Distributed Velocity Controllers of the Individuals of Emerging Swarm Clusters

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Abstract— **This paper presents the distributed velocity-based control laws of $n \in \mathbb{N}$ individuals considered rigid bodies, which gives rise to swarm clusters in a partially known environment. The motion of the individuals is based on Reynold's rules of separation, alignment, and cohesion. If two individuals are in the detection range of each other, there is an attraction between the two for alignment. There is a short-range repulsion to avoid the inter-individual collision. A total potential function is developed using attractive and repulsive potential functions, representing general anisotropic swarms. The decentralized velocity-based controllers of the individuals, which gives rise to a gradient system, are derived from the total potential function. The effectiveness of the decentralized velocity-based controllers is validated through computer simulations carried out using the Mathematica software.**

I. Introduction

In robotics, the problem of controlling multiple autonomous robots such that they behave cooperatively in a cohesive manner is of current importance [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19]. The principle of swarming (see, for example, [4], [5], [15], [16], [20], [14]) is increasingly used to solve this problem because swarming induces self-organization and emergent patterns which allow members of the swarm to work or move about cooperatively and cohesively [21], [22], [23]. There are many opportunities for integrating the swarming principles into the industry as swarm formations frequently play an influential role in several disciplines such as robotics, computer science, surveillance, military, economics, biology, and industrial automation [24], [9], [10], [14]. Some applications or possible applications of the swarming principle are: the possible use of swarm robots for carrying out deep mining operations in hazardous environments [25], the use of swarm unmanned aerial vehicles (UAVs) in the monitoring of; air pollution caused by the gases released due to industries [26], large farms for precision agriculture [27], and exclusive economic zone (EEZ) [19], and using swarms of robots for cooperative transportation and geological surveys [28]. Swarms of robots can also be utilized to monitor defects in civil infrastructure by the construction industry. In the energy production industry, swarms of UAVs or unmanned ground vehicles (UGVs) are also used for monitoring power lines, oil and gas pipes, as it may be dangerous for a human to conduct an inspection.

Over the last twenty years, the attempts of the researchers to comprehend swarming can be categorized into two different modeling approaches: the *Eulerian* and the *Lagrangian* approaches [29], [30], [31], [32], [33], [14]. In the Eulerian approach, the swarm is considered a *continuum* described by its density in one-, two- or three-dimensional space. In the Lagrangian approach, the state (position, instantaneous velocity, and instantaneous acceleration) of each individual and its relationship with other individuals in the swarm is studied; it is an *individual-based* approach, in which the velocity and acceleration can be influenced by spatial coordinates of the individual. Comprehensive reviews of these approaches and their advantages and disadvantages can also be found in [34] and [35].

A fundamental problem in swarm robotics is to develop distributed local control laws of swarm individuals such that the individuals have a continuous path, and the individuals motion is only influenced by motion of the individuals in their neighborhood. In this paper, we want to develop the distributed velocity-based control laws of $n \in \mathbb{N}$ individuals in a partially known environment, which gives rise to swarm clusters. The development of swarm clusters or multiple sub swarms is of great importance for completing different tasks. For instance, exploring and exploiting different areas to decrease search time in search and rescue operations. We begin by developing a system representing multiple rigid bodies and describe its configuration in planar space. The motion of the rigid bodies is based on Reynold's rules [36], which are (1) collision avoidance with neighbors, (2) matching velocity of the neighbors, and (3) staying close to the neighbors. An individual will be stagnant if there is no other individual in its sensing range; that is, there is no interaction between that individual and any other individual in a given workplace. Since this current research involves the state (position and instantaneous velocity) space of each individual and its relationship with other individuals, a Lagrangian swarm model is developed for the rigid bodies. The swarm model is based on the hypothesis that swarming is an interplay between *long-range attraction* and *short-range repulsion* between the individuals which are in the sensing zone of its neighbours.

For individuals in the sensing range of the other individuals, attractive and repulsive potential functions that are part of a total potential function are formed using the artificial potential field technique. The decentralized velocity-based control laws are then derived from the total potential function, which gives rise to a gradient system.

The remainder of the paper is organized as follows: Section II gives a brief description of a two-dimensional swarm model. In Section III, the decentralized velocity-based

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controllers are derived for $n \in \mathbb{N}$ individuals from a total potential function, which is developed using attractive and repulsive potential functions. In Section IV, the simulation studies are presented, and the research is concluded with a brief on future undertakings in sections V.

II. A Two-Dimensional Swarm Model

Lets consider $n \in \mathbb{N}$ individuals as rigid bodies in a workspace. Let the position of the i^{th} individual at time $t \geq 0$ be $(x_i(t), y_i(t))$, for all $i \in \{1, 2, 3, \ldots, n\}$ with $(x_i(t_0), y_i(t_0)) = (x_{i0}, y_{i0})$ as the initial conditions.

Thus, $\mathbf{x}_i := (x_i, y_i) \in \mathbb{R}^2$ is the configuration vector for the i^{th} individual and $\mathbf{x} := (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n) \in \mathbb{R}^{2n}$ becomes the configuration vector for n individuals with the initial conditions vector denoted by $\mathbf{x_0} := (\mathbf{x}_1(0), \mathbf{x}_2(0), \mathbf{x}_3(0), \ldots, \mathbf{x}_n(0)) \in \mathbb{R}^{2n}$.

Definition 2.1: The i^{th} individual is a point mass residing in a disk with center (x_i, y_i) and radius $r_i > 0$. It is described as the set

$$B_i := \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - x_i)^2 + (z_2 - y_i)^2 \leq r_i^2\}. \quad (1)$$

The i^{th} individual has an omni-directional detecting sensor situated at (x_i, y_i) with detection range of r_d as shown in Figure 1.

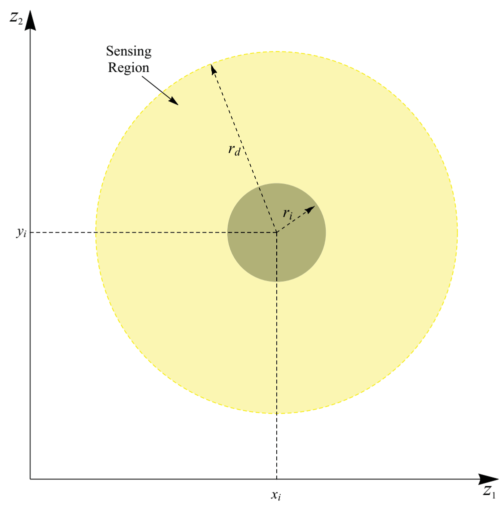


Fig. 1. The i^{th} individual with an omni-directional detecting sensor situated at (x_i, y_i) with detection range of r_d.

There will be communication between the i^{th} individual and j^{th} if and only if they are in the detection range of each other. This means that the behaviour of the i^{th} individual is influenced by its neighbours only. There is no communication between those individuals, which are not in the detection range of each other. Thus, the motion of those individuals will not be influenced by each other. At $t \geq 0$, let $(v_i(t), w_i(t)) := (x_i'(t), y_i'(t))$ be the instantaneous velocities of the i^{th} individual. Using the above notations, we have thus a system of first-order ODEs for the i^{th} individual, assuming the initial condition at $t = t_0 \geq 0$:

$$x_i'(t) = v_i(t),\ y_i'(t) = w_i(t),\ x_{i0} := x_i(t_0),\ y_{i0} := y_i(t_0). \quad (2)$$

Suppressing t, we let $\mathbf{x}_i := (x_i, y_i) \in \mathbb{R}^2$ be our state vectors. Also let $\mathbf{x}_0 = \mathbf{x}(t_0) = \underbrace{(x_{10}, y_{10}, x_{20}, y_{20}, \ldots, x_{n0}, y_{n0})}_{\text{2n terms}}$.

If the instantaneous velocity (v_i, w_i) has a state feedback law of the form, for $i \in \{1, 2, 3, \ldots, n\}$,

$$(v_i(t), w_i(t)) = (-\mu_i f_i(\mathbf{x}(t)), -\varphi_i g_i(\mathbf{x}(t))),$$

for some scalars $\mu_i, \varphi_i > 0$ and some functions $f_i(\mathbf{x}(t))$ and $g_i(\mathbf{x}(t))$, to be constructed appropriately later, and if we define $\mathbf{g}_i(\mathbf{x}) := (-\mu_i f_i(\mathbf{x}), -\varphi_i g_i(\mathbf{x})) \in \mathbb{R}^2$ and $\mathbf{G}(\mathbf{x}) := (\mathbf{g}_1(\mathbf{x}), \ldots, \mathbf{g}_n(\mathbf{x})) \in \mathbb{R}^{2n}$, then the swarm or sub-swarms of $m \leq n$ individuals is

$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}), \ \ \mathbf{x}(t_0) = \mathbf{x}_0. \quad (3)$$

III. Distributed Velocity Controllers of the Individuals

A. Components the Total Potential Function

In the total potential function to be proposed, the following potential functions will be included.

1) Long Range Attraction: To ensure there that the i^{th}, and j^{th} individuals which are neighbours and are in the detection range of each other converge to the centroid of the i^{th}, and j^{th} individuals, $j \neq i$, $i, j \in \{1, 2, 3, \ldots n\}$, a radically unbounded attraction potential function for the i^{th} individual is designed as follows

$$U_{i,j_{att}}(\mathbf{x}) := \frac{1}{8}\alpha_{i,j} q^2, \quad (4)$$

where $\alpha_{i,j} \geq 0$ is the strength of communication between the i^{th} and j^{th} individuals and could be regarded as convergence parameters, and $q = \|\mathbf{x}_i - \mathbf{x}_j\|$ is the distance between the i^{th}, and j^{th} individual at any arbitrary time. To ensure that the is an element of decentralised control, $\alpha_{i,j}$ is defined as

$$\alpha_{i,j} = \begin{cases} \lambda_{i,j}(r_d^2 - d_{i,j}^2)^2, & \text{if } d_{i,j} \leq r_d \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

in which $\lambda_{i,j} > 0$ and $d_{i,j} = q$. The above particular form of $\alpha_{i,j}$ indicates that the i^{th}, and j^{th} individual are navigating in a partially known environment and the it will ensure that the velocity-based controllers to be proposed are continuous. Note that $\dot{\alpha}_{i,j} = 0$.

2) Short Range Repulsion: To ensure that there is inter-agent collision avoidance between the i^{th} and the j^{th} individual which are neighbours and are in the detection range of each other, $j \neq i$, $i, j \in \{1, 2, 3, \ldots n\}$, we consider the function

$$Q_{i,j}(\mathbf{x}) = \frac{1}{2}\left[q^2 - (2r_i)^2\right]. \quad (6)$$

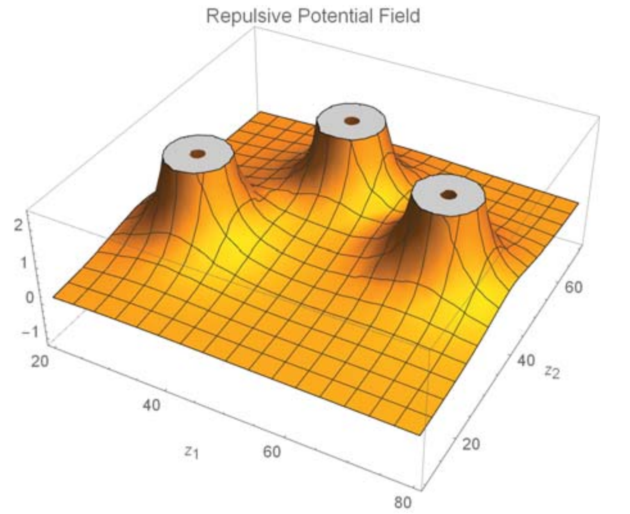
Thus, the repulsive potential field due to j^{th} individual on the i^{th} individual for $j \neq i$, $i, j \in \{1, 2, 3, \ldots n\}$ is given by

$$U_{i,j_{rep}}(\mathbf{x}) = \frac{\beta_{i,j}}{Q_{i,j}(\mathbf{x})} \quad (7)$$

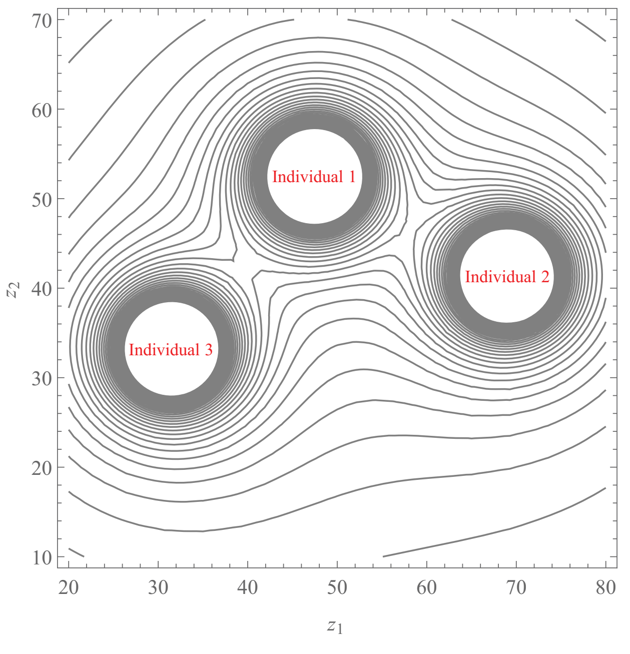
where $\beta_{i,j}$ gives the strength of communication between the i^{th} and j^{th} individuals to avoid collision and is defined as

$$\beta_{i,j} = \begin{cases} \gamma_{i,j}(r_d^2 - d_{i,j}^2)^2, & \text{if } d_{i,j} \leq r_d \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

in which $\gamma_{i,j} > 0$. The above particular form of $\beta_{i,j}$ indicates that the i^{th}, and j^{th} individual are navigating in a partially known environment and the it will ensure that the velocity-based controllers to be proposed are continuous. Note that $\dot{\beta}_{i,j} = 0$. An illustration of the total repulsive potentials for three randomly generated individuals for the function (7) is shown in Figure 2(a), while Figure 2(b) shows the corresponding contour plot generated over a workspace $20 < z_1 < 80$ and $10 < z_2 < 70$.



(a) 3D visualization



(b) Contour Plot

Fig. 2. The repulsive potential fields and the corresponding contour plot generated using the repulsive potential function governed by equation (7). For the parameters, $\gamma_{i,j}$ were randomized between 30 and 70.

B. A Total Potential Function

Using the attractive and repulsive potential together, a total potential function for the i^{th} individual for $i, j \in \{1, 2, 3, \ldots, n\}$ is

$$L_i(\mathbf{x}) = \sum_{\substack{j=1, \\ j \neq i}}^{n} \left(U_{i,j_{att}}(\mathbf{x}) + U_{i,j_{rep}}(\mathbf{x}) \right)$$

Consider a total potential function for the system (3),

$$L(\mathbf{x}) = \sum_{i=1}^{n} L_i(\mathbf{x}) \qquad (9)$$

In the Lyapunov-like function to be proposed, the following potential functions will be included.

Remark 3.1: **Isotropic and Anisotropic Swarm Systems** The total potential function (9) is that of an isotropic system if there is no restriction on the detection range and there is identical inter-individual communication strength that is $\alpha_{i,j} = \beta_{i,j} = 1$. An isotropic system was studied in [37]. However, if $\alpha_{i,j} = \beta_{i,j} \geq 0$ then (9) is the total potential of an anisotropic system. An anisotropic system was studied in [38].

Remark 3.2: **Reciprocal and Nonreciprocal Swarms** The total potential function (9) is that of a reciprocal swarm if there is no restriction on the detection range and the inter-individual communication strength between any two individuals are the same, that is, $\alpha_{i,j} = \alpha_{j,i}$ and $\beta_{i,j} = \beta_{j,i}$ for all i, j. A reciprocal swarm was analyzed in [37]. However, if there is no restriction on the detection range and the inter-individual communication strength between any two individuals are different, that is, $\alpha_{i,j} \neq \alpha_{j,i}$ and $\beta_{i,j} \neq \beta_{j,i}$ for all i, j, then (9) is the total potential of an nonreciprocal swarm. A nonreciprocal swarm was analyzed in [37].

C. Velocity Controllers

Along a trajectory of system (3),

$$\dot{L}(\mathbf{x}) = \nabla L(\mathbf{x}) = f(\mathbf{x})\dot{x} + g(\mathbf{x})\dot{y}. \qquad (10)$$

Let there be scalars $\mu_i > 0$ and $\varphi_i > 0$. Then the velocity controllers of system (3) are

$$\varpi_i = -\mu_i f_i(\mathbf{x}) \text{ and } \omega_i = -\varphi_i g_i(\mathbf{x}) \qquad (11)$$

where

$$f_i(\mathbf{x}) = \sum_{\substack{j=1, \\ j \neq i}}^{n} \left(\frac{\alpha_{i,j}}{2} - \frac{2U_{i,j_{rep}}(\mathbf{x})}{Q_{i,j}} \right) (x_i - x_j), \qquad (12)$$

and

$$g_i(\mathbf{x}) = \sum_{\substack{j=1, \\ j \neq i}}^{n} \left(\frac{\alpha_{i,j}}{2} - \frac{2U_{i,j_{rep}}(\mathbf{x})}{Q_{i,j}} \right) (y_i - y_j). \qquad (13)$$

IV. Simulation Results

Simulations were generated using Wolfram Mathematica 11.3 software. To achieve the desired results a number of sequential Mathematica commands were executed. We numerically simulated system (3) using RK4 method (Runge-Kutta Method). At $t = 0$, the initial positions $(x_{i0}(0), y_{i0}(0))$ were randomly generated.

Example 4.1: In this example, 20 point-mass rigid bodies is considered. Their initial positions at time $t = 0$ are shown in Fig. 3 using circles in colour red. The rigid bodies form three sub-swarm clusters as time evolves as shown in Fig. 3. As time evolves Fig. 4 shows that two of the three sub-swarm clusters join to form a new swarm cluster. As time evolves further, the two clusters that are shown in Fig. 4 join to form a swarm whose individuals are moving in circular motion that shows the natural phenomena of milling as shown in Fig. 5. Usually, natural swarms utilize the milling patterns to confuse its predators so that a particular individual is not made a target. The velocities of the individuals are shown in Figure 6. As time evolves it is evident that the velocities of the i^{th} individual matches the velocities of the individuals in its neighbourhood. For this example, $r_i = 1$, $r_d = 10$, $\mu_i = \varphi_i = 0.01$, $\lambda_{i,j}$ is randomized between 2 and 5, and $\gamma_{i,j}$ is randomized between 0.01 and 3.

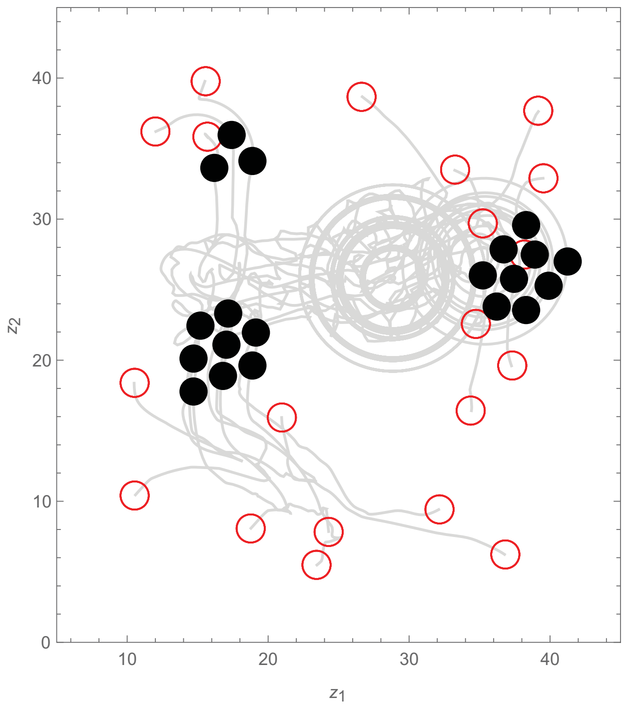


Fig. 3. The positions of the individuals at $t = 149$ shows three swarm clusters.

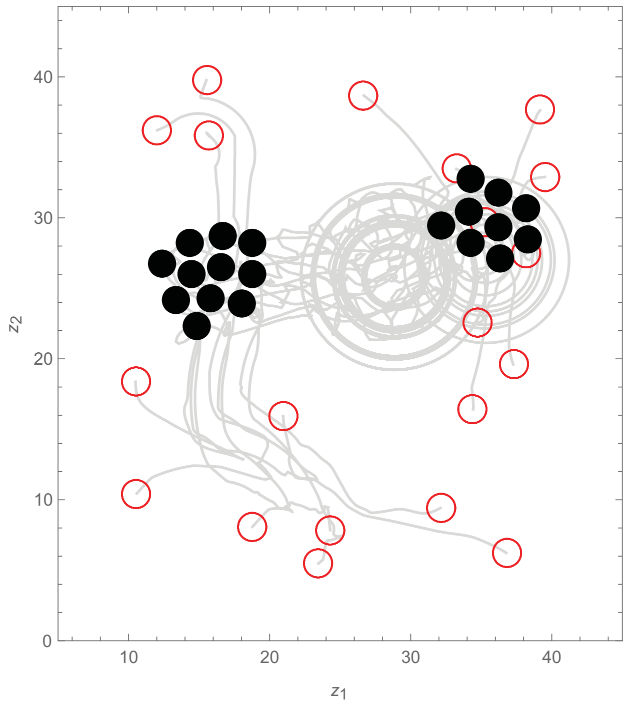


Fig. 4. The positions of the individuals at $t = 191$ shows two swarm clusters.

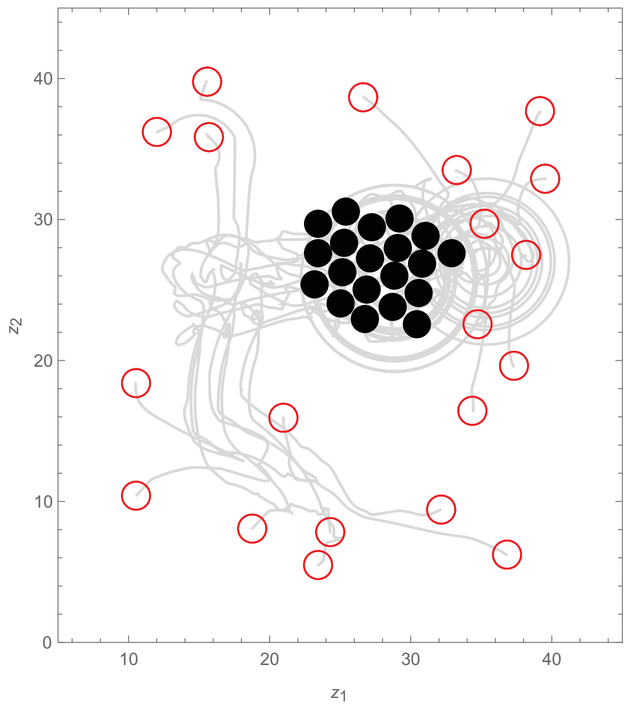


Fig. 5. The positions of the individuals at $t = 308$ shows that the swarm clusters have joined to form a single swarm.

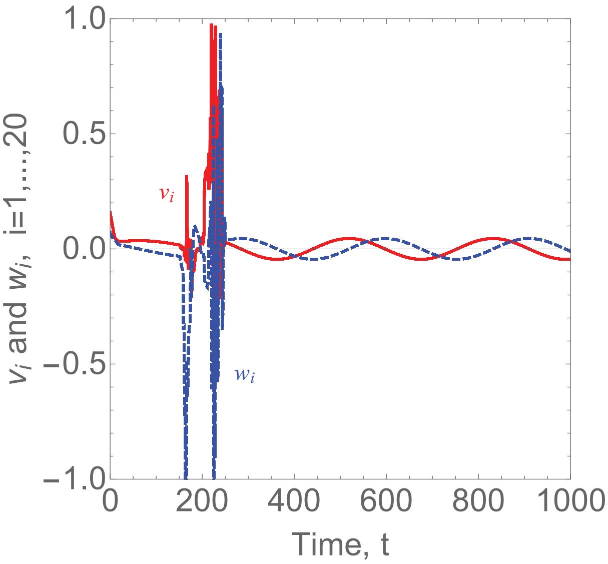


Fig. 6. The velocities of the individuals.

Example 4.2: In this example, 30 point-mass rigid bodies is considered. Their initial positions at time $t = 0$ are shown in Fig. 7 using circles in colour red. The rigid bodies form three sub-swarm clusters as time evolves initially as shown in Fig. 7. As time evolves Fig. 8 shows that two of the three sub-swarm clusters join to form a new swarm cluster. As time evolves further Fig. 9 shows that the two sub-swarm clusters join to form a bigger swarm. The evolution of the velocities of the individuals are similar to that shown in Fig. 6. For this example, $r_i = 1$, $r_d = 5$, $\mu_i = \varphi_i = 0.05$, $\lambda_{i,j} = 0.5$, and $\gamma_{i,j}$ is randomized between 0.1 and 0.5.

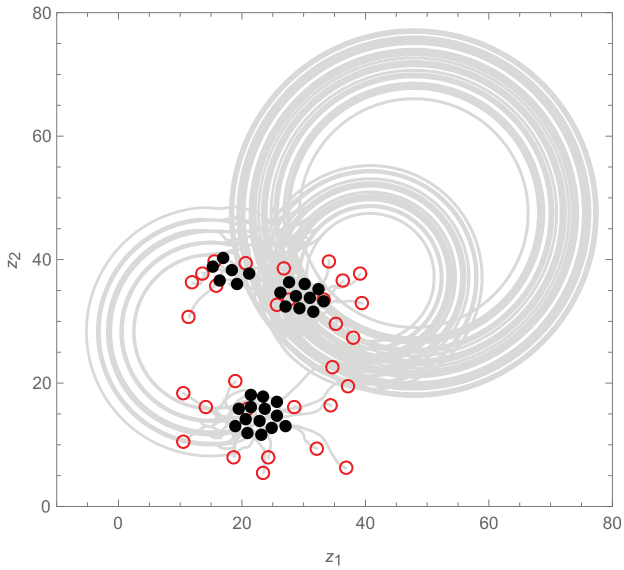


Fig. 7. The positions of the individuals at $t = 64$ shows three swarm clusters.

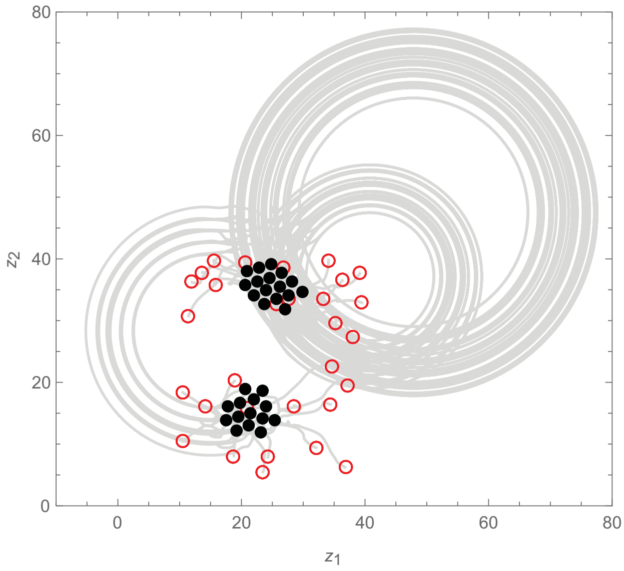


Fig. 8. The positions of the individuals at $t = 80$ shows two swarm clusters.

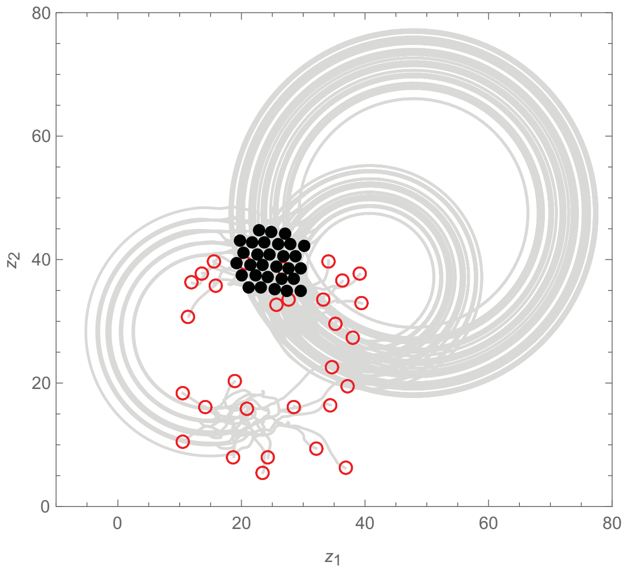


Fig. 9. The positions of the individuals at $t = 773$ shows that the swarm clusters have joined to form a single swarm.

V. Conclusion and Further Work

This paper presents the formulation of a total potential function suitable for anisotropic swarm clusters from attractive and repulsive potential functions. The distributed velocity-based control laws of $n \in \mathbb{N}$ individuals considered rigid bodies in a partially known environment are derived from the total potential function. Engaging computer simulations verified the control laws. The results here provide further scope for developing a swarm cluster system in which the clusters exhibit distinct tasks.

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