) \geq 0$. Note that $\tilde{\eta}_\epsilon$ is no longer K-bi-invariant in this neighbourhood. Let us construct from this an endomorphism function $\eta_\epsilon \in C^\infty(G, \tau, \tau)$ by

$$\eta_\epsilon(g) := \int_{K \times K} \tilde{\eta}_\epsilon(k_1 g k_2)\tau(k_1 k_2)\, dk_1\, dk_2, \quad g \in G. \tag{2.37}$$

Then, we get the following observation.

Corollary 2.23. *For each $\epsilon > 0$, let $\eta_\epsilon \in C_c^\infty(G, \tau, \tau)$ be the K-bi-invariant endomorphism function (2.37). Then, its Fourier transform ${}_\tau\mathcal{F}_\tau(\eta_\epsilon)$ converges uniformly on compact sets C on $\mathfrak{a}_\mathbb{C}^*$ to the identity map:*

$${}_\tau\mathcal{F}_\tau\eta_\epsilon(\lambda) \to \mathrm{Id}, \quad \lambda \in C$$

when $\epsilon \to 0$.

Proof. Consider $\eta_\epsilon \in C^\infty(G, \tau, \tau)$, then for $g \in G$:

$$\eta_\epsilon(g) = \int_K \int_K \tilde{\eta}_\epsilon(k_1 g k_2)\tau(k_1 k_2)\, dk_1\, dk_2 = \int_K \int_K \tilde{\eta}_\epsilon(k_1 g l k_1^{-1})\tau(l)\, dk_1\, dl = \int_K \overline{\eta}_\epsilon(gl)\tau(l)\, dl,$$

where we did a change of variable and set $\overline{\eta}_\epsilon(g) := \int_K \tilde{\eta}_\epsilon(k_1 g k_1^{-1})\ dk_1$. Here, $\tilde{\eta}_\epsilon \in C_c^\infty(U_\epsilon)$ as above (2.36). By computing its Fourier transform, we obtain, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

$$\begin{aligned}
{}_\tau\mathcal{F}_\tau\eta_\epsilon(\lambda) \overset{(2.32)}{=} \int_G e^\tau_{\lambda,1}(g)\eta_\epsilon(g)\ dg &= \int_G \Big(\int_K e^\tau_{\lambda,1}(g)\overline{\eta}_\epsilon(gl)\tau(l)\ dl \Big)\ dg \\
&\overset{(2.25)}{=} \int_G \Big(\int_K e^\tau_{\lambda,1}(gl)\overline{\eta}_\epsilon(gl)\ dl \Big)\ dg \\
&= \int_G e^\tau_{\lambda,1}(g)\overline{\eta}_\epsilon(g)\ dg \\
&= \int_{U_\epsilon} \overline{\eta}_\epsilon(g)(e^\tau_{\lambda,1}(g) - \mathrm{Id})dg + \mathrm{Id}.
\end{aligned}$$

Now, consider a compact set C on $\mathfrak{a}_{\mathbb{C}}^*$ and $\delta > 0$, then there exists $\epsilon > 0$ such that

$$|e^\tau_{\lambda,1}(g) - \mathrm{Id}| < \delta \text{ for } g \in U_\epsilon, \lambda \in C.$$

Thus, this implies that ${}_\tau\mathcal{F}_\tau(\eta_\epsilon)$ converges uniformly on compact sets to Id, when ϵ converges to 0. □

In order to define the Paley-Wiener spaces and the corresponding theorems for sections, we need, first of all, adapt Delorme's intertwining conditions $(D.a)$ in Def. 2.4, for our levels. This will be established in the following subsection.

2.1.3 Intertwining conditions and Paley-Wiener theorems for sections

In this part, we will focus on establishing and proving Delorme's intertwining condition $(D.a)$ in Def. 2.4 for sections as well as the equivalence between our different *levels*, i.e.,

(Level 1) $\iff$ (Level 2) $\iff$ (Level 3).

To do this, we firstly need some preparations. In the previous subsection, we proved in Lem. 2.18, the identification (2.29). Let us now take a closer look. Consider the Frobenius-reciprocity (2.14) with its dual (2.15) and define the map

$$I : \bigoplus_{\sigma\in\hat{M}} H^\sigma_\infty \otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau) \longrightarrow H^{\tau|_M}_\infty$$

by $I(\alpha) = d_\sigma \sum_{i=1}^{m(\tau,\sigma)} s_i\alpha_i$, for $\alpha = \sum_{i=1}^{m(\tau,\sigma)} \alpha_i \otimes S_i \in H^\sigma_\infty \otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau)$, where $s_i = Frob(S_i)$ runs a basis through $\mathrm{Hom}_M(E_\sigma, E_\tau)$, for all i. Here, $m(\tau,\sigma)$ stands for the dimension of the multiplicity space $\mathrm{Hom}_K(H^\sigma_\infty, E_\tau)$. For $T \in \mathrm{Hom}_K(E_\tau, H^\sigma_\infty)$, let

$$\langle \alpha, T\rangle := \sum_{i=1}^{m(\tau,\sigma)} \alpha_i \cdot \mathrm{Tr}_\tau(S_i \circ T).$$

Now, by using the identification $[\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K \overset{(1.8),j}{\cong} H^\sigma_\infty \otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau)$, we can define the map

$$J : \bigoplus_{\sigma\in\hat{M}} [\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K \longrightarrow H^{\tau|_M}_\infty \tag{2.38}$$

by $J = I \circ j$. In addition, for $\beta = \sum_{i=1}^{d_\tau} \beta_i \otimes v_i \in \bigoplus_{\sigma\in\hat{M}} [\mathrm{End}(H_\infty^\sigma) \otimes E_\tau]^K$ and $T \in \mathrm{Hom}_K(E_\tau, H_\infty^\sigma)$, let

$$\langle \beta, T\rangle := \sum_{i=1}^{d_\tau} \beta_i \circ T(v_i) \in H_\infty^\sigma, \tag{2.39}$$

where $\{v_i, i = 1, \ldots, d_\tau\}$ runs a vector basis of E_τ. One checks that $\langle \beta, T\rangle = \langle j(\beta), T\rangle$.

Proposition 2.24. *With the previous notations, let $f := \sum_i^{d_\tau} f_i \otimes v_i \in C_c^\infty(X, \mathbb{E}_\tau)$. Denote by $\mathcal{F}_\tau(f)$ its Fourier transform in $H_\infty^{\tau|_M}$ given in (2.21).*
Then, for $T \in \mathrm{Hom}_K(E_\tau, H_\infty^\sigma)$ and $t = \widetilde{Frob}^{-1}(T) \in \mathrm{Hom}_M(E_\tau, E_\sigma)$, we obtain

(1) $\langle \alpha, T\rangle = t \circ I(\alpha)$,

(2) $\langle \mathcal{F}_{\sigma,\lambda}(f), T\rangle = t \circ \mathcal{F}_\tau f(\lambda, \cdot) \in H_\infty^{\sigma,\lambda}$, for $\lambda \in \mathfrak{a}_\mathbb{C}^*$,

(3) $\mathcal{F}_\tau f(\lambda, \cdot) = J(\bigoplus_{\sigma\in\hat{M}} \mathcal{F}_{\sigma,\lambda}(f))$, for $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

Proof. (1) It is sufficient to prove it for only one summand in α, hence let $\alpha = \alpha_1 \otimes S$. For $T = \widetilde{Frob}(t) \in \mathrm{Hom}_K(E_\tau, H_\infty^\sigma)$ and $S = Frob(s) \in \mathrm{Hom}_K(H_\infty^\sigma, E_\tau)$, we thus obtain

$$\begin{aligned}
\langle \alpha, T\rangle = \alpha_1 \mathrm{Tr}_\tau(S \circ T) &\overset{\text{Lem. 2.11+(2.16)}}{=} \alpha_1 \mathrm{Tr}_\tau\Big(v \mapsto \int_K \tau(k) s \circ t(\tau(k^{-1}))v \, dk\Big), \; v \in E_\tau \\
&= \alpha_1 \mathrm{Tr}_\tau\Big(\int_K \tau(k) s \circ t\tau(k^{-1}) \, dk\Big) \\
&= \alpha_1 \mathrm{Tr}_\tau(s \circ t) \\
&= \mathrm{Tr}_\sigma(t \circ s)\alpha_1.
\end{aligned}$$

Since $\sigma \in \hat{M}$ is irreducible and $t \circ s \in \mathrm{End}_M(E_\sigma)$, by Schur's lemma, we have that $t \circ s = \lambda \cdot \mathrm{Id}$, for some $\lambda \in \mathbb{C}$ and thus $\mathrm{Tr}_\sigma(t \circ s) = \lambda$. Hence

$$\langle \alpha, T\rangle = (t \circ s)(\alpha_1) = t(I(\alpha)).$$

(2) By computation, we obtain

$$\begin{aligned}
\langle \mathcal{F}_{\sigma,\lambda}(f), T\rangle = \sum_{i=1}^{d_\tau} \mathcal{F}_{\sigma,\lambda}(f_i) \circ T(v_i) &\overset{(2.16)}{=} \sum_{i=1}^{d_\tau} \mathcal{F}_{\sigma,\lambda}(f_i)(t\tau(\cdot)(v_i)) \\
&\overset{\text{Def. 2.1}}{=} \sum_{i=1}^{d_\tau} \int_G f_i(g) \pi_{\sigma,\lambda}(g)(t\tau(\cdot)(v_i)) \, dg \\
&= \sum_{i=1}^{d_\tau} \int_G f_i(g) (\pi_{\sigma,\lambda}(g)\varphi_i)(\cdot) \, dg.
\end{aligned}$$

In the last line, we set $\varphi_i(k) := t\tau(k^{-1})(v_i)$, for $k \in K$. Fix $k \in K$, by applying (2.3), we have $(\pi_{\sigma,\lambda}(g)\varphi_i)(k) = a(g^{-1}k)^{-(\lambda+\rho)} \varphi(\kappa(g^{-1}k))^{-1}$.

Thus

$$\begin{aligned}\sum_{i=1}^{d_\tau}\int_G f_i(g)t\tau(\kappa(g^{-1}k))^{-1}a(g^{-1}k)^{-(\lambda+\rho)}v_i\,dg &= \sum_{i=1}^{d_\tau}\int_G f_i(g)te^\tau_{\lambda,k}(g)v_i\,dg\\ &= t\circ\int_G\sum_{i=1}^{d_\tau}e^\tau_{\lambda,k}(g)f_i(g)v_i\,dg\\ &= t\circ\int_G e^\tau_{\lambda,k}(g)f(g)\,dg = t\circ\mathcal{F}_\tau(f)(\lambda,k).\end{aligned}$$

(3) By rewritting (1) and (2) in the following way:

(1') $\mathrm{Tr}_\tau(I^{-1}(\alpha)\circ T) = t\circ\alpha$,

(2') $\mathrm{Tr}_\tau(\mathcal{F}_{\sigma,\lambda}(f)\circ T) = t\circ\mathcal{F}_\tau f(\lambda,\cdot)$

we get that

$$\mathrm{Tr}_\tau(J^{-1}(\mathcal{F}_\tau f(\lambda,\cdot))\circ T) = \mathrm{Tr}_\tau(I^{-1}(\mathcal{F}_\tau f(\lambda,\cdot))\circ T) \overset{(1')}{=} t\circ\mathcal{F}_\tau f(\lambda,\cdot) \overset{(2')}{=} \mathrm{Tr}_\tau(\mathcal{F}_{\sigma,\lambda}(f)\circ T).$$

By taking only the σ-component of $\bigoplus_{\sigma\in\hat{M}}[\mathrm{End}(H^\sigma_\infty)\otimes E_\tau]^K$, we have that the parining in Tr_σ is non-degenerate, thus $J^{-1}(\mathcal{F}_\tau f(\lambda,\cdot)) = \bigoplus_{\sigma\in\hat{M}}\mathcal{F}_{\sigma,\lambda}(f)$. □

Let us formulate Delorme's intertwining condition (Def. 2.4) in (Level 1) for function in $H^{\tau|_M}_\infty$.

Definition 2.25. *Consider $\tau\in\hat{K}$.*

(1) We say that a function

$$\phi\in\prod_{\sigma\in\hat{M}}[\mathrm{Hol}(\mathfrak{a}^*_\mathbb{C},\mathrm{End}(H^\sigma_\infty))\otimes E_\tau]^K \cong \bigoplus_{\sigma\subset\tau|_M}\mathrm{Hol}(\mathfrak{a}^*_\mathbb{C},[\mathrm{End}(H^\sigma_\infty)\otimes E_\tau]^K)$$

satisfies the intertwining condition, if for each $\tilde{v}\in E_{\tilde{\tau}}$:

$$\langle\phi,\tilde{v}\rangle_\tau\in\prod_{\sigma\in\hat{M}}\mathrm{Hol}(\mathfrak{a}^*_\mathbb{C},\mathrm{End}(H^\sigma_\infty))$$

satisfies the intertwining condition in Def. 2.4.

Proposition 2.26. *Let $\phi\in\prod_{\sigma\in\hat{M}}[\mathrm{Hol}(\mathfrak{a}^*_\mathbb{C},\mathrm{End}(H^\sigma_\infty))\otimes E_\tau]^K$ as in Def. 2.25 and (ξ,W) the intertwining data defined in Def. 2.4.*

(D.1) Then, ϕ satisfies the intertwining condition (1) of Def. 2.25 if, and only if, for each intertwining datum (ξ,W) and $T\in\mathrm{Hom}_K(E_\tau,W)\subset\mathrm{Hom}_K(E_\tau,H_\xi)$, the induced element $\phi_\xi\in[\mathrm{End}(H_\xi)\otimes E_\tau]^K$ satisfies

$$\langle\phi_\xi,T\rangle\in W.$$

Proof. For each $i\in\{1,\ldots,d_\tau\}$, consider $f_i\in\mathrm{End}(H_\xi)$ so that for each intertwining datum (ξ,W), we have $f_i(W)\subseteq W$. Consider

$$\phi_\xi = \sum_{i=1}^{d_\tau} f_i\otimes v_i\in[\mathrm{End}(H_\xi)\otimes E_\tau]^K$$

as in Thm. 2.28. It is sufficient to show that for each i and $T \in \mathrm{Hom}_K(E_\tau, W)$, we have $f_i \circ T \in W$ if, and only if, $\langle \phi_\xi, T\rangle \in W$, $\forall T \in \mathrm{Hom}_K(E_\tau, W)$.

The right implication is obvious. By using the definition of the brackets $\langle\cdot,\cdot\rangle$ as in (2.39), we have

$$\langle \phi_\xi, T\rangle = \sum_{i=1}^{d_\tau} f_i \circ T(v_i) \in W$$

since for $v_i \in E_\tau$, $T(v_i) \in W \subset H_\xi$.

For the left implication, write $f_i = \langle \phi_\xi, \tilde{v}_i\rangle_\tau$, for all $i \in \{1, \ldots, d_\tau\}$, where $\tilde{v}_i$ runs a dual basis of $\mathbb{E}_{\tilde{\tau}}$. Consider the mapping $A_{ij} \in \mathrm{End}(E_\tau)$ such that $v_i \mapsto v_j$ and $v_k \mapsto 0,\ k \neq i$. Then, for all $i, j \in \{1, \ldots, d_\tau\}$, we have

$$f_i \circ T(v_j) = \langle \phi_\xi, T(v_j)\cdot \tilde{v}_i\rangle = \langle \phi_\xi, T \circ A_{ij}\rangle = \langle \phi_\xi, p_K(T \circ A_{ij})\rangle,$$

where $p_K : \mathrm{Hom}(E_\tau, W) \to \mathrm{Hom}_K(E_\tau, W)$ is the orthogonal projection. Note that $T(v_j)\cdot\tilde{v}_i \in \mathrm{Hom}(E_\tau, W)$, for all i, j. By setting, now in the last line $T'_{ij} := p_K(T \circ A_{ij})$, we get that $\langle \phi_\xi, T'_{ij}\rangle \in W$. Thus, for all $i \in \{1, \ldots, d_\tau\}$, we have $f_i \circ T \in W$. □

Similar, we state Delorme's intertwining conditions for our setting.

Definition 2.27 (Intertwining conditions in (Level 2) and (Level 3)). *Let $\tau, \gamma \in \hat{K}$ and consider the map J defined in (2.38).*

(2) We say that a function $\psi \in \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, H_\infty^{\tau|_M})$ satisfies the intertwining condition, if

$$J^{-1}\psi \in \bigoplus_{\sigma \subset \tau|_M} \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, [\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K)$$

satisfies the intertwining condition (1) *in Def. 2.25.*

(3) We say that a function $\varphi \in \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, \mathrm{Hom}_M(E_\gamma, E_\tau))$ satisfies the intertwining condition, if

$$\varphi(\lambda)\gamma(k^{-1})w \in \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, H_\infty^{\tau|_M})$$

for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}, k \in K$ and $w \in E_\gamma$, satisfies the above intertwining condition (2)*.*

Note that in (2)*, the function ψ is holomorphic in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and smooth in K/M.*

Combining the above result together with Frobenius-reciprocity, we have finally the equivalence of the intertwining conditions between each level.

Theorem 2.28 (Equivalence of the intertwining conditions in three levels). *Let $\overline{\Xi}$ be the set of all 2-tuples (λ, m) with $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $m \in \mathbb{N}_0$. For $\tau \in \hat{K}$ and each intertwining datum (ξ, W), consider*

$$\begin{aligned} D^\tau_W \;:=\;& \{t \in \bigoplus_{i=1}^{s} \mathrm{Hom}_M(E_\tau, E_{\sigma_i})^{\lambda_i}_{(m_i)} \mid T = \widetilde{Frob}^{-1}(t) \in \mathrm{Hom}_K(E_\tau, W) \subset \mathrm{Hom}_K(E_\tau, H_\xi)\} \\ \subset\;& \bigoplus_{i=1}^{s} \mathrm{Hom}_M(E_\tau, E_{\sigma_i})^{\lambda_i}_{(m_i)} \end{aligned} \tag{2.40}$$

and a finite sequence $\overline{\xi} = (\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_s) \in \overline{\Xi}^s$.

(D.2) *(Level 2) Then,* $\psi \in \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, H^{\tau|_M}_\infty)$ *satisfies the intertwining condition (2) of Def. 2.27 if, and only if, for each intertwining datum* (ξ, W) *and each* $t = (t_1, t_2, \ldots, t_s) \in D^\tau_W \neq \{0\}$, *the induced element* $\psi_{\overline{\xi}} \in \bigoplus_{i=1}^s H^{\tau|_M, \lambda_i}_{\infty,(m_i)} =: H^{\tau|_M}_{\overline{\xi}}$ *satisfies*

$$t \circ \psi_{\overline{\xi}} = (t_1 \circ \psi_1, \ldots, t_2 \circ \psi_s) \in W.$$

(D.3) *(Level 3) Then,* $\varphi \in \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, \mathrm{Hom}_M(E_\gamma, E_\tau))$ *satisfies the intertwining condition (3) of Def. 2.27 if, and only if, for each intertwining datum* (ξ, W) *and each* $t = (t_1, t_2, \ldots, t_s) \in D^\tau_W \neq \{0\}$, *the induced element* $\varphi_{\overline{\xi}} \in \bigoplus_{i=1}^s \mathrm{Hom}_M(E_\gamma, E_\tau)^{\lambda_i}_{(m_i)} =: H^{\gamma,\tau}_{\overline{\xi}}$ *satisfies*

$$t \circ \varphi_{\overline{\xi}} = (t_1 \circ \varphi_1, \ldots, t_2 \circ \varphi_s) \in D^\gamma_W.$$

Here, the m-th derived representation $\mathrm{Hom}_M(E_\tau, E_\sigma)^\lambda_{(m)}$ *as well as* $H^{\tau|_M,\lambda}_{\infty,(m)}$ *are defined similar as in (2.5) in Def. 2.3.*

Proof. We obtain directly the equivalence between $(D.1)$ and $(D.2)$ by applying the Frobenius-reciprocity, Prop. 2.26 and Prop. 2.24 (2).

Concerning $(D.2) \iff (D.3)$, one implication is trivial. For the other one, we have, by the inverse dual Frobenius-reciprocity, that

$$W \ni t \circ \psi_{\overline{\xi}} = t \circ \widetilde{Frob}^{-1}(\varphi_{\overline{\xi}})(w)(k) \overset{(2.15)}{=} t \circ \varphi_{\overline{\xi}} \circ \gamma(k^{-1})w, \quad \forall t \in D^\tau_W$$

for $w \in E_\gamma$ and $k \in K$. This means that $\widetilde{Frob}^{-1}(t \circ \varphi_{\overline{\xi}})(w) \in \mathrm{Hom}_K(E_\gamma, W)$ and hence by applying the dual Frobenius-reciprocity $\mathrm{Hom}_K(E_\gamma, W) \overset{\widetilde{Frob}}{\cong} D^\gamma_W$, this implies that $t \circ \varphi_{\overline{\xi}} \in D^\gamma_W$. □

Example 2.29. (a) Consider $s = 1$ and $m = 0$. Let $\xi := (\sigma, \lambda, 0) \in \Xi$ and $W \subset H^{\sigma,\lambda}_\infty$. Consider $D^\tau_W \subset \mathrm{Hom}_M(E_\tau, E_\sigma)$ as in Thm. 2.28. Then, we have the following intertwining conditions in the corresponding levels:

(D.2a) (Level 2) For each intertwining datum (ξ, W) and $t \in D^\tau_W \neq \{0\}$, we have

$$t \circ \psi(\lambda, \cdot) \in W.$$

Note that for each $\overline{\xi} \in \overline{\Xi}$, the induced element $\psi_{\overline{\xi}} = \psi(\lambda, \cdot)$.

(D.3a) (Level 3) For each intertwining datum (ξ, W) and $t \in D^\tau_W \neq \{0\}$, we have

$$t \circ \varphi(\lambda) \in D^\gamma_W.$$

Note that for each $\overline{\xi} \in \overline{\Xi}$, the induced element $\varphi_{\overline{\xi}} = \varphi(\lambda)$.

(b) Consider now $s = 2$ and $m_1 = m_2 = 0$. Let $L : H^{\sigma_1,\lambda_1}_\infty \longrightarrow H^{\sigma_2,\lambda_2}_\infty$ be an intertwining operator between the two principal series representations. Let $\xi := ((\sigma_1, \lambda_1, 0), (\sigma_2, \lambda_2, 0)) \in \Xi^2$ and $W = graph(L) \subset H^{\sigma_1,\lambda_1}_\infty \oplus H^{\sigma_2,\lambda_2}_\infty$. Moreover, define $l^\tau : \mathrm{Hom}_M(E_\tau, E_{\sigma_1}) \longrightarrow \mathrm{Hom}_M(E_\tau, E_{\sigma_2})$ by

$$l^\tau(t)(v) = L(t\tau(\cdot)^{-1}v)(e)$$

for $v \in E_\tau$ and $t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})$. Then

$$\begin{aligned} D^\tau_W = \{(t_1, t_2) \mid t_2 = l^\tau(t_1)\} &= \{(t, l^\tau(t)) \mid t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})\} \\ &\subset \mathrm{Hom}_M(E_\tau, E_{\sigma_1}) \oplus \mathrm{Hom}_M(E_\tau, E_{\sigma_2}). \end{aligned}$$

In this situation, we have the following intertwining conditions.

(D2.b) (Level 2) For each intertwining datum (ξ, W) and $t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})$, we have for $\psi(\lambda_i, \cdot) \in H_\infty^{\tau|_M,\cdot}$, $i = 1, 2$

$$L(t \circ \psi(\lambda_1, \cdot)) = l^\tau(t) \circ \psi(\lambda_2, \cdot). \tag{2.41}$$

(D3.b) (Level 3) For each intertwining datum (ξ, W) and $t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})$, we have for $\varphi(\lambda_i) \in \mathrm{Hom}_M(E_\gamma, E_\tau)$, $i = 1, 2$

$$l^\gamma(t \circ \varphi(\lambda_1)) = l^\tau(t) \circ \varphi(\lambda_2). \tag{2.42}$$

Now we are in the position to state Delorme's Paley-Wiener space for sections.

Definition 2.30 (Paley-Wiener space for sections in (Level 2) and (Level 3))**.**

(a) For $r > 0$, let $PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$ be the space of sections $\psi \in C^\infty(\mathfrak{a}_\mathbb{C}^* \times K/M, \mathbb{E}_{\tau|_M})$ be such that

(2.i) *the section ψ is holomorphic in $\lambda \in \mathfrak{a}_\mathbb{C}^*$.*

(2.ii)$_r$ *(growth condition) for all $Y \in \mathcal{U}(\mathfrak{k})$ and $N \in \mathbb{N}_0$, there exists a constant $C_{r,N,Y} > 0$ such that*

$$||l_Y \psi(\lambda, k)||_{E_\tau} \leq C_{r,N,Y}(1 + |\lambda|^2)^{-N} e^{r|\mathrm{Re}(\lambda)|}, \quad k \in K,$$

where $||\cdot||_{E_\tau}$ denotes the norm on finite-dimensional vector space E_τ (for convenience, we often denotes it by $|\cdot|$).

(2.iii) *(intertwining condition) (D.2) from Thm. 2.28.*

(b) By considering an additional K-type, let ${}_\gamma PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^*)$ be the space of functions

$$\mathfrak{a}_\mathbb{C}^* \ni \lambda \mapsto \varphi(\lambda) \in \mathrm{Hom}_M(E_\gamma, E_\tau)$$

be such that

(3.i) *the function φ is holomorphic in $\lambda \in \mathfrak{a}_\mathbb{C}^*$.*

(3.ii)$_r$ *(growth condition) for all $N \in \mathbb{N}_0$, there exists a constant $C_{r,N} > 0$ such that*

$$||\varphi(\lambda)||_{op} \leq C_{r,N}(1 + |\lambda|^2)^{-N} e^{r|\mathrm{Re}(\lambda)|},$$

where $||\cdot||_{op}$ denotes the operator norm on the corresponding space.

(3.iii) *(intertwining condition) (D.3) from Thm. 2.28.*

The inequalities provide semi-norms $||\cdot||_{r,N,Y}$ (resp. $||\cdot||_{r,N}$) on $PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$ (resp. ${}_\gamma PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^*)$) and made the vector space $PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$ (resp. ${}_\gamma PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^*)$) to Fréchet space, e.g. one can compare Lem. 10 of Delorme [Del05].
Combining Delorme's Paley-Wiener theorem with the above identifications and observations, this leads us to the Paley-Wiener theorem for our purposes.

Theorem 2.31 (Topological Paley-Wiener theorem for sections in (Level 2) and (Level 3))**.** *Let (τ, E_τ) be a K-representation with associated homogeneous vector bundle $\mathbb{E}_\tau$. For $r > 0$, then the Fourier transform*

$$C_r^\infty(X, \mathbb{E}_\tau) \ni \psi \mapsto \mathcal{F}_\tau(\psi)(\lambda, k) \in PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M), \quad (\lambda, k) \in \mathfrak{a}_\mathbb{C}^* \times K$$

is a topological isomorphism between $C_r^\infty(X, \mathbb{E}_\tau)$ and $PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$.
Moreover, by considering an additional K-representation (γ, E_γ) with associated homogeneous vector bundle $\mathbb{E}_\gamma$, then the Fourier transform

$$C_r^\infty(G, \gamma, \tau) \ni \varphi \mapsto {}_\gamma\mathcal{F}_\tau(\varphi)(\lambda) \in {}_\gamma PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^*), \quad \lambda \in \mathfrak{a}_\mathbb{C}^*$$

is a topological isomorphism between $C_r^\infty(G, \gamma, \tau)$ and ${}_\gamma PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^*)$. □

Furthermore, by taking the union of all $r > 0$, the Paley-Wiener space $PW_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$ is defined as

$$PW_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M) := \bigcup_{r>0} PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$$

similar for ${}_\gamma PW_\tau(\mathfrak{a}_\mathbb{C}^*)$. Hence, by the above result (Thm. 2.31), we also have a linear topological Fourier transform isomorphism from $C_c^\infty(X, \mathbb{E}_\tau)$ (resp. $C_c^\infty(G, \gamma, \tau)$) onto $PW_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$ (resp. ${}_\gamma PW_\tau(\mathfrak{a}_\mathbb{C}^*)$).

Remark 2.32. Helgason proved Thm. 2.31 in (Level 2) for trivial K-representation ([Hel89], Thm. 5.1.). In 1966, he even proved the topological isomorphism between the Paley-Wiener space in (Level 3) and the smooth space of K-bi-invariant functions on G defined in (2.13) for trivial K-representations and G of real rank one or with complex structure ([Hel20], Thm. 7.1). Gangolli [Gan71] completed the proof for arbitrary G.

2.2 Harish-Chandra inversion and the Plancherel theorem for sections

It is well-known that the Harish-Chandra inversion formula (2.44), also called Plancherel theorem, for smooth compactly function on G is given by the following theorem. A complete proof as well as the structure theory behind can be found for example in Wallach's book ([Wal92], Chap. 13).

Theorem 2.33 (Harish-Chandra inversion, [HC76])**.** *Let $\mathcal{Q}$ be a complete set of representatives of association classes of cuspidal parabolic subgroups $Q = M_Q A_Q N_Q$ with $Q \supset P = MAN$ and $A_Q \subset A$. Then, $\mathfrak{a}^* = \mathfrak{a}_Q^* \oplus \mathfrak{a}_{mQ}^*$.*
For each $Q \in \mathcal{Q}$ and any discrete series representations ξ of the corresponding group $\hat{M}_{Q,d}$, there exsits a meromorphic function of polynomial growth

$$\mu_\xi : i\mathfrak{a}_Q^* \longrightarrow [0, \infty]$$

known as the Plancherel density, which is regular, non-negative on $i\mathfrak{a}^*$, such that for each $f \in C_c^\infty(G) \subset L^2(G)$, we have

$$f(e) = \sum_{Q \in \mathcal{Q}} \sum_{\xi \in \hat{M}_{Q,d}} \int_{i\mathfrak{a}_Q^*} \mathrm{Tr}(\pi_{\xi,\lambda}(f)) \mu_\xi(\lambda)\, d\lambda. \tag{2.43}$$

Note that the Plancherel measures $\mu_\xi(\lambda)d\lambda$ depends on the normalization of the Haar measure dg. □

Moreover, for $f, \varphi \in C_c^\infty(G)$, the Plancherel formula in L^2 is given by

$$\langle f, \varphi \rangle_{L^2} := \int_G f(g)\overline{\varphi(g)}\, dg = \sum_{Q \in \mathcal{Q}} \sum_{\xi \in \hat{M}_{Q,d}} \int_{i\mathfrak{a}_Q^*} \mathrm{Tr}(\pi_{\xi,\lambda}(f)\pi_{\xi,\lambda}(\varphi)^*)\mu_\xi(\lambda)\, d\lambda,$$

where by φ^* we mean the adjoint of φ, and thus

$$\int_G \langle f(g), \varphi(g) \rangle\, dg = \sum_{Q \in \mathcal{Q}} \sum_{\xi \in \hat{M}_{Q,d}} \int_{i\mathfrak{a}_Q^*} \mathrm{Tr}(\pi_{\xi,\lambda}(f)\pi_{\tilde{\xi},-\lambda}(\varphi)^t)\mu_\xi(\lambda)\, d\lambda, \tag{2.44}$$

where $\tilde{\xi}$ is the dual of ξ and φ^t is the transpose of φ.

Nevertheless, we want to adapt the Harish-Chandra Plancherel inversion formula (2.44) for sections over homogeneous vector bundles. This will be useful later for the proof of the surjectivity and topological part of the Paley-Wiener theorem for distributions.

The following theorem is a consequence of Harish-Chandra Plancherel Thm. 2.33 and Casselman's embedding theorem (e.g. [Wal88], Thm. 3.8.3). More precisely, Camporesi ([Ca97], Thm. 3.4 & Thm. 4.3) have already proved a similar Harish-Chandra's inversion formulas as below for homogeneous vector bundles over X.

Theorem 2.34. *Let (τ, E_τ) be a finite-dimensional K-representation and $\mathcal{Q}$ as in Thm. 2.33. Then, there exists a finite set $A_Q^\tau \subset \mathfrak{a}_{m_Q}^* \subset \mathfrak{a}^*$ and for $\nu \in A_Q^\tau$, there exists an analytic function of at most polynomial growth*

$$\mu_\nu^Q : i\mathfrak{a}_Q^* \longrightarrow \mathrm{End}_M(E_\tau)$$

such that for each $f \in C_c^\infty(X, \mathbb{E}_\tau)$, we have

$$f(e) \;=\; \sum_{Q \in \mathcal{Q}} \sum_{\nu \in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \tau(k)\mu_\nu^Q(\lambda)\mathcal{F}_\tau(f)(\nu+\lambda, k)\, dk\, d\lambda.$$

Note that $A_P^\tau = \{0\}$. We postpone the proof of Thm. 2.34 and first derive the following corollary from it.

Corollary 2.35. *With the notations above, let $f \in C_c^\infty(X, \mathbb{E}_\tau)$ and $\varphi \in C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$. Then*

$$\int_G \langle \varphi(g), f(g) \rangle_\tau\, dg = \sum_{Q \in \mathcal{Q}} \sum_{\nu \in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \langle \mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda, k), \mu_\nu^Q(\lambda)\mathcal{F}_\tau(f)(\nu+\lambda, k)\rangle_\tau\, dk\, d\lambda. \tag{2.45}$$

Proof. Let $\{\tilde{v}_i, i = 1, \ldots, d_\tau\}$ be a vector basis of $E_{\tilde{\tau}}$. We write $\varphi = \sum_{i=1}^{d_\tau} \varphi_i \cdot \tilde{v}_i$ with $\varphi_i \in C_c^\infty(G)$. For $h \in C_c^\infty(G)$, we set $h^\vee(g) := h(g^{-1})$. Then

$$\int_G \langle \varphi(g), f(g) \rangle\, dg = \sum_{i=1}^{d_\tau} \langle (\varphi_i^\vee * f)(e), \tilde{v}_i \rangle,$$

where we used the usual convolution defined in (2.34). Note that $h * f = l(h)f$, where l is the (left) regular representation of G on $C_c^\infty(X, \mathbb{E}_\tau)$. By the G-equivariance of the

Fourier transform, we have by (2.35): $\mathcal{F}_\tau(h*f)(\lambda,k) = \pi_{\tau,\lambda}(h)(\mathcal{F}_\tau(f)(\lambda,\cdot))(k)$. By applying Thm. 2.34, we obtain for all $i \in \{1,\ldots,d_\tau\}$

$$\langle \tilde{v}_i, (\varphi_i^\vee * f)(e)\rangle = \sum_{Q,\nu}\int_{i\mathfrak{a}_Q^*}\int_K \langle \tilde{v}_i, \tau(k)\mu_\nu^Q(\lambda)\pi_{\tau,\nu+\lambda}(\varphi_i^\vee)(\mathcal{F}_\tau(f)(\nu+\lambda,\cdot))(k)\rangle\, dk\, d\lambda.$$

Using that μ_ν^Q commutes with $\pi_{\tau,\nu+\lambda}$ and that integration over K gives a G-equivariant pairing between $H_\infty^{\tau,\nu+\lambda}$ and $H_\infty^{\tilde{\tau},-(\nu+\lambda)}$, we obtain that the K-integral equals

$$\begin{aligned}\int_K \quad & \langle \tilde{\tau}(k^{-1})\tilde{v}_i, \quad \pi_{\tau,\nu+\lambda}(\varphi_i^\vee)\mu_\nu^Q(\lambda)(\mathcal{F}_\tau(f)(\nu+\lambda,\cdot))(k)\rangle\, dk \\ = \quad & \int_K \langle (\pi_{\tilde{\tau},-(\nu+\lambda)}(\varphi_i)\tilde{\tau}(\cdot)^{-1}\tilde{v}_i)(k), \quad \mu_\nu^Q(\lambda)\mathcal{F}_\tau(f)(\nu+\lambda,k)\rangle\, dk.\end{aligned}$$

Now

$$\begin{aligned}(\pi_{\tilde{\tau},-(\nu+\lambda)}(\varphi_i)\tilde{\tau}(\cdot)^{-1}\tilde{v}_i)(k) &= \int_G \varphi_i(g)\tilde{\tau}(\kappa(g^{-1}k))^{-1}a(g^{-1}k)^{\nu+\lambda-\rho}\tilde{v}_i\, dg \\ &= \int_G \varphi_i(g)e^{\tilde{\tau}}_{-(\nu+\lambda),k}(g)\tilde{v}_i\, dg.\end{aligned}$$

The sum over all i equals to $\mathcal{F}_{\tilde{\tau}}(\varphi)(-(\nu+\lambda),k)$. Combining all the previous formulas, we obtain the corollary. $\square$

Proof of Thm. 2.34. Our starting point is Harish-Chandra's Plancherel formula (2.43) for $f \in C_c^\infty(G)$. It remains valid for $f \in C_c^\infty(G,\mathbb{E}_\tau) \cong [C_c^\infty(G)\otimes E_\tau]^K$, if we set (with a slight abuse of notation) for $f = \sum_{i=1}^{d_\tau} f_i \otimes v_i \in [C_c^\infty(G)\otimes E_\tau]^K$, where $\{v_i, i=1,\cdots,d_\tau\}$ is a vector basis of E_τ:

$$\pi_{\xi,\lambda}(f) = \sum_{i=1}^{d_\tau}\pi_{\xi,\lambda}(f_i)\otimes v_i \in [\mathrm{End}(H_\infty^\xi)\otimes E_\tau]^K$$

and thus

$$\mathrm{Tr}(\pi_{\xi,\lambda}(f)\otimes v_i) = \sum_{i=1}^{d_\tau}\mathrm{Tr}(\pi_{\xi,\lambda}(f_i))\cdot v_i \in E_\tau.$$

Note that, because of the right K-finiteness of f_i, $\pi_{\xi,\lambda}(f_i)$ is an operator of finite rank, which has a well-defined trace independent of the theory of trace class operators or nuclear operators ([Wal88], 8.A.1). Moreover, for fixed τ and any $Q \in \mathcal{Q}$, there are only finitely many $\xi \in \hat{M}_{Q,d}$ really appearing in (2.43). Namely, for all but finitely many $\xi \in \hat{M}_{Q,d}$, we have $[\mathrm{End}(H_\infty^\xi)\otimes E_\tau]^K = \{0\}$. In fact, by Frobenius-reciprocity (2.14)

$$[\mathrm{End}(H_\infty^\xi)\otimes E_\tau]^K \overset{(1.8)}{\cong} H_\infty^\xi \otimes \mathrm{Hom}_K(H_\infty^\xi, E_\tau) \overset{Frob}{\cong} H_\infty^\xi \otimes \mathrm{Hom}_{K\cap M_Q}(E_\xi, E_\tau),$$

where E_ξ denotes the representation space for $\xi \in \hat{M}_{Q,d}$. By a result of Harish-Chandra ([Wal88], Cor. 7.7.3), there are only finitely many discrete series representations (ξ, E_ξ) of M_Q containing a given $K\cap M_Q$-type. This together with the finite dimension of E_τ gives the above assertion.

Second, we want to rewrite (2.43) entirely in terms of the minimal parabolic P. Casselman's embedding theorem says that any irreducible $(\mathfrak{g}, K)$-module E can be $(\mathfrak{g}, K)$-equivariantly embedded into a principal series representation $H_\infty^{\sigma,\lambda}$, for $(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}_\mathbb{C}^*$, corresponding to the minimal parabolic P of G. If E is the underlying $(\mathfrak{g}, K)$-module of a discrete series representation, then λ is necessarily real, i.e., $\lambda \in \mathfrak{a}^*$. This follows from the fact that E has integral, in particular real, infinitesimal character.
We apply this theorem to M_Q instead of G and to $E = E_{\xi,K\cap M_Q}$ by the $(m_Q, K \cap M_Q)$-module of $(K \cap M_Q)$-finite elements in E_ξ. We observe that $P \cap M_Q$ is a minimal parabolic of M_Q with 'M'-part being our original $M = M_P \subset P \subset G$. Then, we fix $\sigma = \sigma(\xi) \in \hat{M}$, $\nu = \nu(\xi) \in \mathfrak{a}_{m_Q}^*$ (in general not uniquely determined) such that there exists a $(m_Q, K \cap M_Q)$-equivariant embedding

$$E_{\xi,K\cap M_Q} \hookrightarrow {}^{M_Q}H_\infty^{\sigma,\nu},$$

where ${}^{M_Q}H_\infty^{\sigma,\nu}$ is the principal series representation for minimal parabolic of M_Q. Let ${}^{M_Q}W_\xi$ be the closure of the image of this embedding. It carries a M_Q-representation. By the Casselman-Wallach globalization theorem ([Wal92], Thm. 11.6.6.) this M_Q representation is isomorphic to the space of smooth vectors $E_{\xi,\infty}$.
Now we fix $\lambda \in i\mathfrak{a}_Q^*$. This equips ${}^{M_Q}H_\infty^{\sigma,\nu}$ (and hence also ${}^{M_Q}W_\xi$) with the structure of a Q-representation ($Q = M_Q A_Q N_Q$ and N_Q acts trivally). Now we induce from Q to G (smooth double induction):

$$\mathrm{Ind}_Q^G({}^{M_Q}H_\infty^{\sigma,\nu}) \cong H_\infty^{\sigma,\nu+i\lambda},$$

where $H_\infty^{\sigma,\nu+i\lambda}$ is the principal series representation of G and

$$W^\xi := \mathrm{Ind}_Q^G({}^{M_Q}W_\xi) \cong H_\infty^{\xi,\lambda}$$

is a closed G-equivariant subspace of $H_\infty^{\sigma,\nu+i\lambda}$. We will view it in the compact picture $W^\xi \subset H_\infty^\sigma$. Then, it is really independent of $\lambda \in i\mathfrak{a}_Q^*$.
We will only consider the finitely many $\xi \in \hat{M}_{Q,d}$ with $\mathrm{Hom}_K(H_\infty^{\xi,\lambda}, E_\tau) \neq \{0\}$, i.e., $\mathrm{Hom}_K(W^\xi, E_\tau) \neq \{0\}$, which implies by Frobenius-reciprocity

$$\mathrm{Hom}_M(E_{\sigma(\xi)}, E_\tau) \neq \{0\}.$$

The above discussion implies that we can replace $\mathrm{Tr}(\pi_{\xi,\lambda}(f))$ in (2.43) by

$$\mathrm{Tr}_{W^\xi}(\pi_{\sigma(\xi),\nu(\xi)+\lambda}(f)), \tag{2.46}$$

where Tr_{W^ξ} denotes the trace of the restriction of W^ξ of an operator leaving W^ξ invariant.

As a last step, we make the connection to the Fourier transform $\mathcal{F}_\tau$. The crucial observation is given in Lem. 2.36. We remark that for $W = H_\infty^{\sigma,\lambda}$, we have $p_W = p_\sigma$, where p_σ is the orthogonal projection to the M-isotypic component $E_\tau(\sigma) \subset E_\tau$.
For $\nu \in A_Q^\tau$ and $\lambda \in i\mathfrak{a}_Q^*$, we set

$$\mu_\nu^Q(\lambda) := \sum_B \frac{\mu_\xi(\lambda)}{d_{\sigma(\xi)}} \cdot p_{W^\xi} \in \mathrm{End}_M(E_\tau),$$

where $B := \{\xi \in \hat{M}_{Q,d} \mid \nu(\xi) = \nu, \mathrm{Hom}_{K\cap M_Q}(E_\xi, E_\tau) \neq \{0\}\}$. Then, (2.43) with the replacement (2.46) together with Lem. 2.36, gives the desired formula for $f(e)$. □

Lemma 2.36. *Consider $f \in C_c^\infty(X, \mathbb{E}_\tau)$. Let $(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}_{\mathbb{C}}^*$ and $W \subset H_\infty^{\sigma,\lambda}$ be a closed G-invariant subspace.*
Then, there exists an orthogonal projection $p_W \in \mathrm{End}_M(E_\tau)$ with values in $E_\tau(\sigma)$ such that

$$\mathrm{Tr}_W(\pi_{\sigma,\lambda}(f)) = \frac{1}{d_\sigma} \int_K \tau(k) p_W(\mathcal{F}_\tau(f)(\lambda, k))\, dk.$$

Proof. As in Thm. 2.28, we consider the space

$$D_W^\tau = \{t \in \mathrm{Hom}_M(E_\tau, E_\sigma) \mid T = \widetilde{Frob}^{-1}(t) \in \mathrm{Hom}_K(E_\tau, W) \subset \mathrm{Hom}_K(E_\tau, H_\infty^\sigma)\}.$$

We set $A_W := \{v \in E_\tau \mid t(v) = 0, \forall t \in D_W^\tau\}$ and $B_W := A_W^\perp \subset E_\tau$. Note that $B_W \subset E_\tau(\sigma)$. Let p_W be the orthogonal projection to B_W. We have a natural isomorphism:

$$\pi_{\sigma,\lambda}(f) \in [\mathrm{End}(H_\infty^\sigma) \otimes E_\tau]^K \overset{(1.8)+Frob}{\cong} \mathrm{Hom}_M(E_\sigma, E_\tau) \otimes H_\infty^\sigma.$$

We choose a basis $s_1, \ldots, s_k, s_{k+1}, \ldots, s_l$ of $\mathrm{Hom}_M(E_\sigma, E_\tau)$ such that

$$\begin{aligned} \mathrm{Im}(s_j) &\subset B_W, j = 1, \ldots, k \\ \mathrm{Im}(s_j) &\subset A_W, j = k+1, \ldots, l. \end{aligned} \tag{2.47}$$

Let $\sum_{j=1}^l s_j \otimes \varphi_j \in \mathrm{Hom}_M(E_\sigma, E_\tau) \otimes H_\infty^\sigma$ be the element corresponding to $\pi_{\sigma,\lambda}(f)$. Since W is G-invariant by Thm. 2.28, we have $\varphi_j \in W$, for $j = 1, \ldots, k$. Also, by (2.19), we have

$$p_\sigma(\mathcal{F}_\tau(f)) = d_\sigma \sum_{j=1}^l s_j \cdot \varphi_j. \tag{2.48}$$

Now let $\{v_i, i = 1 \ldots, d_\tau\}$ be a vector basis of E_τ with dual basis $\tilde{v}_i$ of $E_{\tilde{\tau}}$. We write $f = \sum_{i=1}^{d_\tau} f_i \otimes v_i$, for $f_i \in C_c^\infty(G)$ and $\pi_{\sigma,\lambda}(f) = \sum_{i=1}^{d_\tau} \pi_{\sigma,\lambda}(f_i) \otimes v_i$.
Then, by Lem. 2.11, the operator $\pi_{\sigma,\lambda}(f_i)$ is given by the operator

$$H_\infty^\sigma \ni \psi \mapsto \sum_{j=1}^l \langle S_j(\psi), \tilde{v}_i \rangle \varphi_j =: \sum_{j=1}^l A_{ij}(\psi), \quad \forall i = 1 \ldots, d_\tau,$$

where $S_j(\psi) := \int_K \tau(k) s_j \psi(k)\, dk$.

We claim that, for $j = k+1, \ldots, l$, we have $S_j\big|_W = 0$. Indeed, let $T \in \mathrm{Hom}_K(E_\tau, W)$ with corresponding $t = \widetilde{Frob}(T) \in D_W^\tau$. As in the proof of Prop. 2.24 (a), we obtain

$$\mathrm{Tr}_\tau(S_j \circ T) = \mathrm{Tr}_\sigma(t \circ s_j).$$

The right hand side vanishes since by (2.47), we have $t \circ s_j = 0$. Since the pairing between $\mathrm{Hom}_K(W, E_\tau)$ and $\mathrm{Hom}_K(E_\tau, W) : S, T \mapsto \mathrm{Tr}_\tau(S \circ T)$ is non-degenerate, we obtain $S_j\big|_W = 0$. The claim follows.

The claim implies that $\mathrm{Tr}_\tau(\pi_{\sigma,\lambda}(f)) = \sum_{i=1}^{d_\tau} \sum_{j=1}^k \mathrm{Tr}_W A_{ij} \cdot v_i$. Note that A_{ij} is an operator of rank one (or zero) with image spanned by $\varphi_j \in W$. Hence, $\mathrm{Tr}_W A_{ij} = \mathrm{Tr} A_{ij} = \langle S_j(\varphi_j), \tilde{v}_i \rangle$. We obtain

$$\begin{aligned} \mathrm{Tr}_W(\pi_{\sigma,\lambda}(f)) = \sum_{j=1}^k \int_K \tau(k) s_j \varphi_j(k)\, dk &= \sum_{j=1}^l \int_K \tau(k) p_W s_j \varphi_j(k)\, dk \\ &\overset{(2.48)}{=} \frac{1}{d_\sigma} \int_K \tau(k) p_W p_\sigma \mathcal{F}_\tau(f)\, dk \\ &= \frac{1}{d_\sigma} \int_K \tau(k) p_W(\mathcal{F}_\tau(f))\, dk, \end{aligned}$$

where $p_W p_\sigma = p_W$. □

2.3 Distributional topological Paley-Wiener theorem

So far, we have seen that the Paley-Wiener theorem for sections (Thm. 2.31) asserts that the Fourier transform is a linear topological isomorphism from $C_c^\infty(X, \mathbb{E}_\tau)$ onto $PW_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$ (resp. $C_c^\infty(G, \gamma, \tau)$ onto ${}_\gamma PW_\tau(\mathfrak{a}_\mathbb{C}^*)$). Nevertheless, the theorem has a counterpart, the Paley-Wiener-Schwartz theorem, where smooth functions are replaced by distributions and the growth condition by a condition of 'slow' growth. We use, here, the concept of 'distributions' for generalized functions.

We start in Subsection 2.3.1 by bringing up a framework for dual spaces equipped with their natural topologies. This will be useful later, to establish the topological Fourier isomorphism between the Paley-Wiener-Schwartz space and a space of smooth compactly supported distributions.

In Subsection 2.3.2 we introduce the Fourier transform for distributions as well as the Paley-Wiener-Schwartz spaces with their corresponding topologies.

The interesting part lies in Subsection 2.3.3, where we prove the topological Paley-Wiener-Schwartz theorem (Thm. 2.40) using the Fourier inversion formula, introduced in Section 2.2. Helgason's distributional topological Paley-Wiener theorem on X ([Hel89], Cor. 5.9., Chap. 3) as well as van den Ban and Schlichtkrull's paper [vdBS06] are the main sources of inspiration for our proof.

2.3.1 Dual spaces and their corresponding topologies

In Sect. 1.3, we already have seen that the space of compactly supported distributions $C_c^{-\infty}(X, \mathbb{E}_\tau) := (C^\infty(X, \mathbb{E}_{\tilde{\tau}}))'$ is the topological linear dual of $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$. Likewise for space of distributions, we have

$$C^{-\infty}(X, \mathbb{E}_\tau) := \ (C_c^\infty(X, \mathbb{E}_{\tilde{\tau}}))'.$$

We provide $C_c^{-\infty}(X, \mathbb{E}_\tau)$ with the *strong dual topology*. Actually, we know that $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$ is a Fréchet space with semi-norm

$$||h||_{\Omega,Y} := \sup_{g\in\Omega} |l_Y h(g)|, \quad h \in C^\infty(X, \mathbb{E}_{\tilde{\tau}}), \tag{2.49}$$

where $Y \in \mathcal{U}(\mathfrak{g})$ and Ω is a compact subset of G. Furthermore, a subset B is called bounded, if it is a subset of $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$, then for each $\Omega \subset G$ and $Y \in \mathcal{U}(\mathfrak{g})$ there exists a constant $C_{\Omega,Y} > 0$ such that $\sup_{\varphi\in B} ||\varphi||_{\Omega,Y} \leq C_{\Omega,Y}$. Thus, every semi-norm is bounded on B.
The strong dual topology on $C_c^{-\infty}(X, \mathbb{E}_\tau)$ is a locally convex topology vector space given by the semi-norm system

$$p_B(T) := ||T||_B = \sup_{\varphi\in B} |T(\varphi)| = \sup_{\varphi\in B} |\langle T, \varphi\rangle|, \quad T \in C_c^{-\infty}(X, \mathbb{E}_\tau), \tag{2.50}$$

where B belongs to the family of all bounded subsets of $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$.
As an immediate consequence of theses dualities, the topologies on $C_c^{-\infty}(X, \mathbb{E}_\tau)$ and $C^{-\infty}(X, \mathbb{E}_\tau)$ induce the same topology on the space of distributions supported in a fixed

compact subset Ω of G ([vdBS06], Sect. 14). For example, one can take $\Omega = \overline{B}_r(o)$. Later, we will use the weak-* topology that is defined in a similar way, except that the bounded subsets are replaced by finite one.

A subset $B' \subset C_c^{-\infty}(X, \mathbb{E}_\tau)$ is bounded in the strong dual topology, if for each bounded $B \subset C^\infty(X, \mathbb{E}_{\tilde{\tau}})$, we have

$$\sup_{T \in B'} p_B(T) = \sup_{T \in B', \varphi \in B} |T(\varphi)| < \infty. \tag{2.51}$$

Since, by Schaefer ([Sch71], Cor. 1.6, p. 127), we know that all such sets B' are equicontinuous, this means that there exist a continuous semi-norm p on $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$ and a constant $C > 0$ such that

$$B' \subset \{T \in C_c^{-\infty}(X, \mathbb{E}_\tau) \mid |T(\varphi)| \leq Cp(\varphi), \forall \varphi \in C^\infty(X, \mathbb{E}_{\tilde{\tau}})\}.$$

Let $Y_1, \ldots, Y_n$ be a basis, then for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n), \alpha_j \in \mathbb{N}_0$, we have

$$Y_\alpha := Y_1^{\alpha_1} \cdots Y_n^{\alpha_n} \in \mathcal{U}(\mathfrak{g}).$$

We may assume that this semi-norm p has the form

$$p(\varphi) := \sum_{|\alpha| \leq m} ||\varphi||_{\Omega, \alpha} \overset{(2.49)}{=} \sum_{|\alpha| \leq m} \sup_{g \in \Omega} |l_{Y_\alpha} \varphi(g)|, \qquad \varphi \in C^\infty(X, \mathbb{E}_{\tilde{\tau}}), \forall \alpha \tag{2.52}$$

for some $m \in \mathbb{N}_0$ and compact $\Omega \subset G$.

It is interesting to notice that $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$ is a reflexive Fréchet space, even a Montel space, that is, it is reflexive and a subset is bounded if, and only if, it is relatively compact ([Sch71], p. 147).
Thus, since $C_c^{-\infty}(X, \mathbb{E}_\tau)$ is the strong dual space of a Montel space $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$, we can deduce by Cor. 1 in ([Sch71], p. 154) that $C_c^{-\infty}(X, \mathbb{E}_\tau)$ is a bornological space, that is a locally convex space on which each semi-norm p_B, which is bounded on bounded subsets, is continuous ([Sch71], Chap. 2.8, p. 61).
This observation leads us to the following general result, which will play an imporant role in the topological statement of the main theorem. For bornological spaces, bounded linear maps are continuous ([Sch71], Thm. 8.3., p. 62), hence, we obtain the following.

Lemma 2.37. *Let W be any locally convex topological vector space and consider a linear map $A : C_c^{-\infty}(X, \mathbb{E}_\tau) \to W$. Then A is continuous if, and only if, $A(B')$ is bounded in W, for every bounded subset $B' \subset C_c^{-\infty}(X, \mathbb{E}_\tau)$.* □

Analogously, we equip the space $C_c^{-\infty}(G, \gamma, \tau)$ with the strong dual topology of $C^\infty(G, \tilde{\gamma}, \tilde{\tau})$.

2.3.2 Distributional Fourier transform and Paley-Wiener-Schwartz spaces

In a similar way as in Def. 2.14, we define the Fourier transform for distributional functions on sections.

Definition 2.38 (Fourier transform on distributions). *Let $e^{\tau}_{\lambda,k} \in C^{\infty}(X, \mathbb{E}_{\tilde{\tau}}) \otimes E_{\tau}$ be the 'exponential' function (2.20). The Fourier transform for distributional function $T \in C_c^{-\infty}(X, \mathbb{E}_{\tau})$ is defined by*

$$\mathcal{F}_{\tau}T(\lambda, k) := \langle T, e^{\tau}_{\lambda,k}\rangle = T(e^{\tau}_{\lambda,k}) \in E_{\tau}, \quad (\lambda, k) \in \mathfrak{a}^*_{\mathbb{C}} \times K/M.$$

Similarly, by considering an additional finite-dimensional, not necessary irreducible, K-representation (γ, E_{γ}), we define the Fourier transform for $S \in C_c^{-\infty}(G, \gamma, \tau)$ by

$${}_{\gamma}\mathcal{F}_{\tau}S(\lambda) := \langle S, e^{\tau}_{\lambda,1}\rangle, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}.$$

Analogously as for smooth compactly functions (2.34), we define the convolution for distributions $T \in C_c^{-\infty}(X, \mathbb{E}_{\tau})$ by

$$(T * \varphi)(g) := T(l_g \varphi^{\vee}) = \langle T, l_g \varphi^{\vee}\rangle, \quad g \in G, \varphi \in C_c^{\infty}(G, \tau, \tau),$$

where $\varphi^{\vee} \in C_c^{\infty}(X, \mathbb{E}_{\tilde{\tau}}) \otimes E_{\tau}$ is given by $\varphi^{\vee}(g) := \varphi(g^{-1}), g \in G$. Note that the results obtained in Prop, 2.21, its Cor. 2.23 and Remark 2.22 can be applied for distributions $T \in C_c^{-\infty}(X, \mathbb{E}_{\tau})$ as well.

Now let us define the Paley-Wiener space in (Level 2) and (Level 3) for distributions.

Definition 2.39 (Paley-Wiener-Schwartz space for sections). *For $r \geq 0$, the Paley-Wiener-Schwartz space $PWS_{\tau,r}(\mathfrak{a}^*_{\mathbb{C}} \times K/M)$ for sections over homogeneous vector bundle $\mathbb{E}_{\tau}$ in (Level 2), is the vector space which contains all smooth sections $\psi \in C^{\infty}(\mathfrak{a}^*_{\mathbb{C}} \times K/M, \mathbb{E}_{\tau_{|M}})$ satisfying conditions $(2.i)$, $(2.iii)$ of Def. 2.30 and with the following slow growth condition:*

$(2.iis)_r$ *for all multi-indices α, there exist $N \in \mathbb{N}_0$ and a positive constant $C_{r,N,\alpha}$ such that*

$$||l_{Y_{\alpha}}\psi(\lambda, k)||_{E_{\tau}} \leq C_{r,N,\alpha}(1 + |\lambda|^2)^{N+\frac{|\alpha|}{2}} e^{r|\mathrm{Re}(\lambda)|}, \quad k \in K.$$

The Paley-Wiener-Schwartz space in (Level 3), for two K-representations, is denoted by ${}_{\gamma}PWS_{\tau,r}(\mathfrak{a}^*_{\mathbb{C}})$. Here, the condition $(3.ii)_r$ in Def. 2.30 is replaced by

$(3.iis)_r$ *there exist $N \in \mathbb{N}_0$ and a positive constant $C_{r,N}$ such that*

$$||\varphi(\lambda)||_{op} \leq C_{r,N}(1 + |\lambda|^2)^N e^{r|\mathrm{Re}(\lambda)|}, \quad \varphi \in \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, \mathrm{Hom}_M(E_{\gamma}, E_{\tau})).$$

Next, we will provide $PWS_{\tau}(\mathfrak{a}^*_{\mathbb{C}} \times K/M) := \bigcup_{r \geq 0} PWS_{\tau,r}(\mathfrak{a}^*_{\mathbb{C}} \times K/M)$ (similar for ${}_{\gamma}PWS_{\tau}(\mathfrak{a}^*_{\mathbb{C}})$) with the *inductive limit topology*. We will only explain the topology for the space $PWS_{\tau}(\mathfrak{a}^*_{\mathbb{C}} \times K/M)$, since the procedure is the same for ${}_{\gamma}PWS_{\tau}(\mathfrak{a}^*_{\mathbb{C}})$.
For all $N \in \mathbb{N}_0$ and $r \geq 0$, we consider the Paley-Wiener-Schwartz space:

$$PWS_{\tau,r,N} := \{\psi \in PWS_{\tau,r}(\mathfrak{a}^*_{\mathbb{C}} \times K/M) \mid ||\psi||_{r,N,\alpha} < \infty, \forall \alpha\}.$$

It is not difficult to see that $PWS_{\tau,r,N}$ equipped with the semi-norm

$$||\psi||_{r,N,\alpha} := \sup_{\lambda \in \mathfrak{a}^*_{\mathbb{C}},\, k \in K/M} (1 + |\lambda|^2)^{-(N+\frac{|\alpha|}{2})} e^{-r|\mathrm{Re}(\lambda)|} ||l_{Y_{\alpha}}\psi(\lambda, k)||_{E_{\tau}}, \quad \forall \alpha, k \in K$$

is a Fréchet space (e.g. see [vdBS06], Lem. 15.2.).

Now, by following Schaefer's requirements ([Sch71], p. 57) for the construction of an inductive limit topology, consider $PWS_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^* \times K/M) = \bigcup_{N\in\mathbb{N}_0} PWS_{\tau,r,N}$ such that $PWS_{\tau,r,N_1} \neq PWS_{\tau,r,N_2}$ for $N_1 \neq N_2 \in \mathbb{N}_0$, directed under inclusion. Moreover, on each PWS_{τ,r,N_1} consider a (Hausdorff) locally convex topology $\mathcal{T}_{N_1}$, so that, whenever $N_1 \leq N_2$, the topology induced by $\mathcal{T}_{N_2}$ on PWS_{τ,r,N_2} is finer than $\mathcal{T}_{N_1}$. Thus, for $N_1 \leq N_2$, we have that the canonical embedding

$$PWS_{\tau,r,N_1} \overset{i_{N_1,N_2}}{\hookrightarrow} PWS_{\tau,r,N_2}$$

is continuous. In these circumstances, the family of spaces $PWS_{\tau,r,N}$ indexed by $N \in \mathbb{N}_0$, is thus a directed family and we can give $PWS_{\tau,r}$ the finest locally convex topology, called the inductive limit topology, of Fréchet spaces for the union over N.
Notice, that it is not a strict inductive limit, that is, if $\mathcal{T}_{N_2}$ induces $\mathcal{T}_{N_1}$ on PWS_{τ,r,N_1}, whenever $N_1 \leq N_2$, and that for each $N \in \mathbb{N}_0$, the space $PWS_{\tau,r,N}$ is closed.
Furthermore, this topology is characterized by a linear continuous map
$A : PWS_{\tau,r} \to W$, where W is any locally convex space if, and only if,

$$PWS_{\tau,r,N} \overset{i_N}{\hookrightarrow} PWS_{\tau,r} \xrightarrow{A} W$$

is continuous, i.e., $A \circ i_N$ is continuous. As next step, we provide $\bigcup_{r\geq 0} PWS_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^* \times K/M) = PWS_{\tau}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ with the inductive limit topology, in the same way as above.

2.3.3 On topological Paley-Wiener-Schwartz theorem and its proof

We are now in the position to state the main theorem.

Theorem 2.40 (Topological Paley-Wiener-Schwartz theorem for sections)**.**

(a) *Let (τ, E_τ) be a finite-dimensional K-representation with associated homogeneous vector bundle $\mathbb{E}_\tau$.*
Then, for each $r \geq 0$, the Fourier transform $\mathcal{F}_\tau$ is a linear bijection between the two spaces $C_r^{-\infty}(X, \mathbb{E}_\tau)$ and the Paley-Wiener-Schwartz space $PWS_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$. Moreover, it is a linear topological isomorphism from $C_c^{-\infty}(X, \mathbb{E}_\tau)$ onto $PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$.

(b) *Similarly, if we consider an additional finite-dimensional K-representation (γ, E_γ) with associated homogeneous vector bundle $\mathbb{E}_\gamma$. Then, the Fourier transform ${}_\gamma\mathcal{F}_\tau$ is a linear bijection between the two spaces $C_r^{-\infty}(G, \gamma, \tau)$ and ${}_\gamma PWS_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^*)$, for each $r \geq 0$, and a linear topological isomorphism from $C_c^{-\infty}(G, \gamma, \tau)$ onto ${}_\gamma PWS_\tau(\mathfrak{a}_{\mathbb{C}}^*)$.*

Remark 2.41. Delorme proved in his paper [Del05], the Paley-Wiener theorem in (Level 1) for Hecke algebra

$$\mathcal{H}(G, K) := C_{r=0}^{-\infty}(G)_K \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} C^\infty(K)_K, \tag{2.53}$$

which consists of all $K \times K$-finite distributions on G supported by $K \subset G$.

Let us first prove the injectivity and surjectivity of the Fourier transform.

Proposition 2.42. *Consider a finite-dimensional K-representation (τ, E_τ).*

(a) Let $T \in C_c^{-\infty}(X, \mathbb{E}_\tau)$ such that its Fourier transform $\mathcal{F}_\tau(T) = 0$, then $T = 0$.

(b) For $\tilde{T} \in PWS_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$, there exists $T \in C_c^{-\infty}(X, \mathbb{E}_\tau)$ such that $\tilde{T} = \mathcal{F}_\tau(T)$.

(c) For $r \geq 0$, let $T \in C_r^{-\infty}(X, \mathbb{E}_\tau)$, then its Fourier transform $\mathcal{F}_\tau(T)$ satisfies the conditions of the Paley-Wiener-Schwartz space $PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$ in Def. 2.39.

Proof. For each $\epsilon > 0$, consider a K-bi-invariant endomorphism function $\eta_\epsilon \in C^\infty(G, \tau, \tau)$ with compact support on the closed ball $\overline{B}_\epsilon(o)$ as in Cor. 2.23. Let $T \in C_c^{-\infty}(X, \mathbb{E}_\tau)$ be a distribution, then

$$T_\epsilon := T * \eta_\epsilon \in C_c^\infty(X, \mathbb{E}_\tau).$$

Moreover, by using the same arguements as in the proof of Cor. 2.23, we have that $T_\epsilon \xrightarrow{\epsilon \to 0} T$ (weakly). Hence, by the Paley-Wiener Thm. 2.31, this implies that $\mathcal{F}_\tau(T_\epsilon) \in PW_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$. Note that $\mathcal{F}_\tau(T_\epsilon)$ is holomorphic on $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and it satisfies the conditions $(2.i)$ and $(2.ii)_r$ of Def. 2.30. Furthermore, by Prop. 2.21, we have

$$\mathcal{F}_\tau(T_\epsilon)(\lambda, k) = {}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\lambda)\mathcal{F}_\tau(T)(\lambda, k), \quad (\lambda, k) \in \mathfrak{a}_\mathbb{C}^* \times K/M. \tag{2.54}$$

Due to Cor. 2.23, ${}_\tau\mathcal{F}_\tau(\eta_\epsilon)$ converges uniformly on compact subsets of $\mathfrak{a}_\mathbb{C}^*$ to the identity map, whenever ϵ tends to 0. Hence, $\lim_{\epsilon\to 0} \mathcal{F}_\tau(T_\epsilon) = \mathcal{F}_\tau(T)$ uniformly on compact sets on $\mathfrak{a}_\mathbb{C}^*$.

(a) Now assume that $\mathcal{F}_\tau(T) = 0$. By (2.54), we have that $\mathcal{F}_\tau(T_\epsilon) = 0$. By applying the Paley-Wiener Thm. 2.31, this implies that $T_\epsilon = 0$. Hence, since $T_\epsilon \xrightarrow{\epsilon\to 0} T$ weakly, we have that $T = 0$. This means that $T \mapsto \mathcal{F}_\tau(T)$ is injective on $C_c^{-\infty}(X, \mathbb{E}_\tau)$.

(b) Consider $\psi \in PWS_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$. For each $\epsilon > 0$ and $h \in C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$, let T_ϵ be the functional given by

$$\begin{aligned} T_\epsilon(h) \quad := \quad & \sum_{Q\in\mathcal{Q}} \sum_{\nu\in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \langle \mathcal{F}_{\tilde{\tau}}(h)(-\nu-\lambda, k)\,, \\ & \mu_\nu^Q(\lambda)\, {}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\nu+\lambda)\psi(\nu+\lambda, k)\rangle\; dk\; d\lambda \end{aligned} \tag{2.55}$$

under the same notations introduced in Thm. 2.34. Notice that, since $\mathrm{supp}(\eta_\epsilon) \subset \overline{B}_\epsilon(o)$ and ψ satisfies the slow growth condition $(2.iis)_r$ of Def. 2.39, for all $r \geq 0$, this implies that for each multi-index $\alpha \in \mathbb{N}_0$ and $N \in \mathbb{N}_0$, there exists a constant $C_{r,N,\alpha} > 0$ such that

$$|l_{Y_\alpha}\, {}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\lambda)\psi(\lambda, k)| \leq C_{r,N,\alpha}(1+|\lambda|^2)^{-N} e^{(r+\epsilon)|\mathrm{Re}(\lambda)|}, \quad (\lambda, k) \in \mathfrak{a}_\mathbb{C}^* \times K. \tag{2.56}$$

In addition, for each intertwining datum (ξ, W), the induced operator $({}_\tau\mathcal{F}_\tau(\eta_\epsilon)\psi)_{\overline{\xi}} = {}_\tau\mathcal{F}_\tau(\eta_\epsilon)_{\overline{\xi}}\psi_{\overline{\xi}} \in H_\infty^{\tau|_M}$ satisfies the intertwining condition $(2.iii)$ of Def. 2.30. In fact, for $t \in D_W^\tau$, we have $t \circ {}_\tau\mathcal{F}_\tau(\eta_\epsilon)_{\overline{\xi}} \in D_W^\tau$ and since $\psi \in PWS_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$, this implies that

$$(t \circ {}_\tau\mathcal{F}_\tau(\eta_\epsilon)_{\overline{\xi}}) \circ \psi_{\overline{\xi}} \in W.$$

Therefore, by the Paley-Wiener Thm. 2.31, we have that ${}_\tau\mathcal{F}_\tau(\eta_\epsilon)\psi$ is the Fourier transform of a function $f_\epsilon \in C_c^\infty(X, \mathbb{E}_\tau)$, i.e.,

$$\mathcal{F}_\tau(f_\epsilon) := {}_\tau\mathcal{F}_\tau(\eta_\epsilon)\psi.$$

On the other side, by (2.55) and Cor. 2.35, we have $T_\epsilon = f_\epsilon$. By (2.56), we have that $\operatorname{supp}(T_\epsilon) \subset \overline{B}_{r+\epsilon}(o)$. Thus, by Cor. 2.23, this implies that

$$T_\epsilon(h) \xrightarrow{\epsilon\to 0} T(h) := \sum_{Q\in\mathcal{Q}} \sum_{\nu\in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \langle \mathcal{F}_{\tilde{\tau}}(h)(-\nu-\lambda, k),\ \mu_\nu^Q(\lambda)\psi(\nu+\lambda, k)\rangle\, dk\, d\lambda \tag{2.57}$$

and thus $\operatorname{supp}(T) \subset \overline{B}_r(o)$. Note that μ_ν^Q has maximal polynomial growth, thus T is continuous. Since T is compactly supported, we can set $h := e_{\lambda,k}^\tau$. In conclusion, we have found a distribution $T \in C_c^{-\infty}(X, \mathbb{E}_\tau)$ such that

$$\begin{aligned}\mathcal{F}_\tau(T)(\lambda,k) = T(e_{\lambda,k}^\tau) \overset{(2.57)}{=} \lim_{\epsilon\to 0} T_\epsilon(e_{\lambda,k}^\tau) = \lim_{\epsilon\to 0}\mathcal{F}_\tau(f_\epsilon)(\lambda,k) &= \lim_{\epsilon\to 0} {}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\lambda)\psi(\lambda,k)\\ &= \psi(\lambda,k).\end{aligned}$$

(c) Let us check that for $r \geq 0$, $\mathcal{F}_\tau(T) \in PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$. This means that we need to verify that the Fourier transform of $T \in C_r^{-\infty}(X, \mathbb{E}_\tau)$ satisfies the conditions $(2.i) - (2.iii)$ of Def. 2.30.
The condition $(2.i)$ is immediate. By Prop. 2.16, we know that $e_{\lambda,k}^\tau$ is an entire function with respect to $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and since $T \in C_r^{-\infty}(X, \mathbb{E}_\tau)$, this implies that $\mathcal{F}_\tau(T) = T(e_{\lambda,k}^\tau)$ is smooth in $(\lambda, k) \in \mathfrak{a}_\mathbb{C}^* \times K$ and holomorphic in $\lambda \in \mathfrak{a}_\mathbb{C}^*$.
Concerning the intertwining condition $(2.iii)$, in order to show that for each intertwining datum (ξ, W) and $t \in D_W^\tau$, we have

$$t \circ (\mathcal{F}_\tau(T))_{\overline{\xi}} \in W \subseteq H_\xi,$$

we will use a similar convolution argument as above, except that now we are interested to the convolution on the left instead on the right. For each $\epsilon > 0$, let $\delta_\epsilon \in C_r^\infty(G)$ be a delta-sequence such that $\lim_{\epsilon\to 0}\delta_\epsilon = \delta_0$. Hence, $\lim_{\epsilon\to 0}\delta_\epsilon * T = T$, for $T \in C_r^{-\infty}(X, \mathbb{E}_\tau)$. Moreover, for all representations $(\pi_{\tau,\lambda}, H)$ with Fréchet space H and $v \in H$, we have $\pi_{\tau,\lambda}(\delta_\epsilon)v \xrightarrow{\epsilon\to 0} v$. By taking the Fourier transform on $\delta_\epsilon * T \in C_r^\infty(X, \mathbb{E}_\tau)$, it is sufficient to prove that for each intertwining datum (ξ, W) and $t \in D_W^\tau$:

$$\lim_{\epsilon\to 0}(t \circ \mathcal{F}_\tau(\delta_\epsilon * T)_{\overline{\xi}}) \in W.$$

In fact, we have

$$\begin{aligned} t \circ \mathcal{F}_\tau(\delta_\epsilon * T)_{\overline{\xi}} &\overset{\text{Remark 2.22}}{=} t \circ (\pi_{\tau,\cdot}(\delta_\epsilon)\mathcal{F}_\tau(T))_{\overline{\xi}}\\ &= (\ldots, t_i \circ (\pi_{\tau,\lambda_i}(\delta_\epsilon)\mathcal{F}_\tau(T)(\lambda_i, \cdot))_{(m_i)}, \ldots)\\ &= (\ldots, t_i \circ \pi_{\xi_i}(\delta_\epsilon)\mathcal{F}_\tau(T)_{\overline{\xi}_i}, \ldots)\\ &= (\ldots, \pi_{\xi_i}(\delta_\epsilon)(t_i \circ \mathcal{F}_\tau(T)_{\overline{\xi}_i}), \ldots)\\ &= \pi_\xi(\delta_\epsilon)(t \circ \mathcal{F}_\tau(T)_{\overline{\xi}}) \in W, \end{aligned}$$

where $(\pi_{\sigma_1,\lambda_1}^{(m_1)}(\delta_\epsilon), \ldots, \pi_{\sigma_s,\lambda_s}^{(m_s)}(\delta_\epsilon)) = \pi_\xi(\delta_\epsilon) \in W \subset H_\xi$. Hence, by taking $\epsilon \to 0$ and since W is closed, we obtain that $t \circ (\mathcal{F}_\tau(T))_{\overline{\xi}} \in W$.

It remains to check that $\mathcal{F}_\tau(T)$ statisfies the slow growth condition $(2.iis)_r$. Fix $r \geq 0$. We need to show that for each multi-indices α, there exist $N \in \mathbb{N}_0$ and a constant $C_{r,N,\alpha} > 0$ such that

$$|l_{Y_\alpha}\mathcal{F}_\tau(T)(\lambda, k)| \leq C_{r,N,\alpha}(1 + |\lambda|^2)^{N+\frac{|\alpha|}{2}} e^{r|\mathrm{Re}(\lambda)|}.$$

Note that $l_{Y_\alpha}\mathcal{F}_\tau(T) = \mathcal{F}_\tau(l_{Y_\alpha}T)$. Let $T \in C_r^{-\infty}(X, \mathbb{E}_\tau)$ be a distribution of order $m \in \mathbb{N}_0$ and $h \in C^\infty(X, \mathbb{E}_{\tilde{\tau}})$. Write $X_\beta \in \mathcal{U}(\mathfrak{n})$ and $H_\gamma \in \mathcal{U}(\mathfrak{a})$ for all multi-indices β, γ. Since $G/K \cong NA$ and $\mathcal{U}(\mathfrak{n} \oplus \mathfrak{a}) \cong \mathcal{U}(\mathfrak{n})\mathcal{U}(\mathfrak{a})$, then, for all multi-indices β and γ, there exist a constant $C > 0$ and $m \in \mathbb{N}_0$ such that

$$|T(h)| \leq C \sum_{|\beta|+|\gamma| \leq m} \sup_{g \in \overline{B}_r(o)} |(l_{X_\beta}(l_{H_\gamma}h))(g)|. \tag{2.58}$$

Next, we want to apply it to $h = e^\tau_{\lambda,1}$. We observe that

$$l_{Y_\alpha}\mathcal{F}_\tau(T)(\lambda, k) = \mathcal{F}_\tau(l_{Y_\alpha}T)(\lambda, k) = l_{Y_\alpha}T(e^\tau_{\lambda,k}) \overset{(2.22)}{=} T(l_k l_{Y_\alpha} e^\tau_{\lambda,1}) = (l_k l_{Y_\alpha}T)(h),$$

where $l_k l_{Y_\alpha}$ is a distribution of order $m + |\alpha|$.

Moreover, h is annihilated by each l_{X_β} without constant terms and it is an eigenfunction of each l_{H_γ} with eigenvalue a polynomial in $\lambda \in \mathfrak{a}^*_\mathbb{C}$ of most degree $m \in \mathbb{N}_0$, i.e.

$$|l_{Y_\alpha}\mathcal{F}_\tau(T)(\lambda, k)| = |(l_k l_{Y_\alpha}T)(e^\tau_{\lambda,1})| \leq C_{r,N,\alpha}(1 + |\lambda|^2)^{N+\frac{|\alpha|}{2}} e^{r|\mathrm{Re}(\lambda)|},$$

where we set $N := [\frac{m}{2}]$. Since K is compact and operates constantly on $C_r^{-\infty}(X, \mathbb{E}_\tau)$, the constant $C_{r,N,\alpha}$ is independent of K. □

Consequently, by (2.57), the *inverse Fourier transform* of $\psi \in PWS_\tau(\mathfrak{a}^*_\mathbb{C} \times K/M)$ for a test function $h \in C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$ is given by

$$\langle \mathcal{F}_\tau^{-1}(\psi), h\rangle := \sum_{Q \in \mathcal{Q}} \sum_{\nu \in A^\tau_Q} \int_{i\mathfrak{a}^*_Q} \int_K \langle \mathcal{F}_{\tilde{\tau}}(h)(-\nu - \lambda, k),\ \mu^Q_\nu(\lambda)\psi(\nu + \lambda, k)\rangle \, dk \, d\lambda.$$

Finally, we discuss the topology on the image space by which the Fourier transform becomes a topological isomorphism.

Lemma 2.43. *(a) The Fourier transform $\mathcal{F}_\tau : C_c^{-\infty}(X, \mathbb{E}_\tau) \longrightarrow PWS_\tau(\mathfrak{a}^*_\mathbb{C} \times K/M)$ is continuous with respect to the strong dual topology of $C_c^{-\infty}(X, \mathbb{E}_\tau)$.*

(b) Its inverse Fourier transform

$$\mathcal{F}_\tau^{-1} : PWS_\tau(\mathfrak{a}^*_\mathbb{C} \times K/M) \longrightarrow C^{-\infty}(X, \mathbb{E}_\tau) \tag{2.59}$$

is continuous with respect to the strong dual topology of $C^{-\infty}(X, \mathbb{E}_\tau)$.

Proof. (a) We will show that for each bounded $B' \subset C_c^{-\infty}(X, \mathbb{E}_\tau)$, there exist $r \geq 0$ and $N \in \mathbb{N}_0$ such that $\mathcal{F}_\tau(B')$ is contained as a bounded set in $PWS_{\tau,r,N}$. Since $PWS_{\tau,r,N} \hookrightarrow PWS_\tau(\mathfrak{a}^*_\mathbb{C} \times K/M)$ is continuous, by definition of inductive limit, then $\mathcal{F}_\tau(B')$ is also bounded in $PWS_\tau(\mathfrak{a}^*_\mathbb{C} \times K/M)$. By Lem. 2.37, we will have that $\mathcal{F}_\tau$ is continuous.

Now let $B' \subset C_c^{-\infty}(X, \mathbb{E}_\tau)$ be bounded. Since B' is equicontinuous and because of (2.52), there exist $r \geq 0, m \in \mathbb{N}_0$ and a constant $C > 0$ such that (2.58) holds uniformly for all $T \in B'$:

$$\sup_{T \in B'} p_B(T) = \sup_{T \in B',\, \varphi \in B} |T(\varphi)| \leq C \sum_{|\alpha| \leq m} \sup_{g \in \overline{B}_r(o)} |l_{Y_\alpha} \varphi(g)|.$$

Now by arguing as in the proof of Prop. 2.42 (c), we obtain, for $N = [\frac{m}{2}]$ that

$$||\mathcal{F}_\tau(T)||_{r,N,\alpha} \leq \infty, \quad \forall T \in B'$$

i.e., $\mathcal{F}_\tau(B') \subset PWS_{\tau,r,N}$ is bounded. Hence the Fourier transform is continuous.

(b) Concerning the last assertion. If, for all $r \geq 0$ and $N \in \mathbb{N}_0$

$$\mathcal{F}_\tau^{-1} : PWS_{\tau,r,N}(\mathfrak{a}_\mathbb{C}^* \times K/M) \longrightarrow C^{-\infty}(X, \mathbb{E}_\tau) \tag{2.60}$$

is continuous, then by constructing of the inductive limit topology, we have that (2.59) is continuous.
Fix $r \geq 0$ and $N \in \mathbb{N}_0$. We want to show that (2.60) is continuous. For that, it suffices to show that for every bounded $\tilde{B} \subset C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$, we have

$$p_{\tilde{B}}(\mathcal{F}_\tau^{-1}(\psi)) \leq C||\psi||_{r,N,0}(< \infty), \quad \psi \in PWS_{\tau,r,N},$$

where $p_{\tilde{B}}(\cdot)$ is the seminorm as in (2.50) and C is a positive constant. Since $\tilde{B}$ is bounded subset in $C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$, there exsits $R \geq 0$ so that the support of all $\varphi \in \tilde{B}$ are in $\overline{B}_R(o)$. Thus, for $\psi \in PWS_{\tau,r,N}$, we have that

$$\begin{aligned}
& p_{\tilde{B}}(\mathcal{F}_\tau^{-1}(\psi)) \\
\overset{(2.50)}{=} \;& \sup_{\varphi \in \tilde{B}} |\langle \mathcal{F}_\tau^{-1}(\psi), \varphi \rangle| \\
\overset{(2.45)}{=} \;& \sup_{\varphi \in \tilde{B}} \Big| \sum_{Q \in \mathcal{Q}} \sum_{\nu \in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \langle \mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu - \lambda, k) \;,\; \mu_\nu^Q(\lambda) \psi(\nu + \lambda, k) \rangle \, dk d\lambda \Big| \\
\leq \;& \sup_{\varphi \in \tilde{B}} \sum_{Q \in \mathcal{Q}} \sum_{\nu \in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \Big| \langle \mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu - \lambda, k) \;,\; \mu_\nu^Q(\lambda) \psi(\nu + \lambda, k) \rangle \Big| \, dk d\lambda.
\end{aligned}$$

Fix now $Q \in \mathcal{Q}$ and $\nu \in A_Q^\tau$. Set

$$d_{Q,\nu} := \sup_{\varphi \in \tilde{B}} \int_{i\mathfrak{a}_Q^*} \int_K \Big| \langle \mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu - \lambda, k) \;,\; \mu_\nu^Q(\lambda) \psi(\nu + \lambda, k) \rangle \Big| \, dk d\lambda.$$

It suffices to show that $d_{Q,\nu} \leq C||\psi||_{r,N,0}$. We have

$$\begin{aligned}
d_{Q,\nu} \;\leq\; & \sup_{\varphi \in \tilde{B}} \int_{i\mathfrak{a}_Q^*} \int_K (1 + |\nu{+}\lambda|^2)^{-d_Q} (1 + |\nu{+}\lambda|^2)^{d_Q} |\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu - \lambda, k)| \\
& \qquad |\mu_\nu^Q(\lambda) \psi(\nu + \lambda, k)| \, dk d\lambda \\
\leq\; & C \sup_{\substack{\varphi \in \tilde{B} \\ k \in K, \lambda \in i\mathfrak{a}_Q^*}} (1 + |\nu{+}\lambda|^2)^{d_Q} |\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu - \lambda, k)| \, |\mu_\nu^Q(\lambda) \psi(\nu + \lambda, k)|,
\end{aligned}$$

where $C := \int_{i\mathfrak{a}_Q^*}(1+|\nu+\lambda|^2)^{-d_Q}\,d\lambda < \infty$ and $(1+|\nu+\lambda|^2)^{d_Q}$ is a weight factor with some $d_Q \in \mathbb{N}_0$ depending on the dimension of $i\mathfrak{a}_Q^*$. For some positive constant N and growth constant $m \in \mathbb{N}_0$, we get

$$\begin{aligned} d_{Q,\nu} &\leq C \sup_{\substack{\varphi\in\tilde{B} \\ k\in K,\lambda\in i\mathfrak{a}_Q^*}} (1+|\nu+\lambda|^2)^{d_Q+N+m}|\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \\ &\quad \cdot \sup_{k\in K,\lambda\in i\mathfrak{a}_Q^*} (1+|\nu+\lambda|^2)^{-(N+m)}|\mu_\nu^Q(\lambda)\psi(\nu+\lambda,k)| \\ &\leq C' \sup_{\substack{\varphi\in\tilde{B} \\ k\in K,\lambda\in i\mathfrak{a}_Q^*}} (1+|\nu+\lambda|^2)^{d_Q+N+m}|\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \\ &\quad \cdot \sup_{k\in K,\lambda\in i\mathfrak{a}_{\mathbb{C}}^*} (1+|\nu+\lambda|^2)^{-N}|\psi(\nu+\lambda,k)|, \end{aligned}$$

where $||\mu_\nu^Q(\lambda)||_{\mathrm{op}} \leq C'(1+|\nu+\lambda|^2)^m$ is of at most polynomial growth of $m \in \mathbb{N}_0$. Thus

$$\begin{aligned} d_{Q,\nu} &\leq C'' \sup_{\substack{\varphi\in\tilde{B} \\ k\in K,\lambda\in i\mathfrak{a}_Q^*}} e^{R|\nu|}(1+|\nu+\lambda|^2)^{d_Q+N+m}|\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \\ &\quad \cdot \sup_{k\in K,\lambda\in i\mathfrak{a}_Q^*} e^{r|\nu|}(1+|\nu+\lambda|^2)^{-N}|\psi(\nu+\lambda,k)| \\ &= C'' \sup_{\varphi\in\tilde{B}} ||\mathcal{F}_{\tilde{\tau}}(\varphi)||_{R,d_Q+N+m}||\psi||_{r,N,0}, \end{aligned}$$

where we set $\xi := \nu+\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. By the Paley-Wiener Thm. 2.31, $\mathcal{F}_{\tilde{\tau}}$ is continuous, thus $\sup_{\varphi\in\tilde{B}} ||\mathcal{F}_{\tilde{\tau}}(\varphi)||_{R,d_Q+N+m} < C < \infty$. Therefore, $d_{Q,\nu} \leq C'''|\psi||_{r,N,0}$ and hence the inverse Fourier transform is continuous. □

End of the proof of Thm. 2.40. The isomorphism of the Fourier transform map outcomes from Lem. 2.42 and the continuity and topology statement results from Lem. 2.43, hence this completes the proof.
Analogously, we obtain the topological Fourier isomorphism in (Level 3) by taking $C_c^{-\infty}(G,\gamma,\tau)$ instead of $C_c^{-\infty}(X,\mathbb{E}_\tau)$. □

2.4 The impact of invariant differential operators on the Fourier range

The Paley-Wiener(-Schwartz) Theorems (Thm. 2.31 resp. Thm. 2.40) state a topological isomorphism through the Fourier transform on the space of sections $C_c^{\pm\infty}(X,\mathbb{E}_\tau)$ and the corresponding Paley-Wiener(-Schwartz) space. One can use its impact to compute explicitly all invariant differential operators.

We restrict to the vector space of distributional sections supported at the origin $o = eK \in X$

$$C_{\{o\}}^{-\infty}(X,\mathbb{E}_*) := \{\psi \in C_c^{-\infty}(X,\mathbb{E}_*) \mid \mathrm{supp}(\psi) \subset \{o\}\}, \quad * = \gamma,\tau.$$

Since $g \cdot o \neq o$, G does not act on $C^{-\infty}_{\{o\}}(X, \mathbb{E}_*)$, but K as well as $\mathfrak{g}$ do, thus $C^{-\infty}_{\{o\}}(X, \mathbb{E}_*)$ is a $(\mathfrak{g}, K)$-module ([Wal88], 3.3.1). Moreover, it is generated by the so-called *vector-valued Dirac delta-distributions* δ_v at $v \in E_\tau$:

$$\delta_v(f) = \langle v, f(e)\rangle_\tau, \quad \text{with test function } f \in C^\infty_{(c)}(X, \mathbb{E}_{\tilde{\tau}}),$$

where $\langle \cdot, \cdot \rangle_*$ denotes the pairing in E_*. The following result follows directly from the corresponding theorem for $\mathbb{R}^n$ ([Rud91], Thm. 6.25.). It states that each distribution on a point is the mapping of a δ-distribution.

Lemma 2.44. *In the previous notation, one has the following isomorphism of* $(\mathfrak{g}, K)$*-modules*

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} E_* \overset{\beta}{\cong} C^{-\infty}_{\{o\}}(X, \mathbb{E}_*)$$

given by $\beta(Z \otimes v)(f) := \langle r_Z f(e), v\rangle_*$, *for* $Z \in \mathcal{U}(\mathfrak{g}), v \in E_*, f \in C^\infty(X, \mathbb{E}_{\tilde{*}})$, *with actions* $Y(Z \otimes v) = YZ \otimes v$, *and* $k(Z \otimes v) = \mathrm{Ad}(k)Z \otimes *(k)v$, *for* $Y \in \mathfrak{k}$ (*or* $\mathcal{U}(\mathfrak{k})$), $k \in K$. $\square$

In particular, a linear invariant differential operator D may be viewed as a linear map

$$D : C^{-\infty}_{\{o\}}(X, \mathbb{E}_\gamma) \longrightarrow C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau).$$

In addition, every invariant differential operator $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ can be seen as an element

$$H_D \in \mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau)) \overset{(1.8)}{\cong} [C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau) \otimes E_{\tilde{\gamma}}]^K$$

given by

$$H_D(v) := D(\delta_v) \in C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau), \quad v \in E_\gamma, \delta_v \in C^{-\infty}_{\{o\}}(X, \mathbb{E}_\gamma). \tag{2.61}$$

In other words

$$\langle H_D(v), f\rangle_\tau \overset{(2.61)}{=} \langle \delta_v, D^t(f)\rangle_\gamma = \langle v, D^t(f)(1)\rangle_\gamma, \tag{2.62}$$

where $D^t \in \mathcal{D}_G(\mathbb{E}_{\tilde{\tau}}, \mathbb{E}_{\tilde{\gamma}})$ is the adjoint invariant differential operator of D defined in (1.11). Now by using Lem. 2.44 and (1.9), we deduce the following isomorphism.

Lemma 2.45. *Let* $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ *be the vector space of all invariant differential operators. Then, we have an isomorphism*

$$\begin{array}{ccc} \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau) & \xrightarrow{\sim} & \mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau)) \\ D & \mapsto & H_D \end{array}$$

between the spaces $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ *and* $\mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau))$.

Proof. The injectivity can be shown directly. For $v \in E_\gamma$ and $f \in C^\infty_{(c)}(X, \mathbb{E}_{\tilde{\tau}})$, we have, by (2.62), that $H_D = 0$ implies $\langle v, D^t(f)(1)\rangle_\gamma = 0$. Thus $D^t(f)(1) = 0$. Due the G-invariance in $g \in G$, we then have

$$D^t(f)(g) = l_{g^{-1}} D^t(f)(1) = D^t(l_{g^{-1}} f)(1) = 0,$$

which implies that $D^t = 0$. Therefore by taking again the transpose of it, this leads us to $(D^t)^t = D = 0$.

To prove the surjectivity, it is sufficient to show that the corresponding graded spaces coincide. In fact, from Chap. 1.1, more precisely (1.3), we know that $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ is a filtered space by degree, with graded space

$$Gr(\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)) \cong [S(\mathfrak{p}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)]^K.$$

On the other side, the space $\mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau))$ is filtered by the order of the distribution. Since $C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau)$ is generated by the δ_v-distributions, $v \in E_\tau$, we thus obtain that $Gr(C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau)) \cong S(\mathfrak{p}) \otimes E_\tau$, and hence

$$Gr(\mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau))) \cong \mathrm{Hom}_K(E_\gamma, S(\mathfrak{p}) \otimes E_\tau) \overset{(1.8)}{\cong} [S(\mathfrak{p}) \otimes \mathrm{Hom}(E_\gamma, E_\tau)]^K.$$

Fuhtermore, since the mapping $D \mapsto H_D$ preserves the filtration, this induces also a mapping on the graded spaces. One can check themselves that this induced map is the identity. By a general and simple inductive fact, we can conclude that a filtration perserving map is injective or surjective, if its induced graded map is so. □

Consequently, we have

$$\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau) \cong \mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{0\}}(X,\mathbb{E}_\tau)) \cong C^{-\infty}_{\{0\}}(G,\gamma,\tau).$$

Hence, by applying the Fourier transform in (Level 3) and the Paley-Wiener-Schwartz Thm. 2.40 (b), we have

$$\begin{aligned} {}_\gamma\mathcal{F}_\tau(\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)) &\cong {}_\gamma PWS_{\tau,0}(\mathfrak{a}^*_\mathbb{C}) \\ &:= \{P \in \mathrm{Pol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{Hom}_M(E_\gamma, E_\tau)) \mid P \text{ satisfies } (3.iii) \text{ of Def. } 2.30\}. \end{aligned}$$

In particular, we have the following result.

Theorem 2.46. *Let $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ be an invariant linear differential operator. For $f \in C^{\pm\infty}_c(X,\mathbb{E}_\gamma)$, we then have that*

$$\mathcal{F}_\tau(Df)(\lambda,k) = {}_\gamma\mathcal{F}_\tau(H_D)(\lambda)\mathcal{F}_\gamma(f)(\lambda,k), \qquad \lambda \in \mathfrak{a}^*_\mathbb{C}, k \in K, \tag{2.63}$$

where ${}_\gamma\mathcal{F}_\tau(H_D) \in \mathrm{Pol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{Hom}_M(E_\gamma, E_\tau))$ is a polynomial in $\lambda \in \mathfrak{a}^*_\mathbb{C}$ with values in $\mathrm{Hom}_M(E_\gamma, E_\tau)$.

By combining now this Thm. 2.46 together with Paley-Wiener(-Schwartz) theorem for sections (Thm. 2.31 resp. Thm. 2.40), one can find explicitly the set of all polynomials, which occur under the Fourier image of invariant differential operators, for irreducible K-types γ and τ. Roughly speaking, we will use the knowledge and the '*power*' of Delorme's Paley-Wiener(-Schwartz) theorem for sections in (Level 3) to compute through the corresponding Fourier transform all the inviariant differential operators $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ without knowing them explicitly before.

Remark 2.47. (a) Consider an additional, not necessary, irreducible K-representation (δ, E_δ). Then, for $D_1 \in \mathcal{D}_G(\mathbb{E}_\tau, \mathbb{E}_\delta)$ and $D_2 \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$, Thm. 2.46 implies the mapping

$$D_1 \circ D_2 \mapsto {}_\gamma\mathcal{F}_\delta(H_{D_1} \circ H_{D_2}) = {}_\tau\mathcal{F}_\delta(H_{D_1}) \circ {}_\gamma\mathcal{F}_\tau(H_{D_2}).$$

(b) By bringing into play the Hecke algebra (2.53), in combination with van den Ban's and Souaifi's Lem. 5.3 and Cor. 5.4 in [vdBS14], one can prove the converse of the above comment, where we fused Thm. 2.46 and Thm. 2.31.
This $(\gamma, \tilde{\tau})$-isotypic component of the Hecke algebra $\mathcal{H}(G,K)(\gamma \otimes \tilde{\tau})$ is essentially the invariant differential operators $\mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau) \otimes \mathrm{Hom}_K(E_\tau, E_\gamma)$ between vector bundles. The Fourier transform then becomes the Harish-Chandra homomorphism (see Def. 1.10). In other words, given all invariant differential operators $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ satisfying some relations, one can find the corresponding Paley-Wiener space with their corresponding polynomials.

Proof of Thm. 2.46. We know that the Fourier transform of a distribution $H_D \in \mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X, \mathbb{E}_\tau))$ is defined by ${}_\gamma\mathcal{F}_\tau(H_D)(\lambda)(v) = \langle H_D(v), e^\tau_{\lambda,1}\rangle$, for $v \in E_\gamma$ and where $e^\tau_{\lambda,1} \in C^\infty(G, \tau, \tilde{\tau})$. Hence by (2.62), we obtain

$$ {}_\gamma\mathcal{F}_\tau(H_D)(\lambda)(v) = \langle H_D(v), e^\tau_{\lambda,1}\rangle_\tau \overset{(2.62)}{=} \langle v, D^t(e^\tau_{\lambda,1})(1)\rangle_\gamma = (D^t(e^\tau_{\lambda,1})(1))v, \quad \lambda \in \mathfrak{a}^*_\mathbb{C}. \tag{2.64} $$

Now, by considering a function $f \in C^\infty_c(X, \mathbb{E}_\gamma)$, we conclude, via 'partial integration', that (2.63) holds. In fact

$$ \begin{aligned} \mathcal{F}_\tau(Df)(\lambda, k) = \int_G e^\tau_{\lambda,k}(g) D(f(g))\, dg \quad &\overset{\text{def. of } D^t}{=} \quad \int_G D^t(e^\tau_{\lambda,k}(g)) f(g)\, dg \\ &\overset{(2.24)}{=} \quad \int_G D^t(e^\tau_{\lambda,1}(1)) \circ e^\gamma_{\lambda,k}(g) f(g)\, dg \\ &= \quad D^t(e^\tau_{\lambda,1}(1)) \circ \mathcal{F}_\gamma(f)(\lambda, k) \\ &\overset{(2.64)}{=} \quad {}_\gamma\mathcal{F}_\tau(H_D)(\lambda) \circ \mathcal{F}_\gamma(f)(\lambda, k). \end{aligned} $$

The same computation remains true for $f \in C^{-\infty}_c(X, \mathbb{E}_\gamma)$, by using the pairing $\langle \cdot, \cdot \rangle$ instead of the integration. $\square$

Chapter 3

Examples for Delorme's intertwining conditions

In the previous chapter, we have formulated Delorme's intertwining conditions (Def. 2.30, (2.iii) resp. (3.iii)) for our purposes. However, these intertwining conditions are very difficult to check, in practise, even for special K-types.
The most important source of such conditions are the Knapp-Stein and Želobenko intertwining operators, as well as the embedding of discrete series into principal series $H_\infty^{\sigma,\lambda}$, for $(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}_\mathbb{C}^*$. Therefore, in this part, we rewrite them in a more *accessible* way involving such intertwining operators and the Harish-Chandra **c**-functions.
Moreover, we show that only a part of them is already sufficient for semi-simple Lie groups of real rank one. This will be illustrated on three special examples, namley in $SL(2,\mathbb{R})$, $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and in $SL(2,\mathbb{C})$.

The first Section 3.1 introduces the two well-known intertwining operators, the Knapp-Stein and Želobenko ones. We adopt the same notation as in [Olb95].

In Section 3.2, we present the Harish-Chandra **c**-functions and their relations with the intertwining operators.

Then, in Section 3.3 we prove in Thm 3.13, that for some special cases an approachable subset of these conditions is sufficient to define the Paley-Wiener space for $\mathrm{rk}_\mathbb{R}(G) = 1$. This is already implicitly contained in Delorme's proof ([Del05], Thm. 2). For our proof, we essentially use Delorme's results and its induction procedure on the length of minimal K-types of a generalized principal series representation, similar as Delorme's proposition ([Del05], Prop. 2).

Finally, the last three Sections 3.4, 3.5 and 3.6 give a nice interpretation and understanding of Delorme's intertwining conditions for the three examples. The choice of these three particular examples lies in the fact that they are semi-simple Lie groups of real rank one (respectively two) and their structure of principal series representations are well known.
In fact, for $G = SL(2,\mathbb{R})$, in Section 3.4, by drawing its principal series representations $H_\infty^{\pm,\lambda}$ by '*box-pictures*' (Fig. 3.1), we can see in which closed G-submodule of $H_\infty^{\pm,\lambda}$ there is an intertwining condition in (Level 2) (Thm. 3.18). Afterwards we can deduce the corresponding results also for the other levels (Thms. 3.20 & 3.17).
In addition, in Section 3.5, we show that the intertwining condition of the semi-simple Lie group of rank two, $G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ are the 'same' as for $G = SL(2,\mathbb{R})$ (Thm. 3.25).

As last example, in Section 3.6, we consider $G = SL(2, \mathbb{C})$. The description of its intertwining conditions (Thms. 3.28, 3.29 & 3.35) are more difficult then for the previous examples.

3.1 Knapp-Stein and Želobenko intertwining operators

Let (π, H_1) and (π_2, H_2) be two representations of G on the topological vector spaces. From the representation theory, we know that a linear map $L : H_1 \longrightarrow H_2$ is an intertwining operator between (π_1, H_1) and (π_2, H_2), if

$$L \circ \pi_1(g) = \pi_2(g) \circ L, \quad \text{for every } g \in G.$$

It turns out that every intertwining operator between principal series representations of G provides key information about the *compatibility or intertwining condition* for Delorme's Paley-Wiener space. In this part, we will present two familiar intertwining operators. Namely the Knapp-Stein intertwining operators, which were introduced by Knapp and Stein in 1971 ([KSt71] & [KSt80]), and the Želobenko operators [Zelo76], which are also known as the Bernstein, Gel'fand and Gel'fand (short BGG) resolutions ([BGG71], [BGG75] & [BGG76]). The construction of the last one, is based on the theory of Verma modules and their duals. We refer for example to ([Kna02], Chap. V) for more details and to ([Olb95], Sect. 2.3.2) for the definition of Želobenko operators.

Knapp-Stein intertwining operators

Note that the analytic Weyl group $W_A = N_K(\mathfrak{a})/M$ acts on $\mathfrak{a}^*_{\mathbb{C}}$ as also on $\hat{M}$. Let $w \in W_A$ be represented by $m_w \in M' := N_K(\mathfrak{a})$ and $\sigma \in \hat{M}$. We realise σ on the vector space E_σ. We define a new representation $w\sigma \in \hat{M}$ of M acting on the vector space E_σ

$$w\sigma : M \longrightarrow GL(E_\sigma), \qquad w\sigma(m) := \sigma(m_w^{-1} m m_w), \ \ m \in M.$$

This equivalence class only depends on W_A and not on the choice of m_w.

Definition 3.1 (Knapp-Stein intertwining operator, [KSt71] & [KSt80])**.** *Let $\Delta^-_{\mathfrak{a}} := -\Delta^+_{\mathfrak{a}}$ and $\overline{N}$ be the unipotent subgroup coming from the associated Iwasawa decomposition of $\Delta^-_{\mathfrak{a}}$. Write $\overline{N}_w := N \cap w\overline{N}w^{-1}$. For $(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}^*_{\mathbb{C}}$ with $(\mathrm{Re}(\lambda), \alpha) > 0$, for all $\alpha \in \Delta^+_{\mathfrak{a}} \cap w^{-1}\Delta^-_{\mathfrak{a}}$ and for a fixed representation $m_w \in M'$, we define the intertwining operator*

$$J_{w,\sigma,\lambda} : H^{\sigma,\lambda}_\infty \longrightarrow H^{w\sigma,w\lambda}_\infty$$

by the convergent integral

$$J_{w,\sigma,\lambda}((\varphi)(g)) := \int_{\overline{N}_w} \varphi(g\overline{n}m_w)\, d\overline{n}, \ \ g \in G, \varphi \in H^{\sigma,\lambda}_\infty,$$

which depends holomorphically on $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. This operator has a meromorphic continuation on the whole $\mathfrak{a}^*_{\mathbb{C}}$.

Remark 3.2. (i) Whenever $J_{w,\sigma,\lambda}$ is defined, it intertwines $H^{\sigma,\lambda}_\infty$ and $H^{w\sigma,w\lambda}_\infty$. Note that $J_{w,\sigma,\lambda}$ depends on the choice of the repesentative $m_w \in M'$ and not of $w \in W_A$.

(ii) $J_{w,\sigma,\lambda}$ is invertible, if λ is an element of the complenent set of all zeros of a meromorphic function which is not identically zero ([KSt80], Prop. 7.3-7.5 and Thm. 7.6).

For $\tau \in \hat{K}$, we consider a convergent integral

$$j_{w,\tau,\lambda} := \int_{\overline{N}_w} a(\overline{n})^{\lambda-\rho}\tau(\kappa(\overline{n})^{-1})\, d\overline{n} \in \mathrm{End}_M(E_\tau).$$

The scalar function $j_{w,\tau,\lambda}$ is a meromorphic function on the whole $\mathfrak{a}_{\mathbb{C}}^*$, non identically zero. Notice, that in contrary to the intertwining operators $J_{w,\sigma,\lambda}$, the j-functions $j_{w,\tau,\lambda}$ are defined without the choice of the representatives m_w of $w \in W_A$.
However, one can express the Knapp-Stein intertwining operator $J_{w,\sigma,\lambda}$ in terms of the j-functions.

Lemma 3.3 (e.g. [Olb95], Lem. 3.12)**.** *Let $w \in W_A$ and $m_w \in M'$ the representation of w. Consider for $t \in \mathrm{Hom}_M(E_\tau, E_\sigma)$ and $v \in E_\tau$, the function $\phi_v(t) \in H_\infty^{\sigma,\lambda}$, for any λ, given by*

$$\phi_v(t)(k) := t\tau(k^{-1})v, \quad k \in K. \tag{3.1}$$

Then, we have that

$$J_{w,\sigma,\lambda}(\phi_v(t)) = \phi_v(t \circ j_{w,\tau,\lambda} \circ \tau(m_w^{-1})). \tag{3.2}$$

Proof. In the following, all the computation will be done in the convergent part of the defined integral:

$$\begin{aligned}
J_{w,\sigma,\lambda}(\phi_v(t))(k) &= \int_{\overline{N}_w} \phi_v(t)(km_w\kappa(\overline{n}))a(\overline{n})^{\lambda-\rho}\, d\overline{n} \\
&\overset{(3.1)}{=} \int_{\overline{N}_w} t\tau(\kappa(\overline{n})^{-1})\tau(m_w^{-1})\tau(k^{-1})a(\overline{n})^{\lambda-\rho}v\, d\overline{n} \\
&= t\Big(\int_{\overline{N}_w} a(\overline{n})^{\lambda-\rho}\tau(\kappa(\overline{n})^{-1})\, d\overline{n}\Big)\tau(m_w^{-1})\tau(k^{-1})v \\
&= t \circ j_{w,\tau,\lambda} \circ \tau(m_w^{-1}) \circ \tau(k^{-1})v \\
&\overset{(3.1)}{=} \phi_v(t \circ j_{w,\tau,\lambda})(m_w k) \\
&= \phi_v(t \circ j_{w,\tau,\lambda} \circ \tau(m_w^{-1}))(k),
\end{aligned}$$

where in the first equation, we used the Iwasawa decomposition $\overline{n} = \kappa(\overline{n})a(\overline{n})n(\overline{n})$ and $k = km_w\kappa(\overline{n})$. □

Želobenko intertwining opertators

Now, let us choose a Cartan subalgebra $\mathfrak{t}$ on the Lie algebra $\mathfrak{m}$ of M. Hence $\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan-subalgebra of $\mathfrak{g}$. Fix a positive root system $\Delta^+_{\mathfrak{h}_{\mathbb{C}}}$ of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ so that

$$\{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta^+_{\mathfrak{h}_{\mathbb{C}}}\} =: \Delta^+_{\mathfrak{a}}$$

and consider by $\mathcal{W}$ the Weyl group of this root system.

For some values of $(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}_{\mathbb{C}}^*$, the intertwining operators are not only induced for the analytic Weyl group $W_A = W(\mathfrak{g}, \mathfrak{a})$ but also for the Weyl group $\mathcal{W} = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$

of $\Delta^+_{\mathfrak{h}_\mathbb{C}}$. Moreover $\mathcal{W}$ is generated by the reflections $s_\alpha, \alpha \in \Delta^+_{\mathfrak{h}_\mathbb{C}}$ at the root hyperplanes. For each of these reflections, we assign an intertwining operator. Set

$$\delta_m := \frac{1}{2} \sum_{\alpha \in \Delta^+_{\mathfrak{h}_\mathbb{C}}} \alpha, \quad \text{and} \quad \rho_\mathfrak{m} := \delta_m - \rho, \tag{3.3}$$

where $\rho \in \mathfrak{a}^*$ is extened by 0 on $\mathfrak{h}$. We denote by $\mathfrak{u}$ (resp. $\overline{\mathfrak{u}}$) the sum of all positive (resp. negative) root subspace of $\mathfrak{h}$ on $\mathfrak{g}_\mathbb{C}$. Consider $\mathfrak{g} = \mathfrak{b} \oplus \overline{\mathfrak{u}}$, with $\mathfrak{b} := \mathfrak{h}_\mathbb{C} \oplus \mathfrak{u} \subset \mathfrak{g}$ and an one-dimensional representation of $\mathfrak{b}$ given by

$$\begin{array}{rcll} Hz &=& (\Lambda - \delta)(H)z, & \text{for } H \in \mathfrak{h}, z \in \mathbb{C} \\ Uz &=& 0, & \text{for } U \in \mathfrak{u}, \quad \text{(trivial action).} \end{array}$$

Then, $\mathbb{C}$ can be viewed as a left $\mathcal{U}(\mathfrak{b})$ module $\mathbb{C}_{\Lambda-\delta}$. For $\Lambda \in \mathfrak{h}^*_\mathbb{C}$, the Verma module ([Kna02], p.285) is given by

$$M(\Lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\Lambda-\delta}.$$

Let $\alpha \in \Delta^+_{\mathfrak{h}_\mathbb{C}}$ and $n = 2\frac{\langle \Lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{N}$, then there exist an element $\theta_{\alpha,n} \in \mathcal{U}(\overline{\mathfrak{u}})$ and a non-trivial $\mathfrak{g}$-equivariant embedding

$$S_\alpha : M(\Lambda - n\alpha) \longrightarrow M(\Lambda)$$

defined by $S_\alpha(u \otimes z) := u\theta_{\alpha,n} \otimes z$ for $u \in \mathcal{U}(\mathfrak{g})$ and $z \in \mathbb{C}$. Notice that S_α as well as $\theta_{\alpha,n}$ are unique up to the scalar factor.
Let $\mu_{\tilde\sigma} \in i\mathfrak{t}^* \subset \mathfrak{h}^*_\mathbb{C}$ be the highest weight of the representation $\tilde\sigma$ of M. To relate the above terminology with the principal series representations, consider for $\Lambda = \mu_{\tilde\sigma} + \rho_\mathfrak{m} + \lambda$, the $\mathfrak{g}$-mapping

$$\phi_\Lambda : M(\Lambda) \longrightarrow (H^{\sigma,\lambda}_\infty)^*$$

given by $\phi_\Lambda(u \otimes z) := z\langle v_{\tilde\sigma}, (r_u f)(e)\rangle$, for $u \in \mathcal{U}(\mathfrak{g}), z \in \mathbb{C}, f \in H^{\sigma,\lambda}_\infty$ and where the vector of highest weight $v_{\tilde\sigma}$ is in $E_{\tilde\sigma}$. Here, $\langle \cdot, \cdot \rangle$ is a $\mathcal{W}$-invariant scalar-product on $\mathfrak{h}^*_\mathbb{C}$. Roughly speaking, for S_α there exists an 'adjoint' operator $L_{\alpha,\sigma,\lambda}$, which can be viewed as a G-equivariant operator between principal series representations.

Definition 3.4 (Želobenko intertwining opertators, e.g. [Olb95], Def. 2.21)**.** *Let $\alpha \in \Delta^+_{\mathfrak{h}_\mathbb{C}}$ and set $\Lambda := \mu_{\tilde\sigma} + \rho_\mathfrak{m} + \lambda \in \mathfrak{h}^*_\mathbb{C}$. Consider*

$$\Sigma_{\alpha,n} := \left\{ (\sigma, \lambda) \in \hat{M} \times \mathfrak{a}^*_\mathbb{C} \mid 2\frac{\langle \Lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} = n \in \mathbb{N} \textit{ and } \mu_{\tilde\sigma} - n\alpha|_\mathfrak{t} \textit{ is a highest weight of a rep. of } M \right\}.$$

For $(\sigma, \lambda) \in \Sigma_{\alpha,n}$, we define $(\sigma_\alpha, \lambda_\alpha) \in \hat{M} \times \mathfrak{a}^*_\mathbb{C}$ by

- $\lambda_\alpha := \lambda - n\alpha|_\mathfrak{a}$,
- *$\sigma_\alpha|_{M_0}$ is the dual of the M_0-representation with highest weight $\mu_{\tilde\sigma} - n\alpha|_\mathfrak{t}$. Moreover, due to $M = Z(M)M_0$, σ_α is well-defined. Here, M_0 is the identity component of M, and $Z(M)$ the center of M, which acts through the same character as σ.*

The Želobenko intertwining opertator is a G-mapping

$$L_{\alpha,\sigma,\lambda} : H_\infty^{\sigma,\lambda} \longrightarrow H_\infty^{\sigma_\alpha,\lambda_\alpha}$$

defined by

$$\langle v_{\tilde{\sigma}_\alpha}, (L_{\alpha,\sigma,\lambda} f)(g)\rangle := \langle v_{\tilde{\sigma}}, (r_{\theta_{\alpha,n}} f)(g)\rangle, \quad f \in H_\infty^{\sigma,\lambda}, g \in G.$$

Here, $v_{\tilde{\sigma}_\alpha}$ and $v_{\tilde{\sigma}}$ are again the highest weight-vector of the corresponding M-representations and $\theta_{\alpha,n} \in \mathcal{U}(\overline{\mathfrak{u}})$ is as above.

Furthermore, $L_{\alpha,\sigma,\lambda}$ is a differential operator and if $\alpha|_{\mathfrak{a}} = 0$, then $L_{\alpha,\sigma,\lambda} = 0$, this means that we only consider Weyl-reflections, which are not coming from M.

At the hand of these examples, we can see that the intertwining condition $(D.a)$ in Def. 2.4 can be simplified in some cases.

Example 3.5. (a) For $w \in W_A$ and fixed $(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}_{\mathbb{C}}^*$, we consider the Knapp-Stein intertwining operator $J_{w,\sigma,\lambda}$ as in Def. 3.1 and let $\phi \in \prod_{\sigma\in\hat{M}} \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}(H_\infty^\sigma))$. Then, the condition

$$J_{w,\sigma,\lambda} \circ \phi(\sigma, \lambda) = \phi(w\sigma, w\lambda) \circ J_{w,\sigma,\lambda} \tag{3.4}$$

is a special intertwining condition of $(D.a)$ in Def. 2.4 with $s = 2$ and $m_1 = 0 = m_2$, for each intertwining datum (ξ, W) with $\xi = ((\sigma, \lambda, 0), (w\sigma, w\lambda, 0))$ and with closed G-submodule

$$W = graph(J_{w,\sigma,\lambda}) \subset H_\infty^{\sigma,\lambda} \oplus H_\infty^{w\sigma,w\lambda}.$$

Similarly, by considering the Želobenko intertwining opertator $L_{\alpha,\sigma,\lambda}$, for $\alpha \in \Delta_{\mathfrak{h}_{\mathbb{C}}}^+$ and $(\sigma, \lambda) \in \Sigma_{\alpha,n}$, defined in Def. 3.4, we have the special condition

$$L_{\alpha,\sigma,\lambda} \circ \phi(\sigma, \lambda) = \phi(\sigma_\alpha, \lambda_\alpha) \circ L_{\alpha,\sigma,\lambda} \tag{3.5}$$

for each intertwining datum (ξ, W) with $W = graph(L_{\alpha,\sigma,\lambda}) \subset H_\infty^{\sigma,\lambda} \oplus H_\infty^{\sigma_\alpha,\lambda_\alpha}$.

(b) An irreducible unitary representation (π, E_π) of G is called a representation of the discrete series if there is a G-invariant embedding $E_\pi \hookrightarrow L^2(G)$. Here, $L^2(G)$ denote the space of all square integrable functions with respect to invariant measure dg on G. Write $\hat{G}_d$ the set of equivalence classes of discrete series representations of G. Let H_π be any Hilbert space , where the representations $\pi \in \hat{G}_d$ are realized. For every representation of discrete series $\pi \in \hat{G}_d$, we choose an embedding

$$i_\pi : H_\pi \hookrightarrow H_\infty^{\sigma_\pi,\lambda_\pi}$$

into some principal series representation (Casselman's representation embedding result, [Wal88], Thm. 3.8.3. & Casselman's and Wallach's globalization Thm. [Wal92], Chap. 12) and set

$$W_\pi := i_\pi(H_\pi) \subset H_\infty^{\sigma_\pi,\lambda_\pi}.$$

It is a closed G-invariant subspace. Hence, the condition

$$\phi(\sigma_\pi, \lambda_\pi)(W_\pi) \subset W_\pi, \quad \pi \in \hat{G}_d \tag{3.6}$$

is also of the form $(D.a)$, with $s = 1$ and $m = 0$, and it permits us to define that

$$\phi(\pi) := \phi(\sigma_\pi, \lambda_\pi)|_{W_\pi} \in \mathrm{End}(W_\pi). \tag{3.7}$$

However, one can rewrite the intertwining operators, defined above, by involving the so-called Harish-Chandra **c**-functions, which will be introduced in the next section of this chapter.

3.2 Harish-Chandra c-functions and functional equations

In this part, we want to study the relationship between the (Knapp-Stein) intertwining operators and the Harish-Chandra **c**-function.

Definition 3.6 (Harish-Chandra **c**-function, e.g. [Olb95], Def. 3.8). *Let $\tau \in \hat{K}, w \in W_A$, $\overline{N}_w$ as in Def. 3.1, and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $(\mathrm{Re}(\lambda), \alpha) > 0$, for all $\alpha \in \Delta_{\mathfrak{a}}^+ \cap w^{-1}\Delta_{\mathfrak{a}}^-$. The $\mathbf{c}$-function is defined by*

$$\mathbf{c}_{w,\tau}(\lambda) := \int_{\overline{N}_w} a(\overline{n})^{-(\lambda+\rho)}\tau(\kappa(\overline{n}))\, d\overline{n} \in \mathrm{End}_M(E_\tau),$$

which can be extended to a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$.
Furthermore, we have $\mathbf{c}_{w,\tau}(\sigma, \lambda) := pr_\sigma \circ \mathbf{c}_{w,\tau}(\lambda) \circ pr_\sigma \in \mathrm{End}_M(E_\tau(\sigma))$, where $pr_\sigma : E_\tau \longrightarrow E_\tau(\sigma)$ is the projection on the σ-isotypic compenent.
Consider now $w \in W_A$ as a Weyl element with maximal length, then, we set

$$\mathbf{c}_\tau(\lambda) := \mathbf{c}_{w,\tau}(\lambda) \qquad \text{and} \qquad \mathbf{c}_\tau(\sigma, \lambda) := \mathbf{c}_{w,\tau}(\sigma, \lambda).$$

The definition of the **c**-function does not differ much from the definition of the j-function (Def. 3.1, (b)). In fact, Olbrich showed that there is a relationship between them. The following statement should be read as meromorphic functions identity.

Proposition 3.7 ([Olb95], Satz 3.13). *For $\tau \in \hat{K}, \lambda \in \mathfrak{a}_{\mathbb{C}}^*, w \in W_A$ and $m_w \in N_K(\mathfrak{a})$, we have*

$$j_{w,\tau,\lambda} = \tau(m_w^{-1})\mathbf{c}_{w^{-1},\tau}(w\lambda)\tau(m_w).\Box \tag{3.8}$$

By combining, the above relation (3.8) in (3.2), we get

$$J_{w,\sigma,\lambda}(\phi_v(t)) = \phi_v(t \circ \tau(m_w^{-1})\mathbf{c}_{w^{-1},\tau}(w\lambda)) \tag{3.9}$$

for $\phi_v(t) \in H_\infty^{\sigma,\lambda}$ as in (3.1).

Now, by using the identification (3.9), we can prove the following statement.

Proposition 3.8 (Knapp-Stein intertwining condition in (Level 2)). *With the previous notations, consider the intertwining operator $J_{w,\sigma,\lambda}$ as in Def. 3.1 and $\psi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, H_\infty^{\tau|_M})$. Then, for all $t \in \mathrm{Hom}_M(E_\tau, E_\sigma)$, we have*

$$J_{w,\sigma,\lambda}(t \circ \psi(\lambda, \cdot)) = t \circ \tau(m_w^{-1})\mathbf{c}_{w^{-1},\tau}(w\lambda)\psi(w\lambda, \cdot), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, w \in W_A. \tag{3.10}$$

Proof. We have, for every intertwining datum $((\sigma, \lambda, 0), (w\sigma, w\lambda, 0), W = graph(J_{w,\sigma,\lambda}))$ and $t \in \mathrm{Hom}_M(E_\tau, E_\sigma)$, the intertwining condition (2.41). Now, by using (3.9), we obtain

$$\begin{aligned} J_{w,\sigma,\lambda}(t \circ \psi(\lambda, \cdot)) &= J_{w,\sigma,\lambda}(t \circ \tau(\cdot)^{-1}v)(1) \circ \psi(w\lambda, \cdot) \\ &= \phi_v(t \circ \tau(m_w^{-1})\mathbf{c}_{w^{-1},\tau}(w\lambda)) \circ \psi(w\lambda, \cdot) \\ &\overset{(3.1)}{=} t \circ \tau(m_w^{-1})\mathbf{c}_{w^{-1},\tau}(w\lambda)v \circ \psi(w\lambda, \cdot) \end{aligned}$$

for $\psi(\lambda, \cdot) \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, H_\infty^{\tau|_M})$ and $v \in E_\tau$. $\Box$

Example 3.9. Let τ be a trivial K-representation. Consider a smooth function β in K/M and holomorphic on $\mathfrak{a}_{\mathbb{C}}^*$, which satisfies some growth condition. Helgason showed in ([Hel89], Thm. 5.1.) that the intertwining condition in (Level 2)

$$\int_{K/M} e^{\tau}_{w\lambda,k}(g)\beta(w\lambda,k)dk = \int_{K/M} e^{\tau}_{\lambda,k}(g)\beta(\lambda,k)dk, \qquad w \in W_A \tag{3.11}$$

is sufficient and enough. One can even show that Helgason's intertwining condition is equivalent to (3.4). In fact, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, consider the Poisson transform (e.g. [Olb95], Def. 3.2) $P_{\tau,\lambda} : H_{\infty}^{\tau|_M,\lambda} \longrightarrow C^{\infty}(X, \mathbb{E}_{\tau})$ given by

$$\mathcal{P}_{\tau,\lambda}(f)(g) := \int_K \tau(k)tf(gk)\, dk = \int_K e^{\tau}_{\lambda,k}(g)tf(k)\, dk, \quad g \in G, t \in \mathrm{Hom}_K(E_{\tau}, E_{\tau|_M}).$$

Then, Helgason's condition (3.11) can be expressed in terms of Poisson transform

$$\mathcal{P}_{\tau,\lambda} \circ \beta_{\lambda} = \mathcal{P}_{\tau,w\lambda} \circ \beta_{w\lambda}, \qquad w \in W_A, \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

where $\beta_{\lambda} := \beta(\lambda,\cdot)$.

Now, if we consider an additional K-type (γ, E_{γ}), we obtain a similar relation as above. Note that the Knapp-Stein intertwining operator will completely disappear.

Lemma 3.10 (Knapp-Stein intertwining operator in (Level 3))**.** *With the previous notations, let $\varphi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{Hom}_M(E_{\gamma}, E_{\tau}))$ as in Def. 2.30. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $w \in W_A$, we then have*

$$t \circ \varphi(\lambda)\gamma(m_w^{-1})\mathbf{c}_{w^{-1},\gamma}(w\lambda) = t\tau(m_w^{-1})\mathbf{c}_{w^{-1},\tau}(w\lambda) \circ \varphi(w\lambda), \forall t \in \mathrm{Hom}(E_{\tau}, E_{\sigma}), \sigma \subset \gamma. \tag{3.12}$$

Proof. Let $J_{w,\sigma,\lambda}$ be the Knapp-Stein intertwining operator as in Def. 3.1. For every intertwining datum $((\sigma,\lambda,0),(w\sigma,w\lambda,0),W)$ and $t \in \mathrm{Hom}_M(E_{\tau}, E_{\sigma})$ we have (2.42). By using (3.9) on both side, we obtain the desired equation (3.12). $\square$

Example 3.11. Consider a function $\beta \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*)$ which satisfies some growth condition. Helgason and Gangolli ([Gan71] & [Hel20], Thm. 7.1) proved that the intertwining condition:

$$\beta \text{ is } W_A\text{-invariant, if } \beta(\lambda) = \beta(w\lambda), \text{ for } \lambda \in \mathfrak{a}_{\mathbb{C}}^*, w \in W_A$$

for trivial left and right K-representations in (Level 3), is sufficient and enough.

3.3 Adequateness of Delorme's intertwining conditions for rank 1

In this part, we want to reduce the amount of condition $(D.a)$ of Def. 2.4 to a minimum. Let G with real rank one. In this situation a further simplification can be made, the set of $\Delta_{\mathfrak{a}}^+$ of positive restricted roots consists of at most two elements, namely α and possibly 2α. Moreover, the element H of $\mathfrak{a}$ satisfies $\alpha(H) = 1$ and its choice together with α and the exponential map allow us to identify $\mathfrak{a}$ with $\mathbb{R}$. The Weyl group is reduced to $\{-1, 1\}$ acting on $\mathbb{R}$ by multiplication.

We have already observed two 'special' intertwining conditions in Example 3.5 (a) and (b). Thus, for $r > 0$, we define the 'special' Paley-Wiener space $PW_r^+(G)$ by replacing Delorme's intertwining condition $(D.a)$ by the conditions (3.4) and (3.6) only.

Let $w \in W_A$ be the non-trivial element. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $(\mathrm{Re}(\lambda), \alpha) > 0$, let $m \in \mathbb{N}_0$ be the maximal order of the zeros of $J_{w,\sigma,\mu}(f_\mu)$ at $\mu = \lambda$, where $\mu \mapsto f_\mu \in H_\infty^{\sigma,\mu}$ runs over all germs of holomorphic functions at λ with $f_\lambda \neq 0$.
We consider the induced operator

$$J_{w,\sigma,\lambda}^{(m-1)} : H_{\infty,(m-1)}^{\sigma,\lambda} \longrightarrow H_{\infty,(m-1)}^{w\sigma,-\lambda} \tag{3.13}$$

and the corresponding kernel

$$\mathrm{Ker}(J_{w,\sigma,\lambda}^{(m-1)}) \subset H_{\infty,(m-1)}^{\sigma,\lambda}.$$

By convention, we set $H_{\infty,(-1)}^{\sigma,\lambda} = \{0\}$, for $m = 0$. Notice that due condition (3.4), we have for $\phi^{(m-1)}(\sigma,\lambda) \in \mathrm{End}(H_{\infty,(m-1)}^{\sigma,\lambda})$

$$\phi^{(m-1)}(\sigma,\lambda)(\mathrm{Ker}(J_{w,\sigma,\lambda}^{(m-1)})) \subset \mathrm{Ker}(J_{w,\sigma,\lambda}^{(m-1)}) \subset H_{\infty,(m-1)}^{\sigma,\lambda}, \quad (\sigma,\lambda) \in \hat{M} \times \mathfrak{a}_{\mathbb{C}}^*.$$

Each K-type has a highest weight $\mu \in i\mathfrak{t}^*$, where $\mathfrak{t} \subset \mathfrak{k}$ is a Cartan subalgebra of maximal torus $T \subset K$. We define

$$2\rho_c := \sum_{\alpha \in \Delta^+(\mathfrak{k},\mathfrak{t})} \alpha \in i\mathfrak{t}^*$$

the sum of all positive roots of complex subspace $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{k}_{\mathbb{C}}$.
For $\sigma \in \hat{M}$ (resp. $\pi \in \hat{G}_d$), we let $|\sigma| \in [0,\infty)$ (resp. $|\pi| \in [0,\infty)$) be the length, i.e.

$$|\sigma| := \min_{\{\tau\} \in H_\infty^\sigma(\tau) \neq \{0\}} ||\mu_\tau + 2\rho_c|| \quad (\text{resp. } |\pi| := \min_{\{\tau\} \in H_\pi(\tau) \neq \{0\}} ||\mu_\tau + 2\rho_c||)$$

of 'the' minimal K-type τ of H_∞^σ (resp. H_π), ([Del05], Sect. 1.3). Denote by $B(\sigma) \subset \hat{K}$ the finite set of all minimal K-types.

Example 3.12. Let $G = SL(2,\mathbb{R})$ and $K = SO(2)$ its maximal compact subgroup. With the notations introduced in Sect. 3.4, we have that $\hat{K} \cong \mathbb{Z}$ and $\rho_c = 0$.

(i) Let $M = \{\pm 1\}$, thus

- if σ is trivial, then $B(\sigma) = \{0\} \subset \mathbb{Z}$, (trivial K-type) and $|\sigma| = 0$,
- if σ is non-trivial, then $B(\sigma) = \{+1,-1\} \subset \mathbb{Z}$ and $|\sigma| = 1$.

(ii) Let $\pi = D_k, k \in \mathbb{Z}\backslash\{0\}$ be the discrete series representation of $M = G$, then

$$B(D_k) = \begin{cases} \{k+1\}, & k > 0, \\ \{k-1\}, & k < 0 \end{cases}$$

and $|\pi| = k + 1$.

We can prove the following result, which tells us that the intertwining conditions (3.4) and (3.6) with an additional 'vanishing' condition are sufficient for semi-simple Lie group G of real rank one.

Theorem 3.13. *With the previous notations, let* $\mathrm{rk}_{\mathbb{R}}(G) = 1$. *For* $r > 0$, *let* A *be a linear closed and* $K \times K$ *invariant subspace of* $PW_r^+(G)$ *satisfying* $\mathcal{F}_{\sigma,\lambda}(C_r^\infty(G)) \subset A$ *and*

(D.b) let $\phi \in A$ *such that for all* $\sigma \in \hat{M}$

(i) $\phi(\sigma', \lambda) = 0$, *for all* $\sigma' \in \hat{M}$ *with* $|\sigma'| > |\sigma|$ *and* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,

(ii) $\phi(\pi) = 0$, *for all* $\pi \in \hat{G}_d$ *with* $|\pi| > |\sigma|$.

Then, for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *with* $(\mathrm{Re}(\lambda), \alpha) > 0$, ϕ *induces the zero-operator* $\phi^{(m-1)}(\sigma, \lambda)$ *on* $\mathrm{Ker}(J_{w,\sigma,\lambda}^{(m-1)})$:

$$\phi^{(m-1)}(\sigma, \lambda)\Big|_{\mathrm{Ker}(J_{w,\sigma,\lambda}^{(m-1)})} = 0.$$

Here, $(m-1)$ *depends on* (σ, λ) *as defined above (3.13) and* $\phi(\pi)$ *is defined in (3.7).*

Then,

$$A = PW_r(G) \cong \mathcal{F}_{\sigma,\lambda}(C_r^\infty(G)).$$

Proof of Thm. 3.13. By Delorme's Paley-Wiener Thm. 2.7, we already know that $PW_r(G) \cong \mathcal{F}_{\sigma,\lambda}(C_r^\infty(G))$ is a closed and $K \times K$ invariant subspace of $PW_r^+(G)$

$$PW_r(G) \cong \mathcal{F}_{\sigma,\lambda}(C_r^\infty(G)) \subset PW_r^+(G).$$

Therefore, it suffices to show that $\mathcal{F}_{\sigma,\lambda}(C_r^\infty(G)) \subset A$ is dense. Thus for every $K \times K$-finite element $\phi \in A$, we need to find a function $f \in C_r^\infty(G)_{K\times K}$ such that

$$\pi_{\sigma,\lambda}(f) = \phi(\sigma, \lambda), \quad \forall (\sigma, \lambda) \in \hat{M} \times \mathfrak{a}_{\mathbb{C}}^*.$$

Let $\phi \in A_{K\times K}$. It is given by a collection $(\phi_\sigma), \sigma \in \hat{M}$. By $K \times K$-finiteness, only finitely many ϕ_σ are non-zero. Similar, by $K \times K$-finiteness, $\phi(\pi) = 0$, for all but finitely many $\pi \in \hat{G}_d$. Indeed, for any given K-type τ, there are only finitely many $\pi \in \hat{G}_d$, with $H_\pi(\tau) \neq 0$ (e.g. [Wal88], Cor. 7.7.3).
We define $l(\phi) \in [0, \infty)$ by

$$l(\phi) := \max\{|\sigma|, |\pi| \mid \sigma \in \hat{M}, \phi_\sigma \neq 0; \pi \in \hat{G}_d, \phi(\pi) \neq 0\}.$$

We can now imitate the inductive proof of Prop. 2 in Delorme's paper [Del05].
Assume, as induction hypothesis, that for all $\psi \in A$ with $l(\psi) < l(\phi)$, there are $f \in C_r^\infty(G)$ with $\mathcal{F}(f) = \psi$. We enumerate

$$\{\sigma \in \hat{M} \mid |\sigma| = l(\phi)\} = \{\sigma_1, \ldots, \sigma_n\} \cup \{w\sigma_1, \ldots, w\sigma_n\}$$

and

$$\{\pi \in \hat{G}_d \mid |\pi| = l(\phi)\} = \{\pi_1, \ldots, \pi_s\}.$$

Condition (ii) together with (3.4) says, in particular, that ϕ_{σ_i} belongs to a space that Delorme denotes by $\mathcal{K}_{\sigma_i}$ ([Del05], Def. 1). Strictly speaking Delorme has a condition for $(\mathrm{Re}(\lambda), \alpha) > 0$. But if $\mathrm{rk}_{\mathbb{R}}(G) = 1$, only $(\mathrm{Re}(\lambda), \alpha) > 0$ matters. Note, that $\phi(\pi_j)$ belongs automatically to $\mathcal{K}_{\pi_j}$. We can apply Prop. 1 together with Eq. (1.38) of Delorme's paper [Del05], to deduce the existence of $f_1, f_2, \dots, f_n \in C_r^\infty(G)$ and $g_1, g_2, \dots, g_s \in C_r^\infty(G)$ with

$$\begin{aligned} \pi_{\sigma_i,\lambda}(f_i) &= \phi(\sigma_i, \lambda), \quad i \in \{1, 2, \dots, n\}, \\ \pi_j(g_j) &= \phi(\pi_j), \qquad j \in \{1, 2, \dots, s\}, \end{aligned}$$

for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Moreover, the discussion after Eq. (3.9) in ([Del05], p.1018), makes clear that we can choose the f_i and g_j such that

(i) $l(\mathcal{F}(f_i)) = l(\mathcal{F}(g_j)) = l(\phi), \forall i \in \{1, 2, \dots, n\}, j \in \{1, 2, \dots, s\}$ and

(ii) $\pi_{\sigma_k,\lambda}(f_i) = 0, \forall k \neq i$,

(iii) $\pi_{\sigma_i,\lambda}(g_j) = 0, \forall i, j$,

(iv) $\pi_j(f_i) = 0, \forall i, j$,

(v) $\pi_k(g_j) = 0, \forall k \neq j$.

Now, we set

$$\psi := \phi - \sum_{i=1}^n \mathcal{F}(f_i) - \sum_{j=1}^s \mathcal{F}(g_j).$$

Then, by (i)-(v) we have $l(\psi) < l(\phi)$. Thus, by induction hypothesis $\psi = \mathcal{F}(f_0)$. We conclude that $\phi = \mathcal{F}(f)$ with $f = f_0 + f_1 + \cdots + f_n + g_1 + g_2 + \cdots + g_s$. □

Remark 3.14. The result above can be extended to higher (real) rank. It induces representations for all cuspidal parabolic subgroups P as well as the Knapp-Stein intertwining operator for them.

3.4 The case $G = SL(2, \mathbb{R})$

We consider $G = SL(2,\mathbb{R}) = \left\{ g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) \;\middle|\; \det(g) = 1 \right\}$ the special linear group of dimension 3 with Iwasawa decomposition $G = KAN$, where

$$K = SO(2) = \left\{ k_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in \mathbb{R} \right\}, \quad A = \left\{ a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\},$$

$$N = \left\{ n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}.$$

It is a connected and simple Lie group with maximal compact subgroup K of dimension one. Clearly, K is isomorphic to the unit circle S^1. Hence

$$\hat{K} := \{\delta_n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}, \;\; \delta_n(k_\theta) := e^{in\theta} \in GL(1,\mathbb{C}) \cong \mathbb{C}\backslash\{0\}$$

is the set of all irreducible one-dimensional representations of K. Its representation space E_{δ_n} is one-dimensional equal to $\mathbb{C}$.

Moreover, we denote by $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ the Lie algebra of G and by $\mathfrak{a}$ the Lie algebra of A. If $H = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \in \mathfrak{a}$, then the positive root α is given by $\alpha(H) = 2t$ and $\rho(H) = t$, for all $t \in \mathbb{R}$. We identify $\mathfrak{a}_{\mathbb{C}}^*$ isometric to $\mathbb{C}$ with respect to the norm of the invariant twice trace form on $\mathfrak{g}$, i.e., $z\alpha \mapsto z$ and $\rho \mapsto \frac{1}{2}$.

Let $M = \{\pm \mathrm{Id}\}$ and $\hat{M} = \{\pm\} \cong \mathbb{Z}/2\mathbb{Z}$. For $\sigma = \pm \in \hat{M}$ and $\lambda \in \mathbb{C} \cong \mathfrak{a}_{\mathbb{C}}^*$, we write $(\pi_{\pm,\lambda}, H_\infty^{\pm,\lambda})$ the principal series representations of G, where

$$H_\infty^{\pm,\lambda} = \{f \in C^\infty(G,\mathbb{C}) \mid f(gma_tn_x) = e^{-(2\lambda+1)t}\sigma(m)^{-1}(f(g)), g \in G\}.$$

Its restriction to K is the set of finite Fourier series on S^1 with only non-zero even or odd Fourier coefficients

$$H_\infty^{\pm} = \{f \in C^\infty(K/M,\mathbb{C}) \mid f \text{ even or odd }\} \overset{K}{\cong} \bigoplus_{n \text{ even or odd}} \delta_n.$$

Note that the irreducible K-representations δ_n, contained in the G-representation of $H_\infty^{\pm}$, are the K-types. In order to classify the irreducible G-representations, this means that if all K-types occur with finite multiplicities, we will from now on denote, for convenience, for an exact, not splitting, module sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

shortly by drawing a '*boxes-picture*'

$$B = \begin{array}{|c|}\hline C \\ \hline A \\ \hline\end{array}\,.$$

A proof of the following classical result can be found for example in ([Wal88], 5.6) or in ([La75], Ch. VI). Note that the referenced proof is also valid for G-representations of smooth vectors instead of $(\mathfrak{g}, K)$-modules, if we apply Casselman's and Wallach's globalization theorem.

Theorem 3.15 (Structure of principal series representations of $SL(2,\mathbb{R})$)**.** *The principal series representations $H_\infty^{\pm,\lambda}$ of $SL(2,\mathbb{R})$ is exactly reducible, if*

$$\lambda \in I_\pm := \begin{cases} \frac{1}{2} + \mathbb{Z}, & \sigma = +, \\ \mathbb{Z}, & \sigma = -. \end{cases}$$

For $\lambda = \frac{k}{2} \in I_\pm, k \in \mathbb{N}$, we have

$$H_\infty^{\pm,-\frac{k}{2}} = \begin{array}{|c|}\hline D_{-k} \oplus D_k \\ \hline F_k \\ \hline\end{array} \qquad H_\infty^{\pm,\frac{k}{2}} = \begin{array}{|c|}\hline F_k \\ \hline D_{-k} \oplus D_k \\ \hline\end{array}$$

where $F_k := \bigoplus_{l=-(k-1)}^{k-1} \delta_{2l}$ are the finite-dimensional $SL(2,\mathbb{R})$-representation of dimension k and D_k resp. D_{-k} are smooth vectors of a representation of the discrete series, which are characterized by the K-type decomposition $D_{\pm k} = \overline{\bigoplus}_{j\geq 0}\delta_{\pm(k+1+2j)}$. Furthermore, for $\lambda = 0$, we have

$$H_\infty^{-,0} = \boxed{D_- \oplus D_+}$$

where $D_\pm = \overline{\bigoplus}_{j\geq 0}\delta_{\pm(1+2j)}$ are the limits of the discrete series. □

Remark 3.16. Let W_λ be a proper closed invariant G-submodule of $H_\infty^{\pm,\lambda}$, for $\lambda \in I_\pm$, as in Thm. 3.15. Then, one can observe that

- for $\lambda > 0$, $W_\lambda \in \{D_{-k}, D_k, D_{-k} \oplus D_k\}$,
- for $\lambda < 0$, $W_\lambda \in \left\{F_k, \begin{array}{|c|}\hline D_{-k} \\ \hline F_k \\ \hline\end{array}, \begin{array}{|c|}\hline D_k \\ \hline F_k \\ \hline\end{array}\right\}$,
- while for $\lambda = 0$ and $\sigma = -$, $W_\lambda \in \{D_+, D_-\}$.

To describe the intertwining conditions for $G = SL(2,\mathbb{R})$ in the three levels, we first need some preparation. The Harish-Chandra $\mathbf{c}$-function for G is denoted by $\mathbf{c}_n(\lambda)$, for $n \in \mathbb{Z}$. Due Cohn ([Co74], App. 1), it is given explicitly in terms of gamma function $\Gamma(\cdot)$, by the formula

$$\mathbf{c}_n(\lambda) = \mathbf{c}_{-n}(\lambda) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\lambda)\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\frac{1+n}{2})\Gamma(\lambda+\frac{1-n}{2})}, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*. \tag{3.14}$$

Let $n \equiv m \in \mathbb{Z}$, not necessary distinct, then using the gamma function recurrence formula

$$\Gamma(\lambda + a) = (\lambda + (a-1))\Gamma(\lambda + (a-1)), \quad a \in \mathbb{Z}, \lambda \in \mathfrak{a}_{\mathbb{C}}^* \tag{3.15}$$

repeatedly, the quotient of the $\mathbf{c}$-functions is given by

$$\frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_m(\lambda)} = \frac{\Gamma(\lambda+\frac{1+m}{2})\Gamma(\lambda+\frac{1-m}{2})}{\Gamma(\lambda+\frac{1+n}{2})\Gamma(\lambda+\frac{1-n}{2})} = \begin{cases} 1, & \text{for } |n| = |m| \\ \frac{(\lambda-\frac{|n|-1}{2})(\lambda-\frac{|n|-3}{2})\cdots(\lambda-\frac{|m|+1}{2})}{(\lambda+\frac{|n|-1}{2})(\lambda+\frac{|n|-3}{2})\cdots(\lambda+\frac{|m|+1}{2})}, & \text{for } |n| > |m| \\ \frac{(\lambda+\frac{|m|-1}{2})(\lambda+\frac{|m|-3}{2})\cdots(\lambda+\frac{|n|+1}{2})}{(\lambda-\frac{|m|-1}{2})(\lambda-\frac{|m|-3}{2})\cdots(\lambda-\frac{|n|+1}{2})}, & \text{for } |n| < |m|. \end{cases} \tag{3.16}$$

Note that the quotient has zeros $\lambda \in \{\frac{|n|-1}{2}, \frac{|n|-3}{2}, \ldots, \frac{|m|+1}{2}\}$ and poles in $\{-\frac{|n|-1}{2}, -\frac{|n|-3}{2}, \ldots, -\frac{|m|+1}{2}\}$, for $|n| > |m|$, and inversely for $|n| < |m|$. Subsequently, we know that the matrix coefficient of the Knapp-Stein intertwining operator $J_{-,\pm,\lambda} : H_\infty^{\pm,\lambda} \longrightarrow H_\infty^{\pm,-\lambda}$ corresponds to the Fourier decomposition of $H_\infty^{\pm,\lambda} \cong C^\infty(S^1)$.

Theorem 3.17 (Intertwining conditions in (Level 1))**.** *For $r > 0$, let A be the space of all $\phi \in \prod_{\sigma\in\hat{M}} \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}(H_\infty^\pm))$ such that ϕ satisfies the growth condition $(1.ii)_r$ of Def. 2.5 as well as the two intertwining conditions (3.4) and*

(D.b') ϕ leaves every proper closed G-submodule W_λ of $H_\infty^\pm$, listed in Remark 3.16, invariant.

Then, A satisfies the conditions of Thm. 3.13, this means that $A = PW_r(G)$.

Proof. Note first, that the space A is $K \times K$ invariant and linear closed, due the intertwining conditions (3.4) and $(D.b')$.
We have that $(D.b)$ of Thm. 3.13 gives a condition for each $\sigma \in \hat{M} = \{\pm\}$. Let us first consider $\sigma = \{+\} \in \hat{M}$. By Example 3.12, we have $|+| = 0$ and $|\pi| = k+1 > 0$. Now let $\phi \in A$ satisfying the assumption $(D.b)(ii)$, i.e., in particular

$$\phi^{(0)}\left(+, \frac{k}{2}\right)\Big|_{D_{-k}\oplus D_k} = 0, \quad k \in 2\mathbb{Z}+1.$$

Let us check that:

(a) for $\mathrm{Re}(\lambda) > 0$, the intertwining operator $J_{-,+,\lambda}$ has zeros of order at most one

(b) the kernel of $J_{-,+,\lambda}$ is equal to 0 or $D_{-k} \oplus D_k$ for $\mathrm{Re}(\lambda) > 0$.

Consider a K-representation $n \in 2\mathbb{Z}$ and the Harish-Chandra $\mathbf{c}$-function $\mathbf{c}_n$ as in (3.14). If $n = 0$, then $\mathbf{c}_0(\lambda) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})}$ and we see that $\mathbf{c}_0(\lambda)$, for $\mathrm{Re}(\lambda) > 0$, has no zeros and no poles. Thus, we can consider the quotient

$$\frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_0(\lambda)} = \frac{\Gamma(\lambda+\frac{1}{2})^2}{\Gamma(\lambda+\frac{1+n}{2})\Gamma(\lambda+\frac{1-n}{2})} = \frac{(\lambda - \frac{|n|-1}{2}) \cdots (\lambda - \frac{1}{2})}{(\lambda + \frac{|n|-1}{2}) \cdots (\lambda + \frac{1}{2})}.$$

It has zeros $\lambda \in \{\frac{1}{2}, \cdots, \frac{|n|-1}{2}\}$ of first order. Due to (3.9), we know that the intertwining operator $J_{-,+,\lambda}$ is in relation with the $\mathbf{c}$-function. If on all K-types, we have zeros of first order, then $J_{-,+,\lambda}$ should also have zeros of first order. Hence $J_{-,+,\lambda}$ has zeros of at most order one, this proves the first assertation (a) of the claim.

Concerning (b), we need to check for which K-type n, the quotient $\frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_0(\lambda)}$ has a zero, for fixed $\mathrm{Re}(\lambda) > 0$. It is clear that, if $\lambda \notin I_+$, then $\frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_0(\lambda)}$ has no zeros, i.e. that $\mathrm{Ker}(J_{-,+,\lambda}) = 0$. For fixed $\lambda = \frac{k}{2}, k \in 2\mathbb{Z}$, the $\mathbf{c}$-quotient $\frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_0(\lambda)}$ has zeros if, and only if, n is a K-type of D_{-k} and D_k, i.e. $\mathrm{Ker}(J_{-,+,\lambda}) = D_{-k} \oplus D_k$. Thus, this implies (b).

By (b) and the assumption, we have that the operator $\phi^{(0)}(+,\lambda)$ annihiliates $\mathrm{Ker}(J_{-,+,\lambda})$ for $\sigma = \{+\} \in \hat{M}$ and $\mathrm{Re}(\lambda) > 0$. By (a), we have that the order m is equal to the one, thus this condition is sufficient.
By arguing in a similar way as above for $\sigma = \{-\} \in \hat{M}$ with $|-| = 1$, and $\pi = k-1 \in \hat{G}_d$ with $|\pi| = k-1, k \in \mathbb{Z}\backslash\{0\}$, we can conclude that A satisfies the condition $(D.b)$ of Thm. 3.13. □

Now let us move to (Level 2).

Theorem 3.18 (Intertwining conditions in (Level 2))**.** *Let $m \in \mathbb{Z}$. Then, $\psi \in \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, H_\infty^{\tau|_M})$ satisfies the intertwining condition (2) of Def. 2.27 if, and only if,*

(2.a) $J_{-,\pm,\lambda}\psi(\lambda,\cdot) = (-1)^{m/2}\mathbf{c}_m(\lambda)\psi(-\lambda,\cdot)$, *for all* $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$,

(2.b) $\psi(\lambda,\cdot) \in W_\lambda$, *where W_λ is the invariant (colored in blue) G-submodule represented by the boxes-pictures in Fig. 3.1*

are satisfied.

Notice that if W_λ is the whole colored blue box, then there are no intertwining condition and, thus, it is no more a proper invariant G-submodule of $H^{\pm}_\infty$.

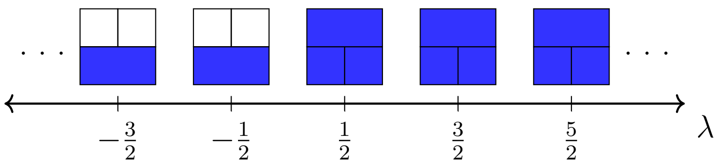
Proof of Thm. 3.18. We need to show that the conditions $(2.a)$ and $(2.b)$ correspond to the condition $(D.2)$ in Thm. 2.28.

In fact, by Prop. 3.8, we have that $(2.a)$ is a special case of (2.41), hence of $(D.2)$ in Thm. 2.28.

Concerning $(2.b)$, condition $(D.2)$ says that for each W we have an intertwining condition corresponding to $(D.b')$ in Thm. 3.17. Now we need to extract in which of these W_λ, there is an intertwining condition. If the K-type m is in a closed G-submodule W_λ of $H^{\pm,\lambda}_\infty$, then D^m_W is one-dimensional. Hence, by Thm. 2.28 and (2.41),

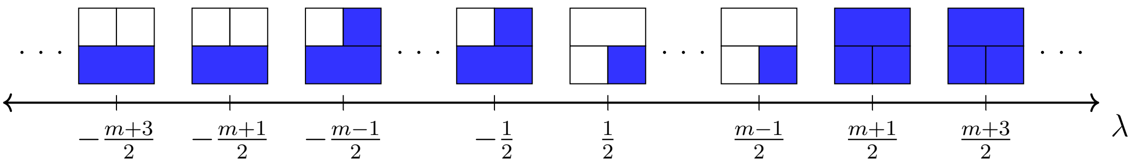
ψ has values in this G-submodule W_λ. By $(D.b')$ in Thm. 3.17, we thus take the smallest closed proper invariant G-submodule of them. Otherwise, if the K-type is not in a closed G-submodule W_λ of $H_\infty^{\pm,\lambda}$, then $D^m_{W_\lambda} = \{0\}$ and thus there are no intertwining conditions. Consequently, we obtain the boxes-pictures in Fig. 3.1. $\square$

- for $m = 0$:

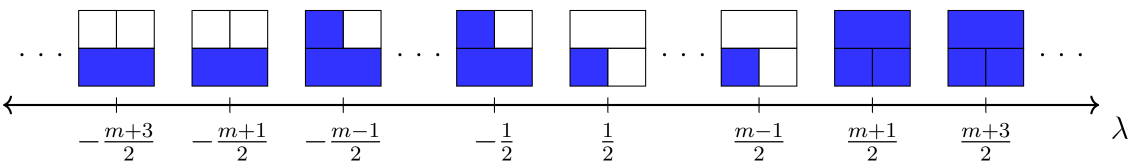


- for $m \in 2\mathbb{Z}$:

$\cdot$ $m > 0$:

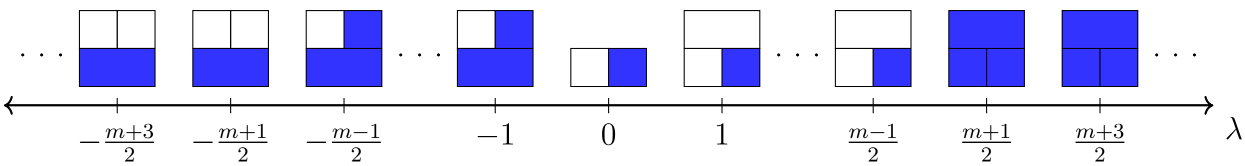


$\cdot$ $m < 0$:



- for $m \in 2\mathbb{Z} + 1$:

$\cdot$ $m > 0$:



$\cdot$ $m < 0$:

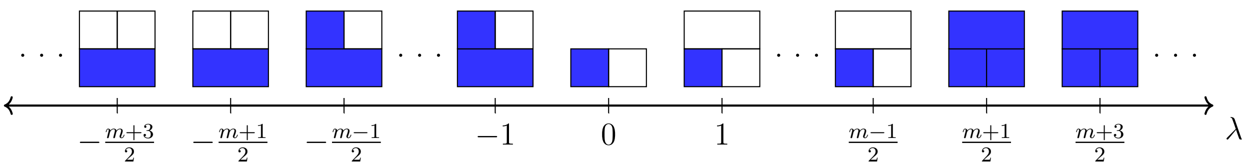


Figure 3.1: Boxes-pictures for $G = SL(2, \mathbb{R})$.

The final step will be to move in (Level 3).

Definition 3.19. *Let $n, m \in \mathbb{Z}$ be two integers. We define the polynomial $q_{n,m}$ in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with values in $\mathrm{Hom}_M(E_n, E_m)$ by*

$$q_{n,m}(\lambda) := \begin{cases} 1, & \text{if } n = m, \\ (\lambda + \frac{|m|+1}{2})(\lambda + \frac{|m|+3}{2}) \cdots (\lambda + \frac{|n|-1}{2}), & \text{if } |n| > |m| \text{ and same signs}, \\ (\lambda - \frac{|n|+1}{2})(\lambda - \frac{|n|+3}{2}) \cdots (\lambda - \frac{|m|-1}{2}), & \text{if } |n| < |m| \text{ and same signs}, \\ (\lambda + \frac{|n|-1}{2})(\lambda + \frac{|n|-3}{2}) \cdots (\lambda - \frac{|m|-1}{2}), & \text{else, with different signs.} \end{cases} \tag{3.17}$$

Theorem 3.20 (Intertwining conditions in (Level 3))**.** *Let $n, m \in \mathbb{Z}$ be two, not necessary distinct, K-types. Then, $\varphi \in \mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}}, \mathrm{Hom}_M(E_n, E_m))$ satisfies the intertwining condition (3) of Def. 2.27 if, and only if, there exists an even holomorphic function $h \in \mathrm{Hol}(\lambda^2)$ such that*

$$\varphi(\lambda) = h(\lambda) \cdot q_{n,m}(\lambda), \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}, \tag{3.18}$$

where $q_{n,m}$ is the polynomial (3.17).

Proof. By Thm. 2.28 and Thm. 3.18, it is sufficient to prove that the conditions (2.a) and (2.b) correspond to (3.18).

In particular, we want to show that (2.b) is satisfied if, and only if, $\varphi(\lambda)$ has zeros on the zeros of the polynomial $q_{n,m}$. From Thm. 3.18 (2.b), we know that the invariant G-submodule W_λ are represented by the boxes-pictures in Fig. 3.1. Thus, we need to check, where the K-type n is not in the colored blue invariant G-submodule W_λ. We leave it to the reader to check that this happens exactly at the zeros of $q_{n,m}$. Thus, we can deduce that φ is of the form (3.18) with h an arbitrary holomorphic function.

Concerning the correspondance between the conditions (2.a) and (3.18), by Lem. 3.10, we proved that (2.a) is a special case of (2.42) for the Knapp-Stein intertwining operator:

$$(-1)^{(m-n)/2} \frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_m(\lambda)} \varphi(\lambda) = \varphi(-\lambda), \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}, n, m \in \mathbb{Z}. \tag{3.19}$$

By using Def. 3.19, we observe that $\frac{q_{n,m}(-\lambda)}{q_{n,m}(\lambda)} = (-1)^{(m-n)/2} \frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_m(\lambda)}$, for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $n, m \in \mathbb{Z}$. Hence, we obtain

$$\frac{\varphi(\lambda)}{q_{n,m}(\lambda)} = \frac{\varphi(-\lambda)}{q_{n,m}(-\lambda)}.$$

This means, (3.19) is satisfied if, and only if, $h(\lambda) = h(-\lambda)$, for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. □

For both levels, we have completely determined the Paley-Wiener-(Schwartz) theorems for $G = SL(2, \mathbb{R})$.
Moreover, by multiplying two polynomials (3.17) together we obtain the following useful relation.

Lemma 3.21. *Let $n, m, l \in \mathbb{Z}$ be integers, not necessary distinct, and $q_{n,m}$ (resp. $q_{l,n}$) be a polynomial in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ as in (3.17). If we multiply these two polynomials together, then we obtain*

$$q_{n,m} \cdot q_{l,n} = r^l_{n,m} \cdot q_{l,m}, \tag{3.20}$$

where $r^l_{n,m} \in \mathbb{C}[\lambda^2]$ is a symmetric polynomial. In particular, for different or same signs of l, n, m, we have

$$r^l_{n,m} = \begin{cases} 1, & \text{if } l \leq n \leq m \text{ or } m \leq n \leq l, \\ q_{n,m} \cdot q_{m,n}, & \text{if } l \leq m < n \text{ or } n < m \leq l, \\ q_{n,l} \cdot q_{l,n}, & \text{if } m < l < n \text{ or } n < l < m. \end{cases}$$

Remark 3.22. (i) Note that $r^l_{n,m}$ depends only on the variable l when it lies inside the open interval (m, n) resp. (n, m) and we have finitely many of them.

(ii) For $n, m \in \mathbb{Z}$, not necessary distinct, we observe that

(a) $q_{n,-m} = q_{-n,m}$,

(b) $q_{0,m} = q_{0,-m}$ and $q_{n,0} = q_{-n,0}$,

(c) $q_{n,m}(\lambda) = -q_{m,n}(-\lambda)$, for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

Proof of Lem. 3.21. Consider three integers n, m, l, which are not necessary distinct under each other. Assume that $q_{l,m} \neq 0$ and write (3.20) in the following form

$$r^l_{n,m} = \frac{q_{n,m} \cdot q_{l,n}}{q_{l,m}}.$$

If l lies outside the different intervals $[n, m]$ (resp. $[m, n]$) or $(n, m]$ (resp. $[m, n)$), then by using Def. 3.19, we easily see that for

(1) $n = m = l$, we get $r^l_{n,m} = \frac{q_{n,n} \cdot q_{n,n}}{q_{n,m}} = 1$,

(2) $n = m$ and $\{|l| < |m|, |m| < |l|$ or different signs$\}$, we have $r^l_{n,m} = \frac{q_{m,m} \cdot q_{l,m}}{q_{l,m}} = 1$,

(3) $l = n$ and $\{|n| < |m|, |m| < |n|$ or different signs$\}$, we obtain $r^l_{n,m} = \frac{q_{n,m} \cdot q_{n,n}}{q_{l,m}} = \frac{q_{n,m}}{q_{n,m}} = 1$,

(4) $l = m$ and $\{|n| < |m|, |m| < |n|$ or different signs$\}$, we get $r^l_{n,m} = \frac{q_{n,m} \cdot q_{m,n}}{q_{m,m}} = q_{n,m} \cdot q_{m,n}$.

Now, if l lies between the open interval (m, n) resp. (n, m), then

$$r^l_{n,m} = \frac{q_{n,m} \cdot q_{l,n}}{q_{l,m}} = \frac{(q_{n,l} \cdot q_{l,m}) \cdot q_{l,n}}{q_{l,m}} = q_{n,l} \cdot q_{l,n},$$

where we used the relation (3) to decompose $q_{n,m}$ in terms of $q_{n,l}$ and $q_{l,m}$. □

3.5 The case $G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$

Let

$$G := G' \times G' = SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) = \{g = (g_1, g_2) \mid g_1, g_2 \in G'\}$$

be the Cartesian product of two copies of semi-simple non-compact connected real Lie groups $G' = SL(2, \mathbb{R})$ with finite center. Since $K' = SO(2)$ is the maximal compact subgroup of G', hence it also applies to $K := K' \times K'$ for G.

Consider two reducible one-dimensional K'-representations (n_1, E_{n_1}) and (n_2, E_{n_2}). We define the exterior tensor product ([Wal73], Sect. 2.36) by

$$(n_1 \boxtimes n_2)(k_1, k_2) := n_1(k_1) \otimes n_2(k_2), \quad \text{for } (k_1, k_2) \in K' \times K'.$$

Let $\{v_i, i = 1, \dots, d_{n_1}\}$ and $\{w_j, j = 1 \dots, d_{n_2}\}$ be two orthonormal bases of E_{n_1} and E_{n_2}, respectively. Then, we attribute to the vector space $E_{n_1} \otimes E_{n_2}$, the Hilbert space structure that makes the basis $\{v_i \otimes w_j\}$ orthonormal. Thus, $(n_1 \boxtimes n_2, E_{n_1} \otimes E_{n_2})$ is a representation of K and is known as the *exterior tensor product representation.*
Note that $(n_1 \boxtimes n_2, E_{n_1} \otimes E_{n_2})$ is unitary, if (n_1, E_{n_1}) and (n_2, E_{n_2}) are, that is, if $n_1(k)$ (resp. $n_2(k)$) is a unitary operator for each k in K.
The following known result tells us that the exterior product tensor of two irreducible unitary one-dimensional representations of K' is a unitary irreducible one-dimensional K-representation, short

$$\hat{K'} \times \hat{K'} \cong \hat{K}.$$

Proposition 3.23 ([Wal73], Prop. 2.3.7)**.** *In the previous notations, let (n, E_n) be a unitary one-dimensional representation of K.*
Then, (n, E_n) is equivalent to the exterior tensor product $(n_1 \boxtimes n_2, E_{n_1} \otimes E_{n_2})$ of two unitary representations of K'. This means that there is a continuous linear isomorphism A of E_n onto $E_{n_1} \otimes E_{n_2}$ so that $A \circ n(k) = (n_1 \boxtimes n_2)(k) \circ A$, for all $k = (k_1, k_2) \in K$. □

Write by tuples of integers

$$n := (n_1, n_2) \text{ and } m := (m_1, m_2) \text{ in } \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}^2 \cong \hat{K}$$

the K-types with their corresponding vector space $E_n := E_{n_1} \otimes E_{n_2}$ respectively $E_m := E_{m_1} \otimes E_{m_2}$. Their associated homogeneous line bundles are given by $\mathbb{E}_n := \mathbb{E}_{n_1} \otimes \mathbb{E}_{n_2}$ (resp. $\mathbb{E}_m := \mathbb{E}_{m_1} \otimes \mathbb{E}_{m_2}$) over $X := X' \times X'$. Since the K' representations are one-dimensional, we identify the vector spaces and the associated homogeneous line bundles with $\mathbb{C}$.

Before we proceed to specify Delorme's intertwining conditions on the three levels, we first need to adapt the Fourier transform for our framework.
Fix an additional irreducible K-type $l = (l_1, l_2) \in \mathbb{Z}^2$ and let $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*$. The Fourier transform for $f \in \mathrm{Hom}_K(\mathbb{E}_l, C_c^{\pm\infty}(X, \mathbb{E}_n)) \cong C_c^{\pm\infty}(G, l, n)$ is given by

$$\begin{aligned} {}_l\mathcal{F}_n f(\lambda) &= \int_{G'} \Big(\int_{G'} e^n_{\lambda,1}(g_1, g_2) f(g_1, g_2)\, dg_1 \Big) dg_2 \\ &= \int_{X'} \Big(\int_{X'} e^n_{\lambda,1}(g_1, g_2) f(g_1, g_2)\, dg_1 \Big) dg_2, \end{aligned} \tag{3.21}$$

where $e^n_{\lambda,k}$ is defined as in (2.20). Note that in the last line, we used the fact that the integrand is right $K' \times K'$-invariant. Moreover, by using the Iwasawa decomposition of $g = (g_1, g_2) \in G$, the 'exponential' function $e^n_{\lambda,k}$ can be rewritten as follows.

Proposition 3.24. *For fixed $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*$ and $k = (k_1, k_2) \in K$, the function $e^n_{\lambda,k}$ defined as in (2.20), can be decomposed in terms of $e^{n_1}_{\lambda_1,k_1}$ and $e^{n_2}_{\lambda_2,k_2}$:*

$$e^n_{\lambda,k}(g_1, g_2) = e^{n_1}_{\lambda_1,k_1}(g_1) \cdot e^{n_2}_{\lambda_2,k_2}(g_2), \quad (g_1, g_2) \in G.$$

Proof. Consider the Iwasawa decomposition of

$$g = (g_1, g_2) = (n_1' a_1 k_1', n_2' a_2 k_2') = n' a k' \in G$$

so that $n' = (n_1', n_2') \in N, a = (a_1, a_2) \in A$ and $k' = (k_1', k_2') \in K$. One can easily deduce that for $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*$

$$e^{(\lambda+\rho)\log(a)} = e^{(\lambda_1+\rho)\log(a_1)+(\lambda_2+\rho)\log(a_2)} = a_1^{\lambda_1+\rho} \cdot a_2^{\lambda_2+\rho}.$$

Hence, for $g \in G$, we then have

$$\begin{aligned} e^n_{\lambda,k}(g) = e^n_{\lambda,k}(g_1, g_2) \overset{(2.25)}{=} a_1^{\lambda_1+\rho} a_2^{\lambda_2+\rho} e^n_{\lambda,1}(k_1, k_2) &= a_1^{\lambda_1+\rho} a_2^{\lambda_2+\rho} n_1(k_1) n_2(k_2) \\ &= a_1^{\lambda_1+\rho} a_2^{\lambda_2+\rho} e^{n_1}_{\lambda_1,1}(k_1) e^{n_2}_{\lambda_2,1}(k_2) \\ &\overset{(2.20)}{=} e^{n_1}_{\lambda_1,k_1}(g_1) \cdot e^{n_2}_{\lambda_2,k_2}(g_2). \qquad \square \end{aligned}$$

Aslike in Sect. 3.4, more precisely Thm. 3.20, we want to prove the adequateness of the intertwining condition in (Level 3) for G. By using Def. 3.19 for $l, n \in \mathbb{Z}^2$, the polynomial $q_{l,n}$ is given by

$$q_{l,n}(\lambda_1, \lambda_2) := q_{l_1,n_1}(\lambda_1) \cdot q_{l_2,n_2}(\lambda_2), \quad (\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*, \tag{3.22}$$

where $q_{l_i,n_i}(\lambda_i)$ is the 'intertwining' polynomial (3.17) for $\lambda_i \in \mathfrak{a}_{\mathbb{C}}^*, i = 1, 2$.

Theorem 3.25 (Intertwining condition in (Level 3))**.** *Let $l, n \in \mathbb{Z}^2$ be two tuples of integers. Then, $\varphi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*, \mathrm{Hom}_M(E_n, E_m))$ satisfies the intertwining condition (3) of Def. 2.27 if, and only if, there exists a symmetric holomorphic function $h \in \mathrm{Hol}(\lambda_1^2, \lambda_2^2)$:*

$$h(\lambda_1, \lambda_2) = h(-\lambda_1, \lambda_2) = h(\lambda_1, -\lambda_2)$$

such that

$$\varphi(\lambda_1, \lambda_2) := q_{l,n}(\lambda_1, \lambda_2) \cdot h(\lambda_1, \lambda_2), \tag{3.23}$$

where $q_{l,n}$ is the polynomial (3.22) in $(\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*$.

To prove Thm. 3.25, we need first a density argument, which permits us to approximate the symmetric holomorphic function h in (3.23) by two even holomorphic functions.

Lemma 3.26. *Let $s \in \mathbb{N}$ and consider*

$$S := \left\{ \sum_{i=1}^{s} f_i(\lambda_1) g_i(\lambda_2) \in \mathrm{Hol}(\mathbb{C}^2) \mid f_i(\lambda_1) = f_i(-\lambda_1), g_i(\lambda_2) = g_i(-\lambda_2), \forall i, \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

a subset of $A := \{h \in \mathrm{Hol}(\mathbb{C}^2) \mid h(\lambda_1, \lambda_2) = h(-\lambda_1, \lambda_2) = h(\lambda_1, -\lambda_2), \forall (\lambda_1, \lambda_2) \in \mathbb{C}^2\}$. Then, $S \subset A$ is dense in A.

Proof. Let $h(\lambda_1, \lambda_2) \in A$ be a holomorphic function. Consider a Taylor series of order $|\alpha| \leq m \in \mathbb{N}_0$ at the point $0 = (0, 0)$ in two variables $(\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*$:

$$\sum_{|\alpha| \leq m} a_\alpha \lambda^\alpha = \sum_{\alpha_1, \alpha_2} a_{\alpha_1,\alpha_2} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2},$$

where a_α are constants with multi-index $\alpha = (\alpha_1, \alpha_2), \alpha_j \in \mathbb{N}_0$ and $|\alpha| = \sum_{j=1}^2 \alpha_j$ is the length. The series converges uniformely on each compact ball $B_r(0)$, this means that, for each $r \geq 0$ and $\epsilon > 0$, there exists an integer k such that

$$\Big| h(\lambda_1, \lambda_2) - \sum_{|\alpha| \leq k} a_\alpha \lambda^\alpha \Big| < \epsilon$$

for $(\lambda_1, \lambda_2) \in B_r(0)$. Thus, h can be approximated uniformely on each $B_r(0)$ by the Taylor series, which corresponds to a function in S. This completes the proof. □

Proof of Thm. 3.25. It suffices to show that

(a) every function $\varphi \in {}_l PW_{n,H}(\mathfrak{a}^*_\mathbb{C} \times \mathfrak{a}^*_\mathbb{C})$ is of the form (3.23) and

(b) (inversely) if φ is of the form (3.23), then it is in ${}_l PW_{n,H}(\mathfrak{a}^*_\mathbb{C} \times \mathfrak{a}^*_\mathbb{C})$.

Let $\varphi \in {}_l PW_{n,H}(\mathfrak{a}^*_\mathbb{C} \times \mathfrak{a}^*_\mathbb{C})$ and consider $f \in C_c^\infty(G, l, n)$ with its Fourier transform (3.21). By Fubini's theorem and Prop. 3.24, we have

$$\begin{aligned} \int_{X'} e^{n_1}_{\lambda_1,1}(g_1) \Big(\int_{X'} e^{n_2}_{\lambda_2,1}(g_2) f(g_1, g_2)\, dg_2 \Big)\, dg_1 \quad &= \quad \int_{X'} e^{n_1}_{\lambda_1,1}(g_1) \tilde{f}_{\lambda_2}(g_1)\, dg_1 \\ &=: \; {}_{l_1}\mathcal{F}_{n_1} \tilde{f}_{\lambda_2}(\lambda_1), \end{aligned} \tag{3.24}$$

where we set $\tilde{f}_{\lambda_2}(g_1) := \int_{X'} e^{n_2}_{\lambda_2,1}(g_2) f(g_1, g_2)\, dg_2 \in \mathbb{E}_{n_1} \otimes \mathbb{E}_{\tilde{l_1}}$. Similar, if we fix $\lambda_1 \in \mathfrak{a}^*_\mathbb{C}$,

$${}_{l_2}\mathcal{F}_{n_2} \tilde{f}_{\lambda_1}(\lambda_2) := \int_{X'} e^{n_2}_{\lambda_2,1}(g_2) \tilde{f}_{\lambda_1}(g_2)\, dg_2, \tag{3.25}$$

with $\tilde{f}_{\lambda_1}(g_2) := \int_{X'} e^{n_1}_{\lambda_1,k_1}(g_1) f(g_1, g_2)\, dg_1 \in \mathbb{E}_{n_2} \otimes \mathbb{E}_{\tilde{l_2}}$. Note that ${}_{l_i}\mathcal{F}_{n_i} \tilde{f}_{\lambda_j}(\lambda_i)$ is the Fourier transform for $\tilde{f}_{\lambda_j} \in \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{Hom}_M(E_{l_i}, E_{n_i}))$ in (Level 3), for fixed λ_j and each $i, j = 1, 2$ distinct (Def. 2.19). Thus, by Thm. 3.20, they have the form

$${}_{l_1}\mathcal{F}_{n_1} \tilde{f}_{\lambda_2}(\lambda_1) = h_{\lambda_2}(\lambda_1) \cdot q_{l_1,n_1}(\lambda_1) \text{ and } {}_{l_2}\mathcal{F}_{n_2} \tilde{f}_{\lambda_1}(\lambda_2) = h_{\lambda_1}(\lambda_2) \cdot q_{l_2,n_2}(\lambda_2), \tag{3.26}$$

where h_{λ_i} is a holomorphic even function in $\lambda_i \in \mathfrak{a}^*_\mathbb{C}$ and q_{l_i,n_i} is the 'intertwining' polynomial in $\lambda_i \in \mathfrak{a}^*_\mathbb{C}$, defined in (3.17). By (3.22), we deduce that

$${}_l\mathcal{F}_n f(\lambda) = h(\lambda_1, \lambda_2) \cdot q_{l,n}(\lambda_1, \lambda_2), \quad \lambda = (\lambda_1, \lambda_2) \in \mathfrak{a}^*_\mathbb{C} \times \mathfrak{a}^*_\mathbb{C}.$$

Now, by applying the Paley-Wiener Thm. 2.31, we have ${}_l\mathcal{F}_n(f) \in {}_l PW_{n,H}(\mathfrak{a}^*_\mathbb{C} \times \mathfrak{a}^*_\mathbb{C})$. Hence, we conclude that $\varphi(\lambda_1, \lambda_2)$ is ${}_l\mathcal{F}_n(f)(\lambda_1, \lambda_2)$.

Concerning (b), let φ of the form (3.23). By (3.22), we have that

$$\varphi(\lambda_1, \lambda_2) = h(\lambda_1, \lambda_2) q_{l_1,n_1}(\lambda_1) q_{l_2,n_2}(\lambda_2)$$

for $(\lambda_1, \lambda_2) \in \mathfrak{a}^*_\mathbb{C} \times \mathfrak{a}^*_\mathbb{C}$. By Lem. 3.26, we can approximate $h(\lambda_1, \lambda_2)$ by the product of two holomorphic functions $h_1(\lambda_1)$ and $h_2(\lambda_2)$. Hence, by Thm. 3.20

$$h_i(\lambda_i) q_{l_i,n_i}(\lambda_i) =: {}_{l_i}\mathcal{F}_{n_i}(f_i)(\lambda_i) \in {}_{l_i}PW_{n_i}(\mathfrak{a}^*_\mathbb{C})$$

is the image of the Fourier transform of the function $f_i \in C_c^\infty(G', l_i, n_i), i = 1, 2$. Consider now the tensor product of these two functions:

$$f_1 \otimes f_2 \in C_c^\infty(G, l_1 \boxtimes l_2, , n_1 \boxtimes n_2) = C_c^\infty(G, l, n).$$

By taking the Fourier transform and using the previous computations at the beginning of the proof together with Fubini's theorem, we obtain that

$$ {}_l\mathcal{F}_n(f_1 \otimes f_2)(\lambda_1, \lambda_2) = {}_{l_1}\mathcal{F}_{n_1}(f_1)(\lambda_1) \times {}_{l_2}\mathcal{F}_{n_2}(f_2)(\lambda_2). $$

By using Thm. 2.31, we have ${}_l\mathcal{F}_n(f_1 \otimes f_2) \in {}_lPW_{n,H}(\mathfrak{a}^*_{\mathbb{C}} \times \mathfrak{a}^*_{\mathbb{C}})$. Hence, we conclude that φ, which can be approximate by Lem. 3.26, is the Fourier transform image of $f_1 \otimes f_2$ and thus it is in ${}_lPW_{n,H}(\mathfrak{a}^*_{\mathbb{C}} \times \ \mathfrak{a}^*_{\mathbb{C}})$. $\square$

In a similar way, we can obtain the previous results for distributional functions. By Thm. 3.25, we explicitly determined the Paley-Wiener(-Schwartz) theorem for $G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ in (Level 3).
Moreover, all the previous results can be generalized for finite copies $G = SL(2,\mathbb{R})^d, d \geq 2$.

3.6 The case $G = SL(2,\mathbb{C})$

Let $G = SL(2,\mathbb{C}) = \{g \in GL(2,\mathbb{C}) \mid \det(g) = 1\}$ be the special linear group over complex numbers with maximal compact subgroup

$$ K = SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in GL(2,\mathbb{C}) \,\Big|\, |\alpha|^2 + |\beta|^2 = 1 \right\}. $$

Note that K is homeomorphic with the 3-sphere and it is simply connected. Furthermore, the one-dimensional abelian subgroup A of G is the same as in Sect. 3.4 and the one-dimensional nilpotent subgroup N as well as its conjugate $\overline{N}$ of G are given by

$$ N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \,\Big|\, t \in \mathbb{C} \right\}, \qquad \overline{N} = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \,\Big|\, t \in \mathbb{C} \right\}. $$

Here, $\mathfrak{a}^*_{\mathbb{C}} \cong \mathbb{C}$, by identifying λ with $\lambda(H)$, for $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}$ with $\rho(H) = 2$.

For the complex representations of K, we know from ([Wal88], Sect. 5.7) that

$$ \hat{K} = \{\delta_n \mid n \in \mathbb{N}_0\} \cong \mathbb{N}_0 \text{ with } d_{\delta_n} := \dim(\delta_n) = n + 1. $$

Since the tensor product of two irreducible K-representations is in general not irreducible, we use Clebsch-Gordan rule (e.g. [Wal88], 5.7.1 (1)) to decompose it into irreducible pieces:

$$ \delta_n \otimes \delta_m = \bigoplus_{0 \leq j \leq \min(n,m)} \delta_{n+m-2j}, \qquad n, m \in \mathbb{N}_0. \tag{3.27} $$

In addition, we define by A and K, $M = \left\{ m_\theta := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \,\Big|\, \theta \in \mathbb{R} \right\}$ which is abelian and the maximal torus in K. We parametrize $\hat{M} := \{\sigma_l \mid l \in \mathbb{Z}\} \cong \mathbb{Z}$ by the integers with $\sigma_l(m_\theta) = e^{il\theta} \in \mathbb{Z}$. Moreover, let $\chi : M \to \mathbb{C}$ denotes the character of a finite-dimensional irreducible representation (δ_n, E_n) of K. Then, the Weyl character formula for $m_\theta \in M$ (e.g. [Kna02], Chap. V.6):

$$ \chi(m_\theta) = \mathrm{Tr}(\delta_n(m_\theta)) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin((n+1)\theta)}{\sin(\theta)} = e^{-in\theta} + e^{-i(n-2)\theta} + \cdots + e^{in\theta} $$

tells us that the weights of (δ_n, E_n) have the form $-n, -(n-2), \dots, n-2, n$, each with multiplicity one, with $n \in \mathbb{N}_0$ a highest weight.

The following important result clarify the irreducibility of the principal series representations in $SL(2,\mathbb{C})$. We refer for example to Wallach's book ([Wal88], Sect. 5.7) for a proof. Note that Wallach's proof is also valid for G-representation of smooth vectors (see remark before Thm. 3.15).

Theorem 3.27 (Structure of principal series representations of $SL(2,\mathbb{C})$)**.** *The principal series representations $H_\infty^{\sigma,\lambda}$ of $SL(2,\mathbb{C})$ is exactly reducible, if λ is real and*

$$|\lambda| > |\sigma| \quad \& \quad |\lambda| - |\sigma| \ \textit{even}.$$

Then, in this case for $\lambda > 0$, there is an unique irreducible subrepresentation $R^{\sigma,\lambda}$ of each $H_\infty^{\sigma,\lambda}$. Then, we have

$$H_\infty^{-\sigma,-\lambda} = \begin{array}{|c|}\hline R^{\sigma,\lambda} \\ \hline F_{m,n} \\ \hline\end{array} \qquad H_\infty^{\sigma,\lambda} = \begin{array}{|c|}\hline F_{m,n} \\ \hline R^{\sigma,\lambda} \\ \hline\end{array}$$

where $m = \frac{\sigma+\lambda}{2} - 1, n = \frac{\lambda-\sigma}{2} - 1$ and $F_{m,n}$ is a finite-dimensional G-representation that has a K-representation isomorphic to $\delta_m \otimes \delta_n$. Moreover, there is a Želobenko intertwining opertator

$$L_{\sigma,\lambda} : H_\infty^{-\lambda,-\sigma} \longrightarrow H_\infty^{\sigma,\lambda}$$

so that $\mathrm{Ker}(J_{-,\sigma,\lambda}) = \mathrm{Im}(L_{\sigma,\lambda}) = R^{\sigma,\lambda}$. *In particular $R^{\sigma,\lambda}$ is isomorphic to $H_\infty^{-\lambda,-\sigma}$. Here, $J_{-,\sigma,\lambda} : H^{\sigma,\lambda} \longrightarrow H^{-\sigma,-\lambda}$ denotes the Knapp-Stein intertwining operator defined in Def. 3.1 with $w = -1$.*

More precisely, from the Želobenko intertwining opertator $L_{\sigma,\lambda}$, we can deduce the other intertwining operators:

$$\begin{tikzcd}
& H_\infty^{\lambda,\sigma} \arrow[dr, dashed, "L'_{\sigma,\lambda}"] \arrow[dd, "J_{-,\sigma,\lambda}" near end] & \\
H_\infty^{-\sigma,-\lambda} \arrow[ur, "\tilde{L}_{\sigma,\lambda}"] \arrow[rr, "J_{-,\sigma,\lambda}" near start] \arrow[dr, dashed, "\tilde{L}'_{\sigma,\lambda}"'] & & H_\infty^{\sigma,\lambda} \\
& H_\infty^{-\lambda,-\sigma} \arrow[ur, "L_{\sigma,\lambda}"'] &
\end{tikzcd}$$

In Fig. 3.2, we illustrate the principal series representations in a grid, where the horizontal axis represents the values of $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and the vertical one the values of $\sigma \in \hat{M}$. Note that inside the region $\{\pm\sigma > \pm\lambda\}$, we have the 'special' principal series representations $H_\infty^{-\lambda,-\sigma}$ respectively $H_\infty^{\lambda,\sigma}$ colored in gray and outside the 'normal' one, colored in black.

Theorem 3.28 (Intertwining condition in (Level 1))**.** *For $r > 0$, let A be the space of all $\phi \in \prod_{\sigma\in\hat{M}} \mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, \mathrm{End}(H_\infty^\sigma))$ such that ϕ satisfies the growth condition $(1.ii)_r$ of Def. 2.5 as well as the two intertwining conditions (3.4) and (3.5). Then, A satisfies the conditions of Thm. 3.13, this means that $A = PW_r(G)$.*

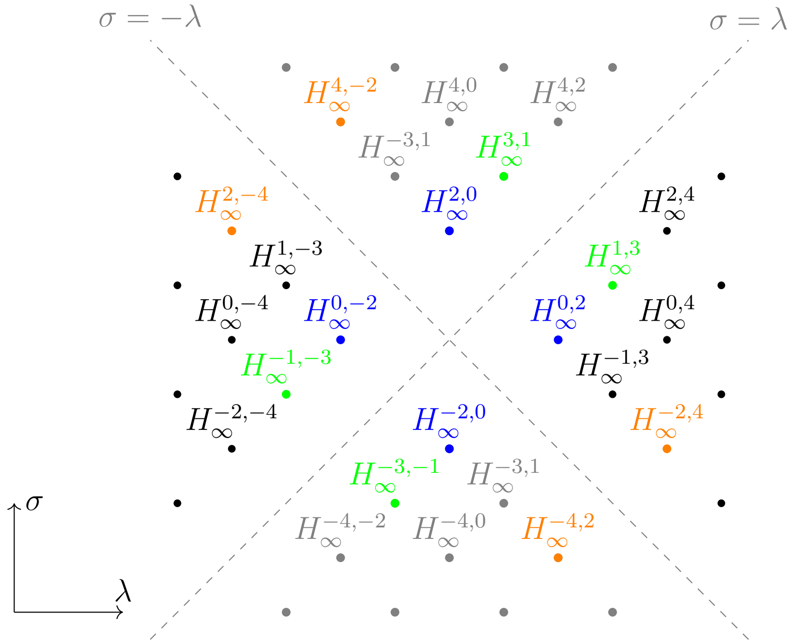


Figure 3.2: Principal series representations in (Level 1), where the colored one indacte the intertwining relations that occur between each others with the same colors.

Before we proceed with the proof of Thm. 3.28, let us first introduce the Harish-Chandra **c**-function ([Co74], App. 2) for $G = SL(2, \mathbb{C})$, which is given by the following formula, for $n \geq |\sigma|$:

$$\mathbf{c}_{n,\sigma}(\lambda) = \frac{\Gamma(\frac{1}{2}(\lambda+\sigma))\Gamma(\frac{1}{2}(\lambda-\sigma))}{\Gamma(\frac{1}{2}(\lambda+n+2))\Gamma(\frac{1}{2}(\lambda-n))}, \quad \lambda \in \mathfrak{a}_\mathbb{C}^*, n \in \mathbb{N}_0, \sigma \in \mathbb{Z}.$$

Consider an additional, not necessary distinct, K-type m and fix $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Then, by using repeatly the relation (3.15), we obtain for $n \equiv m \pmod 2$, the following quotient:

$$\frac{\mathbf{c}_{n,\sigma}(\lambda)}{\mathbf{c}_{m,\sigma}(\lambda)} = \frac{\Gamma(\frac{\lambda}{2}+\frac{m}{2}+1)\Gamma(\frac{\lambda}{2}-\frac{m}{2})}{\Gamma(\frac{\lambda}{2}+\frac{n}{2}+1)\Gamma(\frac{\lambda}{2}-\frac{n}{2})} = \begin{cases} 1, & \text{if } n=m \\ \frac{(\lambda+m)(\lambda+m-2)(\lambda+m-4)\cdots(\lambda+n+2)}{(\lambda-m)(\lambda-(m+2))(\lambda-(m+4))\cdots(\lambda-(n+2))}, & \text{if } n<m \\ \frac{(\lambda-n)(\lambda-(n+2))(\lambda-(n+4))\cdots(\lambda-(m+2))}{(\lambda+n)(\lambda+n-2)(\lambda+n-4)\cdots(\lambda+m+2)}, & \text{if } n>m. \end{cases} \tag{3.28}$$

Hence, we can directly see that the quotient has zeros in $\{-m, -m+2, \dots, -n-2\}$ and poles in $\{m, m+2, \dots, n+2\}$ for $n < m$ and inversely for $n > m$.

Proof of Thm. 3.28. Note that A is a $K \times K$ invariant and linear closed space, due the intertwining conditions. We proceed similar as in the proof of Thm. 3.17. Consider $\phi \in A$ such that for each $\sigma \in \hat{M}$, the assumption $(D.b)(i)$ of Thm. 3.13:

$$\phi_{\sigma'} = 0, \text{ for all } \sigma' \in \hat{M} \text{ with } |\sigma'| > |\sigma|$$

is satisfied. We want to prove that for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ with $\mathrm{Re}(\lambda) > 0$

$$\phi_\sigma^{(0)}(\lambda)\Big|_{R^{\sigma,\lambda}} = 0, \quad \text{for } |\lambda| > |\sigma|, |\lambda| - |\sigma| \text{ even}.$$

Analogous to the proof of Thm. 3.17, we need to check that:

(a) for $\mathrm{Re}(\lambda) > 0$, the intertwining opertator $J_{-,\sigma,\lambda}$ has a zero of order at most one.

(b) the kernel of $J_{-,\sigma,\lambda}$ is equal to 0 or $R^{\sigma,\lambda}$, for $\mathrm{Re}(\lambda) > 0$

The minimal K-type of the principal series representation $H_\infty^{\sigma,\lambda}$ is $n = |\sigma|$, hence its Harish-Chandra $\mathbf{c}$-function (3.6) $\mathbf{c}_{\sigma,\sigma}$ is regular for $\mathrm{Re}(\lambda) > 0$. Let $n \in \hat{K}$, then we see that for $n \geq |\sigma|$ and for $\mathrm{Re}(\lambda) > 0$:

$$\frac{\mathbf{c}_{n,\sigma}(\lambda)}{\mathbf{c}_{\sigma,\sigma}(\lambda)} = \frac{(\lambda - n)(\lambda - (n+2))(\lambda - (n+4)) \cdots (\lambda - (\sigma + 2))}{(\lambda + n)(\lambda + n - 2)(\lambda + n - 4) \cdots (\lambda + \sigma + 2)}$$

is also regular and has no poles, but zeros $\lambda \in \{n, n+2, \ldots, \sigma+2\}$ of first order. Hence due (3.9), $J_{-,\sigma,\lambda}$ has zeros of order one, this proves the first assertion (a) of the claim.

Concerning (b), we need to check for which K-type n, the function $\frac{\mathbf{c}_{n,\sigma}(\lambda)}{\mathbf{c}_{\sigma,\sigma}(\lambda)}$ has a zero for fixed $\mathrm{Re}(\lambda) > 0$. In fact, for fixed $\mathrm{Re}(\lambda) > 0$, $n \geq |\sigma|$ and $|\lambda| - |\sigma|$ even, the function has zeros if, and only if, n is a K-type of $\mathrm{Ker}(J_{-,\sigma,\lambda})$. By Thm. 3.27, we have $\mathrm{Ker}(J_{-,\sigma,\lambda}) = \mathrm{Im}(L_{\sigma,\lambda}) = R^{\sigma,\lambda}$, thus, this implies (b).

Now, by putting everyting together and using the intertwining condition (3.5), we have, for each $\mathrm{Re}(\lambda) > 0$ with $|\lambda| = |-\lambda| > |\sigma|$, that

$$\phi_\sigma(\lambda) \circ L_{\sigma,\lambda} = L_{\sigma,\lambda} \circ \underbrace{\phi_{-\sigma}(-\lambda)}_{=0} = 0.$$

Thus, by (b) and the assumption, we deduce that the operator $\phi_\sigma^{(0)}(\lambda)$ annihiliates $\mathrm{Ker}(J_{-,\sigma,\lambda}) = \mathrm{Im}(L_{\sigma,\lambda}) = R^{\sigma,\lambda}$, for $|\lambda| > |\sigma|$ and $|\lambda| - |\sigma|$ even. Moreover, by (a), this condition is sufficient since the order of m is one. This completes the proof. $\square$

Now let us move to (Level 2) and state the corresponding intertwining conditions there. More precisely, we will determine explicitly the Paley-Wiener(-Schwartz) theorem for $G = SL(2, \mathbb{C})$ in (Level 2).

Theorem 3.29 (Intertwining condition in (Level 2))**.** *Let $n \in \mathbb{N}_0$ be a K-type and $k, l \in \hat{M} \cong \mathbb{Z}$ so that $n \geq |k|, |l|$ and $k \equiv l \equiv n$ (mod 2). Consider the operator*

$$l^n : \mathrm{Hom}_M(E_n, E_{-l}) \longrightarrow \mathrm{Hom}_M(E_n, E_k)$$

defined as in Example 2.29. Then, $\psi \in \mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, H_\infty^{n|_M})$ satisfies the intertwining condition (2) of Def. 2.27 if, and only if,

(2.a) $J_{-,k,\lambda}\psi_k(\lambda) = (-1)^{n/2}\mathbf{c}_{n,k}(-\lambda)\psi_{-k}(-\lambda)$, for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$,

(2.b) $L_{k,l}\psi_{-l}(-k) = d_n\psi_k(l)$, for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and where d_n is a constant depending on n and $\mathrm{Hom}_M(E_n, E_{-l})$

are satisfied.

Proof of Thm. 3.29. By Prop. 3.8, we have that $(2.a)$ corresponds to (2.41), hence it corresponds to the intertwining condition $(D.2)$ in Thm. 2.28. Similar for $(2.b)$, which is, by Example 2.29 (2.41), a special intertwining condition of $(D.2)$ in Thm. 2.28. Hence, we have equivalence between the conditions $(2.a)$ & $(2.b)$ and $(D.2)$ in Thm. 2.28. $\square$

In order to move to the next level, let us first consider the case where the two K-type (n, E_n) and (m, E_m) are equal and then progress to the general case for distincts K-types. The idea and motivation behind this, is that in this way we can choose a (canonical) basis for $\mathrm{End}_M(E_m)$ on the vector space E_m, which is not the case for distinct K-types. The problem is that for $\mathrm{Hom}_M(E_n, E_m)$, for $n \neq m$, we can not take the projection on the M-isotypic components, there will be no natural basis.

Initial case: The K-type m and n are equal

Consider a (canonical) basis on the vector space $\mathrm{End}_M(E_m)$ given by the projection on the M-isotypic components of $\sigma \in \mathbb{Z}$.

Definition 3.30. *Let (m, E_m) be a fixed K-type and $k = -m, -(m-2), \ldots, m-2, m$. We define ${}_m\mathcal{A}_m$ the space of all elements in $\mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}_M(E_m))$, which are given by holomorphic functions $\varphi_k : \mathfrak{a}_{\mathbb{C}}^* \to \mathbb{C}$, ordered to a $(m+1) \times (m+1)$ matrix*

$$\varphi := \begin{pmatrix} \varphi_m(\lambda) & 0 & \cdots & 0 \\ 0 & \varphi_{m-2}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_{-m}(\lambda) \end{pmatrix} \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}_M(E_m))$$

such that

$$\begin{aligned} \varphi_k(\lambda) &= \varphi_{-k}(-\lambda), \quad \text{for } \lambda \in \mathfrak{a}_{\mathbb{C}}^* \\ \varphi_k(l) &= \varphi_l(k), \quad \text{for } k \equiv l \equiv m \ (\text{mod } 2) \ \& \ |k|, |l| \leq m. \end{aligned} \tag{3.29}$$

Note that ${}_m\mathcal{A}_m \subset \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}_M(E_m))$ is even an algebra.

Theorem 3.31. *With the previous notations, we have that*

$${}_m\mathcal{A}_m \cong {}_mPWS_{m,H}(\mathfrak{a}_{\mathbb{C}}^*).$$

Proof. We need to check that the intertwining conditions (3.29) of ${}_m\mathcal{A}_m$ correspond to the intertwining condition (3.*iii*) of Def. 2.30 of ${}_mPWS_{m,H}(\mathfrak{a}_{\mathbb{C}}^*)$. More precisely, by Thm. 2.28 and Example 2.29, it suffices to show that the intertwining condition (2.42) corresponds to (3.29).

Consider the Knapp-Stein intertwining operator $J_{-,\sigma,\lambda}$ as in Thm. 3.27. We know from Lem. 3.10 with $m = \gamma = \tau \in \mathbb{N}_0$, that for each intertwining datum $((\sigma, \lambda, 0), (-\sigma, -\lambda, 0); W)$ and $t \in \mathrm{Hom}_M(E_m, E_\sigma)$, we have

$$(-1)\varphi(\lambda) = \varphi(-\lambda), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \varphi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}_M(E_m)),$$

where the $\mathbf{c}$-functions, on both side, are cancelled, since we have commutativity. Note that the complex hull of $\mathfrak{a}$ is the sum of $\mathfrak{a}$ and $\mathfrak{m}$, where $\mathfrak{m}$ is in the maximal torus M. This means that the Weyl group W_A acts on $\mathfrak{a}$ as well as on $\mathfrak{m}$ by -1. Thus, also on $i\mathfrak{m}^*$ by -1. Let $-k, -(k-2), \ldots, k-2, k$ be the weights of the representation $m \in \mathbb{N}_0$. Then, by the Weyl invariance, every k-component is sent to $-k$-component. The matrix $\mathrm{diag}(\varphi_m(\lambda), \ldots, \varphi_{-m}(\lambda))$ is reversed, i.e., we get $\mathrm{diag}(\varphi_{-m}(-\lambda), \ldots, \varphi_m(-\lambda))$. Hence, $\varphi_{-k}(-\lambda) = \varphi_k(\lambda)$, for all $|k| \leq m$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Now, for convenience, consider the Želobenko intertwining operator $L'_{\sigma,\lambda}$, which is induced by $L_{\sigma,\lambda}$ by Thm. 3.27. Let $|l|, |k| \leq m$ and $k \equiv l \equiv m$ (mod 2). We know that for every intertwining datum $((k,l,0),(l,k,0);W)$ and $t \in \mathrm{Hom}_M(E_m,E_k)$, we have (2.42) for $m=\gamma=\tau\in\mathbb{N}_0$:

$$l^m(t\circ\varphi(l)) = l^m(t)\circ\varphi(k).$$

However, there is only one $t\in\mathrm{Hom}_M(E_m,E_k)$, this means that $\varphi_k(l)=t\circ\varphi(l)$. Since the operator $l^m(t)\in\mathrm{Hom}_M(E_m,E_l)$ is a multiple of the dual, we have that $l^\tau(t)=c\cdot l$, where $c\in\mathbb{C}$. Besides, $c\neq 0$. In fact, by Thm. 3.27, the intertwining operator $L'_{k,l}$ on the K-type is not zero, this means that $L'_{k,l}$ has no kernel, hence l^m too. Consequently, we have

$$\begin{aligned} l^m(t\circ\varphi(l)) = l^m(t)\circ\varphi(k) \iff l^m(\varphi_k(l)) = c\cdot\varphi_l(k) &\iff c\cdot\varphi_k(l)=c\cdot\varphi_l(k)\\ &\iff \varphi_k(l)=\varphi_l(k). \end{aligned}$$

This completes the proof. □

We also consider the corresponding situation in polynomial functions:

$${}_m\mathrm{Pol}_m := \{\varphi\in\mathrm{Pol}(\mathfrak{a}^*_\mathbb{C},\mathrm{End}_M(E_m)) \mid \varphi_k \text{ satisfies (3.29)},\ \forall|k|\leq m\}.$$

It follows directly that ${}_m\mathrm{Pol}_m$ is isomorphic to the vector space ${}_mPWS_{m,0}(\mathfrak{a}^*_\mathbb{C})$.

Theorem 3.32. *With the notations above, let $k=-m,\dots,m$ and $\lambda\in\mathfrak{a}^*_\mathbb{C}$. Then, the algebra ${}_m\mathcal{A}_m$ is a free* Hol$(\mathbb{C})$*-module with $m+1$ generators $(k\lambda)^l$, for $l=0,\dots,m$. Furthermore, we have*

$${}_m\mathcal{A}_m \cong \mathrm{Hol}(\mathbb{C})\otimes_{\mathrm{Pol}(\mathbb{C})} {}_m\mathrm{Pol}_m. \tag{3.30}$$

Analogously for the polynomial functions, ${}_m\mathrm{Pol}_m$ is a free Pol$(\mathbb{C})$*-module with same generators as ${}_m\mathcal{A}_m$.*

Here, Hol$(\mathbb{C})$ acts on ${}_m\mathcal{A}_m$ by

$$(h\cdot\varphi)_k(\lambda) = h(\lambda^2+k^2)\varphi_k(\lambda),\quad h\in\mathrm{Hol}(\mathbb{C}),\varphi\in{}_m\mathcal{A}_m.$$

Similar for Pol$(\mathbb{C})$ acting on ${}_m\mathrm{Pol}_m$. In addition, $\psi=(k\lambda)^l$ means $\psi_k(\lambda):=(k\lambda)^l$. We even have that ${}_m\mathrm{Pol}_m$ is a free finitely generated commutative $\mathbb{C}$-algebra with generators λ^2+k^2-4 and $k\lambda$. Consequently, we can show that each generator of ${}_m\mathrm{Pol}_m$ form a Hol$(\mathbb{C})$-basis for ${}_m\mathcal{A}_m$. More precisely, the generators $(k\lambda)^l$, for $l=0,1,\dots,m$, of ${}_m\mathcal{A}_m$ under Hol$(\mathbb{C})$ generate all and are free.

Proof. Consider $\varphi\in{}_m\mathcal{A}_m$. It is sufficient to show the existence and the uniqueness of holomorphic functions $h_0,\dots,h_m\in\mathrm{Hol}(\mathbb{C})$ so that

$$\varphi_k(\lambda) := \sum_{l=0}^m h_l(\lambda^2+k^2)\cdot(k\lambda)^l,\qquad \text{for } k=-m,\dots,m. \tag{3.31}$$

Then, ${}_m\mathcal{A}_m$ is a free Hol$(\mathbb{C})$-module with generators $(k\lambda)^l, l=0,\dots,m$. We have that Hol $\otimes_{\mathrm{Pol}}$ Pol $\cong$ Hol. Since, we have $m+1$ free generators

$${}_m\mathcal{A}_m \cong \mathrm{Hol}(\mathbb{C})^{m+1} \text{ and } {}_m\mathrm{Pol}_m \cong \mathrm{Pol}(\mathbb{C})^{m+1}$$

we thus have that (3.31) is isomorphic to (3.30).

We proceed by induction on two steps on $m \in \mathbb{N}_0$, for the existence of $h_l \in \mathrm{Hol}(\mathbb{C})$, $l = 0, \ldots, m$.
Starting with the inital step $m = 0$, we see immediately that there is exactly one even holomorphic function $\varphi_0(\lambda) = h_0(\lambda^2)$ in

$$_0\mathcal{A}_0 = \{\varphi_0 \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*) \mid \varphi_0(\lambda) = \varphi_0(-\lambda), \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*\}.$$

Thus, φ_0 is an even polynomial function in ${}_0\mathrm{Pol}_0$, if and only, if $h_0(\lambda^2)$ is one.
For $m = 1$, we have that ${}_1\mathcal{A}_1 = \{\varphi_1, \varphi_{-1} \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*) \mid \varphi_{-1}(\lambda) = \varphi_1(-\lambda), \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*\}$ and $\varphi_1(\lambda)$ can be decomposed into an even and odd part as follows:

$$\varphi_1(\lambda) = \varphi_1^{\text{even}}(\lambda) + \varphi_1^{\text{odd}}(\lambda) = h_0(\lambda^2 + 1) + \lambda h_1(\lambda^2 + 1).$$

Similar for $\varphi_{-1}(\lambda) = \varphi_1(-\lambda) = h_0(\lambda^2+1) - \lambda h_1(\lambda^2+1)$. Hence, this leads us to desired relation $\varphi_k(\lambda) = h_0(\lambda^2 + k^2) + (k\lambda)h_1(\lambda^2 + k^2)$, for all $|k| \le 1$. Since $h_0(\lambda^2 + k^2)$ and $h_1(\lambda^2 + k^2)$ are polynomials, $\varphi_k(\lambda)$ is a polynomial as well.
Assume now that the relation (3.31) holds true for $m - 2$. Let $\varphi \in {}_m\mathcal{A}_m$. By induction hypothesis, we have $\overline{\varphi} := \mathrm{diag}(\varphi_{m-2}(\lambda), \varphi_{m-4}(\lambda), \ldots, \varphi_{-(m-4)}(\lambda), \varphi_{-(m-2)}(\lambda)) \in {}_{m-2}\mathcal{A}_{m-2}$ so that there exsits $h_l \in \mathrm{Hol}(\mathbb{C})$ with

$$\varphi_k(\lambda) = \sum_{l=0}^{m-2} h_l(\lambda^2 + k^2)(k\lambda)^l \quad \text{for } |k| \le m - 2.$$

Consider $\tilde{\varphi} := \mathrm{diag}(\tilde{\varphi}_m(\lambda), \varphi_{m-2}(\lambda), \ldots, \varphi_{-(m-2)}(\lambda), \tilde{\varphi}_{-m}(\lambda)) \in {}_m\mathcal{A}_m$ with

$$\tilde{\varphi}_{\pm m}(\lambda) := \sum_{l=0}^{m-2} h_l(\lambda^2 + m^2)(\pm m\lambda)^l.$$

By taking the difference of φ and $\tilde{\varphi}$, we get that $\varphi - \tilde{\varphi} = \mathrm{diag}(\varphi_m^+(\lambda), 0, \ldots, 0, \varphi_{-m}^+(\lambda)) \in {}_m\mathcal{A}_m$, where we have set $\varphi_{\pm m}^+(\lambda) := \varphi_{\pm m}(\lambda) - \tilde{\varphi}_{\pm m}(\lambda)$. Notice that $\varphi_{\pm m}^+(l) = 0$ for $|l| \le m - 2$. Let us consider a polynomial function, which has zeros, if $|k|, |\lambda| \le m - 2$:

$$p_m(\lambda, k) = \prod_{\substack{|l| \le m-2 \\ l \equiv m \pmod 2}} (k - l)(\lambda - l) \in {}_m\mathrm{Pol}_m.$$

Moreover, if $k = m$, then $p_m(\lambda, m) = c_m \prod_{\substack{|l| \le m-2 \\ l \equiv m \pmod 2}} (\lambda - l)$, where c_m is a non-zero constant depending on the integer m. In addition, $(k - l)(\lambda - l) = (k\lambda) - l(k + \lambda) + l^2$ and thus $p_m(\lambda, k)$ is of the form with even powers of $(k\lambda)$ and maximum power $m - 1$:

$$p_m(\lambda, k) = (k\lambda)^{m-1} + \sum_{\substack{l=0 \\ l \equiv m-1 \pmod 2}}^{m-3} p_l^m(\lambda^2 + k^2)(k\lambda)^l, \tag{3.32}$$

where $p_l^m(\lambda^2 + k^2)$ are some polynomials. We already remarked that for $|k| \le m - 2$, $p_m(\lambda, k) = 0$, hence we can write φ_m^+ as

$$\varphi_m^+(\lambda) = [h_0^+(\lambda^2 + m^2) + h_1^+(\lambda^2 + m^2)(m\lambda)]p_m(\lambda, m).$$

Thus, for all $|k| \leq m$, we define

$$\begin{aligned} \varphi_k^+(\lambda) &:= [h_0^+(\lambda^2+k^2) + h_1^+(\lambda^2+k^2)(k\lambda)]p_m(\lambda,k) \\ &\overset{(3.32)}{=} h_0^+(\lambda^2+k^2)(k\lambda)^{m-1} + h_0^+(\lambda^2+k^2) \sum_{\substack{l=0 \\ l\equiv m-1 \pmod 2}}^{m-3} p_l^m(\lambda^2+k^2)(k\lambda)^l \\ &\quad + h_1^+(\lambda^2+k^2)(k\lambda)^m + h_1^+(\lambda^2+k^2) \sum_{\substack{l=0 \\ l\equiv m-1 \pmod 2}}^{m-3} p_l^m(\lambda^2+k^2)(k\lambda)^{l+1}. \end{aligned}$$

Then, $\varphi_k^+(\lambda) = 0$, for $|k| \leq m-2$. This means that, for all $|k| \leq m$, we have found $h_0, \ldots, h_m \in \mathrm{Hol}(\mathbb{C})$ so that $\varphi_k = \varphi_k^+ + \tilde{\varphi}_k$ is of the form (3.31).

Concerning the uniqueness, we need to show that

$$\varphi_k(\lambda) = 0 \text{ implies } h_l(\lambda^2+k^2) = 0 \quad \forall |k| \leq m, l = 0, \ldots, m, \lambda \in \mathfrak{a}_{\mathbb{C}}^*. \tag{3.33}$$

We proceed again by induction on two steps on $m \in \mathbb{N}_0$. For the initial case $m = 0$, since we have only one holomorphic and even function $\varphi_0(\lambda) = h_0(\lambda^2)$, this directly implies that $h_0(\lambda^2) = 0$.
Assume that (3.33) holds true for m and let us prove that it holds for $m-2$. By using the polynomial function (3.32), consider, for $|k| \leq m-2$:

$$\begin{aligned} \varphi_k(\lambda) = \sum_{\substack{l=0 \\ l\equiv m \pmod 2}}^{m} h_l(\lambda^2+k^2)(k\lambda)^l \quad &- h_{m-1}(\lambda^2+k^2) \sum_{\substack{l=0 \\ l\equiv m-1 \pmod 2}}^{m-3} p_l^m(\lambda^2+k^2)(k\lambda)^l \\ &- h_m(\lambda^2+k^2) \sum_{\substack{l=0 \\ l\equiv m-1 \pmod 2}}^{m-3} p_l^m(\lambda^2+k^2)(k\lambda)^{l+1} \end{aligned}$$

in ${}_{m-2}\mathcal{A}_{m-2}$. By induction hypothesis, we have that $\varphi_k(\lambda) = 0$ implies that for $|l|, |k| \leq m-2$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$:

$$h_l(\lambda^2+k^2) = \begin{cases} h_{m-1}(\lambda^2+k^2)p_{l-1}^m(\lambda^2+k^2) \\ h_m(\lambda^2+k^2)p_{l-1}^m(\lambda^2+k^2). \end{cases}$$

Moreover, for $k = m$, we have

$$\varphi_{\pm m}(\lambda) = [h_{m-1}(\lambda^2+m^2) + h_m(\lambda^2+m^2)(\pm m\lambda)]p_m(\lambda, \pm m) = 0$$

since $p_m(\lambda, \pm m)$ is not identical zero on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. This means that h_{m-1} and h_m are zero and thus this implies that $h_l = 0$ for $|l| \leq m-2$. This completes the proof. $\square$

General case: The K-type n and m are distinct

Consider now two distinct K-types (n, E_n) and (m, E_m). In a similar way as in Def. 3.30, we define

$${}_n\mathcal{A}_m \cong {}_nPWS_{m,H}(\mathfrak{a}_{\mathbb{C}}^*) \subset \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{Hom}_M(E_n, E_m))$$

as well as the situation for polynomial functions ${}_n\mathrm{Pol}_m \cong {}_nPWS_{m,0}(\mathfrak{a}^*_{\mathbb{C}})$. Note that after a choice of a basis, we can write the element in this way:

$$\varphi := \begin{cases} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \\ \varphi_n(\lambda) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varphi_{-n}(\lambda) \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{(m+1)\times(n+1)} & \text{if } n < m, \\ \begin{pmatrix} 0 & \cdots & 0 & \varphi_m(\lambda) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \varphi_{-m}(\lambda) & 0 & \cdots & 0 \end{pmatrix}_{(m+1)\times(n+1)} & \text{if } n > m. \end{cases}$$

For instance, consider (m, E_m) and $(m+2, E_{m+2})$.

Theorem 3.33. *Let m be a non-zero K-type. There exists an unique, up to normalization, operator of first order q_m^+ in ${}_m\mathrm{Pol}_{m+2}$ (resp. q_m^- in ${}_{m+2}\mathrm{Pol}_m$ by taking the adjoint), which correspond to*

$$(\lambda + m + 2) \qquad \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \\ 0 & \cdots & 0 \end{pmatrix}_{(m+2)\times m}$$

$$\left(\textit{resp. } (\lambda - (m+2)) \qquad \begin{pmatrix} 0 & d(m,m) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & d(m,-m) & 0 \end{pmatrix}_{m\times(m+2)} \right)$$

under some appropriate basis choice and where $d(m,k) = (m+2)^2 - k^2$, for all $|k| \leq m$. Moreover, the algebra ${}_m\mathrm{Pol}_{m+2}$ (resp. ${}_{m+2}\mathrm{Pol}_m$) is a free ${}_m\mathrm{Pol}_m$-module with generator q_m^+ (resp. q_m^-). Similarly, we have that the algebra ${}_m\mathcal{A}_{m+2}$ (resp. ${}_{m+2}\mathcal{A}_m$) is a free ${}_m\mathcal{A}_m$-module with same generator q_m^+ (resp. q_m^-).

Note that ${}_m\mathrm{Pol}_{m+2}$ is isomorphic to $\mathcal{D}_G(\mathbb{E}_m, \mathbb{E}_{m+2})$, the set of all invariant differential operators.

Proof. Let $\mathfrak{p} \cong \mathbb{R}^3$ be a subset of $\mathfrak{g}$ and $S(\mathfrak{p})$ the symmetric algebra defined as in Sect. 1.2. From (1.3) we deduce $Gr({}_m\mathrm{Pol}_{m+2}) \cong [S(\mathfrak{p})\otimes\mathrm{Hom}_M(E_m, E_{m+2})]^K$. Moreover, by Helgason's result ([Hel20], III.1), $S(\mathfrak{p})$ is a free $S(\mathfrak{p})^K$-module with generator $H(\mathfrak{p})$:

$$S(\mathfrak{p}) = S(\mathfrak{p})^K \otimes H(\mathfrak{p}), \tag{3.34}$$

where $H(\mathfrak{p}) \overset{K}{\cong} \bigoplus_{l=0}^{\infty} E_{2l}$ is the set of all K-harmonic polynomials in $S(\mathfrak{p})$ and each E_{2l} has degree $l \in \mathbb{N}_0$. Hence, by using Clebsch-Gordan rule (3.27), we have

$$\begin{aligned}[S^{\leq 1}(\mathfrak{p}) \otimes \mathrm{Hom}(E_m, E_{m+2})]^K &\overset{(3.34)}{\cong} [(S^0(\mathfrak{p}) \oplus S^1(\mathfrak{p})) \otimes (E_2 \oplus E_4 \oplus \cdots \oplus E_{2m+2})]^K \\ &\cong [S^1(\mathfrak{p}) \otimes E_2]^K,\end{aligned}$$

where $[S^1(\mathfrak{p}) \otimes E_2]^K$ is one-dimensional. This means that ${}_m\mathrm{Pol}_{m+2}$ contains exactly one element q_m^+ of filter degree 1. By choosing an appriopate basis for $\mathrm{Hom}_M(E_m, E_{m+2})$, we can embeed $E_m \overset{M}{\hookrightarrow} E_{m+2}$ and thus

$$\mathrm{Hom}_M(E_m, E_m) \hookrightarrow \mathrm{Hom}_M(E_m, E_{m+2}).$$

Since $q_{m,k}^+(\lambda)$ is of first order (but the individual components could be constants), this means that all elements, under normalization, are of the form

$$q_{m,k}^+(\lambda) = \begin{cases} c(m,k), & |k| \leq m \\ (\lambda + c(m,k)), & |k| \leq m, \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \end{cases}$$

and there are independent. Here, $c(m,k)$ is a non-zero constant depending on k and m. Consider $\varphi_k(\lambda) \in {}_m\mathcal{A}_{m+2}$, for $|k| \leq m$. By Lem. 3.10 and Harish-Chandra $\mathbf{c}$-functions (3.28), we have that

$$\begin{aligned}\varphi_{-k}(-\lambda) = (-1)^{(m+2-m)/2}\frac{\mathbf{c}_{m+2,\sigma}(\lambda)}{\mathbf{c}_{m,\sigma}(\lambda)}\varphi_k(\lambda) &= (-1)\frac{\Gamma(\frac{\lambda}{2}+\frac{m}{2}+1)\Gamma(\frac{\lambda}{2}-\frac{m}{2})}{\Gamma(\frac{\lambda}{2}+\frac{m}{2}+2)\Gamma(\frac{\lambda}{2}-\frac{m}{2}-1)}\varphi_k(\lambda) \\ &= (-1)\frac{(\lambda-(m+2))}{(\lambda+(m+2))}\varphi_k(\lambda) \\ &= \frac{(-\lambda+(m+2))}{(\lambda+(m+2))}\varphi_k(\lambda).\end{aligned}$$

Moreover, let $\tilde{\varphi}_k(\lambda) \in {}_m\mathcal{A}_m$, for $|k| \leq m$, we know that

$$\varphi_k(\lambda) = q_{m,k}^+(\lambda)\tilde{\varphi}_k(\lambda) \tag{3.35}$$

and that $q_{m,k}^+(\lambda)$ is not zero. Thus

$$\varphi_{-k}(-\lambda) = q_{m,-k}^+(-\lambda)\tilde{\varphi}_{-k}(-\lambda) \overset{(3.29)}{=} q_{m,-k}^+(-\lambda)\tilde{\varphi}_k(\lambda) \overset{(3.35)}{=} \frac{q_{m,-k}^+(-\lambda)}{q_{m,k}^+(\lambda)}\varphi_k(\lambda).$$

Hence, $\frac{q_{m,-k}^+(-\lambda)}{q_{m,k}^+(\lambda)} = \frac{-\lambda+(m+2)}{\lambda+(m+2)}$ and it has a zero $m+2$. This means that $q_{m,k}^+(\lambda)$ is not a constant, thus we conclude that $c(m,k) = m+2$, for all $|k| \leq m$.

Concerning q_m^-, one can check with the same arguments as for q_m^+ that

$$q_m^-(\lambda,k) = (\lambda - c(m,k))d(m,k), \forall |k| \geq m, \lambda \in \mathfrak{a}_{\mathbb{C}}^*$$

with $c(m,k) = m+2$ and $d(m,k) = C((m+2)^2 - k^2)$, where C is a constant. In fact, observe that

$$q_m^-(\lambda,k)q_m^+(\lambda,k) = d(m,k)(\lambda^2 - c(m,k)^2) = d(m,k)(\lambda^2 - (m+2)^2) \in {}_m\mathrm{Pol}_m.$$

By taking $\lambda = 0$ and considering $k = 0$, we get

$$d(m,k)(-(m+2)^2) = d(m,0)(k^2-(m+2)^2) \iff d(m,k) = \frac{d(m,0)}{(m+2)^2}((m+2)^2 - k^2).$$

By setting $C := \frac{d(m,0)}{(m+2)^2}$, we conclude that $d(m,k) = C((m+2)^2 - k^2)$, for all $|k| \leq m$. $\square$

Observe that

$$\begin{aligned} q_m^+ q_m^- &\in {}_m\mathrm{Pol}_{m+2}\,{}_{m+2}\mathrm{Pol}_m = {}_{m+2}\mathrm{Pol}_{m+2} \\ q_m^- q_m^+ &\in {}_{m+2}\mathrm{Pol}_m\,{}_m\mathrm{Pol}_{m+2} = {}_m\mathrm{Pol}_m. \end{aligned}$$

Consequently, by combining the previous observations and results (Thm. 3.32 and Thm. 3.33) together, this leads us to the following generalization for distinct integers n and m.

Definition 3.34. *Let $n, m \in \mathbb{N}$. Consider q_m^+ and q_n^- as in Thm. 3.33. We define the polynomial $q_{n,m} \in {}_n\mathrm{Pol}_m$ in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by*

$$q_{n,m} = \begin{cases} q_{m-2}^+ \cdot q_{m-4}^+ \cdots q_{n+2}^+ \cdot q_n^+, & \text{if } n < m \\ q_m^- \cdot q_{m+2}^- \cdots q_{n-4}^- \cdot q_{n-2}^-, & \text{if } n > m \\ \mathrm{Id}, & \text{if } n = m. \end{cases} \tag{3.36}$$

The following theorem, gives explicitly the Paley-Wiener(-Schwartz) theorem in (Level 3) for $G = SL(2,\mathbb{C})$.

Theorem 3.35 (Intertwining condition in (Level 3))**.** *Let $n, m \in \mathbb{N}_0$ be two K-types, which are not necessary distinct, and let $l := \min(n, m)$. Then, the algebra ${}_n\mathrm{Pol}_m$ (resp. ${}_n\mathcal{A}_m$) is a free ${}_l\mathrm{Pol}_l$ (resp. ${}_l\mathcal{A}_l$)-module with generator $q_{n,m}$. This means that there exists an unique function $h \in {}_l\mathcal{A}_l$ such that*

$${}_n\mathcal{A}_m \ni \varphi(\lambda) = \begin{cases} h(\lambda) q_{n,m}(\lambda), & \text{if } m < n, \\ q_{n,m}(\lambda) h(\lambda), & \text{if } m > n, \end{cases} \quad \text{for } \lambda \in \mathfrak{a}_{\mathbb{C}}^*. \tag{3.37}$$

Moreover, if $L = \max(n, m)$, then ${}_n\mathrm{Pol}_m$ (resp. ${}_n\mathcal{A}_m$) is a ${}_L\mathrm{Pol}_L$ (resp. ${}_L\mathcal{A}_L$)-module with generator $q_{n,m}$.

Proof. Consider the case $m < n$, it suffices to prove that

(a) there exists a unique $h \in {}_m\mathcal{A}_m$ such that $\varphi = h \cdot q_{n,m} \in {}_n\mathcal{A}_m$,

(b) there exists a $\tilde{h} \in {}_n\mathcal{A}_n$ such that $\varphi = q_{n,m} \cdot \tilde{h} \in {}_n\mathcal{A}_m$.

Consider

$$q_{m,n}\,{}_n\mathcal{A}_m \subset \{g \in {}_n\mathcal{A}_n \mid g_k = 0,\ \forall |k| > m\} \subset {}_n\mathcal{A}_n.$$

We also have zeros between the lines $-m$ and m, this means $g_l(k) = 0, \forall |k| > m$. Let $\varphi \in {}_n\mathcal{A}_m$. We identify $q_{m,n}$ and $q_{m,n}$ with corresponding scalar polynomials. The polyniomal $q_{m,n}$ has zeros on the left side and $q_{n,m}$ has zeros on the other side:

$$g = q_{m,n} \cdot \varphi \iff \frac{g}{q_{m,n}} = \frac{q_{m,n}\varphi}{q_{m,n}} \iff \frac{g}{q_{m,n}q_{n,m}} = \frac{q_{m,n}\varphi}{q_{m,n}q_{n,m}} \iff \frac{g}{q_{m,n}q_{n,m}} = \frac{\varphi}{q_{n,m}}.$$

Hence, there exists a unique $h := \frac{g}{q_{m,n}q_{n,m}} \in {}_m\mathcal{A}_m$ so that

$$\varphi = h \cdot q_{n,m}.$$

In fact, since g satisfies the intertwining condition $g_k(l) = g_l(k)$ as well as the symmetry one $g_k(\lambda) = g_{-k}(-\lambda)$, and

$$\begin{aligned} q_{m,n}(k,\lambda) q_{n,m}(k,\lambda) &= q_{m,n}(-k,-\lambda) q_{n,m}(-k,-\lambda) \\ q_{m,n}(k,l) q_{n,m}(k,l) &= q_{m,n}(l,k) q_{n,m}(l,k) \end{aligned}$$

we then have that $h_l(k) = h_l(k)$ and $h_k(\lambda) = h_{-k}(-\lambda)$ for $k \equiv l \equiv m \pmod 2$, $|k| \leq m$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. This proves (a).

For (b), we know from (a) that $h \in {}_m\mathcal{A}_m$ is in the small space, thus we need to find

$$\tilde{h} = \mathrm{diag}(\tilde{h}_n, \ldots, \tilde{h}_{m+2}, h_m, \ldots, h_{-m}, \tilde{h}_{-(m-2)}, \ldots, \tilde{h}_{-n}) \in {}_n\mathcal{A}_n$$

so that $\tilde{h}_k(l) = \tilde{h}_l(k)$ and $\tilde{h}_k(\lambda) = \tilde{h}_{-k}(-\lambda)$ for $|l|, |k| > m$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. By using the interpolation polynomial for each $|k| > m$, we define recursively for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$:

$$\begin{aligned}
h_{m+2}(\lambda) &= \sum_{\substack{i=-m \\ i \equiv m \ (\mathrm{mod}\ 2)}}^{m} h_i(m) \prod_{\substack{l=-m \\ l \neq i}}^{m} \left(\frac{\lambda - l}{i - l}\right) \\
h_{m+4}(\lambda) &= \sum_{\substack{i=-(m+2) \\ i \equiv m+2 \ (\mathrm{mod}\ 2)}}^{m+2} h_i(m+2) \prod_{\substack{l=-(m+2) \\ l \neq i}}^{m+2} \left(\frac{\lambda - l}{i - l}\right) \\
&\ \vdots \\
h_{n-2}(\lambda) &= \sum_{\substack{i=-(n-4) \\ i \equiv n-4 \ (\mathrm{mod}\ 2)}}^{n-4} h_i(n-4) \prod_{\substack{l=-(n-4) \\ l \neq i}}^{n-4} \left(\frac{\lambda - l}{i - l}\right) \\
h_n(\lambda) &= \sum_{\substack{i=-(n-2) \\ i \equiv n-2 \ (\mathrm{mod}\ 2)}}^{n-2} h_i(n-2) \prod_{\substack{l=-(n-2) \\ l \neq i}}^{n-2} \left(\frac{\lambda - l}{i - l}\right).
\end{aligned}$$

The intertwining relations are also satisfies for $|l|, |k| > m$ due $h \in {}_m\mathcal{A}_m$. Concerning the case $n < m$, the proof is analogous. □

Note that Thm. 3.32, in particular relation (3.30), is also true in general for distinct K-type n and m.

Corollary 3.36. *With the notations above,* ${}_n\mathcal{A}_m \cong \mathrm{Hol}(\mathbb{C}) \otimes_{\mathrm{Pol}(\mathbb{C})} {}_n\mathrm{Pol}_m$.

Proof. We can use the same arguments as in the beginning of the proof of Thm. 3.32, except that, here, we have $m+1$ generators (λl) times $q_{n,m}$, i.e., ${}_n\mathcal{A}_m \cong \mathrm{Hol}(\mathbb{C})^{m+1} \cdot q_{n,m}$. Hence, ${}_n\mathcal{A}_m$ is a free $\mathrm{Hol}(\mathbb{C})$-module with

$$\begin{aligned}
{}_n\mathcal{A}_m = q_{n,m} \cdot {}_n\mathcal{A}_n = {}_m\mathcal{A}_m \cdot q_{n,m} &\overset{(3.30)}{\cong} \mathrm{Hol}(\mathbb{C}) \otimes_{\mathrm{Pol}(\mathbb{C})} {}_m\mathrm{Pol}_m \cdot q_{n,m} \\
&\cong \mathrm{Hol}(\mathbb{C}) \otimes_{\mathrm{Pol}(\mathbb{C})} {}_n\mathrm{Pol}_m. \quad \square
\end{aligned}$$

Observe that, if $n \not\equiv m \pmod 2$, then ${}_n\mathrm{Pol}_m = 0$.

By multiplying two polynomials (3.36) together, we obtain the similar relation as in Lem. 3.21.

Lemma 3.37. *Let $n, m, l \in \mathbb{N}_0$ be integers, no necessary distinct. Then, we obtain the relation*

$$q_{n,m} \cdot q_{l,n} = \begin{cases} r^l_{n,m} \cdot q_{l,m}, & \text{if } m \leq l \\ q_{l,m} \cdot r^l_{n,m}, & \text{if } l < m, \end{cases} \tag{3.38}$$

where $r^l_{n,m}$ *is a polynomial of the form:*

$$r^l_{n,m} = \begin{cases} \mathrm{Id}, & \text{if } l \leq n \leq m \text{ or } m \leq n \leq l \\ q_{n,m} \cdot q_{m,n} \in {}_m\mathrm{Pol}_m, & \text{if } l \leq m < n \text{ or } n < m \leq l \\ q_{n,l} \cdot q_{l,n} \in {}_l\mathrm{Pol}_l, & \text{if } m < l < n \text{ or } n < l < m. \end{cases}$$

Proof. The proof is similar as for Lem. 3.21 except that, here, we use (3.36). □

Chapter 4

On solvability and general strategy

We now converge to the end of our preparations for the proof of Conjecture 1. In this chapter, we merge all the obtained dominant results, namely the topological Paley-Wiener(-Schwartz) theorems (Thm. 2.31 & Thm. 2.40) with Delorme's intertwining conditions (Thm. 2.28) for sections in (Level 2) and (Level 3), as well as their adequateness for Lie group of real rank one (Thm. 3.13). Furthermore, the consequence of the Paley-Wiener-Schwartz theorem to the invariant differential operators in the Fourier image (Thm. 2.46) takes also a major place in our strategy.
In addition, we explain and expose a possible strategy on how to attack the conjecture. A key role is to play with the two levels (Level 2) and (Level 3). More precisely, the plan is to solve the conjecture in (Level 3) and then go up to the desired (Level 2). The idea of this 'level playing' came up by making use of Hörmander's estimates results ([Hör73], Thm. 7.6.11. & Cor. 7.6.12.) in (Level 3).

The relations of these Hörmander's estimates are covered in the first Section 4.1 of this chapter. Here, we first present the relevant results in (Level 3) (Hyp. 1, Hyp. 2 & Hyp. 3) before moving to (Level 2) (Hyp. 4 & Thm. 4.1). However, a general proof for these results is not provided, since Delorme's intertwining conditions ($3.iii$) in Def. 2.30 are very difficult to control and maniplutate, even for general Lie group of real rank one. Although, in the next Chapter 5, we picked three examples, where a complete (respectively partial) proof of these results is furnished.

In the last Section 4.2, we complete the proof of the conjecture (Thm. 4.10 & Cor. 4.11) by employing abstract function analytic criteria for closedness and density of images of operators (Prop. 4.7 & Thm. 4.9).

4.1 Estimates theorems for systems of polynomials equations

Let us remind that the initial goal is to show the exactness in the middle of the sequence (1.12). By taking the toplogical dual of (1.12), we are behaving with compact supported distributional sections $C_c^{-\infty}(X, \mathbb{E}_{\tilde{*}})$, for $* \in \{\delta, \tau, \gamma\}$, with strong topology. The strategy consits to apply the Fourier transform and the topological Paley-Wiener-Schwartz Thm. 2.40 for sections over homogeneous vector bundles to prove the exactness of the dual sequence in the middle, as well as, the closed range of our transposed invariant differential operators $D^t \in \mathcal{D}_G(\mathbb{E}_{\tilde{\tau}}, \mathbb{E}_{\tilde{\gamma}})$. In other words,

$$\begin{tikzcd}
C^\infty(X,\mathbb{E}_\gamma) \arrow[r, "D"] \arrow[d, dashed] & \underset{\text{exact}}{C^\infty(X,\mathbb{E}_\tau)} \arrow[r, "\widetilde{D}"] \arrow[d, dashed] & C^\infty(X,\mathbb{E}_\delta) \arrow[d, dashed] \\
C_c^{-\infty}(X,\mathbb{E}_{\tilde{\delta}}) \arrow[r, "\widetilde{D}^t"] \arrow[d, "\mathcal{F}_{\tilde{\delta}}"] & \underset{\text{exact}}{C_c^{-\infty}(X,\mathbb{E}_{\tilde{\tau}})} \arrow[r, "D^t"] \arrow[d, "\mathcal{F}_{\tilde{\tau}}"] & \underset{\text{closed range}}{C_c^{-\infty}(X,\mathbb{E}_{\tilde{\gamma}})} \arrow[d, "\mathcal{F}_{\tilde{\gamma}}"] \\
PWS_{\tilde{\delta}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M) \arrow[r, "Q"] & \underset{\text{exact}}{PWS_{\tilde{\tau}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)} \arrow[r, "P"] & \underset{\text{closed range}}{PWS_{\tilde{\gamma}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M).}
\end{tikzcd} \tag{4.1}$$

Moreover, in Sect. 2.4, we have seen that the Fourier transform of an invariant differential operator, which we denote by Q respectively P, is a matrix of polynomials in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with values in the corresponding homomorphism of the homogeneous vector bundles. Therefore, we can reformulate the initial Conj. 1 in terms of action of Q on $PWS_{\tilde{\tau}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ and action of P on $PWS_{\tilde{\gamma}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$, respectively.

However, the problem remains still difficult. The idea is to relate Hörmander's results and estimates ([Hör73], Thm. 7.6.11. and Cor. 7.6.12.) on the '$\mathfrak{a}$'-part under some conditions on the K-type. More precisely, we fix an additional irreducible K-type (μ, E_μ) on the left while on the right, it is the bundle $\mathbb{E}_{\tilde{*}} \to X$, $* \in \{\delta, \gamma, \tau\}$. In terms of our framework, introduced in Chap. 2, we moved from (Level 2) to (Level 3). Thus, the plan is to first solve the main problem in (Level 3), as illustrated in the following diagram:

$$\begin{tikzcd}
PWS_{\tilde{\delta}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M) \arrow[r, "Q"] \arrow[d, dashed, leftrightarrow] & \underset{\text{exact}}{PWS_{\tilde{\tau}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)} \arrow[r, "P"] \arrow[d, dashed, leftrightarrow] & \underset{\text{closed range}}{PWS_{\tilde{\gamma}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)} \arrow[d, dashed, leftrightarrow] & (\textit{Level } 2) \arrow[d, dashed, leftrightarrow] \\
{}_{\mu}PWS_{\tilde{\delta}}(\mathfrak{a}_{\mathbb{C}}^*) \arrow[r, "Q"] & \underset{\text{exact}}{{}_{\mu}PWS_{\tilde{\tau}}(\mathfrak{a}_{\mathbb{C}}^*)} \arrow[r, "P"] & \underset{\text{closed range}}{{}_{\mu}PWS_{\tilde{\gamma}}(\mathfrak{a}_{\mathbb{C}}^*)} & (\textit{Level } 3)
\end{tikzcd}$$

Consider the Paley-Wiener-Schwartz space ${}_{\mu}PWS_{\tilde{*},H}(\mathfrak{a}_{\mathbb{C}}^*)$ without the slow growth condition $(3.iis)_r$ in Def. 2.38. The analog of Hörmander's result ([Hör73], Lem. 7.6.5), is given by the following hypothesis.

Hypothesis 1. *With the notations above, let* $P \in {}_{\tilde{\tau}}PWS_{\tilde{\gamma},0}(\mathfrak{a}_{\mathbb{C}}^*)$ *be the Fourier transform of the invariant differential operator* $D^t \in \mathcal{D}(\mathbb{E}_{\tilde{\tau}}, \mathbb{E}_{\tilde{\gamma}})$ *and* $Q \in {}_{\tilde{\delta}}PWS_{\tilde{\tau},0}(\mathfrak{a}_{\mathbb{C}}^*)$ *of* $\widetilde{D}^t \in \mathcal{D}(\mathbb{E}_{\tilde{\delta}}, \mathbb{E}_{\tilde{\tau}})$ *as in Sect. 1.3.*
Then, there exists $g \in {}_{\mu}PWS_{\tilde{\delta},H}(\mathfrak{a}_{\mathbb{C}}^*)$ *such that* $f = Qg$*, if and only, if* $Pf = 0$*, for given* $f \in {}_{\mu}PWS_{\tilde{\tau},H}(\mathfrak{a}_{\mathbb{C}}^*)$*. In other words, we have*

$$\mathrm{Im}(Q) = \mathrm{Ker}(P) \ \textit{in} \ {}_{\mu}PWS_{\tilde{\tau},H}(\mathfrak{a}_{\mathbb{C}}^*).$$

Now we want to construct a holomorphic function v in ${}_{\mu}PWS_{\tilde{\tau}}(\mathfrak{a}_{\mathbb{C}}^*)$ in such a way that it verifies the intertwining conditions $(3.iii)$ of Def. 2.30 for $\tilde{\tau}$ and fixed μ in $\hat{K}$ as well as provides us with some 'nice' properties.

Hypothesis 2. *With the notations above, fix $r \geq 0$ and $N \in \mathbb{N}_0$.*
Then, there exist constants $M \in \mathbb{N}_0$ and $C_{r,N} \in \mathbb{N}_0$ so that for each function ${}_{\mu}u \in {}_{\mu}PWS_{\tilde{\tau},H}(\mathfrak{a}^*_{\mathbb{C}})$ such that $||P\,{}_{\mu}u||_{r,N} < \infty$, one can find a function ${}_{\mu}v \in {}_{\mu}PWS_{\tilde{\tau}}(\mathfrak{a}^*_{\mathbb{C}})$ with

(i) $P\,{}_{\mu}u = P\,{}_{\mu}v$ *and*

(ii) $||\,{}_{\mu}v||_{r,N+M} \leq C_{r,N}||P\,{}_{\mu}u||_{r,N}$.

Here, $||\cdot||_{r,N} := \sup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}}\left\{(1+|\lambda|^2)^{-N}e^{-r|\mathrm{Re}(\lambda)|}||\cdot||_{op}\right\}$ *denotes the semi-norm.*

Ideally, we would like to have constants which are independent on the K-type μ.

Hypothesis 3. *With the notations in Hyp. 2, the constant M is independent of the K-type μ as well as $N \in \mathbb{N}_0$ and $C_{r,N}$ is of at most of polynomial growth in $|\mu| \in [0,\infty)$, the length of μ.*

The final step would be to move back to our 'initial' statement in (Level 2) by using some convergence arguments. Thus, Hyp. 2 and Hyp. 3 can be converted in (Level 2) as follows.

Hypothesis 4. *With the notations above, let P as in Hyp. 1.*
Then, for all multi-index $\beta \in \mathbb{N}_0$, there exist a multi-index $\alpha \in \mathbb{N}_0$, $M \in \mathbb{N}_0$, for every $r \geq 0, N \in \mathbb{N}_0$, and a constant $C_{r,N+M,\alpha+\beta} \in \mathbb{N}_0$ so that for each function $u \in PWS_{\tilde{\tau},H}(\mathfrak{a}^*_{\mathbb{C}} \times K/M)$ such that $||Pu||_{r,N,\alpha} < \infty$, one can find a function $v \in PWS_{\tilde{\tau}}(\mathfrak{a}^*_{\mathbb{C}} \times K/M)$ with

(i) $Pu = Pv$ *and*

(ii) $||v||_{r,N+M,\alpha+\beta} \leq C_{r,N+M,\alpha+\beta}||Pu||_{r,N,\alpha}$.

Theorem 4.1. *Assume that Hyp. 2 and Hyp. 3 are satisfied. Then, Hyp. 4 holds true.*

Before being able to prove Thm. 4, we need some preparations. Consider, for instance, the L^2-space $V := L^2(K/M, \mathbb{E}_{\tau|_M})$ and for $k \in K$, let $\overline{\chi_\mu(k)} = \chi_\mu(k^{-1})$ be the character of $\mu \in \hat{K}$. Write by $\{e_i, i = 1,\dots,d_\mu\}$ the basis of E_μ and by $\{\tilde{e}_i, i = 1,\dots,d_\mu\}$ its dual. Then, by Peter-Weyl's theorem, we have, for $f \in V$ and $x \in K$:

$$\begin{aligned} f(x) = \sum_{\mu\in\hat{K}} d_\mu \sum_{i=1}^{d_\mu} \int_K \langle \mu(k^{-1})e_i, \tilde{e}_i\rangle f(k^{-1}x)\,dk &= \sum_{\mu\in\hat{K}} d_\mu \sum_{i=1}^{d_\mu} \int_K \langle \mu(kx^{-1})e_i, \tilde{e}_i\rangle f(k^{-1})\,dk \\ &= \sum_{\mu\in\hat{K}} d_\mu \sum_{i=1}^{d_\mu} \int_K f(k)\tilde{\mu}(k^{-1})\tilde{e}_i\,dk\;\mu(x^{-1})e_i. \end{aligned}$$

Since $\tilde{\mu}(k^{-1})$ is a functional $E_\mu \to \mathbb{C}$, we have that ${}_{\mu}f_i := \int_K f(k)\tilde{\mu}(k^{-1})\tilde{e}_i\,dk$ is in $\mathrm{Hom}_M(E_\mu, E_\tau)$. Now, take $k = x$ and consider $w \in PWS_{\tilde{*}}(\mathfrak{a}^*_{\mathbb{C}} \times K/M)$, for each $* \in \{\delta,\gamma,\tau\}$. Then, its Fourier decomposition is given by

$$w(\lambda, k) = \sum_{\mu\in\hat{K}} d_\mu \sum_{i=1}^{d_\mu} {}_{\mu}w_i(\lambda)\mu(k^{-1})e_i, \quad (\lambda, k) \in \mathfrak{a}^*_{\mathbb{C}} \times K/M, \tag{4.2}$$

where

$$ {}_{\mu}w_i = \int_K w(k)\tilde{\mu}(k^{-1})\tilde{e}_i\, dk \in {}_{\mu}PWS_{\tilde{*}}(\mathfrak{a}^*_{\mathbb{C}}), \quad \forall i = 1,\dots,d_\mu \tag{4.3}$$

are the Fourier coefficients. Now, we let converge the obtained estimates in (Level 3) so that a global estimate for the Fourier series is attained.

Lemma 4.2. *Fix $r \geq 0$ and $N \in \mathbb{N}_0$. For each $\mu \in \hat{K}$ and $i = 1,\dots,d_\mu$, let ${}_{\mu}w_i \in {}_{\mu}PWS_{\tilde{*}}(\mathfrak{a}^*_{\mathbb{C}})$ be the Fourier coefficients of $w \in PWS_{\tilde{*}}(\mathfrak{a}^*_{\mathbb{C}} \times K/M)$, $* \in \{\gamma, \tau, \delta\}$. Consider the Casimir operator of $k \in K$([Kna02], (5.23)):*

$$\mathbf{C}_k = -\sum_i X_i^2,$$

where $\{X_i\}$ is an orthonormal basis corresponding to the K-invariants. Then, for each integer p we have

$$||\,{}_{\mu}w_i||_{r,N+p} \leq (1 + ||\mu||^2)^{-p}||w||_{r,N,Y_p}, \quad \forall i \tag{4.4}$$

where $Y_p := (1 - \mathbf{C}_k)^p \in \mathcal{U}(\mathfrak{k})$.

Proof. We know that for all $\mu \in \hat{K}$, the Fourier coefficients ${}_{\mu}w$ of $w \in PWS_{*,H}(\mathfrak{a}^*_{\mathbb{C}} \times K/M)$ can be estimated, in general, by the supremum or operator norm as follows:

$$||\,{}_{\mu}w||_{\mathrm{op}} \overset{(4.3)}{\leq} \int_K ||w(k)\tilde{\mu}(k^{-1})\tilde{e}_i||_{\mathrm{op}}\, dk = \int_K ||w(k)||_{\mathrm{op}}\, dk \leq \sup_{k\in K} ||w(k)||_\infty =: ||w(k)||_{K,\infty}.$$

Note that the operator norm of $\tilde{\mu}(k^{-1})$ is one.
Concerning the Casimir operators, by ([Kna02], Prop. 5.28 (b)), we know that

$${}_{\mu}\mathbf{C}_k = ||\mu + \rho_k||^2 - ||\rho_k||^2,$$

where μ is the highest weight and ρ_k stands for the half sum of the positive roots $\Delta^+(\mathfrak{k}, \mathfrak{t})$, $\mathfrak{t} \subset \mathfrak{k}$ Lie algebra of maximal torus. Since $||\mu+\rho_k||^2 - ||\rho_k||^2 = ||\mu||^2 + 2\langle \mu, \rho_k\rangle \geq ||\mu||^2$, we then have that

$$C_2(1 + ||\mu||^2) \leq {}_{\mu}(1 + \mathbf{C}_k) \leq C_1(1 + ||\mu||^2),$$

where C_1 and C_2 are positive constants. Without loss of generality, take $C_1 = C_2 = 1$. Hence, for each $p \in \mathbb{N}_0$:

$$||(1 + \mathbf{C}_k)^p w||_\infty \geq ||\,{}_{\mu}((1 + \mathbf{C}_k)^p w_i)||_{\mathrm{op}} = {}_{\mu}(1 + \mathbf{C}_k)^p||\,{}_{\mu}w_i||_{\mathrm{op}} \geq (1 + ||\mu||^2)^p||\,{}_{\mu}w_i||_{\mathrm{op}}.$$

Therefore, by considering the dependence of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and setting $Y_p := (1 + \mathbf{C}_k)^p$, we obtain

$$\sup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}} ||\,{}_{\mu}w_i(\lambda)||_{\mathrm{op}} \leq (1 + ||\mu||^2)^{-p} \sup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}} ||Y_p w(\lambda, k)||_{K,\infty}.$$

Now by multiplying the last inequality, with a weight factor $(1 + |\lambda|^2)^{-(N+p)}e^{-r|\mathrm{Re}(\lambda)|}$ on both side, we finally get (4.4). ☐

Proof of Thm. 4.1. Consider $w \in PWS_{\tilde{\gamma}}(\mathfrak{a}^*_{\mathbb{C}} \times K/M)$ with Fourier decomposition (4.2), where ${}_{\mu}w_i \in {}_{\mu}PWS_{\tilde{\gamma}}(\mathfrak{a}^*_{\mathbb{C}})$ are the Fourier coefficients, for all $i = 1,\dots,d_\mu$.
Set ${}_{\mu}w_i := P\,{}_{\mu}u_i$, then by Hyp. 2 and Hyp. 3, there exists ${}_{\mu}v_i \in {}_{\mu}PWS_{\tilde{\tau}}(\mathfrak{a}^*_{\mathbb{C}})$ such that $P\,{}_{\mu}v_i = {}_{\mu}w_i$ and with estimate condition:

$$||\,{}_{\mu}v_i||_{r,N+M} \leq C_\mu||\,{}_{\mu}w_i||_{r,N} \tag{4.5}$$

for some constants $M \in \mathbb{N}_0$ independent of $N \in \mathbb{N}_0$ and C_μ is of maximal growth in μ, i.e., $C_\mu \leq C(1+||\mu||^2)^d$, with $C \in \mathbb{N}_0$. Note that $d \in \mathbb{N}_0$ is independent of N. Now choose $p \in \mathbb{N}_0$ so that $\sum_{\mu\in\hat{K}} d_\mu^2(1+||\mu||^2)^{d-p} < \infty$. Thus, we have

$$\begin{aligned} ||v||_{r,N+M+p,1} \overset{(4.2)}{\leq} \sum_{\mu\in\hat{K}} d_\mu \sum_{i=1}^{d_\mu} ||{}_\mu v_i||_{r,N+M+p} &\overset{\text{Hyps. 2 \& 3}}{\leq} \sum_{\mu\in\hat{K}} d_\mu \sum_{i=1}^{d_\mu} C(1+||\mu||^2)^d ||{}_\mu w_i||_{r,N+p} \\ &\overset{\text{Lem. 4.2}}{\leq} \Big(\sum_{\mu\in\hat{K}} d_\mu^2 C(1+||\mu||^2)^{d-p}\Big) ||w||_{r,N,Y_p} \\ &\leq C'||w||_{r,N,Y_p}, \end{aligned}$$

where C' is some positive constant. In fact, since the space is complete, the sum $\sum_{\mu\in\hat{K}} d_\mu^2 C(1+||\mu||^2)^{d-p}$ is summable, this means that the series converges absolutely on the Fréchet space with corresponding semi-norm $||\cdot||_{r,N+M+p,1}$ to a function.
Now for higher derivatives, we obtain an analogously inequality:

$$\begin{aligned} ||v||_{r,N+M+p,Y_l} &\leq \sum_{\mu\in\hat{K}} d_\mu \sum_{i=1}^{d_\mu} ||{}_\mu v_i||_{r,N+M+p+l}(1+||\mu||^2)^l \\ &\overset{\text{Hyps. 2 \& 3}}{\leq} \sum_{\mu\in\hat{K}} d_\mu \sum_{i=1}^{d_\mu} C(1+||\mu||^2)^{d+l} ||{}_\mu w_i||_{r,N+p+l} \\ &\overset{\text{Lem. 4.2}}{\leq} \Big(\sum_{\mu\in\hat{K}} d_\mu^2 C(1+||\mu||^2)^{-(p+l)}(1+||\mu||^2)^{d+l}\Big) ||w||_{r,N,Y_{p+l}} \\ &\leq C'||w||_{r,N,Y_{p+l}}. \end{aligned}$$

For the last inequality, we used the same arguments as above. Note that the constants C and C' depend on l. In conclusion, we obtained the desired estimate (ii) in (Level 2). Moreover, the existence of v is assured by ${}_\mu v$. Thus, it is holomorphic in $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and satisfies the intertwining condition as well. □

4.2 On closed range and density

It remains to show that D has a closed and dense range in $\mathrm{Ker}(\widetilde{D})$. More precisely, in terms of dual, we need to prove that D^t is injective modulo $\mathrm{Ker}(\widetilde{D})$ with closed range on the strong dual topology $C_c^{-\infty}(X, \mathbb{E}_{\tilde{\gamma}})$. For comfort, we prove the density and closedness separately. First of all, let us recall some notions and important results of functional analysis theory.

We consider a continuous linear operator $A : V_1 \longrightarrow V_2$ between locally convex Hausdorff vector spaces and its adjoint $A^t : V_2' \to V_1'$. For linear subspaces $W \subset V$, where V is a locally convex Hausdorff vector space, we consider its annihiliator $W^\perp \subset V'$ given by

$$W^\perp := \{\tilde{v} \in V' \mid \langle \tilde{v}, w\rangle = 0,\ \forall w \in W\}.$$

Similarly, for $\widetilde{W} \subset V$ a linear subspace $\widetilde{W}^\perp := \{v \in V \mid \langle \tilde{w}, v\rangle = 0,\ \forall \tilde{w} \in \widetilde{W}\}$.
Let V_α' be V' equipped with the weak-* topology, $\forall\alpha$. It is well-known that $(V_\alpha')' = V$ (as vector spaces) and hence by Hahn-Banach theorem

$$(\widetilde{W}^\perp)^\perp = \overline{\widetilde{W}} \quad \text{(weak-* dense)}.$$

We can now easily derive the following well-known elementary results.

Lemma 4.3. *With the notations above, we have*

(1) $\mathrm{Im}(A)^{\perp} = \mathrm{Ker}(A^t)$.

(2) $\mathrm{Ker}(A)^{\perp} = \overline{\mathrm{Im}(A^t)}$ *(weak-* dense).* □

Proof. (1) is elementary, indeed

$$\begin{aligned} \mathrm{Im}(A)^{\perp} &= \{\tilde{v}_2 \in V_2' \mid \langle \tilde{v}_2, Av_1\rangle = 0, \forall v_1 \in V_1\} \\ &= \{\tilde{v}_2 \in V_2' \mid \langle A^t\tilde{v}_2, v_1\rangle = 0, \forall v_1 \in V_1\} \\ &\overset{\text{H.-B.Thm}}{=} \{\tilde{v}_2 \in V_2' \mid A^t\tilde{v}_2 = 0\} = \mathrm{Ker}(A^t). \end{aligned}$$

Similarly, for (2), we obtain

$$\begin{aligned} \mathrm{Im}(A^t)^{\perp} &= \{v_1 \in V_1 \mid \langle A^t\tilde{v}_2, v_1\rangle = 0, \forall \tilde{v}_2 \in V_2'\} \\ &= \{v_1 \in V_1 \mid \langle \tilde{v}_2, Av_1\rangle = 0, \forall \tilde{v}_2 \in V_2'\} \\ &\overset{\text{H.-B.Thm}}{=} \{v_1 \in V_1 \mid \mid A^t v_1 = 0\} = \mathrm{Ker}(A). \end{aligned}$$

Using the discussion at the beginning, we obtain $\mathrm{Ker}(A)^{\perp} = (\mathrm{Im}(A^t)^{\perp})^{\perp} = \overline{\mathrm{Im}(A^t)}$ (weak-* dense). □

Lemma 4.4. *We consider two operators between locally convex Hausdorff vector spaces*

$$V_1 \xrightarrow{A} V_2 \xrightarrow{B} V_3$$

with $B \circ A = 0$, *i.e.* $\mathrm{Im}(A) \subset \mathrm{Ker}(B)$.
Then, $\mathrm{Im}(A) \subset \mathrm{Ker}(B)$ *is dense if, and only if,* $\mathrm{Im}(B^t) \subset \mathrm{Ker}(A^t)$ *is weak-* dense.*

Proof. By Hahn-Banach theorem, we have that $\mathrm{Im}(A) \subset \mathrm{Ker}(B)$ is dense if, and only if, $\mathrm{Im}(A)^{\perp} = \mathrm{Ker}(B)^{\perp}$. Hence by applying Lem. 4.3, this is equivalent to $\mathrm{Ker}(A^t) = \overline{\mathrm{Im}(B^t)}$, i.e. $\mathrm{Im}(B^t) \subset \mathrm{Ker}(A^t)$ is weak-* dense. □

Corollary 4.5. *Assume that* $\mathrm{Im}(B^t) \subset \mathrm{Ker}(A^t)$ *is dense in the strong dual topology. Then,* $\mathrm{Im}(A) \subset \mathrm{Ker}(B)$ *is dense.*

Proof. Since strongly density implies weak-* density, by applying Lem. 4.4, we directly obtain the result. □

Remark 4.6. (a) If V_2 is (semi-) reflexive, then we even get an equivalence in Cor. 4.5 by the Hahn-Banach theorem.

(b) Setting $B = 0$ in Lem. 4.4, we obtain the well-known result:

$$A \text{ has dense range in } V_2 \iff A^t \text{ is injective.}$$

Proposition 4.7. *Assume that Hyp. 1 and Hyp. 2 are true. Then* $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ *has dense range in* $\mathrm{Ker}(\widetilde{D}) \subset C^\infty(X, \mathbb{E}_\tau)$.

Proof. Let P and Q as in Hyp. 1. In view of Cor. 4.5 and the Paley-Wiener-Schwartz Thm. 2.40, it suffices to show that $\mathrm{Im}(Q) \subset \mathrm{Ker}(P) \subset PWS_{\tilde{\tau}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ is dense. Consider $f \in \mathrm{Ker}(P)$ with its Fourier decomposition:

$$f(\lambda, k) = \sum_{\mu \in \hat{K}} d_\mu \sum_{i=1}^{d_\mu} {}_\mu f_i(\lambda) \mu(k^{-1}) e_i, \quad (\lambda, k) \in \mathfrak{a}_{\mathbb{C}}^* \times K/M, e_i \in E_\mu,$$

where ${}_\mu f_i \in {}_\mu PWS_{\tilde{\tau},H}(\mathfrak{a}_{\mathbb{C}}^*)$. It can be approximated by finite Fourier series

$$f_n(\lambda, k) = \sum_{|\mu| \leq n} d_\mu \sum_{i=1}^{d_\mu} {}_\mu f_i(\lambda) \mu(k^{-1}) e_i, \quad (\lambda, k) \in \mathfrak{a}_{\mathbb{C}}^* \times K/M, e_i \in E_\mu, \forall n.$$

By applying Hyp. 1, we have ${}_\mu f_i = Q\, {}_\mu \tilde{g}_i$, where ${}_\mu \tilde{g}_i \in {}_\mu PWS_{\tilde{\delta},H}(\mathfrak{a}_{\mathbb{C}}^*)$. By Hyp. 2 we find ${}_\mu g_i$ in the Paley-Wiener-Schwartz space ${}_\mu PWS_{\tilde{\delta}}(\mathfrak{a}_{\mathbb{C}}^*)$. Thus, ${}_\mu f_i \in \mathrm{Im}(Q)$ and hence $f_n \in \mathrm{Im}(Q)$. This implies that $\mathrm{Im}(Q)$ is dense, now by summing everything up, we have that $\mathrm{Im}(Q) \subset \mathrm{Ker}(\tilde{P})$ is dense. □

Now it remains to show that D has closed range in $\mathrm{Ker}(\widetilde{D})$, i.e.

$$\overline{\mathrm{Im}(D)} \subset \mathrm{Ker}(\widetilde{D}) \subset C^\infty(X, \mathbb{E}_\tau). \tag{4.6}$$

For this, we will again use some known results from Helgason ([Hel89], Chap. 2) respectively Schaefer [Sch71]. In fact, by the well-known criterion for closed range ([Hel89], Thm. 2.16 (ii) or [Sch71], Thm. 7.7) we have to show that

$$\mathrm{Im}(D^t) \subset C_c^{-\infty}(X, \mathbb{E}_{\tilde{\gamma}})$$

has closed range in the weak-* topology. But first we need a compactness lemma.

Write by V the corresponding Paley-Wiener-Schwartz space $PWS_{\tilde{*}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$, $* \in \{\delta, \gamma, \tau\}$ and $V_{r,N}$ for its subspaces so that

$$V = \bigcup_{r \geq 0} \bigcup_{N \in \mathbb{N}_0} V_{r,N}.$$

$V_{r,N}$ is a Fréchet space with the generating system of semi-norms $|| \cdot ||_{r,N,\alpha}$, for all multi-index α, and V carries the corresponding locally convex inductive limit topology.

Lemma 4.8. *Let $B \subset V_{r,N}$ be a bounded subset, with respect to the corresponding Fréchet topology. Then, B is relatively compact in $V_{r,N+1}$, i.e. the closure of B in $V_{r,N+1}$ is compact.*

Proof. We consider the space

$$W := C^\infty(\mathfrak{a}_{\mathbb{C}}^* \times K/M, \mathbb{E}_{\tau|_M}) \subset C^\infty(\mathfrak{a}_{\mathbb{C}}^* \times K, \mathbb{E}_\tau)$$

with its natural Fréchet topology. As the space of smooth sections of a vector bundle, it is also a Montel space. We have a continuous injection $V_{r,N} \hookrightarrow W$. Thus B, viewed as a subspace of W, is relatively compact.
Now let (f_n) be a sequence in B. We have to show that it has a subsequence (f_{n_l}) that converges in $V_{r,N+1}$.

By the above remarks, (f_n) has a subsequence (f_{n_l}) that converges in W to some $f \in W$, i.e. itself and all its derivatives converge uniformely on sets of the form $C \times K \subset \mathfrak{a}_{\mathbb{C}}^* \times K$, where $C \subset \mathfrak{a}_{\mathbb{C}}^*$ is compact. This implies that f is holomorphic in λ and satisfies the growth condition as well as Delorme's intertwining condition, i.e. $f \in V_{r,N} \subset V_{r,N+1}$.
However, (f_{n_l}) may not converge to f in the (r, N)-topology. Nevertheless, it converges in the $(r, N+1)$-topology. This follows easily from the estimate. Let $g \in V_{r,N}$. Then for any $R > 0$, we have

$$||g||_{r,N+1,\alpha} \leq \max\Big(||l_Y g||_{\overline{B}_R(0)\times K,\infty}\ , \frac{1}{1+R^2}||g||_{r,N,\alpha}\Big), \tag{4.7}$$

where $\overline{B}_R(0)$ denotes the closed ball of radius R in $\mathfrak{a}_{\mathbb{C}}^*$. Indeed, let

$$C := \sup_{g\in B} ||g||_{r,N,\alpha} < \infty.$$

Then, also $||f||_{r,N,\alpha} \leq C$ and thus $||f_{n_l} - f||_{r,N,\alpha} \leq 2C$.
Let $\epsilon > 0$. Choose R large enough such that $\frac{2C}{1+R^2} \leq \epsilon$ and l_0 large enough such that

$$||f_{n_l} - f||_{r,N+1,\alpha} \overset{(4.7)}{\leq} \epsilon \quad \text{for } l \geq l_0.$$

This shows that $||f_{n_l} - f||_{r,N+1,\alpha} \overset{l\to\infty}{\longrightarrow} 0$ for all multi-index α. Hence, $f_{n_l} \to f$ in $V_{r,N+1}$. $\square$

Theorem 4.9. *Assume that Hyp. 4 is true. Then, $D \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$ has closed range in* $\mathrm{Ker}(\widetilde{D})$.

Proof. We first show, using Hyp. 4 for $P = {}_{\tilde{\tau}}\mathcal{F}_{\tilde{\delta}}(D^t)$, that

$$\mathrm{Im}(P) \cap PWS_{\tilde{\gamma},r,N}$$

is closed in the (r, N)-topology, for every $N \in \mathbb{N}_0$ and $r \geq 0$.
In fact, consider a sequence $w_n := Pu_n$ in $\mathrm{Im}(P) \cap PWS_{\tilde{\gamma},r,N}$ so that w_n converges to $w \in PWS_{\tilde{\gamma},r,N}$, whenever n tends to ∞. Then (w_n) is a Cauchy sequence with respect to the semi-norm $||\cdot||_{r,N,\alpha}$.
By Hyp. 4, there exists a sequence $v_n \in PWS_{\tilde{\tau},r,N}$ so that $Pv_n = w_n$ and with estimate (ii) in Hyp. 4. This implies that (Pv_n) is a Cauchy-sequence in $PWS_{\tilde{\gamma},r,N}$. Since, $PWS_{\tilde{\gamma},r,N}$ is complete, there exists $Pv \in \mathrm{Im}(P)$ such that $Pv_n \overset{n\to\infty}{\longrightarrow} Pv$.
Hence, we have that $Pv = w$, i.e., $w \in \mathrm{Im}(P) \subset PWS_{\tilde{\gamma},r,N}$. Therefore, we conclude that $\mathrm{Im}(P) \cap PWS_{\tilde{\gamma},r,N}$ is closed in the (r, N)-topology with respect to the inductive limit topology.

Secondly, we have to show, by using Helgason's results ([Hel89], Chap. 2, Thm. 2.17 (i) and Prop. 2.8), that for every $B' \subset C_c^{-\infty}(X, \mathbb{E}_{\tilde{\gamma}})$, which is weak-* bounded and weak-* closed, that

$$B' \cap \mathrm{Im}(D^t) \quad \text{is weak-* closed.}$$

In other words, $B' \cap \mathrm{Im}(D^t)$ is closed in B'_α, where B'_α is B' equipped with the weak-* topology. However, B' is also strongly bounded (holds in the dual of any Fréchet space, [Hel20] Thm. 2.17 (ii)) and of course also strongly closed. Since $C_c^{-\infty}(X, \mathbb{E}_{\tilde{\gamma}})$ as the dual of a Montel space, is itself Montel in the strong dual topology, we conclude that

B'_β, i.e. equipped with the strong topology, is compact.
Now, assume, for a moment, that $B'_\beta \cap \mathrm{Im}(D^t)$ is closed, then it is compact. Since the identity map

$$B'_\beta \longrightarrow B'_\alpha$$

is continuous, we conclude that $B'_\alpha \cap \mathrm{Im}(D^t)$ is compact, i.e. $B' \cap \mathrm{Im}(D^t)$ is compact in the weak-* topology, in particular it is weak-* closed.
Thus, it suffices to show that $B' \cap \mathrm{Im}(D^t)$ is closed in the strong dual topology, for every strongly bounded and closed B'.

Now we fix such a strongly bounded and closed $B' \subset C_c^{-\infty}(X, \mathbb{E}_{\tilde{\gamma}})$. We take the Fourier transforms. Then

$$\mathcal{F}(B' \cap \mathrm{Im}(D^t)) = \mathcal{F}(B') \cap \mathrm{Im}(P).$$

By the continuity of $\mathcal{F}$ we have that

$$\mathcal{F}(B') \subset PWS_{\tilde{\gamma}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$$

is compact, in particular closed. By the definition of the locally convex inductive limit topology, we see that $\mathcal{F}(B') \subset PWS_{\tilde{\gamma},r,N}$ is closed in every $PWS_{\tilde{\gamma},r,N}$ with (r, N)-topology.
On the other hand, we have seen in the proof of continuity of $\mathcal{F}$ (Proof of Lem.2.43), that there exists $N_0 \in \mathbb{N}_0$ such that $\mathcal{F}(B') \subset PWS_{\tilde{\gamma},r,N_0}$ and that $\mathcal{F}(B')$ is bounded in the (r, N_0)-topology. By Lem. 4.8

$$\mathcal{F}(B') \subset PWS_{\tilde{\gamma},r,N_0+1}$$

is relatively compact. This means that, $\mathcal{F}(B') \subset PWS_{\tilde{\gamma},r,N_0+1}$ is closed in the (r, N_0+1)-topology. Hence, it is compact in the (r, N_0+1)-topology.
Now let $N := N_0 + 1$. By the beginning of the proof, $\mathrm{Im}(P) \cap PWS_{\tilde{\gamma},r,N}$ is closed in the (r, N)-topology. It follows that

$$\mathcal{F}(B') \cap \mathrm{Im}(P) \subset \mathrm{Im}(P) \cap PWS_{\tilde{\gamma},r,N}$$

is compact and closed in the (r, N)-topology. Since the injection

$$PWS_{\tilde{\gamma},r,N} \hookrightarrow PWS_{\tilde{\gamma}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$$

is continuous, we conclude that $\mathcal{F}(B') \cap \mathrm{Im}(P)$ is compact, in particular closed, in $PWS_{\tilde{\gamma}}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$. By applying now the inverse Fourier transform, which is a topological isomorphism by the Paley-Wiener-Schwartz Thm. 2.40, we find that $B' \cap \mathrm{Im}(D^t)$ is strongly closed in $C_c^{-\infty}(X, \mathbb{E}_{\tilde{\gamma}})$. By (4.6), this implies the desired result. □

Thm. 4.9 is remains true if we assume that Hyp. 2 holds true instead of Hyp. 4.

Finally, by combining all the obtained results together, this implies the general solvability in X, i.e. Conjecture 1 is true.

Theorem 4.10. *Let $X = G/K$ be a symmetric space of non-compact type and for $r \geq 0, x \in X$ consider the ball $B_r(x) \subset X$. Assume that Hyp. 1 and Hyp. 4 are true. Then, we have (local and global) solvability in X and in the ball $B_r(x) \subset X$.*

Proof. For $x \in X$, consider an open neighbourhood U of x. Let $g \in C^\infty(U, \mathbb{E}_\tau)$ with $\tilde{D}g = 0$, then we need to show that there exist a neighbourhood $V \subset U$ in x and $f \in C^\infty(V, \mathbb{E}_\gamma)$ with $Df = g\big|_V$.
In fact, let g and U as above, then there exists $r \geq 0$ with $B_r(x) \subset U$. By the solvability on $B_r(x)$, there exists $f \in C^\infty(B_r(x), \mathbb{E}_\gamma)$ with $Df = g\big|_{B_r(x)}$.

Concerning the solvability on balls. We first translate the ball to its origin, then we dualize:

$$\begin{aligned}
(C^\infty(B_R(0), \mathbb{E}_\tau))' \quad &:= \quad \{\varphi \in C_c^{-\infty}(X, \mathbb{E}_{\tilde{\tau}}) \mid \mathrm{supp}(\varphi) \subset B_R(0)\} \\
&= \quad \bigcup_{0 \leq r < R} C_r^{-\infty}(X, \mathbb{E}_{\tilde{\tau}}) \\
&\overset{\text{Thm. 2.40}}{\cong} \quad \bigcup_{0 \leq r < R} \bigcup_{N \in \mathbb{N}_0} PWS_{\tilde{\tau}, r, N}.
\end{aligned} \tag{4.8}$$

The solvability in X is achieved in the same way as in $B_r(0)$, by applying Prop. 4.7 together with Thm. 4.9 for (4.8), if we assume that Hyp. 1 and Hyp. 4 are true. By translating back to $B_R(x)$, we have also solvability there. □

By combining Thm. 4.1 and Thm. 4.10, we obtain directly the following consequence.

Corollary 4.11. *Assume that Hyp. 1, Hyp. 2 and Hyp. 3 are satisfied. Then, the Conj. 1 is true.* □

Note that, if Hyp. 1 and Hyp. 2 are true, then by Prop 4.7 and Thm. 4.9, we have solvability for K-finite elements, thus solvability in (Level 3).

Remark 4.12. Concerning a possible proof of Hyp. 1. If we would have Conj. 2, then we can argue with the same arguments for the proof of Hyp. 1.

Conjecture 2. *Let $\gamma, \tau \in \hat{K}$ and $\mathfrak{h}_\mathbb{C}^*$ be a Cartan algebra of $\mathfrak{g}_\mathbb{C}$. Consider the Weyl group $\mathcal{W} = W(\mathfrak{g}, \mathfrak{h})$ as in Sect. 3.1. Then, ${}_\gamma PWS_{\tau,H}(\mathfrak{a}_\mathbb{C}^*)$ is generated as* $\mathrm{Hol}(\mathfrak{h}_\mathbb{C}^*)^{\mathcal{W}}$ *by ${}_\gamma PWS_{\tau,0}(\mathfrak{a}_\mathbb{C}^*)$. Moreover*

$$\mathrm{Hol}(\mathfrak{h}_\mathbb{C}^*)^{\mathcal{W}} \otimes_{\mathrm{Pol}(\mathfrak{h}_\mathbb{C}^*)^{\mathcal{W}}} {}_\gamma PWS_{\tau,0}(\mathfrak{a}_\mathbb{C}^*) \cong {}_\gamma PWS_{\tau,H}(\mathfrak{a}_\mathbb{C}^*).$$

Delorme [Del05] proved a similar result for subsets. Note that $\mathrm{Hol}(\mathfrak{h}_\mathbb{C}^*)^{\mathcal{W}} \cong \mathrm{Hol}(\mathbb{C}^{\dim(\mathfrak{h})})$ and similar for $\mathrm{Pol}(\mathfrak{h}_\mathbb{C}^*)^{\mathcal{W}} \cong \mathrm{Pol}(\mathbb{C}^{\dim(\mathfrak{h})})$. Here ${}_\gamma PWS_{\tau,H}(\mathfrak{a}_\mathbb{C}^*)$ is not necessary a free module.

Chapter 5

Three examples

In this part, we demonstrate by three examples how one can solve the main problem (Conj. 1) of this dissertation with aid of the proposed preparations and techniques exhibited in the previous chapters. More precisely, we present a complete proof for the estimate's hypotheses, Hyp. 1, Hyp. 2 and Hyp. 3 (except for Hyp. 3 in $\mathbb{H}^2 \times \mathbb{H}^2$ and Hyp. 2 and Hyp. 3 for $\mathbb{H}^3$) by using the nice description of Delorme's intertwining conditions in Chap. 3.

We start, in the first Section 5.1, with the classical example, the upper half plane $\mathbb{H}^2$ of $\mathbb{C}$, which is isomorphic to the symmetric space of non-compact type $X = SL(2,\mathbb{R})/SO(2)$. As we already have seen in Sect. 3.4, the special linear group $SL(2,\mathbb{R})$ is the semi-simple non-compact Lie group of the smallest possible dimension. The proofs of the hypotheses (Thm. 5.1 & Thm. 5.2) in this case are simple.

As second example, in Section 5.1, we consider the product of $SL(2,\mathbb{R})$ with itself, which is an example of real rank two. Even beyond, if we generalize it to $SL(2,\mathbb{R})^d$, with $d \in \mathbb{N}$. Since Delorme's intertwining condition (Thm. 3.25) remains the 'same' as in $SL(2,\mathbb{R})$ (Thm. 3.20), we can use similar arguements as for $SL(2,\mathbb{R})$ (Thm. 5.4 & Thm. 5.5). However, Hyp. 3 is more difficult to achieve, we are not able to prove it yet.

Last but not least, we consider, in Section 5.3, the symmetric space of non-compact type $X = SL(2,\mathbb{C})/SU(2)$, which can be identify with the model of hyperbolic 3-space $\mathbb{H}^3$. We present a complete proof of Hyp. 1 (Thm. 5.7). The proof of Hyp. 2 and Hyp. 3 are work in progress.

5.1 Solvability on the upper half-plane

Under the previous notations introduced in Sect. 3.4, let $G = SL(2,\mathbb{R})$ and $K = SO(2)$ its maximal compact subgroup. Consider three, not necessary irreducible, K-representations $(\delta, E_\delta), (\tau, E_\tau)$ and (γ, E_γ) with

$$E_\delta = \bigoplus_{k=1}^{d_\delta} E_{s_k}, \quad E_\tau = \bigoplus_{j=1}^{d_\tau} E_{n_j}, \quad E_\gamma = \bigoplus_{i=1}^{d_\gamma} E_{m_i},$$

where s_k, n_j and m_i are integers, $\forall k, j, i$. Write by $\mathbb{E}_* \to X$ their corresponding homogeneous line bundles over $X = G/K$ induced by the vector space E_*, for $* \in \{\delta, \tau, \gamma\}$.

Moreover, we can identify X with the upper half-plane of $\mathbb{C}$, also known as the hyperbolic two-space

$$\mathbb{H}^2 = \{z \in \mathbb{C}|\ \mathrm{Im}(z) > 0\}$$

by $gK \to g(i)$, where

$$g \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}, \quad g \in SL(2,\mathbb{R}), z \in \mathbb{H}^2. \tag{5.1}$$

This means that G coincides with the group of rigid motions of $\mathbb{H}^2$ preserving orientation and the stabilizer of $i \in \mathbb{H}^2$ concides with K. In particular, G acts transitively on $\mathbb{H}^2$ by this homography (5.1). In other words, each orbit under G is the whole $\mathbb{H}^2$. In addition, one can easily check that it preserves the metric $\frac{dx^2+dy^2}{y^2}$ and volume form $\frac{dxdy}{y^2}$ of $\mathbb{H}^2$.

Now fix an additional irreducible K-type (l, E_l) and let us prove, in the following, the hypotheses of Chap. 4. We first start to show Hyp. 1.

Theorem 5.1. *The short sequence*

$$_lPWS_{\delta,H}(\mathfrak{a}^*_{\mathbb{C}}) \xrightarrow{Q} {_lPWS_{\tau,H}}(\mathfrak{a}^*_{\mathbb{C}}) \xrightarrow{P} {_lPWS_{\gamma,H}}(\mathfrak{a}^*_{\mathbb{C}}) \tag{5.2}$$

is exact in the middle, that means that $\mathrm{Im}(Q) = \mathrm{Ker}(P)$.

Proof. From Sect. 1.3, we found a candidate $\tilde{D}^t$ for D^t so that their composition is 0. Now by applying the Paley-Wiener-Schwartz Thm. 2.40, we obtain through the Fourier transform and the l-isotopic component projection, the short sequence

$$_lPWS_{\delta,0}(\mathfrak{a}^*_{\mathbb{C}}) \xrightarrow{Q} \underset{\text{exact}}{{_lPWS_{\tau,0}}(\mathfrak{a}^*_{\mathbb{C}})} \xrightarrow{P} {_lPWS_{\gamma,0}}(\mathfrak{a}^*_{\mathbb{C}}),$$

where Q (resp. P) is the Fourier transform of $\widetilde{D}^t$ (resp. D^t). Note that $_lPWS_{*,0}(\mathfrak{a}^*_{\mathbb{C}})$ is a subspace of $_lPWS_{*,H}(\mathfrak{a}^*_{\mathbb{C}})$, for $* \in \{\delta, \tau, \gamma\}$. Moreover due Thm. 3.20, we have that each $_lPWS_{*,H}(\mathfrak{a}^*_{\mathbb{C}})$ is uniquely determined and it is a free finite generated $\mathrm{Pol}(\lambda^2)$-module with the same generators $q_{l,*}$ as $_lPWS_{*,0}(\mathfrak{a}^*_{\mathbb{C}})$. Thus

$$_lPWS_{*,H}(\mathfrak{a}^*_{\mathbb{C}}) \cong \mathrm{Hol}(\lambda^2) \otimes_{\mathrm{Pol}(\lambda^2)} {_lPWS_{*,0}}(\mathfrak{a}^*_{\mathbb{C}}).$$

In addition, $\mathrm{Hol}(\lambda^2)$ is a torsion free model over the principal ideal domain $\mathrm{Pol}(\lambda^2)$. This means that $\mathrm{Hol}(\lambda^2)$ is a flat $\mathrm{Pol}(\lambda^2)$-module. Hence, $\mathrm{Hol}(\lambda^2)$ over the ring $\mathrm{Pol}(\lambda^2)$ preserves the exactness of the sequence. This completes the proof. □

Consider the Fourier series $w \in PWS_\gamma(\mathfrak{a}^*_{\mathbb{C}} \times K/M)$

$$w(\lambda, k_\theta) = \sum_{l\in\mathbb{Z}} {_lw}(\lambda)e^{il\theta}, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}, k_\theta \in SO(2).$$

Then, its Fourier coefficients $_lw$ are also holomorphic on $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and statisfy the intertwining condition for $\gamma \in \hat{K}$ and fixed l too. Hence, they are of the form:

$$_lw = {_lb} \cdot q_{l,\gamma}, \tag{5.3}$$

where $_lb \in \mathrm{Hol}(\lambda^2)$ is a scalar-valued holomorphic even function in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $q_{l,\gamma}$ is the polynomial given in (3.17). Thus, we can prove Hyp. 2 and Hyp. 3 by the following theorem.

Theorem 5.2 (Estimate result in (Level 3))**.** *Let $P \in {}_{\tau}PWS_{\gamma,0}(\mathfrak{a}_{\mathbb{C}}^*)$ be the Fourier transform of the transposed invariant differential operator $D^t \in \mathcal{D}_G(\mathbb{E}_\tau, \mathbb{E}_\gamma)$.*
Then, there exist $M \in \mathbb{N}_0$, for all $r \geq 0$ and $N \in \mathbb{N}_0$, as well as a constant $C_{r,N+M} > 0$ so that for each ${}_l u \in {}_l PWS_{\tau,H}(\mathfrak{a}_{\mathbb{C}}^*)$ such that $||P\,{}_l u||_{r,N} < \infty$, one can find a ${}_l v \in {}_l PWS_\tau(\mathfrak{a}_{\mathbb{C}}^*)$ with

(i) $P\,{}_l u = P\,{}_l v$ *and*

(ii) $||\,{}_l v||_{r,N+M} \leq C_{r,N+M}||P\,{}_l u||_{r,N}$.

The constants $C_{r,N+M}$ and M can be chosen to be independent of the integer l.

Before proceeding with the proof of this theorem, we need a helpful standard result, which will allow us to estimate the Fourier coefficients later on.

Lemma 5.3. *Consider $p(z) := \sum_{n=0}^k a_n z^n$ a polynomial in one variable such that the leading coefficient a_k is not zero. For $r > 1$, let f be a holomorphic function in $B_r(0) \subset \mathfrak{a}_{\mathbb{C}}^*$. Then*

$$|f(0)| \leq |a_k|^{-1} \sup_{|z|=1} |f(z)p(z)|. \tag{5.4}$$

Here, $B_r(0)$ denotes an open ball of radius r centered at 0 in $\mathfrak{a}_{\mathbb{C}}^*$.

Note that (5.4) implies that

$$|f(\lambda)| \leq C_p \sup_{|z|=1} |f(\lambda+z)p(\lambda+z)|, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, f \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*),$$

where the constant C_p depends on the polynomial p, but not on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Proof of Lem. 5.3. Let $q(z) := \sum_{n=0}^k \overline{a}_{k-n} z^n$ be a polynomial such that the product $q \cdot f$ is holomorphic on $B_r(0)$. Then, by the maximum principle for $B_1(0)$, we have

$$|q(0)f(0)| \leq \sup_{|z|=1} |q(z)f(z)|,$$

where $q(0) = \overline{a}_k$. If $|z| = 1$, then we obtain

$$\overline{p(z)} = \sum_{n=0}^k \overline{a}_n \overline{z}^n = \sum_{n=0}^k \overline{a}_n z^{-n} = z^{-k} \sum_{n=0}^\infty \overline{a}_n z^{k-n} = z^{-k} q(z),$$

where we used the fact that $\overline{z} = z^{-1}$. Therefore, for $|z| = 1$, we have $|p(z)| = |q(z)|$ and thus the lemma follows. □

For the proof of Thm. 5.2, we used the same approach as the proof of Hörmander's proposition ([Hör73], Prop. 7.65.).

Proof of Thm. 5.2. We proceed by induction on the dimension d_γ of the vector space E_γ. For instance, fix $l \in \mathbb{Z}$.

Initial case: Let us show that the theorem is true when $d_\gamma = 1$.
Set $\sum_{j=1}^{d_\tau} P_j\,{}_l u_j =: {}_l w$. The aim is to construct a holomorphic Fourier coefficient ${}_l v_j$ of the form:

$${}_l v_j(\lambda) = {}_l h_j(\lambda) \cdot q_{l,n_j}(\lambda), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \forall j = 1, \ldots, d_\tau \tag{5.5}$$

such that ${}_l w = \sum_{j=1}^{d_\tau} P_j\, {}_l v_j$ and which can be estimated later on. Whereas q_{l,n_j} is the polynomial in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, defined in (3.17). To do so, assume that we already have (5.5) and let us instead build a 'new' even function ${}_l\tilde{h}_j \in \mathrm{Hol}(\lambda^2)$. Note that this function will depend on P_j as well as ${}_l w$.
Due Thm. 3.20 together with Thm. 2.46, each P_j has the form

$$P_j := a_j \cdot q_{n_j,m}, \quad \forall j \tag{5.6}$$

where $a_j \in \mathrm{Hol}(\lambda^2)$ is a holomorphic function and $q_{n_j,m}$ is the polynomial (3.17). By putting the equations (5.6), (5.3) and (5.5) in $\sum_{j=1}^{d_\tau} P_j\, {}_l v_j = {}_l w$, we obtain, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$:

$$\begin{aligned}
\sum_{j=1}^{d_\tau} P_j(\lambda)\, {}_l v_j(\lambda) = {}_l w(\lambda) \quad &\iff \quad \sum_{j=1}^{d_\tau} a_j(\lambda) q_{n_j,m}(\lambda) q_{l,n_j}(\lambda)\, {}_l h_j(\lambda) = {}_l b(\lambda) q_{l,m}(\lambda) \\
&\overset{\text{Lem. 3.21}}{\iff} \quad \sum_{j=1}^{d_\tau} a_j(\lambda) r^l_{n_j,m}(\lambda)\, {}_l h_j(\lambda) = {}_l b(\lambda) \\
&\iff \quad \sum_{j=1}^{d_\tau} {}_l\tilde{a}_j(\lambda)\, {}_l h_j(\lambda) = {}_l b(\lambda),
\end{aligned}$$

where in the last equivalence, we set ${}_l\tilde{a}_j(\lambda) := a_j(\lambda) r^l_{n_j,m}(\lambda), \forall j$.
Next, to find out all the common zeros of all the polynomials ${}_l\tilde{a}_j$, for each j, we take the greatest common divisor, gcd, in $\mathbb{C}[\lambda^2]$ of them

$$\gcd_{\mathbb{C}[\lambda^2]}({}_l\tilde{a}_1(\lambda), \ldots, {}_l\tilde{a}_{d_\tau}(\lambda)) =: {}_l\tilde{p}(\lambda) \in \mathbb{C}[\lambda^2].$$

Thus, we have that

$${}_l b(\lambda) := {}_l\beta(\lambda)\, {}_l\tilde{p}(\lambda), \tag{5.7}$$

where ${}_l\beta \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*)$ with ${}_l\beta(\lambda) = {}_l\beta(-\lambda)$. Since $\mathbb{C}[\lambda^2]$ is a principal ideal domain and ${}_l\tilde{a}_j(\lambda)$ are elements of $\mathbb{C}[\lambda^2]$, by Bézout's identity theorem, we can find, for each j, polynomials ${}_l R_j \in \mathbb{C}[\lambda^2]$ such that

$$\sum_{j=1}^{d_\tau} {}_l\tilde{a}_j(\lambda)\, {}_l R_j(\lambda) = {}_l\tilde{p}(\lambda).$$

Now by taking ${}_l b = {}_l\tilde{p}$, we have found a new ${}_l\tilde{h}_j \in \mathrm{Hol}(\lambda^2)$ of the form:

$${}_l\tilde{h}_j(\lambda) := {}_l\beta(\lambda)\, {}_l R_j(\lambda), \quad \text{for } \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \forall j.$$

Therefore, we conclude that ${}_l v_j \in {}_l PWS_{n_j,H}(\mathfrak{a}_{\mathbb{C}}^*)$ and

$${}_l v_j(\lambda) = {}_l\tilde{h}_j(\lambda) q_{l,n_j}(\lambda) = {}_l\beta(\lambda)\, {}_l R_j(\lambda) q_{l,n_j}(\lambda), \text{ for } \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \forall j. \tag{5.8}$$

Concerning the estimate, from the equations (5.3), (5.7), (5.8) and the relation (3.20), we obtain

$$r^l_{n_j,m}(\lambda)\, {}_l R_j(\lambda)\, {}_l w(\lambda) = {}_l v_j(\lambda)\, {}_l\tilde{p}(\lambda) q_{n_j,m}(\lambda).$$

Thus, due Lem. 5.3, this leads us to

$$|{}_l v_j(\lambda)| \le C_{{}_l\tilde{p}, q_{n_j,m}} \sup_{|z|\le 1} \Big\{ |r^l_{n_j,m}(\lambda+z)\, {}_l R_j(\lambda+z)\, {}_l w(\lambda+z)| \Big\},$$

where $C_{_l\tilde{p},q_{n_j,m}}$ is a non-zero constant depending on the two polynomials $_l\tilde{p}$ and $q_{n_j,m}$. Moreover, since we have such polynomials $r^l_{n_j,m}$ and $_lR_j$, we can choose $M \in \mathbb{N}_0$ for each $r \geq 0$ and $N \in \mathbb{N}_0$, so that

$$|r^l_{n_j,m}(\lambda)\,{}_lR_j(\lambda)| \leq C_l(1+|\lambda|^2)^M, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},$$

where the non-zero constant C_l depends on the integer l. However, by Lem. 3.21, there are finitely many $r^l_{n_j,m}$, same for $_lR_j$ and $_l\tilde{p}$. Hence, we can take the maximum over all $l \in \mathbb{Z}$, i.e., $C = \max_{l\in\mathbb{Z}} C_l$. Coming back to our inequality, we get

$$\begin{aligned} ||\,{}_lv(\lambda)||_{\mathrm{op}} &\leq \max\{|\,{}_lv_j(\lambda)| \mid j = 1, \ldots, d_\tau\} \\ &\leq \max_{j,l}\{C'_{_l\tilde{p},q_{n_j,m}}\} \sup_{|z|\leq 1} \left\{(1+|\lambda+z|^2)^M|\,{}_lw(\lambda+z)|\right\} \\ &\leq C'(1+|\lambda|^2)^M||\,{}_lw(\lambda)||_{\mathrm{op}}, \end{aligned}$$

where $C' > 0$ is a constant independent of l. Here, we used the triangle-inequality, i.e.

$$|\lambda + z| \leq |\lambda| + |z| \leq |\lambda| + 1 \leq 1 + 2|\lambda|$$

since $|z| \leq 1$. Now by multiplying both side by a weight factor $(1+|\lambda|^2)^{-N}e^{-r|\mathrm{Re}(\lambda)|}$ and taking $(1+|\lambda|^2)^M$ on the left hand side, we obtain the desired estimate (ii).

Inductive step: It remains to show that the theorem is true for $d_\gamma > 1$.
By induction hypothesis, assume that the statement is already proved for systems involving a smaller number d_γ of equations. Write $P = (P_1, P_{d_{\gamma-1}})^T$.
In particular, we then can consider the equations $P_1\,{}_lv_1 = P_1\,{}_lu$ and conclude that it has a solution $_lv_1 \in {}_lPWS_n(\mathfrak{a}^*_{\mathbb{C}})$ such that

$$||\,{}_lv_1||_{r,N+M} \leq C||P_1\,{}_lu||_{r,N}. \tag{5.9}$$

Now, set $_lv := {}_lv_1 + {}_lh$, where we have to find $_lh$ so that $P\,{}_lh = P({}_lu - {}_lv_1)$ and $_lh$ can be estimated.
By applying Thm. 5.1 to the systems of equations $P_1({}_lu - {}_lv_1) = 0$, we can write

$${}_lu - {}_lv_1 = Q\,{}_lf, \tag{5.10}$$

where $_lf \in {}_lPWS_{\delta,H}(\mathfrak{a}^*_{\mathbb{C}})$. In particular, we need to find $_lh$ of the form $_lh = Q\,{}_lg$. Moreover, the equations $P\,{}_lh = P({}_lu - {}_lv_1)$, then become

$$PQ\,{}_lg = PQ\,{}_lf.$$

Of these d_γ equations, the first satisfied automatically, in view of the definition of Q_1. Hence, we only have $d_\gamma - 1$ equations, so by induction hypothesis, there exists $_lg \in {}_lPWS_\delta(\mathfrak{a}^*_{\mathbb{C}})$ so that with some constants M' and C':

$$\begin{aligned} ||\,{}_lg||_{r,N+M'} \leq C'_{r,N+M'}||PQ\,{}_lf||_{r,N} &\overset{(5.10)}{=} C'_{r,N+M'}||P({}_lu - {}_lv_1)||_{r,N} \\ &\leq C''_{r,N+M'}||P\,{}_lu||_{r,N}\;||P\,{}_lv_1||_{r,N} \\ &\overset{(5.9)}{\leq} C''_{r,N+M'}||P\,{}_lu||_{r,N}C(1+|\lambda|^2)^{M+d_1}||P\,{}_lu||_{r,N}, \end{aligned}$$

where d_1 stands for the upper bound for the degree of the polynomial P. Hence

$$||\,{}_lg||_{r,N+M'+M+d_1} \leq C_{r,N+M'+M}||P\,{}_lu||_{r,N}. \tag{5.11}$$

Therefore, we obtain with ${}_lv := {}_lv_1 + Q\,{}_lg$

$$\begin{aligned} ||\,{}_lv||_{r,N+M} = ||\,{}_lv_1 + {}_lh||_{r,N+M} &\leq ||\,{}_lv_1||_{r,N+M}\ ||\,{}_lh||_{r,N+M} \\ &\overset{(5.9)}{\leq} C||P\,{}_lu||_{r,N}\ ||Q\,{}_lg||_{r,N+M} \\ &\overset{(5.11)}{\leq} C||P\,{}_lu||_{r,N}\ C_{r,N+M'+M}(1+|\lambda|^2)^{M'+d_1+d_2}||P\,{}_lu||_{r,N}, \end{aligned}$$

where d_2 denotes the upper bound for the degree of the polynomial Q. Thus, by taking $(1+|\lambda|^2)^{M'+d_1+d_2}$ on the left hand side, we get the desired estimate

$$||\,{}_lv||_{r,N+M+M'+d_1+d_2} \leq C'_{r,N+M'+M}||P\,{}_lu||_{r,N}.$$

Forbye, $P\,{}_lv = P\,{}_lv_1 + PQ\,{}_lg = P\,{}_lv_1 + P({}_lu - {}_lv_1) = P\,{}_lu$, hence this finally completes the proof. □

Now by applying Thm. 4.10 and Cor. 4.11, we obtain the (local) solvability in $\mathbb{H}^2$.

5.2 Solvability in $\mathbb{H}^2 \times \mathbb{H}^2$

Let $G := G' \times G' = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and $K := K' \times K'$ with $K' = SO(2)$ the maximal compact subgroup of G' as in Sect. 3.5. We consider three, not necessary irreducible, K'-representations $(\delta', E_{\delta'}), (\tau', E_{\tau'})$ and $(\gamma', E_{\gamma'})$ as in Sect. 5.1 and write

$$(\delta = \delta'\boxtimes\delta', E_\delta = E_{\delta'}\otimes E_{\delta'}),\quad (\tau = \tau'\boxtimes\tau', E_\tau = E_{\tau'}\otimes E_{\tau'}),\quad (\gamma = \gamma'\boxtimes\gamma', E_\gamma = E_{\gamma'}\otimes E_{\gamma'})$$

the K-representations with their associated homogeneous vector bundles $\mathbb{E}_\delta, \mathbb{E}_\tau$ and $\mathbb{E}_\gamma$ over X. Analogously as in Sect. 5.1, we identify $\mathbb{H}^2 \times \mathbb{H}^2$ with the symmetric space of non-compact type $X = G'/K' \times G'/K'$.

Consider now an additional irreducible K-type (l, E_l), then, Hyp. 1 is proved by the following theorem.

Theorem 5.4. *With the notations above, the short sequence*

$$_lPWS_{\delta,H}(\mathfrak{a}^*_\mathbb{C} \times \mathfrak{a}^*_\mathbb{C}) \xrightarrow{Q} {}_lPWS_{\tau,H}(\mathfrak{a}^*_\mathbb{C} \times \mathfrak{a}^*_\mathbb{C}) \xrightarrow{P} {}_lPWS_{\gamma,H}(\mathfrak{a}^*_\mathbb{C} \times \mathfrak{a}^*_\mathbb{C}) \qquad (5.12)$$

is exact in the middle, that means that $\mathrm{Im}(Q) = \mathrm{Ker}(P)$.

Proof. By applying Thm. 3.25:

$$_lPWS_{*,H}(\mathfrak{a}^*_\mathbb{C} \times \mathfrak{a}^*_\mathbb{C}) \cong \mathrm{Hol}(\lambda_1^2, \lambda_2^2) \otimes_{\mathrm{Pol}(\lambda_1^2,\lambda_2^2)} {}_lPWS_{*,0}(\mathfrak{a}^*_\mathbb{C} \times \mathfrak{a}^*_\mathbb{C}),\quad * = \delta, \tau, \gamma$$

the proof is analogous to the proof of Thm. 5.1, except that here $\mathbb{C}[\lambda_1^2, \lambda_2^2]$ is not a principal ideal domain, since the ideal $\langle\lambda_1^2, \lambda_2^2\rangle$ is not principal. However, by using Hörmander's result ([Hör73], Lem. 7.6.4), $\mathrm{Hol}(\lambda_1^2, \lambda_2^2)$ is a flat module over $\mathrm{Pol}(\lambda_1^2, \lambda_2^2)$. Hence, we obtain the exactness in the middle of (5.12). □

Concerning the estimate result in (Level 3) (Hyp. 2), the proof is given by the following theorem. Note that, here, $M := M' \times M' = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Theorem 5.5 (Estimate result in (Level 3)). *Fix $l = (l_1, l_2) \in \mathbb{Z}^2$. Let $P \in {}_{\tau}PWS_{\gamma,0}(\mathfrak{a}^*_{\mathbb{C}} \times \mathfrak{a}^*_{\mathbb{C}})$ be of the form*

$$P(\lambda_1, \lambda_2) := a(\lambda_1, \lambda_2) \cdot q_{\tau,\gamma}(\lambda_1, \lambda_2), \quad (\lambda_1, \lambda_2) \in \mathfrak{a}^*_{\mathbb{C}} \times \mathfrak{a}^*_{\mathbb{C}}, \tag{5.13}$$

where the polynomial $a \in \mathrm{Hol}(\lambda_1^2, \lambda_2^2)$ is the symmetric function and $q_{\tau,\gamma}$ a polynomial defined in (3.22).
Then, there exist constants $M \in \mathbb{N}_0$, for all $r \geq 0$, $N \in \mathbb{N}_0$, and $C_{r,N+M,l} > 0$ so that for each ${}_lu \in {}_lPWS_{\tau,H}(\mathfrak{a}^*_{\mathbb{C}} \times \mathfrak{a}^*_{\mathbb{C}})$ such that

$$||P\,{}_lu||_{r,N} := \sup_{(\lambda_1,\lambda_2)\in\mathfrak{a}^*_{\mathbb{C}}\times\mathfrak{a}^*_{\mathbb{C}}} \Big\{(1 + |\lambda_1|^2 + |\lambda_2|^2)^{-N} e^{-r|\mathrm{Re}(\lambda_1,\lambda_2)|} ||\,{}_l(Pu)(\lambda_1, \lambda_2)||_{op}\Big\} < \infty$$

one can find ${}_lv \in {}_lPWS_{\tau}(\mathfrak{a}^*_{\mathbb{C}} \times \mathfrak{a}^*_{\mathbb{C}})$ with

(i) $P\,{}_lu = P\,{}_lv$ and

(ii) $||\,{}_lv||_{r,N+M} \leq C_{r,N+M,l}||P\,{}_lu||_{r,N}$.

The constant $C_{r,N+M,l}$ depends of $l \in \mathbb{Z}^2$.

Before we start to show Thm. 5.5, we need to adapt Lem. 5.3 for our situation. Consider a polynomial p in two variables $z = (z_1, z_2) \in \mathfrak{a}^*_{\mathbb{C}} \times \mathfrak{a}^*_{\mathbb{C}}$, it can be decomposed into homogeneous components: $p = p_k + p_{k-1} + \dots p_0 = \sum_{l=0}^{k} p_k$, where $p_k \neq 0$. For all $s \in \mathbb{C}\backslash\{0\}$, we have

$$p_l(s \cdot z) = s^l p_l(z), \quad \forall l = 0, \dots, k. \tag{5.14}$$

If $p_k \neq 0$, then there exists $\tilde{v} \in \mathbb{C}^2\backslash\{0\}$ such that $p_k(\tilde{v}) \neq 0$. Set $v := \frac{\tilde{v}}{|\tilde{v}|}$ with $|v| = 1$. Then, $p_k(v) \overset{(5.14)}{=} |\tilde{v}|^{-k} p_k(\tilde{v}) \neq 0$. Fix now $\lambda \in \mathfrak{a}^*_{\mathbb{C}} \times \mathfrak{a}^*_{\mathbb{C}}$ and let $p_\lambda : \mathbb{C} \to \mathbb{C}$ be defined by

$$p_\lambda(z) := p(\lambda + z \cdot v).$$

Lemma 5.6. *With the previous notations, p_λ is a polynomial of degree k with largest coefficients $a_k = a_{k,\lambda} \neq 0$ and independent of $\lambda \in \mathfrak{a}^*_{\mathbb{C}} \times \mathfrak{a}^*_{\mathbb{C}}$. In fact, $a_k = p_k(v)$.*

Proof. We have

$$\begin{aligned} a_k = \lim_{z\to\infty} z^{-k} p_\lambda(z) = \lim_{z\to\infty} z^{-k} \sum_{l=0}^{k} p_l(\lambda + z \cdot v) \quad &= \quad \lim_{z\to\infty} z^{-k} \sum_{l=0}^{k} p_l\Big(z\Big(\frac{\lambda}{z} + v\Big)\Big) \\ &\overset{(5.14)}{=} \quad \lim_{z\to\infty} \sum_{l=0}^{k} z^{l-k} p_l\Big(\frac{\lambda}{z} + v\Big) \\ &= \quad p_k(v). \end{aligned}$$ ☐

Set $f_\lambda(z) := f(\lambda + z \cdot v)$. By Lem. 5.3, we get $|f_\lambda(0)| \leq \frac{1}{|a_k|} \sup_{|z|=1} |p_\lambda(z) f_\lambda(z)|$, i.e.

$$|f(\lambda)| \leq \underbrace{\frac{1}{|a_k|}}_{=:C_p} \sup_{|z|=1} |p(\lambda + z \cdot v) f(\lambda + z \cdot v)|, \tag{5.15}$$

where the constant C_p depends on the highest homogeneous part of p. Back to the proof of Thm. 5.5.

Proof of Thm. 5.5. We wish to construct a function ${}_lv_j \in {}_lPWS_{n_j}(\mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*)$ of the form

$$ {}_lv_j(\lambda_1, \lambda_2) = {}_lh_j(\lambda_1, \lambda_2) \cdot q_{l,n_j}(\lambda_1, \lambda_2), \quad (\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*, \ \ \forall j = 1, \dots, d_\tau \tag{5.16} $$

such that $P_{ij}\, {}_lv_j = {}_lw_i$ and which can be estimate later one. Whereas

$$ {}_lw_i(\lambda_1, \lambda_2) = {}_lb_i(\lambda_1, \lambda_2) \cdot q_{l,m_i}(\lambda_1, \lambda_2), \quad (\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*, \forall i = 1, \dots, d_\gamma $$

is a given function and ${}_lh_j \in \mathrm{Hol}(\lambda_1^2, \lambda_2^2)$ (resp. ${}_lb_i$) is a symmetric holomorphic function with q_{l,n_j} (resp. q_{l,m_i}) a polynomial function defined in (3.17).
This build is done as follows. Assume that we already have found a ${}_lv_j \in {}_lPWS_{n_j}(\mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*)$ of the form (5.16), then we obtain

$$ \sum_{i=1}^{d_\gamma} \sum_{j=1}^{d_\tau} P_{ij}(\lambda_1, \lambda_2)\, {}_lv_j(\lambda_1, \lambda_2) = {}_lw_i(\lambda_1, \lambda_2) \overset{(3.20)}{\Longleftrightarrow} \sum_{i=1}^{d_\gamma} \sum_{j=1}^{d_\tau} {}_l\tilde{a}_{ij}(\lambda_1, \lambda_2)\, {}_lh_j(\lambda_1, \lambda_2) = {}_lb_i(\lambda_1, \lambda_2), $$

where ${}_l\tilde{a}_{ij}(\lambda_1, \lambda_2) := a_{ij}(\lambda_1, \lambda_2) r^l_{n_j,m_i}(\lambda_1, \lambda_2)$, $\forall i, j$, is a polynomial in two variables. Hörmander's estimation result ([Hör73], Thm. 7.6.11) tells us that for every ${}_lh_j \in \mathrm{Hol}(\lambda_1^2, \lambda_2^2)$ with $||\, {}_l\tilde{a}_{ij}\, {}_lh_j||_{r,\tilde{N}} < \infty$, for some $\tilde{N} \in \mathbb{N}_0$, $r \geq 0$, one can find ${}_l\tilde{h}_j \in \mathrm{Hol}(\lambda_1^2, \lambda_2^2)$ such that

$$ {}_l\tilde{a}_{ij}\, {}_l\tilde{h}_j = {}_l\tilde{a}_{ij}\, {}_lh_j = {}_lb_i, \quad \forall i, j $$

and there exist constants $\tilde{M} \in \mathbb{N}_0$ and $C_{r,\tilde{N}+\tilde{M}} \in \mathbb{N}_0$ such that

$$ ||\, {}_l\tilde{h}_j||_{r,\tilde{N}+\tilde{M}} \leq C_{r,\tilde{N}+\tilde{M}} ||\, {}_lb_i||_{r,\tilde{N}}, $$

where the constants $C_{r,\tilde{N}+\tilde{M}}$ and $\tilde{M}$ are independent of the integer $l \in \mathbb{Z}^2$. Moreover, by setting

$$ {}_l\tilde{h}'_j(\lambda_1, \lambda_2) := \frac{1}{4}({}_l\tilde{h}_j(\lambda_1, \lambda_2) + {}_l\tilde{h}_j(-\lambda_1, \lambda_2) + {}_l\tilde{h}_j(\lambda_1, -\lambda_2) + {}_l\tilde{h}_j(-\lambda_1, -\lambda_2)), $$

for $(\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*$, we have that ${}_l\tilde{h}'_j(\lambda_1, \lambda_2)$ is a holomorphic and symmetric function on $\mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*$. Note that ${}_l\tilde{h}'_j$ is still a solution of $\sum_{i=1}^{d_\gamma} \sum_{j=1}^{d_\tau} {}_l\tilde{a}_{ij}\, {}_l\tilde{h}'_j = {}_lb_i$, since the coefficients of P_{ij} are even functions. Thus, we have

$$ ||\, {}_l\tilde{h}'_j||_{r,\tilde{N}'+\tilde{M}'} \leq C'_{r,\tilde{N}'+\tilde{M}'} ||\, {}_lb_i||_{r,\tilde{N}}. $$

By using Lem. 5.6, in particular (5.15), we have

$$ ||\, {}_l\tilde{h}'_j||_{r,\tilde{N}'+\tilde{M}'} \leq C'_{r,q_{l,m_i},\tilde{N}'+\tilde{M}'} ||\, {}_lb_i q_{l,m_i}||_{r,\tilde{N}} = C'_{r,q_{l,m_i},\tilde{N}'+\tilde{M}'} ||\, {}_lw_i||_{r,\tilde{N}}. \tag{5.17} $$

Here, the constant $\tilde{M}'$ depends on the degree of q_{l,m_i}. Therefore, we can deduce the existence of a new function ${}_l\tilde{v}_j \in {}_lPWS_{n_j}(\mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*)$ of the form

$$ {}_l\tilde{v}_j(\lambda_1, \lambda_2) = {}_l\tilde{h}'_j(\lambda_1, \lambda_2) \cdot q_{l,n_j}(\lambda_1, \lambda_2), \quad (\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^*, \forall j $$

such that $P_{ij}\, {}_l\tilde{v}_j = {}_lw_i$ and with estimate

$$ \begin{aligned} ||\, {}_l\tilde{v}||_{r,N+M} &\leq \max_{j=1,\dots,d_\tau} ||\, {}_l\tilde{v}_j||_{r,N+M} \\ &= \max_{j=1,\dots,d_\tau} ||\, {}_l\tilde{h}'_j q_{l,n_j}||_{r,N+M} \\ &\overset{(5.17)}{\leq} \max_{i=1,\dots,d_\gamma} C_{r,q_{l,m_i},N+M} (1 + |\lambda_1|^2 + |\lambda_2|^2)^{l d_2} ||\, {}_lw_i||_{r,N}, \end{aligned} $$

where in the last inequality, we estimate for each j, the the polynomial q_{l,n_j} by

$$\sup_{\lambda\in\mathfrak{a}_{\mathbb{C}}^*\times\mathfrak{a}_{\mathbb{C}}^*} |q_{l,n_j}(\lambda_1,\lambda_2)| \leq C(1+|\lambda_1|^2+|\lambda_2|^2)^{{}_ld_2},$$

for some integer ${}_ld_2$. Hence $||{}_l\tilde{v}||_{r,N+M+{}_ld_2} \leq C'_{r,q_{l,m},N+M}||{}_lw||_{r,N}$, where the constants $C'_{r,q_{l,m},N+M}$ and M depend on the integer $l\in\mathbb{Z}^2$. □

Since we are dealing with functions depending on two variables, it turns out to be a complicate task to achieve that the constants $C_{r,N+M,l}$ in (ii) in Thm. 5.5 and M do not depend too much on the K-types l. Thus, we can not prove the solvability in $\mathbb{H}^2\times\mathbb{H}^2$ yet, since Hyp. 3 is not fullfilled. However, we can prove the solvability for $K\times K$-finite elements by adapting the arguments of Chap. 4. In particular, only finite series are involved. Note that this study can be generalized for

$$SL(2,\mathbb{R})^d = \underbrace{SL(2,\mathbb{R})\times\cdots\times SL(2,\mathbb{R})}_{d \text{ times}}, \quad d\geq 2.$$

5.3 Solvability on hyperbolic 3-space

With the notations introduced in Sect. 3.6, let $G = SL(2,\mathbb{C})$ and $K = SU(2)$ its maximal compact subgroup. The quotient $X = G/K$ can be identify as a model of a hyperbolic 3-space $\mathbb{H}^3$. Using Cor. 3.36, we can prove Hyp. 1, by the following theorem.

Theorem 5.7. *We consider the three, not necessary irreducible, K-representations (δ,E_δ), (τ,E_τ) and (γ,E_γ). Fix an irreducible K-type (l,E_l) from the left. Then, the short sequence*

$${}_l\mathcal{A}_\delta(\mathfrak{a}_{\mathbb{C}}^*) \xrightarrow{Q} {}_l\mathcal{A}_\tau(\mathfrak{a}_{\mathbb{C}}^*) \xrightarrow{P} {}_l\mathcal{A}_\gamma(\mathfrak{a}_{\mathbb{C}}^*)$$

is exact in the middle, that means that $\mathrm{Im}(Q)=\mathrm{Ker}(P)$.

Proof. We argue with the same arguments as in the proof of Thm. 5.1, except that we use Cor. 3.36 instead of Thm. 3.20, i.e.

$${}_l\mathcal{A}_*(\mathfrak{a}_{\mathbb{C}}^*) \cong \mathrm{Hol}(\lambda^2)\otimes_{\mathrm{Pol}(\lambda^2)} {}_l\mathrm{Pol}_*(\mathfrak{a}_{\mathbb{C}}^*), \quad *=\delta,\gamma,\tau.$$ □

By Thm. 5.7 and Thm. 3.31, we proved that the equation $Qf = g$ is solvable in ${}_lPWS_{\delta,H}(\mathfrak{a}_{\mathbb{C}}^*)$, for given $g\in {}_lPWS_{\tau,H}(\mathfrak{a}_{\mathbb{C}}^*)$ if, and only if, $Pg=0$ in ${}_lPWS_{\tau,H}(\mathfrak{a}_{\mathbb{C}}^*)$. However, for the complete proof of Conj. 1, we need the proof of Hyp. 2 and Hyp. 3, which are work in progress.

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