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ADVANCED PROPERTIES OF SOME NONLOCAL OPERATORS

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Abstract

In this thesis[1], we deal with problems, related to nonlocal operators. In particular, we introduce a suitable notion of integral operators acting on functions with minimal requirements at infinity. We also present results of stability under the appropriate notion of convergence and compatibility results between polynomials of different orders. The theory is developed not only in the pointwise sense, but also in viscosity setting. Moreover, we discover the main properties of extremal type operators, with some applications. Then using the notion of viscosity solutions and Ishii-Lions technique, we give a different proof of the regularity of the solutions to equations involving fully nonlinear nonlocal operators. In the last part of the thesis we deal with domain variation solutions and with notions of a viscosity solution to two phase free boundary problem. We are looking at minima of energy functionals, the latter involving $p(x)$-Laplace operator or a non-negative matrix. Apart from the Riemannian case, we also consider the related Bernoulli functional in noncommutative framework. Finally, we formulate the suitable definition of a viscosity solution in Carnot groups.

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Introduction

Overview of the thesis

This thesis is mainly focused on the nonlocal integro-differential operators and some equations related to such operators. A classical line of investigation in mathematical analysis and mathematical physics consists in the study of integro-differential operators. The motivations for this stream of research come both from theoretical mathematics (such as harmonic analysis, singular integral theory, fractional calculus, etc.) and concrete problems in applied sciences (with questions related to water waves, crystal dislocations and the classical model, option pricing in finance, optimization, minimal surfaces, etc.): see e.g. the introduction in [20] and the references therein for a number of explicit motivations and examples.

A special focus of this stream of research deals with integro-differential operators of the form

$$Au(x) = \text{P.V.} \int_{\mathbb{R}^n} (u(x)-u(y))K(x,y)\,dy = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} (u(x)-u(y))K(x,y)\,dy. \tag{0.0.1}$$

The notation "P.V." above (which will be omitted in the rest of this paper for the sake of simplicity) means "in the principal value sense" and takes into account possible integral cancellations. The action of such operator is to "weight" the oscillations of the function u according to the kernel K. To make sense of the expression above, two types of assumptions need to be accounted for:

✓ if the kernel K is singular when $x = y$, the function u needs to be regular enough near the point x (to allow integral cancellations and take advantage of the principal value in (0.0.1)),

✓ the function u needs to be sufficiently well-behaved at infinity (namely, its growth has to be balanced by the kernel K to obtain in (0.0.1) a convergent integral at infinity).

Roughly speaking, these two conditions correspond to the request that the integral in (0.0.1) converges both in the vicinity of the given point x and at infinity. With respect to this, the regularity condition is necessary for the local convergence of the integral and it is common to differential (rather than integral) operators: in a sense, for differential problems the regularity of u ensures that incremental quotients converge to derivatives and, somewhat similarly, for integral problems the regularity of u allows the increment inside the integral to compensate the possible singularity of the kernel. Instead, the second assumption on the behavior of u at infinity is needed only to guarantee the tail convergence, it is a merely nonlocal feature and has no counterpart for the case of differential operators.

Conditions "at infinity" are also technically more difficult to deal with. First of all, they are more expensive to be computed, since they need to account for virtually all the values of the given function (while regularity ones deal with the values in an arbitrarily small region). Furthermore, these conditions are typically lost after one analyzes the problem at a small scale (since blow-up procedures alter the behavior of the solutions at infinity, with the aim of detecting the local patterns). Moreover, it is sometimes difficult to detect optimal assumptions for nonlocal problems even in very basic and fundamental questions (see e.g. the open problem after Theorem 3.2 in [45]), hence any theory based only on "essential" assumptions is doomed to have promising future developments.

It would be therefore very desirable to develop a theory of integral operators that does not heavily rely on the conditions at infinity (in spite of the striking fact that these conditions are needed even in the basic definition of the operator itself!). To this end, a theory of "fractional Laplacian operators up to polynomials" has been developed in [32, 33], i.e. when $K(x, y) = |x - y|^{-n-2s}$ for $s \in (0, 1)$, to address the case of functions with polynomial growth (see also [53] for related approaches). By B_R we will denote the open ball of radius R centered at zero and now we will explain the gist of this method. We consider the family of cut-offs

$$\chi_R(x) := \begin{cases} 1, & \text{if } x \in B_R; \\ 0, & \text{otherwise} \end{cases} \tag{0.0.2}$$

and apply the operator to the function $\chi_R u$. Of course, in general, it is not possible to send $R \to +\infty$, since the operator is not well-defined on u, nevertheless it is still possible to perform such an operation once an appropriate polynomial is "taken out" from the equation. Given the "rigidity" of the space of polynomials (which is finite dimensional and easily computable) the method is flexible and solid, it produces interesting results and can be efficiently combined with blow-up procedures, see [13, 1].

Furthermore, we discuss the nonlocal fully nonlinear equations of the form

$$Fu(x) = f(x), \tag{0.0.3}$$

while the right hand side in the expression above is assumed to be bounded. Equations of this type are widely spread in literature, consider for instance [18], where the authors adapted techniques, introduced in [15]. Since the operator F is nonlocal, it possesses the nonlocal features, i.e. long-range interactions. Moreover, this operator is characterized as a fully nonlinear one, by this we intend that F is nonlinear with respect to the highest order derivatives. Usually these derivatives are not given explicitly but represented by a differential quotient of second order.

It is not known a priori if it is possible to compute the operator in the classical sense since the considered function u can be not smooth enough. In order to be able to handle these problems, some useful instruments like weak solutions were established. In particular, viscosity theory, built by M.C. Crandall and P.-L. Lions ([25], [24]), as we know it now. The idea of the notion of viscosity solution is to "pass" derivatives to smooth test functions. This approach turned out to be extremely helpful, since the only condition for a potential candidate function, i.e. viscosity solution, is only to be continuous. Of course, one cannot have all the goods, and there is price to pay: any smooth function touching the solution from above or below must satisfy the corresponding differential inequality.

A special place in science occupy free boundary problems. This kind of problems describe in general a qualitative change of the medium, thus they represent a phase transition. The most typical example is a melting block of ice. A boundary formed from the interface of ice and water is called the free boundary and usually is unknown in advance so it must be found as part of the solution. Besides physics, free bouundary problems arise naturally in biology (switching a state of a cell from active to inactive), finance (procedures of buying and selling assets) and many others.

On the long term, we are interested in finding a suitable way to formulate one or two phase free boundary problem in case when nonlocal operators are involved. Unfortunately, we will not able to achieve the desirable result in this work but here we make first steps towards the main goal. In other words, we study the local variational case, taking inspiration from [3], looking at eventually degenerate operators.

We examine the celebrated paper [3], where the authors look for solutions to some problems dealing with nonlinear functionals, in the sense of the variation domain. More precisely, they find the Euler-Lagrange equations that govern the underlying problem associa-ted with the considered Bernoulli functional: the so called homogeneous elliptic two phase free boundary problem. The minima to such functional are endowed with few regularity properties like global Lipschitz continuity coming from an application of the monotonicity formula proved in [3].

We point out that very recently a new contribution following the mainstream of [3] appeared in [26]. In addition, we remark that a different approach about the inner regularity, that does not use any monotonicity formulas, has been discussed in [51] about the $p-$Laplace case.

In our opinion the approach described in [3] is highly not trivial and at the same time reveals some interesting details that are particularly useful for further generalizations.

In fact, as a consequence of the ideas contained in the previous seminal papers, there have been many other achievements about the viscosity solutions of two phase free boundary problems. From this point of view, talking about the regularity of the free boundary, we recall [37, 21, 41, 5], respectively dedica-ted to homogeneous fully nonlinear operators, homogeneous linear operators with variable coefficients, homogeneous linear operators with bounded first order terms and homogeneous fully nonlinear operators with flat boundaries.

Successively, after the fundamental contribution introduced in [27], many other inhomogeneous cases about two phase problems have been faced in [28, 29, 30]. The technique used in [27] turned out to be particularly flexible and has been extended to other one phase inhomogeneous cases that are not covered by previously cited papers yet, see the recent progress contained in [56, 55] in the variational one phase case.

Other important achievements concerning the viscosity approach to nonlinear degenerate operators, still adapting the ideas introduced in [27], can be found, for example, in [56].

In Chapter 1, we recall some basic facts and results about different types of nonlocal operators. Specifically, we will discuss the Marchaud derivative and the Riesz fractional derivative in Section 1.1. Then we concentrate on the fractional Laplace operator and its properties in 1.2, comparing similarities and differences with other fractional derivatives. Later, in Section 1.3, we will talk about the viscosity approach of solving differential equations and necessary tools for dealing with solutions which are not regular enough. Finally, preliminary notions of noncommutative framework are presented in Section 1.4.

In Chapter 2, we discuss the described "up to a polynomial" setting. Precisely, in the forthcoming Section 2.1 we face the core of this work introducing the main definitions for integral operators "up to a polynomial" and present their fundamental properties. The corresponding Dirichlet problem will be discussed in Section 2.2. While Sections 2.1 and 2.2 focus on the pointwise definition of this generalized notion of operators, Section 2.3 is devoted to the corresponding viscosity theory. Finally, in Section 2.4, we put in evidence the future possible developments of the introduced theory in a noncom-mutative framework that will be the goal of a forthcoming project.

In Chapter 3, we prove interior regularity of solutions to fully nonlinear nonlocal equations. First, we recall the extremal type operators in Section 3.1. Secondly, in Section 3.2 we examine the suitable notion of uniform ellipticity and then demonstrate the Lipschitzianity of solutions and their higher regularity, see Theorem 3.2.5 and Theorem 3.2.6, respectively.

In Chapter 4, we try to find the correct formulation of viscosity solution to a two phase free boundary problem in noncommutative groups, that we also mainly presented in [36]. We put in evidence that local operators may be understood as a particular case of nonlocal operators, as shown in [31] and (1.2.8). In Section 4.1 we recall the definition of a viscosity solution to two phase free boundary problem. Then, we look at the simplest one dimensional Euclidean case in Section 4.2 and discover the modulus of continuity of solutions in the Heisenberg group in Section 4.3. Later, the technique is applied to a functional containing a nonnegative matrix in Section 4.4, while Section 4.5 and Section 4.6 are dedicated to the study of the problem in Carnot groups and a nonlinear Euclidean case, respectively. We conclude the chapter with some outcomes and a summary in Section 4.7.

Notations

- $\mathbb{R}^n$ is the space of reference, $n \in \mathbb{N}$ is the dimention of the space.

- α is the multi-index $(\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbb{N}$ for $i = 1, \ldots, n$. We usually denote multi-indices by Greek letters unless otherwise specified.

- $B_R(x_0)$ is the n-dimentional open ball centered at x_0 of radius $R > 0$, i.e.
$$B_R(x_0) = \{x \in \mathbb{R}^n \ \text{ s.t. } \ |x - x_0| < R\}.$$
B_R is the ball $B_R(x_0)$ centered at $x_0 = 0$.

- $\mathbb{N}_0$ the set of natural numbers with zero, that is $\mathbb{N} \cup 0$.

- Ω^c is the complementary set $\mathbb{R}^n \setminus \Omega$ for a given $\Omega \subseteq \mathbb{R}^n$.

- $(fg)(x)$ is the short way to write $f(x)g(x)$.

- ω_{n-1} denotes the $(n-1)$-dimentional measure of the unit sphere S^{n-1}, namely
$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}. \tag{0.0.4}$$

- $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of smooth functions rapidly decaying at infinity with all their derivatives, precisely
$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \beta \in \mathbb{N}_0 \ \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_\beta f(x)| < +\infty \right\}.$$

- We use standard notations for the Fourier and inverse Fourier trasform. To be clear, we denote by $\xi \in \mathbb{R}^n$ the frequency variable and define the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ as
$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx.$$
Similarly, inverse Fourier transform is given by
$$\mathcal{F}^{-1}f(x) = \check{f}(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi) e^{ix \cdot \xi} d\xi.$$

- Given two colomn vectors $h, q \in \mathbb{R}^n$, we denote by $h \otimes q = hq^T$ their outer product, defined as a matrix. In index notation $(h \otimes q)_{ij} = (a)_{ij} = h_i q_j$.

- $\Gamma(z)$ stands for the Gamma function

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad \text{for any } z \in (0, +\infty).$$

- $\mathrm{B}(x, y)$ denotes the Beta function

$$\mathrm{B}(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

or an alternative integral representation

$$\mathrm{B}(x, y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt \tag{0.0.5}$$

for any positive x and y. Moreover, there exists a representation formula via Gamma function

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{0.0.6}$$

which can be sometimes extremely useful.

- By $F(a, b, c, z)$ we mean the Gauss hypergeometric function, given by the series

$$F(a, b, c, z) := 1 + \sum_{k=1}^\infty \frac{a \cdots (a+k-1) b \cdots (b+k-1)}{c \cdots (c+k-1)} \frac{z^k}{k!},$$

which is convergent for $|z| \le 1$.

- A function $u : \Omega \to (-\infty, \infty]$ is called lower semicontinuous and we denote it by $u \in \mathrm{LSC}(\Omega)$ if

$$\liminf_{y \to x} u(y) := \lim_{r \to 0} \inf_{y \in B_r(x) \setminus \{x\}} u(y) \ge u(x).$$

Analogously, we call a function $u : \Omega \to [-\infty, \infty)$ upper semicontinuous and we denote it by $u \in \mathrm{USC}(\Omega)$ if

$$\limsup_{y\to x} u(y) := \lim_{r\to 0} \sup_{y\in B_r(x)\setminus\{x\}} u(y) \le u(x).$$

Chapter 1

Preliminaries

1.1 Fractional derivatives

The appeared notion of differentiation immediately gave birth to a thought that there might be a way to define differentiation $\frac{d^s f(x)}{dx^s}$ of noninteger orders s. In the end of seventeenth century famous mathematicians like G. W. Leibniz and J. Bernoulli asked themselves whether it is possible to take a derivative of a function of order 1/2. In the sequel, L. Euler, P.-S. marquis de Laplace and then J.-B. J. Fourier proposed diverse ideas for fractional differentiation, among which various integral representations, like

$$\frac{d^s f(x)}{dx^s} = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \lambda^s d\lambda \int\limits_{-\infty}^{\infty} f(t) \cos\left(\lambda x - t\lambda + s\frac{\pi}{2}\right) dt.$$

A true pioneer N. H. Abel introduced, as it is clear to us now, the operation of fractional differentiation

$$\int\limits_a^x \frac{\varphi(t)dt}{(x-t)^s} = f(x) \quad \text{for } s \in (0,1) \text{ and } x > a,$$

thus beginning a new epoch of integro-differentiation. It was followed by a number of papers by J. Liouville, where he suggested a fractional integration formula for all functions $f(x) = \sum_{i=1}^{\infty} a_i e^{b_i x}$, representable by a series. Later, he suggested a definition of the fractional derivative to be a limit of a difference quotient. In the middle of nineteenth century G. F. B. Riemann published a work containing one of the main formulas of fractional integration since that time, given by

$$\frac{1}{\Gamma(s)}\int_0^x \frac{\varphi(t)}{(x-t)^{1-s}}dt$$

for any positive x. Later, Hj. Holmgren and A. K. Grünwald with A. V. Letnikov independently investigated and developed the introduced by Riemann definition of fractional integration, being

$$D^s f(x) = \lim_{y\to 0}\frac{(\Delta_y^s f)(x)}{y^s} := \lim_{y\to 0}\sum_{j=0}^{\infty}\frac{-s\Gamma(j-s)}{\Gamma(1-s)\Gamma(j+1)}f(x-jy)\frac{1}{y^s}.$$

A further development was made by N. Y. Sonin, who found an extension for the Cauchy formula for fractional powers in the second part of nineteenth century. At the end of twentieth century J. S. Hadamard brought new fresh ideas to fractional calculus, defining the differentiation in complex plane in terms of an analytic function's Taylor series. The exact definition is the following expression

$$D^s f(z) = \sum_{j=0}^{\infty} c_j \frac{\Gamma(j+1)}{\Gamma(j+1-s)}(z-z_0)^{j-s},$$

where $c_j := \frac{1}{j!}f^{(j)}(z_0)$. Soon after this, the new century began with a contribution of H. K. H. Weyl, which consisted in dealing with fractional integration for periodic functions, accomplished by a convolution with a special function. An important role in fractonal calculus history played a paper by A. Marchaud, where the author considered for the first time a new form of fractional differentiation.

Here we finish exploring briefly the historical part and address an interested reader to [60] for a meaningful historic outline about the fractional calculus.

We continue by recalling a basic fractional derivative called Marchaud fractional derivative (refer to [66]). Precisely, take $s \in (0,1)$ and consider functions $f \in L^\infty(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})$, where $s < \alpha \leq 1$. Thus left and right Machaud fractional derivative of order s are defined by

$$\mathbb{D}_{\pm}^s f(x) = \frac{s}{\Gamma(1-s)}\int_0^{\infty}\frac{f(x)-f(x\mp y)}{y^{1+s}}dy. \tag{1.1.1}$$

If $f \in L^\infty(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})$, then Marchaud fractional derivatives are well defined. Indeed, denoting $k(s) := \frac{s}{\Gamma(1-s)}$, we have that, for the left Marchaud fractional derivative,

$$\begin{aligned}&\left|k(s)\int\limits_0^\infty \frac{f(x)-f(x-y)}{y^{1+s}}dy\right|\\ &\leq \|f\|_{L^\infty(\mathbb{R})}k(s)\int\limits_0^\infty \frac{1}{y^{1+s}}dy \;+\; \|f\|_{C^{0,\alpha}}k(s)\int\limits_0^1 \frac{1}{y^{1+s-\alpha}} < +\infty.\end{aligned}$$

It is possible to extend the notion of Marchaud fractional derivatives for any $s \in \mathbb{R}_+$. Indeed, one recalls the following expression for left and right Marchaud fractional derivatives of order s:

$$\mathbb{D}^s_\pm f(x) := \frac{\{s\}}{\Gamma(1-\{s\})}\int\limits_0^\infty \frac{f^{[s]}(x)-f^{[s]}(x\mp y)}{y^{1+\{s\}}}dy, \tag{1.1.2}$$

where $[s]$ and $\{s\}$ denote the integer and fractional parts of s, respectively.

As a possible generalization of the Marchaud fractional derivative, one could consider the hypersingular integrals (see [52], chapter 2). For this, we first introduce a centered finite difference of a function $f(x)$ of order l with a step $y \in \mathbb{R}^n$ centered at the point $x \in \mathbb{R}^n$, defined as

$$\Delta^l_y f(x) = \sum_{j=0}^{l}(-1)^j\binom{l}{j} f\left(x+\left(\frac{l}{2}-j\right)y\right). \tag{1.1.3}$$

One could define the non-centered difference in a similar way, namely,

$$\Delta^l_y f(x) = \sum_{j=0}^{l}(-1)^j\binom{l}{j} f(x-jy).$$

In this work we deal only with centered differences to avoid complications in writing although it is possible also to consider non-centered differences as well.

The introduced notion of a finite difference is helpful when one defines

$$\mathbb{D}^s f(x) := \frac{1}{d_n(l,s)}\int\limits_{\mathbb{R}^n} \frac{\Delta^l_y f(x)}{|y|^{n+2s}}dy, \quad s \in \left(0, \frac{l}{2}\right), \tag{1.1.4}$$

which can be seen as a generalization to multiple dimensions of the Marchaud fractional derivative. The constant $d_n(l,s)$ is given by

$$d_n(l,s) := \int\limits_{\mathbb{R}^n} \frac{(e^{i\frac{\zeta_1}{2}} - e^{-i\frac{\zeta_1}{2}})^l}{|\zeta|^{n+2s}} d\zeta. \tag{1.1.5}$$

When the difference is centered, we assume its order l to be always an even number. This fact and why the hypersingular integrals are also called Riesz fractional derivatives of order $2s$ can be explained by the following lemma.

Lemma 1.1.1. *It holds that*

$$\mathbb{D}^s f(x) = \mathcal{F}^{-1}(|\xi|^{2s}\hat{f}(\xi)). \tag{1.1.6}$$

Proof. Taking the Riesz derivative (1.1.4) and applying the Fourier transform, we get

$$\begin{aligned} \mathcal{F}(\mathbb{D}^s f(x)) &= \frac{1}{d_n(l,s)} \sum_{j=0}^{l} (-1)^j \binom{l}{j} \int\limits_{\mathbb{R}^n} \frac{\mathcal{F}(f(x + (\frac{l}{2} - j)y))}{|y|^{n+2s}} dy \\ &= \frac{1}{d_n(l,s)} \sum_{j=0}^{l} (-1)^j \binom{l}{j} \hat{f}(\xi) \int\limits_{\mathbb{R}^n} \frac{e^{i\xi\cdot(j-\frac{l}{2})y}}{|y|^{n+2s}} dy. \end{aligned} \tag{1.1.7}$$

Notice that

$$(e^{-i\xi\cdot\frac{y}{2}} - e^{i\xi\cdot\frac{y}{2}})^l = \sum_{j=0}^{l} (-1)^j \binom{l}{j} e^{i\xi\cdot(j-\frac{l}{2})y},$$

and from (1.1.7) it yields

$$\mathcal{F}(\mathbb{D}^s f(x)) = \frac{1}{d_n(l,s)} \hat{f}(\xi) \int\limits_{\mathbb{R}^n} \frac{(e^{-i\xi\cdot\frac{y}{2}} - e^{i\xi\cdot\frac{y}{2}})^l}{|y|^{n+2s}} dy.$$

Now, we change the variable $z = |\xi| y$ and get

$$\mathcal{F}(\mathbb{D}^s f(x)) = \frac{1}{d_n(l,s)} |\xi|^{2s} \hat{f}(\xi) \int\limits_{\mathbb{R}^n} \frac{(e^{-i\frac{\xi}{|\xi|}\cdot\frac{z}{2}} - e^{i\frac{\xi}{|\xi|}\cdot\frac{z}{2}})^l}{|z|^{n+2s}} dz.$$

Since the integral in the right hand side is rotationally invariant with respect to ξ, we are allowed to consider a rotation $Je_1 := \xi/|\xi|$, which sends $e_1 = (1,0,\ldots,0)$ into $\xi/|\xi|$. Then, denoting by J^T the transpose of rotation and

$\zeta := J^T z$, we find

$$\begin{aligned}\mathcal{F}(\mathbb{D}^s f(x)) &= \frac{1}{d_n(l,s)}|\xi|^{2s}\hat{f}(\xi)\int\limits_{\mathbb{R}^n}\frac{(e^{-iJe_1\cdot\frac{z}{2}}-e^{iJe_1\cdot\frac{z}{2}})^l}{|z|^{n+2s}}dz\\ &= \frac{1}{d_n(l,s)}|\xi|^{2s}\hat{f}(\xi)\int\limits_{\mathbb{R}^n}\frac{(e^{-ie_1\cdot\frac{\zeta}{2}}-e^{ie_1\cdot\frac{\zeta}{2}})^l}{|z|^{n+2s}}dz\\ &= \frac{1}{d_n(l,s)}|\xi|^{2s}\hat{f}(\xi)\int\limits_{\mathbb{R}^n}\frac{(e^{-i\frac{\zeta_1}{2}}-e^{i\frac{\zeta_1}{2}})^l}{|\zeta|^{n+2s}}d\zeta\end{aligned} \tag{1.1.8}$$

Now we set

$$d_n(l,s) := \int\limits_{\mathbb{R}^n}\frac{(e^{-i\frac{\zeta_1}{2}}-e^{i\frac{\zeta_1}{2}})^l}{|\zeta|^{n+2s}}d\zeta \tag{1.1.9}$$

and check that $d_n(l,s)$ never turns out to be zero. Using the Euler's formula $e^{i\theta} = \cos\theta + \ i\sin\theta$, we rewrite the integral in (1.1.9) in the following form:

$$\begin{aligned}d_n(l,s) &= \int\limits_{\mathbb{R}^n}\left(-2i\sin\frac{\zeta_1}{2}\right)^l\frac{d\zeta}{|\zeta|^{n+2s}}\\ &= (-1)^l 2^{l-2s} i^l\int\limits_{\mathbb{R}^n}\sin^l\tilde{\zeta}\,\frac{d\tilde{\zeta}}{|\tilde{\zeta}|^{n+2s}}.\end{aligned} \tag{1.1.10}$$

Equality (1.1.10) shows that the integral vanishes when l is an odd number. Thus, taking finite differences of only even order, we see the validity of (1.1.6).

□

The explicit expression for $d_n(l,s)$ is known and we provide it here for the reader's convenience.

Lemma 1.1.2. *The constant $d_n(l,s)$ in (1.1.4) is explicitly given by*

$$d_n(l,s) = \frac{\pi^{1+n/2}\,k_l(2s)}{2^{2s}\Gamma(1+s)\Gamma(\frac{n}{2}+s)\sin(s\pi)}, \tag{1.1.11}$$

and $k_l(s)$ is

$$k_l(s) := 2\sum_{j=0}^{l/2}(-1)^{j-1}\binom{l}{j}\left(\frac{l}{2}-j\right)^s. \tag{1.1.12}$$

Proof. We start by splitting $d_n(l,s)$ in lower dimensional integrals, i.e.

$$\begin{aligned} d_n(l,s) &= \int_{\mathbb{R}} (e^{i\frac{\zeta_1}{2}} - e^{-i\frac{\zeta_1}{2}})^l d\zeta_1 \int_{\mathbb{R}^{n-1}} \frac{d\tilde{\zeta}}{(\zeta_1^2 + |\tilde{\zeta}|^2)^{\frac{n+2s}{2}}} \\ &= \int_{\mathbb{R}} (e^{i\frac{\zeta_1}{2}} - e^{-i\frac{\zeta_1}{2}})^l |\zeta_1|^{-1-2s} d\zeta_1 \int_{\mathbb{R}^{n-1}} \frac{d\omega}{(1+|\omega|^2)^{\frac{n+2s}{2}}}, \end{aligned} \tag{1.1.13}$$

where we used the change of variable $\omega = \tilde{\zeta}/\zeta_1$ in the last equality. The $(n-1)$-dimensional integral can be found by passing to polar coordinates,

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \frac{d\omega}{(1+|\omega|^2)^{\frac{n+2s}{2}}} &= \omega_{n-2} \int_0^\infty \frac{\tilde{r}^{n-2}}{(1+\tilde{r}^2)^{\frac{n+2s}{2}}} d\tilde{r} \\ &= \frac{\omega_{n-2}}{2} \int_0^\infty \frac{\tilde{r}^{\frac{n-3}{2}}}{(1+r)^{\frac{n+2s}{2}}} dr \\ &= \frac{\omega_{n-2}}{2} \,\mathrm{B}\left(\frac{n-1}{2}, \frac{1+2s}{2}\right), \end{aligned} \tag{1.1.14}$$

where in the last identity we used the definition of beta function in the integral form (0.0.5). Thus, recalling the explicit value of the measure of the sphere (0.0.4), we combine (1.1.13) with (1.1.14) to get

$$\begin{aligned} d_n(l,s) &= \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{n+2s}{2})} \int_{\mathbb{R}} \frac{(e^{i\frac{\zeta_1}{2}} - e^{-i\frac{\zeta_1}{2}})^l}{|\zeta_1|^{1+2s}} d\zeta_1 \\ &= \frac{2\pi^{\frac{n-1}{2}}\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{n+2s}{2})} \int_0^\infty \frac{(e^{i\frac{\zeta_1}{2}} - e^{-i\frac{\zeta_1}{2}})^l}{\zeta_1^{1+2s}} d\zeta_1 \\ &= \frac{2^{l+1} i^l \pi^{\frac{n-1}{2}}\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{n+2s}{2})} \int_0^\infty \frac{\sin^l \frac{\zeta_1}{2}}{\zeta_1^{1+2s}} d\zeta_1 \\ &= \frac{2^{l+1-2s} i^l \pi^{\frac{n-1}{2}}\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{n+2s}{2})} \int_0^\infty \frac{\sin^l \zeta}{\zeta^{1+2s}} d\zeta. \end{aligned} \tag{1.1.15}$$

It remains to simplify the integral, involving sine. To do this, we exploit the binomial theorem and obtain that

$$\sin^l \zeta = \frac{1}{(2i)^l} \sum_{k=0}^{l} \binom{l}{k} e^{i(l-2k)\zeta} = \frac{1}{(2i)^l} \sum_{k=0}^{l} \binom{l}{k} (\cos{(l-2k)\zeta} + i \sin{(l-2k)\zeta}).$$

Thanks to the identity for binomial coefficients

$$\sum_{j=0}^{l}(-1)^j\binom{l}{j}=0$$

and their symmetry, we get

$$\begin{aligned}
\sin^l\zeta &= \frac{1}{(2i)^l}\sum_{k=0}^{l}(-1)^k\binom{l}{k}\cos 2\left(\frac{l}{2}-k\right)\zeta \\
&= \frac{1}{(2i)^l}\sum_{j=-l/2}^{l/2}(-1)^{l/2-j}\binom{l}{l/2-j}\cos 2j\zeta \\
&= 2^{-l}\binom{l}{l/2}-2^{1-l}\sum_{j=1}^{l/2}(-1)^{j-1}\binom{l}{l/2-j}\cos 2j\zeta \\
&= 2^{1-l}\sum_{j=1}^{l/2}(-1)^{j-1}\binom{l}{l/2-j}(1-\cos 2j\zeta).
\end{aligned} \tag{1.1.16}$$

Therefore, we can integrate by parts the integral in (1.1.15) and get

$$\int_0^\infty \frac{\sin^l\zeta}{\zeta^{1+2s}}d\zeta = 2^{1-l}\sum_{j=0}^{l/2}(-1)^{j-1}\binom{l}{l/2-j}\frac{j}{s}\int_0^\infty \frac{\sin 2j\zeta}{\zeta^{2s}}d\zeta. \tag{1.1.17}$$

Now we claim that

$$\int_0^\infty \frac{\sin j\zeta}{\zeta^{2s}}d\zeta = \frac{\pi j^{2s-1}}{2\sin(\pi s)\Gamma(2s)}. \tag{1.1.18}$$

Indeed, the definition of gamma function yields

$$\begin{aligned}
\int_0^\infty \frac{\sin j\zeta}{\zeta^{2s}}d\zeta &= \frac{1}{2i}(-ij)^{2s-1}\Gamma(1-2s)-\frac{1}{2i}(ij)^{2s-1}\Gamma(1-2s) \\
&= -\frac{\pi}{\sin(2\pi s)\Gamma(2s)}\frac{j^{2s-1}(i^{2s-1}-(-i)^{2s-1})}{2i} \\
&= -\frac{\pi}{\sin(2\pi s)\Gamma(2s)}j^{2s-1}\sin\frac{\pi}{2}(2s-1) \\
&= \frac{\pi j^{2s-1}}{2\sin(\pi s)\Gamma(2s)},
\end{aligned} \tag{1.1.19}$$

which proves (1.1.18), as stated. Thus, (1.1.18) together with (1.1.17) give

$$\int_0^\infty \frac{\sin^l \zeta}{\zeta^{1+2s}} = \frac{2^{2s-l}\pi}{\sin(\pi s)\Gamma(1+2s)} \sum_{j=0}^{l/2} (-1)^{j-1} \binom{l}{l/2-j} j^{2s}. \tag{1.1.20}$$

Furthermore, we observe that

$$\sum_{j=1}^{l/2} (-1)^{j-1} \binom{l}{l/2-j} j^{2s} = \frac{1}{2}(-1)^{l/2} k_l(2s),$$

which produces

$$\int_0^\infty \frac{\sin^l \zeta}{\zeta^{1+2s}} = \frac{(-1)^{l/2} 2^{2s-l-1}\pi}{\sin(\pi s)\Gamma(1+2s)} k_l(2s). \tag{1.1.21}$$

Hence (1.1.11) immediately follows by inserting (1.1.21) into (1.1.15). □

1.2 Fractional Laplace operator

In this section we recall the basic notations and useful results related to the fractional Laplace operator and fractional Sobolev spaces. An interested reader can find more details on the topic in [12] or in [31]. The fractional Laplace operator of a function $u : \mathbb{R}^n \to \mathbb{R}$ can be defined in several ways (see [54]), here we give the one in principal value sense, that is, for any parameter $s \in (0, 1)$ we consider

$$\begin{aligned} (-\Delta)^s u(x) : &= c(n,s)\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \\ &= c(n,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \end{aligned} \tag{1.2.1}$$

where $c(n, s)$ is a positive dimentional constant, given by

$$c(n,s) := \frac{2^{2s}\Gamma(\frac{n}{2}+s)}{\pi^{n/2}|\Gamma(-s)|}. \tag{1.2.2}$$

Observe that there is a singularity in the integral above when y approaches x, and in this case one cannot expect to have integrability. To be the operator

(1.2.1) well posed, we take u in Schwartz space $\mathcal{S}(\mathbb{R}^n)$, so that

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}}dy &= \int_{B_1(x)} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}}dy + \int_{B_1^c(x)} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}}dy \\ &\leq C \int_{B_1(x)} \frac{|x-y|^2}{|x-y|^{n+2s}}dy + 2\|u\|_{L^\infty(\mathbb{R}^n)} \int_{B_1^c(x)} \frac{1}{|x-y|^{n+2s}}dy \\ &= C\int_0^1 \frac{1}{\rho^{2s}}d\rho + 2\|u\|_{L^\infty(\mathbb{R}^n)} \int_1^\infty \frac{1}{\rho^{1+2s}}d\rho < +\infty. \end{aligned} \tag{1.2.3}$$

It is worth to mention that definition (1.2.1) is well posed for less regular functions, namely $u \in L^1_s(\mathbb{R}^n)$ where the space

$$L^1_s(\mathbb{R}^n) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) \text{ s.t. } \int_{\mathbb{R}^n} \frac{|u(t)|}{1+|t|^{n+2s}}dt < +\infty \right\} \tag{1.2.4}$$

is endowed naturally with the norm

$$\|u\|_{L^1_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(t)|}{1+|t|^{n+2s}}dt.$$

Locally u should be $C^{2s+\varepsilon}$ for some small $\varepsilon > 0$ when $s \in (0, 1/2)$ and $C^{1,2s+\varepsilon-1}$ if $s \in [1/2, 1)$. Another expression for the fractional Laplace operator can be obtained from the given earlier definition (1.2.1). Precisely, one does the changes of variables $\tilde{z} = x - z$ and $\tilde{t} = t - x$ to get

$$\begin{aligned}(-\Delta)^s u(x) &= c(n,s) \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy \\ &= \frac{c(n,s)}{2} \lim_{\varepsilon\to 0} \left(\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(z)}{|x-z|^{n+2s}}dz + \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(t)}{|x-t|^{n+2s}}dt \right) \\ &= \frac{c(n,s)}{2} \lim_{\varepsilon\to 0} \left(\int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{u(x)-u(x-\tilde{z})}{|\tilde{z}|^{n+2s}}d\tilde{z} + \int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{u(x)-u(x+\tilde{t})}{|\tilde{t}|^{n+2s}}d\tilde{t} \right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{c(n,s)}{2} \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^n \setminus B_\varepsilon} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} dy \\
&= \frac{c(n,s)}{2} \int\limits_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} dy.
\end{aligned} \tag{1.2.5}
$$

Formula (1.2.5) shows that fractional Laplacian is a special case of Riesz derivative (1.1.4) when the finite difference is of the second order, namely $l = 2$. Indeed, using the properties of gamma function we check the value of the constants:

$$
\begin{aligned}
\frac{1}{d_n(2,s)} &= -\frac{2^{2s+1}\Gamma(1+s)\Gamma(\frac{n}{2}+s)\sin(s\pi)}{\pi^{1+n/2}} \\
&= -\frac{2^{2s+1}\Gamma(1+s)\Gamma(\frac{n}{2}+s)}{\pi^{n/2}\Gamma(s)\Gamma(1-s)} = \frac{2^{2s+1}\Gamma(\frac{n}{2}+s)}{\pi^{n/2}\Gamma(-s)} = -\frac{c(n,s)}{2},
\end{aligned}
$$

as stated.

Using the standard notions of Fourier and inverse Fourier transform, we recall a basic fact about the classical Laplace operator. Indeed, operating with only definitions, we see that, for some $u \in \mathcal{S}(\mathbb{R})$,

$$
\begin{aligned}
-\Delta u(x) &= -\Delta(\mathcal{F}^{-1}(\hat{u}))(x) = -\Delta \left(\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \hat{u}(\xi) e^{ix\cdot\xi} d\xi \right) \\
&= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} |\xi|^2 \hat{u}(\xi) e^{ix\cdot\xi} d\xi = \mathcal{F}^{-1}(|\xi|^2 \hat{u}(\xi)),
\end{aligned} \tag{1.2.6}
$$

which shows that the classical Laplacian acts as a multiplier of $|\xi|^2$ in a Fourier space. Another way to see the nature of the fractional Laplacian is to characterize it via the Fourier transform.

Lemma 1.2.1. *For any $u \in \mathcal{S}(\mathbb{R}^n)$ it holds that*

$$
(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s} \hat{u}(\xi)). \tag{1.2.7}
$$

In other words, the identity (1.2.7) tells that the fractional Laplace operator can be seen as a pseudo-differential operator with symbol $|\xi|^{2s}$. Thus (1.2.6) with (1.2.7) explain the connection between the classical and the fractional

Laplacians. Presicely, for any $u \in \mathcal{S}(\mathbb{R}^n)$, it holds

$$\begin{aligned} &\lim_{s\to 1}(-\Delta)^s u = -\Delta u \\ \text{and} \quad &\lim_{s\to 0}(-\Delta)^s u = u, \end{aligned} \tag{1.2.8}$$

giving that the classical Laplace operator is the limit case of the fractional one as $s \to 1$.

In the celebrated paper [17] the authors give description of the fractional Laplace operator viewed as a solution to a harmonic extension problem to the upper half space which maps the boundary condition of the Dirichlet type to the Neumann condition. Roughly speaking, the fractional Laplacian, being a nonlocal operator, can be seen as a local operator in the higher-dimensional half-space. In other terms, we call $U : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}$ the extension function that solves

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla U(x,y)) = 0, & (x,y) \in \mathbb{R}^n \times (0,+\infty), \\ U(x,0) = u(x). \end{cases}$$

Then it turns out that the solution also satisfies

$$-c(s) \lim_{y\to 0^+} y^{1-2s}\partial_y U = (-\Delta)^s u,$$

where the constant $c(s)$ depends only on s and explicitly given by (refer to [69])

$$c(s) = 2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}.$$

1.3 Viscosity solutions

This section is dedicated to the known results and definitions in viscosity theory, which will be much used throughout the current work. We introduce a family of linear nonlocal operators as

$$Lu(x) = \int_{\mathbb{R}^n} (2u(x) - u(x+t) - u(x-t))K(t)dt = \int_{\mathbb{R}^n} \Delta_t^2 u(x)K(t)dt, \tag{1.3.1}$$

where the integral can be possibly evaluated in the principle value sense. Notice that assuming the symmetry condition on the kernel repeating the calculations, similar to (1.2.5), one recognizes that the considered family of operators (1.3.1) is a special case of the family (0.0.1).

Definition 1.3.1. *A function u, continuous in $\overline{\Omega}$, is said to be a subsolution (supersolution) to $Lu = f$, and we write $Lu \leq f$ ($Lu \geq f$), if every time all the following happen*

- *x is any point of Ω;*
- *U is a neighborhood of x in Ω;*
- *φ is some C^2 function in $\overline{U}$;*
- *$\varphi(x) = u(x)$;*
- *$\varphi(y) \geq u(y)$ ($\varphi(y) \leq u(y)$) in U;*

then if we let

$$v := \begin{cases} \varphi, & \text{in } U \\ u, & \text{in } \mathbb{R}^n \setminus U \end{cases} \tag{1.3.2}$$

we have $Lv(x) \leq f(x)$ (resp. $Lv(x) \geq f(x)$). A solution is a function u which is both a subsolution and a supersolution.

Definition of a viscosity solution is continuously connected with the notion of a function touching from above or below, that we give immediately.

Definition 1.3.2. *We say that a function φ touches function u from above (below) at a point $x \in \Omega$, if*

$$\begin{aligned} &\varphi(x) = u(x) \\ \text{and } \ &\varphi(y) \geq u(y) \ (\varphi(y) \leq u(y)) \text{ in case } y \neq x. \end{aligned}$$

Smooth touching from above or below functions are often called test functions. It is also worth noting that whenever $u-\varphi$ attains a local maximum at a point $x_0 \in \Omega$, it holds $(u - \varphi)(x_0) = 0$.

Remark 1.3.3. *Let us give an example of the object of our interest in the simplest possible case in one dimension. Set $\Omega = (-1, 1)$ and*

$$Fu(x) = -u''.$$

Then the function $u(x) = ax + b$ is a solution, being

$$Fu(x) = -(ax + b)'' = 0.$$

While $w(x) = ax^2 + bx + c$, with $a \geq 0$, is a subsolution, since

$$Fw(x) = -(ax^2 + bx + c)'' = -2a \leq 0,$$

we have that $v(x) = -ax^2 + bx + c$ is a supersolution, since

$$Fv(x) = -(-ax^2 + bx + c)'' = 2a \geq 0.$$

It may happen that only the terms up to the second order play the role in the definition of viscosity solutions. For this reason, tools like semi jets have been developed in the literature.

Definition 1.3.4. *Let $(p, X) \in \mathbb{R}^n \times \mathbb{S}^n$ and $\hat{x} \in \Omega$. We say that $(p, X) \in J^{2,+}u(\hat{x})$ if it holds that*

$$u(x) \leq u(\hat{x}) + p \cdot \langle x - \hat{x} \rangle \frac{1}{2} \langle X(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2).$$

Similarly, $(p, X) \in J^{2,-}u(\hat{x})$ if

$$u(x) \geq u(\hat{x}) + p \cdot \langle x - \hat{x} \rangle \frac{1}{2} \langle X(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2)$$

holds true.

Sometimes it is convenient also to consider a kind of a "closure" of a semi jet. This leads us to another

Definition 1.3.5. *We say that*

$$(p, X) \in \overline{J}^{+,2}u(\hat{x})$$

if there exist a sequence $(p_i, X_i) \in J^{2,+}u(x_i)$ such that $(x_i, p_i, X_i) \to (\hat{x}, p, X)$. Analogously, one defines

$$(p, X) \in \overline{J}^{-,2}u(\hat{x})$$

if there exist a sequence $(p_i, X_i) \in J^{2,-}u(x_i)$ such that $(x_i, p_i, X_i) \to (\hat{x}, p, X)$.

For the reader's convenience, we state the Theorem of Sums (refer to [23] or [24]), which is the main tool in the method which we call the Ishii-Lions method. The precise statement of the theorem reads as follows.

Theorem 1.3.6. *Let* Ω_i *be a locally compact subset of* $\mathbb{R}^{n_i}$ *for* $i = 1, \dots, k$,

$$\Omega = \Omega_1 \times \cdots \times \Omega_k,$$

u_i *is* $\mathrm{USC}(\Omega_i)$, *and* φ *be twice continuously differentiable in a neighborhood of* Ω. *Set*

$$h(x) = u_1(x_1) + \cdots + u_k(x_k) \quad \text{for } x = (x_1, \cdots, x_n) \in \Omega,$$

and suppose $\hat{x} = (\hat{x}_1, \dots, \hat{x}_k) \in \Omega$ *is a local maximum of* $h - \varphi$ *relative to* Ω. *Then for each* $\varepsilon > 0$ *there exists* $X_i \in \mathbb{S}^{n_i}$ *such that*

$$(D_{x_i}\varphi(\hat{x}), X_i) \in \overline{J}^{2,+} u_i(\hat{x}_i) \quad \text{for } i = 1, \dots, k,$$

and the block diagonal matrix with entries X_i *satisfies*

$$-\left(\frac{1}{\varepsilon} + \|D^2\varphi(\hat{x})\|\right) I \leq \begin{pmatrix} X_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & X_k \end{pmatrix} \leq D^2\varphi(\hat{x}) + \varepsilon (D^2\varphi(\hat{x}))^2,$$

where $n = n_1 + \cdots + n_k$.

In the theorem above by the norm of a symmetric matrix we mean the supremum of moduli of all eigenvalues of the matrix. To be exact,

$$\|D^2\varphi(\hat{x})\| = \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } D^2\varphi(\hat{x})\}.$$

1.4 Carnot groups

This section is dedicated to the basic information about Carnot groups and the Heisenberg group, being the simplest possible case, although it is not easy at all. We denote by $\mathbb{H}^n$ the set $\mathbb{R}^{2n+1}$, $n \in \mathbb{N}$, $(x, y, t) \in \mathbb{R}^{2n+1}$, endowed with the noncommutative inner law such that for every $(x_1, y_1, t_1) \in \mathbb{R}^{2n+1}$, $(x_2, y_2, t_2) \in \mathbb{R}^{2n+1}$, $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^n$, $i = 1, 2$:

$$(x_1, y_1, t_1) \circ (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(x_2 \cdot y_1 - x_1 \cdot y_2)),$$

and $x_i \cdot y_j$ denote the usual inner product in $\mathbb{R}^n$. We call $\mathbb{H}^n$ the Heisenberg group of order n.

Let $X_i = (e_i, 0, 2y_i)$ and $Y_i = (0, e_i, -2x_i)$, $i = 1, \dots, n$, where $\{e_i\}_{1\le i\le n}$ is the canonical basis for $\mathbb{R}^n$. We use the same symbol to denote the vector fields associated with the introduced vectors so that for $i = 1, \dots, n$

$$X_i = \partial_{x_i} + 2y_i\partial_t,$$

$$Y_i = \partial_{y_i} - 2x_i\partial_t.$$

The commutator between the vector fields is

$$[X_i, Y_i] = -4\partial_t,$$

otherwise it is equal to 0. The intrinsic gradient of a smooth function u in a point P is

$$\nabla_{\mathbb{H}^n} u(P) = \sum_{i=1}^{n} (X_i u(P) X_i(P) + Y_i u(P) Y_i(P)).$$

There exists a unique metric on $H\mathbb{H}^n(P) = \text{span}\{X_1, \dots, X_n, Y_i, \dots, Y_n\}$ that makes orthonor-mal the set of vectors $\{X_1, \dots, X_n, Y_i, \dots, Y_n\}$. Thus, for every $P \in \mathbb{H}^n$ and for every $U, W \in H\mathbb{H}^n(P)$, $U = \sum_{j=1}^n (\alpha_{1,j} X_j(P) + \beta_{1,j} Y_j(P))$, $V = \sum_{j=1}^n (\alpha_{2,j} X_j(P) + \beta_{2,j} Y_j(P))$ the scalar product of two vector fields is defined by

$$\langle U, V\rangle = \sum_{j=1}^{n} (\alpha_{1,j}\alpha_{2,j} + \beta_{1,j}\beta_{2,j}).$$

In particular, we get a norm associated with the metric on
span$\{X_1, \dots, X_n, Y_i, \dots, Y_n\}$ and

$$|U| = \sqrt{\sum_{j=1}^{n} \left(\alpha_{1,j}^2 + \beta_{1,j}^2\right)}.$$

For example, the norm of the intrinsic gradient of the smooth function u in P is

$$|\nabla_{\mathbb{H}^n} u(P)| = \sqrt{\sum_{i=1}^{n} ((X_i u(P))^2 + (Y_i u(P))^2)}.$$

Moreover, if $\nabla_{\mathbb{H}^n} u(P) \neq 0$ the norm of

$$\frac{\nabla_{\mathbb{H}^n} u(P)}{|\nabla_{\mathbb{H}^n} u(P)|}$$

is equal to one.

If $\nabla_{\mathbb{H}^n} u(P) = 0$ then we say that the point P is characteristic for the smooth surface $\{u = u(P)\}$. Hence for every point $P \in \{u = u(P)\}$, that it is not characteristic, it is well defined the intrinsic normal to the surface $\{u = u(P)\}$ as follows:

$$\nu(P) = \frac{\nabla_{\mathbb{H}^n} u(P)}{|\nabla_{\mathbb{H}^n} u(P)|}.$$

We introduce in the Heisenbeg group $\mathbb{H}^n$ the following gauge norm:

$$d_G(x, y, t) \equiv ||(x, y, t)|| = \sqrt[4]{(|x|^2 + |y|^2)^2 + t^2}. \tag{1.4.1}$$

In particular for every positive number r the gauge ball of radius r centered in 0 is

$$B(0, r) = \{P \in \mathbb{H}^n : \ \|P\| < r\}.$$

In the Heisenberg group a group of dilation is also defined as follows: for every $r > 0$ and for every $P \in \mathbb{H}^n$ let

$$\delta_r(P) = (rx, ry, r^2 t).$$

If $P \in \mathbb{H}^n$ and

$$V \in \mathfrak{g} = \text{span(Lie)}\{X_i, Y_j, [X_i, Y_j] : \ i, j = 1, \ldots, n\}$$

we set $\vartheta_{(V,P)}(s) := \exp[sV](P)\,(s \in \mathbb{R})$, i.e., $\vartheta_{(V,P)}$ denotes the integral curve of V starting from P and it turns out to be a 1-parameter subgroup of $\mathbb{H}^n$. The Lie group exponential map is defined by

$$\exp : \mathfrak{g} \longmapsto \mathbb{H}^n, \quad \exp(V) := \exp[V](1).$$

The map exp is an analytic diffeomorphism between $\mathfrak{g}$ and $\mathbb{H}^n$. One has

$$\vartheta_{(V,P)}(s) = P \circ \exp(sV) \quad \forall\ s \in \mathbb{R}.$$

In particular we remark that if $U \in H\mathbb{H}^n(P)$, then

$$\vartheta_{(U,P)}(t) = P \circ \exp(sU)$$

is horizontal.

Indeed, we say that a path $\varphi : [-\tau, \tau] \to \mathbb{H}^n$ in the Heisenberg group is horizontal if $\varphi'(s) \in H\mathbb{H}^n(\varphi(s))$ for almost all $s \in [-\tau, \tau]$.

Concerning the natural Sobolev spaces to consider in the Heisenberg group $\mathbb{H}^n$, we refer to the literature, see for instance [46]. Here, we only recall that

$$\mathcal{L}^{1,2}(\Omega) := \{f \in L^2(\Omega) : X_i f,\ \ Y_i f \in L^2(\Omega),\ \ i = 1, \dots, n\}$$

is a Hilbert space with the norm

$$|f|_{\mathcal{L}^{1,2}(\Omega)} = \left(\int_\Omega \Big(\sum_i^n (X_i f)^2 + (Y_i f)^2 \Big) + |f|^2 dx \right)^{\frac{1}{2}}.$$

Moreover,

$$H^1_{\mathbb{H}^n}(\Omega) = \overline{C^\infty(\Omega) \cap \mathcal{L}^{1,2}(\Omega)}^{|\cdot|_{\mathcal{L}^{1,2}(\Omega)}}.$$

$$H^1_{\mathbb{H}^n,0}(\Omega) = \overline{C_0^\infty(\Omega)}^{|\cdot|_{\mathcal{L}^{1,2}(\Omega)}}.$$

Of course, on the Sobolev-Poincaré inequalites there exists a wide literature, see e.g., [50, 44, 58]. However, here we shall recall only the following one in the Heisenberg group for every $u \in H^1_{\mathbb{H}^n,0}(B_r)$, namely

$$\int_{B_r} |u(x)| dx \le Cr \int_{B_r} |\nabla_{\mathbb{H}^n} u(x)| dx,$$

see also [46] for isoperimetric and Sobolev inequalities in more general situations.

In general, this presentation makes sense also for a larger set of stratificated noncommutative structures: the Carnot groups. In fact, let $(\mathbb{G}, \circ)$ be a group and there exist $\{\mathfrak{g}_i\}_{1 \le i \le m}$, $m \in \mathbb{N}$, $m \le N \in \mathbb{N}$, vector spaces such that,

$$\mathfrak{g}_1 \bigoplus \mathfrak{g}_2 \bigoplus \cdots \bigoplus \mathfrak{g}_m = \mathfrak{g} \equiv \mathbb{R}^N \equiv \mathbb{G}$$

and

$$[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2, \quad [\mathfrak{g}_1, \mathfrak{g}_2] = \mathfrak{g}_3, \quad \dots, [\mathfrak{g}_1, \mathfrak{g}_{m-1}] = \mathfrak{g}_m,$$

where

$$[\mathfrak{g}_1, \mathfrak{g}_m] = 0.$$

In this case we say that $\mathbb{G}$ is a stratified Carnot group of step m.

Moreover, for every

$$x \in \mathbb{G} \equiv \mathbb{R}^N = \mathbb{R}^{k_1} \times \dots \mathbb{R}^{k_m}, \quad \sum_{j=1}^{m} k_i = N,$$

and for every $\lambda > 0$ is defined the anisotropic dilation

$$\delta_\lambda(x) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^m x^{(m)}), \quad \text{where} \quad x^{(j)} \in \mathbb{R}^{k_j}, \quad j = 1, \dots, m$$

such that, if $Z_1, \dots, Z_{k_1} \in \mathfrak{g}_1$ are left invariant vector fields and $Z_j(0) = \frac{\partial}{\partial x_j}_{|x=0}$, $j = 1, \dots, k_1$, then

$$\text{rank}(\text{Lie}\{Z_1, \dots, Z_{k_1}\})(x) = N, \quad \text{(Hörmander condition)}$$

for every $x \in \mathbb{R}^N \equiv \mathbb{G}$. Let us consider the sublaplacian on the stratified Carnot group $\mathbb{G}$ given by

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{k_1} X_j^2. \tag{1.4.2}$$

In particular, there exists a $N \times k_1$ matrix σ such that $\sigma \cdot \sigma^T$ is a $N \times N$ matrix such that

$$\text{div}(\sigma \cdot \sigma^T \nabla \cdot) = \Delta_{\mathbb{G}}. \tag{1.4.3}$$

Moreover,

$$\sigma^T \nabla u = \sum_{j=1}^{k_1} X_j u X_j \equiv \nabla_{\mathfrak{g}_1} u,$$

the so called horizontal gradient of u. Hence

$$A = \sigma \cdot \sigma^T.$$

The Heisenberg group $\mathbb{H}^1$ is an example of Carnot group of step 2. In fact,

$$\mathfrak{g}_1 = \text{span}\{X, Y\}, \quad \mathfrak{g}_2 = \text{span}\{[X, Y]\}, \quad [\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}$$

and the Lie algebra of the Heisenberg group is obtained as

$$\mathfrak{g} = \text{span(Lie)}\{X, Y, [X, Y]\} = \mathfrak{g}_1 \bigoplus \mathfrak{g}_2.$$

Exploiting the cited representation of the sublaplacians (1.4.3), it results that

$$\langle A\nabla u, \nabla u\rangle_{\mathbb{R}^N} = \langle \sigma \cdot \sigma^T \nabla u, \nabla u\rangle_{\mathbb{R}^N} = \langle \sigma^T \nabla u, \sigma^T \nabla u\rangle_{\mathbb{R}^{k_1}} \equiv \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} u\rangle_{\mathbb{R}^{k_1}},$$

where $\nabla_{\mathbb{G}} u \equiv \sigma^T \nabla u = \sum_{j=1}^{k_1} X_j u X_j$ is the horizontal gradient in the Carnot group $\mathbb{G}$. Thus the definition by completion of the Sobolev spaces with respect to the norm

$$||u||_{\mathbb{H}_{1,2}(\mathbb{G})} := \sqrt{\int_\Omega \langle A\nabla u, \nabla u\rangle_{\mathbb{R}^N} + \int_\Omega u^2} \equiv \sqrt{\int_\Omega \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} u\rangle_{\mathbb{R}^{k_1}} + \int_\Omega u^2}$$

is the same.

We spend few words about the Carnot-Charathéodory distance. To do this goal, we recall, see e.g. [9], that if $\{X_1, \dots X_N\}$ are vector fields in $\mathbb{R}^n$, a piecewise regular path $\eta : [0, T] \to \mathbb{R}^n$ is said subunit, with respect to the family $\{X_1, \dots X_N\}$, if for every $\xi \in \mathbb{R}^n$

$$\langle \eta'(t), \xi\rangle^2 \leq \sum_{j=1}^{N} \langle X_j(\eta(t)), \xi\rangle^2, \quad \text{for a.e. } t \in [0, T].$$

Let us denote by $\mathcal{D} := \mathcal{D}(\{X_1, \dots X_N\})$ the set of all the subunit paths.

Proposition 1.4.1 (Chow-Rashevsky)**.** *Let $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ be a Carnot group with the Lie algebra $\mathfrak{g}$ and let $\{X_1, \dots X_N\}$ be a family of vector fields in $\mathbb{R}^n$. If*

$$\mathfrak{g} = Lie\{X_1, \dots X_N\},$$

then for every $x, y \in \mathbb{R}^n$ there exists $\eta \in \mathcal{D}$ such that $\eta(0) = x$, $\eta(T) = y$, moreover

$$d_{CC}(x, y) := \inf\{T > 0 : \textit{ there exists } \eta : [0, T] \to \mathbb{R}^n, \eta \in \mathcal{D}, \eta(0) = x, \eta(T) = y\}$$

is a distance called the Carnot-Charathéodory distance associated with $\{X_1, \dots X_N\}$ and $d_{CC}(\cdot, 0)$ is a homogeneous norm on $\mathbb{G}$.

In the case of the Heisenberg group, there exist positive constants $C_1, C_2 > 0$

such that for every $P \in \mathbb{H}^n$

$$C_1 \|P\|_{\mathbb{H}^n} \leq d_{CC}(P, 0) \leq C_2 \|P\|_{\mathbb{H}^n}.$$

The same equivalence may be extended to Carnot groups, simply by considering the right homogeneous norm versus the Carnot-Charathéodory distance in the considered group $\mathbb{G}$. In addition, a strong maximum principle holds, see [10], even if $\Delta_{\mathbb{G}}$ is a degenerate operator.

Proposition 1.4.2. *Let u be such that $\Delta_{\mathbb{G}} u \geq 0$ in $\Omega \subset \mathbb{G}$ is an open set and $\mathbb{G}$ is a group whose Lie algebra $\mathfrak{g}$ satisfies the Hörmander condition. Then the supremum of u can not be realized in Ω unless u is constant.*

Chapter 2

"Up to a polynomial" framework

The goal of this chapter is twofold: on the one hand, we review and extend the theory developed in [32, 33], on the other hand, we generalize the previous setting in order to include much more general classes of kernels (in particular, kernels which are not necessarily scaling invariant). Besides its interest in pure mathematics, this generalization has a concrete impact on the study of interaction potentials of interatomic type arising in molecular mechanics and materials science, such as the Morse potential [62]

$$K(x,y) = e^{-2(|x-y|-1)} - e^{-(|x-y|-1)}, \tag{2.0.1}$$

the Buckingham potential [11]

$$K(x,y) = e^{-|x-y|} - \frac{1}{|x-y|^6}, \tag{2.0.2}$$

as well as their desingularized forms obtained by setting

$$K_\varepsilon(x,y) := \min\left\{\frac{1}{\varepsilon},\, K(x,y)\right\}.$$

Other classical potentials arising in probability and modelization include also the Gauss kernel

$$K(x,y) = e^{-|x-y|^2}, \tag{2.0.3}$$

the Abel kernel

$$K(x,y) = e^{-|x-y|}, \tag{2.0.4}$$

the mollification kernel

$$K(x,y) = \begin{cases} e^{-\frac{1}{1-|x-y|^2}} & \text{if } |x-y|<1, \\ 0 & \text{if } |x-y|\geq 1 \end{cases} \tag{2.0.5}$$

and the class of kernels comparable to that of the fractional Laplacian

$$\frac{\lambda}{|x-y|^{n+2s}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+2s}} \tag{2.0.6}$$

for $s \in (0,1)$ and $\Lambda \geq \lambda > 0$.

The theory of integral operators that we develop is broad enough to include the kernels above (and others as well) into a unified setting. The operators will be suitably defined "up to a polynomial", in a sense that will be made precise in Definition 2.1.7. This framework relies on a suitable decomposition of the integral operator with respect to cut-off functions that is showcased in Theorem 2.1.2. This setting is stable under the appropriate notion of convergence, as it will be detailed in Proposition 2.1.17, and it presents nice compatibility results between polynomials of different orders, as it will be pointed out in Corollary 2.1.19 and Lemma 2.1.20. We also stress that the generalized notion of operators that we deal with is "as good as the classical one" in terms of producing solutions for the associated Dirichlet problem: indeed, as it will be clarified in Theorem 2.2.1, the solvability of the classical Dirichlet problem in the class of functions with nice behavior at infinity is sufficient to ensure the solvability of the generalized Dirichlet problem for the operator defined "up to a polynomial".

To develop this theory, we mainly focused on the case of sufficiently smooth (though not necessarily well-behaved at infinity) functions. This choice was dictated by three main reasons. First of all, we aimed at developing the core of the theory by focusing on its essential features, rather than complicating it by additional difficulties. Moreover, we intended to split the complications arising from the possible lack of smoothness of the solutions with those produced by their behavior at infinity, consistently with the initial discussion presented right after (0.0.1). Additionally, we stress that the generality of kernels addressed by our theory goes far beyond the ones of "elliptic" type, therefore a comprehensive regularity theory does not hold in such an extensive framework.

However, one can also recast our theory in terms of viscosity solutions.

For this, since viscosity theory relates to maximum principles, one needs the additional assumption that the kernel has a sign. In particular, in this context one can obtain a viscosity definition of operators "up to a polynomial" (see Definition 2.3.2) and discuss its stability properties under uniform convergence (see Lemma 2.3.7) and the consistency properties with respect of polynomials of different degree (see Corollary 2.3.8 and Lemma 2.3.9). When the structure is compatible with both settings, the pointwise framework and the viscosity one are essentially equivalent (see Lemma 2.3.6). Furthermore, for kernels comparable with that of the fractional Laplacian a complete solvability of the Dirichlet problem can be obtained (see Theorem 2.3.13).

2.1 Definitions and main properties

The mathematical setting in which we work is the following. For every $\vartheta \in [0,2]$, we define $\mathcal{C}_\vartheta$ as the set of functions $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$u \in \begin{cases} C(B_4) \cap L^\infty(B_4) & \text{if } \vartheta = 0, \\ C^\vartheta(B_4) & \text{if } \vartheta \in (0,1), \\ C^{0,1}(B_4) & \text{if } \vartheta = 1, \\ C^{1,\vartheta-1}(B_4) & \text{if } \vartheta \in (1,2), \\ C^{1,1}(B_4) & \text{if } \vartheta = 2. \end{cases} \tag{2.1.1}$$

Furthermore, for all $m \in \mathbb{N}_0$ and all $\vartheta \in [0,2]$, we introduce $\mathcal{K}_{m,\vartheta}$ as the space of kernels $K = K(x,y)$ such that

$$\text{for all } y \in B_3^c, \text{ the map } x \in B_1 \mapsto K(x,y) \text{ is } C^m(B_1) \tag{2.1.2}$$

$$\text{and } \int_{\mathbb{R}^n} \min\{|x-y|^\vartheta, 1\}\, |K(x,y)|\, dy < +\infty \quad \text{for all } x \in B_1. \tag{2.1.3}$$

If $\vartheta \in (1,2]$ we require additionally that every $K \in \mathcal{K}_{m,\vartheta}$ satisfies

$$K(x, x+z) = K(x, x-z) \quad \text{for all } x,\, z \in B_1. \tag{2.1.4}$$

Given $K \in \mathcal{K}_{m,\vartheta}$, we consider the space $\mathcal{C}_{\vartheta,K}$ of all the functions $u \in \mathcal{C}_\vartheta$ for which

$$\sum_{|\alpha|\le m-1} \int_{B_R\setminus B_3} |u(y)|\,|\partial_x^\alpha K(x,y)|\,dy < +\infty \qquad \text{for all } R>3 \text{ and } x \in B_1 \tag{2.1.5}$$

$$\text{and } \int_{B_3^c} |u(y)| \sup_{\substack{|\alpha|=m\\ x\in B_1}} |\partial_x^\alpha K(x,y)|\,dy < +\infty. \tag{2.1.6}$$

Remark 2.1.1. *We observe that these assumptions are satisfied by the kernel in* (2.0.1) *for every* n*, by the kernel in* (2.0.2) *with* $n = 5$ *and by all the corresponding desingularized kernels for every* n*. The kernels in* (2.0.3), (2.0.4) *and* (2.0.5) *also satisfy these assumptions for every* n*. The kernel in* (2.0.6) *satisfies* (2.1.3) *for every* n *and every* $\vartheta \in (2s,2]$*.*

In this setting, we have the following results that show the role played by a cut-off function in the computation of the operator in (0.0.1) on functions in $\mathcal{C}_{\vartheta,K}$:

Theorem 2.1.2. *Let* $m \in \mathbb{N}_0$*,* $\vartheta \in [0,2]$*,* $K \in \mathcal{K}_{m,\vartheta}$ *and* $u \in \mathcal{C}_{\vartheta,K}$*. Let also* $\tau : \mathbb{R}^n \to [0,1]$ *be compactly supported and such that* $\tau = 1$ *in* B_3*.*

Then, there exist a function $f_{u,\tau} : \mathbb{R}^n \to \mathbb{R}$ *and a polynomial* $P_{u,\tau}$ *of degree at most* $m-1$ *such that*

$$A(\tau u) = P_{u,\tau} + f_{u,\tau} \tag{2.1.7}$$

in B_1*. In addition,* $f_{u,\tau}$ *can be written in the following form: there exists* $\psi : B_1 \times B_3^c \to \mathbb{R}$*, with*

$$\sup_{x\in B_1} |\partial_x^\gamma \psi(x,y)| \le C \sup_{\substack{x\in B_1\\ m\le|\eta|\le m+|\gamma|}} |\partial_x^\eta K(x,y)|, \tag{2.1.8}$$

for every $\gamma \in \mathbb{N}^n$ *and for a suitable constant* $C > 0$ *depending on* m*,* n *and* $|\gamma|$*, such that*

$$f_{u,\tau} = f_{1,u} + f_{2,u} + f^*_{u,\tau}, \tag{2.1.9}$$

where

$$\begin{aligned} f_{1,u}(x) &:= \int_{B_3} \big(u(x)-u(y)\big)K(x,y)\,dy, \\ f_{2,u}(x) &:= u(x)\int_{B_3^c} K(x,y)\,dy \qquad (2.1.10) \\ \textit{and} \qquad f^*_{u,\tau}(x) &:= \int_{B_3^c} \tau(y)u(y)\psi(x,y)\,dy. \end{aligned}$$

Proof. We observe that

$$f_{1,u} \text{ and } f_{2,u} \text{ are well-defined and finite for every } x \in B_1. \qquad (2.1.11)$$

To check this, we first consider the case in which $\vartheta \in [0,1]$. In this case, for every $x \in B_1$ and $y \in B_3$,

$$|u(x)-u(y)| \le C|x-y|^\vartheta,$$

for some $C>0$, and thus

$$\begin{aligned} |f_{1,u}(x)| &\le \int_{B_3} \big|u(x)-u(y)\big|\,|K(x,y)|\,dy \le C\int_{B_3} |x-y|^\vartheta\,|K(x,y)|\,dy \\ &\le C\left(\int_{B_3\cap B_1(x)} |x-y|^\vartheta\,|K(x,y)|\,dy + \int_{B_3\setminus B_1(x)} |K(x,y)|\,dy\right), \end{aligned}$$

up to renaming $C>0$, and this shows that $f_{1,u}$ is well-defined and finite, thanks to (2.1.3).

If instead $\vartheta \in (1,2]$ we claim that, since $u \in \mathcal{C}_\vartheta$, there exists a constant $L>0$ such that for all $|z|$ sufficiently small (say $z \in B_1$) we have that

$$|2u(x)-u(x+z)-u(x-z)| \le L|z|^\vartheta. \qquad (2.1.12)$$

Indeed, in this case we know that $u \in C^{1,\vartheta-1}(B_4)$ and thus

$$\begin{aligned} |2u(x)-u(x+z)-u(x-z)| &= |(u(x)-u(x+z))+(u(x)-u(x-z))| \\ &= \left|-\int_0^1 \nabla u(x+tz)\cdot z\,dt + \int_0^1 \nabla u(x-tz)\cdot z\,dt\right| \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 |\nabla u(x+tz) - \nabla u(x-tz)|\,|z|\,dt \\
&\leq C\int_0^1 |x+tz-(x-tz)|^{\vartheta-1}|z|\,dt = C\int_0^1 t^{\vartheta-1}|z|^{\vartheta-1}|z|\,dt = C|z|^{\vartheta},
\end{aligned} \tag{2.1.13}$$

up to relabeling C at every step. This establishes (2.1.12).

Now, we notice that

$$\begin{aligned}
f_{1,u}(x) &= \int_{B_3} \big(u(x)-u(y)\big)K(x,y)\,dy \\
&= \int_{B_1(x)} \big(u(x)-u(y)\big)K(x,y)\,dy + \int_{B_3\setminus B_1(x)} \big(u(x)-u(y)\big)K(x,y)\,dy \\
&=: I_1 + I_2.
\end{aligned} \tag{2.1.14}$$

Using (2.1.4) and (2.1.12), we obtain that

$$\begin{aligned}
&\left|\int_{B_1(x)} \big(u(x)-u(y)\big)K(x,y)\,dy\right| \\
&= \frac{1}{2}\left|\int_{B_1} \big(u(x)-u(x+z)\big)K(x,x+z)\,dz \right.\\
&\qquad \left. + \int_{B_1} \big(u(x)-u(x-z)\big)K(x,x-z)\,dz\right| \\
&= \frac{1}{2}\left|\int_{B_1} \big(2u(x)-u(x+z)-u(x-z)\big)K(x,x+z)\,dz\right| \\
&\leq \frac{1}{2}\int_{B_1} \big|2u(x)-u(x+z)-u(x-z)\big|\,|K(x,x+z)|\,dz \\
&\leq \frac{L}{2}\int_{B_1} |z|^{\vartheta}\,|K(x,x+z)|\,dz \\
&\leq \frac{L}{2}\int_{B_1} |z|^{\vartheta}\,|K(x,x+z)|\,dz.
\end{aligned}$$

As a consequence,

$$|I_1| \leq \frac{L}{2}\int_{B_1} |z|^{\vartheta}\,|K(x,x+z)|\,dz, \tag{2.1.15}$$

which is finite, thanks to (2.1.3).

Furthermore,

$$\begin{aligned}|I_2| &\le \int_{B_3\setminus B_1(x)} \big(|u(x)|+|u(y)|\big)\,|K(x,y)|\,dy\\ &\le 2\|u\|_{L^\infty(B_4)}\int_{B_3\setminus B_1(x)} |K(x,y)|\,dy,\end{aligned}$$

which is finite, in light of (2.1.3). This, together with (2.1.14) and (2.1.15), proves that $f_{1,u}$ is well-defined and finite in the case $\vartheta\in(1,2]$.

Also, $f_{2,u}$ is well-defined and finite for every $\vartheta\in[0,2]$, thanks to (2.1.3). These observations establish (2.1.11).

As a consequence, for any $x\in B_1$, we can write

$$\begin{aligned}&A(\tau u)(x)\\ =&\int_{B_3}\big((\tau u)(x)-(\tau u)(y)\big)K(x,y)\,dy+\int_{B_3^c}\big((\tau u)(x)-(\tau u)(y)\big)K(x,y)\,dy\\ =&\int_{B_3}\big(u(x)-u(y)\big)K(x,y)\,dy+u(x)\int_{B_3^c}K(x,y)\,dy-\int_{B_3^c}(\tau u)(y)K(x,y)\,dy\\ =&\,f_{1,u}(x)+f_{2,u}(x)-\int_{B_3^c}(\tau u)(y)K(x,y)\,dy.\end{aligned} \tag{2.1.16}$$

Now, in light of the assumption in (2.1.2), we are allowed to use Proposition 5.34 in [22] (see also e.g. Theorem 4 on page 461 of [72]) and we find that

$$K(x,y)=\sum_{|\alpha|\le m-1}\partial_x^\alpha K(0,y)\frac{x^\alpha}{\alpha!}-\psi(x,y),$$

where

$$\psi(x,y):=-\sum_{|\alpha|=m}\frac{m\,x^\alpha}{\alpha!}\int_0^1(1-t)^{m-1}\partial_x^\alpha K(tx,y)\,dt. \tag{2.1.17}$$

As a consequence,

$$\begin{aligned}\int_{B_3^c}(\tau u)(y)K(x,y)\,dy &= \int_{B_3^c}(\tau u)(y)\left(\sum_{|\alpha|\le m-1}\partial_x^\alpha K(0,y)\frac{x^\alpha}{\alpha!}-\psi(x,y)\right)dy\\ &= \sum_{|\alpha|\le m-1}\left(\int_{B_3^c}(\tau u)(y)\partial_x^\alpha K(0,y)\,dy\right)\frac{x^\alpha}{\alpha!}-\int_{B_3^c}(\tau u)(y)\psi(x,y)\,dy.\end{aligned} \tag{2.1.18}$$

Now, we set, for every $|\alpha|\le m-1$,

$$\theta_{\tau,\alpha} := \int_{B_3^c}(\tau u)(y)\frac{\partial_x^\alpha K(0,y)}{\alpha!}\,dy. \tag{2.1.19}$$

Suppose that the support of τ is contained in some ball B_R with $R>3$, and thus

$$|\theta_{\tau,\alpha}| \le \int_{B_R\setminus B_3}\left|(\tau u)(y)\frac{\partial_x^\alpha K(0,y)}{\alpha!}\right|dy \le \frac{1}{\alpha!}\int_{B_R\setminus B_3}|u(y)|\,\left|\partial_x^\alpha K(0,y)\right|dy. \tag{2.1.20}$$

We stress that the coefficients $\theta_{\tau,\alpha}$ are well-defined, thanks to (2.1.5).

Hence, setting

$$P_{u,\tau}(x) := -\sum_{|\alpha|\le m-1}\theta_{\tau,\alpha}x^\alpha \tag{2.1.21}$$

we have that $P_{u,\tau}$ is a polynomial in x of degree at most $m-1$. Plugging this information into (2.1.18), we obtain that

$$\int_{B_3^c}(\tau u)(y)K(x,y)\,dy = -P_{u,\tau}(x)-\int_{B_3^c}(\tau u)(y)\psi(x,y)\,dy.$$

Now, we notice that, for all $x\in B_1$ and all $y\in B_3^c$,

$$|(\tau u)(y)\psi(x,y)| \le C|u(y)|\sup_{\substack{|\alpha|=m\\ z\in B_1}}|\partial_x^\alpha K(z,y)|,$$

for some $C>0$, possibly depending on m and n. The last function lies in $L^1(B_3^c)$, thanks to (2.1.6), and therefore, recalling the definition of $f^*_{u,\tau}$ in (2.1.10), we have that

$$f^*_{u,\tau} \text{ is well-defined and finite.} \tag{2.1.22}$$

With this setting, we have that

$$\int_{B_3^c} (\tau u)(y)K(x,y)\,dy = -P_{u,\tau}(x) - f^*_{u,\tau}(x),$$

and therefore, plugging this information into (2.1.16) and recalling (2.1.9), we obtain (2.1.7).

Hence, to complete the proof of Theorem 2.1.2, it remains to check (2.1.8). For this, recalling the definition of ψ in (2.1.17), we have that, for all $x \in B_1$ and all $y \in B_3^c$,

$$\begin{aligned}
\partial_x^\gamma \psi(x,y) &= \sum_{|\alpha|=m} \int_0^1 c_\alpha(t)\,\partial_x^\gamma \left(x^\alpha\,\partial_x^\alpha K(tx,y)\right)\,dt \\
&= \sum_{|\alpha|=m} \int_0^1 c_\alpha(t) \sum_{\beta\le\gamma} \binom{\gamma}{\beta} \partial_x^\beta(x^\alpha)\,\partial_x^{\gamma-\beta}(\partial_x^\alpha K(tx,y))\,dt \\
&= \sum_{|\alpha|=m} \int_0^1 c_\alpha(t) \sum_{\beta\le\gamma} \binom{\gamma}{\beta} \partial_x^\beta(x^\alpha)\,t^{|\gamma-\beta|}\,\partial_x^{\alpha+\gamma-\beta} K(tx,y)\,dt,
\end{aligned}$$

where $c_\alpha(t) := \frac{m}{\alpha!}(1-t)^{m-1}$. Here $\beta \le \gamma$ means that $\beta_1 \le \gamma_1, \cdots, \beta_n \le \gamma_n$ and we used the notation

$$\binom{\gamma}{\beta} = \binom{\gamma_1}{\beta_1} \times \cdots \times \binom{\gamma_n}{\beta_n}.$$

Hence,

$$|\partial_x^\gamma \psi(x,y)| \le C \sup_{\substack{z\in B_1\\ m\le|\eta|\le m+|\gamma|}} |\partial_x^\eta K(z,y)|,$$

for some $C > 0$ depending on m, n and $|\gamma|$. This establishes (2.1.8). □

Remark 2.1.3. *For $m = 0$, the result in Theorem 2.1.2 holds true, simply by taking $P_{u,\tau} := 0$ and $\psi(x,y) := K(x,y)$.*

Remark 2.1.4. *Notice that the right hand side of* (2.1.8) *may be infinite and in this case* (2.1.8) *is obviously true. On the contrary, if the right hand side of* (2.1.8) *is finite, then the quantity on the left hand side of* (2.1.8) *is bounded as well.*

Corollary 2.1.5. *Let $m \in \mathbb{N}_0$, $\vartheta \in [0,2]$, $K \in \mathcal{K}_{m,\vartheta}$, $u \in \mathcal{C}_{\vartheta,K}$ and $R > 3$. Let*

$\tau_R : \mathbb{R}^n \to [0,1]$ *be supported in* B_R*, with* $\tau_R = 1$ *in* B_3*, and such that*

$$\lim_{R\to+\infty} \tau_R = 1 \ a.e. \ in \ \mathbb{R}^n. \tag{2.1.23}$$

Then, there exist a function $f_u : \mathbb{R}^n \to \mathbb{R}$ *and a family of polynomials* P_{u,τ_R}*, which have degree at most* $m-1$*, such that*

$$\lim_{R\to+\infty} [A(\tau_R u)(x) - P_{u,\tau_R}(x)] = f_u(x) \tag{2.1.24}$$

for any $x \in B_1$*. More precisely, we have that*

$$f_u = f_{1,u} + f_{2,u} + f_{3,u}, \tag{2.1.25}$$

where $f_{1,u}$ *and* $f_{2,u}$ *are as in* (2.1.10) *and*

$$f_{3,u}(x) := \int_{B_3^c} u(y)\psi(x,y)\,dy. \tag{2.1.26}$$

Proof. We apply Theorem 2.1.2 with $\tau := \tau_R$ for any fixed R, and then send $R \to +\infty$. Indeed, by (2.1.8) (used here with $\gamma := 0$), for any $x \in B_1$ and $y \in B_3^c$ we have

$$\big|(\tau_R u)(y)\psi(x,y)\big| \le C|u(y)| \sup_{\substack{|\eta|=m\\ z\in B_1}} |\partial_x^\eta K(z,y)|,$$

for some $C > 0$ and the latter function of y lies in $L^1(B_3^c)$, thanks to (2.1.6).

Consequently, we use (2.1.10), (2.1.23) and the Dominated Convergence Theorem, thus obtaining that

$$\lim_{R\to+\infty} f^*_{u,\tau_R} = \lim_{R\to+\infty} \int_{B_3^c} (\tau_R u)(y)\psi(x,y)\,dy = \int_{B_3^c} u(y)\psi(x,y)\,dy = f_{3,u}(x).$$

Accordingly, taking the limit in (2.1.7) we obtain (2.1.24). Also, the claims in (2.1.25) and (2.1.26) follow from (2.1.10). □

Remark 2.1.6. *It is also interesting to point out that, when* $\tau_R := \chi_R$*, the limit in* (2.1.24) *is uniform for* $x \in B_1$*. Indeed, by* (2.1.7) *and* (2.1.8)*, for* *all* $R_2 > R_1 > 4$,

$$\begin{aligned} & \sup_{x\in B_1} \Big| [A(\tau_{R_1}u)(x) - P_{u,\tau_{R_1}}(x)] - [A(\tau_{R_2}u)(x) - P_{u,\tau_{R_2}}(x)] \Big| \\ \leq\; & \sup_{x\in B_1} \int_{B_{R_2}\setminus B_{R_1}} |u(y)|\, |\psi(x,y)|\, dy \\ \leq\; & C \int_{B_{R_2}\setminus B_{R_1}} |u(y)| \sup_{\substack{|\eta|=m \\ z\in B_1}} |\partial_x^\eta K(z,y)|, \end{aligned}$$

which is as small as we wish, owing to (2.1.6)*. This observation will be further expanded in Lemma 2.1.13.*

We are now ready to introduce the formal setting to deal with general operators defined "up to a polynomial":

Definition 2.1.7. *Let* $m \in \mathbb{N}_0$, $\vartheta \in [0,2]$, $K \in \mathcal{K}_{m,\vartheta}$, $u \in \mathcal{C}_{\vartheta,K}$ *and* $f : B_1 \to \mathbb{R}$ *be bounded and continuous. We say that*

$$Au \overset{m}{=} f \qquad \text{in } B_1$$

if there exist a family of polynomials P_R*, with* $\deg P_R \leq m-1$*, and functions* $f_R : B_1 \to \mathbb{R}$ *such that*

$$A(\chi_R u) = f_R + P_R \tag{2.1.27}$$

in B_1*, with*

$$\lim_{R\to+\infty} f_R(x) = f(x). \tag{2.1.28}$$

Remark 2.1.8. *In order to show the connection of the introduced setting with the developed theories in the literature, we will compute a simple example. Take* $m = 2$, $\vartheta = 2$ *in dimension* $n = 1$*, together with* $K(x,y) = |x-y|^{-1-2s}$ *and* $s = 1/2$*, so that* $Au = (-\triangle)^{1/2}u$*. We consider* $u(x) = x^2$*, then for all*

$x \in (-1,1)$ and a fixed $R > 2$ we get

$$
\begin{aligned}
(-\triangle)^{1/2}(\chi_R(x)x^2) &= \int_{-R}^{R} \frac{\chi_R(x)x^2 - \chi_R(y)y^2}{|x-y|^2} dy + \int_{\mathbb{R}\setminus(-R,R)} \frac{\chi_R(x)x^2 - \chi_R(y)y^2}{|x-y|^2} dy \\
&= \int_{-R}^{R} \frac{x^2 - y^2}{(x-y)^2} dy + \int_{-\infty}^{-R} \frac{x^2}{(x-y)^2} dy + \int_{R}^{\infty} \frac{x^2}{(x-y)^2} dy \\
&= \underbrace{-2R}_{P_R} \underbrace{-2x \ln\left|\frac{R-x}{R+x}\right| + x^2\left(\frac{1}{R+x} + \frac{1}{R-x}\right)}_{f_R}.
\end{aligned}
$$

Therefore $f_R \to 0$ as $R \to +\infty$, and we obtain

$$
(-\triangle)^{1/2} x^2 \overset{2}{=} 0.
$$

Remark 2.1.9. We observe that (2.1.27) is considered here in the pointwise sense. This is possible, since the setting in (2.1.1) suffices for writing the equation pointwise (recall (2.1.11) and (2.1.22)). A viscosity theory is also possible by appropriate modifications of the setting (in particular, to pursue a viscosity theory, to be consistent with the elliptic framework, one would need the additional assumption that the kernel is nonnegative). For instance, for fractional elliptic equations a viscosity approach is useful to establish existence results by the Perron method, which combined with fractional elliptic regularity theory for viscosity solutions often provides the existence of nice solutions for the Dirichlet problem (see e.g. [68]). The viscosity setting will be discussed in Section 2.3.

Remark 2.1.10. From Definition 2.1.7 one immediately sees that for all $j \in \mathbb{N}$ and $K \in \mathcal{K}_{m,\vartheta} \cap \mathcal{K}_{m+j,\vartheta}$,

$$
\text{if } Au \overset{m}{=} f, \text{ then } Au \overset{m+j}{=} f,
$$

in B_1, since polynomials of degree at most $m-1$ are also polynomials of degree at most $m+j-1$.

Remark 2.1.11. From Definition 2.1.7 and Corollary 2.1.5 (used here with $\tau_R := \chi_R$, in the notation of (0.0.2)), we can write $Au \overset{m}{=} f_u$ in B_1 for any $K \in \mathcal{K}_{m,\vartheta}$ and $u \in \mathcal{C}_{\vartheta,K}$.

Remark 2.1.12. We observe that from Definition 2.1.7 it follows that any polynomial of degree less than or equal to $m-1$ can be arbitrarily added to f_R and subtracted from P_R in (2.1.27), hence, for any polynomial P with $\deg P \le m-1$ we have that

$$\text{if } Au \overset{m}{=} f, \text{ then } Au \overset{m}{=} f + P$$

in B_1.

We now investigate in further detail the convergence properties of the approximating source term f_R.

Lemma 2.1.13. *Let $m \in \mathbb{N}_0$, $\vartheta \in [0,2]$ and $K \in \mathcal{K}_{m,\vartheta}$. Let $u \in \mathcal{C}_{\vartheta,K}$, f and f_R be as in Definition 2.1.7.*

Then, if $R' > R > 4$ we have that

$$\inf \|f_{R'} - f_R - P\|_{L^\infty(B_1)} \le \int_{B_R^c} |u(y)| \sup_{\substack{|\alpha|=m \\ x\in B_1}} |\partial_x^\alpha K(x,y)|\, dy, \tag{2.1.29}$$

where the inf *in* (2.1.29) *is taken over all the polynomials P with degree at most $m-1$.*

Proof. We define $v := (1-\chi_4)u$. In this way $v = 0$ in B_4 and $|v| \le |u|$, so

$$v \in \mathcal{C}_{\vartheta,K}. \tag{2.1.30}$$

Moreover, if $R > 4$,

$$(\chi_R - \chi_4)u = (\chi_R - \chi_4)v.$$

Hence, from (2.1.27),

$$A((\chi_R-\chi_4)v) = A((\chi_R-\chi_4)u) = f_R - f_4 + P_R - P_4 = f_R - f_4 + \tilde{P}_R, \tag{2.1.31}$$

where $\tilde{P}_R := P_R - P_4$ is a polynomial of degree at most $m-1$.

We also remark that, due to (2.1.30), we can use Theorem 2.1.2 here on the function v. More specifically, using Theorem 2.1.2 on the function v (twice, once with $\tau := \chi_R$ and once with $\tau := \chi_4$), we obtain that

$$\begin{aligned} A((\chi_R - \chi_4)v) &= P_{v,\chi_R} - P_{v,\chi_4} + f_{v,\chi_R} - f_{v,\chi_4} \\ &= \bar{P}_{v,\chi_R} + (f_{1,v} + f_{2,v} + f^*_{v,\chi_R}) - (f_{1,v} + f_{2,v} + f^*_{v,\chi_4}) \\ &= \bar{P}_{v,\chi_R} + f^*_{v,\chi_R} - f^*_{v,\chi_4} \\ &= \bar{P}_{v,\chi_R} + \int_{B_R \setminus B_4} u(y)\psi(x,y)\,dy \end{aligned} \tag{2.1.32}$$

in B_1, where $\bar{P}_{v,\chi_R} := P_{v,\chi_R} - P_{v,\chi_4}$ is a polynomial of degree at most $m-1$. Comparing the right hand sides of (2.1.31) and (2.1.32), we obtain that in B_1

$$f_R = f_4 + P^*_R + \int_{B_R \setminus B_3} u(y)\psi(x,y)\,dy,$$

where $P^*_R := \bar{P}_{v,\chi_R} - \tilde{P}_R$ is a polynomial of degree at most $m-1$.

Therefore, for any $R' > R$,

$$\begin{aligned} f_{R'} - P^*_{R'} - f_R + P^*_R &= \left(f_4 + \int_{B_{R'} \setminus B_3} u(y)\psi(x,y)\,dy \right) \\ &\quad - \left(f_4 + \int_{B_R \setminus B_3} u(y)\psi(x,y)\,dy \right) \\ &= \int_{B_{R'} \setminus B_R} u(y)\psi(x,y)\,dy \end{aligned}$$

and, as a consequence,

$$\|f_{R'} - P^*_{R'} - f_R + P^*_R\|_{L^\infty(B_1)} = \|\Psi_{R',R}\|_{L^\infty(B_1)}, \tag{2.1.33}$$

where

$$\Psi_{R',R}(x) := \int_{B_{R'} \setminus B_R} u(y)\psi(x,y)\,dy.$$

From (2.1.8) and Remark 2.1.6, we know that

$$\begin{aligned} \|\Psi_{R',R}\|_{L^\infty(B_1)} &\le \sup_{x \in B_1} \int_{B_{R'} \setminus B_R} |u(y)||\psi(x,y)|\,dy \\ &\le \int_{B_{R'} \setminus B_R} |u(y)| \sup_{\substack{|\alpha|=m \\ x \in B_1}} |\partial^\alpha_x K(x,y)|\,dy \le \int_{B^c_R} |u(y)| \sup_{\substack{|\alpha|=m \\ x \in B_1}} |\partial^\alpha_x K(x,y)|\,dy. \end{aligned}$$

This and (2.1.33) imply that

$$\|f_{R'} - P^*_{R'} - f_R + P^*_R\|_{L^\infty(B_1)} \le \int_{B^c_R} |u(y)| \sup_{\substack{|\alpha|=m \\ x\in B_1}} |\partial^\alpha_x K(x,y)|\, dy,$$

which gives (2.1.29). □

Next result deals with the stability of the equation under uniform convergence (and this can be seen as an adaptation to our setting of the result contained e.g. in Lemma 5 of [19]).

Lemma 2.1.14. *Let $\vartheta \in [0,2]$ and $K \in \mathcal{K}_{0,\vartheta}$. For every $k \in \mathbb{N}$, let $u_k \in \mathcal{C}_{\vartheta,K}$ and f_k be bounded and continuous in B_1. Assume that*

$$Au_k = f_k \tag{2.1.34}$$

in B_1, that

$$f_k \text{ converges uniformly in } B_1 \text{ to some function } f \text{ as } k \to +\infty, \tag{2.1.35}$$

that

$$u_k \text{ converges in } B_4 \text{ to some function } u \text{ as } k \to +\infty$$
$$\text{in the topology of } \begin{cases} L^\infty(B_4) & \text{if } \vartheta = 0, \\ C^\vartheta(B_4) & \text{if } \vartheta \in (0,1), \\ C^{0,1}(B_4) & \text{if } \vartheta = 1, \\ C^{1,\vartheta-1}(B_4) & \text{if } \vartheta \in (1,2), \\ C^{1,1}(B_4) & \text{if } \vartheta = 2. \end{cases}$$

and that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n\setminus B_3} \big|u(y) - u_k(y)\big| |K(x,y)|\, dy = 0, \tag{2.1.36}$$

for every $x \in B_1$.

Then,

$$Au = f$$

in B_1.

Proof. Let $x_0 \in B_1$ and $\rho > 0$ such that $B_\rho(x_0) \Subset B_1$. We claim that

$$\lim_{k\to+\infty} \int_{B_\rho(x_0)} \big(u_k(x_0) - u_k(y)\big) K(x_0, y)\, dy = \int_{B_\rho(x_0)} \big(u(x_0) - u(y)\big) K(x_0, y)\, dy. \tag{2.1.37}$$

For this, we distinguish two cases. If $\vartheta \in [0, 1]$, we observe that

$$\begin{aligned} &\left| \int_{B_\rho(x_0)} \big(u_k(x_0) - u_k(y)\big) K(x_0, y)\, dy - \int_{B_\rho(x_0)} \big(u(x_0) - u(y)\big) K(x_0, y)\, dy \right| \\ \le\;& \int_{B_\rho(x_0)} \big|(u_k - u)(x_0) - (u_k - u)(y)\big|\, |K(x_0, y)|\, dy \\ \le\;& \|u_k - u\|_{C^\vartheta(B_4)} \int_{B_\rho(x_0)} |x_0 - y|^\vartheta\, |K(x_0, y)|\, dy \\ \le\;& C\, \|u_k - u\|_{C^\vartheta(B_4)} \end{aligned}$$

for some $C > 0$, thanks to (2.1.3), and this proves (2.1.37) in this case.

Hence, to complete the proof of (2.1.37), we now assume that $\vartheta \in (1, 2]$. In this case, we recall (2.1.4) and we see that, for k sufficiently large,

$$\begin{aligned} &\int_{B_\rho(x_0)} \big(u_k(x_0) - u_k(y)\big) K(x_0, y)\, dy \\ =\;& \frac{1}{2} \int_{B_\rho} \big(u_k(x_0) - u_k(x_0 + z)\big) K(x_0, x_0 + z)\, dz \\ +\;& \frac{1}{2} \int_{B_\rho} \big(u_k(x_0) - u_k(x_0 - z)\big) K(x_0, x_0 - z)\, dz \\ =\;& \frac{1}{2} \int_{B_\rho} \big(2u_k(x_0) - u_k(x_0 + z) - u_k(x_0 - z)\big) K(x_0, x_0 + z)\, dz, \end{aligned}$$

and a similar computation holds with u instead of u_k. Consequently, recalling also (2.1) (used here with $u_k - u$ in place of u),

$$\begin{aligned} &\left| \int_{B_\rho(x_0)} \big(u_k(x_0) - u_k(y)\big) K(x_0, y)\, dy - \int_{B_\rho(x_0)} \big(u(x_0) - u(y)\big) K(x_0, y)\, dy \right| \\ &\le \frac{1}{2} \int_{B_\rho} \big|2(u_k - u)(x_0) - (u_k - u)(x_0 + z) - (u_k - u)(x_0 - z)\big|\, |K(x_0, x_0 + z)|\, dz \\ &\le \frac{\|u_k - u\|_{C^{1,\vartheta-1}(B_4)}}{2} \int_{B_\rho} |z|^\vartheta\, |K(x_0, x_0 + z)|\, dz \\ &\le C\, \|u_k - u\|_{C^{1,\vartheta-1}(B_4)}, \end{aligned}$$

for some $C > 0$, thanks to (2.1.3), and this completes the proof of (2.1.37).

We now claim that

$$\lim_{k\to+\infty} \int_{B_3\setminus B_\rho(x_0)} \big(u_k(x_0)-u_k(y)\big) K(x_0,y)\,dy = \int_{B_3\setminus B_\rho(x_0)} \big(u(x_0)-u(y)\big) K(x_0,y)\,dy. \tag{2.1.38}$$

To prove it, we use (2.1.3) to conclude that

$$\begin{aligned}
&\left| \int_{B_3\setminus B_\rho(x_0)} \big(u_k(x_0) - u_k(y)\big) K(x_0,y)\,dy - \int_{B_3\setminus B_\rho(x_0)} \big(u(x_0) - u(y)\big) K(x_0,y)\,dy \right| \\
&\le C\,\|u_k - u\|_{L^\infty(B_4)} \int_{B_3\setminus B_\rho(x_0)} |K(x_0,y)|\,dy \\
&\le C\,\|u_k - u\|_{L^\infty(B_4)}
\end{aligned}$$

up to renaming $C > 0$ from line to line, and this establishes (2.1.38).

Furthermore, using (2.1.3),

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n\setminus B_3} \big(u_k(x_0) - u_k(y)\big) K(x_0,y)\,dy - \int_{\mathbb{R}^n\setminus B_3} \big(u(x_0) - u(y)\big) K(x_0,y)\,dy \right| \\
\le\; & \int_{\mathbb{R}^n\setminus B_3} \big|u_k(x_0) - u(x_0)\big|\,|K(x_0,y)|\,dy + \int_{\mathbb{R}^n\setminus B_3} \big|u_k(y) - u(y)\big||K(x_0,y)|\,dy \\
\le\; & C\,\|u_k - u\|_{L^\infty(B_4)} + \int_{\mathbb{R}^n\setminus B_3} \big|u_k(y) - u(y)\big||K(x_0,y)|\,dy,
\end{aligned}$$

which is infinitesimal thanks to (2.1.36).

Gathering together this, (2.1.37) and (2.1.38), we conclude that $Au_k(x_0) \to Au(x_0)$ as $k \to +\infty$. From this, (2.1.34) and (2.1.35) we obtain the desired result. □

Remark 2.1.15. *We observe that condition* (2.1.36) *cannot be dropped from Lemma 2.1.14. Indeed, if* $s \in (0,1)$ *and*

$$\mathbb{R} \ni x \mapsto u_k(x) := -\frac{\chi_{(k,k^2)}(x)\, x^{2s}}{\log k}$$

we have that $u_k \to 0 =: u$ *locally uniformly and that, for each* $x \in (-1,1)$,

$$\int_{\mathbb{R}} \frac{u_k(x) - u_k(y)}{|x-y|^{1+2s}}\,dy = \frac{1}{\log k} \int_k^{k^2} \frac{y^{2s}}{(y-x)^{1+2s}}\,dy =: f_k(x).$$

We stress that, if $x \in (-1,1)$ and $y > k$,

$$y - x \geq y - 1 = \frac{k-1}{k}\, y + \frac{y}{k} - 1 \geq \frac{k-1}{k}\, y$$

and

$$y - x \leq y + 1 = \frac{k+1}{k}\, y - \frac{y}{k} + 1 \leq \frac{k+1}{k}\, y.$$

As a result, if $x \in (-1,1)$,

$$f_k(x) \leq \frac{k^{1+2s}}{(k-1)^{1+2s}} \frac{1}{\log k} \int_k^{k^2} \frac{y^{2s}}{y^{1+2s}}\, dy = \frac{k^{1+2s}}{(k-1)^{1+2s}}$$

and

$$f_k(x) \geq \frac{k^{1+2s}}{(k+1)^{1+2s}} \frac{1}{\log k} \int_k^{k^2} \frac{y^{2s}}{y^{1+2s}}\, dy = \frac{k^{1+2s}}{(k+1)^{1+2s}},$$

thus $f_k \to 1 =: f$ uniformly in $(-1,1)$. This example shows that

$$(-\Delta)^s u_k = f_k \to f = 1 \neq 0 = (-\Delta)^s u.$$

A natural question is whether the stability result in Lemma 2.1.14 carries over directly to the setting introduced in Definition 2.1.7. The answer is in general negative, as pointed out by the following counterexample:

Proposition 2.1.16. *Let $k \in \mathbb{N}$. Let*

$$u_k(x) := \begin{cases} 0 & \text{if } x \in (-\infty, k], \\ kx & \text{if } x \in (k, +\infty) \end{cases}$$

and

$$f_k(x) := \frac{kx}{k-x} + k \log \frac{k}{k-x}.$$

Then,

$$\sqrt{-\Delta} u_k \stackrel{1}{=} f_k \text{ in } (-1,1), \tag{2.1.39}$$

$$u_k \text{ converges to zero locally uniformly,} \tag{2.1.40}$$

$$\lim_{k \to +\infty} \int_{\mathbb{R} \setminus (-1,1)} \frac{|u_k(y)|}{|y|^{2+a}}\, dy = 0 \text{ for all } a > 1, \tag{2.1.41}$$

$$f_k(x) \text{ converges to } 2x \text{ uniformly in } [-1,1]. \tag{2.1.42}$$

Proof. Let $R > k$ and $u_{k,R}(x) := u_k(x)\chi_{(-R,R)}(x)$. Then, if $x \in (-1,1)$,

$$\begin{aligned}
\int_{\mathbb{R}} \frac{u_{k,R}(x) - u_{k,R}(y)}{|x-y|^2}\,dy &= k\int_k^R \frac{y}{(y-x)^2}\,dy \\
&= \frac{kx}{k-x} - \frac{kx}{R-x} + k\log(R-x) - k\log(k-x) \\
&= \frac{kx}{k-x} - \frac{kx}{R-x} + k\log\frac{R-x}{R} + k\log R + k\log\frac{k}{k-x} - k\log k \\
&= f_k(x) + k\log\frac{R-x}{R} - \frac{kx}{R-x} + k\log R - k\log k.
\end{aligned}$$

Since the term $k\log R - k\log k$ is a constant in x (hence a polynomial of degree zero) and the function $k\log\frac{R-x}{R} - \frac{kx}{R-x}$ goes to zero as $R \to +\infty$, the identity above proves (2.1.39).

Additionally, the claim in (2.1.40) is obvious and, if $a > 1$,

$$\lim_{k\to+\infty}\int_{\mathbb{R}\setminus(-1,1)} \frac{|u_k(y)|}{|y|^{2+a}}\,dy = \lim_{k\to+\infty} k\int_k^{+\infty}\frac{dy}{y^{1+a}} = \frac{1}{a}\lim_{k\to+\infty}\frac{1}{k^{a-1}} = 0,$$

thus establishing (2.1.41).

Furthermore, for every $x \in [-1,1]$, if k is large enough,

$$\begin{aligned}
|f_k(x) - 2x| &\le \left|\frac{kx}{k-x} - x\right| + \left|k\log\frac{k}{k-x} - x\right| = \left|\frac{x^2}{k-x}\right| + \left|k\int_1^{\frac{k}{k-x}}\frac{dt}{t} - x\right| \\
&\le \frac{1}{k-1} + \left|k\int_1^{1+\frac{x}{k-x}}\frac{dt}{t} - \frac{kx}{k-x}\right| + \left|\frac{kx}{k-x} - x\right| \\
&\le \frac{2}{k-1} + k\left|\int_1^{1+\frac{x}{k-x}}\frac{dt}{t} - \int_1^{1+\frac{x}{k-x}} dt\right| \le \frac{2}{k-1} + k\int_{1-\frac{1}{k-1}}^{1+\frac{1}{k-1}}\frac{|1-t|}{t}\,dt \\
&\le \frac{6}{k-1},
\end{aligned}$$

which gives (2.1.42). $\square$

Concerning the example in Proposition 2.1.16, notice in particular that, in $(-1,1)$,

$$\sqrt{-\Delta}u_k \overset{1}{=} f_k \xrightarrow{k\to+\infty} 2x \overset{1}{\neq} 0 = \sqrt{-\Delta}0,$$

showing that some care is necessary to pass Definition 2.1.7 to the limit and additional assumptions are needed to exchange the order in which different limits are taken.

From the positive side, as an affirmative counterpart of the counterexample in Proposition 2.1.16, we provide the following stability result for the setting of Definition 2.1.7:

Proposition 2.1.17. *Let $m \in \mathbb{N}_0$, $\vartheta \in [0,2]$ and $K \in \mathcal{K}_{m,\vartheta}$. For every $k \in \mathbb{N}$, let $u_k \in \mathcal{C}_{\vartheta,K}$ and f_k be bounded and continuous in B_1. Assume that*

$$Au_k \stackrel{m}{=} f_k \quad \text{in } B_1, \tag{2.1.43}$$

that

$$f_k \text{ converges uniformly in } B_1 \text{ to some function } f \text{ as } k \to +\infty,$$

that

$$\begin{aligned} & u_k \text{ converges in } B_4 \text{ to some function } u \text{ as } k \to +\infty \\ & \text{in the topology of } \begin{cases} L^\infty(B_4) & \text{if } \vartheta = 0, \\ C^\vartheta(B_4) & \text{if } \vartheta \in (0,1), \\ C^{0,1}(B_4) & \text{if } \vartheta = 1, \\ C^{1,\vartheta-1}(B_4) & \text{if } \vartheta \in (1,2), \\ C^{1,1}(B_4) & \text{if } \vartheta = 2, \end{cases} \end{aligned} \tag{2.1.44}$$

that

$$\lim_{k\to+\infty} \sup_{\substack{x\in B_1\\ R>4}} \int_{B_R\setminus B_1(x)} \big((u-u_k)(x)-(u-u_k)(y)\big)K(x,y)\,dy = 0 \tag{2.1.45}$$

and that

$$\lim_{R\to+\infty} \sup_{k\in\mathbb{N}} \int_{\mathbb{R}^n\setminus B_R} |u_k(y)| \sup_{\substack{|\eta|=m\\ z\in B_1}} |\partial_x^\eta K(z,y)|\,dy = 0. \tag{2.1.46}$$

Then,

$$Au \stackrel{m}{=} f \quad \text{in } B_1. \tag{2.1.47}$$

To prove Proposition 2.1.17, we establish a uniqueness result in the spirit of Lemma 1.2 of [32]:

Lemma 2.1.18. *Let $m \in \mathbb{N}_0$, $\vartheta \in [0,2]$, $K \in \mathcal{K}_{m,\vartheta}$ and $u \in \mathcal{C}_{\vartheta,K}$. Let f_1*

and f_2 be bounded and continuous in B_1. Suppose that

$$Au \overset{m}{=} f_1 \text{ and } Au \overset{m}{=} f_2 \text{ in } B_1. \tag{2.1.48}$$

Then, there exists a polynomial of degree at most $m-1$ such that $f_1 - f_2 = P$.

Proof. In light of (2.1.48) and Definition 2.1.7, we have that there exist two families of polynomials P^1_R and P^2_R, with degree at most $m-1$, such that, for every $x \in B_1$,

$$\lim_{R\to+\infty} \big(A(\chi_R u)(x) - P^1_R(x)\big) = f_1(x)$$
$$\text{and} \qquad \lim_{R\to+\infty} \big(A(\chi_R u)(x) - P^2_R(x)\big) = f_2(x).$$

As a consequence, for every $x \in B_1$,

$$\begin{aligned} f_1(x) - f_2(x) &= \lim_{R\to+\infty} \big(A(\chi_R u)(x) - P^1_R(x)\big) - \lim_{R\to+\infty} \big(A(\chi_R u)(x) - P^2_R(x)\big) \\ &= \lim_{R\to+\infty} \big(P^2_R(x) - P^1_R(x)\big). \end{aligned}$$

We remark that $P^2_R - P^1_R$ is a polynomial of degree at most $m-1$. Accordingly, we can use Lemma 2.1 of [32] to conclude that $f_1 - f_2$ is a polynomial of degree at most $m-1$. This establishes the desired result. □

Proof of Proposition 2.1.17. We exploit the setting of Corollary 2.1.5 with $\tau_R := \chi_R$. In this way, for each k, we find a function $f_{u_k} : \mathbb{R}^n \to \mathbb{R}$ and a family of polynomials $P_{u_k,R}$, which have degree at most $m-1$, such that, in B_1,

$$\lim_{R\to+\infty} [A(\chi_R u_k)(x) - P_{u_k,R}(x)] = f_{u_k}(x). \tag{2.1.49}$$

As a matter of fact (recall the footnote on page 46), we see that, for every $x \in B_1$ and every $k \in \mathbb{N}$,

$$\Big| A(\chi_R u_k)(x) - P_{u_k,R}(x) - f_{u_k}(x) \Big| \le C \int_{\mathbb{R}^n \setminus B_R} |u_k(y)| \sup_{\substack{|\eta|=m \\ z\in B_1}} |\partial^\eta_x K(z,y)| \le \frac{1}{R}, \tag{2.1.50}$$

as long as R is sufficiently large, thanks to (2.1.46).

Comparing (2.1.49) with Definition 2.1.7, we thus conclude that $Au_k \overset{m}{=} f_{u_k}$ in B_1. This and (2.1.43), together with the uniqueness result in Lemma 2.1.18, yield that $f_{u_k} = f_k + \tilde{P}_k$ for a suitable polynomial $\tilde{P}_k$ of degree at most $m-1$.

Consequently, setting $\tilde{P}_{u_k,R} := P_{u_k,R} + \tilde{P}_k$, we have that, by (2.1.50),

$$\left|A(\chi_R u_k)(x) - \tilde{P}_{u_k,R}(x) - f_k(x)\right| \le \frac{1}{R}. \tag{2.1.51}$$

Now, we claim that, given $R > 4$,

$$\lim_{k\to+\infty} \sup_{B_1} \left|A\big((u-u_k)\chi_R\big)\right| = 0. \tag{2.1.52}$$

To this end, we calculate that, for each $x \in B_1$,

$$\begin{aligned}
&\left|A\big((u-u_k)\chi_R\big)(x)\right| \\
=\;& \Bigg|\int_{B_1(x)} \big((u-u_k)(x) - (u-u_k)(y)\big)K(x,y)\,dy \\
&+ \int_{B_R\setminus B_1(x)} \big((u-u_k)(x) - (u-u_k)(y)\big)K(x,y)\,dy \\
&+ \int_{\mathbb{R}^n\setminus B_R} (u-u_k)(x)\,K(x,y)\,dy\Bigg| \\
\le\;& \left|\int_{B_1(x)} \big((u-u_k)(x) - (u-u_k)(y)\big)K(x,y)\,dy\right| \\
&+ \left|\int_{B_R\setminus B_1(x)} \big((u-u_k)(x) - (u-u_k)(y)\big)K(x,y)\,dy\right| \\
&+ \|u-u_k\|_{L^\infty(B_1)} \int_{\mathbb{R}^n\setminus B_R} |K(x,y)|\,dy
\end{aligned}$$

and hence (2.1.52) follows from (2.1.3), (2.1.37) (used here with $\rho := 1$; notice that we can use (2.1.37) in this setting in light of (2.1.44)) and (2.1.45).

Thus, in light of (2.1.52), given $R > 4$ we can find $k_R \in \mathbb{N}$ such that

$$\sup_{B_1} \left|A\big((u-u_{k_R})\chi_R\big)\right| \le \frac{1}{R}. \tag{2.1.53}$$

We define $f_R := f_{k_R}$ and $P_R := \tilde{P}_{u_{k_R},R}$. We stress that P_R is a polynomial of degree at most $m-1$. Moreover, for every $x \in B_1$,

$$\begin{aligned}
&\Big|A(\chi_R u)(x) - f_R(x) - P_R(x)\Big| \\
&\quad \le \Big|A\big((u-u_{k_R})\chi_R\big)(x)\Big| + \Big|A(\chi_R u_{k_R})(x) - f_R(x) - P_R(x)\Big| \\
&\quad \le \frac{1}{R} + \Big|A(\chi_R u_{k_R})(x) - f_{k_R}(x) - \tilde{P}_{k_R}(x)\Big| \le \frac{2}{R}
\end{aligned}$$

thanks to (2.1.50) and (2.1.53), which proves (2.1.47). □

A consequence of Lemmata 2.1.13 and 2.1.14 is the following equivalence result:

Corollary 2.1.19. *Let $\vartheta \in [0,2]$, $K \in \mathcal{K}_{0,\vartheta}$ and $u \in \mathcal{C}_{\vartheta,K}$. Let f be bounded and continuous in B_1.*

Then

$$Au = f \ \textit{in}\ B_1$$

is equivalent to

$$Au \overset{0}{=} f \ \textit{in the sense of Definition 2.1.7.}$$

Proof. Suppose that $Au = f$ in B_1. Then, for $R > 10$,

$$\begin{aligned}
A(\chi_{R/2}u)(x) &= Au(x) - A((1-\chi_{R/2})u)(x) \\
&= f(x) + \int_{\mathbb{R}^n} (1-\chi_{R/2}(y))u(y)K(x,y)\,dy
\end{aligned} \tag{2.1.54}$$

for every $x \in B_1$. Now, we set

$$w := (\chi_R - \chi_{R/2})u.$$

We observe that $w = 0$ in B_4, so we can exploit Theorem 2.1.2 to w (applied here with $m = 0$) and get that, for any $x \in B_1$,

$$\begin{aligned}
&A((\chi_R - \chi_{R/2})u)(x) = Aw(x) \\
&\quad = f_{1,w} + f_{2,w} + f^*_{w,\chi_R}(x) = \int_{B_R\setminus B_3} w(y)\psi(x,y)\,dy = \int_{B_R\setminus B_{R/2}} u(y)\psi(x,y)\,dy.
\end{aligned} \tag{2.1.55}$$

Hence, from (2.1.54) and (2.1.55), we find that

$$\begin{aligned} A(\chi_R u)(x) &= A((\chi_R - \chi_{R/2})u)(x) + A(\chi_{R/2}u)(x) \\ &= \int_{B_R \setminus B_{R/2}} u(y)\psi(x,y)\,dy + f(x) + \int_{\mathbb{R}^n} (1-\chi_{R/2}(y))u(y)\psi(x,y)\,dy \\ &=: f_R(x) \end{aligned} \tag{2.1.56}$$

for every $x \in B_1$. We remark that $f_R \to f$ in B_1 as $R \to +\infty$, thanks to (2.1.8) (used here with $m = 0$ and $\gamma = (0, \dots, 0)$) and (2.1.6).

Now we recall Definition 2.1.7 (here with $m = 0$ and $P_R = 0$) and we conclude that $Au \stackrel{0}{=} f$ in B_1, as desired.

Conversely, we now suppose that $Au \stackrel{0}{=} f$ in B_1. From Definition 2.1.7 and the fact that $m = 0$, we have that P_R is identically zero, and so we can write that $A(\chi_R u) = f_R$ in B_1, with $f_R \to f$ in B_1 as $R \to +\infty$. We observe that $\chi_R u$ approaches u locally uniformly in $\mathbb{R}^n$. Also, we can use here Lemma 2.1.13: in this way, we find that

$$\|f_{R'} - f_R\|_{L^\infty(B_1)} \le \int_{B_R^c} |u(y)| \sup_{x \in B_1} |K(x,y)|\,dy.$$

Therefore, we send $R' \to +\infty$ and obtain that, for any $x \in B_1$,

$$\begin{aligned} |f(x) - f_R(x)| &= \lim_{R' \to +\infty} |f_{R'}(x) - f_R(x)| \\ &\le \lim_{R' \to +\infty} \|f_{R'} - f_R\|_{L^\infty(B_1)} \le \int_{B_R^c} |u(y)| \sup_{x \in B_1} |K(x,y)|\,dy. \end{aligned}$$

As a consequence, recalling (2.1.6) (here with $m = 0$) we have that f_R converges to f uniformly in B_1 as $R \to +\infty$. From this, we can exploit Lemma 2.1.14 and conclude that $Au = f$, as desired. □

A natural question deals with the consistency of the operator setting for functions that are sufficiently well-behaved to allow definitions related to two different indices: roughly speaking, in the best possible scenario, if we know that $Au \stackrel{m}{=} f$ and $j \le m$, can we say that $Au \stackrel{j}{=} f$? Posed like this, the answer to this question is negative, since, after all, in light of Lemma 2.1.18, the function f is uniquely defined only "up to a polynomial". Nevertheless, the answer becomes positive if we take into account this additional polynomial normalization, as stated in the next result:

Lemma 2.1.20. *Let $j, m \in \mathbb{N}_0$, with $j \le m$, $\vartheta \in [0,2]$ and $K \in \mathcal{K}_{j,\vartheta} \cap \mathcal{K}_{m,\vartheta}$. Let f be bounded and continuous in B_1 and let $u \in \mathcal{C}_{\vartheta,K}$ such that*

$$Au \stackrel{m}{=} f \tag{2.1.57}$$

in B_1.

Then, there exist a function $\bar{f}$ and a polynomial P of degree at most $m-1$, such that $\bar{f} = f + P$ and $Au \stackrel{j}{=} \bar{f}$ in B_1.

Proof. Let $v := (1-\chi_4)u$ and $w := \chi_4 u$. We notice that v, $w \in \mathcal{C}_{\vartheta,K}$. Hence, since $K \in \mathcal{K}_{j,\vartheta}$, recalling Remark 2.1.11, (2.1.10), (2.1.25) and (2.1.26), we can write that

$$Av \stackrel{j}{=} f_v = \int_{B_4^c} u(y)\psi(x,y)$$

in B_1. That is, by Definition 2.1.7,

$$A(\chi_R v) = \int_{B_4^c} u(y)\psi(x,y)\,dy + \tilde{\varphi}_R + Q_R, \tag{2.1.58}$$

for some $\tilde{\varphi}_R$ such that $\tilde{\varphi}_R \to 0$ in B_1 as $R \to +\infty$ and a polynomial Q_R of degree at most $j-1$.

Furthermore, by (2.1.57), and recalling Definition 2.1.7, we get that

$$A(\chi_R u) = f + \varphi_R + P_R, \tag{2.1.59}$$

for some φ_R such that $\varphi_R \to 0$ if $R \to +\infty$ and a polynomial P_R with $\deg P_R \le m-1$. Therefore, subtracting (2.1.58) from (2.1.59), we obtain

$$f+\varphi_R+P_R-\int_{B_4^c} u(y)\psi(x,y)\,dy-\tilde{\varphi}_R-Q_R = A(\chi_R(u-v)) = A(\chi_R w). \tag{2.1.60}$$

We notice that, for every $x \in B_1$ and every $R > 4$,

$$\begin{aligned} A(\chi_R w)(x) &= \int_{\mathbb{R}^n} (\chi_R w(x) - \chi_R w(y))\, K(x,y)\, dy \\ &= \int_{B_R} (w(x) - w(y))\, K(x,y)\, dy + \int_{\mathbb{R}^n \setminus B_R} w(x)\, K(x,y)\, dy \\ &= \int_{B_4} (w(x) - w(y))\, K(x,y)\, dy + \int_{\mathbb{R}^n \setminus B_4} w(x)\, K(x,y)\, dy \\ &= \int_{\mathbb{R}^n} (\chi_4 w(x) - \chi_4 w(y))\, K(x,y)\, dy = A(\chi_4 w)(x). \end{aligned}$$

As a consequence of this and (2.1.60), we have that, in B_1,

$$f + \varphi_R + P_R - \int_{B_4^c} u(y)\psi(x,y)\, dy - \tilde{\varphi}_R - Q_R = A(\chi_4 w).$$

This shows that the limit

$$\lim_{R \to +\infty} (\varphi_R + P_R - \tilde{\varphi}_R - Q_R)$$

exists. As a result, the limit

$$\lim_{R \to +\infty} (P_R - Q_R)$$

exists. Then, exploiting Lemma 2.1 in [32] we conclude that

$$\lim_{R \to +\infty} (P_R - Q_R) = P,$$

for some polynomial P of degree at most $m-1$.

Now we set $\bar{f} := f + P$ and $S_R := \varphi_R + P_R - Q_R - P$, and we see that $S_R \to 0$ as $R \to +\infty$. Thus, from (2.1.59) we obtain that

$$A(\chi_R u) = \bar{f} + S_R + Q_R$$

in B_1. Since the degree of Q_R is at most $j-1$, this shows that $Au \stackrel{j}{=} \bar{f}$ in B_1, as desired. □

2.2 The Dirichlet problem

In this section we consider the existence problem for equations involving general operators that are defined "up to a polynomial". The main result is the following:

Theorem 2.2.1. *Let* $m \in \mathbb{N}_0$, $\vartheta \in [0,2]$, $K \in \mathcal{K}_{0,\vartheta} \cap \mathcal{K}_{m,\vartheta}$ *and* $u \in \mathcal{C}_{\vartheta,K}$. *Assume that* $u_0 \in L^1_{\mathrm{loc}}(B_1^c)$ *satisfies* (2.1.5) *and* (2.1.6) *and* $f : B_1 \to \mathbb{R}$ *is bounded and continuous in* B_1.

Additionally, suppose that for any $\tilde{f} : B_1 \to \mathbb{R}$ *which is bounded and continuous in* B_1 *and any* $\tilde{u}_0 \in L^1(B_1^c)$ *there exists a unique solution* $\tilde{u} \in \mathcal{C}_\vartheta$ *to the Dirichlet problem*

$$\begin{cases} A\tilde{u} = \tilde{f} & \text{in } B_1, \\ \tilde{u} = \tilde{u}_0 & \text{in } B_1^c. \end{cases} \tag{2.2.1}$$

Then, there exists a function $u \in \mathcal{C}_{\vartheta,K}$ *such that*

$$\begin{cases} Au \overset{m}{=} f & \text{in } B_1, \\ u = u_0 & \text{in } B_1^c. \end{cases} \tag{2.2.2}$$

Also, the solution to (2.2.2) *is not unique, since the space of solutions of* (2.2.2) *has dimention* N_m, *with*

$$N_m := \sum_{j=0}^{m-1} \binom{j+n-1}{n-1}. \tag{2.2.3}$$

Proof. To begin with, we prove the existence of solutions for (2.2.2). To do this, we define

$$u_1 := \chi_{B_4^c}\, u_0 \qquad \text{and} \qquad \tilde{u}_0 := \chi_{B_4 \setminus B_1}\, u_0.$$

Since u_1 vanishes in B_4 and $K \in \mathcal{K}_{m,\vartheta}$, we can write $Au_1 \overset{m}{=} f_{u_1}$ in B_1, for some function f_{u_1}, due to Remark 2.1.11.

We now consider the solution of (2.2.1) with $\tilde{f} := f - f_{u_1}$. Therefore, using

Remark 2.1.10 and Corollary 2.1.19 we obtain that

$$\begin{cases} A\tilde{u} \overset{m}{=} \tilde{f} & \text{in } B_1, \\ \tilde{u} = \tilde{u}_0 & \text{in } B_1^c. \end{cases}$$

Then, we set $u := u_1 + \tilde{u}$ and we get that $Au = Au_1 + A\tilde{u} \overset{m}{=} f_{u_1} + \tilde{f} = f$ in B_1. Moreover, we have that $u = u_1 + \tilde{u}_0 = u_0$ in B_1^c, that is $u \in \mathcal{C}_{\vartheta,K}$ is solution of (2.2.2). This establishes the existence of solution for (2.2.2).

Now we prove that solutions of (2.2.2) are not unique and determine the dimension of the corresponding linear space. For this, we notice that for any polynomial P with $\deg P \leq m-1$ there exists a unique solution $\tilde{u}_P \in \mathcal{C}_\vartheta$ of the problem

$$\begin{cases} A\tilde{u}_P = P & \text{in } B_1, \\ \tilde{u}_P = 0 & \text{in } B_1^c, \end{cases} \tag{2.2.4}$$

due to the existence and uniqueness assumption for (2.2.1). This is equivalent to say that $A\tilde{u}_P \overset{0}{=} P$ in B_1, thanks to Corollary 2.1.19. Using Remark 2.1.10, we obtain that $A\tilde{u}_P \overset{m}{=} P$ in B_1. Thus, applying Remark 2.1.12, we obtain that $\tilde{u}_P$ is a solution of

$$\begin{cases} A\tilde{u}_P \overset{m}{=} 0 & \text{in } B_1, \\ \tilde{u}_P = 0 & \text{in } B_1^c. \end{cases} \tag{2.2.5}$$

From this it follows that if u is a solution of (2.2.2), then $u + \tilde{u}_P$ is also a solution of (2.2.2).

Viceversa, if u and v are two solutions of (2.2.2), then $w := u - v$ is a solution of

$$\begin{cases} Aw \overset{m}{=} 0 & \text{in } B_1, \\ w = 0 & \text{in } B_1^c. \end{cases}$$

Here we can apply Lemma 2.1.20 with $j := 0$ thus obtaining that $Aw \overset{0}{=} P$ in B_1, where P is a polynomial of $\deg P \leq m-1$. Using again Corollary 2.1.19, one deduces that

$$\begin{cases} Aw = P & \text{in } B_1, \\ w = 0 & \text{in } B_1^c. \end{cases} \tag{2.2.6}$$

Therefore, the uniqueness of the solution of (2.2.6), confronted with (2.2.4), gives us that $w = \tilde{u}_P$, and thus $v = u + \tilde{u}_P$.

This reasoning gives that the space of solutions of (2.2.2) is isomorphic to the space of polynomials with degree less than or equal to $m-1$, which has exactly dimension N_m, given by (2.2.3) (see e.g. [34]). □

2.3 A viscosity approach

Up to now, we focused our attention on the case of equations defined pointwise. In principle, this requires functions that are "sufficiently regular" for the equation to be satisfied at every given point. However, a less restrictive approach adopted in the classical theory of elliptic equations is to consider weaker notions of solutions (and possibly recover the pointwise setting via an appropriate regularity theory): in this spirit, a convenient setting, which is also useful in case of fully nonlinear equations, is that of viscosity solutions, which does not require a high degree of regularity of the solution itself since the equation is computed pointwise only at smooth functions touching from either below or above (see e.g. [15] for a thorough discussion on viscosity solutions).

In this section, we recast the setting of general operators defined "up to a polynomial" into the viscosity solution framework. To this end, we proceed as follows. For all $m \in \mathbb{N}_0$ and $\vartheta \in [0,2]$, we define $\mathcal{K}^+_{m,\vartheta}$ as the space of kernels $K = K(x,y)$ verifying (2.1.2), (2.1.3) and (2.1.4), and such that

$$K(x,y) \geq 0 \text{ for all } x \in B_1 \text{ and } y \in \mathbb{R}^n. \tag{2.3.1}$$

Given $K \in \mathcal{K}^+_{m,\vartheta}$, we consider the space $\mathcal{V}_K$ of all the functions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap C(B_4) \cap L^\infty(B_4)$ for which

$$\sum_{|\alpha|\leq m-1} \int_{B_R \setminus B_3} |u(y)|\, |\partial^\alpha_x K(x,y)|\, dy < +\infty \qquad \text{for all } R > 3 \text{ and } x \in B_1 \tag{2.3.2}$$

$$\text{and} \int_{B_3^c} |u(y)| \sup_{\substack{|\alpha|=m \\ x\in B_1}} |\partial^\alpha_x K(x,y)|\, dy < +\infty. \tag{2.3.3}$$

Remark 2.3.1. Notice that if (2.3.1) holds true and $K \in \mathcal{K}_{m,\vartheta}$ for some $m \in \mathbb{N}_0$ and some $\vartheta \in [0,2]$, then $K \in \mathcal{K}^+_{m,\vartheta}$.

In the viscosity framework we introduce the following definition.

Definition 2.3.2. *Let $m \in \mathbb{N}_0$, $\vartheta \in [0,2]$, $K \in \mathcal{K}^+_{m,\vartheta}$, $u \in \mathcal{V}_K$ and $f : B_1 \to \mathbb{R}$ be bounded and continuous. We say that*

$$Au \overset{m}{=} f \quad \text{in } B_1 \text{ in the viscosity sense}$$

if there exist a family of polynomials P_R, with $\deg P_R \le m-1$, and bounded and continuous functions $f_R : B_1 \to \mathbb{R}$ such that

$$A(\chi_R u) = f_R + P_R \tag{2.3.4}$$

in B_1 in the viscosity sense, with

$$\lim_{R\to+\infty} f_R(x) = f(x) \quad \text{uniformly in } B_1. \tag{2.3.5}$$

Remark 2.3.3. We point out that the limit in (2.3.5) is assumed to hold uniformly (this is a stronger assumption than the one in (2.1.28) that was assumed for the pointwise setting, and it is taken here to make the setting compatible with the viscosity method, see e.g. the proof of the forthcoming Corollary 2.3.8). See also [1] for related observations.

Remark 2.3.4. We observe that, for all $j \in \mathbb{N}$ and $K \in \mathcal{K}^+_{m,\vartheta} \cap \mathcal{K}^+_{m+j,\vartheta}$,

$$\text{if } Au \overset{m}{=} f, \text{ then } Au \overset{m+j}{=} f$$

in B_1 in the viscosity sense of Definition 2.3.2.

Remark 2.3.5. From Definition 2.3.2 it follows that any polynomial of degree less than or equal to $m-1$ can be arbitrarily added to f_R and subtracted from P_R in (2.3.4), hence, for any polynomial P with $\deg P \le m-1$ we have that

$$\text{if } Au \overset{m}{=} f, \text{ then } Au \overset{m}{=} f + P$$

in B_1 in the viscosity sense of Definition 2.3.2.

We now establish that when the structure is compatible with the both the settings in Definitions 2.1.7 and 2.3.2, the pointwise and viscosity frameworks are equivalent:

Lemma 2.3.6. *Let $m \in \mathbb{N}_0$, $\vartheta \in [0,2]$, $K \in \mathcal{K}_{m,\vartheta}$, $u \in \mathcal{C}_{\vartheta,K}$ and $f : B_1 \to \mathbb{R}$*

be bounded and continuous. If (2.3.1) *holds true and u is a solution of*

$$Au \overset{m}{=} f \text{ in the sense of Definition 2.1.7,}$$

then $K \in \mathcal{K}^+_{m,\vartheta}$, $u \in \mathcal{V}_K$ and it is a solution of

$$Au \overset{m}{=} f \text{ in the viscosity sense of Definition 2.3.2.}$$

Conversely, let $m \in \mathbb{N}_0$, $\vartheta \in [0,2]$, $K \in \mathcal{K}^+_{m,\vartheta}$, $u \in \mathcal{V}_K$ and $f : B_1 \to \mathbb{R}$ be bounded and continuous. If $K \in \mathcal{K}_{m,\vartheta}$ and $u \in \mathcal{C}_{\vartheta,K}$ is a solution of

$$Au \overset{m}{=} f \text{ in the viscosity sense of Definition 2.3.2,}$$

then u is a solution of

$$Au \overset{m}{=} f \text{ in the sense of Definition 2.1.7.}$$

Proof. Assume that u is a solution of $Au \overset{m}{=} f$ in B_1 in the sense of Definition 2.1.7. It follows that there exist a family of polynomials P_R with $\deg P_R \le m-1$ and functions f_R such that

$$A(\chi_R u) = f_R + P_R \tag{2.3.6}$$

pointwise in B_1. Since $u \in \mathcal{C}_{\vartheta,K}$ we have that also $u \in \mathcal{V}_K$. Moreover, we have that $K \in \mathcal{K}^+_{m,\vartheta}$, due to (2.3.1) and Remark 2.3.1.

Now we observe that if $v \in \mathcal{C}_\vartheta$ and vanishes outside B_2, g is a bounded and continuous function, and $Av = g$ pointwise in B_1, then also

$$Av = g \text{ in } B_1 \text{ in the viscosity sense.} \tag{2.3.7}$$

To check this, let φ be a smooth function touching v from below at some point $x_0 \in B_1$. Then, we have that $v(y) - \varphi(y) \ge 0 = v(x_0) - \varphi(x_0)$ for all $y \in \mathbb{R}^n$ and therefore, by (2.3.1),

$$\begin{aligned} A\varphi(x_0) &= \int_{\mathbb{R}^n} (\varphi(x_0) - \varphi(y))K(x_0,y)\,dy \\ &\ge \int_{\mathbb{R}^n} (v(x_0) - v(y))K(x_0,y)\,dy = Av(x_0) = g(x_0). \end{aligned}$$

Similarly, if φ touches v from above, one obtains the opposite inequality, and these observations complete the proof of (2.3.7).

As a consequence of (2.3.6) and (2.3.7), we obtain that

$$A(\chi_R u) = f_R + P_R$$

in B_1 in the viscosity sense.

Hence, to finish the first part of the proof, we show that

$$f_R \to f \text{ uniformly in } B_1. \tag{2.3.8}$$

For this, we observe that, in light of Remark 2.1.11, we can write $Au \overset{m}{=} f_u$ in B_1 in the sense of Definition 2.1.7. That is, recalling Corollary 2.1.5, there exist functions $f_{u,R}$ and a family of polynomials P_{u,τ_R}, which have degree at most $m-1$, such that

$$A(\chi_R u) = f_{u,R} + P_{u,\chi_R} \tag{2.3.9}$$

pointwise in B_1.

Furthermore, using Lemma 2.1.13, we have that, if $R' > R > 4$, there exists a polynomial $P_{R,R'}$ of degree at most $m-1$ such that

$$\|f_{u,R'} - f_{u,R} - P_{R,R'}\|_{L^\infty(B_1)} \le \int_{B_R^c} |u(y)| \sup_{\substack{|\alpha|=m \\ x\in B_1}} |\partial_x^\alpha K(x,y)|\, dy. \tag{2.3.10}$$

We now claim that

$$P_{R,R'} = 0. \tag{2.3.11}$$

Indeed, by a careful inspection of the proof of Lemma 2.1.13, one can notice that the polynomial $P_{R,R'}$ is explicit and, denoting by $v := (1-\chi_4)u$, it is equal to

$$P_{v,\chi_{R'}} - P_{u,\chi_{R'}} - P_{v,\chi_R} + P_{u,\chi_R},$$

where we have used the notation of Theorem 2.1.2. In particular, recalling the notation in formulas (2.1.19) and (2.1.21), we have that the coefficients of the polynomial $P_{R,R'}$ are given by

$$\begin{aligned}
&\int_{B_3^c} (\chi_{R'} v)(y) \frac{\partial_x^\alpha K(0,y)}{\alpha!}\, dy - \int_{B_3^c} (\chi_{R'} u)(y) \frac{\partial_x^\alpha K(0,y)}{\alpha!}\, dy \\
&\qquad - \int_{B_3^c} (\chi_R v)(y) \frac{\partial_x^\alpha K(0,y)}{\alpha!}\, dy + \int_{B_3^c} (\chi_R u)(y) \frac{\partial_x^\alpha K(0,y)}{\alpha!}\, dy \\
= &\int_{B_{R'}\setminus B_4} u(y) \frac{\partial_x^\alpha K(0,y)}{\alpha!}\, dy - \int_{B_{R'}\setminus B_3} u(y) \frac{\partial_x^\alpha K(0,y)}{\alpha!}\, dy \\
&\qquad - \int_{B_R\setminus B_4} u(y) \frac{\partial_x^\alpha K(0,y)}{\alpha!}\, dy + \int_{B_R\setminus B_3} u(y) \frac{\partial_x^\alpha K(0,y)}{\alpha!}\, dy \\
= & - \int_{B_4\setminus B_3} u(y) \frac{\partial_x^\alpha K(0,y)}{\alpha!}\, dy + \int_{B_4\setminus B_3} u(y) \frac{\partial_x^\alpha K(0,y)}{\alpha!}\, dy \\
= &\ 0,
\end{aligned}$$

for every $|\alpha| \leq m-1$, which proves (2.3.11).

Hence, using the information of formula (2.3.11) into (2.3.10), we obtain that $f_{u,R}$ converges to f_u uniformly in B_1.

Now, as a consequence of (2.3.6) and (2.3.9), we have that

$$\lim_{R\to+\infty} \big(P_{u,\chi_R} - P_R\big) = \lim_{R\to+\infty} \big(f_R - f_{u,R}\big) = f - f_u \tag{2.3.12}$$

in B_1. Therefore, in light of Lemma 2.1 in [32], we have that the convergence in (2.3.12) is uniform in B_1. Furthermore,

$$\|f_R - f\|_{L^\infty(B_1)} \leq \|f_R - f_{u,R} + f_u - f\|_{L^\infty(B_1)} + \|f_{u,R} - f_u\|_{L^\infty(B_1)}.$$

These considerations prove (2.3.8) and therefore, the first part of the proof is complete.

Now take $K \in \mathcal{K}^+_{m,\vartheta}$ and a solution $u \in \mathcal{C}_{\vartheta,K}$ to $Au \stackrel{m}{=} f$ in B_1 in the viscosity sense of Definition 2.3.2. We have that there exist a family of polynomials P_R with $\deg P_R \leq m-1$ and functions f_R such that

$$A(\chi_R u) = f_R + P_R \text{ in } B_1 \text{ in the viscosity sense.} \tag{2.3.13}$$

Our objective is now to check that

$$\text{the equation in (2.3.13) holds true in the pointwise sense as well.} \tag{2.3.14}$$

Indeed, once this is established, we can send $R \to +\infty$ and conclude that $Au \overset{m}{=} f$ in the pointwise sense of Definition 2.1.7. To prove (2.3.14), we use a convolution argument. We pick $\rho \in (0,1)$ and we define v_ε to be the convolution of $\chi_R u$ against a given mollifier η_ε. We also denote by g_ε the convolution of $f_R + P_R$ against η_ε and we remark that, if ε is small enough, then $Av_\varepsilon = g_\varepsilon$ in B_ρ in the viscosity sense, and actually also in the pointwise sense, since v_ε is smooth and can be used itself as a test function in the viscosity definition. Hence, we can take any point $x_0 \in B_\rho$ and conclude that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}(v_\varepsilon(x_0)-v_\varepsilon(y))K(x_0,y)\,dy\\&\qquad=\lim_{\varepsilon\to 0}Av_\varepsilon(x_0)=\lim_{\varepsilon\to 0}g_\varepsilon(x_0)=f_R(x_0)+P_R(x_0).\end{aligned}\tag{2.3.15}$$

We now claim that, for all $R > 5$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}(v_\varepsilon(x_0)-v_\varepsilon(y))K(x_0,y)\,dy=\int_{\mathbb{R}^n}(\chi_R(x_0)u(x_0)-\chi_R(y)u(y))K(x_0,y)\,dy.\tag{2.3.16}$$

We stress that, once this is proved, then (2.3.14) would follow directly from (2.3.15). Hence, our goal now is to check (2.3.16). We perform the argument when $\vartheta \in (1,2]$ (the argument when $\vartheta \in [0,1]$ being similar and simpler, not requiring any additional symmetrization). We exploit (2.1.4) to see that

$$\begin{aligned}&2\int_{B_1(x_0)}(v_\varepsilon(x_0)-v_\varepsilon(y))K(x_0,y)\,dy\\&\quad=\int_{B_1}(v_\varepsilon(x_0)-v_\varepsilon(x_0+z))K(x_0,x_0+z)\,dz\\&\quad+\int_{B_1}(v_\varepsilon(x_0)-v_\varepsilon(x_0-z))K(x_0,x_0-z)\,dz\\&\quad=\int_{B_1}\big(2v_\varepsilon(x_0)-v_\varepsilon(x_0+z)-v_\varepsilon(x_0-z)\big)K(x_0,x_0+z)\,dz.\end{aligned}\tag{2.3.17}$$

Also, since $u \in \mathcal{C}_{\vartheta,K}$ (and we are supposing $\vartheta \in (1,2]$), for all $z \in B_1$,

$$\begin{aligned}&|2v_\varepsilon(x_0)-v_\varepsilon(x_0+z)-v_\varepsilon(x_0-z)|=\left|\int_0^1\Big(\nabla v_\varepsilon(x_0+tz)-\nabla v_\varepsilon(x_0-tz)\Big)\cdot z\,dt\right|\\&\le|z|\int_0^1\Big|\nabla v_\varepsilon(x_0+tz)-\nabla v_\varepsilon(x_0-tz)\Big|\,dt\\&=|z|\int_0^1\left|\int_{B_\varepsilon}\nabla(\chi_R u)(x_0+tz-\zeta)\eta_\varepsilon(\zeta)\,d\zeta-\int_{B_\varepsilon}\nabla(\chi_R u)(x_0-tz-\zeta)\eta_\varepsilon(\zeta)\,d\zeta\right|\,dt\end{aligned}$$

$$\begin{aligned}
&\le |z| \int_0^1 \int_{B_\varepsilon} |\nabla(\chi_R u)(x_0 + tz - \zeta) - \nabla(\chi_R u)(x_0 - tz - \zeta)|\, \eta_\varepsilon(\zeta)\, d\zeta\, dt \\
&\le C|z|^\vartheta \int_0^1 \int_{B_\varepsilon} \eta_\varepsilon(\zeta)\, d\zeta\, dt \\
&= C|z|^\vartheta,
\end{aligned}$$

for some $C > 0$.

From this and the Dominated Convergence Theorem, recalling (2.1.3), we deduce from (2.3.17) that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} 2 \int_{B_1(x_0)} (v_\varepsilon(x_0) - v_\varepsilon(y)) K(x_0, y)\, dy \\
&\qquad = \int_{B_1} \lim_{\varepsilon\to 0} \big(2v_\varepsilon(x_0) - v_\varepsilon(x_0 + z) - v_\varepsilon(x_0 - z)\big) K(x_0, x_0 + z)\, dz \\
&\qquad = \int_{B_1} \big(2(\chi_R u)(x_0) - (\chi_R u)(x_0 + z) - (\chi_R u)(x_0 - z)\big) K(x_0, x_0 + z)\, dz \\
&\qquad = 2 \int_{B_1(x_0)} \big((\chi_R u)(x_0) - (\chi_R u)(y)\big) K(x_0, y)\, dy.
\end{aligned}$$

Using again the Dominated Convergence Theorem, one deduces (2.3.16) from the previous equation, as desired. □

Next result shows the stability of the equation under the uniform convergence in the viscosity sense. To this end, we will also assume other mild conditions on the kernel. First of all, we assume a continuity hypothesis in the first variable, that is we suppose that

$$\text{for all } y \in \mathbb{R}^n \text{ and all } x_0 \in B_3 \setminus \{y\}, \quad \lim_{x\to x_0} K(x, y) = K(x_0, y). \tag{2.3.18}$$

Additionally, we assume a local integrability condition outside a possible singularity of the kernel and a locally uniform version of condition (2.1.3), namely we suppose that

$$\text{for all } x_0 \in B_3 \text{ and all } r > 0, \quad \int_{\mathbb{R}^n \setminus B_r(x_0)} \sup_{x\in B_3\setminus B_r(y)} |K(x, y)|\, dy < +\infty \tag{2.3.19}$$

and

$$\int_{B_3} \sup_{x\in B_1} |x - y|^2\, |K(x, y)|\, dy < +\infty. \tag{2.3.20}$$

Lemma 2.3.7. *Let* $\vartheta \in [0,2]$. *Let* $K \in \mathcal{K}^+_{0,\vartheta}$ *satisfying* (2.3.18), (2.3.19) *and* (2.3.20). *For every* $k \in \mathbb{N}$, *let* $u_k \in \mathcal{V}_K$ *and* f_k *be bounded and continuous in* B_1. *Assume that*

$$Au_k = f_k \tag{2.3.21}$$

in B_1 *in the viscosity sense, that*

$$f_k \text{ converges uniformly in } B_1 \text{ to some function } f \text{ as } k \to +\infty,$$

that

$$u_k \text{ converges uniformly in } B_4 \text{ to some function } u \in \mathcal{V}_K \text{ as } k \to +\infty \tag{2.3.22}$$

and that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n\setminus B_3} \big|u(y) - u_k(y)\big| \sup_{x\in B_1} |K(x,y)|\, dy = 0. \tag{2.3.23}$$

Then,

$$Au = f$$

in B_1 *in the viscosity sense.*

Proof. Let $x_0 \in B_1$ and $\rho > 0$ such that $B_\rho(x_0) \Subset B_1$. Let $\varphi \in C^2(\overline{B_\rho(x_0)})$ with $\varphi = u$ outside $B_\rho(x_0)$. Suppose that $v := \varphi - u$ has a local maximum at x_0.

We define, for every $k \in \mathbb{N}$,

$$\varepsilon_k := \|u - u_k\|_{L^\infty(B_1)} + \frac{1}{k}$$

and

$$\varphi_k(x) := \begin{cases} \varphi(x) - \sqrt{\varepsilon_k}\,|x - x_0|^2 & \text{in } \overline{B_\rho(x_0)}, \\ u_k(x) & \text{in } \mathbb{R}^n \setminus \overline{B_\rho(x_0)}. \end{cases}$$

We let $v_k := \varphi_k - u_k$ and $x_k \in \overline{B_\rho(x_0)}$ be such that

$$v_k(x_k) = \max_{\overline{B_\rho(x_0)}} v_k.$$

We observe that

$$\begin{aligned}
|x_k - x_0|^2 &= \frac{\varphi(x_k) - \varphi_k(x_k)}{\sqrt{\varepsilon_k}} \\
&= \frac{v(x_k) - v_k(x_k) + u(x_k) - u_k(x_k)}{\sqrt{\varepsilon_k}} \\
&\le \frac{v(x_0) - v_k(x_0) + \varepsilon_k}{\sqrt{\varepsilon_k}} \\
&= \frac{\varphi(x_0) - \varphi_k(x_0) - u(x_0) + u_k(x_0) + \varepsilon_k}{\sqrt{\varepsilon_k}} \\
&= \frac{-u(x_0) + u_k(x_0) + \varepsilon_k}{\sqrt{\varepsilon_k}} \\
&\le 2\sqrt{\varepsilon_k}.
\end{aligned}$$

Thus, since ε_k is infinitesimal due to (2.3.22), we have that x_k converges to x_0 as $k \to +\infty$ and, in particular, the function v_k has an interior maximum at x_k. This and (2.3.21) give that

$$0 \le A\varphi_k(x_k) - f_k(x_k) \le A\varphi_k(x_k) - f(x_0) + |f(x_0) - f(x_k)| + \|f - f_k\|_{L^\infty(B_1)}. \tag{2.3.24}$$

Now we claim that

$$\lim_{k\to+\infty} \int_{B_\rho(x_0)} \big(\varphi_k(x_k) - \varphi_k(y)\big) K(x_k, y)\, dy = \int_{B_\rho(x_0)} \big(\varphi(x_0) - \varphi(y)\big) K(x_0, y)\, dy. \tag{2.3.25}$$

To this end, we first observe that

$$\begin{aligned}
\varphi_k(x_k) - \varphi_k(y) &= \varphi(x_k) - \varphi(y) + \sqrt{\varepsilon_k}|y - x_0|^2 - \sqrt{\varepsilon_k}|x_k - x_0|^2 \\
&= \varphi(x_k) - \varphi(y) + \sqrt{\varepsilon_k}(2x_0 - x_k - y) \cdot (x_k - y).
\end{aligned} \tag{2.3.26}$$

We define

$$F(y) := \sup_{x\in B_1} |x - y|^2\, |K(x, y)| \tag{2.3.27}$$

and we observe that $F \in L^1(B_3)$, thanks to (2.3.20). Accordingly, by the absolute continuity of the Lebesgue integrals, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if the Lebesgue measure of a set $Z \subset B_3$ is less than δ, then

$$\int_Z F(y)\, dy \le \varepsilon. \tag{2.3.28}$$

We recall (2.1.4) and we see that, for k sufficiently large,

$$\begin{aligned}
&\int_{B_{\rho/2}(x_k)} \big(\varphi_k(x_k)-\varphi_k(y)\big)K(x_k,y)\,dy \\
=\;& \frac{1}{2}\int_{B_{\rho/2}} \big(\varphi_k(x_k)-\varphi_k(x_k+z)\big)K(x_k,x_k+z)\,dz \\
&+\frac{1}{2}\int_{B_{\rho/2}} \big(\varphi_k(x_k)-\varphi_k(x_k-z)\big)K(x_k,x_k-z)\,dz \\
=\;& \frac{1}{2}\int_{B_{\rho/2}} \big(2\varphi_k(x_k)-\varphi_k(x_k+z)-\varphi_k(x_k-z)\big)K(x_k,x_k+z)\,dz \\
=\;& \frac{1}{2}\int_{B_{\rho}(x_0)} \chi_{B_{\rho/2}(x_k)}(y)\big(2\varphi_k(x_k)-\varphi_k(y)-\varphi_k(2x_k-y)\big)K(x_k,y)\,dy.
\end{aligned}$$

Thus, for all $y\in B_\rho(x_0)$ we define

$$\zeta_k(y) := \frac{1}{2}\chi_{B_{\rho/2}(x_k)}(y)\big(2\varphi_k(x_k)-\varphi_k(y)-\varphi_k(2x_k-y)\big)K(x_k,y)$$

and we point out that

$$\begin{aligned}
&|\zeta_k(y)| \le \frac{1}{2}\Big|\big(\varphi_k(x_k)-\varphi_k(y)\big)+\big(\varphi_k(x_k)-\varphi_k(2x_k-y)\big)\Big|\,|K(x_k,y)| \\
&\quad= \frac{1}{2}\left|\left(\int_0^1 \nabla\varphi_k\big(tx_k+(1-t)y\big)\,dt\right.\right. \\
&\qquad\qquad\left.\left. -\int_0^1 \nabla\varphi_k\big(tx_k+(1-t)(2x_k-y)\big)\,dt\right)\cdot(x_k-y)\right|\,|K(x_k,y)| \\
&\quad= \left|\int_0^1 \left((1-t)\int_0^1 D^2\varphi_k\Big(\tau\big(tx_k+(1-t)y\big)\right.\right. \\
&\qquad\left.\left. +\,(1-\tau)\big(tx_k+(1-t)(2x_k-y)\big)\Big)\,d\tau\right)\,dt\,(x_k-y)\cdot(x_k-y)\right|\,|K(x_k,y)| \\
&\quad\le C\,|x_k-y|^2\,|K(x_k,y)| \\
&\quad\le C\,F(y),
\end{aligned}$$

where the notation in (2.3.27) was used. In particular, since $F\in L^1(B_3)$ by (2.3.20), we can exploit the absolute continuity of the Lebesgue integrals (see (2.3.28)) and deduce that for all $\varepsilon>0$ there exists $\delta>0$ such that if the Lebesgue measure of a set $Z\subset B_\rho(x_0)$ is less than δ, then

$$\int_Z |\zeta_k(y)|\,dy \le C\varepsilon.$$

Hence, recalling (2.3.18), we utilize the Vitali Convergence Theorem and obtain that

$$\begin{aligned}\lim_{k\to+\infty}\int_{B_\rho(x_0)}\zeta_k(y)\,dy &= \int_{B_\rho(x_0)}\lim_{k\to+\infty}\zeta_k(y)\,dy\\ &= \frac{1}{2}\int_{B_{\rho/2}(x_0)}\big(2\varphi(x_0)-\varphi(y)-\varphi(2x_0-y)\big)K(x_0,y)\end{aligned}$$

which proves that

$$\lim_{k\to+\infty}\int_{B_{\rho/2}(x_0)}\big(\varphi_k(x_k)-\varphi_k(y)\big)K(x_k,y)\,dy=\int_{B_{\rho/2}(x_0)}\big(\varphi(x_0)-\varphi(y)\big)K(x_0,y)\,dy. \tag{2.3.29}$$

Now, for every $y\in\mathbb{R}^n\setminus B_{\rho/2}(x_0)$ we define

$$\eta_k(y):=\big(\varphi_k(x_k)-\varphi_k(y)\big)K(x_k,y).$$

We observe that, if k is sufficiently large, for every $y\in B_3\setminus B_{\rho/2}(x_0)$,

$$|\eta_k(y)|\le C\sup_{x\in B_3\setminus B_{\rho/4}(y)}|K(x,y)|,$$

which belongs to $L^1(B_\rho(x_0)\setminus B_{\rho/2}(x_0))$ due to (2.3.19).

Consequently, by (2.3.18) and the Dominated Convergence Theorem,

$$\begin{aligned}\lim_{k\to+\infty}\int_{B_\rho(x_0)\setminus B_{\rho/2}(x_0)}\big(\varphi_k(x_k)-\varphi_k(y)\big)K(x_k,y)\,dy &= \lim_{k\to+\infty}\int_{B_\rho(x_0)\setminus B_{\rho/2}(x_0)}\eta_k(y)\,dy\\ =\int_{B_\rho(x_0)\setminus B_{\rho/2}(x_0)}\lim_{k\to+\infty}\eta_k(y)\,dy &= \int_{B_\rho(x_0)\setminus B_{\rho/2}(x_0)}\big(\varphi(x_0)-\varphi(y)\big)K(x_0,y)\,dy.\end{aligned}$$

This and (2.3.29) show the validity of (2.3.25).

Having completed the proof of (2.3.25), we now claim that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n\setminus B_\rho(x_0)}\big(\varphi_k(x_k)-\varphi_k(y)\big)K(x_k,y)\,dy=\int_{\mathbb{R}^n\setminus B_\rho(x_0)}\big(\varphi(x_0)-\varphi(y)\big)K(x_0,y)\,dy. \tag{2.3.30}$$

Indeed, for all $y\in\mathbb{R}^n\setminus\overline{B_\rho(x_0)}$ we set

$$\mu_k(y):=\big(\varphi_k(x_k)-\varphi_k(y)\big)K(x_k,y)=\big(\varphi(x_k)-\sqrt{\varepsilon_k}|x_k-x_0|^2-u_k(y)\big)K(x_k,y).$$

If $y \in B_3 \setminus \overline{B_\rho(x_0)}$ and k is sufficiently large, then, by (2.3.22),

$$|\mu_k(y)| \le C\,(1+|u(y)|)\,|K(x_k,y)| \le C\,(1+\|u\|_{L^\infty(B_3)}) \sup_{x\in B_3\setminus B_{\rho/2}(y)} |K(x,y)|$$

and the latter function belongs to $L^1(B_3 \setminus \overline{B_\rho(x_0)})$ owing to (2.3.19).

This, (2.3.18), (2.3.22) and the Dominated Convergence Theorem lead to

$$\begin{aligned}
&\lim_{k\to+\infty} \int_{B_3\setminus B_\rho(x_0)} \big(\varphi_k(x_k)-\varphi_k(y)\big)K(x_k,y)\,dy \\
&\quad= \int_{B_3\setminus B_\rho(x_0)} \lim_{k\to+\infty} \big(\varphi(x_k)-\sqrt{\varepsilon_k}|x_k-x_0|^2-u_k(y)\big)K(x_k,y)\,dy \qquad (2.3.31)\\
&\quad= \int_{B_3\setminus B_\rho(x_0)} \big(\varphi(x_0)-u(y)\big)K(x_0,y)\,dy.
\end{aligned}$$

In light of (2.3.3) and (2.3.23) we also remark that, for large k,

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B_3} |u_k(y)| \sup_{x\in B_1} K(x,y)\,dy \\
&\quad\le \int_{\mathbb{R}^n\setminus B_3} \big|u(y)-u_k(y)\big| \sup_{x\in B_1} K(x,y)\,dy + \int_{\mathbb{R}^n\setminus B_3} |u(y)| \sup_{x\in B_1} K(x,y)\,dy \\
&\quad\le 1+\int_{\mathbb{R}^n\setminus B_3} |u(y)| \sup_{x\in B_1} K(x,y)\,dy \le C,
\end{aligned} \qquad (2.3.32)$$

up to renaming C once again. Furthermore, if $y \in \mathbb{R}^n \setminus B_3$ and $\delta_k := \big|\varphi(x_k) - \varphi(x_0)\big| + \sqrt{\varepsilon_k}$, we have that δ_k is infinitesimal as $k \to +\infty$ and

$$\begin{aligned}
&\Big|\big(\varphi_k(x_k)-\varphi_k(y)\big)K(x_k,y)-\big(\varphi(x_0)-u(y)\big)K(x_0,y)\Big| \\
\le\ & \Big(\big|\varphi(x_k)-\varphi(x_0)\big|+\sqrt{\varepsilon_k}+|u_k(y)-u(y)|\Big)|K(x_k,y)| \\
&+\big|\varphi(x_0)-u(y)\big|\,\big|K(x_0,y)-K(x_k,y)\big| \\
\le\ & \delta_k \sup_{x\in B_1}|K(x,y)| + |u_k(y)-u(y)| \sup_{x\in B_1}|K(x,y)| \\
&+C\,(1+|u(y)|)\,\big|K(x_0,y)-K(x_k,y)\big| \\
\le\ & \delta_k \sup_{x\in B_1\setminus B_1(y)}|K(x,y)| + |u_k(y)-u(y)| \sup_{x\in B_1}|K(x,y)| \\
&+C\,(1+|u(y)|)\,\big|K(x_0,y)-K(x_k,y)\big|.
\end{aligned}$$

Gathering this information, (2.3.19) and (2.3.23), we find that

$$\begin{aligned}&\lim_{k\to+\infty}\int_{\mathbb{R}^n\setminus B_3}\Big|\big(\varphi_k(x_k)-\varphi_k(y)\big)K(x_k,y)-\big(\varphi(x_0)-u(y)\big)K(x_0,y)\Big|\,dy\\&\qquad\le\lim_{k\to+\infty}C\int_{\mathbb{R}^n\setminus B_3}(1+|u(y)|)\,\big|K(x_0,y)-K(x_k,y)\big|\,dy.\end{aligned}\tag{2.3.33}$$

We also point out that, if $y\in\mathbb{R}^n\setminus B_3$,

$$(1+|u(y)|)\,\big|K(x_0,y)-K(x_k,y)\big|\le 2(1+|u(y)|)\sup_{x\in B_1\setminus B_1(y)}|K(x,y)|$$

and the latter function belongs to $L^1(\mathbb{R}^n\setminus B_3)$, thanks to (2.3.3) and (2.3.19). The Dominated Convergence Theorem and (2.3.18) thereby give that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n\setminus B_3}(1+|u(y)|)\,\big|K(x_0,y)-K(x_k,y)\big|\,dy=0.$$

This and (2.3.33) yield that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n\setminus B_3}\Big|\big(\varphi_k(x_k)-\varphi_k(y)\big)K(x_k,y)-\big(\varphi(x_0)-u(y)\big)K(x_0,y)\Big|\,dy=0,$$

which, combined with (2.3.31), proves (2.3.30).

By combining (2.3.25) and (2.3.30), we deduce that $A\varphi_k(x_k)\to A\varphi(x_0)$ as $k\to+\infty$. As a result, by passing to the limit in (2.3.24), we conclude that $A\varphi(x_0)\ge f(x_0)$.

Similarly, one sees that if the function $\varphi-u$ has a local minimum at x_0 then $A\varphi(x_0)\le f(x_0)$. □

Corollary 2.3.8. *Let $\vartheta\in[0,2]$. Let $K\in\mathcal{K}^+_{0,\vartheta}$ satisfying* (2.3.18), (2.3.19) *and* (2.3.20) *and $u\in\mathcal{V}_K$. Let f be bounded and continuous in B_1.*

Then

$$Au=f\ \text{ in } B_1$$

in viscosity sense is equivalent to

$$Au\overset{0}{=}f\ \text{ in } B_1$$

in the viscosity sense of Definition 2.3.2.

Proof. Suppose first that $Au = f$ in B_1 in viscosity sense. For every $R > 5$, we define

$$f_R(x) := f(x) + \int_{B_R^c} u(y)K(x,y)\,dy.$$

Notice that

$$\sup_{x\in B_1} |f_R(x) - f(x)| \le \int_{B_R^c} |u(y)| \sup_{x\in B_1} K(x,y)\,dy,$$

which is infinitesimal, thanks to (2.3.3) (used here with $m := 0$), and thus f_R converges to f uniformly in B_1.

Our objective is to prove that $A(\chi_R u) = f_R$ in B_1 in the viscosity sense (from which we obtain that $Au \stackrel{0}{=} f$ in B_1 in the viscosity sense of Definition 2.3.2).

To check this claim, we pick a point $x_0 \in B_1$ and touch $\chi_R u$ from below by a test function φ at x_0, with $\varphi = \chi_R u$ outside B_2. We define $\psi := \varphi + (1-\chi_R)u$ and we observe that ψ touches u by below at x_0 and that $\psi = u$ outside B_2. As a result, $A\psi(x_0) \ge f(x_0)$ and therefore

$$\begin{aligned}
f_R(x_0) &= f(x_0) + \int_{B_R^c} u(y)K(x_0,y)\,dy \\
&\le A\psi(x_0) + \int_{B_R^c} u(y)K(x_0,y)\,dy \\
&= \int_{\mathbb{R}^n} (\psi(x_0) - \psi(y))K(x_0,y)\,dy + \int_{B_R^c} u(y)K(x_0,y)\,dy \\
&= \int_{\mathbb{R}^n} (\varphi(x_0) - \psi(y))K(x_0,y)\,dy + \int_{B_R^c} u(y)K(x_0,y)\,dy \\
&= \int_{B_R} (\varphi(x_0) - \varphi(y))K(x_0,y)\,dy + \int_{B_R^c} (\varphi(x_0) - \psi(y))K(x_0,y)\,dy \\
&\qquad + \int_{B_R^c} u(y)K(x_0,y)\,dy \\
&= A\varphi(x_0) + \int_{B_R^c} (\varphi(y) - \psi(y))K(x_0,y)\,dy + \int_{B_R^c} u(y)K(x_0,y)\,dy \\
&= A\varphi(x_0).
\end{aligned}$$

Similarly, if φ touches $\chi_R u$ from above, then $A\varphi(x_0) \le f_R(x_0)$. These observations entail that $A(\chi_R u) = f_R$ in B_1 in the viscosity sense.

This proves one of the implications of Corollary 2.3.8. To prove the other, we assume now that $Au \stackrel{0}{=} f$ in B_1 in the viscosity sense of Definition 2.3.2. Then, we find $f_R : B_1 \to \mathbb{R}$ such that $A(\chi_R u) = f_R$ in B_1 in viscosity sense, with f_R converging to f uniformly in B_1. We remark that

$$\lim_{R\to+\infty}\int_{\mathbb{R}^n\setminus B_3}\big|u(y)-(\chi_R u)(y)\big|\sup_{x\in B_1}|K(x,y)|\,dy=0,$$

thanks to (2.3.3) (used here with $m := 0$) and the Dominated Convergence Theorem.

We can therefore apply Lemma 2.3.7 and conclude that $Au = f$ in B_1 in the sense of viscosity, as desired. □

In the next result we state the viscosity counterpart of Lemma 2.1.20 (its proof is omitted since it is similar to the one of Lemma 2.1.20, just noticing that the functions v and $\chi_R w - \chi_4 w$ vanish in B_4, hence the viscous and pointwise setting would equally apply to them).

Lemma 2.3.9. *Let $j, m \in \mathbb{N}_0$, with $j \le m$, $\vartheta \in [0,2]$ and $K \in \mathcal{K}^+_{j,\vartheta} \cap \mathcal{K}^+_{m,\vartheta}$. Let f be bounded and continuous in B_1 and let $u \in \mathcal{V}_K$ such that*

$$Au \overset{m}{=} f \tag{2.3.34}$$

in B_1.

Then, there exist a function $\bar f$ and a polynomial P of degree at most $m-1$ such that $\bar f = f + P$ and $Au \overset{j}{=} \bar f$ in B_1.

2.3.1 Applications

Next, as a possible application, we show a specific case in which the existence of solution to a Dirichlet problem is guaranteed. For this, we consider a family of kernels comparable to the fractional Laplace operator, as follows. For any $s \in (0,1)$, given real numbers $\Lambda \ge \lambda > 0$, we consider the family of kernels K as defined in (2.0.6). We suppose that

$$\text{there exists } m \in \mathbb{N}_0 \text{ such that condition (2.1.2) is satisfied.} \tag{2.3.35}$$

We also assume that condition (2.1.4) holds true and that K is translation invariant, i.e.

$$K(x+z, y+z) = K(x,y) \text{ for any } x,\ y,\ z \in \mathbb{R}^n. \tag{2.3.36}$$

With this, we have that K belongs to $\mathcal{K}^+_{m,\vartheta}$, with m as in (2.3.35) and for every $\vartheta \in (2s, 2]$. Moreover, it also satisfies (2.3.19) and (2.3.20).

We introduce the fractional Sobolev space

$$\mathbb{H}^s(B_1) := \left\{ u \in L^2(B_1) : \iint_{\mathbb{R}^{2n}\setminus(B_1^c \times B_1^c)} (u(x) - u(y))^2 K(x, y)\, dy < +\infty \right\}$$

and, given $g \in \mathbb{H}^s(B_1)$, the class

$$J_g(B_1) := \Big\{ u \in \mathbb{H}^s(B_1) : \; u = g \;\text{ in } B_1^c \Big\}.$$

We use this class to seek solutions to the Dirichlet problem (see [63]). More precisely, the following result can be proved by using the Direct Methods of the Calculus of Variations and the strict convexity of the functional.

Proposition 2.3.10. *Let K be as in* (2.0.6), (2.1.4), (2.3.35) *and* (2.3.36)*. Let $f \in L^2(B_1)$ and $g \in \mathbb{H}^s(B_1)$. Then, there exists a unique minimizer of the functional*

$$\mathcal{E}(u) := \frac{1}{4} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))^2 K(x, y)\, dx\, dy - \int_{B_1} f(x) u(x)\, dx \qquad (2.3.37)$$

over $J_g(B_1)$.

In addition, $u \in J_g(B_1)$ is a minimizer of (2.3.37) *over $J_g(B_1)$ if and only if it is a weak solution of*

$$\begin{cases} Au = f & \text{in } B_1, \\ u = g & \text{in } B_1^c, \end{cases} \qquad (2.3.38)$$

that is, for every $\varphi \in C_0^\infty(B_1)$,

$$\frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)\, K(x, y)\, dx\, dy = \int_{B_1} f(x)\varphi(x)\, dx.$$

The next result is a generalization of Theorem 2 in [67], which shows the global continuity of weak solutions of an equation which includes the operator of our interest.

Proposition 2.3.11. *Let K be as in* (2.0.6) (2.1.4)*,* (2.3.35) *and* (2.3.36)*. Let*

$f \in L^\infty(B_1)$ and $g \in C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0, \min\{2s, 1\})$. Assume that

$$|g(x)| \le C|x|^\alpha \qquad \text{for all } x \in \mathbb{R}^n \setminus B_1.$$

Let also $u \in J_g(B_1)$ be a weak solution of

$$Au = f \quad \text{in } B_1. \tag{2.3.39}$$

Then, $u \in C(\mathbb{R}^n)$.

Proof. First of all, we exploit Proposition 2.3.10 with $g := 0$ to find a weak solution v of

$$\begin{cases} Av = f & \text{in } B_1, \\ v = 0 & \text{in } B_1^c. \end{cases}$$

By Proposition 7.2 in [65] (see also [47] for related results), we have that $v \in C(\mathbb{R}^n)$.

Let now $w := u - v$. We see that w is a weak solution of

$$\begin{cases} Aw = 0 & \text{in } B_1, \\ w = u = g & \text{in } B_1^c. \end{cases}$$

We thus exploit Theorem 1.4 in [6] and find that $w \in C(\mathbb{R}^n)$. From these observations, we find that $u = v + w \in C(\mathbb{R}^n)$, as desired. □

With this, we can now prove that, in this setting, week solutions are also viscosity solutions.

Proposition 2.3.12. *Let K be as in* (2.0.6) (2.1.4), (2.3.18), (2.3.35) *and* (2.3.36). *Let f be bounded and continuous in B_1 and $g \in C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0, \min\{2s, 1\})$. Assume that*

$$|g(x)| \le C|x|^\alpha \qquad \text{for all } x \in \mathbb{R}^n \setminus B_1. \tag{2.3.40}$$

Let also $u \in J_g(B_1)$ be a weak solution of

$$\begin{cases} Au = f & \text{in } B_1, \\ u = g & \text{in } B_1^c. \end{cases} \tag{2.3.41}$$

Then, u is a viscosity solution of (2.3.41).

Proof. By Proposition 2.3.11, we know that $u \in C(\mathbb{R}^n)$.

Now, we take a point $x_0 \in B_1$ and a function $\rho \in C_0^\infty(B_1,[0,1])$ and we consider an even mollifier $\rho_\varepsilon := \varepsilon^{-n}\rho(x/\varepsilon)$, for any $\varepsilon \in (0,1)$. We set $u_\varepsilon := u * \rho_\varepsilon$ and $f_\varepsilon := f * \rho_\varepsilon$ (where we identified f with its null extension outside B_1).

We claim that

$$Au_\varepsilon = f_\varepsilon \text{ in the weak sense in any ball } B_r(x_0) \text{ such that } B_r(x_0) \Subset B_1. \tag{2.3.42}$$

To prove this, we take a ball $B_r(x_0)$ such that $B_r(x_0) \Subset B_1$ and a function $\varphi \in C_0^\infty(B_\rho(x_0))$. We observe that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{2n}}(u(x+z)-u(y+z))(\varphi(x)-\varphi(y))\rho_\varepsilon(z)K(x,y)\,dx\,dy\right)dz\\
&=\int_{B_\varepsilon}\Bigg(\iint_{\mathbb{R}^{2n}\setminus(B_r^c(x_0)\times B_r^c(x_0))}(u(x+z)-u(y+z))\\
&\qquad\qquad\times(\varphi(x)-\varphi(y))\rho_\varepsilon(z)K(x,y)\,dx\,dy\Bigg)\,dz\\
&\le\frac{\varepsilon^{-n}}{2}\int_{B_\varepsilon}\Bigg(\iint_{\mathbb{R}^{2n}\setminus(B_1^c\times B_1^c)}(u(x)-u(y))^2K(x,y)\,dx\,dy\\
&\qquad\qquad+\iint_{\mathbb{R}^{2n}\setminus(B_1^c\times B_1^c)}(\varphi(x)-\varphi(y))^2K(x,y)\,dx\,dy\Bigg)\,dz\\
&<+\infty,
\end{aligned} \tag{2.3.43}$$

thanks to (2.3.36).

Therefore, Tonelli's Theorem gives us that the function

$$(x,y,z)\in\mathbb{R}^{2n}\times\mathbb{R}^n\mapsto(u(x+z)-u(y+z))(\varphi(x)-\varphi(y))\rho_\varepsilon(z)K(x,y)$$

lies in $L^1(\mathbb{R}^{2n}\times\mathbb{R}^n)$. One can interchange the order of integration in (2.3.43), thanks to Fubini's Theorem, and exploit the definition of u_ε to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{2n}}(u(x+z)-u(y+z))(\varphi(x)-\varphi(y))\rho_\varepsilon(z)K(x,y)\,dx\,dy\right)dz\\
&\quad=\iint_{\mathbb{R}^{2n}}\left(\int_{\mathbb{R}^n}(u(x+z)-u(y+z))(\varphi(x)-\varphi(y))\rho_\varepsilon(z)K(x,y)\,dz\right)dx\,dy\\
&\quad=\iint_{\mathbb{R}^{2n}}(u_\varepsilon(x)-u_\varepsilon(y))(\varphi(x)-\varphi(y))K(x,y)\,dx\,dy.
\end{aligned} \tag{2.3.44}$$

Then, we can use Fubini's Theorem once again to get

$$\begin{aligned}\int_{\mathbb{R}^n} & f_\varepsilon(x)\varphi(x)\,dx \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(x+z)\rho_\varepsilon(z)\varphi(x)\,dx\right)\,dz \\ &= \int_{B_\varepsilon}\left(\int_{\mathbb{R}^n} f(\tilde{x})\varphi(\tilde{x}-z)\rho_\varepsilon(z)d\tilde{x}\right)\,dz\end{aligned} \tag{2.3.45}$$

$$\begin{aligned}&= \frac{1}{2}\int_{B_\varepsilon}\left(\iint_{\mathbb{R}^{2n}}(u(\tilde{x})-u(\tilde{y}))(\varphi(\tilde{x}-z)-\varphi(\tilde{y}-z))\rho_\varepsilon(z)K(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}\right)\,dz \\ &= \frac{1}{2}\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{2n}}(u(x+z)-u(y+z))(\varphi(x)-\varphi(y))\rho_\varepsilon(z)K(x,y)\,dx\,dy\right)\,dz \\ &= \frac{1}{2}\iint_{\mathbb{R}^{2n}}(u_\varepsilon(x)-u_\varepsilon(y))(\varphi(x)-\varphi(y))K(x,y)\,dx\,dy,\end{aligned} \tag{2.3.46}$$

since the kernel K is translation invariant and u satisfies (2.3.41) in weak sense. This shows (2.3.42).

Now, given $B_r(x_0) \Subset B_1$, we show that

$$\text{the map } B_r(x_0) \ni x \mapsto \int_{\mathbb{R}^n}(u_\varepsilon(x)-u_\varepsilon(y))\,K(x,y)\,dy \text{ is continuous.} \tag{2.3.47}$$

For this, we let x_k be a sequence converging to a given point $x \in B_r(x_0)$ and we define

$$\zeta_k(z) := (2u_\varepsilon(x_k)-u_\varepsilon(x_k+z)-u_\varepsilon(x_k-z))\,K(0,z).$$

Since u_ε is smooth and its growth at infinity is controlled via (2.3.40), we know that

$$|2u_\varepsilon(x_k)-u_\varepsilon(x_k+z)-u_\varepsilon(x_k-z)| \le C_\varepsilon \min\{|z|^2,|z|^\alpha\},$$

for some $C_\varepsilon > 0$. For this reason and (2.0.6),

$$|\zeta_k(z)| \le \frac{C_\varepsilon \min\{|z|^2,|z|^\alpha\}}{|z|^{n+2s}},$$

up to renaming C_ε and therefore we are in the position of applying the Dominated

Convergence Theorem and conclude that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}(2u_\varepsilon(x_k)-u_\varepsilon(x_k+z)-u_\varepsilon(x_k-z))\,K(0,z)\,dz$$
$$=\int_{\mathbb{R}^n}(2u_\varepsilon(x)-u_\varepsilon(x+z)-u_\varepsilon(x-z))\,K(0,z)\,dz.$$

In view of (2.1.4) and (2.3.36), this proves (2.3.47).

We also observe that

$$Au_\varepsilon=f_\varepsilon \text{ pointwise in any ball } B_r(x_0) \text{ such that } B_r(x_0)\Subset B_1. \tag{2.3.48}$$

Indeed, by (2.1.4), (2.3.42) and (2.3.47), if $x\in B_r(x_0)$ and $\varphi\in C_0^\infty(B_r(x_0))$,

$$\int_{\mathbb{R}^n} f_\varepsilon(x)\varphi(x)\,dx=\iint_{\mathbb{R}^{2n}}(u_\varepsilon(x)-u_\varepsilon(y))\varphi(x)K(x,y)\,dx\,dy.$$

Since φ is arbitrary, we arrive at

$$f_\varepsilon(x)=\int_{\mathbb{R}^n}(u_\varepsilon(x)-u_\varepsilon(y))K(x,y)\,dy,$$

from which we obtain (2.3.48).

We also have that

$$Au_\varepsilon=f_\varepsilon \text{ in the viscosity sense in any ball } B_r(x_0) \text{ such that } B_r(x_0)\Subset B_1. \tag{2.3.49}$$

For this, we take a smooth function ψ touching, say from below, the function u_ε at some point $p\in B_r(x_0)$. Since the kernel K is positive (thanks to (2.0.6)) and recalling (2.3.48), we have that

$$f_\varepsilon(p)=\int_{\mathbb{R}^n}(u_\varepsilon(p)-u_\varepsilon(y))K(p,y)\,dy=\int_{\mathbb{R}^n}(\psi(p)-u_\varepsilon(y))K(p,y)\,dy$$
$$\le\int_{\mathbb{R}^n}(\psi(p)-\psi(y))K(p,y)\,dy=A\psi(p).$$

This and a similar computation when ψ touches from above give (2.3.49).

We also remark that u_ε and f_ε converge uniformly to u and f, respectively, in any ball $B_r(x_0)\Subset B_1$, due to Theorem 9.8 in [71]. In addition, by (2.3.40), we see that, for every $y \in \mathbb{R}^n \setminus B_{3r}(x_0)$,

$$|u_\varepsilon(y)| \le \int_{B_\varepsilon} |u(y-z)|\rho_\varepsilon(z)\,dz \le C\int_{B_\varepsilon} |y-z|^\alpha \rho_\varepsilon(z)\,dz \le C|y|^\alpha,$$

up to renaming $C > 0$. As a consequence of this and (2.0.6), we have that, for every $y \in \mathbb{R}^n \setminus B_{3r}(x_0)$,

$$|u(y) - u_\varepsilon(y)| \sup_{x\in B_r(x_0)} K(x,y) \le C|y|^\alpha \sup_{x\in B_r(x_0)} \frac{1}{|x-y|^{n+2s}} \le \frac{C}{|y|^{n+2s-\alpha}},$$

up to relabeling $C > 0$. Since $\alpha < \min\{2s, 1\}$, this function is in $L^1(\mathbb{R}^n \setminus B_{3r}(x_0))$, and therefore we exploit the Dominated Convergence Theorem to obtain that

$$\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^n\setminus B_{3r}(x_0)} |u(y)-u_\varepsilon(y)| \sup_{x\in B_1} K(x,y)\,dy = 0.$$

Consequently, condition (2.3.23) is satisfied, and therefore we can apply Lemma 2.3.7, thus obtaining that $Au = f$ in the viscosity sense, as desired. □

With this preliminary work, we can now address the existence of solutions for a Dirichlet problem in a generalized setting.

Theorem 2.3.13. *Let K be as in* (2.0.6) (2.1.4), (2.3.18), (2.3.35) *and* (2.3.36). *Let f be bounded and continuous in B_1 and $g \in C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0, \min\{2s, 1\})$. Assume that*

$$|g(x)| \le C|x|^\alpha \qquad \text{for all } x \in \mathbb{R}^n \setminus B_1.$$

Then, there exists a function $u \in \mathcal{V}_K$ such that

$$\begin{cases} Au \overset{m}{=} f & \text{in } B_1,\\ u = g & \text{in } B_1^c. \end{cases} \tag{2.3.50}$$

Also, the solution to (2.3.50) *is not unique, since the space of solutions of* (2.3.50) *has dimension N_m, with*

$$N_m := \sum_{j=0}^{m-1} \binom{j+n-1}{n-1}.$$

Proof. Firstly, we prove the existence of solutions for (2.3.50). For this goal, we set

$$u_1 := \chi_{B_4^c}\, g \qquad \text{and} \qquad \tilde{g} := \chi_{B_4 \setminus B_1}\, g.$$

Since u_1 is identically zero in B_4 and $K \in \mathcal{K}^+_{m,\vartheta}$, we can write $Au_1 \overset{m}{=} f_{u_1}$ in B_1 in both pointwise and viscosity sense, for some function f_{u_1}, due to Remark 2.1.11 and Lemma 2.3.6.

We now define $\tilde{f} := f - f_{u_1}$ and consider the Dirichlet problem given by

$$\begin{cases} A\tilde{u} = \tilde{f} & \text{in } B_1, \\ \tilde{u} = \tilde{g} & \text{in } B_1^c. \end{cases} \tag{2.3.51}$$

By Proposition 2.3.10, we find that (2.3.51) has a unique weak solution $\tilde{u}$. Moreover, thanks to Proposition 2.3.12, we get that $\tilde{u}$ is a viscosity solution of (2.3.51).

Furthermore, by Remark 2.3.4 and Corollary 2.3.8 we obtain that

$$\begin{cases} A\tilde{u} \overset{m}{=} \tilde{f} & \text{in } B_1, \\ \tilde{u} = \tilde{g} & \text{in } B_1^c. \end{cases}$$

Now, we set $u := u_1 + \tilde{u}$ and we get that $Au = Au_1 + A\tilde{u} \overset{m}{=} f_{u_1} + \tilde{f} = f$ in B_1. Moreover, we have that $u = u_1 + \tilde{g} = g$ in B_1^c. These observations give that is u is solution of (2.3.50). This proves the existence of solution for (2.3.50).

Now, we focus on the second part of the proof. Namely we establish that solutions of (2.3.50) are not unique and we determine the dimension of the corresponding linear space. For this, we notice that, exploiting Propositions 2.3.10 and 2.3.12, one can find a unique solution $\tilde{u}_P$ of the problem

$$\begin{cases} A\tilde{u}_P = P & \text{in } B_1, \\ \tilde{u}_P = 0 & \text{in } B_1^c. \end{cases} \tag{2.3.52}$$

Furthermore, $A\tilde{u}_P \overset{0}{=} P$ in B_1, due to Corollary 2.3.8. Using Remark 2.3.4, we obtain that $A\tilde{u}_P \overset{m}{=} P$ in B_1. Moreover, from Remark 2.3.5, we obtain that $\tilde{u}_P$ is a solution of

$$\begin{cases} A\tilde{u}_P \overset{m}{=} 0 & \text{in } B_1, \\ \tilde{u}_P = 0 & \text{in } B_1^c. \end{cases} \tag{2.3.53}$$

This yields that if u is a solution of (2.3.50), then $u + \tilde{u}_P$ is also a solution of (2.3.50).

Viceversa, if u and v are two solutions of (2.3.50), then $w := u - v$ is a solution of

$$\begin{cases} Aw \overset{m}{=} 0 & \text{in } B_1, \\ w = 0 & \text{in } B_1^c. \end{cases}$$

Here we can apply Lemma 2.3.9 with $j := 0$ thus obtaining that $Aw \overset{0}{=} P$ in B_1, where P is a polynomial of $\deg P \leq m - 1$. We use Corollary 2.3.8 one more time to find that

$$\begin{cases} Aw = P & \text{in } B_1, \\ w = 0 & \text{in } B_1^c. \end{cases} \tag{2.3.54}$$

Therefore, the uniqueness of the solution of (2.3.54), confronted with (2.3.52), gives us that $w = \tilde{u}_P$, and thus $v = u + \tilde{u}_P$.

This reasoning gives that the space of solutions of (2.3.50) is isomorphic to the space of polynomials with degree less than or equal to $m - 1$, which has dimension N_m, given by (2.2.3). □

2.4 Further possible developments

Another interesting application of the setting "up to a polynomial" can be its possible extension to the noncommutative groups, in particular, to the Carnot groups. Now we will introduce the fractional operator of our interest in the spirit of the fractional Laplace operator (1.2.1), leaving apart the subsequent theory that will be cultivated in a forthcoming research project.

In the paper [40], see Theorem 3.11, the authors found a representation for the sub-Laplacian in an arbitrary Carnot group $\mathbb{G}$. Precisely, given a parameter $s \in (0, 1)$, they introduced

$$\tilde{R}_{-2s}(x) := \frac{s}{|\Gamma(-s)|} \int_0^{+\infty} \frac{h(t,x)}{t^{1+s}} dt, \tag{2.4.1}$$

where the heat kernel $h(t, x) : (0, +\infty) \times \mathbb{G} \to [0, +\infty)$ is the fundamental solution of the heat operator $\partial_t - \Delta_{\mathbb{G}}$ (refer to [43]). Moreover, the following result holds true.

Theorem 2.4.1. *Let $s \in (0,1)$ and $u \in \mathcal{S}(\mathbb{G})$. Then*

$$-(-\Delta_{\mathbb{G}})^s u(x) = P.V. \int_{\mathbb{G}} (u(y)-u(x))\tilde{R}_{-2s}(y^{-1}x)dy, \tag{2.4.2}$$

where $\Delta_{\mathbb{G}}$ is defined as in (1.4.2).

Observe that the kernel $R_{-2s}(x)$, being a positive and homogeneous function of degree $-2s-Q$ and smooth in $\mathbb{G}\setminus\{0\}$, satisfies all the properties of a homogeneous norm, if taken to the power $-\frac{1}{Q+2s}$. Hence, one can set

$$\|x\|_s := \tilde{R}_{-2s}^{-\frac{1}{Q+2s}}(x) \tag{2.4.3}$$

and thus

$$-(-\Delta_{\mathbb{G}})^s u(x) = \text{P.V.} \int_{\mathbb{G}} \frac{u(y)-u(x)}{\|y^{-1}x\|_s^{Q+2s}} dy.$$

Remark 2.4.2. *We stress that the norm* (2.4.3) *is equivalent to the associated homogeneous group norm. For example, in the Heisenberg group $\mathbb{H}^n$ the norms* (2.4.3) *and* (1.4.1) *are equivalent.*

Theorem 2.4.3. *Let $u \in \mathcal{S}(\mathbb{G})$ and $W(t,x)$ be a solution of the heat diffusion problem*

$$\begin{cases} \partial_t W = \Delta_{\mathbb{G}} W, & \text{in } (0,+\infty)\times\mathbb{G}, \\ W(0,x) = u(x). \end{cases} \tag{2.4.4}$$

Then

$$-(-\Delta_{\mathbb{G}})^s u(x) = \frac{2s}{|\Gamma(-s)|} \int_0^{+\infty} t^{-s-1}(W(t,x)-u(x))dt. \tag{2.4.5}$$

Proof. We know by [4] that $W(t,x)$ is obtained by the convolution of the heat kernel $h(t,x)$ with u, namely

$$W(t,x) = \int_{\mathbb{G}} h(t,y^{-1}x)u(y)dy$$

is a solution to the problem (2.4.4). Using this and the property of heat kernel

$$\int_{\mathbb{G}} h(t,x)dx = 1 \qquad \text{for any } t>0,$$

we compute

$$\begin{aligned}
\int_0^{+\infty} t^{-s-1}(W(t,x)-u(x))dt &= \int_0^{+\infty} t^{-s-1}\left(\int_{\mathbb{G}} h(t,y)u(y^{-1}x)dy - u(x)\int_{\mathbb{G}} h(t,y)dy\right)dt \\
&= \int_0^{+\infty}\int_{\mathbb{G}} t^{-s-1}h(t,y)(u(y^{-1}x)-u(x))dydt \\
&= \int_{\mathbb{G}} (u(y^{-1}x)-u(x)) \int_0^{+\infty} t^{-s-1}h(t,y)dtdy \\
&= \frac{|\Gamma(-s)|}{s}\int_{\mathbb{G}} (u(y^{-1}x)-u(x))\tilde{R}_{-2s}(y)dy = \frac{|\Gamma(-s)|}{s}\int_{\mathbb{G}} \frac{u(y^{-1}x)-u(x)}{\|y\|_s^{Q+\alpha}}dy \\
&= \frac{|\Gamma(-s)|}{2s}\int_{\mathbb{G}} \frac{u(xy)+u(y^{-1}x)-2u(x)}{\|y\|_s^{Q+\alpha}}dy.
\end{aligned} \tag{2.4.6}$$

Notice, that following the idea of (1.2.5), we obtain that

$$-(-\Delta_{\mathbb{G}})^s u(x) = \int_{\mathbb{G}} \frac{u(xy)+u(y^{-1}x)-2u(x)}{\|y\|_s^{Q+\alpha}}dy.$$

This and (2.4.6) yield

$$\int_0^{+\infty} t^{-s-1}(W(t,x)-u(x))dt = -\frac{|\Gamma(-s)|}{2s}(-\Delta_{\mathbb{G}})^s u(x),$$

thus proving (2.4.5). □

Chapter 3

Nonlinear equations

In this chapter we will take a closer look at fully nonlinear operators. In particular, we consider operators of extremal type and also more general operators later therein.

3.1 Extremal type operators

An important example of a nonlinear operator are extremal operators of Pucci type. This section is dedicated to showcase some properties of these operators and their connection with classical functions.

To start with, let us introduce some notions that will be used throughout the current chapter. For some fixed positive real numbers λ and Λ, such that $0 < \lambda \leq \Lambda$, we define the set of quadratic, symmetric, positively defined matrices as

$$\mathcal{A}_{\lambda,\Lambda} = \{A(x) \in \mathbb{S}^n \mid \quad 0 < \lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n\}.$$

Then, it is convenient to introduce the set

$$\mathcal{E}_r := \{x \in \mathbb{R}^n \mid \quad \|\sqrt{A^{-1}}x\| \leq r\}$$

and the extremal operators of Pucci type

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u(x)) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} Tr(A(x)D^2u(x)),$$

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u(x)) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} Tr(A(x)D^2u(x)).$$

For a fixed $s \in (0,1)$ we define

$$\mathcal{L}_A^s u(x) = c(n,s) \int\limits_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{< A^{-1}y, y >^{\frac{n+2s}{2}}} \frac{dy}{det\sqrt{A}},$$

where $c(n,s)$ is the constant, introduced in (1.2.2). We remark that the introduced operator is indeed the general case of fractional Laplacian, that is, when $\Lambda = \lambda = 1$ we obtain $A = I$ and

$$\frac{1}{2}\mathcal{L}_I^s u(x) = \frac{1}{2}c(n,s) \int\limits_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} dy = (-\Delta)^s u(x). \quad (3.1.1)$$

However, the main objects of our interest are the following extremal operators of Pucci type:

$$\mathcal{P}^{-,s}(u) := - \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} c(n,s) \int\limits_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{< A^{-1}y, y >^{\frac{n+2s}{2}}} \frac{dy}{det\sqrt{A}}$$
$$= - \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \mathcal{L}_A^s u(x), \quad (3.1.2)$$

$$\mathcal{P}^{+,s}(u) := - \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} c(n,s) \int\limits_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{< A^{-1}y, y >^{\frac{n+2s}{2}}} \frac{dy}{det\sqrt{A}}$$
$$= - \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \mathcal{L}_A^s u(x), \quad (3.1.3)$$

Next result shows basic properties of extremal operators and, in particular, the behavior of operators when the parameter s approaches one.

Theorem 3.1.1. *Let $\Lambda \geq \lambda > 0$ and $n \geq 2$. For any $u \in C_0^\infty(\mathbb{R}^n)$ the following statements hold:*

(i) $\mathcal{P}^{+,s}(u) = -c(n,s) \int\limits_{\mathbb{R}^n} \frac{(\frac{\lambda}{\Lambda})^{n/2}\lambda^s(\Delta_y^2 u(x))_+ - (\frac{\Lambda}{\lambda})^{n/2}\Lambda^s(\Delta_y^2 u(x))_-}{|y|^{n+2s}} dy$*;*

(ii) $\mathcal{P}^{-,s}(u) = -c(n,s) \int\limits_{\mathbb{R}^n} \frac{(\frac{\Lambda}{\lambda})^{n/2}\Lambda^s(\Delta_y^2 u(x))_+ - (\frac{\lambda}{\Lambda})^{n/2}\lambda^s(\Delta_y^2 u(x))_-}{|y|^{n+2s}} dy$*;*

(iii) $\mathcal{P}^{-,s}(-u) = -\mathcal{P}^{+,s}(u)$ *and* $\mathcal{P}^{+,s}(-u) = -\mathcal{P}^{-,s}(u)$*;*

(iv) $\mathcal{P}^{-,s}$ *is concave whereas* $\mathcal{P}^{+,s}$ *is convex;*

(v) $\lim\limits_{s\to 1-} \mathcal{P}^{+,s}(u) = -2\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u(x))$;

(vi) $\lim\limits_{s\to 1-} \mathcal{P}^{-,s}(u) = -2\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u(x))$.

Proof. (i) We begin the chain of equalities

$$\mathcal{P}^{+,s}(u) = -\inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathcal{L}^s_A u(x) = -\inf_{A\in\mathcal{A}_{\lambda,\Lambda}} (-\mathcal{L}^s_A(-u)(x)) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathcal{L}^s_A(-u)(x)$$
$$= \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} c(n,s)\int\limits_{\mathbb{R}^n} \frac{(\Delta^2_y u(x))_+ - (\Delta^2_y u(x))_-}{< A^{-1}y, y >^{\frac{n+2s}{2}}} \frac{dy}{det\sqrt{A}}$$
$$= c(n,s)\int\limits_{\mathbb{R}^n} \frac{(\frac{\Lambda}{\lambda})^{n/2}\Lambda^s(\Delta^2_y u(x))_+ - (\frac{\lambda}{\Lambda})^{n/2}\lambda^s(\Delta^2_y u(x))_-}{|y|^{n+2s}} dy.$$

Finally, using the fact that $f_+ = (-f)_-$, we arrive to the last equalities

$$\mathcal{P}^{+,s}(u) = c(n,s)\int\limits_{\mathbb{R}^n} \frac{(\frac{\Lambda}{\lambda})^{n/2}\Lambda^s(\Delta^2_y u(x))_- - (\frac{\lambda}{\Lambda})^{n/2}\lambda^s(\Delta^2_y u(x))_+}{|y|^{n+2s}} dy$$
$$= -c(n,s)\int\limits_{\mathbb{R}^n} \frac{(\frac{\lambda}{\Lambda})^{n/2}\lambda^s(\Delta^2_y u(x))_+ - (\frac{\Lambda}{\lambda})^{n/2}\Lambda^s(\Delta^2_y u(x))_-}{|y|^{n+2s}} dy.$$

Thus (i) is proved as well as (ii) in the same way with infimum and supremum exchanged.
(iv) Using both the definition of $\mathcal{P}^{+,s}$ and the definition of convexity we write for any $t \in [0,1]$

$$\mathcal{P}^{+,s}(tu + (1-t)v)$$
$$= -\inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \frac{c(n,s)}{2}\int\limits_{\mathbb{R}^n} \frac{\Delta^2_y\big(tu + (1-t)v\big)}{< A^{-1}y, y >^{\frac{n+2s}{2}}} \frac{dy}{det\sqrt{A}}$$
$$\leq -\inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \frac{c(n,s)}{2}\int\limits_{\mathbb{R}^n} \frac{2tu(x) - tu(x+y) - tu(x-y)}{< A^{-1}y, y >^{\frac{n+2s}{2}}} \frac{dy}{det\sqrt{A}}$$
$$-\inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \frac{c(n,s)}{2}\int\limits_{\mathbb{R}^n} \frac{2(1-t)v(x) - (1-t)v(x+y) - (1-t)v(x-y)}{< A^{-1}y, y >^{\frac{n+2s}{2}}} \frac{dy}{det\sqrt{A}}$$
$$= t\mathcal{P}^{+,s}(u) + (1-t)\mathcal{P}^{+,s}(v).$$

This shows the convexity of $\mathcal{P}^{+,s}$ and one proves the concavity of $\mathcal{P}^{-,s}$ in the same way only changing the inequality signs.

(v) We notice that

$$\left| \int_{\mathbb{R}^n \setminus \mathcal{E}_1} \frac{2u(x) - u(x+y) - u(x-y)}{< A^{-1}y, y >^{\frac{n+2s}{2}}} \frac{dy}{det\sqrt{A}} \right|$$
$$\leq 4\|u\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus \mathcal{E}_1} \frac{1}{|(\sqrt{A})^{-1}y|^{n+2s}} \frac{dy}{det\sqrt{A}}$$
$$= 4\|u\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_1} \frac{1}{|z|^{n+2s}} dz = 4\|u\|_{L^\infty(\mathbb{R}^n)} \omega_{n-1} \int_1^\infty \frac{1}{\rho^{1+2s}} d\rho = \frac{2\omega_{n-1}}{s} \|u\|_{L^\infty(\mathbb{R}^n)}.$$

Then, knowing the behaviour of the dimentional constant,

$$\lim_{s \to 1-} \frac{c(n,s)}{s(1-s)} = \frac{4n}{\omega_{n-1}}$$

(refer to Corollary 4.2 in [31]), we get

$$\lim_{s \to 1-} \frac{c(n,s)}{2} \int_{\mathbb{R}^n \setminus \mathcal{E}_1} \frac{2u(x) - u(x+y) - u(x-y)}{< A^{-1}y, y >^{\frac{n+2s}{2}}} \frac{dy}{det\sqrt{A}} = 0. \tag{3.1.4}$$

Furthermore, with the help of Cauchy-Swartz inequality, we have that

$$\begin{aligned}
&\left| \int_{\mathcal{E}_1} \frac{2u(x) - u(x+y) - u(x-y) - < D^2u(x)y, y >}{< A^{-1}y, y >^{\frac{n+2s}{2}}} \frac{dy}{det\sqrt{A}} \right| \\
&\leq \|u\|_{C^3(\mathbb{R}^n)} \int_{B_1} \frac{|y|^3}{|(\sqrt{A})^{-1}y|^{n+2s}} \frac{dy}{det\sqrt{A}} = \|u\|_{C^3(\mathbb{R}^n)} \int_{B_1} \frac{|\sqrt{A}z|^3}{|z|^{n+2s}} dz \\
&\leq \|u\|_{C^3(\mathbb{R}^n)} \int_{B_1} \frac{\|\sqrt{A}\|^3 |z|^3}{|z|^{n+2s}} dz = \|u\|_{C^3(\mathbb{R}^n)} \int_{B_1} \frac{\|\sqrt{A}\|^3 |z|^3}{|z|^{n+2s}} dz \\
&= \omega_{n-1} \|u\|_{C^3(\mathbb{R}^n)} \|\sqrt{A}\|^3 \int_0^1 \frac{1}{\rho^{2s-2}} dz = \frac{\omega_{n-1}}{3-2s} \|u\|_{C^3(\mathbb{R}^n)} \|\sqrt{A}\|^3.
\end{aligned} \tag{3.1.5}$$

Recalling again the Taylor formula for u at the point x, we get

$$\lim_{s\to 1-}\frac{c(n,s)}{2}\int_{\mathcal{E}_1}\frac{2u(x)-u(x+y)-u(x-y)}{<A^{-1}y,y>^{\frac{n+2s}{2}}}\frac{dy}{det\sqrt{A}}$$
$$=\lim_{s\to 1-}\frac{c(n,s)}{2}\int_{\mathcal{E}_1}\frac{<D^2u(x)y,y>}{<A^{-1}y,y>^{\frac{n+2s}{2}}}\frac{dy}{det\sqrt{A}}. \quad (3.1.6)$$

On the other hand, we have

$$\int_{\mathcal{E}_1}\frac{<D^2u(x)y,y>}{<A^{-1}y,y>^{\frac{n+2s}{2}}}\frac{dy}{det\sqrt{A}}=\int_{B_1}\frac{<D^2u(x)\sqrt{A}z,\sqrt{A}z>}{|z|^{n+2s}}dz$$
$$=\int_{B_1}\frac{<\sqrt{A}D^2u(x)\sqrt{A}z,z>}{|z|^{n+2s}}dz \quad (3.1.7)$$

Then, denoting $M(x):=\sqrt{A}D^2u(x)\sqrt{A}$, we see that for $i\neq j$,

$$\int_{B_1}m_{ij}(x)y_i\cdot y_j dy=0. \quad (3.1.8)$$

From the other hand, for any fixed i, we have

$$\int_{B_1}\frac{m_{ii}(x)y_i^2}{|y|^{n+2s}}dy=m_{ii}(x)\int_{B_1}\frac{y_i^2}{|y|^{n+2s}}dy=m_{ii}(x)\int_{B_1}\frac{y_1^2}{|y|^{n+2s}}dy$$
$$=\frac{m_{ii}(x)}{n}\sum_{k=1}^{n}\int_{B_1}\frac{y_k^2}{|y|^{n+2s}}dy=\frac{m_{ii}(x)}{n}\int_{B_1}\frac{|y|^2}{|y|^{n+2s}}dy \quad (3.1.9)$$
$$=\frac{m_{ii}(x)\omega_{n-1}}{2n(1-s)}.$$

Finally, putting together (3.1.4), (3.1.6), (3.1.7), (3.1.8) and (3.1.9), we end up with

$$\lim_{s\to 1-}\mathcal{P}^{+,s}(u)=\lim_{s\to 1-}\left(-\inf_{A\in\mathcal{A}_{\lambda,\Lambda}}c(n,s)\int_{\mathcal{E}_1}\frac{2u(x)-u(x+y)-u(x-y)}{<A^{-1}y,y>^{\frac{n+2s}{2}}}\frac{dy}{det\sqrt{A}}\right)$$
$$=\lim_{s\to 1-}\left(-\inf_{A\in\mathcal{A}_{\lambda,\Lambda}}c(n,s)\int_{\mathcal{E}_1}\frac{<D^2u(x)y,y>}{<A^{-1}y,y>^{\frac{n+2s}{2}}}\frac{dy}{det\sqrt{A}}\right)$$

$$
\begin{aligned}
&= \lim_{s\to 1-}\left(-\inf_{A\in\mathcal{A}_{\lambda,\Lambda}} c(n,s)\int\limits_{B_1}\frac{<\sqrt{A}D^2u(x)\sqrt{A}z, z>}{|z|^{n+2s}}dz\right)\\
&= \lim_{s\to 1-}\left(-\inf_{A\in\mathcal{A}_{\lambda,\Lambda}} c(n,s)\int\limits_{B_1}\frac{\sum\limits_{i=1}^{n} m_{ii}(x)z_i^2}{|z|^{n+2s}}dz\right)\\
&= -\inf_{A\in\mathcal{A}_{\lambda,\Lambda}}\lim_{s\to 1-}\frac{c(n,s)\omega_{n-1}}{2n(1-s)}\sum_{i=1}^{n} m_{ii}(x) = -2\inf_{A\in\mathcal{A}_{\lambda,\Lambda}}\sum_{i=1}^{n} m_{ii}(x)\\
&= -2\inf_{A\in\mathcal{A}_{\lambda,\Lambda}} Tr(A(x)D^2u(x)).
\end{aligned}
$$

This is the desired result and (vi) is proved analogously. □

3.1.1 Applications

This section is devoted to possible applications of extremal operators and shows their importance with connection to special functions, such as Gamma and Beta functions (refer to Chapter 6 in [2]).

Let $s\in(0,1)$ and $p>-1$. We define

$$
\begin{aligned}
u_p^{(n)}(x) &= (1-|x|^2)_+^p, \quad x\in\mathbb{R}^n,\\
v_p^{(n)}(x) &= (1-|x|^2)_+^p x_n, \quad x\in\mathbb{R}^n.
\end{aligned}
\tag{3.1.10}
$$

Lemma 3.1.2. *For any $s\in(0,1)$, $p>-1$ and any $x\in\mathbb{R}^n$ it holds that*

$$
\mathcal{P}^{-,s}u_p^{(n)}(x) = -2\frac{c(n,s)B(-s,p+1)\Lambda^s}{\Gamma(\frac{n}{2})}\left(\frac{\Lambda\pi}{\lambda}\right)^{\frac{n}{2}} {}_2F_1\left(s+\frac{n}{2}, -p+s, \frac{n}{2}; |x|^2\right),
$$

$$
\begin{aligned}
&\mathcal{P}^{-,s}v_p^{(n)}(x)\\
&= -2\frac{c(n+2,s)B(-s,p+1)\pi^{n/2+1}\Lambda^s}{\Gamma(\frac{n}{2}+1)}\left(\frac{\Lambda}{\lambda}\right)^{\frac{n}{2}} {}_2F_1\left(s+\frac{n}{2}+1, -p+s, \frac{n}{2}+1; |x|^2\right),
\end{aligned}
$$

Moreover, it holds that

$$
\mathcal{P}^{+,s}u_p^{(n)}(x) = -2\frac{c(n,s)B(-s,p+1)\lambda^s}{\Gamma(n/2)}\left(\frac{\lambda\pi}{\Lambda}\right)^{\frac{n}{2}} {}_2F_1\left(s+\frac{n}{2}, -p+s, \frac{n}{2}; |x|^2\right),
$$

$$\mathcal{P}^{+,s}v_p^{(n)}(x)$$
$$= -2\frac{c(n+2,s)B(-s,p+1)\pi^{n/2+1}\lambda^s}{\Gamma(n/2+1)}\left(\frac{\lambda}{\Lambda}\right)^{\frac{n}{2}} {}_2F_1\left(s+\frac{n}{2}+1, -p+s, \frac{n}{2}+1; |x|^2\right).$$

Proof. We recall the relation between the fractional Laplace operator and the operator $\mathcal{L}_A$ (3.1.1), i.e.

$$(-\triangle)^s u(x) = \frac{1}{2}\mathcal{L}_I^s u(x) = \frac{c(n,s)}{2}\int_{\mathbb{R}^n}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}dy.$$

Using this and Theorem 1 in [35], we compute,

$$\begin{aligned}
\mathcal{P}^{-,s}u_p^{(n)}(x) &= -\sup_A \mathcal{L}_A^s u_p^{(n)}(x)\\
&= -2\sup_A c(n,s)\int_{\mathbb{R}^n}\frac{2u_p^{(n)}(x)-u_p^{(n)}(x+y)-u_p^{(n)}(x-y)}{\langle A^{-1}y,y\rangle^{\frac{n+2s}{2}}}\frac{dy}{det\sqrt{A}}\\
&= -2\left(\frac{\Lambda}{\lambda}\right)^{\frac{n}{2}}\Lambda^s\int_{\mathbb{R}^n}\frac{2u_p^{(n)}(x)-u_p^{(n)}(x+y)-u_p^{(n)}(x-y)}{|y|^{n+2s}}dy\\
&= -2\left(\frac{\Lambda}{\lambda}\right)^{\frac{n}{2}}\Lambda^s(-\triangle)^s u_p^{(n)}(x) =\\
&= -2\frac{c(n,s)B(-s,p+1)\Lambda^s}{\Gamma(n/2)}\left(\frac{\Lambda\pi}{\lambda}\right)^{\frac{n}{2}} {}_2F_1\left(s+\frac{n}{2}, -p+s, \frac{n}{2}; |x|^2\right).
\end{aligned} \tag{3.1.11}$$

In particular, it shows that if $p = s+1$, then

$$\begin{aligned}
\mathcal{P}^{-,s}u_{s+1}^{(n)}(x) &= -2\frac{c(n,s)B(-s,s+2)\Lambda^s}{\Gamma(n/2)}\left(\frac{\Lambda\pi}{\lambda}\right)^{\frac{n}{2}} {}_2F_1\left(s+\frac{n}{2}, -1, \frac{n}{2}; |x|^2\right)\\
&= 2^{2s+1}\left(\frac{\Lambda}{\lambda}\right)^{\frac{n}{2}}\Lambda^s\,\Gamma(s+2)\Gamma\left(\frac{n}{2}+s\right)\Gamma\left(\frac{n}{2}\right)^{-1}\left(1-\left(1+\frac{2s}{n}\right)|x|^2\right).
\end{aligned}$$

If $p = s$ we have

$$\begin{aligned}
\mathcal{P}^{-,s}u_s^{(n)}(x) &= -2\frac{c(n,s)B(-s,s+1)\Lambda^s}{\Gamma(n/2)}\left(\frac{\Lambda\pi}{\lambda}\right)^{\frac{n}{2}} {}_2F_1\left(s+\frac{n}{2}, 0, \frac{n}{2}; |x|^2\right)\\
&= -2\frac{4^s\Gamma\left(\frac{n}{2}+s\right)\Gamma(-s)\Gamma(s+1)}{\pi^{n/2}|\Gamma(-s)|\Gamma(1)\,\Gamma\left(\frac{n}{2}\right)}\left(\frac{\Lambda\pi}{\lambda}\right)^{\frac{n}{2}}\Lambda^s\cdot 1\\
&= 2^{2s+1}\left(\frac{\Lambda}{\lambda}\right)^{\frac{n}{2}}\Lambda^s\,\Gamma(s+1)\Gamma\left(\frac{n}{2}+s\right)\Gamma\left(\frac{n}{2}\right)^{-1}.
\end{aligned} \tag{3.1.12}$$

Moreover, using the same reasoning, we get for $v_p^{(n)}$,

$$\begin{aligned}
&\mathcal{P}^{-,s}v_p^{(n)}(x) = -\sup_A \mathcal{L}_A^s v_p^{(n)}(x)\\
&= -2\sup_A c(n,s)\int_{\mathbb{R}^n}\frac{2v_p^{(n)}(x)-v_p^{(n)}(x+y)-v_p^{(n)}(x-y)}{\langle A^{-1}y,y\rangle^{\frac{n+2s}{2}}}\frac{dy}{det\sqrt{A}}\\
&= -2\Big(\frac{\Lambda}{\lambda}\Big)^{\frac{n}{2}}\Lambda^s\int_{\mathbb{R}^n}\frac{2v_p^{(n)}(x)-v_p^{(n)}(x+y)-v_p^{(n)}(x-y)}{|y|^{n+2s}}dy\\
&= -2\Big(\frac{\Lambda}{\lambda}\Big)^{\frac{n}{2}}\Lambda^s(-\triangle)^s v_p^{(n)}(x)\\
&= -2\frac{c(n+2,s)B(-s,p+1)\pi^{n/2+1}\Lambda^s}{\Gamma(n/2+1)}\Big(\frac{\Lambda}{\lambda}\Big)^{\frac{n}{2}}{}_2F_1\left(s+\frac{n}{2}+1,-p+s,\frac{n}{2}+1;|x|^2\right).
\end{aligned} \tag{3.1.13}$$

When $p=s$ we have

$$\mathcal{P}^{-,s}v_s^{(n)}(x) = 2^{2s+1}\Big(\frac{\Lambda}{\lambda}\Big)^{\frac{n}{2}}\Lambda^s\,\Gamma(s+1)\Gamma\left(\frac{n}{2}+s+1\right)\Gamma\left(\frac{n}{2}+1\right)^{-1}x_n.$$

□

As a consequence of Lemma 3.1.2, we get the next nice corollary.

Corollary 3.1.3. *If $n=1$ and $p=s$ we have*

$$\mathcal{P}^{-,s}(1-x^2)^s = -2\Lambda^s\Gamma(2s+1).$$

Proof. Notice that the matrix A in this case becomes a number a and we get

$$\begin{aligned}
\mathcal{L}_a^s u(x) &= c(1,s)\int_{-\infty}^{+\infty}\frac{2u(x)-u(x+y)-u(x-y)}{\langle a^{-1}y,y\rangle^{\frac{1+2s}{2}}}\frac{dy}{\sqrt{a}}\\
&= \frac{c(1,s)}{a^{\frac{1}{2}}a^{-\frac{1+2s}{2}}}\int_{-\infty}^{+\infty}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{1+2s}}dy = 2a^s(-\triangle)^s u(x).
\end{aligned}$$

Table 1 in [35] tells us that for $x\in(-1,1)$ holds

$$(-\triangle)^s(1-x^2)^s = \Gamma(2s+1).$$

Thus for $x \in (-1,1)$ we calculate

$$\mathcal{P}^{-,s}(1-x^2)^s = -\sup_a \mathcal{L}_a^s(1-x^2)^s = -\sup_a 2a^s(-\triangle)^s(1-x^2)^s = -2\Lambda^s\Gamma(2s+1).$$

□

3.2 Regularity of solutions to fully nonlinear equations

In this section we prove the main products of this chapter, which are the regularity results of solutions to fully nonlinear equations. We consequently prove the Lipschitz and $C^{1,1}$ regularity of solutions, using the method of H. Ishii and P. Lions.

Given $s \in (0,1)$, we consider the space of functions

$$G := \left\{ u : \mathbb{R}^n \to \mathbb{R},\ \text{s.t. } u \in C(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} \frac{|u(t)|}{1+|t|^{n+2s}} dt < +\infty \right\}.$$

Also, we take the family of operators (1.3.1) and set the following class of linear operators for any $s \in (0,1)$ and given $\Lambda \geq \lambda > 0$:

$$\mathcal{L} := \left\{ L \;\middle|\; K \geq 0 \text{ and symmetric: } c(n,s)\frac{\lambda}{|t|^{n+2s}} \leq K(t) \leq c(n,s)\frac{\Lambda}{|t|^{n+2s}} \right\}. \tag{3.2.1}$$

Definition 3.2.1. *Given a family of operators $\mathcal{L}$, we set*

$$M^-_{\mathcal{L}}u(x) := \inf_{L \in \mathcal{L}} Lu(x),$$

$$M^+_{\mathcal{L}}u(x) := \sup_{L \in \mathcal{L}} Lu(x).$$

For a nonlocal operator F we state the uniform ellipticity property and recall some other notions that are crucial to formulate the main results of this section.

Definition 3.2.2. *We say that $F : G \to \mathbb{R}$ is uniformly elliptic in Ω if for* *any $x, y \in \Omega$ and $u, v \in G$ such that if there exist $\varepsilon > 0$ such that*

$$\left| \int_{B_\varepsilon(x)} \frac{\Delta_t^2 u(x)}{|t|^{n+2s}} dt \right| < +\infty \quad \text{and} \quad \left| \int_{B_\varepsilon(y)} \frac{\Delta_t^2 v(y)}{|t|^{n+2s}} dt \right| < +\infty,$$

then we have

$$M_{\mathcal{L}}^-(u(x) - v(y)) \leq Fu(x) - Fv(y) \leq M_{\mathcal{L}}^+(u(x) - v(y)).$$

Definition 3.2.3. *We call an operator $F : G \to \mathbb{R}$ convex if for any two given functions $u, v \in G$ and all $t \in [0,1]$ it holds that*

$$F(tu + (1-t)v) \leq tFu + (1-t)Fv.$$

Definition 3.2.4. *An operator $F : G \to \mathbb{R}$ is said to be nonnegatively homogeneous if for all $r \geq 0$ and a given $u \in G$ it holds that*

$$F(ru) = rFu.$$

Next result shows that the solutions of a fully nonlinear nonlocal equation are Lipschitz, provided that the operator satisfies the homogeneity and uniform ellipticity property.

Theorem 3.2.5. *Let $F : G \to \mathbb{R}$ be nonnegatively homogeneous of degree one and uniformly elliptic in the sense of Definition 3.2.2. Let also f be bounded and continuous in B_1 and $u \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be a viscosity solution to*

$$Fu(x) = f(x) \qquad \text{in} \quad B_1. \tag{3.2.2}$$

Then for all $x, y \in B_{1/2}$

$$|u(x) - u(y)| \leq L|x - y| \tag{3.2.3}$$

for some positive constant L.

Proof. Without loss of generality we assume that

$$0 \leq u \leq 1 \quad \text{ in } B_1.$$

Otherwise divide (3.2.2) by $\|u\|_{L^\infty(B_1)}$, and denote $\tilde{f} := f/\|u\|_{L^\infty(B_1)}$, thus obtaining, thanks to homogeneity,

$$Fv(x) = \tilde{f}(x),$$

which has the structure of the original equation.

Let $L, \tilde{L}$ be positive constants and

$$\varphi(x, y) := L|x - y| - \tilde{L}|x - y|^{3/2}$$

for any $x, y \in \overline{B}_1$.

Now choose $\zeta \in B_{1/4}$ and set, for all $x, y \in \overline{B}_1$,

$$\Phi(x, y) = u(x) - u(y) - \varphi(x, y) - 2|x - \zeta|^2.$$

Now we prove that $\Phi \leq 0$ (refer to [48] or [64] for additional details). For this goal suppose that there exist $\hat{x}, \hat{y} \in B_1$ such that

$$\sup_{\overline{B}_1 \times \overline{B}_1} \Phi(\hat{x}, \hat{y}) > 0. \tag{3.2.4}$$

Observe that $\varphi(\hat{x}, \hat{y}) \neq 0$. Otherwise, take $L > 2\tilde{L}$ and suppose

$$\varphi(\hat{x}, \hat{y}) = L|\hat{x} - \hat{y}| \left(1 - \frac{\tilde{L}}{L}|\hat{x} - \hat{y}|^{1/2}\right) = 0.$$

It follows that $\hat{x} = \hat{y}$ and thus $\Phi \leq 0$, giving us a contradiction with (3.2.4).

Now we notice that if $L > 4\tilde{L}$ is large enough and $(\hat{x}, \hat{y}) \in \partial(B_1 \times B_1)$, then $\varphi(\hat{x}, \hat{y}) > 0$, giving us that

$$\varphi(\hat{x}, \hat{y}) + 2|\hat{x} - \zeta|^2 = L|\hat{x} - \hat{y}| \left(1 - \frac{\tilde{L}}{L}|\hat{x} - \hat{y}|^{1/2}\right) + 2|\hat{x} - \zeta|^2 \geq 1,$$

and thus $\Phi(\hat{x}, \hat{y}) \leq 0$, that yields a contradiction with (3.2.4). As a consequence, we have that $(\hat{x}, \hat{y}) \notin \partial(B_1 \times B_1)$.

Therefore, we can use the Theorem of Sums 1.3.6, i.e. there exist $X, Y \in \mathbb{S}^n$

such that

$$\begin{aligned}
&(D_x\varphi(\hat{x},\hat{y}),X) \in \overline{J}^{2,+}\big(u(\hat{x}) - 2|\hat{x}-\zeta|^2\big),\\
&(-D_y\varphi(\hat{x},\hat{y}),Y) \in \overline{J}^{2,-}u(\hat{y}),\\
&\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le D^2\varphi(\hat{x},\hat{y}) + \mu(D^2\varphi(\hat{x},\hat{y}))^2.
\end{aligned} \tag{3.2.5}$$

By definition, $(D_x\varphi(\hat{x},\hat{y}),X) \in \overline{J}^{2,+}\big(u(\hat{x}) - 2|\hat{x}-\zeta|^2\big)$ can be written as

$$u(x) - 2|x-\zeta|^2 \le u(\hat{x}) - 2|\hat{x}-\zeta|^2 + \langle D_x\varphi(\hat{x},\hat{y}), x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle,$$

$$u(x) \le u(\hat{x}) + \langle D_x\varphi(\hat{x},\hat{y}), x-\hat{x}\rangle + 2\langle x, x-\hat{x}\rangle + 2\langle \hat{x}-2\zeta, x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle,$$

$$\begin{aligned}
u(x) \le u(\hat{x}) + \langle D_x\varphi(\hat{x},\hat{y}), x-\hat{x}\rangle + \langle 2\hat{x}-4\zeta, x-\hat{x}\rangle + \langle 2\hat{x}, x-\hat{x}\rangle + 2|x-\hat{x}|^2 \\
+ \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle,
\end{aligned}$$

$$u(x) \le u(\hat{x}) + \langle D_x\varphi(\hat{x},\hat{y}), x-\hat{x}\rangle + \langle 4\hat{x}-4\zeta, x-\hat{x}\rangle + \frac{1}{2}\langle (X+4I)(x-\hat{x}), x-\hat{x}\rangle.$$

As a consequence, from (3.2.5) we conclude

$$\begin{aligned}
&(D_x\varphi(\hat{x},\hat{y}) + 4\hat{x} - 4\zeta, X+4I) \in \bar{J}^{2,+}u(\hat{x}),\\
&(-D_y\varphi(\hat{x},\hat{y}),Y) \in \bar{J}^{2,-}u(\hat{y}),\\
\text{and}\quad &\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le D^2\varphi(\hat{x},\hat{y}) + \mu(D^2\varphi(\hat{x},\hat{y}))^2.
\end{aligned} \tag{3.2.6}$$

Now we compute, recalling that $|\hat{x}-\hat{y}| \ne 0$,

$$D\varphi(\hat{x},\hat{y}) = \left(\frac{L}{|\hat{x}-\hat{y}|} - \frac{3\tilde{L}}{|\hat{x}-\hat{y}|^{1/2}}\right)\begin{pmatrix} \hat{x}-\hat{y} \\ \hat{y}-\hat{x} \end{pmatrix},$$

$$D^2\varphi(\hat{x},\hat{y}) = \begin{pmatrix} M & -M \\ -M & M \end{pmatrix},$$

where

$$M = \left(\frac{L}{|\hat{x}-\hat{y}|} - \frac{3\tilde{L}}{|\hat{x}-\hat{y}|^{1/2}}\right) I + \left(\frac{3\tilde{L}}{|\hat{x}-\hat{y}|^{1/2}} - \frac{L}{|\hat{x}-\hat{y}|}\right)\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|} \otimes \frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}.$$

Furthermore,

$$(D^2\varphi(\hat{x},\hat{y}))^2 = 2\begin{pmatrix} M^2 & -M^2 \\ -M^2 & M^2 \end{pmatrix},$$

where we denoted

$$\begin{aligned}
M^2 &= \left(\frac{L}{|\hat{x}-\hat{y}|} - \frac{3\tilde{L}}{2|\hat{x}-\hat{y}|^{1/2}}\right)^2 I \\
&\quad + \left(\frac{L}{|\hat{x}-\hat{y}|} - \frac{3\tilde{L}}{2|\hat{x}-\hat{y}|^{1/2}}\right)\left(\frac{3\tilde{L}}{4|\hat{x}-\hat{y}|^{1/2}} - \frac{L}{|\hat{x}-\hat{y}|}\right)\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}\otimes\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|} \\
&\quad + \left(\frac{3\tilde{L}}{4|\hat{x}-\hat{y}|^{1/2}} - \frac{L}{|\hat{x}-\hat{y}|}\right)^2 \frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}\otimes\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|} \\
&= \left(\frac{L}{|\hat{x}-\hat{y}|} - \frac{3\tilde{L}}{2|\hat{x}-\hat{y}|^{1/2}}\right)^2 I \\
&\quad + \left(\frac{L}{|\hat{x}-\hat{y}|} - \frac{3\tilde{L}}{4|\hat{x}-\hat{y}|^{1/2}}\right)\left(\frac{3\tilde{L}}{4|\hat{x}-\hat{y}|^{1/2}}\right)\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}\otimes\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}.
\end{aligned}$$

In particular, taking $\xi \in \mathbb{R}^n$ and inequality (3.2.6), we see that

$$\begin{aligned}
&\begin{pmatrix} \xi & \xi \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \\
&\qquad \le \begin{pmatrix} \xi & \xi \end{pmatrix}\begin{pmatrix} M & -M \\ -M & M \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix} + \mu\begin{pmatrix} \xi & \xi \end{pmatrix}\begin{pmatrix} M^2 & -M^2 \\ -M^2 & M^2 \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix},
\end{aligned}$$

which is exactly

$$\xi^T(X-Y)\xi \le 0. \tag{3.2.7}$$

This means that all the eigenvalues of the matrix $X - Y$ are nonpositive. Observe also that using vectors $\begin{pmatrix} \xi & 0 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & \xi \end{pmatrix}^T$, we get from (3.2.6),

$$\begin{aligned}
\xi^T X\xi &\le \xi^T(M+\mu M^2)\xi, \\
-\xi^T Y\xi &\le \xi^T(M+\mu M^2)\xi.
\end{aligned} \tag{3.2.8}$$

Notice now that, if $\xi = \dfrac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}$, then

$$\begin{aligned} 2\left(\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}\right)^T & (M+\mu M^2)\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|} \\ & = -\frac{3\tilde{L}}{2|\hat{x}-\hat{y}|^{1/2}} + 4\mu\left(\frac{L^2}{|\hat{x}-\hat{y}|^2} - \frac{9L\tilde{L}}{|\hat{x}-\hat{y}|^{3/2}} + \frac{27\tilde{L}^2}{16|\hat{x}-\hat{y}|}\right). \quad (3.2.9) \end{aligned}$$

The factor

$$\frac{L^2}{|\hat{x}-\hat{y}|^2} - \frac{9L\tilde{L}}{|\hat{x}-\hat{y}|^{3/2}} + \frac{27\tilde{L}^2}{16|\hat{x}-\hat{y}|} = \left(\frac{L}{|\hat{x}-\hat{y}|} - \frac{9\tilde{L}}{8|\hat{x}-\hat{y}|^{1/2}}\right)^2 + \frac{27\tilde{L}^2}{64|\hat{x}-\hat{y}|}$$

is positive, which yields that we can choose μ such that

$$4\mu\left(\frac{L^2}{|\hat{x}-\hat{y}|^2} - \frac{9L\tilde{L}}{|\hat{x}-\hat{y}|^{3/2}} + \frac{27\tilde{L}^2}{16|\hat{x}-\hat{y}|}\right) < \frac{\tilde{L}}{2|\hat{x}-\hat{y}|^{1/2}}.$$

This, (3.2.6) and (3.2.9) lead us to the fact that

$$\text{one of the eigenvalues of } X-Y \text{ is negative.} \quad (3.2.10)$$

Now, take $r_1, r_2 > 0$ such that $B_{r_1}(\hat{x}), B_{r_2}(\hat{y}) \Subset B_1$, and set

$$v(x) := \begin{cases} u(\hat{x}) + \langle D_x\varphi(\hat{x},\hat{y}) + 4(\hat{x}-\zeta), x-\hat{x}\rangle & \\ \qquad + \frac{1}{2}\langle (X+4I)(x-\hat{x}), x-\hat{x}\rangle + o(|x-\hat{x}|^2), & \text{if } x \in B_{r_1}(\hat{x}); \\ u(x), & \text{if } x \notin B_{r_1}(\hat{x}). \end{cases}$$

$$w(y) := \begin{cases} u(\hat{y}) - \langle D_y\varphi(\hat{x},\hat{y}), y-\hat{y}\rangle + \frac{1}{2}\langle Y(y-\hat{y}), y-\hat{y}\rangle + o(|y-\hat{y}|^2), & \text{if } y \in B_{r_2}(\hat{y}); \\ u(y), & \text{if } y \notin B_{r_2}(\hat{y}). \end{cases}$$

Since u is a viscosity solution to (3.2.2), we find that

$$Fv(\hat{x}) \le f(\hat{x}), \quad (3.2.11)$$

$$Fw(\hat{y}) \ge f(\hat{y}). \quad (3.2.12)$$

Then (3.2.11), (3.2.12) and the uniform ellipticity property 3.2.2 give

$$\begin{aligned} 0 &\geq Fv(\hat{x}) - Fw(\hat{y}) - f(\hat{x}) + f(\hat{y}) \\ &\geq M^-_{\mathcal{L}}\,(v(\hat{x}) - w(\hat{y})) - 2\|f\|_{L^\infty(B_1)}. \end{aligned} \tag{3.2.13}$$

Now, we fix $r < \min\{r_1, r_2\}$ and use (3.2.13), obtaining

$$\begin{aligned} 0 &\geq \inf_{L\in\mathcal{L}} L(v(\hat{x}) - w(\hat{y})) - 2\|f\|_{L^\infty(B_1)} \\ &\geq \inf_K \left(\int_{B_r} (\Delta^2_t v(\hat{x}) - \Delta^2_t w(\hat{y}))K(t)dt + \int_{B^c_r} (\Delta^2_t v(\hat{x}) - \Delta^2_t w(\hat{z}))K(t)dt \right) \\ &\quad - 2\|f\|_{L^\infty(B_1)}, \end{aligned} \tag{3.2.14}$$

where the notation of the centered difference (1.1.3) has been exploited. Recall that the functions v, w are smooth in small neighborhoods of $\hat{x}$, and $\hat{y}$, respectively. Hence, we can apply the Taylor formula in (3.2.14) and get

$$\begin{aligned} 0 &\geq \inf_K \int_{B_r} \langle (Y - X - 4I)t, t\rangle K(t)dt \,+\, \inf_K \int_{B^c_r} (\Delta^2_t v(\hat{x}) - \Delta^2_t w(\hat{z}))K(t)dt \\ &\quad - 2\|f\|_{L^\infty(B_1)}. \end{aligned} \tag{3.2.15}$$

Set $M = (m_{ij}) := Y - X - 4I$, and observe that for $i \neq j$,

$$\int_{B_r} m_{ij} \cdot t_i \cdot t_j \cdot K(t)dt = 0, \tag{3.2.16}$$

due to positivity and symmetry of the kernel.

In particular, for any fixed i, we have

$$\begin{aligned} \int_{B_r} m_{ii} t_i^2\, K(t)dt &= \int_{B_r} m_{ii} t_i^2\, K(t)dt = \int_{B_r} m_{ii} v(x) t_1^2\, K(t)dt \\ &= \sum_{k=1}^n \int_{B_r} \frac{m_{ii}}{n} t_k^2\, K(t)dt = \int_{B_r} \frac{m_{ii}}{n} |t|^2\, K(t)dt. \end{aligned} \tag{3.2.17}$$

Taking use of (3.2.7) and (3.2.10), and then exploiting (3.2.16) and (3.2.17), we have that (3.2.15) becomes

$$\begin{aligned}
0 \geq\ & \mathrm{Tr}(Y - X - 4I)\frac{1}{n}\inf_K \int_{B_r} |t|^2 K(t)dt \\
& - 8\|u\|_{L^\infty(\mathbb{R}^n)} \sup_K \int_{B_r^c} K(t)dt - 2\|f\|_{L^\infty(B_1)} \\
\geq\ & \frac{\tilde{L}}{|\hat{x}-\hat{y}|^{1/2}} \frac{\lambda c_{n,s}}{n} \int_{B_r} \frac{1}{|t|^{n+2s-2}}dt - 4\lambda c_{n,s} \int_{B_r} \frac{1}{|t|^{n+2s-2}}dt \\
& - 8\|u\|_{L^\infty(\mathbb{R}^n)} \Lambda c_{n,s} \int_{B_r^c} \frac{1}{|t|^{n+2s}}dt - 2\|f\|_{L^\infty(B_1)}.
\end{aligned} \tag{3.2.18}$$

All integrals in (3.2.18) are convergent, due to the fact that $s \in (0,1)$. Moreover, the right hand side of (3.2.18) becomes positive, since

$$\frac{\tilde{L}}{|\hat{x}-\hat{y}|^{1/2}} \to +\infty$$

as $\tilde{L} \to +\infty$, which gives us a contradiction.

Hence, we proved that $\Phi(x,y) = u(x) - u(y) - \varphi(x,y) - 2|x-\zeta|^2 \leq 0$, that yields (3.2.3), setting $\zeta = x$. □

In fact, we are able to prove that the solutions of an equation involving a fully nonlinear operator are even more regular, as the following result shows.

Theorem 3.2.6. *Let $F : G \to \mathbb{R}$ be convex, nonnegatively homogeneous of degree one and uniformly elliptic in the sense of Definition 3.2.2. Let also f be bounded and continuous in B_1 and $u \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be a viscosity solution to*

$$Fu(x) = f(x) \qquad in \quad B_1. \tag{3.2.19}$$

Then for all $x, y \in B_{1/2}$

$$\left| u(x) + u(y) - 2u\left(\frac{x+y}{2}\right)\right| \leq L_1|x-y|^2 \tag{3.2.20}$$

for some positive constant L_1.

Proof. Without loss of generality we assume that

$$0 \leq u \leq 1 \quad \text{ in } B_1.$$

Otherwise one divides (3.2.19) by $\|u\|_{L^\infty(B_1)}$ and denotes $\tilde{f} := f/\|u\|_{L^\infty(B_1)}$, obtaining by virtue of homogeneity

$$Fv(x) = \tilde{f}(x),$$

which has the structure of the original equation.

Let L_1, L_2, L_3, L_4 be positive constants and

$$\varphi(x, y, z) := L_1|x - y|^2 - L_2|x - y|^{5/2} + L_3|x + y - 2z| - L_4|x + y - 2z|^{3/2}$$

for any $x, y, z \in \overline{B}_1$.

Now choose $\zeta \in B_{1/2}$, and set, for all $x, y, z \in \overline{B}_1$,

$$\Phi(x, y, z) = u(x) + u(y) - 2u(z) - \varphi(x, y, z) - 2|x - \zeta|^2.$$

We shall show now that $\Phi \leq 0$. For this goal suppose that there exist $\hat{x}, \hat{y}, \hat{z} \in B_1$ such that

$$\sup_{\overline{B}_1 \times \overline{B}_1 \times \overline{B}_1} \Phi(\hat{x}, \hat{y}, \hat{z}) > 0. \tag{3.2.21}$$

Observe that $\varphi(\hat{x}, \hat{y}, \hat{z}) \neq 0$. Otherwise, take $L_1 > 2L_2$ and $L_3 > 4L_4$. If

$$\begin{aligned} &\varphi(\hat{x}, \hat{y}, \hat{z}) \\ &= L_1|\hat{x} - \hat{y}|^2 \left(1 - \frac{L_2}{L_1}|\hat{x} - \hat{y}|^{1/2}\right) + L_3|\hat{x} + \hat{y} - 2\hat{z}| \left(1 - \frac{L_4}{L_3}|\hat{x} + \hat{y} - 2\hat{z}|^{1/2}\right) = 0, \end{aligned}$$

then it follows that $\hat{x} = \hat{y} = \hat{z}$ and thus $\Phi \leq 0$, giving us a contradiction with (3.2.21).

Now we notice that if L_1, L_3 are large enough and $(\hat{x}, \hat{y}, \hat{z}) \in \partial(B_1 \times B_1 \times B_1)$, then $\varphi(\hat{x}, \hat{y}, \hat{z}) > 0$, giving us that

$$\begin{aligned} \varphi(\hat{x}, \hat{y}, \hat{z}) + 2|\hat{x} - \zeta|^2 &= L_1|\hat{x} - \hat{y}|^2 \left(1 - \frac{L_2}{L_1}|\hat{x} - \hat{y}|^{1/2}\right) \\ &+ L_3|\hat{x} + \hat{y} - 2\hat{z}| \left(1 - \frac{L_4}{L_3}|\hat{x} + \hat{y} - 2\hat{z}|^{1/2}\right) + 2|\hat{x} - \zeta|^2 \geq 2, \end{aligned}$$

and therefore $\Phi(\hat{x}, \hat{y}, \hat{z}) \leq 0$, that yields $(\hat{x}, \hat{y}, \hat{z}) \notin \partial(B_1 \times B_1 \times B_1)$. Furthermore, we can use the Theorem of Sums (1.3.6) and state that there exist $X, Y, Z \in \mathbb{S}^n$ such that

$$(D_x\varphi(\hat{x}, \hat{y}, \hat{z}) + 4(\hat{x} - \zeta), X + 4I) \in \bar{J}^{2,+}u(\hat{x}),$$

$$(D_y\varphi(\hat{x},\hat{y},\hat{z}),Y)\in \bar{J}^{2,+}u(\hat{y}),$$

$$(D_z\varphi(\hat{x},\hat{y},\hat{z}),Z)\in -2\bar{J}^{2,-}(u(\hat{z})),$$

$$\begin{pmatrix} X & 0 & 0\\ 0 & Y & 0\\ 0 & 0 & Z\end{pmatrix}\le D^2\varphi(\hat{x},\hat{y},\hat{z})+\mu(D^2\varphi(\hat{x},\hat{y},\hat{z}))^2. \tag{3.2.22}$$

Now we compute, assuming $|\hat{x}-\hat{y}|\neq 0$ and $|\hat{x}+\hat{y}-2\hat{z}|\neq 0$,

$$D\varphi(\hat{x},\hat{y},\hat{z}) = 2L_1\begin{pmatrix}\hat{x}-\hat{y}\\ \hat{y}-\hat{x}\\ 0\end{pmatrix} - \frac{5}{2}L_2|\hat{x}-\hat{y}|^{1/2}\begin{pmatrix}\hat{x}-\hat{y}\\ \hat{y}-\hat{x}\\ 0\end{pmatrix}$$
$$+L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1}\begin{pmatrix}\hat{x}+\hat{y}-2\hat{z}\\ \hat{x}+\hat{y}-2\hat{z}\\ -2\hat{x}-2\hat{y}+4\hat{z}\end{pmatrix} - \frac{3}{2}L_4|\hat{x}+\hat{y}-2\hat{z}|^{-1/2}\begin{pmatrix}\hat{x}+\hat{y}-2\hat{z}\\ \hat{x}+\hat{y}-2\hat{z}\\ -2\hat{x}-2\hat{y}+4\hat{z}\end{pmatrix}. \tag{3.2.23}$$

We also calculate the Hessian matrix of φ at the point $(\hat{x},\hat{y},\hat{z})$. Namely, we denote

$$\tilde{e}:=\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|} \text{ and } \bar{e}:=\frac{\hat{x}+\hat{y}-2\hat{z}}{|\hat{x}+\hat{y}-2\hat{z}|}$$

and obtain

$$D^2\varphi(\hat{x},\hat{y},\hat{z}) = 2L_1\begin{pmatrix} I & -I & 0\\ -I & I & 0\\ 0 & 0 & 0\end{pmatrix} - \frac{5}{4}L_2|\hat{x}-\hat{y}|^{-3/2}\begin{pmatrix}\hat{x}-\hat{y}\\ \hat{y}-\hat{x}\\ 0\end{pmatrix}\otimes\begin{pmatrix}\hat{x}-\hat{y}\\ \hat{y}-\hat{x}\\ 0\end{pmatrix}$$
$$-\frac{5}{2}L_2|\hat{x}-\hat{y}|^{1/2}\begin{pmatrix} I & -I & 0\\ -I & I & 0\\ 0 & 0 & 0\end{pmatrix}$$
$$-L_3|\hat{x}+\hat{y}-2\hat{z}|^{-3}\begin{pmatrix}\hat{x}+\hat{y}-2\hat{z}\\ \hat{x}+\hat{y}-2\hat{z}\\ -2\hat{x}-2\hat{y}+4\hat{z}\end{pmatrix}\otimes\begin{pmatrix}\hat{x}+\hat{y}-2\hat{z}\\ \hat{x}+\hat{y}-2\hat{z}\\ -2\hat{x}-2\hat{y}+4\hat{z}\end{pmatrix}$$
$$+L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1}\begin{pmatrix} I & I & -2I\\ I & I & -2I\\ -2I & -2I & 4I\end{pmatrix} \tag{3.2.24}$$

$$
\begin{aligned}
&- \frac{3}{2}L_4|\hat{x}+\hat{y}-2\hat{z}|^{-1/2}\begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \\
&+ \frac{3}{4}L_4|\hat{x}+\hat{y}-2\hat{z}|^{-5/2}\begin{pmatrix} \hat{x}+\hat{y}-2\hat{z} \\ \hat{x}+\hat{y}-2\hat{z} \\ -2\hat{x}-2\hat{y}+4\hat{z} \end{pmatrix} \otimes \begin{pmatrix} \hat{x}+\hat{y}-2\hat{z} \\ \hat{x}+\hat{y}-2\hat{z} \\ -2\hat{x}-2\hat{y}+4\hat{z} \end{pmatrix} \\
&= (2L_1 - \frac{5}{2}L_2|\hat{x}-\hat{y}|^{1/2})\begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{5}{4}L_2|\hat{x}-\hat{y}|^{-3/2}\begin{pmatrix} \hat{x}-\hat{y} \\ \hat{y}-\hat{x} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \hat{x}-\hat{y} \\ \hat{y}-\hat{x} \\ 0 \end{pmatrix} \\
&- (L_3|\hat{x}+\hat{y}-2\hat{z}|^{-3} - \frac{3}{4}L_4|\hat{x}+\hat{y}-2\hat{z}|^{-5/2}) \\
&\qquad \times \begin{pmatrix} \hat{x}+\hat{y}-2\hat{z} \\ \hat{x}+\hat{y}-2\hat{z} \\ -2\hat{x}-2\hat{y}+4\hat{z} \end{pmatrix} \otimes \begin{pmatrix} \hat{x}+\hat{y}-2\hat{z} \\ \hat{x}+\hat{y}-2\hat{z} \\ -2\hat{x}-2\hat{y}+4\hat{z} \end{pmatrix} \\
&+ (L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1} - \frac{3}{2}L_4|\hat{x}+\hat{y}-2\hat{z}|^{-1/2})\begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \\
&= \underbrace{(2L_1 - \frac{5}{2}L_2|\hat{x}-\hat{y}|^{1/2})}_{a}\begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} - \underbrace{\frac{5}{4}L_2|\hat{x}-\hat{y}|^{1/2}}_{b}\begin{pmatrix} \tilde{e} \\ -\tilde{e} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \tilde{e} \\ -\tilde{e} \\ 0 \end{pmatrix} \\
&+ \underbrace{(-L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1} + \frac{3}{4}L_4|\hat{x}+\hat{y}-2\hat{z}|^{-1/2})}_{c}\begin{pmatrix} \bar{e} \\ \bar{e} \\ -2\bar{e} \end{pmatrix} \otimes \begin{pmatrix} \bar{e} \\ \bar{e} \\ -2\bar{e} \end{pmatrix} \\
&+ \underbrace{(L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1} - \frac{3}{2}L_4|\hat{x}+\hat{y}-2\hat{z}|^{-1/2})}_{d}\begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}.
\end{aligned}
\tag{3.2.25}
$$

Let us denote

$$
\begin{aligned}
\tilde{a} &:= 8L_1^2 - 20L_1L_2|\hat{x}-\hat{y}|^{1/2} + \frac{25}{2}L_2^2|\hat{x}-\hat{y}|, \\
\tilde{b} &:= -5L_1L_2|\hat{x}-\hat{y}|^{-3/2} + \frac{25}{4}L_2^2|\hat{x}-\hat{y}|^{-1}|\hat{x}-\hat{y}|^2, \\
\tilde{c} &:= \frac{25}{8}L_2^2|\hat{x}-\hat{y}|,
\end{aligned}
$$

and

$$\begin{aligned}
\tilde{p} &:= 6L_3^2|\hat{x}+\hat{y}-2\hat{z}|^{-2} - 9L_3L_4|\hat{x}+\hat{y}-2\hat{z}|^{-3/2} + \frac{27}{8}L_4^2|\hat{x}+\hat{y}-2\hat{z}|^{-1},\\
\tilde{q} &:= -12L_3^2|\hat{x}+\hat{y}-2\hat{z}|^{-2} + 27L_3L_4|\hat{x}+\hat{y}-2\hat{z}|^{-3/2} - \frac{27}{2}L_4^2|\hat{x}+\hat{y}-2\hat{z}|^{-1},\\
\tilde{d} &:= 6L_3^2|\hat{x}+\hat{y}-2\hat{z}|^{-2} - 18L_3L_4|\hat{x}+\hat{y}-2\hat{z}|^{-3/2} + \frac{27}{2}L_4^2|\hat{x}+\hat{y}-2\hat{z}|^{-1}.
\end{aligned}$$

With this notation, we can find the square of the Hessian matrix $(D^2\varphi(\hat{x},\hat{y},\hat{z}))^2$. Indeed, naming by $v := x-y$ and $w := x+y-2z$, we compute,

$$\begin{aligned}
&\begin{pmatrix} I & -I & 0\\ -I & I & 0\\ 0 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix} v\\ -v\\ 0\end{pmatrix}\otimes\begin{pmatrix} v\\ -v\\ 0\end{pmatrix}\\
&\qquad = \begin{pmatrix} v\\ -v\\ 0\end{pmatrix}\otimes\begin{pmatrix} v\\ -v\\ 0\end{pmatrix}\cdot\begin{pmatrix} I & -I & 0\\ -I & I & 0\\ 0 & 0 & 0\end{pmatrix} = 2\begin{pmatrix} vv^T & -vv^T & 0\\ -vv^T & vv^T & 0\\ 0 & 0 & 0\end{pmatrix},
\end{aligned}$$

$$\begin{pmatrix} I & -I & 0\\ -I & I & 0\\ 0 & 0 & 0\end{pmatrix}^2 = 2\begin{pmatrix} I & -I & 0\\ -I & I & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\begin{aligned}
&\begin{pmatrix} I & -I & 0\\ -I & I & 0\\ 0 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix} w\\ w\\ -2w\end{pmatrix}\otimes\begin{pmatrix} w\\ w\\ -2w\end{pmatrix}\\
&\qquad = \begin{pmatrix} w\\ w\\ -2w\end{pmatrix}\otimes\begin{pmatrix} w\\ w\\ -2w\end{pmatrix}\cdot\begin{pmatrix} I & -I & 0\\ -I & I & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},
\end{aligned}$$

$$\begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}$$

$$= \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \cdot \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} v \\ -v \\ 0 \end{pmatrix} \otimes \begin{pmatrix} v \\ -v \\ 0 \end{pmatrix} \cdot \begin{pmatrix} w \\ w \\ -2w \end{pmatrix} \otimes \begin{pmatrix} w \\ w \\ -2w \end{pmatrix}$$

$$= \begin{pmatrix} w \\ w \\ -2w \end{pmatrix} \otimes \begin{pmatrix} w \\ w \\ -2w \end{pmatrix} \cdot \begin{pmatrix} v \\ -v \\ 0 \end{pmatrix} \otimes \begin{pmatrix} v \\ -v \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \cdot \begin{pmatrix} v \\ -v \\ 0 \end{pmatrix} \otimes \begin{pmatrix} v \\ -v \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} v \\ -v \\ 0 \end{pmatrix} \otimes \begin{pmatrix} v \\ -v \\ 0 \end{pmatrix} \cdot \begin{pmatrix} I & -I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \cdot \begin{pmatrix} w \\ w \\ -2w \end{pmatrix} \otimes \begin{pmatrix} w \\ w \\ -2w \end{pmatrix}$$

$$= \begin{pmatrix} w \\ w \\ -2w \end{pmatrix} \otimes \begin{pmatrix} w \\ w \\ -2w \end{pmatrix} \cdot \begin{pmatrix} I & -I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}$$

$$= 6 \begin{pmatrix} ww^T & ww^T & -2ww^T \\ ww^T & ww^T & -2ww^T \\ -2ww^T & -2ww^T & 4ww^T \end{pmatrix},$$

$$\left(\begin{pmatrix} v \\ -v \\ 0 \end{pmatrix} \otimes \begin{pmatrix} v \\ -v \\ 0 \end{pmatrix}\right)^2 = 2 \begin{pmatrix} (vv^T)^2 & -(vv^T)^2 & 0 \\ -(vv^T)^2 & (vv^T)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\left(\begin{pmatrix} w \\ w \\ -2w \end{pmatrix} \otimes \begin{pmatrix} w \\ w \\ -2w \end{pmatrix}\right)^2 = 6 \begin{pmatrix} (ww^T)^2 & (ww^T)^2 & -2(ww^T)^2 \\ (ww^T)^2 & (ww^T)^2 & -2(ww^T)^2 \\ -2(ww^T)^2 & -2(ww^T)^2 & 4(ww^T)^2 \end{pmatrix},$$

$$\begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}^2 = 6 \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}.$$

Thus we get

$$\begin{aligned}
&(D^2\varphi(\hat{x},\hat{y},\hat{z}))^2 \\
&\quad = \tilde{a} \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \tilde{b} \begin{pmatrix} \tilde{e}\tilde{e}^T & -\tilde{e}\tilde{e}^T & 0 \\ -\tilde{e}\tilde{e}^T & \tilde{e}\tilde{e}^T & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad + \tilde{c} \begin{pmatrix} (\tilde{e}\tilde{e}^T)^2 & -(\tilde{e}\tilde{e}^T)^2 & 0 \\ -(\tilde{e}\tilde{e}^T)^2 & (\tilde{e}\tilde{e}^T)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad + \tilde{p} \begin{pmatrix} (\bar{e}\bar{e}^T)^2 & (\bar{e}\bar{e}^T)^2 & -2(\bar{e}\bar{e}^T)^2 \\ (\bar{e}\bar{e}^T)^2 & (\bar{e}\bar{e}^T)^2 & -2(\bar{e}\bar{e}^T)^2 \\ -2(\bar{e}\bar{e}^T)^2 & -2(\bar{e}\bar{e}^T)^2 & 4(\bar{e}\bar{e}^T)^2 \end{pmatrix} \\
&\quad + \tilde{q} \begin{pmatrix} \bar{e}\bar{e}^T & \bar{e}\bar{e}^T & -2\bar{e}\bar{e}^T \\ \bar{e}\bar{e}^T & \bar{e}\bar{e}^T & -2\bar{e}\bar{e}^T \\ -2\bar{e}\bar{e}^T & -2\bar{e}\bar{e}^T & 4\bar{e}\bar{e}^T \end{pmatrix} + \tilde{d} \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}.
\end{aligned} \tag{3.2.26}$$

Notice also that

$$\begin{aligned} \xi^T \tilde{e}\tilde{e}^T \xi &\le |\xi|^2, \\ \xi^T \bar{e}\bar{e}^T \xi &\le |\xi|^2. \end{aligned} \tag{3.2.27}$$

Moreover, all matrices in (3.2.26) are nonnegatively definite. Indeed, for example, taking the matrix

$$\begin{pmatrix} \tilde{e}\tilde{e}^T & -\tilde{e}\tilde{e}^T & 0 \\ -\tilde{e}\tilde{e}^T & \tilde{e}\tilde{e}^T & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we calculate that

$$(\tilde{e}\tilde{e}^T)\tilde{e} = \tilde{e}(\tilde{e}^T\tilde{e}) = 1 \cdot \tilde{e},$$

which shows that the matrix has eigenvalue one of multiplicity one, and eigenvalue zero of multiplicity $3n-1$. This yields that the considered matrix is nonnegatively definite. Similarly, one proves that other matrices in (3.2.26) are nonnegatively definite, as stated.

On the one hand, multiplying both sides of (3.2.22) by the vector $(\xi, \xi, \xi)^T$, we get that

$$X + Y + Z \leq 0. \tag{3.2.28}$$

From the other hand, the inequality (3.2.22) (we use $\begin{pmatrix} \xi \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \xi \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ \xi \end{pmatrix}$, respectively) gives us

$$\begin{aligned} \langle \xi^T X, \xi \rangle &\leq a|\xi|^2 - b|\xi^T\tilde{e}|^2 + c|\xi^T\bar{e}|^2 + d|\xi|^2 \\ &\quad + \mu(\tilde{a}|\xi|^2 + \tilde{b}|\xi^T\tilde{e}|^2 + \tilde{c}|\xi^T\tilde{e}\tilde{e}^T|^2 + \tilde{p}|\xi^T\bar{e}\bar{e}^T|^2 + \tilde{q}|\xi^T\bar{e}|^2 + \tilde{d}|\xi|^2), \\ \langle \xi^T Y, \xi \rangle &\leq a|\xi|^2 - b|\xi^T\tilde{e}|^2 + c|\xi^T\bar{e}|^2 + d|\xi|^2 \\ &\quad + \mu(\tilde{a}|\xi|^2 + \tilde{b}|\xi^T\tilde{e}|^2 + \tilde{c}|\xi^T\tilde{e}\tilde{e}^T|^2 + \tilde{p}|\xi^T\bar{e}\bar{e}^T|^2 + \tilde{q}|\xi^T\bar{e}|^2 + \tilde{d}|\xi|^2), \\ \langle \xi^T Z, \xi \rangle &\leq 4c|\xi^T\bar{e}|^2 + 4d|\xi|^2 \\ &\quad + \mu(4\tilde{p}|\xi^T\bar{e}\bar{e}^T|^2 + 4\tilde{q}|\xi^T\bar{e}\bar{e}^T|^2 + 4\tilde{d}|\xi|^2). \end{aligned} \tag{3.2.29}$$

Observe that, fixed $L_1 > 100L_2, L_3 > 100L_4$, and recalling that

$$|\hat{x} - \hat{y}|^{1/2} < 2 \text{ and } |\hat{x} + \hat{y} - 2\hat{z}| < 4,$$

one has that

$$\begin{aligned} a &= 2L_1 - \frac{5}{2}L_2|\hat{x} - \hat{y}|^{1/2} > 200L_2 - 5L_2 = 195L_2 > 0, \\ d &= L_3|\hat{x} + \hat{y} - 2\hat{z}|^{-1} - \frac{3}{2}L_4|\hat{x} + \hat{y} - 2\hat{z}|^{-1/2} \end{aligned}$$

$$
\begin{aligned}
&= L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1/2}(|\hat{x}+\hat{y}-2\hat{z}|^{-1/2}-\frac{3L_4}{2L_3}),\\
&= \frac{97}{200}L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1/2}>0,\\
c &= -L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1}+\frac{3}{4}L_4|\hat{x}+\hat{y}-2\hat{z}|^{-1/2}\\
&= -L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1/2}(|\hat{x}+\hat{y}-2\hat{z}|^{-1/2}-\frac{3L_4}{4L_3}),\\
&< \frac{197}{400}L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1/2}<0.
\end{aligned}
$$

The same idea can be applied to estimate $(D^2\varphi(\hat{x},\hat{y},\hat{z}))^2$, that is,

$$
\begin{aligned}
&\tilde{a} = 2(2L_1-\frac{5}{2}L_2|\hat{x}+\hat{y}-2\hat{z}|^{1/2})^2 > 2\cdot 195^2 L_2^2>0,\\
&\tilde{b} = -5L_1L_2|x-y|^{1/2}+\frac{25}{4}L_2^2|\hat{x}-\hat{y}| = -L_2|\hat{x}-\hat{y}|^{1/2}(5L_1-\frac{25}{2}L_2^2|x-y|^{1/2})\\
&\quad < -475L_2^2|\hat{x}-\hat{y}|^{1/2}<0,\\
&\tilde{p} = 6(L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1}-\frac{3}{4}L_4|\hat{x}+\hat{y}-2\hat{z}|^{-1/2})^2\\
&\quad > 6\left(\frac{197}{400}\right)^2 L_3^2|\hat{x}+\hat{y}-2\hat{z}|^{-1}>0.
\end{aligned}
\tag{3.2.30}
$$

In a similar way we estimate also the remaining coefficients,

$$
\begin{aligned}
\tilde{q} &= -12L_3|\hat{x}+\hat{y}-2\hat{z}|^{-2}+27L_3L_4|\hat{x}+\hat{y}-2\hat{z}|^{-3/2}-\frac{27}{2}L_4^2|\hat{x}+\hat{y}-2\hat{z}|^{-1}\\
&= -12\left(L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1}-\frac{27}{24}L_4|\hat{x}+\hat{y}-2\hat{z}|^{-1/2}\right)^2\\
&+12\left(\frac{27}{24}\right)^2 L_4^2|\hat{x}+\hat{y}-2\hat{z}|^{-1}-\frac{27}{2}L_4^2|\hat{x}+\hat{y}-2\hat{z}|^{-1}\\
&< -12\left(\frac{1183}{24}\right)^2 L_2^2|\hat{x}+\hat{y}-2\hat{z}|^{-1}+12\left(\frac{27}{24}\right)^2 L_4^2|\hat{x}+\hat{y}-2\hat{z}|^{-1}\\
&-\frac{27}{2}L_4^2|\hat{x}+\hat{y}-2\hat{z}|^{-1}<0,\\
\tilde{d} &= 6\left(L_3|\hat{x}+\hat{y}-2\hat{z}|^{-1}-\frac{3}{2}L_4|\hat{x}+\hat{y}-2\hat{z}|^{-1/2}\right)^2\\
&> 6\left(\frac{197}{200}\right)^2 L_3^2|\hat{x}+\hat{y}-2\hat{z}|^{-1}>0.
\end{aligned}
$$

To sum up, we have that

$$a, b, d, \tilde{a}, \tilde{c}, \tilde{p}, \tilde{d} \text{ are positive, whereas}$$

$$c, \tilde{b}, \tilde{q} \text{ are negative.}$$

This, (3.2.29) and (3.2.27) together yield

$$\begin{aligned}
\langle \xi^T X, \xi \rangle &\leq (a + d + \mu(\tilde{a} + \tilde{c} + \tilde{p} + \tilde{d}))|\xi|^2 + c|\xi^T \bar{e}|, \\
\langle \xi^T Y, \xi \rangle &\leq (a + d + \mu(\tilde{a} + \tilde{c} + \tilde{p} + \tilde{d}))|\xi|^2 + c|\xi^T \bar{e}|, \\
\langle \xi^T Z, \xi \rangle &\leq (4d + \mu(4\tilde{p} + 4\tilde{d}))|\xi|^2 + 4c|\xi^T \bar{e}|.
\end{aligned} \tag{3.2.31}$$

Now, summing inequalities (3.2.31), and then taking $\xi = \bar{e}$, we obtain

$$\begin{aligned}
\langle \bar{e}(X + Y + Z), \bar{e} \rangle &\leq (2a + 6d + \mu(2\tilde{a} + 2\tilde{c} + 6\tilde{p} + 6\tilde{d}))|\bar{e}|^2 + 6c|\bar{e}|^2 \\
&= 2a + 6d + 6c + \mu(2\tilde{a} + 2\tilde{c} + 6\tilde{p} + 6\tilde{d}) \\
&= 4L_1 - 5L_2|\hat{x} - \hat{y}|^{1/2} + 6L_3|\hat{x} + \hat{y} - 2\hat{z}|^{-1} - 9L_4|\hat{x} + \hat{y} - 2\hat{z}|^{-1/2} \\
&\quad - 6L_3|\hat{x} + \hat{y} - 2\hat{z}|^{-1} + \frac{9}{2}L_4|\hat{x} + \hat{y} - 2\hat{z}|^{-1/2} + \mu(2\tilde{a} + 2\tilde{c} + 6\tilde{p} + 6\tilde{d}) \\
&= 4L_1 - 5L_2|\hat{x} - \hat{y}|^{1/2} - \frac{9}{2}L_4|\hat{x} + \hat{y} - 2\hat{z}|^{-1/2} + \mu(2\tilde{a} + 2\tilde{c} + 6\tilde{p} + 6\tilde{d}),
\end{aligned} \tag{3.2.32}$$

where we used $|\bar{e}| = 1$ in the first equality.

Notice, that $2\tilde{a} + 2\tilde{c} + 6\tilde{p} + 6\tilde{d}$ is positive, so we can choose μ such that

$$\mu(2\tilde{a} + 2\tilde{c} + 6\tilde{p} + 6\tilde{d}) < \frac{9}{4}L_4|\hat{x} + \hat{y} - 2\hat{z}|^{-1/2}. \tag{3.2.33}$$

From (3.2.32) and (3.2.33), it follows that, if L_4 is very large, then

$$\langle \bar{e}(X + Y + Z), \bar{e} \rangle < 4L_1 - 5L_2|\hat{x} - \hat{y}|^{1/2} - \frac{9}{4}L_4|\hat{x} + \hat{y} - 2\hat{z}|^{-1/2} < 0, \tag{3.2.34}$$

which shows that one of the eigenvalues of $X + Y + Z$ is negative.

Now, take $r_1, r_2, r_3 > 0$ such that $B_{r_1}(\hat{x}), B_{r_2}(\hat{y}), B_{r_3}(\hat{z}) \Subset B_1$, and set

$$v(x) := \begin{cases} u(\hat{x}) + \langle D_x\varphi(\hat{x}, \hat{y}, \hat{z}) + 2k(\hat{x} - \zeta), x - \hat{x} \rangle & \\ \qquad + \frac{1}{2}\langle (X + 2kI)(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2), & \text{if } x \in B_{r_1}(\hat{x}), \\ u(x), & \text{if } x \notin B_{r_1}(\hat{x}), \end{cases}$$

$$w(y) := \begin{cases} u(\hat{y}) + \langle D_y\varphi(\hat{x},\hat{y},\hat{z}), y-\hat{y}\rangle + \frac{1}{2}\langle Y(y-\hat{y}), y-\hat{y}\rangle + o(|y-\hat{y}|^2), & \text{if } y \in B_{r_2}(\hat{y}),\\ u(y), & \text{if } y \notin B_{r_2}(\hat{y}),\end{cases}$$

$$h(z) := \begin{cases} u(\hat{z}) + \langle D_z\varphi(\hat{x},\hat{y},\hat{z}), z-\hat{z}\rangle + \frac{1}{2}\langle -\frac{1}{2}Z(z-\hat{z}), z-\hat{z}\rangle + o(|z-\hat{z}|^2), & \text{if } z \in B_{r_3}(\hat{z}),\\ u(z), & \text{if } z \notin B_{r_3}(\hat{z}).\end{cases}$$

Since u is a viscosity solution to (3.2.19), we find that

$$Fv(\hat{x}) \le f(\hat{x}), \tag{3.2.35}$$
$$Fw(\hat{y}) \le f(\hat{y}), \tag{3.2.36}$$
$$Fh(\hat{z}) \ge f(\hat{z}). \tag{3.2.37}$$

Then (3.2.35), (3.2.36), (3.2.37) and convexity give

$$\begin{aligned} 0 &\ge Fv(\hat{x}) + Fw(\hat{y}) - 2Fh(\hat{z}) - (f(\hat{x}) + f(\hat{y}) - 2f(\hat{z}))\\ &\ge 2F\left(\frac{v(\hat{x})+w(\hat{y})}{2}\right) - 2Fh(\hat{z}) - 3\|f\|_{L^\infty(B_1)}. \end{aligned} \tag{3.2.38}$$

Moreover, due to the uniform ellipticity property 3.2.2,

$$2F\left(\frac{v(\hat{x})+w(\hat{y})}{2}\right) - 2Fh(\hat{z}) \ge 2M^-_{\mathcal{L}}\left(\frac{v(\hat{x})+w(\hat{y})}{2} - h(\hat{z})\right). \tag{3.2.39}$$

Now, we fix $r < \min\{r_1, r_2, r_3\}$ and use (3.2.38) together with (3.2.39), obtaining

$$\begin{aligned} 0 &\ge \inf_{L\in\mathcal{L}} L(v(\hat{x}) + w(\hat{y}) - 2h(\hat{z})) - 3\|f\|_{L^\infty(B_1)}\\ &\ge \inf_K \left(\int_{B_r} (\Delta_t^2 v(\hat{x}) + \Delta_t^2 w(\hat{y}) - 2\Delta_t^2 h(\hat{z}))K(t)dt\right.\\ &\quad \left. + \int_{B_r^c} (\Delta_t^2 v(\hat{x}) + \Delta_t^2 w(\hat{z}) - 2\Delta_t^2 h(\hat{z}))K(t)dt\right) - 3\|f\|_{L^\infty(B_1)}. \end{aligned} \tag{3.2.40}$$

Recall that the functions v, w, h are C^2 in small neighborhoods of $\hat{x}, \hat{y}$ and $\hat{z}$, respectively. Hence, we can apply the Taylor formula in (3.2.40) and get

$$\begin{aligned} 0 \geq & \inf_K \int_{B_r} \langle -(X + 2kI + Y + Z)t, t \rangle K(t) dt \\ & + \inf_K \int_{B_r^c} (\Delta_t^2 v(\hat{x}) + \Delta_t^2 w(\hat{z}) - 2\Delta_t^2 h(\hat{z})) K(t) dt - 3\|f\|_{L^\infty(B_1)}. \end{aligned} \tag{3.2.41}$$

Set $M = (m_{ij}) := -(X + 4I + Y + Z)$, and observe that for $i \neq j$,

$$\int_{B_r} m_{ij} \cdot t_i \cdot t_j \cdot K(t) dt = 0. \tag{3.2.42}$$

In particular, for any fixed i, we have

$$\begin{aligned} \int_{B_r} m_{ii} t_i^2 \, K(t) dt = \int_{B_r} m_{ii} t_i^2 \, K(t) dt = \int_{B_r} m_{ii} v(x) t_1^2 \, K(t) dt \\ = \sum_{k=1}^n \int_{B_r} \frac{m_{ii}}{n} t_k^2 \, K(t) dt = \int_{B_r} \frac{m_{ii}}{n} |t|^2 \, K(t) dt. \end{aligned} \tag{3.2.43}$$

Taking use of (3.2.28) and (3.2.34), and then exploiting (3.2.42) and (3.2.43), we continue the chain of inequalities (3.2.41),

$$\begin{aligned} 0 \geq & - \operatorname{Tr}(X + Y + Z + 4I) \frac{1}{n} \inf_K \int_{B_r} |t|^2 K(t) dt \\ & - 12\|u\|_{L^\infty(\mathbb{R}^n)} \sup_K \int_{B_r^c} K(t) dt - 3\|f\|_{L^\infty(B_1)} \\ \geq & \left(-4L_1 + 5L_2|\hat{x} - \hat{y}|^{1/2} + \frac{9}{4} L_4 |\hat{x} + \hat{y} - 2\hat{z}|^{-1/2} \right) \frac{\lambda c_{n,s}}{n} \int_{B_r} \frac{1}{|t|^{n+2s-2}} dt \\ & + 4\lambda c_{n,s} \int_{B_r} \frac{1}{|t|^{n+2s-2}} dt - 12\|u\|_{L^\infty(\mathbb{R}^n)} \Lambda c_{n,s} \int_{B_r^c} \frac{1}{|t|^{n+2s}} dt - 3\|f\|_{L^\infty(B_1)}. \end{aligned} \tag{3.2.44}$$

This shows that the right hand side of (3.2.44) becomes positive, since

$$-4L_1 + 5L_2|\hat{x} - \hat{y}|^{1/2} + \frac{9}{4} L_4 |\hat{x} + \hat{y} - 2\hat{z}|^{-1/2} \to +\infty$$

as $L_4 \to +\infty$, which gives us a contradiction.

Hence, we proved that

$$\Phi(x, y, z) = u(x) + u(y) - 2u(z) - \varphi(x, y, z) - 2|x - \zeta|^2 \leq 0,$$

that yields (3.2.20) setting $\zeta = x$. □

Chapter 4

Domain variation solutions

In this final chapter we deal with some advanced properties associated with local operators. In fact, as we pointed out in (1.2.8), local operators may be considered as a special case of nonlocal operators.

4.1 Domain variation solutions

In this section, following the ideas described in [3], we would like to find the right formulation of the two phase free boundary problems arising from Bernoulli type functionals when we consider nonnegative matrices of variable coefficients or a nonlinear dependence both on the gradient of the solutions and on the variable x.

This would be a first step before starting to face the one-phase problems governed by degenerate operators, even possibly defined on noncommutative groups.

We have in mind two concrete examples respectively given by the Kohn-Laplace operator in the Heisenberg group and the $p(x)-$Laplace operator.

Since the $p-$Laplace case has been discussed in [57] as well, so that it results also interesting to understand the behavior of the $p(x)-$Laplace operator. We remind that the $p-$Laplace operator is defined as

$$\Delta_p := \mathrm{div}(|\nabla \cdot|^{p-2}\nabla),$$

while the $p(x)-$Laplace operator is

$$\Delta_{p(x)} := \mathrm{div}(|\nabla \cdot|^{p(x)-2}\nabla),$$

where the function p satisfies $1 < p(x) < \infty$.

Of course, $\Delta_{p(x)} = \Delta_p$ when $p(x)$ is constant and $p(x) \equiv p$.

The Kohn-Laplace operator in $\mathbb{H}^1$ is defined as

$$\Delta_{\mathbb{H}^1} u(x,y,t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 4y\frac{\partial^2 u}{\partial x \partial t} - 4x\frac{\partial^2 u}{\partial y \partial t} + 4(x^2+y^2)\frac{\partial^2 u}{\partial t^2} \tag{4.1.1}$$

and even if it is linear, it results to be degenerate elliptic.

In particular, using an intrinsic interpretation of the geometric objects entering in the description of the noncommutative underlying structure $\mathbb{H}^1$, it is possible to obtain an intrinsic formulation of the two phase problem. We recall that the Kohn-Laplace operator is degenerate. Indeed, its lowest eigenvalue is always zero. As a consequence, it is important to understand what is the right condition to require to put on the free boundary in case we wish to formulate the problem in a viscosity sense.

The theory of the viscosity solutions has been applied to the study of free boundary problems, like

$$\begin{cases} \Delta u = f, & \text{in } \Omega^+(u) := \{x \in \Omega : \ u(x) > 0\}, \\ \Delta u = f, & \text{in } \Omega^-(u) := \mathrm{Int}(\{x \in \Omega : \ u(x) \leq 0\}), \\ (u_n^+)^2 - (u_n^-)^2 = 1 & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{4.1.2}$$

since [14], for homogeneous problems, by Luis Caffarelli. Here $\Omega \subset \mathbb{R}^n$ is an open set, and $f \in C^{0,\alpha} \cap L^\infty(\Omega)$, while u_n^+ formally denotes the normal derivative at the points belonging to $\mathcal{F}(u)$, where n is the unit normal in those points whenever this makes sense, pointing inside $\Omega^+(u)$, as well as u_n^- denotes the normal derivative to the set $\mathcal{F}(u)$ and n is the unit normal to the set $\mathcal{F}(u)$ at the point $x \in \mathcal{F}(u)$ pointing inside $\Omega^-(u)$.

If, in case $\mathcal{F}(u)$ were C^1, even supposing for simplicity that $f \equiv 0$, then u would satisfy $\Delta u = 0$ in $\Omega^+(u) \cup \Omega^-(u)$. On the other hand, $u \in C(\Omega)$ is a viscosity solution, so that $\Delta u = 0$ in $\Omega^+(u)$ and $\Delta u = 0$ in $\Omega^-(u)$ in the classic sense and the problem (4.1.2) may be reduced to two Dirichlet problems. However the assumption on the level set $\mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega$ can not be formulated in a classical fashion, because $\mathcal{F}(u)$ is an unknown of the problem. In principle, the set $\mathcal{F}(u)$ might be very irregular and the notion of solution would not make sense in the classical meaning, so that has to be weakened.

On the contrary, we suppose exactly that the fact itself of knowing that u satisfies the free boundary problem should imply that u is endowed with some further regularity properties. Thus, assuming only that $\mathcal{F}(u)$ is Lipschitz, the solution of the Dirichlet problem in a neighborhood of the free boundary may be a priori no better than a Hölder continuous function until the boundary.

Hence, it appears clear that we can not give a pointwise classical formulation of the problem on the free boundary. For avoiding this loop, in [14] a viscosity notion of solution was introduced. In that case the boundary condition is supposed to be fulfilled only where a weak normal exists, see [16].

The definition of solution of the problem (4.1.2) can be stated, in a viscosity sense, see [28] and the original statement in [14] or in [16] as well, in the following way.

A continuous function u is a solution to (4.1.2) if

(i) $\Delta u = f$ in a viscosity sense in $\Omega^+(u)$ and $\Omega^-(u)$;

(ii) let $x_0 \in \mathcal{F}(u)$. For every function $v \in C(B_\varepsilon(x_0))$, $\varepsilon > 0$ such that $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$, being $B := B_\varepsilon(x_0)$ and $\mathcal{F}(v) \in C^2$, if v touches u from below (resp. above) at $x_0 \in \mathcal{F}(v)$, then

$$(v_n^+(x_0))^2 - (v_n^-(x_0))^2 \leq 1 \quad \text{(resp.} \quad (v_n^+(x_0))^2 - (v_n^-(x_0))^2 \geq 1).$$

Moreover, also the following notion of strict comparison subsolution (supersolution) plays a fundamental role in the regularity theory of one/two-phase free boundary problems, see [27]: a function $v \in C(\Omega)$ is a strict comparison subsolution (supersolution) to (4.1.2) if $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ and

(i) $\Delta v > f$ (resp. $\Delta v < f$) in a viscosity sense in $\Omega^+(v) \cup \Omega^-(v)$;

(ii) for every $x_0 \in \Omega$, if $x_0 \in \mathcal{F}(v)$ then

$$(v_n^+(x_0))^2 - (v_n^-(x_0))^2 > 1 \quad \text{(resp.} \quad (v_n^+(x_0))^2 - (v_n^-(x_0))^2 < 1, \quad v_n^+(x_0) \neq 0).$$

As a consequence, a strict comparison subsolution v cannot touch a viscosity solution u from below at any point in $\mathcal{F}(u) \cap \mathcal{F}(v)$. Analogously, a strict comparison supersolution v cannot touch a viscosity solution u from above at any point in $\mathcal{F}(u) \cap \mathcal{F}(v)$.

We are mainly interested in viscosity solution, but the natural definition of two-phase free boundary problems is usually determined by looking for local minima of functionals like

$$\mathcal{E}(v) = \int_{\Omega} \Big(|\nabla v|^2 + \chi_{\{v>0\}} + 2fv \Big) dx \tag{4.1.3}$$

defined on subsets of $\mathbb{H}^1(\Omega)$ satisfying some fixed conditions, for instance assumed on the boundary of Ω and on the sign of the functions themselves.

In [3] exactly this approach has been followed for functionals, associated with the Laplace operator like (4.1.3), in the homogeneous case. As a consequence, to local minima u of (4.1.3) (supposing $f = 0$) in [3] have been determined the conditions that have to be satisfied on the free boundary, morally the set $\{x \in \Omega : \ u(x) = 0\}$.

Since we are interested in problems governed by other operators with respect to the Euclidean laplacian, like nonlinear ones and, overall possibly degenerate, we wish, at first, to understand what is the right condition to put on the free boundary, for the problem in a non-divergence form, in a degenerate setting.

In fact, the free boundary $\mathcal{F}(u)$ is an unknown of the problem and for this reason we need to start from the energy functional that describes the problem in the variational setting for obtaining the non-divergence case.

With this aim, we discuss the notion of domain variation solution assuming that the energy functionals that we wish to study may be associated with degenerate operators like the $p(x)$-Laplace operator $\Delta_{p(x)}$, that is a generalization of the most popular $p-$Laplace operator when the function $p(x)$ is constant or operators like $\mathrm{div}(A(x)\nabla)$, supposing that the matrix A satisfies $\langle A(x)\xi, \xi \rangle \geq 0$ for every $\xi \in \mathbb{R}^n$ whenever A is a smooth matrix of coefficients. For the notion of solution in the sense of variation domain and applications we refer to [70].

At the end of our discussion we conclude that in any Carnot group the two phase problem assumes the following nonvariational form:

$$\begin{cases} \Delta_{\mathbb{G}} u = f, & \text{in } \Omega^+(u) := \{x \in \Omega : \ u(x) > 0\}, \\ \Delta_{\mathbb{G}} u = f, & \text{in } \Omega^-(u) := \mathrm{Int}(\{x \in \Omega : \ u(x) \leq 0\}), \\ |\nabla_{\mathbb{G}} u^+|^2 - |\nabla_{\mathbb{G}} u^-|^2 = 1, & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{4.1.4}$$

where $\Delta_{\mathbb{G}}$ is a sublaplacian in a Carnot group $\mathbb{G}$, see Section 1.4 for the definitions of Carnot groups and the associated notation, and Section 4.4 for a little more general presentation of the result.

We remark here, however, that now the condition posed on free boundary is governed by an intrinsic jump of gradients, see Section 1.4 and, for the one-phase case, see [42].

Moreover, in the case of the $p(x)-$Laplacian, the functional becomes

$$\mathcal{E}_{p(x)}(u) = \int_{\Omega} \left(|\nabla v|^{p(x)} + \chi_{\{v>0\}} + p(x)fv\right) dx,$$

so that we obtain:

$$\begin{cases} \Delta_{p(x)} u = f, & \text{in } \Omega^+(u) := \{x \in \Omega : \ u(x) > 0\}, \\ \Delta_{p(x)} u = f, & \text{in } \Omega^-(u) := \text{Int}(\{x \in \Omega : \ u(x) \leq 0\}), \\ |\nabla u^+|^{p(x)} - |\nabla u^-|^{p(x)} = \frac{1}{p(x)-1}, & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{4.1.5}$$

see also Section 4.6 for a slightly more general setting of the problem.

We complete our analysis in Section 4.7 stating the suitable notion of viscosity solutions for problems like (4.1.4) and (4.1.5). In the case (4.1.4) the characteristic points introduce new difficulties in the application of the approach used in [27]. Regarding the notion of viscosity solution we refer to [15, 24, 7].

In the next section, for describing the meaning of domain variation solution, we deal with the simplest case in one dimension.

4.2 One dimensional Euclidean case

Before entering the details of our subject we consider the basic heuristic example in the one dimension for the following functional

$$\mathcal{E}(v) = \int_{-1}^{1} (v'^2 + \chi_{\{v>0\}} + 2fv) dx,$$

where

$$\chi_{\{v>0\}} = \begin{cases} 1, & x \in \{v > 0\}, \\ 0, & x \in \{v \leq 0\}, \end{cases}$$

and $v \in K = \{w \in H^1([-1,1]) : \quad w(-1) = a, \; w(1) = b\}$ being a, b assigned values to the boundary. Moreover, we assume, for simplicity, that $f \in C^{0,\gamma}([-1,1])$.

We are interested in those functions which become minima or critical values for $\mathcal{E}$ perturbing the set of definition in a neighborhood of the points where v vanishes. In mathematical language, for every function $v \in K$ and for every function $\varphi \in C_0^\infty(]-1,1[)$ we consider the function $v(x) = v_\varepsilon^\varphi(x + \varepsilon\varphi(x))$. We shall simply write $v_\varepsilon := v_\varepsilon^\varphi$ to avoid the cumbersome notation. It is clear that $\tau_\varepsilon = I + \varepsilon\varphi$ is an application that transforms $[-1,1]$ in itself whenever ε is sufficiently small. We say that v is a variational domain solution whenever

$$\frac{d}{d\varepsilon}\mathcal{E}(v_\varepsilon)\bigg|_{\varepsilon=0} = 0.$$

To do this, we consider

$$\mathcal{E}(v_\varepsilon) = \int_{-1}^{1} (v_\varepsilon'^2(y) + \chi_{\{v_\varepsilon>0\}}(y) + 2f(y)v_\varepsilon(y))dy. \tag{4.2.1}$$

Since τ_ε is invertible whenever ε is small we obtain $(\tau_\varepsilon^{-1})'(y) = (\tau_\varepsilon'(x))^{-1}$, being $x = \tau_\varepsilon^{-1}(y)$ and

$$\tau_\varepsilon'(x) = 1 + \varepsilon\varphi'(x),$$
$$(\tau_\varepsilon^{-1})'(y) = \frac{1}{1 + \varepsilon\varphi'(\tau_\varepsilon^{-1}(y))}.$$

This implies that for $\varepsilon \to 0$

$$(\tau_\varepsilon^{-1})'(y) = 1 - \varepsilon\varphi'(\tau_\varepsilon^{-1}(y)) + o(\varepsilon).$$

We perform the change of variable $y = \tau_\varepsilon(x)$ so that:

$$\begin{aligned}\mathcal{E}(v_\varepsilon) &= \int_{-1}^{1} \left(v_\varepsilon'^2(\tau_\varepsilon(x)) + \chi_{\{v_\varepsilon>0\}}(\tau_\varepsilon(x)) + 2f(\tau_\varepsilon(x))v_\varepsilon(\tau_\varepsilon(x))\right)\tau_\varepsilon'(x)dx \\ &= \int_{-1}^{1} \left(v_\varepsilon'^2(\tau_\varepsilon(x)) + \chi_{\{v_\varepsilon>0\}}(\tau_\varepsilon(x)) + 2f(\tau_\varepsilon(x))v_\varepsilon(\tau_\varepsilon(x))\right)(1 + \varepsilon\varphi'(x))dx\end{aligned} \tag{4.2.2}$$

and, since $v'(x) = v'_\varepsilon(\tau_\varepsilon(x))\tau'_\varepsilon(x) = v'_\varepsilon(\tau_\varepsilon(x))(1+\varepsilon\varphi'(x))$, we get

$$
\begin{aligned}
&\mathcal{E}(v_\varepsilon)\\
&= \int_{-1}^{1} [v'^2(x)(1+\varepsilon\varphi'(x))^{-2} + \chi_{\{v_\varepsilon>0\}}(\tau_\varepsilon(x)) + 2f(\tau_\varepsilon(x))v(x)](1+\varepsilon\varphi'(x))dx\\
&= \int_{-1}^{1} [v'^2(x)(1-\varepsilon\varphi'(x)+o(\varepsilon))^2 + \chi_{\{v_\varepsilon>0\}}(x+\varepsilon\varphi(x))\\
&\quad + 2f(\tau_\varepsilon(x))v(x)](1+\varepsilon\varphi'(x))dx\\
&= \int_{-1}^{1} [v'^2(x)(1-2\varepsilon\varphi'(x)+o(\varepsilon)) + \chi_{\{v_\varepsilon>0\}}(x+\varepsilon\varphi(x))](1+\varepsilon\varphi'(x))dx\\
&\quad + 2\int_{-1}^{1} f(\tau_\varepsilon(x))v(x)(1+\varepsilon\varphi'(x))dx.
\end{aligned}
\tag{4.2.3}
$$

In other words,

$$
\begin{aligned}
&\mathcal{E}(v_\varepsilon)\\
&= \mathcal{E}(v) + \int_{-1}^{1} -\varepsilon v'^2(x)\varphi'(x) + [\chi_{\{v_\varepsilon>0\}}(x+\varepsilon\varphi(x))(1+\varepsilon\varphi'(x)) - \chi_{\{v>0\}}(x)]dx\\
&\quad + 2\varepsilon\int_{-1}^{1} (f(x)v(x)\varphi'(x) + f'(x)v(x)\varphi(x))dx + o(\varepsilon)\\
&= \mathcal{E}(v) + \int_{-1}^{1} -\varepsilon v'^2(x)\varphi'(x) + [\chi_{\{v_\varepsilon>0\}}(x+\varepsilon\varphi(x)) - \chi_{\{v>0\}}(x)]dx\\
&\quad + \varepsilon\int_{-1}^{1} \chi_{\{v_\varepsilon>0\}}(x+\varepsilon\varphi(x))\varphi'(x)dx\\
&\quad + 2\varepsilon\int_{-1}^{1} (f(x)\varphi'(x) + f'(x)\varphi(x))v(x)dx + o(\varepsilon).
\end{aligned}
\tag{4.2.4}
$$

Hence, integrating by parts and recalling that φ is a compactly supported function, we obtain

$$\begin{aligned}\mathcal{E}(v_\varepsilon) &= \mathcal{E}(v) + \int_{-1}^{1} -\varepsilon v'^2(x)\varphi'(x)dx + \varepsilon \int_{-1}^{1} \chi_{\{v_\varepsilon>0\}}(x+\varepsilon\varphi(x))\varphi'(x)dx \\ &+ 2(f(1)\varphi(1)v(1) - f(-1)\varphi(-1)v(-1)) - 2\varepsilon \int_{-1}^{1} f(x)\varphi(x)v'(x)dx + o(\varepsilon) \\ &= \mathcal{E}(v) + \int_{-1}^{1} -\varepsilon v'^2(x)\varphi'(x)dx + \varepsilon \int_{-1}^{1} \chi_{\{v_\varepsilon>0\}}(x+\varepsilon\varphi(x))\varphi'(x)dx \\ &- 2\varepsilon \int_{-1}^{1} f(x)\varphi(x)v'(x)dx + o(\varepsilon).\end{aligned} \tag{4.2.5}$$

As a consequence, if v is a local minimum for the functional $\mathcal{E}$ in K, then

$$\begin{aligned}0 \le \frac{\mathcal{E}(v_\varepsilon) - \mathcal{E}(v)}{\varepsilon} &= -\int_{-1}^{1} v'^2(x)\varphi'(x)dx + \int_{-1}^{1} \chi_{\{v_\varepsilon>0\}}(x+\varepsilon\varphi(x))\varphi'(x)dx \\ &- 2\int_{-1}^{1} f(x)\varphi(x)v'(x)dx + o(1).\end{aligned} \tag{4.2.6}$$

Moreover, it also results that for every $\varphi \in C_0^\infty(]-1,1[)$ we have

$$\lim_{\varepsilon\to 0} \frac{\mathcal{E}(v_\varepsilon) - \mathcal{E}(v)}{\varepsilon} = 0.$$

Hence, if v is a local minimum for $\mathcal{E}$ on K, then v is a domain variational solution.

As a consequence, we have obtained that a local minimum has to satisfy the following relationship:

$$-\int_{-1}^{1} v'^2(x)\varphi'(x)dx - 2\int_{-1}^{1} f(x)\varphi(x)v'(x)dx + \int_{-1}^{1} \chi_{\{v>0\}}(x)\varphi'(x)dx = 0,$$

for every $\varphi \in C_0^\infty(]-1,1[)$.

On the other hand, for every $\varphi \in C_0^\infty(-1,1)$ such that $\mathrm{supp}(\varphi) \subset \{v_\varepsilon > 0\}$ or $\mathrm{supp}(\varphi) \subset \mathrm{int}\{v_\varepsilon \leq 0\}$ it follows from the previous relation that $v'' = f(x)$ in $]-1,1[\setminus\{x \in]-1,1[: \quad v(x) = 0\}$ because v is a local minimum for $\mathcal{E}$, (we will proof this property below in a more general case).

As a consequence,

$$
\begin{aligned}
&-\lim_{\delta\to 0}\int_{-1}^{x_\delta} v'^2(x)\varphi'(x) + 2f(x)\varphi(x)v'(x)dx - \lim_{\varepsilon\to 0}\int_{x_\varepsilon}^{1} v'^2(x)\varphi'(x) + 2f(x)\varphi(x)v'(x)dx \\
&-\lim_{\substack{\varepsilon\to 0\\ \delta\to 0}}\int_{x_\varepsilon}^{x_\delta} v'^2(x)\varphi'(x) + 2f(x)\varphi(x)v'(x)dx + \int_{-1}^{1}\chi_{\{v>0\}}(x)\varphi'(x)dx = 0,
\end{aligned}
\tag{4.2.7}
$$

for every $\varphi \in C_0^\infty(]-1,1[)$ and $\varepsilon, \delta > 0$, we consider the sets $\{v(x) < -\varepsilon\}$ and $\{v(x) > \delta\}$. Then integrating by parts we obtain from (4.2.7), keeping in mind that $\mathrm{meas}_1(\{v = 0\}) = 0$,

$$
\begin{aligned}
&\lim_{\delta\to 0}\int_{-1}^{x_\delta} 2\left(v''(x) - f(x)\right)\varphi(x)v'(x)dx - \lim_{\delta\to 0}[v'^2(x)\varphi(x)]_{x=-1}^{x=x_\delta} \\
&+\lim_{\varepsilon\to 0}\int_{\delta_\varepsilon}^{1} 2\left(v''(x) - f(x)\right)\varphi(x)v'(x)dx - \lim_{\varepsilon\to 0}[v'^2(x)\varphi(x)]_{x=x_\varepsilon}^{x=1} \\
&+\int_{-1}^{1}\chi_{|\{v>0\}}(x)\varphi'(x)dx = 0,
\end{aligned}
\tag{4.2.8}
$$

where $v(x_\varepsilon) = -\varepsilon$ and $v(x_\delta) = \delta$.

Thus, from (4.2.8), we get

$$
-\lim_{\delta\to 0}[v'^2(x)\varphi(x)]_{x=-1}^{x=x_\delta} - \lim_{\varepsilon\to 0}[v'^2(x)\varphi(x)]_{x=x_\varepsilon}^{x=1} + \int_{-1}^{1}\chi_{|\{v>0\}}(x)\varphi'(x)dx = 0,
\tag{4.2.9}
$$

or

$$
-\lim_{\delta\to 0} v'^2(x_\delta)\varphi(x_\delta) + \lim_{\varepsilon\to 0} v'^2(x_\varepsilon)\varphi(x_\varepsilon) + \int_{x_0}^{1}\varphi'(x)dx = 0. \tag{4.2.10}
$$

This implies, for every $\varphi \in C_0^\infty(-1,1)$, denoting by x_0 the free boundary, that is $v(x_0) = 0$,

$$\big(-(v^-)'^2(x_0) + (v^+)'^2(x_0)\big)\varphi(x_0) - \varphi(x_0) = 0.$$

Hence, it results

$$(v^+)'^2(x) - (v^-)'^2(x) = 1, \quad \text{on} \quad \{v = 0\}.$$

In this way, we have obtained the free boundary condition associated with the Euler-Lagrange equations to local minima of the functional $\mathcal{E}$ in the non-homogeneous case, (of course assuming that the free boundary is a set of measure zero). We also proved that, at least in one dimension, the free boundary condition does not depend on the non-homogeneous term f.

4.3 The Bernoulli functional in the Heisenberg group

In this section, following the scheme of [3] we make some computations in the Heisenberg group $\mathbb{H}^n$, but using the same arguments, the final results apply also to Carnot groups. In particular, here we recall that local minima of our functionals are globally continuous. Let

$$J_{\mathbb{H}^n}(v) = \int_\Omega \big(|\nabla_{\mathbb{H}^n} v|^2 + q^2(x)\lambda^2(v) + 2fv\big)\,dx, \quad v \in K$$

be the functional that we will study, where $q^2(x) \neq 0$,

$$\lambda^2(v) = \begin{cases} \lambda_1^2, & \text{if} \quad v < 0, \\ \lambda_2^2, & \text{if} \quad v > 0, \end{cases} \tag{4.3.1}$$

and $\lambda^2(v)$ is lower semicontinuous at $v = 0$; it is assumed that $\lambda_i^2 > 0$ and $\Lambda = \lambda_1^2 - \lambda_2^2 \neq 0$. Here is

$$K = \{v \in L^1_{\text{loc}}(\Omega) : \ \nabla_{\mathbb{H}^n} v \in L^2(\Omega), \ v = u^0 \text{ on } S \subset \partial\Omega\}$$

and $\Omega \subset \mathbb{R}^n$ is a domain.

There exists a unique solution to the following Dirichlet problem

$$\begin{cases} \Delta_{\mathbb{H}^n} v_R = 0, & \text{in } B_R, \\ v_R = u, & \text{on } \partial B_R. \end{cases} \tag{4.3.2}$$

If u realises a minimum for $J_{\mathbb{H}^n}$, then for every ball $B_r \subset \Omega$ we get:

$$\int_{B_r} \left(|\nabla_{\mathbb{H}^n} u|^2 + q^2(x)\lambda^2(u) + 2fu\right) dx \le \int_{B_r} \left(|\nabla_{\mathbb{H}^n} v_r|^2 + q^2(x)\lambda^2(v_r) + 2fv_r\right) dx.$$

Hence by the Poincaré inequality we obtain

$$\begin{aligned} \int_{B_r} \left(|\nabla_{\mathbb{H}^n} u|^2 - |\nabla_{\mathbb{H}^n} v_r|^2\right) dx &\le \int_{B_r} \left(q^2(x)\lambda^2(v_r) - q^2(x)\lambda^2(u)\right) + 2f(v_r - u)dx \\ &\le C(\lambda_1, \lambda_2, Q)r^Q + 2\int_{B_r} f(v_r - u)dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_{B_r} \langle \nabla_{\mathbb{H}^n}(u - v_r), \nabla_{\mathbb{H}^n}(u + v_r)\rangle dx \\ &\qquad = \int_{B_r} |\nabla_{\mathbb{H}^n}(u - v_r)|^2 + 2\int_{B_r} \langle \nabla_{\mathbb{H}^n}(u - v_r), \nabla_{\mathbb{H}^n} v_r\rangle \\ &\qquad = \int_{B_r} |\nabla_{\mathbb{H}^n}(u - v_r)|^2 - 2\int_{B_r} f(u - v_r)dx \end{aligned}$$

and

$$\int_{B_r} \langle \nabla_{\mathbb{H}^n}(u - v_r), \nabla_{\mathbb{H}^n}(u + v_r)\rangle dx = \int_{B_r} \left(|\nabla_{\mathbb{H}^n} u|^2 - |\nabla_{\mathbb{H}^n} v_r|^2\right) dx.$$

Hence

$$\int_{B_r} |\nabla_{\mathbb{H}^n}(u - v_r)|^2 = \int_{B_r} \left(|\nabla_{\mathbb{H}^n} u|^2 - |\nabla_{\mathbb{H}^n} v_r|^2\right) dx + 2\int_{B_r} f(u - v_r)dx.$$

That is, by Hölder inequality

$$\int_{B_r} |\nabla_{\mathbb{H}^n}(u - v_r)|^2 \le C(\lambda_1, \lambda_2, Q)r^Q + 4\|f\|_{L^Q(B_r)}\left(\int_{B_r} |(u - v_r)|^2\right)^{1/2} r^{\frac{Q-2}{2}}$$

and, recalling Sobolev-Poincaré inequality one more time, we get

$$\int_{B_r} |\nabla_{\mathbb{H}^n}(u-v_r)|^2 \leq C(\lambda_1,\lambda_2,Q)r^Q + c'\|f\|_{L^Q(B_r)}\Big(\int_{B_r} |\nabla_{\mathbb{H}^n}(u-v_r)|^2\Big)^{1/2} r^{\frac{Q}{2}}.$$

Thus, applying Cauchy inequality we get for $\varepsilon > 0$

$$\int_{B_r} |\nabla_{\mathbb{H}^n}(u-v_r)|^2 \leq C(\lambda_1,\lambda_2,Q)r^Q + \frac{c'}{2\varepsilon}\|f\|^2_{L^Q(B_r)}r^Q + \frac{c'\varepsilon}{2}\int_{B_r} |\nabla_{\mathbb{H}^n}(u-v_r)|^2$$

that implies

$$\left(1-\frac{c'\varepsilon}{2}\right)\int_{B_r} |\nabla_{\mathbb{H}^n}(u-v_r)|^2 \leq C(\lambda_1,\lambda_2,Q,\bar{\varepsilon},\|f\|_{L^Q(B_r(0))})r^Q,$$

where

$$C(\lambda_1,\lambda_2,Q,\bar{\varepsilon},\|f\|_{L^Q(B_r)}) := C(\lambda_1,\lambda_2,Q) + \frac{c'}{2\varepsilon}\|f\|^2_{L^Q(B_r)}.$$

Thus, by fixing $\bar{\varepsilon} > 0$ such that $1 - \frac{\varepsilon}{2}c' > \frac{1}{2}$ we conclude that there exists a constant $\bar{C} := \bar{C}(\lambda_1,\lambda_2,\bar{\varepsilon},\|f\|_{L^Q(\Omega)},Q)$ such that

$$\int_{B_r} |\nabla_{\mathbb{H}^n}(u-v_r)|^2 \leq \bar{C}r^Q.$$

As a consequence, in analogy with the Euclidean case, we can not expect on u more than a modulus of continuity ruled by the Carnot-Charathéodory distance like, see the argument used by [3, 61, 57]:

$$|u(x)-u(y)| \leq C d_{CC}(x,y)\left|\log\left(\frac{1}{d_{CC}(x,y)}\right)\right|,$$

for every $x, y \in K$, $d_{CC}(x,y) < \frac{1}{2}$.

The existence of a global Lipschitz intrinsic modulus of continuity may be faced having a monotonicity formula. In $\mathbb{H}^1$, see some partial results obtained in [39, 38].

4.4 Domain variation solutions for a non-negative matrix

In this section we face the general case with variable coefficients.

Let us consider the functional

$$\mathcal{E}_A(v) = \int_\Omega \left(\langle A(x)\nabla v, \nabla v\rangle + M^2(v,x) + 2fv\right),$$

where $\langle A(x)\xi, \xi\rangle \geq 0$ for every $x \in \Omega$, for every $\xi \in \mathbb{R}^n$, and

$$M(u,x) = q(x)(\lambda_+\chi_{\{u>0\}} + \lambda_-\chi_{\{u<0\}}),$$

where λ_+, λ_- are non-negative numbers and $q \not\equiv 0$ is a function.

We define $\tau_\varepsilon(x) = x + \varepsilon\varphi(x)$ for some $\varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$. Recalling Section 1.4 we remark that A might be one of the matrices that are associated with a sublaplacian.

Lemma 4.4.1. *Let $u \in K$ be a local minimum of $\mathcal{E}_A$. Then u satisfies $div(A(x)\nabla u(x)) = f$ in $\Omega \setminus \{u = 0\}$.*

Proof. For every $\varphi \in C_0^\infty(\Omega \setminus \{u = 0\})$ and for every $\varepsilon > 0$ sufficiently small, it results

$$\begin{aligned}
\mathcal{E}_A(u + \varepsilon\varphi) = &\int_\Omega \langle A(x)\nabla u, \nabla u\rangle dx + 2\varepsilon \int_\Omega \langle A(x)\nabla u, \nabla\varphi\rangle dx \\
&+ \varepsilon^2 \int_\Omega \langle A(x)\nabla\varphi, \nabla\varphi\rangle dx + \int_\Omega M^2(u+\varepsilon\varphi, x)dx + 2\int_\Omega f(u+\varepsilon\varphi)dx \\
&= \mathcal{E}_A(u) + 2\varepsilon \int_\Omega \langle A(x)\nabla u, \nabla\varphi\rangle dx + 2\varepsilon \int_\Omega f\varphi dx + o(\varepsilon^2).
\end{aligned}$$

As a consequence,

$$\frac{\mathcal{E}_A(u+\varepsilon\varphi) - \mathcal{E}_A(u)}{\varepsilon} = 2\left(\int_\Omega \langle A(x)\nabla u, \nabla\varphi\rangle dx + \int_\Omega f\varphi\right) dx + o(\varepsilon)$$

and

$$\lim_{\varepsilon\to 0+} \frac{\mathcal{E}_A(u+\varepsilon\varphi) - \mathcal{E}_A(u)}{\varepsilon} = 2\left(\int_\Omega \langle A(x)\nabla u, \nabla\varphi\rangle dx + \int_\Omega f\varphi dx\right) = 0, \tag{4.4.1}$$

that is $\mathrm{div}(A(x)\nabla u(x)) = f$ in $\Omega \setminus \{u = 0\}$ in the weak sense. □

Theorem 4.4.2. *Let u be a local minimum of $\mathcal{E}_A$ and $meas_n(\{u = 0\}) = 0$. Then u is a domain variation solution and for every $\varphi \in C_0(\Omega, \mathbb{R}^n)$*

$$\lim_{\varepsilon\to 0} \int\limits_{\partial\{-\varepsilon<u\}} \langle\varphi,\nu\rangle(M^2 - \langle A(x)\nabla u^+, \nabla u^+\rangle)dS + \lim_{\delta\to 0} \int\limits_{\partial\{u<\delta\}} \langle\varphi,\nu\rangle(M^2 - \langle A(x)\nabla u^-, \nabla u^-\rangle)dS = 0.$$

Proof. Denoting by u_ε the function such that $u_\varepsilon(\tau_\varepsilon(x)) = u(x)$ where $\tau_\varepsilon(x) = x + \varepsilon\varphi(x)$, $\varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$ and assuming that A is smooth, we get

$$\begin{aligned} J(u_\varepsilon) &= \int_\Omega \big(\langle A(y)\nabla u_\varepsilon(y), \nabla u_\varepsilon(y)\rangle + M^2(u_\varepsilon(y), y) + 2f(y)u_\varepsilon(y)\big)\, dy \\ &= \int_\Omega \big(\langle A(\tau_\varepsilon(x))\nabla u_\varepsilon(\tau_\varepsilon(x)), \nabla u_\varepsilon(\tau_\varepsilon(x))\rangle + M^2(u(\tau_\varepsilon(x)), \tau_\varepsilon(x)) \\ &+ 2f(\tau_\varepsilon(x))u_\varepsilon(\tau_\varepsilon(x))\big)\, |\mathrm{det} J\tau_\varepsilon| dx. \end{aligned}$$

On the other hand, since

$$J\tau_\varepsilon(x) = I + \varepsilon J\varphi,$$

then

$$\mathrm{det} J\tau_\varepsilon = 1 + \varepsilon \mathrm{Tr}(J\varphi) + o(\varepsilon),$$

for $\varepsilon \to 0$. Moreover,

$$\nabla u(x) = \nabla(u_\varepsilon(\tau_\varepsilon(x)) = \nabla u_\varepsilon(\tau_\varepsilon(x))J\tau_\varepsilon(x),$$

hence

$$J\tau_\varepsilon(x)^{-1}\nabla u(x) = \nabla u_\varepsilon(\tau_\varepsilon(x)).$$

Keeping in mind that

$$J\tau_\varepsilon(x)^{-1} = I - \varepsilon J\varphi + o(\varepsilon),$$

we conclude that

$$J\tau_\varepsilon(x)^{-1}\nabla u(x) = (I - \varepsilon J\varphi + o(\varepsilon))\nabla u(x) = \nabla u(x) - \varepsilon J\varphi\nabla u(x) + o(\varepsilon)$$

and since A is smooth we get

$$A(\tau_\varepsilon(x)) = A(x) + \varepsilon JA(x)\varphi + o(\varepsilon).$$

As a consequence,

$$\begin{aligned}
&\int_\Omega \Big(\langle (A(x) + \varepsilon JA(x)\varphi + o(\varepsilon)) J\tau_\varepsilon(x)^{-1}\nabla u(x), J\tau_\varepsilon(x)^{-1}\nabla u(x)\rangle \\
&+ M^2(u(x), \tau_\varepsilon(x)) + 2f(\tau_\varepsilon(x))u(x)\Big)|\det J\tau_\varepsilon| dx \\
&= \int_\Omega \Big(\langle (A(x) J\tau_\varepsilon(x)^{-1}\nabla u(x), J\tau_\varepsilon(x)^{-1}\nabla u(x)\rangle \\
&+ M^2(u(x), \tau_\varepsilon(x)) + 2f(\tau_\varepsilon(x))u(x)\Big)|\det J\tau_\varepsilon| dx \\
&+ \varepsilon \int_\Omega \left(\langle (JA(x)\varphi + o(\varepsilon)) J\tau_\varepsilon(x)^{-1}\nabla u(x), J\tau_\varepsilon(x)^{-1}\nabla u(x)\rangle\right) |\det J\tau_\varepsilon| dx \\
&= \int_\Omega \left(\langle (A(x)\nabla u(x), \nabla u(x)\rangle + M^2(u(x), \tau_\varepsilon(x)) + 2f(\tau_\varepsilon(x))u(x)\right) |\det J\tau_\varepsilon| dx \\
&- 2\varepsilon \int_\Omega \langle A(x)\nabla u(x), J\varphi\nabla u(x)\rangle |\det J\tau_\varepsilon| dx + \varepsilon \int_\Omega \langle JA(x)\varphi\nabla u, \nabla u\rangle |\det J\tau_\varepsilon| dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\left.\frac{dJ(u_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} &= \int_\Omega \langle (A(x)\nabla u(x), \nabla u(x)\rangle + M^2(u(x), x) + 2f(x)u(x)\rangle)\mathrm{Tr}(J\varphi) dx \\
&- 2\int_\Omega \langle A\nabla u, J\varphi\nabla u\rangle dx + \int_\Omega \langle JA\varphi\nabla u, \nabla u\rangle dx \\
&+ \int_\Omega \left(\langle \nabla_x M^2(u(x), x), \varphi\rangle + 2\langle \nabla f(x), \varphi\rangle u\right) dx \\
&= \int_\Omega \langle (A(x)\nabla u(x), \nabla u(x)\rangle + M^2(u(x), x)\rangle)\mathrm{Tr}(J\varphi) dx \\
&- 2\int_\Omega \langle A\nabla u, J\varphi\nabla u\rangle dx + \int_\Omega \langle JA\varphi\nabla u, \nabla u\rangle dx \\
&+ \int_\Omega \langle \nabla_x M^2(u(x), x), \varphi\rangle dx - 2\int_\Omega f(x)\langle \varphi, \nabla u\rangle dx \\
&= \int_\Omega \mathrm{div}\left(\left(\langle A(x)\nabla u(x), \nabla u(x)\rangle dx + M^2(u, x)\right)\varphi - 2\langle \varphi, \nabla u\rangle A\nabla u\right) dx.
\end{aligned}$$

Since u is a local minimum, then

$$-\left.\frac{dJ(u(x + \varepsilon\varphi(x)))}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{dJ(u_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0,$$

that is u is a domain variation solution. Hence, for every $\varphi \in C^1_0(\Omega, \mathbb{R}^n)$ we have:

$$\begin{aligned}
&\left.\frac{dJ(u_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} \\
&= \int_\Omega \langle (A(x)\nabla u(x), \nabla u(x)\rangle + M^2(u(x),x)\rangle)\mathrm{div}\varphi dx - 2\int_\Omega \langle A\nabla u, J\varphi\nabla u\rangle dx \\
&+ \int_\Omega \langle JA\varphi\nabla u, \nabla u\rangle dx + \int_\Omega \langle \nabla_x M^2(u(x),x), \varphi\rangle dx - 2\int_\Omega f(x)\langle \varphi, \nabla u\rangle dx \\
&= 0.
\end{aligned}$$

Now, let us consider now $\Omega = \{x \in \Omega : \ u < -\varepsilon\} \cup \{x \in \Omega : \ u > \delta\} \cup \{x \in \Omega : \ -\varepsilon \leq u \leq \delta\}$, where $\varepsilon, \delta > 0$. Then, integrating by parts and denoting $\Omega_{\varepsilon,\delta}(u) = \{x \in \Omega : \ -\varepsilon \leq u \leq \delta\}$ as well as

$$\begin{aligned}
R_{\varepsilon,\delta} := &\int_{\Omega_{\varepsilon,\delta}(u)} \langle (A(x)\nabla u(x), \nabla u(x)\rangle + M^2(u(x),x)\rangle)\mathrm{div}\varphi dx \\
&- 2\int_{\Omega_{\varepsilon,\delta}(u)} \langle A\nabla u, J\varphi\nabla u\rangle dx + \int_{\Omega_{\varepsilon,\delta}(u)} \langle JA\varphi\nabla u, \nabla u\rangle dx \\
&+ \int_{\Omega_{\varepsilon,\delta}(u)} \langle \nabla_x M^2(u(x),x), \varphi\rangle dx - 2\int_{\Omega_{\varepsilon,\delta}(u)} f(x)\langle \varphi, \nabla u\rangle dx,
\end{aligned}$$

we get

$$\begin{aligned}
0 = &-\int_{\Omega\cap\{u<-\varepsilon\}} \langle \nabla\langle (A(x)\nabla u(x), \nabla u(x)\rangle + M^2(u(x),x)\rangle), \varphi\rangle dx \\
&+ \int_{\partial\{u<-\varepsilon\}} \langle A(x)\nabla u(x), \nabla u(x)\rangle + M^2(u(x),x)\langle \varphi, \nu\rangle d\sigma \\
&- \int_{\Omega\cap\{u>\delta\}} \langle \nabla\langle (A(x)\nabla u(x), \nabla u(x)\rangle + M^2(u(x),x)\rangle), \varphi\rangle dx \\
&+ \int_{\partial\{u>\delta\}} \langle A(x)\nabla u(x), \nabla u(x)\rangle + M^2(u(x),x)\langle \varphi, \nu\rangle d\sigma \\
&- 2\int_{\Omega\cap(\{u>\delta\}\cup\{u<-\varepsilon\})} \langle A\nabla u, J\varphi\nabla u\rangle dx \\
&+ \int_{\Omega\cap(\{u>\delta\}\cup\{u<-\varepsilon\})} \langle JA\varphi\nabla u, \nabla u\rangle dx + \int_{\Omega\cap(\{u>\delta\}\cup\{u<-\varepsilon\})} \langle \nabla_x M^2(u(x),x), \varphi\rangle dx \\
&- 2\int_{\Omega\cap(\{u>\delta\}\cup\{u<-\varepsilon\})} f(x)\langle \varphi, \nabla u\rangle dx + R_{\varepsilon,\delta}.
\end{aligned}$$

Thus, by recalling that u satisfies $\mathrm{div}(A\nabla u) = f(x)$ in $\Omega \setminus \{u = 0\}$ we get, denoting $u^+ := \sup\{u, 0\}$ and $u^- := \sup\{-u, 0\}$,

$$
\begin{aligned}
0 = &\lim_{\varepsilon\to 0}\int_{\partial\{-\varepsilon<u\}} \langle\varphi,\nu\rangle(\langle A(x)\nabla u^+, \nabla u^+\rangle + M^2)dS \\
&+ \lim_{\delta\to 0}\int_{\partial\{u<\delta\}} \langle\varphi,\nu\rangle(\langle A(x)\nabla u^-, \nabla u^-\rangle + M^2)dS \\
&- 2\left(\lim_{\varepsilon\to 0}\int_{\partial\{-\varepsilon<u\}} \langle\varphi,\nu\rangle\langle A(x)\nabla u^+, \nabla u^+\rangle dS\right. \\
&\left.+ \lim_{\delta\to 0}\int_{\partial\{u<\delta\}} \langle\varphi,\nu\rangle\langle A(x)\nabla u^-, \nabla u^-\rangle dS\right),
\end{aligned}
\tag{4.4.2}
$$

because by hypothesis $\text{meas}_n(\{u = 0\}) = 0$ so that $\lim_{\varepsilon,\delta\to 0} R_{\varepsilon,\delta} = 0$.

Finally (4.4.2) leads to

$$
0 = \lim_{\varepsilon\to 0}\int_{\partial\{-\varepsilon<u\}} \langle\varphi,\nu\rangle(M^2 - \langle A(x)\nabla u^+, \nabla u^+\rangle)dS + \lim_{\delta\to 0}\int_{\partial\{u<\delta\}} \langle\varphi,\nu\rangle(M^2 - \langle A(x)\nabla u^-, \nabla u^-\rangle)dS.
$$

□

In conclusion, we have obtained, whenever $\text{meas}_n\{u = 0\} = 0$, that

$$
\begin{cases}
\text{div}(A(x)\nabla u) = f & \text{in } \Omega^+(u) := \{x \in \Omega : \ u(x) > 0\}, \\
\text{div}(A(x)\nabla u) = f & \text{in } \Omega^-(u) := \text{Int}(\{x \in \Omega : \ u(x) \le 0\}), \\
\langle A\nabla u^+, \nabla u^+\rangle - \langle A\nabla u^- \nabla u^-\rangle = q^2(x)\Lambda & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega.
\end{cases}
\tag{4.4.3}
$$

where $\Lambda := \lambda_+^2 - \lambda_-^2$. In the case of the Heisenberg group this reads as follows (see Section 4.5 for the details and further generalizations),

$$
\begin{cases}
\Delta_{\mathbb{H}^n} u = f & \text{in } \Omega^+(u) := \{x \in \Omega : \ u(x) > 0\}, \\
\Delta_{\mathbb{H}^n} u = f & \text{in } \Omega^-(u) := \text{Int}(\{x \in \Omega : \ u(x) \le 0\}), \\
|\nabla_{\mathbb{H}^n} u^+|^2 - |\nabla_{\mathbb{H}^n} u^-|^2 = q^2(x)\Lambda & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega.
\end{cases}
\tag{4.4.4}
$$

4.5 Some comments about the Heisenberg group and Carnot groups

We compute $\langle A(x)\nabla u, \nabla u\rangle$ assuming that

$$A = \begin{bmatrix} 1, & 0, & 2y \\ 0, & 1, & -2x \\ 2y, & -2x, & 4(x^2+y^2) \end{bmatrix}.$$

Then

$$\begin{aligned}
\langle A\nabla u, \nabla u\rangle &= \begin{bmatrix} Xu \\ Yu \\ 2y\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial y} + 4\frac{\partial u}{\partial t}(x^2+y^2) \end{bmatrix} \cdot \nabla u \\
&= Xu\frac{\partial u}{\partial x} + Yu\frac{\partial u}{\partial y} + \left(2y\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial t} + 4(\frac{\partial u}{\partial t})^2(x^2+y^2) \\
&= (Xu)^2 - 2yXu\frac{\partial u}{\partial t} + (Yu)^2 + 2xYu\frac{\partial u}{\partial t} \\
&+ \left(2y\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial t} + 4\left(\frac{\partial u}{\partial t}\right)^2(x^2+y^2) = (Xu)^2 + (Yu)^2 \\
&= |\nabla_{\mathbb{H}^1} u|^2 = \langle \nabla_{\mathbb{H}^1} u, \nabla_{\mathbb{H}^1} u\rangle_{\mathbb{H}^1}.
\end{aligned}$$

Notice that

$$\operatorname{div}(A(x)\nabla u(x)) = X^2u + Y^2u = \Delta_{\mathbb{H}^1} u = \operatorname{div}_{\mathbb{H}^1}(\nabla_{\mathbb{H}^1} u) = X(Xu) + Y(Yu).$$

It is possible to give another example for the Engel group. In this case we have:

$$g_1 \bigoplus g_2 \bigoplus g_3,$$

where

$$g_1 = \operatorname{span}\{X_1, X_2\}, \quad g_2 = \operatorname{span}\{X_3\}, \quad g_3 = \operatorname{span}\{X_4\},$$

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4,$$

$$X_1 = \frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_3}, \quad X_4 = \frac{\partial}{\partial x_4},$$

$$xy = (x_1 + y_1, x_2 + y_2, x_3 + y_3 - y_1x_2, x_4 + y_4 + \frac{1}{2}y_1^2x_2 - y_1x_3).$$

Moreover,

$$\begin{bmatrix} 1, & 0 \\ 0, & 1 \\ -x_2, & 0 \\ -x_3, & 0 \end{bmatrix} \begin{bmatrix} 1, & 0, & -x_2, & -x_3 \\ 0, & 1, & 0, & 0 \end{bmatrix} = \begin{bmatrix} 1, & 0, & -x_2, & -x_3 \\ 0, & 1, & 0, & 0 \\ -x_2, & 0, & x_2^2, & x_2x_3 \\ -x_3, & 0, & x_2x_3, & x_3^2 \end{bmatrix}.$$

In this case,

$$\Delta_E = X_1^2 + X_2^2.$$

We can generalize this remark. Indeed, see Section 1.5-(A3) in [9], it is well known that every sublaplacian $\Delta_{\mathbb{G}} = \sum_{i=1}^{n_1} Z_i^2$ on a group $\mathbb{G}$ can be written in divergence form as

$$\Delta_{\mathbb{G}} = \mathrm{div}(A(x)\nabla),$$

where

$$A = \sigma(x)\sigma^T(x) \tag{4.5.1}$$

and σ is the $n \times n_1$ matrix whose columns are given by the coefficients of the vector fields $Z_1, \ldots, Z_{n_1}$.

We conclude that the two-phase problems for Carnot sublaplacians have to satisfy, whenever $\mathrm{meas}_{\mathbb{G}}(\{u = 0\}) = 0$, the following condition on the free boundary

$$0 = \lim_{\varepsilon\to 0} \int\limits_{\{-\varepsilon<u\}} \langle\varphi, \nu\rangle(M^2 - |\nabla_{\mathbb{G}}u^+|^2)dS + \lim_{\delta\to 0} \int\limits_{\{u<\delta\}} \langle\varphi, \nu\rangle(M^2 - |\nabla_{\mathbb{G}}u^-|^2)dS,$$

where $|\nabla_{\mathbb{G}} u|^2 = \sum_{i=1}^{n_1} (Z_i u)^2$. Then

$$\begin{cases} \Delta_{\mathbb{G}} u = f, & \text{in } \Omega^+(u) := \{x \in \Omega :\ u(x) > 0\}, \\ \Delta_{\mathbb{G}} u = f, & \text{in } \Omega^-(u) := \mathrm{Int}(\{x \in \Omega :\ u(x) \leq 0\}), \\ |\nabla_{\mathbb{G}} u^+|^2 - |\nabla_{\mathbb{G}} u^-|^2 = q^2(x)(\lambda_+^2 - \lambda_-^2) := q(x)\Lambda, & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{4.5.2}$$

where, whatever the function u is sufficiently smooth, it results:

$$|\nabla_{\mathbb{G}} u|^2 = \langle A(x)\nabla u, \nabla u\rangle = \langle \sigma^T \nabla u, \sigma^T \nabla u\rangle_{\mathbb{R}^{n_1}}$$

and

$$\nabla_{\mathbb{G}} u(x) := \sigma^T(x)\nabla u(x) = \sum_{k=1}^{n_1} Z_k u(x) Z_k(x).$$

In the case of $\mathbb{H}^1$, the functions like $\alpha(ax+by)^+ - \beta(ax+by)^-$, where $a^2 + b^2 > 0$, $a, b \in \mathbb{R}$ are fixed, as well as $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$, satisfy the two-phase homogeneous problem

$$\begin{cases} \Delta_{\mathbb{H}^1} u = 0, & \text{in } \Omega^+(u) := \{x \in \Omega :\ u(x) > 0\}, \\ \Delta_{\mathbb{H}^1} u = 0, & \text{in } \Omega^-(u) := \mathrm{Int}(\{x \in \Omega :\ u(x) \leq 0\}), \\ |\nabla_{\mathbb{H}^1} u^+|^2 - |\nabla_{\mathbb{H}^1} u^-|^2 = (a^2+b^2)(\alpha^2 - \beta^2), & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{4.5.3}$$

In this case the free boundary $\mathcal{F}(u)$ is the set $\{(x,y,t) \in \mathbb{H}^1 :\ ax + by = 0\}$ that does not have characteristic points.

4.6 Nonlinear case: $p(x)-$Laplace operator

Now our attention is attracted by the functional

$$J(u) = \int_\Omega \big(a(|\nabla u|, x) + M^2(u,x) + p(x)f(x)u(x)\big)\, dx,$$

where

$$M(u,x) = q(x)(\lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u<0\}})$$

and a is a function that we shall introduce in a while.

We define $\tau_\varepsilon(x) = x + \varepsilon\varphi(x)$ where $\varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$. Then, denoting by u_ε

the function such that $u_\varepsilon(\tau_\varepsilon(x)) = u(x)$, we get

$$\begin{aligned} J(u_\varepsilon) &= \int_\Omega \big(a(|\nabla u_\varepsilon(y)|, y) + M^2(u_\varepsilon(y), y) + p(y)f(y)u_\varepsilon(y)\big)\, dy \\ &= \int_\Omega \big(a(|\nabla u_\varepsilon(\tau_\varepsilon(x))|, \tau_\varepsilon(x)) + M^2(u(\tau_\varepsilon(x)), \tau_\varepsilon(x)) \\ &\qquad\qquad + p(\tau_\varepsilon(x))f(\tau_\varepsilon(x))u_\varepsilon(\tau_\varepsilon(x)))\, |\mathrm{det} J\tau_\varepsilon| dx. \end{aligned}$$

On the other hand, exploiting the notation of the case, described in Section 4.4, we obtain

$$\begin{aligned} J(u_\varepsilon) &= \int_\Omega \big(a(|J\tau_\varepsilon(x)^{-1}\nabla u(x)|, \tau_\varepsilon(x)) + M^2(u(x), \tau_\varepsilon(x)) \\ &\quad + p(\tau_\varepsilon(x))f(\tau_\varepsilon(x))u(x))\, |\mathrm{det} J\tau_\varepsilon| dx \\ &= \int_\Omega \Big(a(|\nabla u(x) - \varepsilon J\varphi \nabla u(x) + o(\varepsilon)|, \tau_\varepsilon(x)) + M^2(u(x), \tau_\varepsilon(x)) \\ &\quad + p(\tau_\varepsilon(x))f(\tau_\varepsilon(x))u(x)\Big)|\mathrm{det} J\tau_\varepsilon| dx. \end{aligned}$$

In the case when $a(b, c) = b^{p(c)}$, denoting

$$\mathcal{E}_{p(x)}(u) := \int_\Omega \big(|\nabla u|^{p(x)} + M^2(u, x) + p(x)f(x)u(x)\big)\, dx,$$

we get, from the Taylor expansion,

$$\begin{aligned} &a(|\nabla u(x) - \varepsilon J\varphi\nabla u(x) + o(\varepsilon)|, \tau_\varepsilon(x)) = |\nabla u(x) - \varepsilon J\varphi\nabla u(x) + o(\varepsilon)|^{p(\tau_\varepsilon(x))} \\ &= |\nabla u(x) - \varepsilon J\varphi\nabla u(x) + o(\varepsilon)|^{p(x)+\varepsilon\langle\nabla p(x),\varphi(x)\rangle+o(\varepsilon)} \\ &= |\nabla u(x) - \varepsilon J\varphi\nabla u(x) + o(\varepsilon)|^{p(x)}|\nabla u(x) - \varepsilon J\varphi\nabla u(x) + o(\varepsilon)|^{\varepsilon\langle\nabla p(x),\varphi(x)\rangle+o(\varepsilon)} \end{aligned}$$

which leads

$$\begin{aligned} &a(|\nabla u(x) - \varepsilon J\varphi\nabla u(x) + o(\varepsilon)|, \tau_\varepsilon(x)) \\ &=(|\nabla u(x)|^2 - 2\varepsilon\langle J\varphi\nabla u(x), \nabla u(x) + o(1)\rangle + o(\varepsilon))^{\frac{p(x)}{2}} \\ &\quad \times |\nabla u(x) - \varepsilon J\varphi\nabla u(x) + o(\varepsilon)|^{\varepsilon\langle\nabla p(x),\varphi(x)\rangle+o(\varepsilon)} \\ &= \big(|\nabla u(x)|^{p(x)} - \varepsilon p(x)\langle J\varphi\nabla u(x), \nabla u(x)\rangle|\nabla u(x)|^{p(x)-2} + o(\varepsilon)\big) \\ &\quad \times \exp\{\varepsilon(\langle\nabla p(x), \varphi(x)\rangle + o(1)) \log(|\nabla u(x) - \varepsilon J\varphi\nabla u(x) + o(\varepsilon)|)\} \end{aligned}$$

$$
\begin{aligned}
&= \left(|\nabla u(x)|^{p(x)} - \varepsilon p(x)\langle J\varphi \nabla u(x), \nabla u(x)\rangle |\nabla u(x)|^{p(x)-2} + o(\varepsilon)\right) \\
&\quad \times \exp\Big(\varepsilon(\langle \nabla p(x), \varphi(x)\rangle + o(1))\left(\log(|\nabla u(x)|)\right. \\
&\quad \left. + \log\left(1 - \varepsilon\langle J\varphi \nabla u(x), \nabla u(x)\rangle + o(\varepsilon)\right)\right)\Big),
\end{aligned}
$$

that is

$$
\begin{aligned}
&a(|\nabla u(x) - \varepsilon J\varphi \nabla u(x) + o(\varepsilon)|, \tau_\varepsilon(x)) \\
&= \left(|\nabla u(x)|^{p(x)} - \varepsilon p(x)\langle J\varphi \nabla u(x), \nabla u(x)\rangle |\nabla u(x)|^{p(x)-2} + o(\varepsilon)\right) \\
&\quad \times (1 + \varepsilon\langle \nabla p(x), \nabla\varphi(x)\rangle \log|\nabla u(x)| + o(\varepsilon))) \\
&= |\nabla u(x)|^{p(x)} + \varepsilon\left(|\nabla u(x)|^{p(x)}\langle \nabla p(x), \varphi(x)\rangle \log|\nabla u(x)|\right. \\
&\quad \left. - p(x)\langle J\varphi \nabla u(x), \nabla u(x)\rangle |\nabla u(x)|^{p(x)-2}\right) + o(\varepsilon).
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
&\mathcal{E}_{p(x)}(u_\varepsilon) = \int_\Omega \Big(|\nabla u(x)|^{p(x)} + \varepsilon\Big(|\nabla u(x)|^{p(x)}\langle \nabla p(x), \nabla\varphi(x)\rangle \log|\nabla u(x)| \\
&- p(x)\langle J\varphi \nabla u(x), \nabla u(x)\rangle |\nabla u(x)|^{p(x)-2}\Big) + o(\varepsilon)) \\
&+ M^2(u(x), \tau_\varepsilon(x)) + (p(x) + \varepsilon\langle \nabla p(x), \varphi\rangle + o(\varepsilon)) f(\tau_\varepsilon(x)) u(x)\Big) |\mathrm{det} J\tau_\varepsilon| dx,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\mathcal{E}_{p(x)}(u_\varepsilon) \\
&= \int_\Omega \Big(|\nabla u(x)|^{p(x)} + M^2(u(x), x) + p(x) f(x) u(x)\Big)\Big(1 + \varepsilon \mathrm{Tr}(J\varphi) + o(\varepsilon)\Big) dx \\
&+ \varepsilon \int_\Omega \Big(|\nabla u(x)|^{p(x)}\langle \nabla p(x), \varphi(x)\rangle \log|\nabla u(x)| \\
&- p(x)\langle J\varphi \nabla u(x), \nabla u(x)\rangle |\nabla u(x)|^{p(x)-2} + u(x)\langle \nabla p(x), \varphi\rangle f(x)\Big) \\
&\qquad \Big(1 + \varepsilon \mathrm{Tr}(J\varphi) + o(\varepsilon)\Big) dx \\
&+ \varepsilon \int_\Omega \left(p(x)\langle \nabla f(x), \varphi\rangle u(x) + \langle \nabla M^2(u(x), x), \varphi\rangle\right) \\
&\qquad \Big(1 + \varepsilon \mathrm{Tr}(J\varphi) + o(\varepsilon)\Big) dx + o(\varepsilon).
\end{aligned}
$$

Thus it follows that

$$\mathcal{E}_{p(x)}(u_\varepsilon) = \mathcal{E}_{p(x)}(u) + \varepsilon\Big\{ \int_\Omega \Big(|\nabla u(x)|^{p(x)} + M^2(u(x),x) + p(x)f(x)u(x)\Big)\mathrm{Tr}(J\varphi)dx$$
$$+ \int_\Omega \Big(|\nabla u(x)|^{p(x)}\langle \nabla p(x), \varphi(x)\rangle \log|\nabla u(x)| - p(x)\langle J\varphi \nabla u(x), \nabla u(x)\rangle |\nabla u(x)|^{p(x)-2}$$
$$+ \langle \nabla p(x), \varphi\rangle u(x)f(x) + p(x)u(x)\langle \nabla f(x), \varphi\rangle + \langle \nabla M^2(u(x),x),\varphi\rangle\Big)dx\Big\} + o(\varepsilon).$$

Thus, recalling that u is a minimum, we can conclude that

$$\lim_{\varepsilon\to 0} \frac{\mathcal{E}_{p(x)}(u_\varepsilon) - \mathcal{E}_{p(x)}(u)}{\varepsilon} = 0.$$

Thus we deduce, recalling $\mathrm{Tr}(J\varphi) = \mathrm{div}(\varphi)$, that

$$0 = \int_\Omega \Big(|\nabla u(x)|^{p(x)} + M^2(u(x),x) + p(x)f(x)u(x)\Big)\mathrm{div}(\varphi)dx$$
$$+ \int_\Omega \Big(|\nabla u(x)|^{p(x)}\langle \nabla p(x), \varphi(x)\rangle \log|\nabla u(x)| - p(x)\langle J\varphi \nabla u(x), \nabla u(x)\rangle |\nabla u(x)|^{p(x)-2}$$
$$+ \langle \nabla p(x), \varphi\rangle u(x)f(x) + p(x)u(x)\langle \nabla f(x), \varphi\rangle + \langle \nabla M^2(u(x),x),\varphi\rangle\Big)dx, \tag{4.6.1}$$

that is also

$$0 = \int_\Omega \Big(|\nabla u(x)|^{p(x)} + M^2(u(x),x)\Big)\mathrm{div}(\varphi)dx$$
$$+ \int_\Omega \Big(|\nabla u(x)|^{p(x)}\langle \nabla p(x), \varphi(x)\rangle \log|\ \nabla u(x)| - p(x)\langle J\varphi \nabla u(x), \nabla u(x)\rangle |\nabla u(x)|^{p(x)-2}$$
$$+ \langle \nabla M^2(u(x),x),\varphi\rangle - f(x)p(x)\langle \nabla u, \varphi\rangle\Big)dx. \tag{4.6.2}$$

Hence, integrating by parts, recalling that $\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f$ in $\Omega \setminus F(u)$, considering $\Omega = \{x \in \Omega :\ u < -\varepsilon\} \cup \{x \in \Omega :\ u > \delta\} \cup \{x \in \Omega :\ -\varepsilon \leq u \leq \delta\}$, where $\varepsilon, \delta > 0$, recalling that $\Omega_{\varepsilon,\delta}(u) = \{x \in \Omega :\ -\varepsilon \leq u \leq \delta\}$ and, denoting by

$$R_{\varepsilon,\delta} := \int_{\Omega_{\varepsilon,\delta}(u)} \Big(|\nabla u(x)|^{p(x)} + M^2(u(x),x)\Big)\mathrm{div}\varphi - \int_{\Omega_{\varepsilon,\delta}(u)} p(x)\langle \nabla J\varphi \nabla u(x), \nabla u(x)\rangle$$
$$+ \int_{\Omega_{\varepsilon,\delta}(u)} |\nabla u(x)|^{p(x)}\langle \nabla p(x), \varphi(x)\rangle \log|\nabla u(x)| + \int_{\Omega_{\varepsilon,\delta}(u)} \langle \nabla_x M^2(u(x),x),\varphi\rangle$$

$$- \int_{\Omega_{\varepsilon,\delta}(u)} p(x) f(x) \langle \varphi, \nabla u \rangle,$$

we get

$$0 = \lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \Big\{ \int_{\partial\{u<-\varepsilon\}} \langle n, \varphi \rangle \Big((1-p(x)) |\nabla u(x)|^{p(x)} + M^2(u(x),x) \Big) dS$$
$$+ \int_{\partial\{u>\delta\}} (1-p(x)) \langle n, \varphi \rangle \Big(|\nabla u(x)|^{p(x)} + M^2(u(x),x) \Big) dS + R_{\varepsilon,\delta} \Big\}.$$

This result that implies

$$0 = \lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \Big\{ \int_{\partial\{u<-\varepsilon\}} \langle n, \varphi \rangle \Big((1-p(x)) |\nabla u(x)|^{p(x)} + M^2(u(x),x) \Big) dS$$
$$+ \int_{\partial\{u>\delta\}} \langle n, \varphi \rangle \Big((1-p(x)) |\nabla u(x)|^{p(x)} + M^2(u(x),x) \Big) dS \Big\},$$

because we assumed that $\text{meas}_n\{u=0\} = 0$, so that $\lim_{\varepsilon\to 0, \delta \to 0} R_{\varepsilon,\delta} = 0$.

As a consequence, the natural pointwise condition on the free boundary $\{u = 0\}$ is

$$(p(x)-1)|\nabla u^+|^{p(x)} - (p(x)-1)|\nabla u^-|^{p(x)} = q^2(x)(\lambda_+^2 - \lambda_-^2).$$

Usually, previous condition is written as well as

$$(u_n^+)^{p(x)} - (u_n^-)^{p(x)} = q^2(x) \frac{\lambda_+^2 - \lambda_-^2}{p(x)-1},$$

where u_n^+ and u_n^- denote the normal derivatives, computed considering n pointing inside to $\Omega^+(u)$ and $\Omega^-(u)$ respectively, at the points of the set $\{u = 0\}$, of course whenever this fact makes sense. In fact for every $x \in \{u = 0\}$, and such that $\nabla u(x) \neq 0$, we have:

$$u_n(x) = \langle \nabla u(x), \frac{\nabla u(x)}{|\nabla u(x)|} \rangle = |\nabla u(x)|.$$

In conclusion, the two phase problem can be formulated in viscosity sense as:

$$\begin{cases} \Delta_{p(x)}u = f, & \text{in } \Omega^+(u) := \{x \in \Omega : \ u(x) > 0\}, \\ \Delta_{p(x)}u = f, & \text{in } \Omega^-(u) := \mathrm{Int}(\{x \in \Omega : \ u(x) \leq 0\}), \\ |\nabla u^+|^{p(x)} - |\nabla u^-|^{p(x)} = q^2(x)\frac{\Lambda}{p(x)-1}, & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{4.6.3}$$

being $\Lambda := \lambda_+^2 - \lambda_-^2$.

4.7 Conclusions

Starting from the condition on the free boundary that we have obtained, in Carnot groups for the two phase problems, we ask ourselves if a comparison result may work in this framework. Following the mentioned viscosity approach introduced in [27, 28, 29, 30], the first thing to prove seems to be the existence of a comparison result. From this point of view, it is natural to recall the properties arising from the Hopf maximum principle. About this subject in Carnot groups, we cite [8], for a detailed study, for a discussion in the Heisenberg group, and [59] for a generalization to the Carnot groups. In fact in [8], see Lemma 2.1, the authors remark that if a set Ω satisfies the inner intrinsic ball property, namely if $P_0 \in \partial\Omega$ is such that there exists a Koranyi ball $B_R^{\mathbb{H}^1}(Q) \subset \Omega$, such that $P_0 = \partial B_R^{\mathbb{H}^1}(Q) \cap \partial\Omega$, u satisfies $\Delta_{\mathbb{H}^1}u(P) \geq 0$ and $u(P) > u(P_0)$ for every $P \in B_R^{\mathbb{H}^1}(P_0) \cap \Omega$, then

$$\lim_{t\to 0+} \frac{f(P_0) - f(P_0 - th)}{t} < 0,$$

where h denotes any exterior direction to $\partial\Omega$ at P_0; moreover, in case if $\frac{\partial f(P_0)}{\partial h}$ exists, then $\frac{\partial f(P_0)}{\partial h} < 0$. In this order of ideas the right definition of a viscosity solution for (4.1.4) may be the following one.

Unfortunately, if the contact point between the set and the ball is realized in a characteristic point, then $\frac{\partial f}{\partial h} = 0$ at the characteristic points along all the horizontal admissible directions $h \in H\mathbb{H}^n$, that is $\nabla_{\mathbb{H}^n} f = 0$ at the characteristic points.

We denote by ν the intrinsic normal to $\mathcal{F}(v)$ at $x_0 \in \mathcal{F}(v)$ and, as usual, $v_\nu^+(x_0)$ and $v_\nu^-(x_0)$ represent the horizontal derivatives with respect to the inner intrinsic normal ν to $\Omega^+(v)$ and to $\Omega^-(v)$ respectively.

We are in position to state the definition of solution of a two-phase free boundary problem in a simpler case like (4.1.4) as follows:

Definition 4.7.1. *We say that* $u \in C(\Omega)$ *is a solution to* (4.1.4) *if:*

(i) $\Delta_{\mathbb{G}} u = f$ *in a viscosity sense in* $\Omega^+(u)$ *and* $\Omega^-(u)$*;*

(ii) let $x_0 \in \mathcal{F}(u)$*. For every function* $v \in C(B_\varepsilon(x_0))$*,* $\varepsilon > 0$ *such that* $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$*, being* $B := B_\varepsilon(x_0)$ *and* $\mathcal{F}(v) \in C^2$*, if* v *touches* u *from below (resp. above) at* $x_0 \in \mathcal{F}(v)$*, and* x_0 *is not characteristic for* $\mathcal{F}(v)$*, then*

$$(v_\nu^+(x_0))^2 - (v_\nu^-(x_0))^2 \leq 1 \quad (\textit{resp.} \quad (v_\nu^+(x_0))^2 - (v_\nu^-(x_0))^2 \geq 1).$$

Moreover, the following notion of strict comparison subsolution (supersolution) plays a fundamental role, at least in the Euclidean setting, see [27, 28]. Here below we state it in the framework of Carnot groups.

Definition 4.7.2. *We say that a function* $v \in C(\Omega)$ *is a strict comparison subsolution (supersolution) to* (4.1.4) *if:* $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ *and*

(i) $\Delta_{\mathbb{G}} v > f$ *(resp.* $\Delta_{\mathbb{G}} v < f$*) in a viscosity sense in* $\Omega^+(v) \cup \Omega^-(v)$*;*

(ii) for every $x_0 \in \mathcal{F}(v)$*, if* x_0 *is not characteristic for* $\mathcal{F}(v)$*, then*

$$(v_\nu^+(x_0))^2 - (v_\nu^-(x_0))^2 > 1 \quad (\textit{resp.} \quad (v_\nu^+(x_0))^2 - (v_\nu^-(x_0))^2 < 1.$$

As a consequence, we obtain the following result.

Theorem 4.7.3. *No strict viscosity subsolution* v *of* (4.1.4) *can touch a solution* u *from below at no point in* $\mathcal{F}(u) \cap \mathcal{F}(v)$ *that is noncharacteristic for* $\mathcal{F}(v)$*. Analogously, no strict comparison supersolution* v *of* (4.1.4) *can touch a viscosity solution* u *from above at points belonging to* $\mathcal{F}(u) \cap \mathcal{F}(v)$ *that are noncharacteristic for* $\mathcal{F}(v)$*.*

Proof. It follows by the definitions of solution and strict sub/supersolution in $\mathbb{G}$. □

Corollary 4.7.4. *Let* v *and* u *be respectively a strict subsolution and a solution of* (4.1.4) *in* $\mathbb{G}$*. If* $v \leq u$ *in* Ω *and* $\mathcal{F}(v)$ *is a noncharacteristic set then* $v < u$ *in* Ω*.*

Let w *and* u *be respectively a strict supersolution and a solution of*(4.1.4) *in* $\mathbb{G}$*. If* $w \geq u$ *in* Ω *and* $\mathcal{F}(w)$ *is a noncharacteristic set, then* $w > u$ *in* Ω*.*

Proof. Suppose that strict subsolution of (4.1.2) such that $v \leq u$. Then such point x_0 can not be inside $\Omega^+(u) \cup \Omega^-(u)$ because, on the contrary, from

$$\Delta_{\mathbb{G}} v - \Delta_{\mathbb{G}} u \geq f(x) - f(x) = 0$$

in $\Omega^+(u) \cup \Omega^-(u)$ and $v - u$ realizing a maximum at x_0 we would introduce a contradiction with the maximum principle. Then this contact point $x_0 \in \mathcal{F}(u) \cap \mathcal{F}(v)$, and, by the definition of strict subsolution, this fact can not happen. □

As a consequence there might exist solutions u, v of (4.1.4) such that $v \leq u$, $u \not\equiv v$ but u, v might touch in a characteristic point $x_0 \in \mathcal{F}(u) \cap \mathcal{F}(v)$. In fact it is well known that a Hopf maximum principle in the Heisenberg group formulated simply substituting to the normal derivative at a boundary point the intrinsic (horizontal) normal derivative fails, since they may exist characteristic points on a C^1 surface. For instance, sets with genus 0 (without holes) having smooth boundary have always characteristic points belonging to the boundary. As a consequence, there can not exist solutions of (4.1.2) satisfying flux condition pointwise on the free boundary, when $\mathcal{F}(u)$ is the boundary of a set of genus 0.

Here we give some examples of solutions in $\mathbb{H}^1$. Let u be a solution of a two phase problem (4.1.2) in a set $A \subset \mathbb{R}^2$ satisfying the same condition $|\nabla u^+|^2 - |\nabla u^-|^2 = 1$ (in the Euclidean setting) on $\mathcal{F}(u) := A \cap \partial A(u)$. Then $\tilde{u}(x, u, t) = u(x, y)$ is a solution of (4.1.4) in the cylinder $\Omega = A \times (a, b)$, when $\mathbb{G} = \mathbb{H}^1$.

In the case of the $p(x)-$Laplace operator characteristic points do not exist. So that the definition of solution of the simpler problem (4.1.5), in the viscosity sense, is the following one, keeping in mind that we denote by n the normal to $\mathcal{F}(v)$ at $x_0 \in \mathcal{F}(v)$ and, by $v_n^+(x_0)$ and $v_n^-(x_0)$ we denote the normal derivatives with respect to the inner normal n to $\Omega^+(v)$ and to $\Omega^-(v)$ respectively.

Definition 4.7.5. *Let $u \in C(\Omega)$. We say that u is a solution to* (4.1.5) *if:*

(i) $\Delta_{p(x)} u = f$ *in a viscosity sense in* $\Omega^+(u)$ *and* $\Omega^-(u)$;

(ii) for every $x_0 \in \mathcal{F}(u)$ *and for every function* $v \in C(B_\varepsilon(x_0))$, $\varepsilon > 0$ *such that* $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$, *being* $B := B_\varepsilon(x_0)$ *and* $\mathcal{F}(v) \in C^2$ *and* $\nabla v(x_0) \neq 0$,

if v touches u from below (resp. above) at $x_0 \in \mathcal{F}(v)$, then

$$(v_n^+(x_0))^2 - (v_n^-(x_0))^2 \leq 1 \quad (\textit{resp.} \quad (v_n^+(x_0))^2 - (v_n^-(x_0))^2 \geq 1).$$

In this case, even if we consider only non-degenerate points where $\nabla v \neq 0$ on $\mathcal{F}(u)$, the Hopf maximum principle holds in the classical sense, so that, we introduce the following strict comparison notion of subsolution/supersolution.

Definition 4.7.6. $v \in C(\Omega)$ *is a strict comparison subsolution (supersolution) to* (4.1.5) *if:* $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ *and*

(i) $\Delta_{p(x)} v > f$ *(resp.* $\Delta_{p(x)} v < f$*) in a viscosity sense in* $\Omega^+(v) \cup \Omega^-(v)$*;*

(ii) for every $x_0 \in \mathcal{F}(v)$*, if* $\nabla v(x_0) \neq 0$*, then*

$$(v_n^+(x_0))^2 - (v_n^-(x_0))^2 > 1 \quad (\textit{resp.} \quad (v_n^+(x_0))^2 - (v_n^-(x_0))^2 < 1.$$

As a consequence we obtain the following result.

Theorem 4.7.7. *No strict viscosity subsolution v of* (4.1.5) *can touch a solution u from below. Analogously, no strict comparison supersolution v of* (4.1.5) *can touch a viscosity solution u from above.*

Proof. The proof immediately follows applying the definitions (4.7.5), (4.7.6), because of inner maximum principle and via the Hopf maximum principle since, in the last case, the gradient on that contact boundary points can not be 0. □

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