MODELING LONGITUDINAL BUNCHED BEAM DYNAMICS IN HADRON SYNCHROTRONS USING SCALED FOURIER-HERMITE EXPANSIONS

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Abstract

To devise control strategies and to analyze the stability of systems with feedback, a set of few ordinary differential equations (ODEs) describing the underlying dynamics is required. It is deduced by combining two approaches not used in that context before:
(I) Numerical Fourier-Hermite solutions of the Vlasov equation have been studied for over fifty years [1, 2]. The idea to expand the distribution function in Fourier series in space and Hermite functions in velocity is transferred to the dynamics of bunched beams in hadron synchrotrons in this contribution. The Hermite basis is a natural choice for plasmas with Maxwellian velocity profile as well as for particle beams with Gaussian momentum spread. The Fourier basis used for spatially nearly uniform plasmas has to be adapted to bunched beams where the beam profile is not uniform in phase. (II) This is achieved analogously to the deduction of the three term recursion relations to construct orthogonal polynomials, but applied to Fourier series with the weight function taken from the Hamiltonian. The resulting system of ODEs for the expansion coefficients of desired order - dependent on the number of functions retained - is roughly checked against macro particle tracking simulations.

BASICS

Longitudinal Bunched Beam Dynamics

A short summary of longitudinal beam dynamics in hadron synchrotrons is given to set up a common notation and to introduce the assumptions made. For a more detailed discussion see e.g. [3]. Coupled bunch oscillations are not considered. All derivation are done on the case below transition but should be equivalent above transition.

The dynamics of a single particle $m \in \{1, 2, \ldots, M\}$ with relative phase φ_m (reference Φ_R) and scaled energy deviation E_m are described by the Tracking equations

$$\frac{\mathrm{d}\varphi_m}{\mathrm{d}t} = a\,E_m \tag{1a}$$

$$\frac{\mathrm{d}E_m}{\mathrm{d}t} = b\,u(\varphi_m + \Phi_R) - b\,u(\Phi_R) \tag{1b}$$

with $a := \frac{h\omega_R^2\eta_R}{\beta_R^2\gamma_R m_0 c^2} < 0$, $b := \frac{Q\hat{U}}{2\pi}$ and $u(\theta) := \sin(\theta)$ for a single harmonic gap voltage. The synchrotron frequency is $\omega_{syn} = \sqrt{-ab\cos\Phi_R}$, the index R refers to the synchronous reference particle, h is the rf harmonic number, ω the angular revolution frequency, $\eta < 0$ the phase slip factor, β the speed of a particle over the speed of light c, γ is the ratio between the total energy of a particle and its rest mass m_0, Q the charge of a particle and $\hat{U}$ the amplitude of the rf voltage. Due to space restrictions and to keep the formulas succinct only the stationary case $\Phi_R = 0$ is considered in the following.

As the number of particles M is usually large the system can be approximated by a continuous distribution function $f(\varphi, E, t)$. If longitudinal and transverse oscillations are decoupled the system (1) of $2M$ ODEs can be replaced by one partial differential equation (PDE), the Vlasov equation, describing the evolution of f in phase space

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \dot{\varphi}\,\frac{\partial f}{\partial \varphi} + \dot{E}\,\frac{\partial f}{\partial E} \\ &= \frac{\partial f}{\partial t} - \frac{\partial H}{\partial E}\,\frac{\partial f}{\partial \varphi} + \frac{\partial H}{\partial \varphi}\,\frac{\partial f}{\partial E} = 0\end{aligned} \tag{2}$$

where the Hamiltonian is given by

$$H(\varphi, E) = -\tfrac{1}{2}a\,E^2 + b\,(1 - \cos\varphi) \tag{3}$$

with $-\frac{\partial H}{\partial E} = \dot{\varphi} = a\,E$ and $\frac{\partial H}{\partial \varphi} = \dot{E} = b\,\sin\varphi$.

The stationary distribution is a function of the Hamiltonian; for an adiabatically captured beam with an initial Gaussian momentum spread with variance $\frac{1}{-va}$ it follows

$$f(H) \propto \mathrm{e}^{-vH} \tag{4}$$

with v depending on the bunch size.

Three-Term Recurrence Relation

For any given positive weight function $w := w(x) \in L^1(c, d)$ there exists a unique set of monic orthogonal polynomials, that - apart from normalization which is omitted to enhance readability - can be constructed as follows [4]:

$$p_0 = 1,\; p_1 = x - \alpha_1 \;\text{ with } \alpha_1 = \int_c^d wx\,\mathrm{d}x \Big/ \int_c^d w\,\mathrm{d}x \;\text{ and}$$

$$\begin{aligned} p_{n+1} &= (x - \alpha_{n+1})p_n - \beta_{n+1}p_{n-1} \\ \text{with }\; \alpha_{n+1} &= \int_c^d wxp_n^2\mathrm{d}x \Big/ \int_c^d wp_n^2\mathrm{d}x \\ \text{and }\; \beta_{n+1} &= \int_c^d wxp_np_{n-1}\mathrm{d}x \Big/ \int_c^d wp_{n-1}^2\mathrm{d}x \;\text{ for } n \geq 1.\end{aligned} \tag{5}$$

FOURIER-HERMITE EXPANSION

Scaled Hermite functions were used by Schumer and Holloway [5, 6] to improve the convergence when dealing with filamentation.

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Series Expansion of the Distribution Function

The Vlasov Equation (2) can be rewritten in form of a countable systems of ordinary differential equations [7] - which can than be approximated by Galerkin's method - by means of a double series expansion of the distribution function

$$f(\varphi, E, t) = \sum_k \sum_l f^{kl}(t)\ \Phi_k(\varphi)\Psi_l(E) \tag{6}$$

$$f^{kl}(t) = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} f(\varphi, E, t)\ \Phi^k(\varphi)\Psi^l(E)\ \mathrm{d}\varphi\, \mathrm{d}E \tag{7}$$

$$= \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \sum_m \delta(\varphi - \varphi_m(t))\delta(E - E_m(t)) \dots$$

$$\dots \Phi^k(\varphi)\Psi^l(E)\ \mathrm{d}\varphi\, \mathrm{d}E$$

$$= \sum_m \Phi^k(\varphi_m(t))\Psi^l(E_m(t)) \tag{8}$$

where Φ^k, Ψ^l denote the weight and Φ_k, Ψ_l the basis functions. For high accuracy of low order models the unperturbed distribution should be given by $\Phi_0(\varphi)\Psi_0(E)$ such that $f^{kl}(t) \equiv 1$ for $k = l = 0$ and zeros elsewise.

Hermite Polynomials - In Energy

The probabilists' and the physicists' Hermite polynomials follow from the three-term recurrence relation (5) on the interval $(-\infty, \infty)$ with the weight functions $w(x) = \mathrm{e}^{-\frac{x^2}{2}}$ and $w(x) = \mathrm{e}^{-x^2}$ respectively. To match the expansion with the momentum spread scaled Hermite functions based on $w(E) = \mathrm{e}^{-\tilde{v}E^2}$ are used where $\tilde{v} := -\frac{av}{2}$. The polynomials are then given by $p_0 = 1$, $p_1 = E$, $p_{l+1} = Ep_l - \frac{l}{2\tilde{v}}p_{l-1}$ with the derivative $\frac{\mathrm{d}p_l}{\mathrm{d}E} = lp_{l-1}$.

Fourier Polynomials - In Phase

The idea the tree-term recursion relations are based on can also be applied to Fourier series with $w(\varphi) \propto \mathrm{e}^{-\tilde{v}\cos\varphi}$ taken from the Hamiltonian (3) or rather the stationary distribution (4) and $\tilde{v} := bv$. The recurrence relations to provide orthogonality fall into two parts - in the following called cosine and sine type - each with tree terms again. From $c_0 = 1$, $c_1 = \cos\varphi - \hat{\alpha}_1$, $c_{k+1} = \cos\varphi c_k - \hat{\alpha}_{k+1}c_k - \hat{\beta}_{k+1}c_{k-1}$ for the cosine type polynomials it follows

$$\sin\varphi\ c_k = \hat{\mu}_k s_{k-1} + \hat{\nu}_k s_k + s_{k+1} \tag{9a}$$

$$\frac{\mathrm{d}c_k}{\mathrm{d}\varphi} = \hat{\kappa}_k s_{k-1} - k s_k \tag{9b}$$

whereas for the sine type polynomials $s_0 = 0$, $s_1 = \sin\varphi$, $s_{k+1} = \cos\varphi\ s_k - \tilde{\alpha}_{k+1}s_k - \tilde{\beta}_{k+1}s_{k-1}$ gives

$$\sin\varphi\ s_k = \tilde{\mu}_k c_{k-1} + \tilde{\nu}_k c_k - c_{k+1} \tag{10a}$$

$$\frac{\mathrm{d}s_k}{\mathrm{d}\varphi} = \tilde{\kappa}_k c_{k-1} + k c_k \tag{10b}$$

The bunch size dependent constants in the Equations (9) and (10) are dominated by modified Bessel function of the first kind and are plotted in Figure 1 as they account for the final model given in (11).

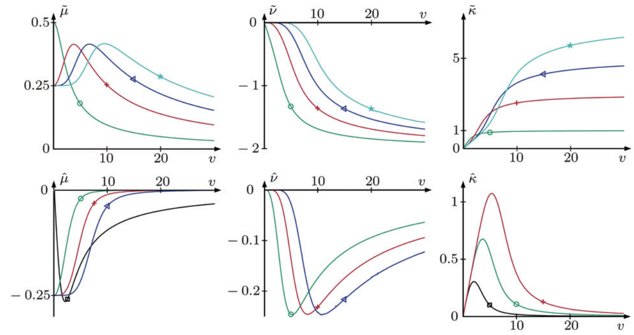

Figure 1: Recurrence coefficients $\tilde{\mu}_k$, $\hat{\mu}_k$, $\tilde{\nu}_k$, $\hat{\nu}_k$, $\tilde{\kappa}_k$ and $\hat{\kappa}_k$ appearing in (9) and (10) over v introduced in (4) and associated with the bunch size. The line styles refer to the indices k as follows: 0 - , 1 - , 2 - , 3 - , 4 - .

Basis and Weight Functions

An asymmetrically weighted representation is chosen such that the states f^{k0} are linear combinations of the measured Fourier coefficients of the beam current which is important for the applicability including the interaction with surrounding impedances. For simulation purposes symmetrically weighted ordinary Fourier-Hermite expansions may be preferable as the square integral of the distribution function was a constant of the motion. Conservation of this square integral provides numerical stability [5] which the physical energy does not as (the numerical approximation of) the distribution function is not necessarily positive [6].

Barring normalization the orthogonal weight and basis functions can now be defined as $\Psi^l(E) = p_l(E)$, $\Phi^{2k}(\varphi) = c_k(\varphi)$, $\Phi^{2k-1}(\varphi) = s_k(\varphi)$, $\Psi_l(E) = p_l(E)\,\mathrm{e}^{av\frac{E^2}{2}}$, $\Phi_{2k}(\varphi) = c_k(\varphi)\,\mathrm{e}^{-bv(1-\cos\varphi)}$ and $\Phi_{2k-1}(\varphi) = s_k(\varphi)\,\mathrm{e}^{-bv(1-\cos\varphi)}$. The calculation of the recurrence relations including the derivatives of these functions is straight forward.

Limits for Short and Long Bunches

The limits for infinite long bunches ($v = 0$) with $w(\varphi) = \frac{1}{2\pi}$ are the terms of an usual Fourier series

$$c_k(\varphi) = 2^{1-k}\cos(k\varphi) \text{ and } s_k(\varphi) = 2^{1-k}\sin(k\varphi).$$

The size dependent constants become $\tilde{\mu}_1 = \frac{1}{2}$, $\tilde{\mu}_k = \frac{1}{4}$ for $k \geq 2$, $\tilde{\nu}_k = 0$, $\tilde{\kappa}_k = 0$, $\hat{\mu}_k = -\frac{1}{4}$, $\hat{\nu}_k = 0$ and $\hat{\kappa}_k = 0$.

For infinitesimal short bunches ($v \to \infty$) with $w(\varphi) = \delta(\varphi)$ the polynomials converge to

$$c_k = (\cos\varphi - 1)^k \text{ and } s_k = (\cos\varphi - 1)^{k-1}\sin\varphi.$$

In this case $\tilde{\mu}_k = 0$, $\tilde{\nu}_k = -2$, $\tilde{\kappa}_k = 2k-1$, $\hat{\mu}_k = 0$, $\hat{\nu}_k = 0$ and $\hat{\kappa}_k = 0$. Apart from normalization p_l approaches E^l like in a moment approach.

System of ODEs

The requested set of ODEs can be accomplished

(a) taking the derivative of the states $f^{kl}(t)$ given in (7) but replacing the function $f(\varphi, E, t)$ by the sum over all

$$\dot{s}_{kl} = lb\left(\tilde{\mu}_k c_{(k-1)(l-1)} + \tilde{\nu}_k c_{k(l-1)} - c_{(k+1)(l-1)}\right) + a\left[\tilde{\kappa}_k\left(c_{(k-1)(l+1)} + \tfrac{l}{-2av} c_{(k-1)(l-1)}\right) + k\left(c_{k(l+1)} + \tfrac{l}{-2av} c_{k(l-1)}\right)\right] \tag{11a}$$

$$\dot{c}_{kl} = lb\left(\hat{\mu}_k s_{(k-1)(l-1)} + \hat{\nu}_k s_{k(l-1)} + s_{(k+1)(l-1)}\right) + a\left[\hat{\kappa}_k\left(s_{(k-1)(l+1)} + \tfrac{l}{-2av} s_{(k-1)(l-1)}\right) - k\left(s_{k(l+1)} + \tfrac{l}{-2av} s_{k(l-1)}\right)\right] \tag{11b}$$

particles represented by Dirac deltas resulting in a summation of the weight functions as shown in (8) thus allowing to use the tracking equations when taking the derivative or (b) by inserting the expansion of the distribution function (6) into the Vlasov Equation (2), multiplication with the weight functions followed by integration using the orthonormality.

The result - equations of motion for the Fourier components of the beam current - is obviously the same, a countable set of linear ODEs. With the states defined as $f^{(2k)l} = c_{kl}$ and $f^{(2k-1)l} = s_{kl}$ the system describing the dynamics of the expansion coefficients is given in (11) on the top of the page.

A finite dimensional system follows when closing the set of coupled ODEs by $f^{kl}(t) \equiv 0$ for $k + l > n$. The order of the system is $4 \cdot n!$ where n states have been eliminated due to conserved quantities. Other truncation schemes are also conceivable to cope with selected constants of motion to be reproduced by the model.

The system separates into two independent subsystems of equal size for the phases and amplitudes of the Fourier decomposition of the beam current or odd and even single bunch oscillation modes respectively. Its eigenvalues are mere imaginary with zero real part. For a single particle ($v \to \infty$) the frequencies are integer multiples of the synchrotron frequency. Nevertheless filamentation due to the synchrotron frequency spread is reproduced within a certain time range as energy can dissipate into higher frequencies for a while. Due to the finite order approximation of the infinite dimensional dynamics this transfer is limited as shown in Figure 2 by means of different approximation orders.

NUMERICAL EVALUATION

The dynamical system is compared to tracking simulations (cf. (1)) performed with $M = 10^4$ macro particles. With less particles, the low accuracy of the tracking simulations dominates the result of the comparison. For longer bunches even more macro particles are necessary. However the error in the presented model increases faster as well due to the larger frequency spread. No quantitative analysis is done as the objective of this work is a low order approximate description of bunched beam dynamics to obtain analytical as opposed to numerical results.

Figure 2 shows the phase of the first harmonic of the beam current for a dipole mode oscillation of a short bunch over the number of synchrotron periods $N = t\frac{\omega_{syn}}{2\pi}$. The first approximation ($n = 1$) is similar to the usually used harmonic oscillator but the frequency adapts to the bunch size as $\sqrt{\tilde{\kappa}_1}\omega_{syn}$. The phase space density reproduced by the last approximation ($n = 5$) is shown in Figure 3, where the synchrotron frequency spread shows up quite clearly, especially in the plots on the right where the stationary distribution was subtracted.

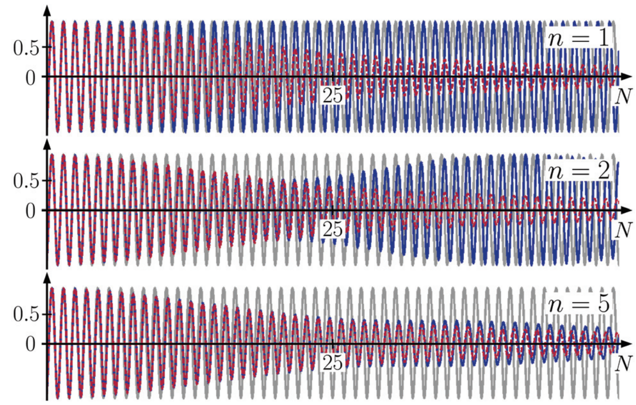

Figure 2: Phase of the first harmonic for different truncation orders n: harmonic oscillator - ⁄, presented model - ⁄, tracking simulations - ⁄.

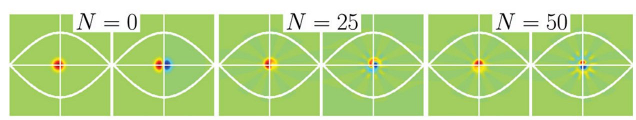

Figure 3: Approximation of the density distribution in phase space corresponding to the last plot in Figure 2.

OUTLOOK

Meanwhile a control input has been added to the homogeneous ODEs presented here resulting in a bilinear system. Thus the influence of beam loading compensation on longitudinal closed loop feedback algorithms has been shown.

Coupled bunch modes can possibly be included by Fourier analysis with respect to the revolution harmonic instead of the rf harmonic. Further analysis may follow.

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