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REGULARITY IN DEGENERATE ELLIPTIC AND PARABOLIC FREE BOUNDARY PROBLEMS

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Abstract

In this thesis, the main topic is the study of some free boundary problems, more precisely the investigation of regularity issues in degenerate elliptic and parabolic ones. Specifically, three different problems are treated. The first one is the one-phase Stefan problem, for which the regularity of flat free boundaries is dealt with by relying on perturbation arguments leading to a linearization of the problem. This approach is inspired by the elliptic counterpart. The second problem concerns the question of the existence of an Alt-Caffarelli-Friedman monotonicity formula in the Heisenberg group. Following the ideas exploited in the Euclidean setting, a necessary condition about the existence of such tool in that noncommutative setting is found. The last problem faced is related to almost minimizers of the p-Laplacian. In particular, the optimal Lipschitz continuity of almost minimizers, for p greater or equal than 2, is proved as well as the regularity of the free boundary is studied.

Contents

Introduction

In this thesis, the main topic is the study of some free boundary problems, more precisely the investigation of regularity issues in degenerate elliptic and parabolic ones.
A free boundary problem is a problem that involves some partial differential equations satisfied in some sense by functions which have to fulfill certain conditions on unknown domains determined by the functions themselves. Hence, these domains are a priori unknown and depend on the problem. The boundaries of such unknown domains, inside the set on which the problem is stated, determine the so-called free boundary of the solution. We will come back later on the precise notion of free boundary, which depends on the problem under consideration. In general, in this kind of problems, we are not only interested in the regularity of the solutions, but also in the study of the free boundary properties. Actually, one of the main mathematical challenges is exactly to understand the regularity of the free boundary.
Free boundary problems naturally arise in several fields, such as physics, industry, biology, finance and other areas. In general, in these applied problems, there is a qualitative change of a medium and thus an appearance of a phase transition, for instance ice to water, liquid to crystal, buying to selling (assets), active to inactive (biology), blue to red (coloring games), disorganized to organized (self-organizing criticality), see [1] for some of such examples. Let us consider a typical free boundary problem given by a block of melting ice. In this case, the free boundary is the moving interphase separating the ice and the water, the PDE controlling the process is given by the

heat equation and its solution is the temperature distribution. Specifically, this problem is in the class of Stefan problems, see Chapter 1.

An important feature of such kind of problems is the distinction between one-phase and two-phase free boundary problems. This characterization depends on whether the solutions of the problem we are looking for are non-negative or sign-changing. We call positive phase the set where the solution is positive and in case of a two-phase problem, we call negative phase the set where the solution is negative as well. An example of a one-phase problem is the classical one-phase elliptic Bernoulli free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega :\ u(x) > 0\}, \\ |\nabla u| = 1 & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$. The two-phase form is instead

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega :\ u(x) > 0\}, \\ \Delta u = 0 & \text{in } \Omega^-(u) := \mathrm{Int}(\{x \in \Omega :\ u(x) \leq 0\}), \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 1 & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{0.2}$$

with $u^+ := \sup\{u, 0\}$ and $u^- := \sup\{-u, 0\}$. In (0.1) and (0.2), $F(u)$ denotes the free boundary of u, while $\Omega^+(u)$ and $\Omega^-(u)$ its positive and negative phase respectively.

Some free boundary problems can be written as variational inequalities, for which the solution is obtained by minimizing a constrained energy functional, see [50], [15] and [71] for an introduction. Specific examples can be found in [2], and also in Chapter 3 for a free boundary problem involving almost minimizers.

Among the free boundary problems in a variational formulation, a canonical example is the classical obstacle problem, whose formulation is the following:

$$\text{minimize} \quad \frac{1}{2}\int_\Omega |\nabla v|^2\, dx \quad \text{among all functions } v \geq \varphi,$$

given a smooth function φ (the "obstacle"), under the further boundary conditions $v|_{\partial\Omega} = g$, where g is a datum. Here, Ω is a bounded domain in

$\mathbb{R}^n$. In $\mathbb{R}^2$, we can think of the solution v as a "membrane" which is elastic and constrained to be above φ. The Euler-Lagrange equation of the minimization problem defining the obstacle problem is

$$\begin{cases} v \geq \varphi & \text{in } \Omega, \\ \Delta v \leq 0 & \text{in } \Omega, \\ \Delta v = 0 & \text{in the set } \{v > \varphi\}, \end{cases} \tag{0.3}$$

together with the boundary conditions $v|_{\partial\Omega} = g$. Alternatively, we may take $u := v - \varphi$ and the problem (0.3) is equivalent to

$$\begin{cases} u \geq 0 & \text{in } \Omega, \\ \Delta u \leq f & \text{in } \Omega, \\ \Delta u = f & \text{in the set } \{u > 0\}, \end{cases} \tag{0.4}$$

where $f := -\Delta\varphi$ (classically defined since φ is smooth). In particular, such solution u can be found minimizing

$$\int_\Omega \left(\frac{1}{2}|\nabla u|^2 + fu\right) dx \quad \text{among all functions } u \geq 0.$$

In other words, we can make the obstacle just zero, by adding a right-hand side f. Coherently, the minimization is now subject to the boundary conditions $u|_{\partial\Omega} = \tilde{g}$, with $\tilde{g} := g - \varphi$.
At this point, the natural question could be: why is the obstacle problem a free boundary problem? Looking at (0.4), we point out that the condition $\Delta u = f$ is required to be satisfied in the set $\{u > 0\}$, whose boundary is unknown, since it depends on the solution u itself. This is exactly the peculiarity of free boundary problems, as stressed in the initial definition. The set $\Gamma_u := \partial\{u > 0\} \cap \Omega$ is the free boundary of u. Specifically, the obstacle problem is one of the most motivating examples in the study of free boundary problems.
Variational inequalities can also be used to define a weak notion of solution to free boundary problems. As often happens in PDEs, even in free boundary problems, a classical notion of solution is not sufficient for a complete

investigation, for instance in facing the question of the existence. So, one or more weak notions are needed. Concerning free boundary problems, the weak notion of viscosity solution has turned out to be very suitable, thanks to its flexibility. For this purpose, we want to mention the classical contribution of Luis A. Caffarelli, who introduced an original geometric approach to the study of the free boundary regularity, exploiting this notion of solution. Specifically, the use of viscosity solutions was one of the most innovative elements of his work.

The previously mentioned Caffarelli's approach turned out to be fundamental to develop the study of free boundary regularity as a research trend in free boundary problems, and more generally in the field of partial differential equations. Precisely, he proved in his pioneer work [11] that Lipschitz free boundaries for problems like (2.2) are $C^{1,\alpha}$, while in [13] he showed that "flat" free boundaries are Lipschitz. The key step of the method in [11, 13] consists in finding a family of comparison subsolutions using supconvolutions on balls of variable radii. To provide a complete bibliography on Caffarelli's contribution, we mention [12] and [15] as well. As pointed out before, his work was a key breakthrough for the comprehension of the free boundary regularity. Indeed, in the homogeneous case as for [11, 13], regularity results in the spirit of [11, 13] were subsequently proved for more general operators. In particular, in [80, 81] Wang considered concave fully nonlinear uniformly elliptic operators of the form $F(D^2u)$. Moreover, [11] was extended by Feldman in [35, 36] to a class on nonconcave fully nonlinear uniformly elliptic operators of the type $F(D^2u, Du)$ and to certain nonisotropic problems. For operators with variable coefficients, we recall the work of Cerutti, Ferrari, Salsa, see [18], and Ferrari, Salsa, see [40, 41]. In addition, Ferrari and then Argiolas, Ferrari in [37, 3] dealt with a class of fully nonlinear operators of the form $F(D^2u, x)$ with Hölder dependence on x. Concerning higher regularity of the free boundary, instead, it follows from the classical work of Kinderlehrer and Nirenberg [59].

More recently, the viscosity theory has been also used by D. De Silva in [26] to improve Caffarelli's approach to obtain the $C^{1,\alpha}$ regularity of flat free boundaries for problems in the class of the one-phase nonhomogeneous Bernoulli problem

$$\begin{cases} \Delta u = f & \text{in } B_1 \cap \{u > 0\}, \\ |\nabla u| = 1 & \text{on } F(u) := B_1 \cap \partial\{u > 0\}, \end{cases} \tag{0.5}$$

where B_1 denotes the open ball of radius 1 and center at 0 in $\mathbb{R}^n$. More generally, $B_r(x_0)$ denotes the open ball of radius r and center at x_0 in $\mathbb{R}^n$. For the sake of simplicity, we denote $B_r = B_r(0)$. We refer to Chapter 1 for the discussion of her approach and its importance in the investigation of the free boundary regularity.

To conclude this introduction, we point out that the one-phase problem in the parabolic setting may take the form

$$\begin{cases} u_t = \Delta u & \text{in } (\Omega \times (0, T]) \cap \{u > 0\}, \\ u_t = |\nabla u|^2 & \text{on } (\Omega \times (0, T]) \cap \partial\{u > 0\}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $u : \Omega \times [0, T] \to \mathbb{R}$, $u \geq 0$. We remark that here the free boundary $\partial\{u > 0\}$ involves the time as well, so that the complexity of the problem increases. We will study in details this problem in Chapter 1.

The thesis is organized as follows. In Chapter 1, we study the regularity of flat free boundaries for the one-phase Stefan problem based on perturbation arguments leading to a linearization of the problem, taking inspiration from the elliptic counterpart developed by De Silva in [26]. In the following chapter, we investigate the existence of an Alt-Caffarelli-Friedman monotonicity formula, see [2], in the Heisenberg group $\mathbb{H}^1$. Lastly, in Chapter 3 we deal with the extension of the results in [30] to a generalization of the classical one-phase (Bernoulli) energy functional to each $p > 1$.

Chapter 1

Free boundary regularity in the one-phase Stefan problem

This chapter focuses on [28], which I have written together with D. De Silva and O. Savin. Here, we study the regularity of flat free boundaries for the one-phase Stefan problem

$$\begin{cases} u_t = \Delta u & \text{in } (\Omega \times (0,T]) \cap \{u > 0\}, \\ u_t = |\nabla u|^2 & \text{on } (\Omega \times (0,T]) \cap \partial\{u > 0\}, \end{cases} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^n$, $u : \Omega \times [0,T] \to \mathbb{R}$, $u \geq 0$.

1.1 Some generalities of the Stefan problem

In this section, we provide some generalities about the Stefan problem, mentioning a bit of the literature on it.
The Stefan problem is one of the most classical and important free boundary problems. It dates back to the 19th century and its name precisely to around 1890, when the physicist Josef Stefan introduced the general class of such problems in a series of four papers relating the freezing of the ground and the formation of sea ice, see [73] for a comprehensive description of the history of the Stefan problem. The importance of this problem lies in the fact

that it has a physical motivation. Precisely, the Stefan problem describes the phase transition between solids and liquids, such as the melting of the ice (or the solidification of the water), see for example [49] and again [73]. In particular, the one-phase form (1.1) is based on an assumption that one of the material phases may be neglected. Typically, this is achieved by assuming that such a phase is everywhere at the phase change temperature and hence any variation from this temperature leads to a change of phase. As a consequence, we can just focus on the behavior of the other phase. In the setting of the melting of the ice, u represents the temperature of the water, the region $\{u = 0\}$ the unmelted region of ice and the free boundary $\partial\{u > 0\}$ the moving interphase separating the ice and the water. Furthermore, the water satisfies the heat equation and the condition $u_t = |\nabla u|^2$ on $\partial\{u > 0\}$ is the law of conservation of energy, which defines the position of the interphase. Here, $\frac{u_t}{|\nabla u|}$ is the speed of $\partial\{u > 0\}$, at t fixed, in the direction $-\nu$, with $\nu := \frac{\nabla u}{|\nabla u|}$.

The main object of interest in the one-phase Stefan problem (1.1) is the behavior of the free boundary $\partial\{u > 0\}$.

In problems of this type, free boundaries may not regularize instantaneously. A two dimensional example in which a Lipschitz free boundary for (1.1) preserves corners can be found for instance in [15], together with a three dimensional one in the more general setting of the two-phase Stefan problem. For this reason, exactly in such framework of the two-phase Stefan problem, Athanasopoulos, Caffarelli, and Salsa studied the regularizing properties of the free boundary under reasonable assumptions. Specifically, in [4] they showed that Lipschitz free boundaries in space-time become smooth provided a nondegeneracy condition holds, while in [5] the same conclusion was established for sufficiently "flat" free boundaries. The techniques were based on the original work of Caffarelli in the elliptic case we recalled in Introduction, see [11] and [13]. For the sake of completeness, we provide the two-phase formulation of the Stefan problem:

$$\begin{cases} u_t = \Delta u & \text{in } (\Omega \times (0,T]) \cap \{u > 0\}, \\ u_t = \Delta u & \text{in } (\Omega \times (0,T]) \cap \{u \le 0\}^0, \\ \dfrac{u_t^+}{|\nabla u^+|} = |\nabla u^+| - |\nabla u^-| & \text{on } (\Omega \times (0,T]) \cap \partial\{u > 0\}, \\ \dfrac{u_t^-}{|\nabla u^-|} = |\nabla u^+| - |\nabla u^-| & \text{on } (\Omega \times (0,T]) \cap \partial\{u > 0\}, \end{cases} \tag{1.2}$$

$\Omega \subset \mathbb{R}^n$. Here, $\frac{u_t^+}{|\nabla u^+|}$ and $\frac{u_t^-}{|\nabla u^-|}$ both represent the speed of the interphase introduced before. The form (1.2) describes the physical scenario in which both the two phases can not be neglected and have non-constant zero temperature.

About the one-phase Stefan problem, we mention, at this level, the contribution given by S. Choi and I. C. Kim, who showed in [19] that solutions regularize instantaneously if the initial free boundary is locally Lipschitz with bounded Lipschitz constant and the initial data has subquadratic growth.

1.2 A recent approach for the free boundary regularity in the one-phase Stefan problem

In this section, we introduce the contents of [28], which we expose in details in the remaining of the chapter. The approach in [28] relies on perturbation arguments leading to a linearization of the problem. This is inspired by the elliptic counterpart developed by De Silva in [26]. There, the author improves Caffarelli's approach to obtain the $C^{1,\alpha}$ regularity of flat free boundaries for problems in the class of the one-phase nonhomogeneous Bernoulli problem (0.5). Precisely, the strategy in [26] consists of showing that the graph of a solution u in this class of problems satisfies an "improvement of flatness" property and then iterating it to achieve the $C^{1,\alpha}$ regularity. As stressed in the introduction of the thesis, De Silva in [26] exploits the theory of viscosity solutions. Before focusing on the flexibility of De Silva's approach, we give an idea of the notion of viscosity solution, roughly saying what is a viscosity solution to (0.5). In this regard, a viscosity solution to (0.5) is essentially a nonnegative continuous function such that its graph can not be touched by above (resp. below) at a point, locally, by the graph of a classical strict supersolution to (0.5) (resp. subsolution). Here, by a classical strict supersolution to (0.5) we mean a sufficiently smooth function which solves (0.5) with $<$ instead of $=$. Similarly, we can define a classical strict subsolution. Moreover, given two continuous functions u, φ, intuitively we say that φ touches u from below (resp. above) at a point if the graph of φ touches the graph of u at such a point and it is below (resp. above) the graph of u in a neighborhood of that point.

As previously anticipated, the techniques in [26] have come out to be very flexible and have been widely generalized to a variety of free boundary problems, including two-phase nonhomogeneous problems, "thin" free boundary problems and minimization problems (see for example [27], [29], [25]), for which regularity results have not been proved using classical Caffarelli's approach yet. As a consequence, it is important to understand how these techniques could be applied in the context of time dependent problems. This has been exactly the motivation with which De Silva, Savin and I have started investigating the regularity of flat free boundaries for (1.1), taking inspiration by [26]. At this level, we want to point out that the methods developed in [28] are suitable to further extensions as well.

Looking now into the details of [28], the result is basically equivalent to the previously mentioned flatness result in [5]. Specifically, the main theorem roughly states that a solution to the Stefan problem in a ball of size λ in space-time which is of size λ and has a "flat free boundary" in space, must have smooth free boundary in the interior provided that a necessary nondegeneracy condition holds. The nondegeneracy condition for u requires that u is bounded below by a small multiple of λ at some point in the domain at distance λ from the free boundary. Precisely, we assume that $u : \Omega \times [0, T] \to \mathbb{R}^+$

solves (1.1) in the viscosity sense. This means that u is continuous and its graph cannot be touched by above (resp. below) at a point (x_0, t_0) in a parabolic cylinder $B_r(x_0) \times (t_0 - r^2, t_0]$, by the graph of a classical strict supersolution φ^+ (resp. subsolution). By a classical strict supersolution we mean that $\varphi(x,t) \in C^2$, $\nabla_x \varphi \neq 0$, and it solves

$$\begin{cases} \varphi_t > \triangle \varphi & \text{in } (\Omega \times (0,T]) \cap \{\varphi > 0\}, \\ \varphi_t > |\nabla \varphi|^2 & \text{on } (\Omega \times (0,T]) \cap \partial\{\varphi > 0\}. \end{cases} \tag{1.3}$$

Similarly we can define a strict classical subsolution.

Throughout the chapter, given a space-time function, ∇, Δ, and D^2 are computed with respect to the space variable x.

The rigorous statement of the main theorem is as follows.

Theorem 1.1. *Fix a constant K (large) and let u be a solution to the one-phase Stefan problem* (1.1) *in $B_\lambda \times [-K^{-1}\lambda, 0]$ for some $\lambda \leq 1$. Assume that*

$$|u| \leq K\lambda, \qquad u(x_0, t) \geq K^{-1}\lambda \quad \textit{for some} \quad x_0 \in B_{\frac{3}{4}\lambda}.$$

There exists ϵ_0 depending only on K and n such that if, for each t, $\partial_x\{u > 0\}$ is ϵ_0-flat in B_λ, then the free boundary $\partial\{u > 0\}$ (and u up to the free boundary) is smooth in $B_{\frac{\lambda}{2}} \times [-(2K)^{-1}\lambda, 0]$.

Here we use the notation $\partial_x\{u > 0\}$ to denote the boundary in $\mathbb{R}^n$ of $\{u(\cdot, t) > 0)\}$, with t fixed. By $\partial_x\{u > 0\}$ is ϵ_0-flat in B_λ we understand that, for each t, $\partial_x\{u > 0\} \cap B_\lambda$ is trapped in a strip of width $\epsilon_0\lambda$ (the region between two parallel hyperplanes at distance $\epsilon_0\lambda$ from each other), and $u = 0$ on one side of this strip while $u > 0$ on the other side.

The assumption that u is of size λ in a domain of size λ around the free boundary is natural, since this eventually holds for all classical solutions by choosing λ small. We point out that in Theorem 1.1 the behavior of the solution depends strongly on the value of λ. If we scale the domain to unit size and keep the function u of size 1, then the rescaled function

$$(x,t) \quad \mapsto \quad \frac{1}{\lambda} u(\lambda x, \lambda t), \qquad (x,t) \in B_1 \times [-K^{-1}, 0],$$

solves a Stefan problem with possibly large diffusion coefficient λ^{-1}

$$\begin{cases} \lambda u_t = \triangle u & \text{in } (B_1 \times (-K^{-1}, 0]) \cap \{u > 0\}, \\ u_t = |\nabla u|^2 & \text{on } (B_1 \times (-K^{-1}, 0]) \cap \partial\{u > 0\}. \end{cases} \tag{1.4}$$

Theorem 1.1 states that nondegenerate solutions of size 1 of (1.4) which have ϵ_0- flat free boundaries in B_1 are smooth up to the free boundary. We remark that ϵ_0 is independent of λ, which means that we need to obtain uniform estimates in λ for the oscillation of the free boundaries of solutions of (1.4). The results in [28] show that the free boundary has a uniform $C^{1,\alpha}$ bound in space. On the other hand, the estimates for u in the set where it is positive depend on the parameter λ. The strategy is to approximate u with a family of explicit functions $l_{a,b}$ which in the direction perpendicular to the free boundary depend on λ while on the tangential directions to the free boundary are independent of the parameter λ.

Formally as $\lambda \to 0^+$, a solution u to (1.4) solves the Hele-Shaw equation. Estimates for this problem by similar methods as those developed in [28] were obtained by H. Chang-Lara and N. Guillen in [CG].

To prove main Theorem 1.1, we show that if a solution u satisfies the hypotheses of Theorem 1.1 then, after a convenient dilation, the flatness assumption can be extended to the whole function u instead of just the free boundary. Then Theorem 1.1 is a consequence of the following result.

Theorem 1.2. *Fix a constant K (large) and let u be a solution to the one-phase Stefan problem* (1.1) *in* $B_{2\lambda} \times [-2\lambda, 0]$ *for some* $\lambda \leq 1$. *Assume that* $0 \in \partial\{u > 0\}$, *and*

$$a_n(t)\ (x_n - b(t) - \epsilon_1\lambda)^+ \ \leq u\ \leq a_n(t)\ (x_n - b(t) + \epsilon_1\lambda)^+\,,$$

with

$$K^{-1} \leq a_n \leq K, \qquad |a_n'(t)| \leq \lambda^{-2}, \qquad b'(t) = -a_n(t),$$

for some small ϵ_1 *depending only on* K *and* n. *Then in* $B_\lambda \times [-\lambda, 0]$ *the free boundary* $\partial\{u > 0\}$ *is a* $C^{1,\alpha}$ *graph in the* x_n *direction.*

The assumption that $b' = -a_n(t)$ means that the approximating linear functions in x, $a_n(t)(x_n - b(t))^+$, satisfy the free boundary condition, while $|a_n'(t)| \leq \lambda^{-2}$ respects the parabolic scaling of the interior equation and represents that a_n can change at most $o(1)$ in a time interval of length $o(\lambda^2)$.

We remark that it suffices to prove Theorem 1.2 under the more relaxed hypotheses

$$\lambda \leq \lambda_0 \quad \text{and} \quad |a_n'(t)| \leq c_0 \lambda^{-2}, \tag{1.5}$$

with λ_0, c_0 small depending on K, n. We end up in this setting by working in balls of size $\tau\lambda$ with τ sufficiently small, and then relabel $\tau\lambda$ by λ and $\epsilon_1 \tau^{-1}$ by ϵ_1.

Theorem 1.2 applies, for example, when u is a perturbation of order $o(1)\lambda$ of a traveling wave solution

$$(e^{ax_n + a^2 t} - 1)^+, \qquad K^{-1} \leq a \leq K.$$

In this case we choose $a_n(t) = a$, $b(t) = -at$, and consider $\lambda \leq \lambda_0$ small so that the difference between the approximating linear part $a_n(t)(x_n - b(t))$ and the exact solution above is less than $\frac{1}{2}\epsilon_1 \lambda$ in B_λ.

The proof of Theorem 1.2 is based on linearization techniques. The linearized equation in our setting has the form of an oblique derivative parabolic problem

$$\begin{cases} \lambda v_t = tr(A(t) D^2 v) & \text{in } \{x_n > 0\}, \\ v_t = \gamma(t) \cdot \nabla v & \text{on } \{x_n = 0\}, \end{cases} \tag{1.6}$$

with $A(t)$ uniformly elliptic and $\gamma_n > 0$. An important task in our analysis is to develop Schauder-type estimates for equation (1.6) with respect to an appropriate distance d_λ and to capture both features of the mixed parabolic/hyperbolic scaling.

The remaining of the chapter is organized as follows. In the next section, we show that Theorem 1.1 can be deduced from Theorem 1.2. In Section 1.4, we use a Hodograph transform to obtain an equivalent quasilinear parabolic equation with oblique derivative boundary condition. In the following section, we state an improvement of flatness result Proposition 1.9 for solutions of such nonlinear problem, then we show how this implies Theorem 1.2. The proof of Proposition 1.9 is presented in Section 1.6, and it relies on various Hölder estimates (with respect to the appropriate distance) for solutions to the linearized problem associated to the nonlinear problem. Sections 1.7 and 1.8 are devoted to the proofs of such Hölder estimates, while Section 1.9 focuses on the one dimensional linear problem, which plays an essential role. The last section contains some general technical results on solutions to the linear problem.

1.3 From flat free boundaries to flat solutions

In this section, we show that Theorem 1.1 can be reduced to Theorem 1.2.

We assume that the function u satisfies the ϵ_0-flatness hypothesis of the free boundary from Theorem 1.1 for some $\lambda \leq 1$, and that $(0,0)$ is a free boundary point. Precisely, by $\partial_x\{u > 0\}$ is ϵ_0-flat in B_λ we understand that, for each t, there exists a direction ν such that

$$\partial_x\{u(\cdot,t) > 0\} \cap B_\lambda \subset \{|\langle x - x_0, \nu\rangle| \leq \epsilon_0\lambda\},$$

and

$$\begin{aligned} u = 0 \quad &\text{in } \{\langle x - x_0, \nu\rangle \leq -\epsilon_0\lambda\},\\ u > 0 \quad &\text{in } \{\langle x - x_0, \nu\rangle \geq \epsilon_0\lambda\}. \end{aligned}$$

First, we show that in a smaller domain $B_{\eta\lambda} \times [-\eta\lambda, 0]$ the whole graph of u is η^β- flat, for some small β, provided that $\epsilon_0 \leq c(\eta, K)$. Then, in this domain the hypotheses of Theorem 1.2 are satisfied by choosing η sufficiently small.

We work with the parabolic rescaling of the function u which is defined in $B_1 \times [-(K\lambda)^{-1}, 0]$ and keeps the function u of unit size:

$$(x,t) \quad \mapsto \quad \frac{1}{\lambda}u(\lambda x, \lambda^2 t), \qquad (x,t) \in B_1 \times [-(K\lambda)^{-1}, 0].$$

By abuse of notation we denote this rescaling by u, and then u solves a Stefan problem with possibly small speed coefficient λ,

$$\begin{cases} u_t = \triangle u & \text{in } (B_1 \times (-(K\lambda)^{-1}, 0]) \cap \{u > 0\}, \\ u_t = \lambda |\nabla u|^2 & \text{on } (B_1 \times (-(K\lambda)^{-1}, 0]) \cap \partial\{u > 0\}. \end{cases} \tag{1.7}$$

We prove the following main lemma. Universal constants only depend on n, K. As usual, in the body of the proofs, constants denoted by C may change from line to line.

Lemma 1.3. *Assume that u solves* (1.7),

$$|u| \leq K, \qquad u(x_0, t) \geq K^{-1} \quad \textit{for some} \quad x_0 \in B_{3/4},$$

$$0 \in \partial_x\{u(\cdot, 0) > 0\}, \qquad \textit{and } \partial_x\{u(\cdot, t) > 0\} \textit{ is } \epsilon_0\textit{-flat in } B_1.$$

Then for all small $\eta > 0$ we have

$$a_n(t)\left(x_n - b(t) - \eta^{1+\beta}\right)^+ \leq u \leq a_n(t)\left(x_n - b(t) + \eta^{1+\beta}\right)^+ \quad \textit{in } B_\eta \times [-\lambda^{-1}\eta, 0],$$

with $\beta = 1/20$ and for $c, C > 0$ universal,

$$c \leq a_n(t) \leq C, \quad |a_n'(t)| \leq \eta^{\beta - 2}, \qquad b'(t) = -\lambda a_n(t), \quad b(0) = 0,$$

provided that $\epsilon_0 \leq c(\eta, K)$.

When we rescale the conclusion back to the original coordinates, we obtain that the hypotheses of Theorem 1.2 are satisfied in the cylinder $B_{\eta\lambda} \times [-\eta\lambda, 0]$ with $\epsilon_1 = \eta^\beta$.

We start by proving a result about the location of the free boundary in time.

Lemma 1.4. *Assume u solves* (1.7) *in $B_2 \times [-K^{-1}, 1]$ and that $0 \leq u \leq K$. If $u(x, 0) = 0$ in B_1, then*

$$u(x, t) \leq C(|x| - 1)^+, \quad \textit{if} \quad t \in [-(2K)^{-1}, 0], \tag{1.8}$$

and

$$u(x, t) = 0 \quad \textit{if } |x| < 1 - C\lambda, \ t \in [0, 1], \tag{1.9}$$

with $C > 0$ universal.

Proof. Since the support of u is increasing with time we deduce that $u = 0$ in B_1 for all $t \in [-K^{-1}, 0]$. Then, in the annular domain $(B_2 \setminus B_1) \times [-K^{-1}, 0]$, by the comparison principle, u is less than a multiple of the solution to the heat equation which equals 0 on $\partial B_1 \times (-K^{-1}, 0]$, and 1 on the remaining part of the parabolic boundary. This, together with the boundary regularity of such solution, implies the estimate (1.8).

Now, for times $t \in [0, 1]$ we compare u with

$$w(x,t) = C_0 \, g(|x| - r(t)), \quad r(t) := 1 - C_0 \lambda t,$$

with g a 1D function such that $g(s) = 0$ if $s \leq 0$, and for positive s is defined by the ODE

$$g''(s) + 2ng'(s) = 0, \quad g(0) = 0, \quad g'(0) = 1.$$

Notice that $g' \in [0, 1]$.

We may assume that $r(t) \geq 1/2$, otherwise the conclusion (1.9) is trivial (say for $C > 2C_0$).

The constant C_0 is chosen large such that $w \geq u$ at time $t = 0$ (by (1.8)) and also on $\partial B_2 \times [0, 1]$. We check that w is a supersolution to (1.7); indeed in $\{w > 0\}$ we have (recall $r(t) \geq 1/2$),

$$w_t = C_0^2 \lambda g' \geq 0, \qquad \triangle w = C_0 \left(g'' + \frac{n-1}{|x|} g' \right) < 0,$$

and on $\partial\{w > 0\}$

$$w_t = \lambda C_0^2 = \lambda |\nabla w|^2.$$

In conclusion, $u \leq w$ which gives the desired conclusion (1.9). □

Now, we turn to the proof of Lemma 1.3.

Proof of Lemma 1.3. We assume that u satisfies (1.7) in $B_1 \times [-(K\lambda)^{-1}, 0]$, and $\partial_x \{u > 0\}$ is ϵ_0-flat in B_1. Suppose that $(0,0) \in \partial\{u > 0\}$ and then, after a rotation,

$$u(x, 0) > 0 \text{ if } x_n > \epsilon_0, \text{ and } u(x, 0) = 0 \text{ if } x_n < -\epsilon_0.$$

From (1.8) in Lemma 1.4 (applied to balls tangent to $\{x_n = -\epsilon_0\}$) we find that $u \le C(x_n + \epsilon_0)^+$ in $B_{1/2} \times [-(2K)^{-1}, 0]$.

We define

$$u_\tau := \frac{1}{\tau} u(\tau x, \tau^2 t), \qquad \text{with} \quad \tau \ge \epsilon_0^{1/2},$$

and, if $\tau \in [\epsilon_0^{1/2}, c]$, then

$$u_\tau \le C(x_n + \tau)^+ \quad \text{in } B_1 \times [-2, 0]. \tag{1.10}$$

Notice that u_τ satisfies (1.7) with $\tau\lambda$ instead of λ. We apply (1.9) of Lemma 1.4 for u_τ and obtain that (since $(0,0) \in \partial\{u_\tau > 0\}$),

$$\partial_x\{u_\tau > 0\} \cap B_{1/2} \text{ intersects } \{x_n \le C\lambda\tau\}, \quad \text{for all } t \in [-1, 0]. \tag{1.11}$$

Moreover, $\partial_x\{u_\tau > 0\}$ is $\tau^{-1}\epsilon_0$-flat in B_1, which combined with (1.11) implies that

$$\partial\{u_\tau > 0\} \cap (B_{1/2} \times [-1, 0]) \text{ is included in } \{x_n \le C(\lambda\tau + \tau^{-1}\epsilon_0)\}. \tag{1.12}$$

In $(B_{1/2} \cap \{x_n > C\tau\}) \times [-1, 0]$ we compare u_τ with the solution w to the heat equation which equals 0 on $\{x_n = C\tau\}$, and equals u_τ on the remaining part of the parabolic boundary. Notice that by (1.12), since $\tau \ge \epsilon_0^{1/2}$, $u_\tau > 0$ on $\{x_n = C\tau\}$. From (1.10) we find $|u_\tau - w| \le C\tau$, and the boundary regularity of w gives

$$|u_\tau - a x_n| \le C\rho^{3/2} + C\tau \le 2C\rho^{3/2} \qquad \text{in} \quad B_{2\rho}^+ \times [-\rho^2, 0], \tag{1.13}$$

for some constant $a < C$, provided that we choose $\tau = \rho^{3/2}$ with ρ small, to be made precise later.

We claim that the nondegeneracy assumption $u(x_0, t) \ge K^{-1}$ for some $x_0 \in B_{3/4}$ implies that $a > c$. For this we use (1.12) which, in terms of the function u, implies that $\partial_x\{u(\cdot, t) > 0\}$, at all times $t = -\tau^2 \le -\epsilon_0$, intersects the x_n axis at distance at most $C(\lambda|t| + \epsilon_0)$ from the origin. As for (1.12), using that $\partial_x\{u > 0\}$ is ϵ_0-flat in B_1, we obtain that $u(x, t) > 0$

if $x_n > C\epsilon_0 + C\lambda|t|$ in $B_{1/2}$. Now we can use the nondegeneracy condition with a Hopf-type lemma for the heat equation and obtain

$$u \geq c(x_n - C(\epsilon_0 + \lambda|t|))^+ \quad \text{in} \quad B_{1/4} \times [-(4K)^{-1}, 0],$$

for some $c > 0$ that depends only on n and K. We use this inequality at time $t = 0$ in (1.13) and conclude $a > c$ since $\tau\rho > 2\tau^2 \geq 2\epsilon_0$. We can restate (1.13) as

$$(ax_n - C\eta^{1+\frac{1}{5}})^+ \leq u \leq (ax_n + C\eta^{1+\frac{1}{5}})^+ \quad \text{in} \quad B_{2\eta} \times [-\eta^2, 0],$$

with $\eta := \tau\rho = \rho^{5/2}$.

Similarly, by looking at the points $(b(t)e_n, t)$ where the free boundary intersects the x_n axis, we obtain that

$$|b(t)| \leq C(\lambda|t| + \epsilon_0) \leq C_0\eta \quad \text{if} \quad t \in [-\lambda^{-1}\eta, 0],$$

and in the domain $B_{2C_0\eta} \times [t - \eta^2, t]$ we have

$$\left(a(t) \cdot (x - b(t)e_n) - C\eta^{\frac{6}{5}}\right)^+ \leq u(x,s) \leq \left(a(t) \cdot (x - b(t)e_n) + C\eta^{\frac{6}{5}}\right)^+$$

with $c \leq |a(t)| \leq C$. The flatness assumption of the free boundary in B_1 implies

$$|a(t) - a_n(t)e_n| \leq C\eta,$$

so we may replace $a(t) \cdot (x - b(t)e_n)$ above by $a_n(t)(x_n - b(t))$.

The bounds on u above imply that $a_n(t)$ can vary at most $C\eta^{1/5}$ in an interval of length η^2. We can regularize $a_n(t)$ by averaging over such intervals (convolving with a mollifier) and the bounds for u still hold after changing the value of the constant C. Hence for all $t \in [-\lambda^{-1}\eta, 0]$, we can find $a_n(t) \in \mathbb{R}$ such that

$$a_n(t)\left(x_n - b(t) - C\eta^{\frac{6}{5}}\right)^+ \leq u \leq a_n(t)\left(x_n - b(t) + C\eta^{\frac{6}{5}}\right)^+ \tag{1.14}$$

in $B_{2C_0\eta} \times [t - \eta^2, t]$ with

$$c \leq a_n(t) \leq C, \qquad |a_n'(t)| \leq C\eta^{\frac{1}{5}-2}, \qquad |b(t)| \leq C_0\eta. \tag{1.15}$$

It remains to show that we can modify b slightly so that it satisfies the ODE $b' = -\lambda a_n$. Precisely, we let

$$\tilde{b}'(t) = -\lambda a_n(t), \quad \tilde{b}(0) = 0,$$

and we show that

$$|b(t) - \tilde{b}(t)| \leq C\eta^{1+\beta} \quad \text{if} \quad t \in [-\lambda^{-1}\eta, 0], \qquad \beta = 1/10. \tag{1.16}$$

For this we perturb the family of evolving planes $a_n(t)(x_n - \tilde{b}(t))^+$ into a subsolution/supersolution. Let

$$d(t) := \tilde{b}(t) + C_1\eta^\beta \lambda t,$$

with C_1 large, to be specified later. We claim that

$$b(t) \geq d(t) - 2\eta^{1+\beta}. \tag{1.17}$$

For this we define the function

$$v := (1 - C_2\eta^\beta)\, a_n(t)\, (h(x - d(t)e_n))^+,$$

with

$$h(x) := x_n - \eta^{\beta-1}(|x'|^2 - 2nx_n^2),$$

and check that it is a subsolution to our problem (1.7) in the domain

$$\Omega := \bigcup_{t\in[-\lambda^{-1}\eta,0]} B_{2\eta}(d(t)e_n) \times \{t\}.$$

Notice that in a ball of radius 2η,

$$h \leq C\eta, \qquad |\nabla h| = 1 + O(\eta^\beta), \tag{1.18}$$

and the constant $C_2 = C_2(n)$ is chosen depending only on n such that

$$v \leq a_n(t)(x_n - d(t))^+, \tag{1.19}$$

with equality at $d(t)e_n$ and moreover, when $x \in \partial B_{2\eta}(d(t)e_n) \cap \{v(x,t) > 0\}$, the difference between the two functions above is greater than $\eta^{1+\beta}$.

Next, we check that v is a strict subsolution. In the interior $\{v > 0\}$, using (1.15), (1.18), the definition of $\tilde{b}$, we have (for η small)

$$|v_t| \leq C|a_n'|\eta + C|d'| \leq C\eta^{-4/5}, \qquad \triangle v \geq c\eta^{\beta-1} > v_t,$$

and on the free boundary (C' depending only on C_2, n),

$$v_t = (1 - C_2\eta^\beta)a_n(-d')h_n, \qquad |\nabla v|^2 \geq (1 - C'\eta^\beta)a_n^2.$$

Since

$$h_n = 1 + O(\eta^\beta), \qquad (-d')a_n = \lambda a_n^2 - C_1\lambda a_n\eta^\beta,$$

we can choose C_1 large such that $v_t < \lambda|\nabla v|^2$.

If

$$b(t_0) < d(t_0) - 2\eta^{1+\beta} \text{ for some } t_0 \in [-\lambda^{-1}\eta, 0],$$

then by (1.14) and (1.19) we find that $v < u$ at time $t = t_0$ in $B_{2\eta}(d(t_0)e_n) \cap \overline{\{v > 0\}}$. On the other hand $v = u$ at the origin $(0, 0)$. This means that as we increase t from t_0 to 0, the graph of $v(\cdot, t)$ in $\overline{B_{2\eta}(d(t)e_n)} \cap \overline{\{v > 0\}}$ will touch by below the graph of u for a first time t, and the contact must be an interior point to $B_{2\eta}(d(t)e_n)$ due to the properties (1.14),(1.19) of u and v (in particular the difference between $a_n(t)\,(x_n - d(t))^+$ and v is greater than $\eta^{1+\beta}$ on $\partial B_{2\eta}(d(t)e_n)$). This contact point is either on the free boundary $\partial\{v > 0\}$ or on the positivity set $\{v > 0\}$, and we reach a contradiction since v is a strict subsolution. The claim (1.17) is proved, hence

$$b(t) \geq \tilde{b}(t) - C\eta^{1+\beta} \quad \text{if} \qquad t \in [-\lambda^{-1}\eta, 0].$$

The opposite inequality is obtained similarly and the claim (1.16) holds. Then from (1.14) we deduce that for all $\eta \leq c$ small

$$a_n(t)\left(x_n - \tilde{b}(t) - \eta^{1+\beta'}\right)^+ \;\leq\; u \;\leq\; a_n(t)\left(x_n - \tilde{b}(t) + \eta^{1+\beta'}\right)^+$$

in $B_\eta \times [-\lambda^{-1}\eta, 0]$ with $\beta' = 1/20$ and

$$c \leq a_n(t) \leq C, \quad |a_n'(t)| \leq \eta^{\beta'-2}, \qquad \tilde{b}'(t) = -\lambda a_n(t), \quad \tilde{b}(0) = 0.$$

□

1.4 The Nonlinear problem

In this section, we use a standard Hodograph transform to reduce our Stefan problem (1.1) to an equivalent nonlinear problem with fixed boundary and oblique derivative boundary condition (see (1.23)).

Here and henceforth, for $n \geq 2$, given $r > 0$ we set

$$Q_r := (-r, r)^n, \qquad Q_r^+ := Q_r \cap \{x_n \geq 0\}, \quad Q_r(x_0) := x_0 + Q_r,$$

$$\mathcal{C}_r := (Q_r \cap \{x_n > 0\}) \times (-r, 0], \quad F_r := \{(x,t) | \; x \in Q_r \cap \{x_n = 0\}, \; t \in (-r, 0]\}.$$

Also, by parabolic cylinders we mean

$$\mathcal{P}_r(x_0, t_0) := Q_r(x_0) \times (t_0 - r^2, t_0].$$

1.4.1 The Hodograph transform

As mentioned above, we use a Hodograph transform to reduce the Stefan problem (1.1) to one with fixed boundary. Precisely, we view the graph of u in $\mathbb{R}^{n+2}$

$$\Gamma := \{(x, x_{n+1}, t) | \quad x_{n+1} = u(x_1, x_2, \ldots, x_n, t)\}$$

as the graph of a possibly multi-valued function $\bar{u}$ with respect to the x_n direction

$$\Gamma := \{(x, x_{n+1}, t) | \quad x_n = \bar{u}(x_1, x_2, \ldots, x_{n-1}, x_{n+1}, t)\}.$$

We use $(y_1, \ldots, y_n)$ to denote the coordinates $(x_1, x_2, \ldots, x_{n-1}, x_{n+1})$. Then, if Du and $D\bar{u}$ denote at some point on the graph Γ the gradients with respect to the first n entries of u and $\bar{u}$, we find

$$Du = -\frac{1}{\bar{u}_n}(\bar{u}_1, \ldots, \bar{u}_{n-1}, -1), \qquad u_t = -\frac{\bar{u}_t}{\bar{u}_n}$$

$$D^2 u = -\frac{1}{\bar{u}_n} \left(A(D\bar{u})\right)^T \; D^2 \bar{u} \; A(D\bar{u}),$$

where $A(D\bar{u})$ is a square matrix which agrees with the identity matrix except on the nth row where the entries are given by the right hand side of Du above.

The Stefan problem (1.1) in terms of $\bar{u}$ can be written abstractly as the following quasilinear parabolic equation with oblique derivative boundary condition:

$$\begin{cases} \bar{u}_t = tr(\bar{A}(\nabla \bar{u})\, D^2 \bar{u}) & \text{in } \{y_n > 0\}, \\ \bar{u}_t = g(\nabla \bar{u}) & \text{on } \{y_n = 0\}, \end{cases} \tag{1.20}$$

with $\bar{A}(p)$ symmetric, positive definite as long as $p_n \neq 0$, and $g_n(p) > 0$.

The free boundary of u is given by the graph of the trace of $\bar{u}$ on $\{y_n = 0\}$. Our goal becomes to show that $\bar{u}$ is $C^{1,\alpha}$ with respect to the y', t variables. Let us assume that u satisfies the hypotheses of Theorem 1.2 (it is now more convenient to work in cubes rather than in balls). Below we denote by c, C various constants depending on K and n. From the flatness assumption

$$\left| u - a_n(t)(x_n - b(t))^+ \right| \leq C\epsilon_1 \lambda \qquad \text{in} \quad Q_\lambda \times [-\lambda, 0], \tag{1.21}$$

and $0 \in \partial\{u > 0\}$ implies $|b(0)| \leq C\epsilon_1\lambda$ which together with $|b'| \leq C\lambda$ gives

$$|b(t)| \leq C(\epsilon_1 + |t|)\lambda.$$

Thus, if $(x,t) \in Q_\lambda \times [-c\lambda, 0]$, then (for ϵ_1 possibly smaller), $|b(t)| \leq \lambda/2$ and by (1.21) the domain of definition of $\bar{u}$ at time t contains $Q^+_{\bar{c}\lambda}$ for $\bar{c}$ small enough. We conclude that $\bar{u}$ is well-defined in $Q^+_{\bar{\lambda}} \times [-\bar{\lambda}, 0]$, with $\bar{\lambda} := c_1\lambda$, c_1 sufficiently small.

Moreover, the graph of $\bar{u}$ in this set is closed in $\mathbb{R}^{n+2}$ (since it is obtained as a rigid motion from the graph of u) and it satisfies equation (1.20) in the viscosity sense, see Definition 1.6 below.

Remark 1.5. We observe that $\bar{u}$ is single-valued in the region $y_n \geq C\epsilon_1\lambda$, and possibly multi-valued near $y_n = 0$. Indeed, similarly as above, if $t \in [t_0 - \lambda^2, t_0 + \lambda^2]$, then using the bound for $|b'|$ and (1.5) for $|a'|$,

$$|a(t) - a(t_0)| \leq c_0, \qquad |b(t) - b(t_0)| \leq C\lambda^2,$$

hence, if λ_0, c_0 are smaller than ϵ_1 then

$$\left| u - a_n(t_0)(x_n - b(t_0))^+ \right| \leq C\epsilon_1\lambda \qquad \text{in} \quad Q_\lambda \times [t_0 - \lambda^2, t_0 + \lambda^2], \tag{1.22}$$

with $|b(t_0)| \leq \lambda/2$. By applying interior gradient estimates in parabolic cylinders included in $\{u > 0\}$ we find from (1.22) that if

$$(x_0, t_0) \quad \text{with} \quad x_0 \in Q_\lambda, \quad t_0 > -c\lambda \quad \text{is in the region} \quad C\epsilon_1\lambda \leq u(x_0, t_0) \leq c\lambda$$

then

$$|\nabla u(x_0, t_0) - a_n(t_0)e_n| \leq (2K)^{-1}.$$

Finally, the main hypotheses of Theorem 1.2 can be written in terms of $\bar{u}$ as

$$|\bar{u} - (\bar{a}_n(t)y_n + \bar{b}(t))| \leq C\epsilon_1\bar{\lambda} \quad \text{in} \quad Q^+_{\bar{\lambda}} \times [-\bar{\lambda}, 0],$$

$$\bar{b}'(t) = g(\bar{a}_n(t)e_n), \qquad K^{-1} \leq \bar{a}_n \leq K,$$

$$\bar{\lambda} \leq \bar{\lambda}_1, \qquad |\bar{a}'_n| \leq \bar{c}_1\bar{\lambda}^{-2}.$$

Our purpose in this paper is to prove an improvement of flatness result for solutions of the nonlinear equation (1.20) as above, provided that ϵ_1, $\bar{\lambda}_1$, $\bar{c}_1$ are chosen small depending on n and K (see Proposition 1.9 in the next section). Then Theorem 1.2 can be obtained by iterating such statement.

1.4.2 Assumptions on the nonlinear problem.

We consider solutions to the following problem (for simplicity of notation we drop the bars in our formulation, and we use x rather than y),

$$\begin{cases} u_t = F(\nabla u, D^2 u) & \text{in } \mathcal{C}_\lambda, \\ u_t = g(\nabla u) & \text{on } \mathcal{F}_\lambda. \end{cases} \tag{1.23}$$

We assume that F is linear in D^2u, that is $F(\nabla u, D^2u) = tr(A(\nabla u)D^2u)$ and $g_n > 0$.

We start by stating precisely the notion of viscosity solution, which can be easily adapted to multi-valued functions u whose graphs are compact sets of $\mathbb{R}^{n+2}$.

Definition 1.6. We say that a continuous function $u : \overline{\mathcal{C}}_\lambda \to \mathbb{R}$ is a *viscosity subsolution* to (1.23) if its graph cannot be touched by above at points in

$\mathcal{C}_\lambda \cup \mathcal{F}_\lambda$ (locally, in parabolic cylinders) by graphs of strict C^2 supersolutions φ of (1.23), i.e.

$$\begin{cases} \varphi_t > F(\nabla\varphi, D^2\varphi) & \text{in } \mathcal{C}_\lambda, \\ \varphi_t > g(\nabla\varphi) & \text{on } \mathcal{F}_\lambda. \end{cases} \tag{1.24}$$

Similarly we can define *viscosity supersolutions* and *viscosity solutions* to (1.23).

We define now a class of linear in x functions that we use throughout this paper to express the flatness condition.

Definition 1.7. We denote by $l_{a,b}(x,t)$ functions which for each fixed t are linear in the x variable, and whose coefficients in the x' variable are independent of t, and also so that $l_{a,b}$ satisfies the boundary condition in (1.23) on $\{x_n = 0\}$. More precisely,

$$l_{a,b}(x,t) := a(t)\cdot x + b(t),$$

with

$$a(t) := (a_1, \dots, a_{n-1}, a_n(t)), \quad a_i \in \mathbb{R}, \;\; i = 1, \dots, n-1,$$

and

$$b'(t) = g(a(t)).$$

Our main result is to show that if u is a viscosity solution of (1.23) which is possibly multi-valued near $\{x_n = 0\}$ and is well approximated by $l_{a,b}$ in a cylinder $\mathcal{C}_\lambda$, i.e.

$$|u - l_{a,b}| \le \epsilon\lambda \quad \text{in} \quad \mathcal{C}_\lambda,$$

then in a smaller cylinder $\mathcal{C}_{\tau\lambda}$ it can be approximated by another function $l_{\tilde{a},\tilde{b}}$ with an error $\epsilon_\tau = \epsilon\tau^\alpha$ that improved by a $C^{1,\alpha}$ scaling.

Before formulating this result rigorously in the next section, we state here the precise hypotheses on F and g. We assume that $F(p,M)$ is uniformly elliptic in M for each fixed slope $p \in \mathbb{R}^n$ with $p_n > 0$ and the ellipticity constants could degenerate as $p_n \to 0^+$ or $|p| \to \infty$. Precisely, for any given constant K large there exists Λ large depending on K such that

$$\Lambda I \ge D_M F(p,M) \ge \Lambda^{-1} I, \qquad \text{if} \quad p \in \mathcal{R}_K, \tag{1.25}$$

with

$$\mathcal{R}_K := B_K \cap \{p_n \geq K^{-1}\} \quad \subset \quad \mathbb{R}^n. \tag{1.26}$$

We choose K sufficiently large such that when p is restricted to the set above we also have

$$|D_p F| \leq \Lambda |M|, \quad \|g\|_{C^1} \leq \Lambda, \quad g_n \geq \Lambda^{-1}. \tag{1.27}$$

From now on we assume that the constants K and Λ have been fixed such that (1.25)-(1.27) hold. In fact, for notational simplicity, by possibly choosing K larger, we can assume that (1.25)-(1.27) hold with $\Lambda = K$. We consider the situation when u is well approximated in $\mathcal{C}_\lambda$ by a function $l_{a,b}$ as above with slopes $a(t)$ belonging to the region $\mathcal{R}_K$.

We suppose in addition that u satisfies the Harnack inequality from scale λ to scale $\sigma\lambda$ where σ is a small parameter. We denote this property for u as property $H(\sigma)$ which is defined in the following way.

Definition 1.8. Given a positive constant σ small, we say that

$$u \text{ has property } H(\sigma) \text{ in } \mathcal{C}_\lambda$$

if u (possibly multi-valued) satisfies the following version of interior Harnack inequality in parabolic cylinders of size $r \in [\sigma\lambda, \lambda]$.

Let l denote a linear function

$$l(x) := a \cdot x + b, \qquad \text{with} \quad a \in \mathbb{R}^n, \;\; b \in \mathbb{R}, \quad |a| \leq K.$$

If

$$u \geq l \quad \text{in} \quad Q_r(x_0) \times [t_0 - r^2, t_0 + r^2] \qquad \subset \quad \mathcal{C}_\lambda,$$

with $r \geq \sigma\lambda$, and

$$(u - l)(x_0, t_0) \geq \mu, \quad \text{for some} \quad \mu \geq 0,$$

then

$$u - l \geq \kappa\mu \quad \text{in} \quad Q_{r/2}(x_0) \times \left[t_0 + \frac{1}{2}r^2, t_0 + r^2\right],$$

for some constant κ depending on n and K (but independent of σ).

Similarly, if $u \leq l$ we require these inequalities to hold for $l - u$ instead of $u - l$.

Property $H(\sigma)$ for all $\sigma > 0$ is a consequence of the parabolic Harnack inequality in the case when u is a viscosity solution of (1.23), and in addition we know that $\nabla u \in \mathcal{R}_K$. However, we will show below that property $H(\sigma)$ for some σ small, is satisfied for solutions u which are well approximated by functions $l_{a,b}$ and are graphical with respect to the e_n direction.

1.5 The iterative statement

In this section, we state the main improvement of flatness result Proposition 1.9, and we show how Theorem 1.2 can be deduced from it. We also describe the strategy of the proof of Proposition 1.9, and its connection to the corresponding linearized problem (1.34).

The improvement of flatness statement reads as follows (we use the notation from Subsection 3.2). The rest of the paper will be devoted to its proof.

Proposition 1.9 (Improvement of flatness)**.** *Fix $K > 0$ large, and assume F,g satisfy* (1.25)-(1.27)*. Assume that u is a viscosity solution to* (1.23) *possibly multi-valued, which satisfies property $H(\epsilon^{1/2})$ and*

$$|u - l_{a,b}| \leq \epsilon\lambda \quad \text{in } \overline{\mathcal{C}}_\lambda, \text{ with } \quad b'(t) = g(a(t)), \tag{1.28}$$

$$a(t) \in \mathcal{R}_K, \quad |a_n'(t)| \leq \delta\epsilon\lambda^{-2},$$

and

$$\epsilon \leq \epsilon_0, \quad \lambda \leq \lambda_0, \quad \lambda \leq \delta\epsilon.$$

Then there exists $l_{\tilde{a},\tilde{b}}$ such that

$$|u - l_{\tilde{a},\tilde{b}}| \leq \frac{\epsilon}{2}\tau\lambda \quad \text{in } \overline{\mathcal{C}}_{\tau\lambda}, \qquad \tilde{b}'(t) = g(\tilde{a}(t)),$$

with

$$|a(t) - \tilde{a}(t)| \leq C\epsilon, \qquad |\tilde{a}_n'(t)| \leq \frac{\delta\epsilon}{2}(\tau\lambda)^{-2}.$$

Here the constants $\epsilon_0, \lambda_0, \delta, \tau > 0$ small and C large depend only on n, and K.

For the remainder of the section constants depending only on n and K are called universal, and denoted by c_i, C_i.

Remark 1.10. We apply the proposition above to the hodograph transform of a solution to the original Stefan problem, hence in our case u is graphical with respect to the e_n direction. Then (1.28) already implies our hypothesis that

$$u \text{ satisfies property } H(\epsilon^{1/2}) \text{ in } \mathcal{C}_\lambda.$$

Indeed, if $t \in [t_0 - \lambda^2, t_0 + \lambda^2]$, then using the bounds for $|a'|$, $|b'|$,

$$|a(t) - a(t_0)| \leq \delta\epsilon, \qquad |b(t) - b(t_0)| \leq C\lambda^2 \leq C\delta\epsilon\lambda,$$

hence

$$|a(t_0) \cdot x + b(t_0) - l_{a,b}| \leq C\delta\epsilon\lambda \quad \text{in} \quad Q_\lambda^+ \times [t_0 - \lambda^2, t_0 + \lambda^2]. \tag{1.29}$$

This shows that u is well approximated in each parabolic cylinder of size λ by a linear function which is constant in t,

$$|u - (a(t_0) \cdot x + b(t_0))| \leq 2\epsilon\lambda \quad \text{in} \quad Q_\lambda^+ \times [t_0 - \lambda^2, t_0 + \lambda^2], \tag{1.30}$$

with $C \geq a_n(t_0) > c$. Since the graph of u coincides with the graph (in the e_n direction) of a solution to the heat equation, we can use the standard Harnack inequality for the heat equation and find that u satisfies property $H(C\epsilon)$ in $\mathcal{C}_\lambda$ (as we used interior regularity in Remark 1.5). Thus u satisfies property $H(\epsilon^{1/2})$ by choosing ϵ_0 smaller if necessary.

This argument shows that if u is graphical with respect to the e_n direction, then it is single-valued away from a $O(\epsilon\lambda)$ neighborhood of $\{x_n = 0\}$.

We now show that Proposition 1.9 implies Theorem 1.2, and the remainder of the paper will be devoted to prove Proposition 1.9.

Proof of Theorem 1.2. As discussed in Subsection 1.4.1, Theorem 1.2 is equivalent to obtaining $C^{1,\alpha}$ estimates on $\{x_n = 0\}$ for the hodograph transform. After relabeling constants if necessary, the hodograph transform does satisfy the hypotheses of Proposition 1.9 with $\epsilon = \epsilon_0$, $\lambda \le \min\{\delta\epsilon_0, \lambda_0\}$, $a_0(t) = (0, 0, \ldots, 0, (a_0)_n(t)) \in \mathcal{R}_{K/2}$. Now Proposition 1.9 can be applied indefinitely in the cylinders $\mathcal{C}_{\lambda_k}$, $\lambda_k := \lambda\tau^k$, with $\epsilon = \epsilon_k := \epsilon_0 2^{-k} = C(\lambda)\lambda_k^\alpha$. The hypothesis that $a_k(t) \in \mathcal{R}_K$ is satisfied (by choosing ϵ_0 smaller if necessary) since

$$|a_k(t) - a_{k-1}(t)| \le C\epsilon_k, \qquad a_0(t) \in \mathcal{R}_{K/2},$$

from which we also deduce that

$$|a_k(t) - \nabla u(0,t)| \le C\epsilon_k. \tag{1.31}$$

Hence

$$|u - l_{a_k,b_k}| \le \epsilon_k\lambda_k \le C(\lambda)\,\lambda_k^{1+\alpha} \qquad \text{in} \quad \mathcal{C}_{\lambda_k},$$

for all $k \ge 0$, and from (1.30) (applied for λ_k) and (1.31) we deduce that

$$|\nabla u(0,t) - \nabla u(0,s)| \le C(\lambda)|t-s|^{\alpha/2},$$

which gives

$$|a_k(t) - a_k(s)| \le C(\lambda)\lambda_k^{\alpha/2} \quad \text{if} \quad t, s \in [-\lambda_k, 0].$$

Using that $b_k' = g(a_k)$ we lastly obtain

$$|u - (a_k(0)\cdot x + b_k'(0)t + b_k(0))| \le C(\lambda)\lambda_k^{1+\frac{\alpha}{2}} \qquad \text{in} \quad \mathcal{C}_{\lambda_k},$$

which is the desired conclusion. □

1.5.1 Strategy of the proof of the improvement of flatness.

We briefly explain the strategy of the proof of Proposition 1.9. The main idea is to linearize the equation near $l_{a,b}$. Define $w(x,t)$ the rescaled error by

$$u(x,t) := l_{a,b}(x,t) + \epsilon\lambda w\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right), \quad (x,t) \in \mathcal{C}_\lambda. \tag{1.32}$$

Then w is defined in $\mathcal{C}_1$, possibly multi-valued near $\{x_n = 0\}$, and satisfies by hypothesis

$$|w| \le 1 \quad \text{in} \quad \mathcal{C}_1,$$

and

$$\begin{cases} \lambda a_n'(\lambda t)x_n + b'(\lambda t) + \epsilon w_t(x,t) = F\left(a(\lambda t) + \epsilon \nabla w, \frac{\epsilon}{\lambda} D^2 w\right) & \text{in } \mathcal{C}_1, \\ \\ b'(\lambda t) + \epsilon w_t = g(a(\lambda t) + \epsilon \nabla w) & \text{on } \mathcal{F}_1. \end{cases} \tag{1.33}$$

We show that w is well approximated by a solution to the linear equation obtained formally by multiplying the first equation by $\lambda\epsilon^{-1}$ and the second by ϵ^{-1} and then letting $\epsilon \to 0$, $\delta \to 0$. Using $|a'| \le \delta\epsilon\lambda^{-2}$, and $\lambda\epsilon^{-1} \le \delta \to 0$ we obtain

$$\begin{cases} \lambda v_t = tr(A_\lambda(t) D^2 v) & \text{in } \mathcal{C}_1, \\ v_t = \gamma_\lambda(t) \cdot \nabla v & \text{on } \mathcal{F}_1, \end{cases} \tag{1.34}$$

with

$$A_\lambda(t) := A(a(\lambda t)), \qquad \gamma_\lambda(t) := \nabla g(a(\lambda t)).$$

Using that $A, g \in C^2(\mathcal{R}_K)$, and that $|a'| \ll \lambda^{-2}$ we find

$$|A_\lambda'(t)| \le \lambda^{-1}, \qquad |\gamma_\lambda'(t)| \le \lambda^{-1}.$$

The next sections are devoted to the study of the linear problem (1.34), and to obtain estimates which are uniform with respect to λ. To this aim, we introduce a distance d between points $(x,t) \in \mathbb{R}^{n+1}$

$$\begin{aligned} & d((x,t),(y,s)) \\ & := \min\{|x'-y'| + |x_n - y_n| + |t-s|^{1/2}, \quad |x'-y'| + |x_n| + |y_n| + |t-s|\}, \end{aligned}$$

which is consistent with the scaling of the equation, so that d is equivalent with the standard Euclidean distance on the hyperplane $x_n = 0$ and with the standard parabolic distance far away from this hyperplane. The various Hölder estimates in the next section are written with respect to this distance d, or after a dilation of factor λ^{-1} with respect to the rescaled distance d_λ.

In particular, this allows us to show that solutions v to the linear problem enjoy an improvement of flatness property in cylinders $\mathcal{C}_{\tau^k}$, which can be transferred further to the solutions of the nonlinear problem (1.33).

The relation between solutions w to (1.33) and v to (1.34) is made precise in the next proposition. It states that w satisfies essentially a comparison principle with C^2 subsolutions/supersolutions v of (1.34) which have bounded derivatives and second derivatives in x.

Proposition 1.11 (Comparison principle). *Let $v \in C^2(\overline{\Omega})$ with $\Omega \subset \mathcal{C}_1$ satisfy*

$$|\nabla v|, |D^2 v| \leq M,$$

for some large constant M and

$$\begin{cases} \lambda v_t \leq tr(A_\lambda(t) D^2 v) - C\delta & \text{in } \Omega, \\ v_t \leq \gamma_\lambda(t) \cdot \nabla v - \delta & \text{on } \mathcal{F}_1 \cap \overline{\Omega}, \end{cases} \tag{1.35}$$

with $A_\lambda(t)$, $\gamma_\lambda(t)$ as above.

Then v is a subsolution to (1.33), as long as C is sufficiently large, universal, and $\epsilon \leq \epsilon_1(\delta, M)$. In particular, if

$$v \leq w \quad \text{on} \quad \overline{\partial\Omega \setminus (\{t=0\} \cup \{x_n = 0\})}$$

then

$$v \leq w \quad \text{in} \quad \Omega.$$

Similarly, we have the same result for supersolutions by replacing $\leq$ by $\geq$ and the $-$ signs in (1.35) by $+$.

Proof. It is straightforward to show that (1.35) implies the corresponding inequalities for v (in place of w) in (1.33). We need to use the hypotheses of Proposition 1.9 and that

$$\lambda \|a'\|_{L^\infty} + \|b'\|_{L^\infty} \leq C, \quad |A(a(\lambda t) + \epsilon \nabla v) - A(a(\lambda t))| \leq C\epsilon M,$$

$$|g(a(\lambda t) + \epsilon \nabla v) - g(a(\lambda t)) - \epsilon \nabla g(a(\lambda t)) \cdot \nabla v| \leq C\epsilon^2 M^2.$$

□

As a consequence, we obtain that if the rescaled error w is close to a C^2 solution v of (1.34) on the *Dirichlet boundary* of a domain $\Omega \subset \mathcal{C}_1$ then v and w remain close to each other in the whole domain Ω.

Corollary 1.12. *Let w be a solution to* (1.33) *and $v \in C^2$ be a solution of* (1.34) *in a domain $\Omega \subset \mathcal{C}_1$, with*

$$|\nabla v|, |D^2 v| \leq M.$$

If $\epsilon \leq \epsilon_1(\delta, M)$ and

$$|v - w| \leq \sigma \quad on \quad \overline{\partial\Omega \setminus (\{t = 0\} \cup \{x_n = 0\})}$$

then

$$|v - w| \leq \sigma + C\delta \quad in \quad \Omega.$$

Proof. This follows immediately by applying Proposition 1.11 to

$$v \pm (C\delta(x_n^2 - t - 2) - \sigma).$$

□

We apply Proposition 1.11 and Corollary 1.12 to functions v for which M is large, universal. In order to apply Corollary 1.12 we need to show that w can be well approximated near the boundary of $\mathcal{C}_{1/2}$ by a solution v to (1.34) with bounded second derivatives in x. We prove that w has essentially a Hölder modulus of continuity (as $\delta \to 0$) with respect to the distance d_λ induced by d, and then we let v be the solution to the Dirichlet problem (1.34) in $\mathcal{C}_{1/2}$ with boundary data which is sufficiently close to w.

We conclude this section by stating a version of interior Harnack inequality for w with respect to constants, which is an immediate consequence of property $H(\epsilon^{1/2})$ of u in $\mathcal{C}_\lambda$, see Definition 1.8.

As in (1.29), the error between $l_{a,b}$ and a linear function independent of t in a time-interval of size $(\lambda r)^2$ is $C\delta\epsilon\lambda r^2$. Then Definition 1.8 implies the following property for $u - l_{a,b}$.

If for some constant ω

$$u - (\omega + l_{a,b}) \geq 0 \quad \text{in} \quad Q_{\lambda r}(x_0) \times [t_0 - (\lambda r)^2, t_0 + (\lambda r)^2] \subset \mathcal{C}_\lambda,$$

with $r \in [\epsilon^{1/2}, 1]$, and

$$(u - (\omega + l_{a,b}))(x_0, t_0) \geq \mu\epsilon\lambda, \quad \text{for some} \quad \mu \geq C\delta r^2,$$

then

$$u - (\omega + l_{a,b}) \geq \frac{\kappa}{2}\mu\epsilon\lambda \quad \text{in} \quad Q_{r\lambda/2}(x_0) \times \left[t_0 + \frac{1}{2}(\lambda r)^2, t_0 + (\lambda r)^2\right],$$

with κ the universal constant from Definition 1.8. In terms of w this can be written as follows.

Interior Harnack inequality for w. If

$$w \geq \omega \quad \text{in} \quad Q_r(x_0) \times [t_0 - \lambda r^2, t_0 + \lambda r^2] \subset \mathcal{C}_1,$$

with ω a constant, $r \geq \epsilon^{1/2}$, and

$$w(x_0, t_0) \geq \omega + \mu, \quad \text{for some} \quad \mu \geq C\delta r^2,$$

then

$$w \geq \omega + \frac{\kappa}{2}\mu \quad \text{in} \quad Q_{r/2}(x_0) \times \left[t_0 + \frac{\lambda}{2}r^2, t_0 + \lambda r^2\right]. \tag{1.36}$$

1.6 The linearized problem

In this section, we state various estimates for the linear problem (1.34) which are uniform in the parameter $\lambda \leq 1$ and we use them to prove the main result Proposition 1.9. We start with introducing the distance d_λ with respect to which such estimates are obtained.

1.6.1 Definition of the distances d, d_λ and the family of balls $\mathcal{B}_r$, $\mathcal{B}_{\lambda,r}$.

We define the following distance in $\mathbb{R}^{n+1}$

$$\begin{aligned}&d((x,t),(y,s))\\&:=\min\{|x'-y'|+|x_n-y_n|+|t-s|^{1/2},\quad |x'-y'|+|x_n|+|y_n|+|t-s|\},\end{aligned}$$

which interpolates between the parabolic distance and the standard one depending on how far points are from $\{x_n=0\}$. It is not too difficult to check that d satisfies the triangle inequality.

For $r\le 1$ and points (y,s) with $y_n\in[0,1]$, we define the family of "balls" of center (y,s) and radius r, which are backwards in time and restricted to $\{x_n\ge 0\}$, and which are consistent with the distance induced by d:

$$\begin{aligned}\mathcal{B}_r(y,s)&:=Q_r(y)\times(s-r^2,s), &&\text{if } r<|y_n|,\\ \mathcal{B}_r(y,s)&:=Q_r^+(y)\times(s-r,s), &&\text{if } 1\ge r\ge |y_n|,\end{aligned}$$

where we recall that

$$Q_r(y):=\{x\in\mathbb{R}^n|\,|x_i-y_i|<r\},\qquad Q_r^+(y):=Q_r(y)\cap\{x_n\ge 0\}.$$

Notice that

$$(x,t)\in\mathcal{B}_{2r}(y,s)\setminus\mathcal{B}_r(y,s)\quad\Longrightarrow\quad d((x,t),(y,s))\sim r.$$

A function $v:\overline{U}\to\mathbb{R}$, with $U\subset\mathcal{C}_1$, is Hölder with respect to the distance d if

$$[v]_{C_d^\alpha}:=\sup_{(x,t)\ne(y,s)}|v(x,t)-v(y,s)|\;d((x,t),(y,s))^{-\alpha}<\infty.$$

Equivalently, $v\in C_d^\alpha(\overline{U})$ if and only if there exists M such that $\forall(x,t)\in\overline{U}$

$$\operatorname{osc}\ v\le Mr^\alpha\quad\text{in}\quad\mathcal{B}_r(x,t)\cap\overline{U}.$$

Rescaling. Assume $\lambda \leq 1$ and we perform a dilation of factor λ^{-1} which maps Q_λ^+ into Q_1^+. We use hyperbolic scaling for the rescaled distance d_λ of d

$$d_\lambda((x,t),(y,s)) := \frac{1}{\lambda} d(\lambda(x,t), \lambda(y,s))$$

$$= \min\{|x'-y'| + |x_n - y_n| + \lambda^{-1/2}|t-s|^{1/2}, |x'-y'| + |x_n| + |y_n| + |t-s|\}.$$

The corresponding family of balls induced by d_λ denoted by $\mathcal{B}_{\lambda,r}$ is obtained by dilating of a factor λ^{-1} the sizes of the balls $\mathcal{B}_r$ above and then relabeling $\lambda^{-1}r$ by r. We find

$$\mathcal{B}_{\lambda,r}(y,s) := Q_r(y) \times (s - \lambda r^2, s), \qquad \text{if } r < |y_n|,$$
$$\mathcal{B}_{\lambda,r}(y,s) := Q_r^+(y) \times (s - r, s), \qquad \text{if } \lambda^{-1} \geq r \geq |y_n|,$$

and notice that $\mathcal{B}_{\lambda,r}(y,s) = \mathcal{B}_r(y,s)$ if $y_n = 0$.

As above a function v is Hölder with respect to the distance d_λ in $\overline{U}$ and write $v \in C^\alpha_{d_\lambda}(\overline{U})$ if there exists M such that

$$\text{osc} \quad v \leq M r^\alpha \quad \text{in} \quad \mathcal{B}_{\lambda,r}(x,t) \cap \overline{U}.$$

1.6.2 Estimates.

Having introduced the distance d_λ, we are now ready to state the estimates for the linear problem

$$\begin{cases} \lambda v_t = tr(A(t)D^2 v) & \text{in } \mathcal{C}_1, \\ v_t = \gamma(t) \cdot \nabla v & \text{on } \mathcal{F}_1, \end{cases} \tag{1.37}$$

with

$$K^{-1} I \leq A(t) \leq KI, \qquad K^{-1} \leq \gamma_n \leq K, \qquad |\gamma| \leq K$$
$$\lambda \in (0,1], \qquad |A'(t)| \leq \lambda^{-1}, \qquad |\gamma'(t)| \leq \lambda^{-1},$$

for some large constant K. Here constants depending on n and K are called universal.

We start with an interior regularity result (see Definition 1.7 of $l_{a,b}$).

Proposition 1.13 (Interior estimates)**.** *Let v be a viscosity solution to* (1.37) *such that $\|v\|_{L^\infty} \le 1$. Then*

$$|\nabla v|,\ \ |D^2 v|\ \le C \qquad in \quad \mathcal{C}_{1/2},$$

and for each $\rho \le 1/2$, there exists $l_{\bar{a},\bar{b}}$ such that

$$|v - l_{\bar{a},\bar{b}}| \le C\rho^{1+\alpha} \qquad in\ \mathcal{C}_\rho,$$

with

$$\bar{b}'(t) = \gamma(t)\cdot \bar{a}, \qquad |\bar{a}_n'| \le C\rho^{\alpha-1}\lambda^{-1}, \qquad |\bar{a}| \le C,$$

with α, C universal.

In terms of the Dirichlet problem for (1.37), we define the *Dirichlet boundary* of $\mathcal{C}_1$ as

$$\partial_D \mathcal{C}_1 := \partial\mathcal{C}_1 \cap \left(\{t=-1\} \cup \{x_n = 1\} \cup_{i=1}^{n-1} \{|x_i| = 1\}\right).$$

Notice that $\partial_D\mathcal{C}_1$ is different from the standard parabolic boundary since the points on $\mathcal{F}_1$ are also excluded.

Proposition 1.14 (The Dirichlet problem)**.** *Let ϕ be a continuous function on $\partial_D\mathcal{C}_1$. Then there exists a unique classical solution $v \in C^{2,1}(\mathcal{C}_1)\cap C^0(\bar{\mathcal{C}}_1)$ to the Dirichlet problem* (1.37) *with $v = \phi$ on $\partial_D\mathcal{C}_1$. Moreover,*

$$|\nabla v|, |D^2 v| \le C(\sigma)\|v\|_{L^\infty} \quad in \quad C_1^\sigma := \{d_\lambda((x,t), \partial_D\mathcal{C}_1) \ge \sigma\},$$

and if ϕ is C^α with respect to the distance d_λ, then v is also C^α up to the boundary and

$$\|v\|_{C^\alpha_{d_\lambda}} \le C\|\phi\|_{C^\alpha_{d_\lambda}},$$

with $C(\sigma)$, C universal constants (independent of λ).

Here

$$\|v\|_{C^\alpha_{d_\lambda}} := \|v\|_{L^\infty} + \sup_{(x,t)\neq(y,s)} |v(x,t) - v(y,s)| d_\lambda((x,t),(y,s))^{-\alpha}.$$

The proofs of Propositions 1.13 and 1.14 are based on a Harnack inequality for solutions to (1.37), which we provide in the next section. The Harnack inequality holds for more general equations of the same type with measurable coefficients. It applies also for solutions w to the nonlinear problem (1.33) up to scale $\epsilon^{1/2}$. To state it, we recall the definition of the maximal Pucci operators

$$\mathcal{M}^+_K(N) = \max_{K^{-1}I\le A\le KI} tr\ AN, \qquad \mathcal{M}^-_K(N) = \min_{K^{-1}I\le A\le KI} tr\ AN. \tag{1.38}$$

Theorem 1.15 (Hölder continuity)**.** *Let v be a viscosity solution to*

$$\begin{cases} \mathcal{M}^+_K(D^2v) \ge \lambda v_t \ge \mathcal{M}^-_K(D^2v) & \text{in } \mathcal{C}_1, \\ \\ K^{-1}v_n^- - Kv_n^+ - K|\nabla_{x'}v| \ge v_t \ge K^{-1}v_n^+ - Kv_n^- - K|\nabla_{x'}v| & \text{on } \mathcal{F}_1. \end{cases} \tag{1.39}$$

Then v is locally Hölder continuous in $\mathcal{C}_{1/2}$ with respect to the metric induced by d_λ, that is

$$\|v\|_{C^\alpha_{d_\lambda}(\mathcal{C}_{1/2})} \le C\|v\|_{L^\infty(\mathcal{C}_1)}.$$

Moreover, if v is continuous up to the boundary and $v = \phi$ on $\partial_D\mathcal{C}_1$ with $\phi \in C^\alpha_{d_\lambda}$ then $v \in C^\alpha_{d_\lambda}$ up to the boundary and

$$\|v\|_{C^\alpha_{d_\lambda}} \le C\|\phi\|_{C^\alpha_{d_\lambda}}.$$

The constants α and C depend only on n and K.

Proposition 1.16 (Harnack inequality for w)**.** *Assume that u satisfies the hypotheses of Proposition* 1.9 *and w is defined as in* (1.32)*. Then*

$$\operatorname*{osc}_{\mathcal{B}_{\lambda,r}(x_0,t_0)} w \le Cr^\alpha, \qquad \forall (x_0,t_0)\in\mathcal{C}_{1/2}, \quad r\ge C(\delta)\epsilon^{1/2},$$

provided that $\delta \le c'$ universal.

1.6.3 Proof of Proposition 1.9.

Using the results above we can complete the proof of Proposition 1.9.

Proof of Proposition 1.9. We divide the proof in two steps.

Step 1. We prove that there exists a solution v to (1.34) which approximates w well in $\mathcal{C}_{1/2}$, that is

$$|v - w| \le C\delta \quad \text{in } \mathcal{C}_{1/2},$$

provided that $\epsilon \le \epsilon_1(\delta)$.

Indeed, by Proposition 1.16 we know that there exists a function ϕ defined in $\mathcal{C}_{1/2}$ such that

$$|w - \phi| \le \delta, \qquad \|\phi\|_{C^\alpha_{d_\lambda}} \le C. \tag{1.40}$$

Let v be the solution to (1.34) in $\mathcal{C}_{1/2}$ with $v = \phi$ on $\partial_D \mathcal{C}_{1/2}$, which exists in view of Proposition 1.14 and satisfies,

$$\|v\|_{C^\alpha_{d_\lambda}} \le C. \tag{1.41}$$

Then, if $d_\lambda((x,t), \partial_D \mathcal{C}_{1/2}) \le \delta^{1/\alpha}$, there exists (y,s) on $\partial_D \mathcal{C}_{1/2}$ so that (using (1.41) and (1.40)),

$$|v(x,t) - \phi(y,s)| \le C\delta, \qquad |w(x,t) - \phi(y,s)| \le C\delta,$$

thus,

$$|v - w| \le C\delta \qquad \text{on} \quad \mathcal{C}_{1/2} \cap \{d_\lambda((x,t), \partial_D \mathcal{C}_{1/2}) \le \delta^{1/\alpha}\}. \tag{1.42}$$

In particular

$$|v - w| \le C\delta \quad \text{on} \quad \partial_D \Omega, \qquad \Omega := \mathcal{C}_{1/2} \cap \{d_\lambda((x,t), \partial_D \mathcal{C}_{1/2}) > \delta^{1/\alpha}\}.$$

On the other hand, by Proposition 1.14,

$$|\nabla v|, |D^2 v| \le C(\delta) \qquad \text{in} \quad \Omega.$$

Thus, using Corollary 1.12,

$$|v - w| \le C\delta \quad \text{in} \quad \Omega,$$

which gives the desired claim.

Step 2. Applying Proposition 1.13, to the solution v above, we find that

$$|w - l_{\bar{a},\bar{b}}| \le C\rho^{1+\alpha} + C\delta \quad \text{in} \quad \mathcal{C}_\rho,$$

and

$$\bar{b}'(t) = \gamma_\lambda(t) \cdot \bar{a}, \quad |\bar{a}_n'| \le C\rho^{\alpha-1}\lambda^{-1}, \qquad |\bar{a}| \le C,$$

with $\gamma_\lambda(t) = \nabla g(a(\lambda t))$. We choose $\rho = \tau$ small, universal, and

$$\delta = \tau^{1+\frac{\alpha}{2}},$$

so that $\delta \le c'$ the constant from Proposition 1.16, and

$$|w - l_{\bar{a},\bar{b}}| \le \frac{1}{4}\tau \quad \text{in } \mathcal{C}_\tau, \qquad |\bar{a}_n'| \le \frac{1}{4}\delta\, \tau^{-2}\lambda^{-1}.$$

In terms of the original function u, this inequality implies

$$\left| u - \left(l_{a,b} + \epsilon\lambda l_{\bar{a},\bar{b}}\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right)\right)\right| = \epsilon\lambda \left| w\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right) - l_{\bar{a},\bar{b}}\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right)\right| \le \frac{\epsilon}{4}\tau\lambda \quad \text{in} \quad \mathcal{C}_{\tau\lambda}.$$

Set

$$\tilde{a}(t) := a(t) + \epsilon\, \bar{a}\left(\frac{t}{\lambda}\right), \quad \hat{b}(t) := b(t) + \epsilon\lambda\, \bar{b}\left(\frac{t}{\lambda}\right),$$

then

$$|u - l_{\tilde{a},\hat{b}}| \le \frac{\epsilon}{4}\tau\lambda \quad \text{in} \quad \mathcal{C}_{\tau\lambda},$$

and

$$|\tilde{a}_n'| \le \frac{\epsilon\delta}{\lambda^2}\left(1 + \frac{1}{4\tau^2}\right) \le \frac{\epsilon\delta}{2(\tau\lambda)^2}.$$

Finally, we define $\tilde{b}$ by the ODE

$$\tilde{b}' = g(\tilde{a}), \qquad \tilde{b}(0) = \hat{b}(0),$$

and then we have

$$\hat{b}' = b' + \epsilon\bar{b}'\left(\frac{t}{\lambda}\right) = g(a(t)) + \epsilon\nabla g(a(t)) \cdot \bar{a}\left(\frac{t}{\lambda}\right) = g(\tilde{a}(t)) + O(\epsilon^2) = \tilde{b}' + O(\epsilon^2).$$

If $t \in [-\tau\lambda, 0]$ then

$$|(\tilde{b} - \hat{b})(t)| \leq C\epsilon^2 |t| \leq \frac{\epsilon}{4}\tau\lambda,$$

which implies the desired conclusion

$$|u - l_{\tilde{a},\tilde{b}}| \leq \frac{\epsilon}{2}(\lambda\tau) \quad \text{in} \quad \mathcal{C}_{\tau\lambda},$$

and $\tilde{a}, \tilde{b}$ satisfy the required bounds. □

1.7 Harnack inequality

In this section, we prove Theorem 1.15 and Proposition 1.16. The key ingredient is to establish a diminishing of oscillation property. As usual, universal constants depend on n, K.

Proposition 1.17. *Assume that v is a viscosity solution of* (1.39) *and* $0 \leq v \leq 1$ *in $\mathcal{C}_1$. Then*

$$\underset{\mathcal{C}_{1/2}}{\operatorname{osc}}\, v \leq 1 - c,$$

with $c > 0$ universal.

In order to prove Proposition 1.17 we start with a lemma. Let Ω be a smooth domain in $\mathbb{R}^n$, $n \geq 2$, such that

$$\bar{Q}^+_{3/4} \subset \bar{\Omega} \subset \bar{Q}^+_{7/8},$$

and call

$$T := \{x_n = 0\} \cap Q_{3/4} \subset \partial\Omega.$$

Define $\eta(x')$ a standard bump function supported on $Q'_{5/8}$ and equal 1 on $Q'_{1/2}$ (here the prime denotes cubes in $\mathbb{R}^{n-1}$). Let ϕ satisfy (see (1.38) for the definition of the Pucci operator),

$$\mathcal{M}^-_K(D^2\phi) = 0 \quad \text{in } \Omega,$$

$$\phi = 0 \quad \text{on } \partial\Omega \setminus T, \quad \phi = \eta \quad \text{on } T,$$

and notice that $0 \leq \phi \leq 1$, $\phi \geq c$ on $Q^+_{1/2}$, and by Hopf lemma $\phi_n > 0$ on $\{x_n = 0\} \cap \{\phi = 0\}$. The following lemma holds.

Lemma 1.18. *Let $v \geq 0$ satisfy*

$$\begin{cases} \mathcal{M}_K^+(D^2 v) \geq \lambda v_t \geq \mathcal{M}_K^-(D^2 v) & \text{in } \mathcal{C}_1, \\ v_t \geq K^{-1} v_n^+ - K v_n^- - K|\nabla_{x'} v| & \text{on } \mathcal{F}_1, \end{cases} \tag{1.43}$$

in the viscosity sense. If for some $t_0 \in (-1, 0]$,

$$v(x, t_0) \geq s_0 \, \phi(x) \quad \text{in } Q_1^+, \quad s_0 \geq 0,$$

then

$$v(x, t) \geq s(t) \, \phi(x) \quad \text{in } Q_1^+ \times [t_0, 0],$$

with

$$s'(t) = -C_0 s(t), \quad s(t_0) = s_0, \qquad C_0 \text{ large universal.}$$

Moreover, if $s_0 \leq c_0$ with c_0 small universal, and

$$v\left(\frac{1}{2} e_n, t_0 + \lambda/4\right) \geq \frac{1}{2}, \tag{1.44}$$

then

$$v(x, t_0 + \lambda) \geq (s_0 + c_0 \lambda) \phi(x).$$

Proof. For the first part of the claim, since $v \geq 0$, it suffices to show that with our choice of s,

$$w(x, t) := s(t) \phi(x),$$

is a subsolution to (1.43) in $\Omega \times [t_0, 0]$, that is

$$\begin{cases} \lambda w_t \leq \mathcal{M}_K^-(D^2 w) & \text{in } \Omega \times (t_0, 0], \\ w_t \leq K^{-1} w_n^+ - K w_n^- - K|\nabla_{x'} w| & \text{on } \{x_n = 0\} \cap (\Omega \times (t_0, 0]). \end{cases}$$

The interior equation is immediately satisfied since $s' \leq 0$ and $s \geq 0$. On $\{x_n = 0\}$, we need to show that

$$C\phi + K^{-1} \phi_n^+ - K \phi_n^- - K|\nabla_{x'} \phi| \geq 0,$$

for some large C. By Hopf lemma $\phi_n > 0$ on $\{\phi = 0\} \cap \{x_n = 0\}$ and moreover $|\nabla_{x'} \phi| = 0$, thus

$$K^{-1} \phi_n^+ - K \phi_n^- - K|\nabla_{x'} \phi| = K^{-1} \phi_n > 0 \quad \text{on} \quad \{\phi = 0\} \cap \{x_n = 0\}.$$

The same holds in a neighborhood of this set by continuity, and then we can choose C sufficiently large so that the desired inequality holds.

For the second part, denote for simplicity

$$t_i := t_0 + i\frac{\lambda}{4}, \quad i = 1, \dots, 4.$$

We define

$$D := \{x \in \Omega | \quad d(x, \partial\Omega) > c\} \subset \Omega,$$

with c small universal such that there exists a C^2 function $\psi \geq 0$ defined in $\Omega \setminus D$ satisfying

$$\mathcal{M}_K^-(D^2\psi) \geq 4 \quad \text{in} \quad \Omega \setminus D,$$

and

$$\psi = 0, \quad |\nabla\psi| \geq 1 \quad \text{on } \partial\Omega, \quad \psi \leq 1 \quad \text{on } \partial D.$$

An example of such a function is given by $\psi = d + Cd^2$ with C sufficiently large, where d is the distance function to $\partial\Omega$. In view of (1.44)

$$v\left(\frac{1}{2}e_n, t_1\right) \geq 1/2.$$

Thus, we can use Harnack inequality (after rescaling) to conclude that

$$v \geq 2c_1 \quad \text{on } D \times [t_2, t_4], \tag{1.45}$$

for some small c_1. We claim that at time $t = t_3$,

$$v(x, t_3) \geq s(t_3)\phi + c_1\psi \quad \text{in} \quad \Omega \setminus D. \tag{1.46}$$

For this we compare v in $(\Omega \setminus D) \times [t_2, t_3]$ with

$$q(x, t) := s(t_3)\phi + c_1\left(\psi + \frac{t - t_3}{t_3 - t_2}\right).$$

The inequality $q \leq v$ holds on the boundary of the domain. Indeed (recall that s is decreasing), on ∂D

$$q(x, t) \leq s(t_3)\phi + c_1 \leq s_0 + c_1 \leq 2c_1 \leq v,$$

where in the last inequality we used (1.45), and on $\partial\Omega$ or at $t = t_2$ we have $q \le s(t_3)\phi \le v$.

It remains to check that q is a subsolution for the interior equation. Indeed,

$$\lambda q_t = 4c_1 \le c_1\,\mathcal{M}^-_K(D^2\psi) \le \mathcal{M}^-_K(D^2 q),$$

where we used that $\mathcal{M}^-_K(N_1) + \mathcal{M}^-_K(N_2) \le \mathcal{M}^-_K(N_1 + N_2)$, and claim (1.46) is proved.

Next, in the domain $(\Omega \setminus D) \times [t_3, t_4]$ we compare v with the subsolution

$$z(x,t) \coloneqq (s(t_3) + c_2(t - t_3))\phi(x) + c_1\psi(x),$$

with c_2 sufficiently small.

The inequality $v \ge z$ is satisfied at time $t = t_3$ by (1.46), and on ∂D we have

$$z \le s_0 + c_2 + c_1 \le 2c_1 \le v,$$

while on $\partial\Omega \setminus \{x_n = 0\}$ we have $z = 0 \le v$. We check that z is a subsolution of our problem. For the interior inequality we have

$$\lambda z_t = c_2\lambda\phi \le c_2 \le c_1\,\mathcal{M}^-_K(D^2\psi) \le \mathcal{M}^-_K(D^2 z).$$

For the boundary condition, on $\{x_n = 0\}$ we get

$$z_t = c_2\phi \le c_2 \le \frac{c_1}{4}K^{-1}\psi_n, \tag{1.47}$$

where in the second inequality we have used that $\psi_n \ge 1$ on $\partial\Omega \cap \{x_n = 0\}$. Moreover, since $\phi_n \ge -C$ on $\partial\Omega \cap \{x_n = 0\}$, we get (for s_0, c_2 small enough),

$$z_n \ge -\left(s_0 + c_2\frac{\lambda}{4}\right)C + c_1\psi_n \ge \frac{c_1}{2}\psi_n,$$

and lastly ($|\nabla_{x'}\psi| = 0$ on $\{x_n = 0\}$)

$$K|\nabla_{x'}z| \le \left(s_0 + \frac{c_2}{4}\right)K|\nabla_{x'}\phi| \le \frac{c_1}{4}K^{-1}\psi_n.$$

Together with (1.47), this gives

$$z_t = c_2\phi \le c_2 \le K^{-1}z_n - K|\nabla_{x'}z| \quad \text{on } \{x_n = 0\}.$$

In conclusion, at time $t = t_4$ we have $v \ge z$ in $\Omega \setminus D$ and $v \ge 2c_1$ in D which gives the desired claim by choosing c_0 sufficiently small. $\square$

Remark 1.19. In the proof above we only used the subsolution property for v

$$\mathcal{M}_K^+(D^2 v) \geq \lambda v_t, \tag{1.48}$$

in order to extend the inequality (1.44) from one point to (1.45) by applying the interior parabolic Harnack inequality. Alternately, it is sufficient to assume that the Harnack inequality holds for v only in a neighborhood of D and not necessarily up to $\{x_n = 0\}$.

The rest of the proof is based on comparing v with the explicit C^2 subsolutions w, q and z which all have bounded second derivatives in the x variable. Thus the hypothesis that v is a viscosity supersolution of (1.43) can be slightly relaxed, and require instead, that v only satisfies the comparison principle with respect to the explicit barriers above.

Remark 1.20. The hypothesis (1.48) can be removed completely if instead of (1.44) we assume a measure estimate

$$\left|\left\{v \geq \frac{1}{4}\right\} \cap \left(Q_1 \times \left[t_0, t_0 + \frac{\lambda}{4}\right]\right)\right| \geq \frac{1}{2}\left|Q_1 \times \left[t_0, t_0 + \frac{\lambda}{4}\right]\right|.$$

Then, the inequality (1.45) follows directly from the supersolution property for v and the weak Harnack inequality (see for example [79]).

We are now ready to prove Proposition 1.17.

Proof of Proposition 1.17. Assume that $0 \leq v \leq 1$, and for half of the values of

$$t_k := -1 + k\lambda, \qquad \text{so that} \quad t_k \in [-1, -1/2), \quad k = 0, 1, 2, \ldots,$$

we have

$$v\left(\frac{1}{2}e_n, t_k + \lambda/4\right) \geq \frac{1}{2}. \tag{1.49}$$

We apply Lemma 1.18 repeatedly to the sequence of times t_k and obtain

$$v(x, t_k) \geq s_k \phi, \quad s_k := s(t_k), \qquad s_0 = 0,$$

with ϕ given in Lemma 1.18, and

$$s_{k+1} \geq s_k + c_0\lambda \qquad \text{if (1.49) holds and } s_k \leq c_0,$$

or

$$s_{k+1} \geq s_k(1 - C_0\lambda) \quad \text{otherwise.}$$

Now it follows that $s_k \geq c_1$ for the last value of k so that $t_k < -1/2$, for c_1 appropriately chosen depending on c_0, C_0. Then we apply the first part of Lemma 1.18 to obtain

$$v(x,t) \geq \bar{c}\phi \quad \text{for all } t \geq -1/2,$$

which gives the desired conclusion, since $\phi > c$ on $Q^+_{1/2}$. □

The same arguments show that a similar statement to that of Proposition 1.17 holds for a solution w of (1.33) defined in (1.32). Below is the key lemma which connects the linear and nonlinear problem and allows us to reduce our analysis mostly to the linear case.

Lemma 1.21. *Assume that u satisfies the hypotheses of Proposition* 1.9 *and let w be defined as in* (1.32), *with $-1 \leq w \leq 1$. Then*

$$\operatorname*{osc}_{\mathcal{C}_{1/2}} w \leq 2(1-c),$$

with c universal, provided that $\delta \leq c'$ and $\epsilon \leq \epsilon_1(\delta)$.

Proof. We may assume as above that $w(e_n/2, t_k + \lambda/4) \geq 0$ for more than half the values of k, and then show that w separates from the lower constraint -1. For this we apply the same argument as above to the function

$$\bar{w} \coloneqq w + 1 + C\delta(2 + t - x_n^2) \geq 0,$$

for which the relaxed hypotheses of Remark 1.19 hold. Indeed, by (1.36), $\bar{w}$ satisfies the required Harnack inequality (1.44) $\Longrightarrow$ (1.45) and, by Proposition 1.11, it satisfies the comparison with the explicit barriers of Lemma 1.18.

We remark that we have only used that u has property $H(c'')$ in $\mathcal{C}_\lambda$ for some c'' small, universal. □

Before we proceed with the proofs of Theorem 1.15 and Proposition 1.16 we provide a boundary version of the diminishing of oscillation Proposition 1.17.

Lemma 1.22. *Assume that U is a space-time domain obtained by the intersection of $n+1$ half spaces in the $x_1, \dots, x_{n-1}, x_n$ and t variables,*

$$U := (-\infty, z_1) \times (-\infty, z_2) \times \cdots \times (-\infty, z_n) \times (-z_{n+1}, \infty) \quad \subset \mathbb{R}^{n+1},$$

with $z_i \in [0, 1]$.

Assume that $v \geq 0$ satisfies

$$\begin{cases} \lambda v_t \geq \mathcal{M}_K^-(D^2 v) & \text{in } \mathcal{C}_1 \cap U, \\ v_t \geq K^{-1} v_n^+ - K v_n^- - K|\nabla_{x'} v| & \text{on } \mathcal{F}_1 \cap U, \\ v \geq \frac{1}{4} & \text{on } \partial U \cap \mathcal{C}_1. \end{cases} \tag{1.50}$$

If $\min z_i \leq \frac{7}{8}$, then

$$v \geq c \quad \text{in } \mathcal{C}_{1/2} \cap U, \qquad c \text{ universal.}$$

Proof. This follows easily from Lemma 1.18. Indeed, we work with the truncation $\tilde{v} := \min\{v, \frac{1}{4}\}$ extended by $\frac{1}{4}$ in $\mathcal{C}_1 \setminus U$. Then $\tilde{v}$ is a supersolution for our problem in $\mathcal{C}_1$.

If $z_{n+1} < 1$, then we can apply directly the first part of Lemma 1.18 for $\tilde{v}$ for some t_0 close to -1 and for s_0 universal, and obtain the desired conclusion.

On the other hand, if $z_{n+1} = 1$, then $z_i \leq \frac{7}{8}$ for some $i \leq n$ hence for each time $t \in [-1, 0]$ we find

$$\left| \left\{ \tilde{v} \geq \frac{1}{4} \right\} \cap Q_1 \right| \geq c|Q_1|.$$

Now the conclusion follows as before, see Remark 1.19. □

We are now ready to prove Theorem 1.15.

Proof of Theorem 1.15. Notice that the rescaling of v

$$v_r(x,t) = v(r\,x, r\,t), \qquad r \le 1,$$

satisfies again the hypotheses of Theorem 1.15 in $\mathcal{C}_1$ with the constant λ replaced by $\lambda_r = \lambda r$. Proposition 1.17 applied to v_r implies that

$$\operatorname*{osc}_{\mathcal{C}_{1/2}} v_r \le (1-c) \operatorname*{osc}_{\mathcal{C}_1} v_r$$

which gives (recall that $\mathcal{B}_{\lambda,r}(y,s) = \mathcal{B}_r(y,s)$ if $y_n = 0$),

$$\operatorname*{osc}_{\mathcal{B}_{r/2}(0,0)} v \le (1-c) \operatorname*{osc}_{\mathcal{B}_r(0,0)} v.$$

Similarly, if $(y,s) \in \overline{\mathcal{C}}_{1/2} \cap \{x_n = 0\}$, then by considering cylinders centered at (y,s) we obtain

$$\operatorname*{osc}_{\mathcal{B}_{r/2}(y,s)} v \le (1-c) \operatorname*{osc}_{\mathcal{B}_r(y,s)} v, \qquad \forall r \le 1/2, \tag{1.51}$$

which proves the desired oscillation decay on $\{x_n = 0\} \cap \overline{\mathcal{C}}_{1/2}$.

If $(y,s) \in \mathcal{C}_{1/2}$, then (1.51) applied at $((y',0),s)$ implies

$$\operatorname*{osc}_{\mathcal{B}_{\lambda,r/8}(y,s)} v \le (1-c) \operatorname*{osc}_{\mathcal{B}_{\lambda,r}(y,s)} v, \qquad \text{if} \quad y_n \le r \le 1/4.$$

In the case when $r < y_n$, then the inequality above follows from the standard parabolic Harnack inequality applied to v in the interior cylinder $\mathcal{B}_{\lambda,r}(y,s)$.

The boundary version follows in the same way. Precisely, if $(y,s) \in \overline{\mathcal{C}}_1 \cap \{x_n = 0\}$ then we find

$$\operatorname*{osc}_{\mathcal{B}_{r/2}(y,s)\cap\overline{\mathcal{C}}_1} v \le (1-c) \operatorname*{osc}_{\mathcal{B}_r(y,s)\cap\overline{\mathcal{C}}_1} v, \qquad \forall r \le 1,$$

by applying either Proposition 1.17 or Lemma 1.22 depending whether or not $\mathcal{B}_{\lambda,r}(y,s)$ intersects the boundary $\partial_D \mathcal{C}_1$.

The inequality above can be deduced at all points $(y,s) \in \overline{\mathcal{C}}_1$ after replacing $r/2$ by $r/8$ on the left hand side. Indeed, if $r \ge y_n$ then it follows from the inequality above applied at the point $((y',0),s)$, and if $r < y_n$ then we can apply the standard parabolic Harnack inequality or its boundary version since $\mathcal{B}_{\lambda,r}(y,s)$ does not intersect $\{x_n = 0\}$. $\square$

We conclude the section with the proof of Proposition 1.16, that is the Harnack inequality for w.

Proof of Proposition 1.16. By Lemma 1.21 we find that, in terms of u, we satisfy again the hypotheses of Proposition 1.9 in $\mathcal{C}_{\lambda/2}$ with λ replaced by $\lambda/2$, ϵ replaced by $2(1-c)\epsilon$, and with δ the same. The function a stays the same while b is modified by a small constant. Moreover, the property $H(\epsilon^{1/2})$ of u in $\mathcal{C}_\lambda$ implies that u satisfies property $H(2\epsilon^{1/2})$ in $\mathcal{C}_{\lambda/2}$. We can iterate this result k times as long as the scale parameter of the property $H(2^k\epsilon^{1/2})$ remains small, universal, and the hypotheses of Lemma 1.21 hold:

$$2^k\epsilon^{1/2} \le c'', \qquad \delta \le c', \qquad 2^k(1-c)^k\epsilon \le \epsilon_1(\delta),$$

with c'' small, universal. This means that we can iterate k times if

$$2^k\epsilon^{1/2} \le \epsilon_2(\delta), \qquad \delta \le c'.$$

In terms of w, we obtain that its oscillation in $\mathcal{C}_{2^{-k}}$ is bounded by $2(1-c)^k$ as long as k satisfies the inequality above. On the other hand for the interior balls $\mathcal{B}_{\lambda,r}$, by (1.36), w satisfies a similar diminishing of oscillation up to scale $r \sim \epsilon^{1/2}$, and the conclusion follows. □

1.8 Proof of Proposition 1.13

In this section, we prove Proposition 1.13 by using Theorem 1.15 and the estimates for the one-dimensional problem which will be proved in Lemma 1.23 of the next section. The constants C in this proof depend on n and K.

Proof of Proposition 1.13. The proof is divided in four steps.

Step 1 - Interior Estimates. Let $(y,s) \in \mathcal{C}_{1/2}$. From Theorem 1.15 we know that

$$\operatorname*{osc}_{\mathcal{B}_r(y,s)} v \le Cr^\alpha, \qquad r = y_n.$$

The rescaling

$$\tilde{v}(x,t) := v(y + rx, s + r^2\lambda t),$$

solves in $Q_1 \times (-1,0)$

$$\tilde{v}_t = tr(\tilde{A}(t)D^2\tilde{v}), \qquad \tilde{A}(t) := A(s + r^2\lambda t).$$

Since $|A'| \le \lambda^{-1}$, we have $|\tilde{A}'(t)| \le C$, and we find by interior estimates that $|\tilde{v}_n(0,0)| \le C \ \operatorname{osc}_{Q_1\times(-1,0)} \ \tilde{v}$, from which we deduce

$$|v_n(y,s)| \le Cr^{\alpha-1} = Cy_n^{\alpha-1}.$$

On the other hand, we prove in appendix that the difference of two viscosity solutions is still a viscosity solution. Thus, the estimates for v can be extended to the derivatives of v in the x_i directions, $i = 1, \dots, n-1$. Indeed, by applying the interior Hölder estimates to discrete differences in the x_i directions, and iterating this we find that

$$\|D^k_{x'}v\| \le C(k) \quad \text{in } \mathcal{C}_{1/2}, \quad \forall k \ge 1.$$

In particular, using also the estimate for v_n above, we obtain

$$\|D^2_{x'}v\| \le C, \qquad |v_{in}| \le Cx_n^{\alpha-1} \quad \text{in } \mathcal{C}_{1/2}.$$

Step 2 - Reduction to 1D. Combining the interior estimates with our assumptions on γ, we obtain that when we restrict v to a two-dimensional space in which we freeze the x' variable, say for simplicity $x' = 0$, then the function $v((0,x_n),t)$ solves in the x_n, t variables the equation

$$\begin{cases} v_t = \frac{1}{\lambda}\{a^{nn}(t)v_{nn} + h(x_n,t)\} & \text{in } \mathcal{C}_1, \\ v_t = \gamma_n(t)v_n + f(t) & \text{on } \mathcal{F}_1, \end{cases} \tag{1.52}$$

with

$$|h| \le Cx_n^{\alpha-1}, \quad |f(t)| \le C,$$

$$h(x_n,t) := \sum_{(i,j)\ne(n,n)} a^{ij}(t)\ v_{ij}((0,x_n),t), \qquad f(t) := \sum_{i<n} \gamma_i(t)\ v_i(0,t).$$

The boundary condition on $\mathcal{F}_1$ is understood in the viscosity sense.

Indeed, if a C^1 function $\varphi(x_n, t)$ touches $v(0, x_n, t)$ by above/below, say at $(0, 0)$, in $\mathcal{B}_r(0, 0) \subset \mathbb{R}^2$, then

$$\varphi(x_n, t) + \sum_{i<n} v_i(0, 0)x_i \pm C|(x, t)|^{1+\alpha}$$

touches v by above/below at the origin in $\mathcal{B}_r(0, 0) \subset \mathbb{R}^{n+1}$. This follows from the C^α continuity of v_i, $i < n$, which implies

$$\left| v(x, t) - \left(v(0, x_n, t) + \sum_{i<n} v_i(0, 0)x_i \right) \right| \leq C|(x, t)|^{1+\alpha}. \tag{1.53}$$

Now, we can use Lemma 1.23 a) for $v(0, x_n, t)$, where we establish $C^{1,\alpha}$ estimates for the 1D problem (1.52). We obtain

$$|v((0, x_n), t) - v(0, t) - v_n(0, t)x_n| \leq C x_n^{1+\alpha},$$

which together with (1.53) gives

$$\left| v - \left(v(0, t) + v_n(0, t)x_n + \sum_{i=1}^{n-1} v_i(0, 0)x_i \right) \right| \leq C\rho^{1+\alpha} \quad \text{in } \mathcal{C}_\rho.$$

This means that

$$|v - l_{a,b}| \leq C\rho^{1+\alpha} \quad \text{in } \mathcal{C}_\rho,$$

with

$$a(t) := (v_1(0, 0), \ldots, v_{n-1}(0, 0), v_n(0, t)), \quad b(t) := v(0, t),$$

and

$$b' = \gamma_n(t)a_n + f(t) = \gamma(t) \cdot a + \sum_{i<n} \gamma_i(t)(v_i(0, t) - v_i(0, 0)).$$

Step 3 - Modifying the linear approximation. Next, we modify a and b slightly into $\bar{a}$, $\bar{b}$ so that

$$|v - l_{\bar{a},\bar{b}}| \leq C\rho^{1+\alpha} \quad \text{in } \mathcal{C}_\rho,$$

and we also satisfy

$$|\bar{a}'(t)| \leq C\lambda^{-1}\rho^{\alpha-2}, \qquad \bar{b}' = \gamma(t) \cdot \bar{a}. \tag{1.54}$$

By Lemma 1.23 we know that

$$|a_n(t) - a_n(s)| \leq C\lambda^{-\frac{\alpha}{2}}|t-s|^{\frac{\alpha}{2}}, \tag{1.55}$$

and by the Hölder continuity of the v_i's,

$$|b' - \gamma(t)\cdot a| \leq \sum_{i<n} |\gamma_i||v_i(0,t) - v_i(0,0)| \leq C|t|^\alpha. \tag{1.56}$$

Thus, a_n oscillates $C\rho^\alpha$ in an interval of length $\lambda\rho^2$. We define $\bar{a}$ by averaging a over intervals of this length. More precisely, let η be a standard mollifier in $\mathbb{R}$ with compact support in $[-1,1]$, and η_τ denote its rescaling with support of size τ. We extend $a_n(t)$ to be constant for $t \geq 0$ and define

$$\bar{a}_n := a_n * \eta_{\lambda\rho^2}, \qquad \bar{a}_i := a_i, \quad i = 1, \ldots, n-1.$$

Then (1.55) implies the inequality (1.54) for $\bar{a}'$ and also

$$|a - \bar{a}| \leq C\rho^\alpha. \tag{1.57}$$

We define $\bar{b}(t)$ for $t \leq 0$ as

$$\bar{b}' = \gamma(t)\cdot\bar{a}, \qquad \bar{b}(0) = b(0).$$

Then, (1.56), (1.57) imply

$$|(\bar{b} - b)'| \leq C\rho^\alpha \quad \Longrightarrow \quad |\bar{b} - b| \leq C\rho^{1+\alpha} \quad \text{in } [-\rho, 0],$$

and the desired conclusion follows.

Step 4 - Conclusion. The tangential derivatives v_i, with $i < n$, satisfy the same estimates as v. We find from Step 2 applied to v_i that the mixed derivatives v_{in} must be bounded by a universal bound. This improves the initial estimate in Step 1, which in turn improves the regularity of f and h in Step 2. More precisely, by Lemma 1.23 we find that v_{in} satisfies the estimate (1.59). This holds also for the tangential derivatives of order up to 2. Then the functions $h(x,t)$ and $f(t)$ in (1.52) satisfy the hypotheses of part b) of Lemma 1.23. This gives that the remaining second derivative v_{nn} is bounded as well, and (1.55) holds for $\alpha + 1$ instead of α. Thus we can replace α by $\alpha + 1$ in the bound (1.54) above, and the proposition is proved. □

1.9 Estimates for the 1D case

In this section, we provide the necessary estimates for solutions to the 1D linear problem. The difference with the higher dimensional case is that now, in the 1D case, the Hölder estimates and the subsequent $C^{1,\alpha}$ and $C^{2,\alpha}$ estimates can be iterated in parabolic cylinders

$$\mathcal{P}_\rho := (0,\rho) \times (-\rho^2, 0],$$

and we can use the standard Hölder parabolic norms with respect to the standard parabolic distance: $d((x,t),(y,s)) := |x-y| + |t-s|^{1/2}$. Following Krylov [60], we denote the corresponding Hölder spaces with respect to this distance with $C^{k,\alpha}_{x,t}$.

Precisely, we prove the following.

Lemma 1.23 (1D-Estimates)**.** *Assume that $\lambda \le 1$ and $w(x,t)$ is a viscosity solution in $\mathcal{C}_1 \subset \mathbb{R}^2$ of the equation*

$$\begin{cases} w_t = \frac{1}{\lambda}\{A(t) w_{xx} + h(x,t)\} & \text{in } \mathcal{C}_1, \\ w_t = \gamma(t)\, w_x + f(t) & \text{on } \mathcal{F}_1, \end{cases} \tag{1.58}$$

with

$$\|w\|_{L^\infty} \le 1, \quad K^{-1} \le A(t),\, \gamma(t) \le K, \qquad |A'(t)| \le K\lambda^{-1}.$$

a) If

$$|h| \le K x^{\alpha-1}, \quad |f(t)| \le K,$$

then $w \in C^{1,\alpha}$ in the x variable, $w \in C^1$ on $\{x = 0\}$, and the free boundary condition is satisfied in the classical sense. More precisely, in $\mathcal{C}_{1/2}$ we have

$$|w(x,t) - (w(0,t) + x w_x(0,t))| \le C x^{1+\alpha}, \qquad |w_x| \le C,$$

and

$$|w(y,t) - w(z,s)| \le C\left(|y-z|^\alpha + \lambda^{-\frac{\alpha}{2}}|t-s|^{\frac{\alpha}{2}}\right),$$
$$|w_x(y,t) - w_x(z,s)| \le C\left(|y-z|^\alpha + \lambda^{-\frac{\alpha}{2}}|t-s|^{\frac{\alpha}{2}}\right), \tag{1.59}$$

with C depending only on K and α.

b) If in addition in $\mathcal{C}_{3/4}$

$$|h(y,t) - h(z,s)| \le K\left(|y-z|^\alpha + \lambda^{-\frac{\alpha}{2}}|t-s|^{\frac{\alpha}{2}}\right),$$

$$|\gamma(t) - \gamma(s)| \le K\lambda^{-\frac{\alpha}{2}}|t-s|^{\frac{\alpha}{2}}, \qquad |f(t) - f(s)| \le K\lambda^{-\frac{\alpha}{2}}|t-s|^{\frac{\alpha}{2}},$$

then in $\mathcal{C}_{1/2}$

$$|w_x(0,t) - w_x(0,s)| \le C\lambda^{-\frac{1+\alpha}{2}}|t-s|^{\frac{1+\alpha}{2}}, \qquad |w_{xx}| \le C. \tag{1.60}$$

After subtracting $F(t) := \int_0^t f(s)ds$ from w and replacing h by $h - \lambda f(t)$ we may assume that $f \equiv 0$. We work with $v(x,t) = w(x,\lambda t)$, and after relabeling λt by t in the arguments of A and h, we obtain

$$\begin{cases} v_t = A(t)v_{xx} + h(x,t) & \text{in } (0,1)\times(-\lambda^{-1},0], \\ v_t = \lambda\gamma(t)\, v_x & \text{on } \{x=0\}, \end{cases} \tag{1.61}$$

with

$$K^{-1} \le A(t),\, \gamma(t) \le K, \qquad |A'(t)| \le K, \quad |h| \le Kx^{\alpha-1}. \tag{1.62}$$

Lemma 1.23 is equivalent to the Lemma 1.24 below, where we establish the corresponding estimates for v using parabolic scaling.

Lemma 1.24. *Assume that v is a viscosity solution of* (1.61) *in $\mathcal{P}_1$ with $\lambda \le 1$, and coefficients that satisfy* (1.62)*. Then*

$$\|v\|_{C^{1,\alpha}_{x,t}(\mathcal{P}_{1/2})} \le C(\|v\|_{L^\infty(\mathcal{P}_1)} + 1), \tag{1.63}$$

and the free boundary condition is satisfied in the classical sense. If in addition

$$\|h\|_{C^{0,\alpha}_{x,t}}, \quad \|\gamma\|_{C_t^{\frac{\alpha}{2}}} \le K,$$

then

$$\|v\|_{C^{2,\alpha}_{x,t}(\mathcal{P}_{1/2})} \le C(\|v\|_{L^\infty(\mathcal{P}_1)} + 1),$$

with C depending only on n, K and α.

Proof. If v solves (1.61) in $\mathcal{P}_\rho$ then the rescaling

$$\tilde{v}(x,t) := \rho^{-\beta} v(\rho x, \rho^2 t)$$

solves (1.61) in $\mathcal{P}_1$ with coefficients

$$\tilde{A}(t) = A(\rho^2 t), \quad \tilde{h}(x,t) = \rho^{2-\beta} h(\rho x, \rho^2 t), \quad \tilde{\lambda} = \rho\lambda, \quad \tilde{\gamma}(t) = \gamma(\rho^2 t). \quad (1.64)$$

Notice that the hypotheses on the coefficients are preserved as long as $\beta \leq 1+\alpha$, and moreover $\tilde{\lambda} \to 0$ as $\rho \to 0$.

We divide the proof in four steps.

Step 1: Hölder estimates. We show that

$$\|v\|_{C^{0,\beta}_{x,t}(\mathcal{P}_{1/2})} \leq C\left(\|v\|_{L^\infty(\mathcal{P}_1)} + 1\right),$$

for some $\beta > 0$ small.

Notice that after an initial dilation, we may assume that $\lambda \leq \lambda_0$ is small. It suffices to prove the following claim.

If v is a viscosity solution of (1.61) then

$$\operatorname*{osc}_{\mathcal{P}_1} v \leq 2 \quad \Longrightarrow \quad \operatorname*{osc}_{\mathcal{P}_\rho} v \leq \frac{3}{2}, \qquad \text{with } \rho = c_0 \text{ small, universal.} \quad (1.65)$$

The Hölder estimate is obtained by iterating this claim in parabolic cylinders centered on the t axis, while for the interior parabolic cylinders (included in $\{x > 0\}$) we can apply directly the diminishing of oscillation for parabolic equations.

In order to prove (1.65), we let $g(x,t)$ be the solution to the 1D heat equation on the real-line

$$g_t = K^{-1} g_{xx}, \quad g(x,0) = \chi_{(0,\infty)} - \chi_{(-\infty,0)}. \quad (1.66)$$

Notice that for all $t > 0$, in $x = 0$ we have

$$g(0,t) = 0, \qquad g_x(0,t) \leq C t^{-1/2},$$

and

$$g_t \leq 0, \quad \text{for } x > 0.$$

We want to show that if $|v| \leq 1$ in $\mathcal{P}_1$, then we can improve the upper bound or lower bound by a fixed amount in the interior, depending on the value of v at $(0,-1)$, i.e.

$$|v| \leq 1 \text{ in } \mathcal{P}_1 \text{ and } v(0,-1) \leq 0, \text{ then } v \leq 1/2 \text{ in } \mathcal{P}_\rho, \text{ with } \rho = c_0.$$

In $\mathcal{P}_1$ we compare v with

$$G(x,t) := C_1 g(x,t+1) + \frac{1}{4}(t+1)^{1/2} - C_2 x^{1+\alpha}.$$

We choose C_2 and then C_1 sufficiently large such that G is a classical supersolution to (1.61) and $G \geq 1$ on the boundary $(0,1] \times \{-1\}$ and $\{1\} \times [-1,0]$, while $G(0,0) = 1/4$. Then we find $v \leq G$ in $\mathcal{P}_1$, which gives the claim (1.65) by choosing c_0 sufficiently small.

Step 2: $C^{1,\alpha}$ estimates. We show that (1.63) holds by first establishing a pointwise $C^{1,\alpha}$ estimate at the origin.

After an initial dilation and after dividing by a large constant, we may assume that $\lambda \leq \delta$, $|h| \leq \delta x^{\alpha-1}$ for some small δ, and $\|v\|_{L^\infty(\mathcal{P}_1)}$ is sufficiently small.

Claim. If a function l_0 (linear in x) of the form

$$l_0 = a_0 x + b_0(t), \qquad b_0' = \lambda\gamma(t) a_0, \qquad |a_0| \leq 1, \tag{1.67}$$

approximates v in $\mathcal{P}_\rho$ to order $1+\alpha$, i.e.

$$|v - l_0| \leq \rho^{1+\alpha} \qquad \text{in } \mathcal{P}_\rho, \qquad \rho \leq \delta,$$

then we can approximate v to order $1+\alpha$ in $\mathcal{P}_{c_1\rho}$ by a function l_1 as above, with $|a_1 - a_0| \leq C\rho^\alpha$, and c_1 small universal. Then the claim can be iterated indefinitely by starting with $l_0 \equiv 0$ in $\mathcal{P}_\delta$.

We prove the claim by compactness. Notice that $v - l_0$ solves (1.61) with a slightly modified h that satisfies $|h| \leq \delta x^{\alpha-1} + C\delta$. This means that the rescaled error

$$\tilde{v}(x,t) := \rho^{-(1+\alpha)}(v - l_0)(\rho x, \rho^2 t),$$

satisfies (1.61) with coefficients as in (1.64). Since $\|\tilde{v}\|_{L^\infty} \leq 1$, by Step 1 we know that

$$\|\tilde{v}\|_{C^{0,\beta}_{x,t}(\mathcal{P}_{1/2})} \leq C.$$

This means that if we consider a sequence of $\delta_n \to 0$ and corresponding solutions v_n in $\mathcal{P}_{\rho_n}$, then we can extract a uniformly convergence subsequence of the rescalings $\tilde{v}_n$ in $\mathcal{P}_{1/2}$ such that

$$\tilde{v}_n \to \bar{v}.$$

Then the Hölder continuous limit function $\bar{v}$ is a viscosity solution of

$$\begin{cases} \bar{v}_t = \bar{A}\,\bar{v}_{xx} & \text{in } \mathcal{P}_{1/2}, \\ \bar{v}_t = 0 & \text{on } \{x = 0\}, \end{cases}$$

with $\bar{A}$ constant. Since $\bar{v}$ is constant on the boundary $\{x = 0\}$, the C^2 estimate for the standard heat equation implies

$$|\bar{v} - (\bar{a}x + \bar{b})| \leq C\tau^2 \leq \frac{1}{2}\tau^{1+\alpha} \qquad \text{in} \quad \mathcal{P}_\tau, \quad \tau \leq c_1.$$

This shows that if δ is chosen sufficiently small, then the rescaling $\tilde{v}$ satisfies the inequality above instead of $\bar{v}$ which implies

$$|v - (a_1 x + b(t))| \leq \frac{3}{4}(\tau\rho)^{1+\alpha} \qquad \text{in} \quad \mathcal{P}_{\tau\rho}, \quad \tau = c_1,$$

with

$$a_1 = a_0 + \rho^\alpha \bar{a}, \quad b(t) = b_0(t) + \rho^{1+\alpha}\bar{b}.$$

We define $b_1(t)$ so that l_1 has the form as in (1.67), that is

$$b_1'(t) = \lambda\gamma(t)a_1, \quad b_1(0) = b(0).$$

Then

$$|(b_1 - b)'| \leq C|\bar{a}\rho^\alpha| \leq C\rho^\alpha \quad \Longrightarrow \quad |b_1 - b| \leq C\rho^\alpha(\tau\rho)^2 \leq \frac{1}{4}(\tau\rho)^{1+\alpha} \quad \text{in} \quad \mathcal{P}_{\tau\rho},$$

where we used $\rho \leq \delta$ sufficiently small. In conclusion,

$$|v - l_1| \leq (\tau\rho)^{1+\alpha} \quad \text{in} \quad \mathcal{P}_{\tau\rho}, \qquad l_1 = a_1 x + b_1(t),$$

and the claim is proved.

We remark that the oscillation of $b_0(t)$ which appears in the approximation function l_0 in (1.67) is less than $C\rho^2$ in $\mathcal{P}_\rho$. Thus we can modify b_0 to be constant in (1.67) and take l_0 to be linear, and then adjust the error $\rho^{1+\alpha}$ by $C\rho^{1+\alpha}$. This pointwise $C^{1,\alpha}$ estimate can be applied at other points on $\{x = 0\}$, which combined with interior $C^{1,\alpha}$ estimates for parabolic equations implies the desired conclusion (1.63).

Step 3. Boundary regularity. We check that v is C^1 on $\{x = 0\}$ and the boundary condition is satisfied in the classical sense.

For this assume by contradiction that there exists a sequence $t_k \to 0^-$ such that

$$\frac{1}{t_k}(v(0,t_k) - v(0,0)) < \mu := \lambda\gamma(0)(v_x(0,0) - \eta), \quad \text{for some } \eta > 0. \quad (1.68)$$

For each k, we look at the contact point where the graph of v is touched by below by a translation of the graph of the classical strict subsolution to (1.61)

$$g(x,t) := v(0,0) + \mu t + x\left(v_x(0,0) - \frac{1}{2}\eta\right) + Cx^{1+\alpha},$$

in the domain $D_k := [0, c(\eta)] \times [t_k, 0]$.

We choose $c(\eta)$ small such that $g_x(x,t) < v_x(x,t)$ in the domain D_k for all large k. This implies that the contact point must occur on $D_k \cap \{x = 0\}$. On the other hand, (1.68) gives

$$v(0,t_k) - v(0,0) > g(0,t_k) - g(0,0)$$

which shows that the contact point is different than $(0, t_k)$ and we reach a contradiction.

Step 4. $C^{2,\alpha}$ estimates. On $\{x = 0\}$ we know that $v_x, \gamma \in C^{\alpha/2}$, and the boundary condition implies that $v(0,t) \in C^{1,\alpha/2}$. Now we can apply the standard $C^{2,\alpha}$ Schauder estimates up to the boundary for the heat equation.

□

1.10 Viscosity solutions for the linear problem

In this section, we collect some general facts about viscosity solutions for the linear problem (1.37) and establish the existence and uniqueness claim in Proposition 1.14 by Perron's method. Similar results for different types of boundary conditions were established by G. Lieberman (see for example [64]). However, we are not aware of an existence result that applies directly to the linear problem (1.37). Therefore, for completeness we provide the details in this case.

Recall that $v \in C(\mathcal{C}_1)$ satisfies

$$\begin{cases} \lambda v_t \leq tr(A(t)D^2 v) & \text{in } \mathcal{C}_1, \\ v_t \leq \gamma(t) \cdot \nabla v & \text{on } \mathcal{F}_1, \end{cases} \tag{1.69}$$

in the viscosity sense if v cannot be touched by above at any point $(x_0, t_0) \in \mathcal{C}_1 \cup \mathcal{F}_1$ in a small neighborhood $\mathcal{B}_r(x_0, t_0)$ by a classical strict supersolution $w \in C^2(\overline{\mathcal{B}_r(x_0, t_0)})$. As usually, this definition is equivalent to the one where we restrict w to belong to the class of quadratic polynomials rather than to the class of C^2 functions.

Another equivalent way is to say that v is a viscosity subsolution of the parabolic equation in $\mathcal{C}_1$, and a viscosity subsolution of the boundary condition on $\mathcal{F}_1$. This last condition means that we cannot touch v locally by above at any point $(x_0, t_0) \in \mathcal{F}_1$ by a function $w \in C^1(\overline{\mathcal{B}_r(x_0, t_0)})$ (or say w is a linear function) that satisfies

$$w_t(x_0, t_0) > \gamma(t_0) \cdot \nabla w(x_0, t_0).$$

The two definitions are the same since, if $w \in C^1$ is as above, and say $(x_0, t_0) = (0, 0)$, then a vertical translation of the quadratic polynomial

$$w(0) + (w_t(0) - \epsilon)t + (\nabla w(0) + \epsilon e_n) \cdot x + M(|x'|^2 - nK^2 x_n^2),$$

must touch v by above at some interior point $(x, t) \in \mathcal{B}_r$. Here r is chosen sufficiently small and M large, appropriately, and then the polynomial is a strict supersolution in $\mathcal{B}_r$.

We state the comparison principle for viscosity solutions.

Lemma 1.25. *Assume v_1 is a viscosity subsolution, and v_2 a viscosity supersolution to* (1.37) *in $\overline{\mathcal{C}_1}$. If $v_1 \le v_2$ on $\partial_D \mathcal{C}_1$ then $v_1 \le v_2$ in $\mathcal{C}_1$.*

Corollary 1.26. *The difference of two viscosity solutions of* (1.37) *is also a viscosity solution of* (1.37).

We work with the rescaling $w(x,t) = v(x, \lambda t)$.

First we prove a preliminary result on the evolution in time of a Lipschitz "trace" $w((x',0),t)$ under specific growth assumptions.

Lemma 1.27. *Assume that $w \le 1$ satisfies*

$$\begin{cases} w_t \le \mathcal{M}^+_K(D^2 w) + 1 & \text{in } (Q_1 \cap \{x_n > 0\}) \times (0,T], \\ \frac{1}{\lambda} w_t \le K w_n^+ - K^{-1} w_n^- + K|\nabla_{x'} w| & \text{on } \{x_n = 0\}, \end{cases} \tag{1.70}$$

and

$$w((x',0),0) \le |x'|^2.$$

Then

$$w(0,t) \le C\lambda(t^{1/2} + t) \quad \text{for } t \ge 0,$$

with C depending on n and K.

Proof. We compare w with

$$G(x,t) := g(x_n,t) + C\lambda(t^{1/2}+t) + |x'|^2 + C(2x_n - x_n^2),$$

where $g(x_n,t)$ is the solution to the 1D heat equation on the real-line (see (1.66))

$$g_t = K^{-1} g_{nn}, \quad g(x_n, 0) = \chi_{(0,\infty)} - \chi_{(-\infty,0)}.$$

It is easy to check that G is a classical supersolution which is above w on the boundary of our domain, and that gives the desired result. □

Lemma 1.28. *Assume that* $w \le 1$ *satisfies* (1.70) *in* $\mathcal{C}_1$ *and the trace of* w *on* $\{x_n = 0\}$ *is Lipschitz, i.e.*

$$|\nabla_{x'} w| \le 1 \qquad on\ \{x_n = 0\}.$$

Then

$$w((x',0),t) \ge w((x',0),0) - C\lambda^{\frac{2}{3}}\,|t|^{\frac{1}{2}} \qquad if\ x' \in Q'_{1/2}.$$

Proof. We prove the inequality for $x' = 0$. Since w is Lipschitz the parabola

$$w(0,t) + Cr^2 + r^{-2}|x'|^2$$

is greater than $w((x',0),t)$, with r to be specified later. Now we can apply the previous lemma to the rescaling

$$\tilde{w}(y,s) := w(ry, t + r^2 s) - w(0,t) - Cr^2,$$

which solves (1.70) with $\tilde{\lambda} = \lambda r$, and obtain that

$$\tilde{w}(0,s) \le C\tilde{\lambda}(s^{1/2} + s).$$

This gives

$$C(r^2 + \lambda|t|^{\frac{1}{2}} + \lambda r^{-1}|t|) \ge w(0,0) - w(0,t),$$

and we choose $r = (\lambda|t|)^{1/3}$ to get

$$w(0,t) \ge w(0,0) - C(\lambda|t|^{\frac{1}{2}} + (\lambda|t|)^{2/3}) \ge w(0,0) - C\lambda^{\frac{2}{3}}\,|t|^{\frac{1}{2}}.$$

□

Remark 1.29. The proof of Lemma 1.28 shows that we can construct a supersolution $\bar{G}(x,t)$ in $\mathcal{C}_1$ such that $\bar{G}((x',0),-1) = |x'|$, $\bar{G} \ge 1$ on the remaining part of $\partial_D \mathcal{C}_1$, and so that $\bar{G}(0,t) \le C\lambda^{\frac{2}{3}}\,|t|^{\frac{1}{2}}$. Similarly, given $\alpha > 0$, we can construct a supersolution with $\bar{G}((x',0),-1) = |x'|^\alpha$, $\bar{G} \ge 1$ on the remaining of $\partial_D \mathcal{C}_1$ and such that $\bar{G}(0,t) \le C(\lambda|t|)^\beta$, for some β depending on α.

We are now ready to prove our main lemma.

Proof of Lemma 1.25. Let $w_i(x,t) = v_i(x, \lambda t)$, $i = 1, 2$, so that w_1 is a subsolution and w_2 a supersolution of

$$\begin{cases} w_t = tr(A(t)D^2 w) & \text{in } \{x_n > 0\}, \\ \frac{1}{\lambda} w_t = \gamma(t) \cdot \nabla w & \text{on } \{x_n = 0\}, \end{cases}$$

and we want to show that w_1 cannot touch w_2 strictly by below at an interior point. Assume by contradiction that this is the case.

The standard viscosity theory of parabolic equations implies that the contact point cannot occur in $\{x_n > 0\}$. Below we denote by C, c various constants that may depend on w_i and λ.

After a translation and a dilation we may assume that in $\mathcal{C}_1$

$$w_1 \leq w_2 + \mu t, \qquad w_1(0,0) = w_2(0,0) = 0,$$

for some $\mu > 0$ small. Without loss of generality we may also assume that w_1/w_2 has a semiconvex/semiconcave trace in the x' variable, that is

$$D^2_{x'} w_1 \geq -I, \qquad D^2_{x'} w_2 \leq I, \tag{1.71}$$

and also

$$\|w_i\|_{L^\infty} \leq 1 \tag{1.72}$$

and each w_i solves the parabolic equation in the interior. This is achieved in the following way. First we replace a subsolution w with the standard regularization using the sup-convolutions in the x' variable

$$w_\epsilon(x,t) = \max_y \left\{ w(y,t) - \frac{1}{2\epsilon}|y' - x'|^2 \right\},$$

then we divide w_ϵ by a large constant, and in the end we solve the parabolic equation in the interior of $\mathcal{C}_1$ by keeping the same boundary values on the parabolic boundary. All these operations maintain the subsolution property of w, and justify the extra assumptions (1.71)-(1.72).

Moreover, after subtracting from each w_i a function of the type $a' \cdot x' + b(t)$ with $\frac{d}{dt}b(t) = \lambda a' \cdot \gamma(t)$ we may assume in addition that

$$w_i(0,0) = 0, \quad \nabla_{x'} w_i(0,0) = 0, \tag{1.73}$$

and the interior parabolic equations have the form

$$\partial_t w_i = tr(A(t)D^2 w_i) + h(t), \qquad |h| \le C.$$

We show that $w_i(0,t)$ are differentiable at the origin in the t variable, and that the derivative of w_1 is less than the derivative of w_2, which would contradict our hypothesis that $w_1 \le w_2 + \mu t$.

To achieve this we apply Lemma 1.28 several times. By (1.71)-(1.72)-(1.73) and Lemma 1.28 we find that

$$w_1 \ge -Cr \quad \text{and} \quad w_2 \le Cr \qquad \text{on} \quad \mathcal{P}_r \cap \{x_n = 0\}. \tag{1.74}$$

Since $w_1 \le w_2$, we can use the pointwise C^α parabolic estimates at the origin and find that, given any $\alpha < 1$, we have

$$\operatorname*{osc}_{\mathcal{P}_r} w_i \le Cr^\alpha \quad \text{for all } r > 0. \tag{1.75}$$

We can iterate this argument, by working with the rescaling

$$\tilde{w}_1(x,t) = r^{-\alpha} w_1(rx, r^2 t),$$

which satisfies a similar equation with $\tilde{\lambda} = \lambda r$, and is such that (1.71)-(1.72) -(1.73) hold for $\tilde{w}_1$. Again by Lemma 1.28 we find

$$\tilde{w}_1((x',0),t) \ge -Cr^{2/3} \qquad \text{if } x' \in Q'_{1/2},$$

hence we improve the estimate (1.74) as

$$w_1 \ge -Cr^{\alpha+\frac{2}{3}} \quad \text{on} \quad \mathcal{P}_r \cap \{x_n = 0\}. \tag{1.76}$$

The same holds for w_2 with $\le$ instead of $\ge$ and $Cr^{\alpha+\frac{2}{3}}$ instead of $-Cr^{\alpha+\frac{2}{3}}$. This in turn shows that w_i are pointwise $C^{\alpha+\frac{2}{3}}$ at the origin.

We modify again each w_i by subtracting the corresponding function $\partial_n w_i(0)x_n + b_i(t)$, with $\frac{d}{dt}b_i = \lambda\gamma_n\partial_n w_i(0)$. Using that $\partial_n(w_1 - w_2)(0) \le 0$, we find that the inequality $w_1 \le w_2 + \mu t$ is still valid on $\{x_n = 0\}$, while (1.75) holds with $r^{\alpha+2/3}$ instead of r^α. The same argument as above implies that (1.76) holds again with $r^{\alpha+4/3}$ instead of $r^{\alpha+2/3}$. Since $\alpha + 4/3 > 2$, this means that $w_1(0,t) \ge -C|t|^{1+\beta}$ and $w_2(0,t) \le C|t|^{1+\beta}$ for all small $t < 0$, which contradicts $w_1(0,t) \le w_2(0,t) + \mu t$. □

We can lastly conclude the proof of Proposition 1.14.

Proof of Proposition 1.14. The interior C^2 estimates in the x variable and the Hölder estimates up to the boundary were already proved in Proposition 1.13 and Theorem 1.15. It remains to prove existence by Perron's method.

We assume for simplicity that the boundary data ϕ is Lipschitz, and the general case follows by approximation. As usual, we define

$$v(x,t) := \sup_{w\in\mathcal{A}} w(x,t),$$

where $\mathcal{A}$ is the class of continuous subsolutions on $\overline{\mathcal{C}}_1$ which have boundary data below ϕ on $\partial_D\mathcal{C}_1$. The conclusion that v solves our problem is easily checked once its continuity has been established.

Claim. For each $(x_0,t_0) \in \partial_D\mathcal{C}_1$ there exists a subsolution $w_{(x_0,t_0)}$ which vanishes at (x_0,t_0), is below the cone $-|(x,t)-(x_0,t_0)|$ on $\partial_D\mathcal{C}_1$ and has a Hölder modulus of continuity at (x_0,t_0).

This can be deduced from the proof of Theorem 1.15, where the Hölder continuity at the boundary was achieved using explicit barriers. More precisely, as in Lemma 1.18 and Lemma 1.22, for all $r \le 1/2$ we can construct a subsolution ϕ_r defined in $\mathcal{B}^\pm_{\lambda,r}(x_0,t_0) \cap \mathcal{C}_1$, where

$$\mathcal{B}^\pm_{\lambda,r}(x_0,t_0) := \{(x,t)| \quad d_\lambda((x,t),(x_0,t_0)) < r\},$$

so that

$$\phi_r = 0 \quad \text{on} \quad \partial\mathcal{B}^\pm_{\lambda,r}(x_0,t_0)\setminus(\partial_D\mathcal{C}_1\cup\mathcal{F}_1), \qquad \phi_r \le 1 \quad \text{on} \quad \partial\mathcal{B}^\pm_{\lambda,r}(x_0,t_0)\cap\partial_D\mathcal{C}_1$$

and

$$\phi_r \geq c_0 \quad \text{on} \quad \partial\mathcal{B}^{\pm}_{\lambda,r/2}(x_0, t_0).$$

Then $w_{(x_0,t_0)}$ is obtained by superposing appropriate multiples of ϕ_r for a dyadic sequence of $r = 2^{-m}$. We omit the details.

Using the claim we can construct a subsolution $\underline{\phi}$ and supersolution $\overline{\phi}$ which are Hölder continuous on $\partial_D\mathcal{C}_1$ and agree with the boundary data ϕ. Thus we can restrict the class $\mathcal{A}$ of subsolutions to satisfy

$$\underline{\phi} \leq w \leq \overline{\phi}. \tag{1.77}$$

This shows that the limit v achieves the boundary data ϕ continuously. Moreover, using (1.77) we can replace each $w \in \mathcal{A}$ by its maximum among appropriate x' translations

$$\max_{y'} \{w(x - (y', 0), t) - C|y'|^{\alpha}\},$$

and remain in the same class. Therefore we may assume that $\mathcal{A}$ contains only subsolutions which are uniformly Hölder continuous in the x' variable. Using this together with Remark 1.29, we find that the trace of v on $\{x_n = 0\}$ is locally Hölder continuous in the x', t variables. This means that the solution $\bar{v}$ to the interior parabolic equation in $\mathcal{C}_1$ with boundary data v is continuous up to the boundary. By the maximum principle $\bar{v} \geq w$ for any $w \in \mathcal{A}$, and it is straightforward to check that $\bar{v} \in \mathcal{A}$, hence $v = \bar{v}$ is continuous in $\overline{\mathcal{C}}_1$. □

Chapter 2

An Alt-Caffarelli-Friedman monotonicity formula in the Heisenberg group

In this chapter, we investigate the question of the existence of an Alt -Caffarelli-Friedman monotonicity formula in the Heisenberg group, on which I have worked together with my advisor Fausto Ferrari. Specifically, Ferrari and I have written the papers “A new glance to the Alt-Caffarelli-Friedman monotonicity formula” [38] and “Some remarks about the existence of an Alt-Caffarelli-Friedman monotonicity formula in the Heisenberg group” [39]. The first one is a review of the classical Alt-Caffarelli-Friedman monotonicity formula, with exactly a look to our result in the Heisenberg group, while the second one contains all the details of our work.

2.1 The classical Alt-Caffarelli-Friedman monotonicity formula and its role in the study of free boundary problems

In this section, we recall the classical Alt-Caffarelli-Friedman monotonicity formula and its importance in the study of free boundary problems.
The Alt-Caffarelli-Friedman monotonicity formula was introduced in [2] as a fundamental tool for studying the main properties of the solutions of two-phase free boundary problems.
Looking into [2], the result roughly says that there exists $r_0 > 0$ such that for every nonnegative $u_1, u_2 \in C(B_1) \cap H^1(B_1)$, if $0 \in F(u_i)$, $\Delta u_i \geq 0$, $i = 1, 2$, and $u_1 u_2 = 0$ in B_1, then

$$\Phi(r) := r^{-4} \int_{B_r} \frac{|\nabla u_1|^2}{|x|^{n-2}} dx \int_{B_r} \frac{|\nabla u_2|^2}{|x|^{n-2}} dx \tag{2.1}$$

is well defined, bounded and monotone increasing in $(0, r_0)$. In [2] the authors used this result for proving the Lipschitz continuity of critical points of a functional like

$$\mathcal{E}_{\lambda_+,\lambda_-}(v) := \int_\Omega \left(|\nabla v|^2 + \lambda_+ \chi_{\{v>0\}} + \lambda_- \chi_{\{v<0\}}\right) dx,$$

with $\lambda_+ - \lambda_- \neq 0$, defined on a set $K \subset H^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a given bounded open set and K is determined by some known conditions on v given on $\partial\Omega$.
In particular, the critical points of the functional $\mathcal{E}_{\lambda_+,\lambda_-}$, with $\lambda_+ = 1$ and $\lambda_- = 0$ satisfy the two-phase free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega : \ u(x) > 0\}, \\ \Delta u = 0 & \text{in } \Omega^-(u) := \mathrm{Int}(\{x \in \Omega : \ u(x) \leq 0\}), \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 1 & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{2.2}$$

see [2], thus the Lipschitz continuity of critical points of $\mathcal{E}_{1,0}$ transfers to solutions of (2.2). More precisely, solutions of (2.2) satisfy, at least in a "weak" sense, see [15] for a more general viscosity meaning, the following property: for every $P \in F(u)$

$$(u_\nu^+(P))^2(u_\nu^-(P))^2 = \lim_{r\to 0^+} \Phi(r) \leq C,$$

where ν, formally, denotes the unit normal vector to $F(u)$ at the points belonging to $F(u)$, with the convention that ν is pointing inside the set $\Omega^+(u)$ for u^+ as well as inside $\Omega^-(u)$ for u^-. Hence, if one of the two phases, let say u^-, is sufficiently regular at $P \in F(u)$, then by Hopf maximum principle (see [54]) it results $u_\nu^-(P) > 0$ so that, as a by-product, $u_\nu^+(P)$ has to be bounded. In this way, the solutions of (2.2) are globally Lipschitz.

As said before, the Alt-Caffarelli-Friedman monotonicity formula turned out to be a key tool in the comprehension of free boundary problems. Indeed, after [2] many other important papers on this topic were written. We quote some of them, without attempting to cite all the literature. Precisely, in [12] it was proved that the monotonicity formula holds for linear uniformly elliptic operators in divergence form with Hölder continuous coefficients, while in [14] a formula for nonhomogeneous free boundary problems was discovered. More recently, in [75] the Riemannian case has been treated and in [69] the nondivergence form case has been faced. In addition, some very partial results have been obtained even in the nonlinear case, in lower dimension, see [32] for the $p-$Laplacian case. Furthermore, this formula has become increasingly popular for other applications as well. Among these, we recall further ones to two-phase problems, see [11] for the elliptic homogeneous case, [6] and [42] for the parabolic homogeneous setting, and [27] for elliptic linear nonhomogeneous problems. Lastly, we mention some segregation problems, see for instance [70], [72], [77] and [76].

In view of such extensions of the classical Alt-Caffarelli-Friedman monotonicity formula, the goal of Ferrari and myself has been to investigate if a formula of this type can hold in the framework of the Heisenberg group. Concerning other similar formulas about sublaplacians, we find in the literature some important contributions, see [51] and in particular [53], where the authors deal with the Almgren frequency function in Carnot groups. Moreover, we quote

[22] and [21] for further papers in noncommutative settings about other free boundary problems, precisely the obstacle problem.

2.2 Positioning of the problem in the Heisenberg group

In this section, we set the problem of the existence of an Alt-Caffarelli -Friedman monotonicity formula in the context of the Heisenberg group. Precisely, it holds, by applying the definition of solution in the sense of the domain variation to the functional

$$\mathcal{E}_{\mathbb{H}^n}(v) := \int_{\Omega} \left(|\nabla_{\mathbb{H}^n} v|^2 + \chi_{\{v>0\}} \right) dx,$$

$\Omega \subset \mathbb{H}^n$, that the parallel two-phase problem to (2.2) is

$$\begin{cases} \Delta_{\mathbb{H}^n} u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega : \ u(x) > 0\}, \\ \Delta_{\mathbb{H}^n} u = 0 & \text{in } \Omega^-(u) := \mathrm{Int}(\{x \in \Omega : \ u(x) \leq 0\}), \\ |\nabla_{\mathbb{H}^n} u^+|^2 - |\nabla_{\mathbb{H}^n} u^-|^2 = 1 & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{2.3}$$

see [34]. We remark that, in this particular noncommutative context, the gradient jump $|\nabla u^+|^2 - |\nabla u^-|^2 = 1$ is governed by the jump of the horizontal gradient $\nabla_{\mathbb{H}^n}$. As a first consequence, in this degenerate case associated with the sublaplacian $\Delta_{\mathbb{H}^n}$, a new geometric problem, that in the Euclidean two-phase problem (2.2) does not exist, appears. As a matter of fact, since classical smooth free boundaries of (2.3), in principle, might have characteristic points, the jump of the horizontal gradient of a solution u to (2.3) on $F(u)$ could not be satisfied pointwise, because the horizontal gradient vanishes on characteristic points, see Section 2.4. Moreover, it has been already proved, see [43, Section 3], that every minimum u of the functional $\mathcal{E}_{\mathbb{H}^n}$ is endowed by a locally bounded horizontal gradient and satisfies $\Delta_{\mathbb{H}^n} u = 0$ in $\Omega^+(u)$, as well as $\Delta_{\mathbb{H}^n} u = 0$ in $\Omega^-(u)$, even if no words have been spent about the behavior of the free boundary of these minima. Specifically, here it is shown an alternative way of proving that a minimum of the functional $\mathcal{E}_{\mathbb{H}^n}$ is intrinsically Lipschitz, instead of using the monotonicity formula. Thus, it seems natural to consider, as a candidate for an Alt-Caffarelli -Friedman monotonicity formula in the Heisenberg group, the function

$$J_{\beta,\mathbb{H}^n}(r) := r^{-\beta} \int_{B_r^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_1|^2}{|\xi|_{\mathbb{H}^n}^{Q-2}} d\xi \int_{B_r^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_2|^2}{|\xi|_{\mathbb{H}^n}^{Q-2}} d\xi, \tag{2.4}$$

where $\beta > 0$ is a suitable fixed exponent and $0 \in F(u_i)$, $i = 1, 2$. This function is indeed the natural one correspondent to (2.1) in the Heisenberg group $\mathbb{H}^n$, derived substituting Euclidean balls with Koranyi balls, the Euclidean gradient with the horizontal one, the Euclidean dimension with the homogeneous one and recalling that the fundamental solution for the Kohn-Laplace operator $\Delta_{\mathbb{H}^n}$ is, up to a constant, $|\xi|_{\mathbb{H}_n}^{2-Q}$, see Section 2.4 for these notions in $\mathbb{H}^n$.
Following the main steps of the Euclidean proof, recalled in Section 2.3, in [39] the main result below is proved.

Theorem 2.1. *If there exists a positive number β for which $J_{\beta,\mathbb{H}^1}$ is monotone increasing in $(0, r_0)$, $r_0 > 0$, for every nonnegative $u_1, u_2 \in C(B_1^{\mathbb{H}^1}(0)) \cap H^1_{\mathbb{H}^1}(B_1^{\mathbb{H}^1}(0))$, such that $\Delta_{\mathbb{H}^1} u_i \geq 0$, $0 \in F(u_i)$, $i = 1, 2$ and $u_1 u_2 = 0$, then $\beta \leq 4$.*

We point out that such result is stated in the first Heisenberg group only. This is due to the fact that a monotonicity formula for all the Heisenberg groups is not proved, but simply that if such a formula holds in $\mathbb{H}^1$, then the right exponent β has to be smaller or equal than 4. Nevertheless, even if the proof in higher Heisenberg groups would require more efforts, it may be obtained following the same ideas. On the other hand, the breakthrough that we would need to conclude that, at least in $\mathbb{H}^1$, the sharp exponent β has to be exactly 4 depends on a long standing open question. As a matter of fact, the profile of the set that realizes the equality in the isoperimetric inequality in the Heisenberg group (and as a byproduct the descendant Polya-Szëgo inequality on the surface of the Koranyi ball of radius one) is still open, see for example [17] for an introduction to this problem. We shall discuss better this topic in Section 2.14. Concerning the same problem in the Euclidean setting, of course, the question is well understood. However, for the sake of completeness, in Section 2.3 we recall the steps useful to prove the classical Alt-Caffarelli-Friedman monotonicity formula and, in particular, we treat a little bit this subject. In the remainder of the chapter, we expose the details of [39].

2.3 The Euclidean setting

In this section, following the original papers [2] and [15], we recall the main steps needed for proving the Alt-Caffarelli-Friedman monotonicity formula in the Euclidean setting.

First, after a straightforward differentiation, it results by (2.1)

$$\Phi'(r) = I_1(r)I_2(r)r^{-5}\left(-4 + r\left(\frac{I_1'}{I_1} + \frac{I_2'}{I_2}\right)\right), \tag{2.5}$$

where

$$I_i(r) := \int_{B_r} \frac{|\nabla u_i|^2}{|x|^{n-2}} dx, \quad i = 1, 2.$$

By a rescaling argument the problem may be reduced to

$$\Phi'(r) = I_1(r)I_2(r)r^{-5}\left(-4 + \frac{\displaystyle\int_{\partial B_1} |\nabla u_1|^2 d\sigma}{\displaystyle\int_{B_1} \frac{|\nabla u_1|^2}{|x|^{n-2}} dx} + \frac{\displaystyle\int_{\partial B_1} |\nabla u_2|^2 d\sigma}{\displaystyle\int_{B_1} \frac{|\nabla u_2|^2}{|x|^{n-2}} dx}\right). \tag{2.6}$$

Precisely, we have

$$I_i(r) = \int_{B_1} \frac{|\nabla u_i(ry)|^2}{|ry|^{n-2}}\, r^n\, dy = r^2 \int_{B_1} \frac{|\nabla u_i(ry)|^2}{|y|^{n-2}}\, dy,$$

and

$$I_i(r) = \int_0^r \left(\int_{\partial B_\rho(0)} \frac{|\nabla u_i(x)|^2}{|x|^{n-2}}\, d\sigma(x)\right) d\rho = \int_0^r \left(\int_{\partial B_1} \frac{|\nabla u_i(\rho y)|^2}{\rho^{n-2}} \right.$$
$$\left. \rho^{n-1} d\sigma(y)\right) d\rho = \int_0^r \rho \left(\int_{\partial B_1} |\nabla u_i(\rho y)|^2\, d\sigma(y)\right) d\rho,$$

where here y denotes the coordinates on ∂B_1. Thus, we get

$$\frac{I_i'}{I_i} = \frac{\dfrac{d}{dr}\displaystyle\int_0^r \rho \left(\int_{\partial B_1} |\nabla u_i(\rho y)|^2 \, d\sigma(y)\right) d\rho}{r^2 \displaystyle\int_{B_1} \frac{|\nabla u_i(ry)|^2}{|y|^{n-2}} \, dy} = \frac{r \displaystyle\int_{\partial B_1} |\nabla u_i(ry)|^2 \, d\sigma(y)}{r^2 \displaystyle\int_{B_1} \frac{|\nabla u_i(ry)|^2}{|y|^{n-2}} \, dy}$$

$$= \frac{1}{r} \frac{\displaystyle\int_{\partial B_1} |\nabla u_i(ry)|^2 \, d\sigma(y)}{\displaystyle\int_{B_1} \frac{|\nabla u_i(ry)|^2}{|y|^{n-2}} \, dy},$$

which implies, if we define

$$(u_i)_r(x) := \frac{u_i(rx)}{r}, \quad x \in B_1,$$

that

$$\frac{I_i'}{I_i} = \frac{1}{r} \frac{\displaystyle\int_{\partial B_1} |\nabla (u_i)_r(y)|^2 \, d\sigma(y)}{\displaystyle\int_{B_1} \frac{|\nabla (u_i)_r(y)|^2}{|y|^{n-2}} \, dy},$$

where $(u_i)_r$ is defined in B_1. As a consequence, if we write $y = x$ and $(u_i)_r = u_i$ the last equality gives

$$r \frac{I_i'}{I_i} = \frac{\displaystyle\int_{\partial B_1} |\nabla u_i(x)|^2 \, d\sigma(x)}{\displaystyle\int_{B_1} \frac{|\nabla u_i(x)|^2}{|y|^{n-2}} \, dx},$$

and so (2.5) becomes (2.6).

Now, if

$$-4 + \frac{\displaystyle\int_{\partial B_1} |\nabla u_1|^2 d\sigma}{\displaystyle\int_{B_1} \frac{|\nabla u_1|^2}{|x|^{n-2}} dx} + \frac{\displaystyle\int_{\partial B_1} |\nabla u_2|^2 d\sigma}{\displaystyle\int_{B_1} \frac{|\nabla u_2|^2}{|x|^{n-2}} dx} \geq 0$$

then, from (2.6), $\Phi'(r) \geq 0$. Hence, in order to prove that previous inequality holds, the following ratios

$$J_i(r) := \frac{\displaystyle\int_{\partial B_1} |\nabla u_i|^2 d\sigma}{\displaystyle\int_{B_1} \frac{|\nabla u_i|^2}{|x|^{n-2}} dx},$$

for $i = 1, 2$, have to be estimated.

Since the gradient may split in two orthogonal parts involving the radial part and the tangential part, respectively denoted by $\nabla^\rho u_i$ and $\nabla^\theta u_i$, it holds

$$|\nabla u_i|^2 = |\nabla^\rho u_i|^2 + |\nabla^\theta u_i|^2.$$

Then, we can rewrite J_i as

$$J_i(r) = \frac{\displaystyle\int_{\partial B_1} \left(|\nabla^\rho u_i|^2 + |\nabla^\theta u_i|^2\right) d\sigma}{\displaystyle\int_{B_1} \frac{|\nabla u_i|^2}{|x|^{n-2}} dx}. \tag{2.7}$$

At this point, we estimate the numerator and denominator of (2.7) separately. About the numerator, we define first

$$\lambda(\Gamma_i) := \inf_{v \in H_0^1(\Gamma_i)} \frac{\displaystyle\int_{\Gamma_i} |\nabla^\theta v|^2 d\sigma}{\displaystyle\int_{\Gamma_i} v^2 d\sigma},$$

where

$$\Gamma_i := \{x \in \partial B_1 : \ u_i(x) > 0\}$$

and $\lambda(\Gamma_i)$, $i = 1, 2$, is the Rayleigh quotient. By definition of $\lambda(\Gamma_i)$, we thus obtain, for every $\beta_i \in (0, 1)$,

$$\begin{aligned}\int_{\partial B_1} \left|\nabla^\theta u_i\right|^2 d\sigma = \int_{\Gamma_i} \left|\nabla^\theta u_i\right|^2 d\sigma \geq \lambda(\Gamma_i) \int_{\Gamma_i} u_i^2 d\sigma = (1 - \beta_i \\ + \beta_i)\lambda(\Gamma_i) \int_{\Gamma_i} u_i^2 d\sigma = \beta_i \lambda(\Gamma_i) \int_{\Gamma_i} u_i^2 d\sigma + (1 - \beta_i)\lambda(\Gamma_i) \int_{\Gamma_i} u_i^2 d\sigma,\end{aligned}$$

hence, by Cauchy inequality, we have

$$\begin{aligned}&\int_{\partial B_1} \left(|\nabla^\rho u_i|^2 + |\nabla^\theta u_i|^2\right) d\sigma \geq \int_{\Gamma_i} |\nabla^\rho u_i|^2 d\sigma + \beta_i \lambda(\Gamma_i) \int_{\Gamma_i} u_i^2 d\sigma \\ &+ (1 - \beta_i)\lambda(\Gamma_i) \int_{\Gamma_i} u_i^2 d\sigma \geq 2 \left(\int_{\Gamma_i} |\nabla^\rho u_i|^2 d\sigma\right)^{1/2} \\ &\left(\beta_i \lambda(\Gamma_i) \int_{\Gamma_i} u_i^2 d\sigma\right)^{1/2} + (1 - \beta_i)\lambda(\Gamma_i) \int_{\Gamma_i} u_i^2 d\sigma.\end{aligned} \tag{2.8}$$

Concerning the denominator, instead, we compute

$$\Delta(u_i^2) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(2u_i \frac{\partial u_i}{\partial x_j}\right) = 2\left(|\nabla u_i|^2 + u_i \Delta u_i\right) \geq 2\,|\nabla u_i|^2,$$

since $u_i \Delta u_i \geq 0$ by the assumptions on u_i.

Consequently, we achieve the following estimate

$$\int_{B_1} \frac{|\nabla u_i|^2}{|x|^{n-2}} dx \leq \left(\int_{\Gamma_i} |\nabla^\rho u_i|^2 \, d\sigma\right)^{\frac{1}{2}} \left(\int_{\Gamma_i} u_i^2 \, d\sigma\right)^{\frac{1}{2}} + \frac{n-2}{2} \int_{\Gamma_i} u_i^2 \, d\sigma. \tag{2.9}$$

Precisely, the previous inequality follows after two integration by parts, using the facts that $|x|^{2-n}$ is, up to a multiplicative constant, the fundamental solution of Δ, and $0 \in F(u_i)$, $i = 1, 2$, and by Hölder inequality, because

$$\begin{aligned}
&\int_{B_1} \frac{|\nabla u_i|^2}{|x|^{n-2}} dx \leq \frac{1}{2} \int_{B_1} \Delta\left(u_i^2\right) |x|^{2-n} dx = \frac{1}{2} \int_{B_1} \operatorname{div}\left(|x|^{2-n} \nabla u_i^2\right) dx \\
&- \int_{B_1} \nabla |x|^{2-n} \cdot \nabla u_i^2 dx = \frac{1}{2}\Bigg(\int_{\partial B_1} |x|^{2-n} \nabla u_i^2 \cdot \frac{x}{|x|} d\sigma - \int_{B_1} \operatorname{div}(u_i^2 \nabla |x|^{2-n}) dx \\
&+ \int_{B_1} u_i^2 \Delta(|x|^{2-n}) dx \Bigg) = \frac{1}{2}\left(\int_{\Gamma_i} 2u_i \nabla^\rho u_i \, d\sigma + (n-2) \int_{\Gamma_i} u_i^2 \, |x|^{1-n} \, d\sigma \right) \\
&\leq \left(\int_{\Gamma_i} |\nabla^\rho u_i|^2 \, d\sigma\right)^{\frac{1}{2}} \left(\int_{\Gamma_i} u_i^2 \, d\sigma\right)^{\frac{1}{2}} + \frac{n-2}{2} \int_{\Gamma_i} u_i^2 \, d\sigma.
\end{aligned}$$

Now, putting together (2.8) and (2.9), we get, in view of (2.7),

$$J_i(r) \geq \frac{2(\beta_i \lambda(\Gamma_i))^{\frac{1}{2}} \xi_i \eta_i + (1-\beta_i)\lambda(\Gamma_i)\eta_i^2}{\xi_i \eta_i + \frac{n-2}{2}\eta_i^2},$$

setting $\xi_i = \left(\int_{\Gamma_i} |\nabla^\rho u_i(x)|^2 \, d\sigma\right)^{\frac{1}{2}}$ and $\eta_i = \left(\int_{\Gamma_i} u_i^2(x) \, d\sigma\right)^{\frac{1}{2}}$, which entails

$$\begin{aligned}
J_i(r) &\geq \frac{2(\beta_i \lambda(\Gamma_i))^{\frac{1}{2}} \xi_i \eta_i + (1-\beta_i)\lambda(\Gamma_i)\eta_i^2}{\xi_i \eta_i + \frac{n-2}{2}\eta_i^2} = \frac{2(\beta_i \lambda(\Gamma_i))^{\frac{1}{2}} + (1-\beta_i)\lambda(\Gamma_i)\frac{\eta_i}{\xi_i}}{1 + \frac{n-2}{2}\frac{\eta_i}{\xi_i}} \\
&\geq \inf_{z \geq 0} \frac{2(\beta_i \lambda(\Gamma_i))^{\frac{1}{2}} + (1-\beta_i)\lambda(\Gamma_i) z}{1 + \frac{n-2}{2} z} = 2 \min\left\{ \frac{\lambda(\Gamma_i)}{n-2}(1-\beta_i), (\beta_i \lambda(\Gamma_i))^{\frac{1}{2}} \right\}.
\end{aligned}$$

The last equality easily follows by elementary arguments.

Now, if it were possible to choose $\beta_i \in (0,1)$ in such a way that

$$\frac{\lambda(\Gamma_i)}{n-2}(1-\beta_i) = (\beta_i \lambda(\Gamma_i))^{\frac{1}{2}}$$

we would realize, by denoting $\alpha_i := (\beta_i \lambda(\Gamma_i))^{\frac{1}{2}}$, that previous equation is satisfied if and only if

$$\alpha_i^2 + (n-2)\alpha_i - \lambda(\Gamma_i) = 0.$$

On the other hand, since a function $u = \rho^\alpha g(\theta)$, $\theta := (\theta_1, \ldots, \theta_{n-1})$, is harmonic in a cone determined by a domain Γ whenever

$$\rho^{\alpha-2}\left((\alpha(\alpha-1) + \alpha(n-1))g(\theta) + \Delta_\theta g\right) = 0,$$

we deduce that there exists α_i such that

$$\alpha_i(\alpha_i - 1) + \alpha_i(n-1) = \lambda(\Gamma_i),$$

namely

$$\alpha_i^2 + (n-2)\alpha_i - \lambda(\Gamma_i) = 0.$$

By the structure of the equation, it immediately comes out that there always exists a strictly positive solution $\alpha_i = \alpha_i(\Gamma_i)$, which is called the characteristic constant of Γ_i.

Therefore, we have to prove the existence of $\beta_i \in (0,1)$ such that

$$\frac{-(n-2) + \sqrt{(n-2)^2 + 4\lambda(\Gamma_i)}}{2} = (\beta_i \lambda(\Gamma_i))^{\frac{1}{2}}. \tag{2.10}$$

Specifically, (2.10) is equivalent to solve

$$\frac{4\lambda(\Gamma_i)}{(n-2) + \sqrt{(n-2)^2 + 4\lambda(\Gamma_i)}} = 2(\beta_i \lambda(\Gamma_i))^{\frac{1}{2}},$$

that is

$$\frac{2\lambda(\Gamma_i)^{\frac{1}{2}}}{(n-2) + \sqrt{(n-2)^2 + 4\lambda(\Gamma_i)}} = \beta_i^{\frac{1}{2}}.$$

Since the continuous positive function defined in $[0, +\infty)$ as

$$z \to \frac{z}{(n-2) + \sqrt{(n-2)^2 + z^2}}$$

is strictly increasing, $\left(\frac{z}{(n-2)+\sqrt{(n-2)^2+z^2}}\right)(0)=0$, and $\sup_{[0,+\infty)}\frac{z}{(n-2)+\sqrt{(n-2)^2+z^2}} = 1$, we conclude that for every $\lambda(\Gamma_i)>0$, there exists β_i such that (2.10) holds. In particular, we get

$$\beta_i=\left(\frac{2\lambda(\Gamma_i)^{\frac{1}{2}}}{(n-2)+\sqrt{(n-2)^2+4\lambda(\Gamma_i)}}\right)^2.$$

Hence, with the previous choice of β_i, if we denote

$$\alpha_i:=\min\left\{\frac{\lambda(\Gamma_i)}{n-2}(1-\beta_i),(\beta_i\lambda(\Gamma_i))^{\frac{1}{2}}\right\},$$

which is also the exponent corresponding to the eigenvalue given by the Rayleigh quotient $\lambda(\Gamma_i)$, we conclude that, whenever

$$\alpha_1+\alpha_2\geq 2, \tag{2.11}$$

then $\Phi'\geq 0$.

So, for completing this proof, we would need to know that (2.11) holds.

To this end, by [74] we know that $\alpha_i(\Gamma_i)\geq\alpha_i(\Gamma_i^*)$, where $\Gamma_i^*\subset\partial B_1$ is a spherical cap, namely a set of the form

$$\Gamma_i^*=\partial B_1\cap\{x_n>s\}, \quad -1<s<1,$$

such that $\mathcal{H}^{n-1}(\Gamma_i)=\mathcal{H}^{n-1}(\Gamma_i^*)$. Here $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure on ∂B_1.

Precisely, it is shown in [74] that if $u\in C^\infty(\partial B_1,\mathbb{R})$, then

$$\begin{cases}\int\limits_{\partial B_1}\|\nabla u^*\|^p\,d\mathcal{H}^{n-1}\leq\int\limits_{\partial B_1}\|\nabla u\|^p\,d\mathcal{H}^{n-1} \quad 1\leq p<\infty,\\ \|\nabla u^*\|_{L^\infty(\partial B_1)}\leq\|\nabla u\|_{L^\infty(\partial B_1)},\end{cases} \tag{2.12}$$

where u^* is the function associated to u so that $u^*\in C^0(\partial B_1,\mathbb{R})$ is Lipschitz, depends only on the latitude of the argument, and its image measure on the Borel sets of $\mathbb{R}$ coincides with that of u. More technically speaking, there exists a monotone decreasing, continuous function g from $[0;\pi]$ to $\mathbb{R}$ such that $u^*(x) = g(\arccos(\langle x, x_0\rangle))$ for $x \in \partial B_1$, with $x_0 \in \partial B_1$ fixed, and $u_\#(\mathcal{H}^{n-1})|_{\mathcal{B}(\mathbb{R})} = u^*_\#(\mathcal{H}^{n-1})|_{\mathcal{B}(\mathbb{R})}$, where $u_\#(\mathcal{H}^{n-1})$ and $u^*_\#(\mathcal{H}^{n-1})$ denote the pushforward measures of u and u^* respectively. This last condition then entails

$$\int_{\partial B_1} \phi \circ u \, d\mathcal{H}^{n-1} = \int_{\partial B_1} \phi \circ u^* \, d\mathcal{H}^{n-1}, \tag{2.13}$$

for any function $\phi : \mathbb{R} \to \mathbb{R}$ μ^*-measurable, where μ^* is the outer measure defined on the power set $\mathbb{P}(\mathbb{R})$ of $\mathbb{R}$ as

$$\mu^*(F) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathcal{B}(\mathbb{R}), F \subset \bigcup_{i=1}^{\infty} A_i \right\},$$

with $\mu = u_\#(\mathcal{H}^{n-1})|_{\mathcal{B}(\mathbb{R})} = u^*_\#(\mathcal{H}^{n-1})|_{\mathcal{B}(\mathbb{R})}$ and $F \in \mathbb{P}(\mathbb{R})$. Hence, choosing $\phi = x^2$ in (2.13), we obtain

$$\int_{\partial B_1} u^2 \, d\mathcal{H}^{n-1} = \int_{\partial B_1} (u^*)^2 \, d\mathcal{H}^{n-1},$$

which gives, together with (2.12), $\lambda(\Gamma_i) \geq \lambda(\Gamma_i^*)$, and thus, using the expression of $\alpha_i(\Gamma_i)$, $\alpha_i(\Gamma_i) \geq \alpha_i(\Gamma_i^*)$, since u^* is defined on Γ_i^*, if u is defined on Γ_i. The fact that $\mathcal{H}^{n-1}(\Gamma_i) = \mathcal{H}^{n-1}(\Gamma_i^*)$ derives from a property of u^* which says that

$$\mathcal{H}^{n-1}(u^{-1}[\rho, \infty)) = \mathcal{H}^{n-1}((u^*)^{-1}[\rho, \infty)), \quad \forall \rho \in \mathbb{R}.$$

On the other hand, from [48] we achieve that $\alpha_i(\Gamma_i^*) \geq \psi(s_i)$, where $s_i = \frac{\mathcal{H}^{n-1}(\Gamma_i^*)}{\mathcal{H}^{n-1}(\partial B_1)}$ and

$$\psi(s) := \begin{cases} \frac{1}{2} \log \dfrac{1}{4s} + \frac{3}{2}, & 0 < s \leq \dfrac{1}{4}, \\ 2(1-s), & \dfrac{1}{4} \leq s < 1. \end{cases} \tag{2.14}$$

is convex and decreasing. Specifically, the proof of this fact is organized in the following steps.

First of all, we set $\alpha(s, n) = \alpha(E)$, where $\alpha(E)$ is the characteristic constant of the spherical cap $E \subset \partial B_1$, $s = \frac{\mathcal{H}^{n-1}(E)}{\mathcal{H}^{n-1}(\partial B_1)}$, and n is the dimension. At this point, Theorem 2 in [48] tells us that $\alpha(s,n)$ is a monotone decreasing function of n for fixed s, so

$$\alpha(s,\infty) = \lim_{n\to\infty} \alpha(s,n) \tag{2.15}$$

is well-defined and satisfies $\alpha(s,\infty) \le \alpha(s,n)$ for every n. It is thus sufficient to show that $\alpha(s,\infty) \ge \psi(s)$ defined in (2.14). To this end, Theorem F in [48], which is taken by [57], gives us that $\alpha \ge \psi(s)$, where

$$s := \int_h^\infty e^{-(1/2)t^2} dt,$$

with $h = h(\alpha)$ the largest real zero of

$$F(x) = e^{-(1/4)x^2} H_\alpha\left(\frac{x}{\sqrt{2}}\right)$$

satisfying

$$\frac{d^2F}{dx^2} + \left(\alpha + \frac{1}{2} - \frac{1}{4}x^2\right) F = 0$$

and

$$\frac{F'(0)}{F(0)} = -2^{1/2}\frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)},$$

where Γ is the Euler gamma function. In particular, $H_\alpha(x)$ is the Hermite's function of order α. We denote α here introduced by $\alpha(s)$. Furthermore, the proof of Theorem 3, again in [48], shows that $\alpha(s,\infty)$ defined in (2.15) is equal to $\alpha(s)$, because

$$\frac{\mathcal{H}^{n-1}(E)}{\mathcal{H}^{n-1}(\partial B_1)} \overset{n\to\infty}{\to} \int_h^\infty e^{-(1/2)t^2} dt.$$

Hence, $\alpha(s,\infty) \ge \psi(s)$, and from the argument exposed above $\alpha(s,n) \ge \psi(s)$ for every n and for all $s \in (0,1)$.
As a consequence, recalling that $s_i = \frac{\mathcal{H}^{n-1}(\Gamma_i^*)}{\mathcal{H}^{n-1}(\partial B_1)}$, $i \in \{1,2\}$, $\frac{s_1+s_2}{2} \le \frac{1}{2}$, because $\Gamma_1^* \cap \Gamma_2^* = \emptyset$, thus, since $\psi(s)$ defined in (2.14) is convex and decreasing, we get

$$\alpha_1+\alpha_2 \ge \psi(s_1)+\psi(s_2) \ge 2\left(\frac{1}{2}\psi(s_1) + \frac{1}{2}\psi(s_2)\right) \ge 2\psi\left(\frac{s_1+s_2}{2}\right) \ge 2\psi\left(\frac{1}{2}\right),$$

which lastly gives (2.11).
An alternative proof of this result is given in [15], where, using [7] and [10], the two authors directly show that $\alpha(s_1)+\alpha(s_2) \geq 2$, exploiting the properties of $\alpha(s)$ of Theorem F in [48], which is the first Dirichlet eigenvalue on $[h, \infty)$ associated to the Hermite operator

$$-\frac{d^2}{dx^2} + \left(\frac{1}{4}x^2 - \frac{1}{2}\right).$$

2.4 The Heisenberg group

In this section, we introduce the main notation in the Heisenberg group which we will employ to describe the problem about the existence of an Alt-Caffarelli-Friedman monotonicity formula in such a group.
We denote by $\mathbb{H}^n$ the n-th Heisenberg group, i.e. the set $\mathbb{R}^{2n+1}$, $n \in \mathbb{N}$, $n \geq 1$, endowed with the following noncommutative inner law: for every $P \equiv (x_1, y_1, t_1) \in \mathbb{R}^{2n+1}$, $M \equiv (x_2, y_2, t_2) \in \mathbb{R}^{2n+1}$, $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^n$, $i = 1, 2$, we have

$$P \circ M := (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(\langle x_2, y_1 \rangle - \langle x_1, y_2 \rangle)),$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$.

Let $X_i = (e_i, 0, 2y_i)$ and $Y_i = (0, e_i, -2x_i)$, $i = 1, \ldots, n$, with $\{e_i\}_{1 \leq i \leq n}$ the canonical basis for $\mathbb{R}^n$.
We exploit the same symbols to denote the vector fields associated with the previous vectors, that is

$$X_i = \partial_{x_i} + 2y_i\partial_t, \quad Y_i = \partial_{y_i} - 2x_i\partial_t, \quad i = 1, \ldots, n. \tag{2.16}$$

The commutator between the vector fields is then

$$[X_i, Y_i] := X_iY_i - Y_iX_i = \begin{cases} -4\partial_t, & i = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Using these vector fields, we can define the intrinsic gradient of a smooth function $u : \mathbb{H}^n \to \mathbb{R}$ in a point P as

$$\nabla_{\mathbb{H}^n} u(P) := \sum_{i=1}^{n} (X_i u(P) X_i(P) + Y_i u(P) Y_i(P)). \tag{2.17}$$

Now, there exists a unique metric on

$$H\mathbb{H}^n_P := \mathrm{Span}\{X_1(P), \ldots, X_n(P), Y_1(P), \ldots, Y_n(P)\}$$

which makes orthonormal the set of vectors $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$. Thus, for every $P \in \mathbb{H}^n$ and for every $U, V \in H\mathbb{H}^n_P$, $U = \sum_{j=1}^{n} (\alpha_{1,j} X_j(P) + \beta_{1,j} Y_j(P))$, $V = \sum_{j=1}^{n} (\alpha_{2,j} X_j(P) + \beta_{2,j} Y_j(P))$, we have

$$\langle U, V \rangle = \sum_{j=1}^{n} (\alpha_{1,j}\alpha_{2,j} + \beta_{1,j}\beta_{2,j}).$$

In particular, we get a norm associated with the metric on the space $H\mathbb{H}^n_P$, which is

$$|U| = \sqrt{\sum_{j=1}^{n} \left(\alpha_{1,j}^2 + \beta_{1,j}^2\right)}.$$

For example, recalling (2.17), we achieve

$$|\nabla_{\mathbb{H}^n} u(P)| = \sqrt{\sum_{i=1}^{n} ((X_i u(P))^2 + (Y_i u(P))^2)}. \tag{2.18}$$

Moreover, if $\nabla_{\mathbb{H}^n} u(P) \neq 0$, then

$$\left| \frac{\nabla_{\mathbb{H}^n} u(P)}{|\nabla_{\mathbb{H}^n} u(P)|} \right| = 1.$$

If $\nabla_{\mathbb{H}^n} u(P) = 0$, instead, we say that the point P is characteristic for the smooth surface $\{u = u(P)\}$. Therefore, for every point $M \in \{u = u(P)\}$ which is not characteristic, it is well defined the intrinsic normal to the surface $\{u = u(P)\}$. Precisely, we have

$$\nu(M) := \frac{\nabla_{\mathbb{H}^n} u(M)}{|\nabla_{\mathbb{H}^n} u(M)|}.$$

Let us introduce in $\mathbb{H}^n$ the following gauge norm as well:

$$|(x, y, t)|_{\mathbb{H}^n} := \sqrt[4]{(|x|^2 + |y|^2)^2 + t^2}. \tag{2.19}$$

For every positive number r, we can set the gauge ball of radius r centered at 0 as

$$B_r^{\mathbb{H}^n}(0) := \{P \in \mathbb{H}^n : \ |P|_{\mathbb{H}^n} < r\}.$$

In the Heisenberg group $\mathbb{H}^n$, a dilation semigroup is defined as follows: for every $r > 0$ and for every $P = (x, y, t) \in \mathbb{H}^n$, let

$$\delta_r(P) := (rx, ry, r^2 t).$$

Taken $P = (x, y, t), O = (0, 0, 0) \in \mathbb{H}^n$, we define

$$d_K(P, O) := |P|_{\mathbb{H}^n}.$$

In addition, for every $P, T \in \mathbb{H}^n$ is well defined

$$d_K(P, T) := |P^{-1} \circ T|_{\mathbb{H}^n}.$$

In other words we have a distance d_K on $\mathbb{H}^n$, known as the Koranyi distance. This distance is left invariant, that is for every $P, T, R \in \mathbb{H}^n$

$$d_K(R \circ P, R \circ T) = d_K(P, T).$$

Coherently, $B_r^{\mathbb{H}^n}(0)$ is known as the Koranyi ball of radius r centered at 0.

At this point, we recall the definition of the fundamental sublaplacian $\Delta_{\mathbb{H}^n}$, also known as the Kohn-Laplace operator, which is the correspondent differential operator on $\mathbb{H}^n$ to the classical Laplacian in $\mathbb{R}^n$.
Precisely, the sublaplacian $\Delta_{\mathbb{H}^n}$ of a smooth function $u : \mathbb{H}^n \to \mathbb{R}$ is

$$\Delta_{\mathbb{H}^n} u := \sum_{i=1}^{n} (X_i^2 u + Y_i^2 u). \tag{2.20}$$

We want to show then a direct computation which somehow justify that $|P|_{\mathbb{H}^n}^{2-Q}$ is, up to a constant, the fundamental solution of the sublaplacian $\Delta_{\mathbb{H}^n}$ in $\mathbb{H}^n$, with the pole in the origin, and $\Gamma(P, R) = c|P^{-1} \circ R|_{\mathbb{H}^n}^{2-Q}$ is the general fundamental solution, see [44] for the details. For the sake of simplicity, we may perform our computation assuming that dealing with $d_K(P,O) = |P|_{\mathbb{H}^n}$, where $O = (0,0,0)$, simply because from the left invariance of d_K we obtain

$$d_K(P,T) = d_K(T^{-1}\circ P, T^{-1}\circ T) = d_K(T^{-1}\circ P, 0).$$

Specifically, for every $i = 1,\dots,n$, we get, according to (2.19) and (2.16), with $P = (x,y,t)$,

$$\begin{aligned} X_i|P|_{\mathbb{H}^n} &= \frac{1}{4}((|x|^2+|y|^2)^2+t^2)^{-\frac{3}{4}}(4(|x|^2+|y|^2)x_i+4y_it)\\ &= |P|_{\mathbb{H}^n}^{-3}((|x|^2+|y|^2)x_i+y_it) \end{aligned}$$

and

$$Y_i|P|_{\mathbb{H}^n} = |P|_{\mathbb{H}^n}^{-3}((|x|^2+|y|^2)y_i-x_it),$$

namely

$$\begin{aligned} X_i|P|_{\mathbb{H}^n} &= |P|_{\mathbb{H}^n}^{-3}((|x|^2+|y|^2)x_i+y_it),\\ Y_i|P|_{\mathbb{H}^n} &= |P|_{\mathbb{H}^n}^{-3}((|x|^2+|y|^2)y_i-x_it). \end{aligned} \tag{2.21}$$

Using these, we achieve

$$\begin{aligned} X_i^2|P|_{\mathbb{H}^n} &= -3|P|_{\mathbb{H}^n}^{-4}X_i|P|_{\mathbb{H}^n}((|x|^2+|y|^2)x_i+y_it)+|P|_{\mathbb{H}^n}^{-3}(2x_i^2+|x|^2+|y|^2\\ &+2y_i^2) = -3|P|_{\mathbb{H}^n}^{-7}((|x|^2+|y|^2)x_i+y_it)^2+|P|_{\mathbb{H}^n}^{-3}(2(x_i^2+y_i^2)+|x|^2+|y|^2) \end{aligned}$$

and

$$Y_i^2|P|_{\mathbb{H}^n} = -3|P|_{\mathbb{H}^n}^{-7}((|x|^2+|y|^2)y_i-x_it)^2+|P|_{\mathbb{H}^n}^{-3}(2(x_i^2+y_i^2)+|x|^2+|y|^2),$$

which give

$$\begin{aligned} X_i^2|P|_{\mathbb{H}^n} &= -3|P|_{\mathbb{H}^n}^{-7}((|x|^2+|y|^2)x_i+y_it)^2+|P|_{\mathbb{H}^n}^{-3}(2(x_i^2+y_i^2)+|x|^2+|y|^2)\\ Y_i^2|P|_{\mathbb{H}^n} &= -3|P|_{\mathbb{H}^n}^{-7}((|x|^2+|y|^2)y_i-x_it)^2+|P|_{\mathbb{H}^n}^{-3}(2(x_i^2+y_i^2)+|x|^2+|y|^2). \end{aligned} \tag{2.22}$$

Therefore, by (2.21), we have, recalling (2.18),

$$\begin{aligned} |\nabla_{\mathbb{H}^n}|P|_{\mathbb{H}^n}|^2 &= |P|_{\mathbb{H}^n}^{-6}\sum_{i=1}^n((|x|^2+|y|^2)^2(x_i^2+y_i^2)+t^2(x_i^2+y_i^2))\\ &= |P|_{\mathbb{H}^n}^{-6}\sum_{i=1}^n((|x|^2+|y|^2)^2+t^2)(x_i^2+y_i^2) = (|x|^2+|y|^2)|P|_{\mathbb{H}^n}^{-2} \end{aligned}$$

i.e.

$$|\nabla_{\mathbb{H}^n}|P|_{\mathbb{H}^n}|^2 = (|x|^2+|y|^2)|P|_{\mathbb{H}^n}^{-2}. \tag{2.23}$$

In parallel, we obtain, by virtue of (2.20), (2.22) and exploiting the computations done to obtain (2.23),

$$\Delta_{\mathbb{H}^n}|P|_{\mathbb{H}^n} = -3|P|_{\mathbb{H}^n}^{-7}\sum_{i=1}^{n}((((|x|^2+|y|^2)x_i+y_it)^2+((|x|^2+|y|^2)y_i-x_it)^2)$$
$$+|P|_{\mathbb{H}^n}^{-3}\sum_{i=1}^{n}(2(|x|^2+|y|^2)+4(x_i^2+y_i^2)) = -3|P|_{\mathbb{H}^n}^{-7}\sum_{i=1}^{n}(((|x|^2+|y|^2)^2(x_i^2+y_i^2)$$
$$+t^2(x_i^2+y_i^2)) + |P|_{\mathbb{H}^n}^{-3}(2n+4)(|x|^2+|y|^2) = (2n+1)(|x|^2+|y|^2)|P|_{\mathbb{H}^n}^{-3}$$

which yields

$$\Delta_{\mathbb{H}^n}|P|_{\mathbb{H}^n} = (2n+1)(|x|^2+|y|^2)|P|_{\mathbb{H}^n}^{-3}. \tag{2.24}$$

Hence, denoting by $Q \coloneqq 2n+2$ the homogeneous dimension, we get, in view of (2.23) and (2.24),

$$\Delta_{\mathbb{H}^n}|P|_{\mathbb{H}^n}^{2-Q} = \sum_{i=1}^{n}(X_i((2-Q)|P|_{\mathbb{H}^n}^{1-Q}X_i|P|_{\mathbb{H}^n}) + Y_i((2-Q)|P|_{\mathbb{H}^n}^{1-Q}Y_i|P|_{\mathbb{H}^n}))$$
$$= \sum_{i=1}^{n}((2-Q)(1-Q)|P|_{\mathbb{H}^n}^{-Q}(X_i|P|_{\mathbb{H}^n})^2 + (2-Q)|P|_{\mathbb{H}^n}^{1-Q}X_i^2|P|_{\mathbb{H}^n} + (2-Q)$$
$$(1-Q)|P|_{\mathbb{H}^n}^{-Q}(Y_i|P|_{\mathbb{H}^n})^2 + (2-Q)|P|_{\mathbb{H}^n}^{1-Q}Y_i^2|P|_{\mathbb{H}^n})$$
$$= (2-Q)(1-Q)|P|_{\mathbb{H}^n}^{-Q}|\nabla_{\mathbb{H}^n}|P|_{\mathbb{H}^n}|^2 + (2-Q)|P|_{\mathbb{H}^n}^{1-Q}\Delta_{\mathbb{H}^n}|P|_{\mathbb{H}^n}$$
$$= (2-Q)(1-Q)|P|_{\mathbb{H}^n}^{-Q}(|x|^2+|y|^2)|P|_{\mathbb{H}^n}^{-2} + (2-Q)|P|_{\mathbb{H}^n}^{1-Q}(2n+1)(|x|^2+|y|^2)$$
$$|P|_{\mathbb{H}^n}^{-3} = (2-Q)|P|_{\mathbb{H}^n}^{-2-Q}(|x|^2+|y|^2)\,(1-Q+2n+1) = 0,$$

that is

$$\Delta_{\mathbb{H}^n}|P|_{\mathbb{H}^n}^{2-Q} = 0.$$

In conclusion, this computation somehow shows that $|P|_{\mathbb{H}^n}^{2-Q}$ is indeed, up to a constant, the fundamental solution of the sublaplacian $\Delta_{\mathbb{H}^n}$, with the pole in the origin, and $\Gamma(P,R) = c|P^{-1}\circ R|_{\mathbb{H}^n}^{2-Q}$ is the general fundamental solution.

Coherently with the classical case of the Laplacian Δ in $\mathbb{R}^n$, the definition of

$\mathbb{H}^n$-subharmonic function, as well as the one of $\mathbb{H}^n$-superharmonic function in a set $\Omega \subset \mathbb{H}^n$, can be stated directly giving conditions on $\Delta_{\mathbb{H}^n}$. Precisely, we require that $\Delta_{\mathbb{H}^n} u(P) \geq 0$ for every $P \in \Omega$ for the $\mathbb{H}^n$-subharmonicity, and that $\Delta_{\mathbb{H}^n} u(P) \leq 0$ for every $P \in \Omega$ for having $\mathbb{H}^n$-superharmonicity. If u satisfies both $\Delta_{\mathbb{H}^n} u(P) \geq 0$ and $\Delta_{\mathbb{H}^n} u(P) \leq 0$ in Ω, which means that $\Delta_{\mathbb{H}^n} u(P) = 0$, we say that u is $\mathbb{H}^n$-harmonic. We refer to [9] for further details.

Concerning the natural Sobolev spaces to consider in $\mathbb{H}^n$, we refer again to the literature, see for instance [52]. Here, we simply recall that

$$\mathcal{L}^{1,2}(\Omega) \coloneqq \{f \in L^2(\Omega) : X_i f,\ \ Y_i f \in L^2(\Omega),\ \ i = 1, \dots, n\}$$

is a Hilbert space with respect to the norm

$$|f|_{\mathcal{L}^{1,2}(\Omega)} \coloneqq \left(\int_\Omega \left(\sum_{i=1}^n ((X_i f)^2 + (Y_i f)^2) + |f|^2 \right) dx \right)^{\frac{1}{2}}.$$

Moreover

$$H^1_{\mathbb{H}^n}(\Omega) \coloneqq \overline{C^\infty(\Omega) \cap \mathcal{L}^{1,2}(\Omega)}^{|\cdot|_{\mathcal{L}^{1,2}(\Omega)}}.$$

Now, if $E \subset \mathbb{H}^n$ is a measurable set, a notion of $\mathbb{H}^n$-perimeter measure $|\partial E|_{\mathbb{H}^n}$ has been introduced in [52]. Actually, in [52] the authors work in a more general setting, but for our purposes we just recall some results in the framework of the Heisenberg group, the simplest nontrivial example of Carnot group. We refer to [52], [45], [47], [46] for a detailed presentation.
Precisely, it is sufficient to recall that, if E has locally finite $\mathbb{H}^n$-perimeter (is a $\mathbb{H}^n$-Caccioppoli set), then $|\partial E|_{\mathbb{H}^n}$ is a Radon measure in $\mathbb{H}^n$, which is invariant under group translations and $\mathbb{H}^n$-homogeneous of degree $Q - 1$. In addition, the following representation theorem holds, see [16].

Proposition 2.2. *If E is a $\mathbb{H}^n$-Caccioppoli set with Euclidean $\mathbf{C}^1$ boundary, then there is an explicit representation of the $\mathbb{H}^n$-perimeter in terms of the Euclidean $2n$-dimensional Hausdorff measure $\mathcal{H}^{2n}$*

$$P^{\Omega,E}_{\mathbb{H}^n}(\partial E) = \int_{\partial E \cap \Omega} \left(\sum_{j=1}^n \left(\langle X_j, n_E \rangle^2_{\mathbb{R}^{2n+1}} + \langle Y_j, n_E \rangle^2_{\mathbb{R}^{2n+1}} \right) \right)^{1/2} d\mathcal{H}^{2n},$$

where $n_E = n_E(x)$ is the Euclidean unit outward normal to ∂E and $\Omega \subset \mathbb{R}^n$ is an open set.

Going into the specific of our notation, in this chapter we will denote with $d\sigma_{\mathbb{H}^n}$ the surface element in the Heisenberg group $\mathbb{H}^n$, whose expression in terms of the Euclidean surface element in $\mathbb{R}^{2n}$ $d\sigma$ is the following, see (2.23):

$$d\sigma_{\mathbb{H}^n}(\xi) = \frac{|\nabla_{\mathbb{H}^n}|\xi|_{\mathbb{H}^n}|}{|\nabla|\xi|_{\mathbb{H}^n}|}\, d\sigma(\xi) = \frac{\sqrt{|x|^2+|y|^2}}{|\xi|_{\mathbb{H}^n}\,|\nabla|\xi|_{\mathbb{H}^n}|}\, d\sigma(\xi).$$

Next, we recall the statement of the divergence theorem in $\mathbb{H}^n$ and the definition of a Friedrichs mollifier in this framework, together with a convergence result for mollifiers of $\mathbb{H}^n$–subharmonic functions, see [9] for the last two statements. Before, we provide the definition of the divergence operator in $\mathbb{H}^n$.
If $b : \mathbb{H}^n \to \mathbb{R}^{2n}$ is smooth, we denote with $\mathrm{div}_{\mathbb{H}^n}$ the operator

$$\operatorname*{div}_{\mathbb{H}^n} b = \operatorname*{div}_{\mathbb{H}^n}(b_1, \dots, b_{2n}) = \sum_{i=1}^{n}(X_i b_i + Y_i b_{n+i}). \tag{2.25}$$

Proposition 2.3. *If E is a regular bounded open set with Euclidean $\mathbf{C}^1$ boundary and ϕ is a horizontal vector field, continuously differentiable on $\overline{\Omega}$, then*

$$\int_E \operatorname*{div}_{\mathbb{H}^n} \phi\, dx = \int_{\partial E} \langle \phi, \nu_{\mathbb{H}^n}\rangle dP^E_{\mathbb{H}^n},$$

where $\nu_{\mathbb{H}^n}(x)$ is the intrinsic horizontal unit outward normal to ∂E, given by the (normalized) projection of $n_E(x)$ on the fiber $H\mathbb{H}^n_x$ of the horizontal fiber bundle $H\mathbb{H}^n$.

Remark 2.4. The definition of $\nu_{\mathbb{H}^n}$ is well-posed, since $H\mathbb{H}^n_x$ is transversal to the tangent space of E at x for $P^E_{\mathbb{H}^n}(\partial E)$-a.e. $x \in \partial E$ (see [65]).

Definition 2.5. Let $u : \mathbb{H}^n \to [-\infty, +\infty)$. Let $J \in C_0^\infty(\mathbb{H}^n)$, $J \geq 0$ such that $\mathrm{supp}(J) \subset B_1^{\mathbb{H}^n}(0)$ and $\int_{\mathbb{H}^n} J = 1$. For every positive number ε, we define u_ε to be the Friedrichs mollifier of u as

$$u_\varepsilon(x) = \varepsilon^{-Q}\int_{\mathbb{H}^n} u(-y \circ x) J(\delta_{\varepsilon^{-1}}(y))dy.$$

Proposition 2.6. *Let* $u : \mathbb{H}^n \to [-\infty, +\infty)$ *be a* $\mathbb{H}^n$*–subharmonic function. Then* $u_\varepsilon \in C^\infty(\mathbb{H}^n)$ *is* $\mathbb{H}^n$*–subharmonic, and* $u_\varepsilon \to u$ *in* $L^1_{\text{loc}}(\mathbb{H}^n)$ *as* $\varepsilon \to 0^+$.

Let us conclude this section with the following fact. Let $\mathcal{S}(\mathbb{H}^n)$ be the set of the $\mathbb{H}^n$-subharmonic functions. Then if $u \in \mathcal{S}(\mathbb{H}^n)$,

$$L_u(\varphi) := \int_{\mathbb{H}^n} u(x)\Delta_{\mathbb{H}^n}\varphi(x)\, d, \quad \varphi \in C_0^\infty(\mathbb{H}^n),\ \varphi \geq 0$$

is positive, i.e. if $u \in \mathcal{S}(\mathbb{H}^n)$ then $\Delta_{\mathbb{H}^n} u \geq 0$ in the distributional sense.

2.5 Some estimates in the Heisenberg group

In this section, we provide some partial steps useful to obtain a monotonicity formula associated with $J_{\beta,\mathbb{H}^n}$.

Lemma 2.7. *There exists a positive constant* $c = c(Q)$ *such that for every nonnegative* $\mathbb{H}^n$*–subharmonic function* $u \in C(B_1^{\mathbb{H}^n}(0))$, *if* $u(0) = 0$, *then*

$$\int_{B_\rho^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u|^2}{|\xi|_{\mathbb{H}^n}^{Q-2}} d\xi \leq c\rho^{-Q} \int_{B_{2\rho}^{\mathbb{H}^n}(0)\setminus B_\rho^{\mathbb{H}^n}(0)} u^2\, d\xi, \quad 0 < \rho < \frac{1}{2}.$$

Proof. For every $\varepsilon > 0$ small, let u_ε be the Friedrichs mollifier of u as in Definition 2.5. Then, by hypothesis and Proposition 2.6, we have

$$\Delta_{\mathbb{H}^n} u_\varepsilon^2 = 2u_\varepsilon \Delta_{\mathbb{H}^n} u_\varepsilon + 2|\nabla_{\mathbb{H}^n} u_\varepsilon|^2 \geq 2|\nabla_{\mathbb{H}^n} u_\varepsilon|^2.$$

Hence, for every test function $\varphi \in C_0^\infty(B_1^{\mathbb{H}^n}(0))$ we get

$$\begin{aligned} &\int_{B_1^{\mathbb{H}^n}(0)} u^2 \Delta_{\mathbb{H}^n}\varphi\, d\xi = \lim_{\varepsilon\to 0} \int_{B_1^{\mathbb{H}^n}(0)} u_\varepsilon^2 \Delta_{\mathbb{H}^n}\varphi\, d\xi = \lim_{\varepsilon\to 0} \int_{B_1^{\mathbb{H}^n}(0)} \varphi \Delta_{\mathbb{H}^n} u_\varepsilon^2\, d\xi \\ &\geq \lim_{\varepsilon\to 0} 2 \int_{B_1^{\mathbb{H}^n}(0)} \varphi |\nabla_{\mathbb{H}^n} u_\varepsilon|^2 d\xi = 2\int_{B_1^{\mathbb{H}^n}(0)} \varphi |\nabla_{\mathbb{H}^n} u|^2 d\xi. \end{aligned} \tag{2.26}$$

As a consequence, $u \in H^1_{\text{loc}}(B_1^{\mathbb{H}^n}(0))$ and $\Delta_{\mathbb{H}^n} u^2 \geq 2|\nabla_{\mathbb{H}^n} u|^2$ as a distribution.

Let now ψ be a cutoff function, $\psi \equiv 1$ in $B_\rho^{\mathbb{H}^n}(0)$, $\psi \equiv 0$ outside $B_{2\rho}^{\mathbb{H}^n}(0)$, $0 < \rho < \frac{1}{2}$. We also set

$$\gamma_\varepsilon = \eta_\varepsilon * \gamma, \tag{2.27}$$

where $\gamma(\xi) = |\xi|_{\mathbb{H}^n}^{2-Q}$ and η_ε is an approximation of the identity. Then, $\psi\gamma_\varepsilon$ is a test function in $B_\rho^{\mathbb{H}^n}(0)$, so, in view of (2.26) and (2.27), we achieve

$$\begin{aligned}
&2\int_{B_\rho^{\mathbb{H}^n}(0)} \psi\gamma_\varepsilon |\nabla_{\mathbb{H}^n} u|^2 d\xi \le \int_{B_{2\rho}^{\mathbb{H}^n}(0)} u^2 \Delta_{\mathbb{H}^n}(\psi\gamma_\varepsilon)\, d\xi = \int_{B_{2\rho}^{\mathbb{H}^n}(0)} u^2 \psi \Delta_{\mathbb{H}^n}\gamma_\varepsilon\, d\xi \\
&+ 2\int_{B_{2\rho}^{\mathbb{H}^n}(0)\setminus B_\rho^{\mathbb{H}^n}(0)} u^2 \langle \nabla_{\mathbb{H}^n}\psi, \nabla_{\mathbb{H}^n}\gamma_\varepsilon\rangle\, d\xi + \int_{B_{2\rho}^{\mathbb{H}^n}(0)\setminus B_\rho^{\mathbb{H}^n}(0)} u^2 \gamma_\varepsilon \Delta_{\mathbb{H}^n}\psi\, d\xi \\
&= \int_{B_{2\rho}^{\mathbb{H}^n}(0)} u^2 \psi(\eta_\varepsilon * \Delta_{\mathbb{H}^n}\gamma)\, d\xi + 2\int_{B_{2\rho}^{\mathbb{H}^n}(0)\setminus B_\rho^{\mathbb{H}^n}(0)} u^2 \langle \nabla_{\mathbb{H}^n}\psi, \eta_\varepsilon * \nabla_{\mathbb{H}^n}\gamma\rangle\, d\xi \\
&+ \int_{B_{2\rho}^{\mathbb{H}^n}(0)\setminus B_\rho^{\mathbb{H}^n}(0)} u^2 \gamma_\varepsilon \Delta_{\mathbb{H}^n}\psi\, d\xi,
\end{aligned}$$

which yields the desired conclusion letting ε go to 0, because γ is, up to a multiplicative constant, the fundamental solution of $\Delta_{\mathbb{H}^n}$ and thus also $\gamma_\varepsilon \to \gamma$ in $L^1_{\text{loc}}(\mathbb{H}^n)$, $u(0) = 0$, and ψ is a cutoff function. $\square$

Lemma 2.8. *For every couple of nonnegative $\mathbb{H}^n$-subharmonic functions $u_i \in C(B_1^{\mathbb{H}^n}(0))$, $i = 1, 2$, such that $u_1 u_2 = 0$ and $u_1(0) = u_2(0) = 0$, we have*

$$\frac{J'_{\beta,\mathbb{H}^n}(1)}{J_{\beta,\mathbb{H}^n}(1)} = \frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_1|^2}{\sqrt{|x|^2+y^2}} d\sigma_{\mathbb{H}^n}(\kappa)}{\displaystyle\int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_1|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa} + \frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_2|^2}{\sqrt{|x|^2+y^2}} d\sigma_{\mathbb{H}^n}(\kappa)}{\displaystyle\int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_2|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa} - \beta.$$

Moreover, if $\frac{J'_{\beta,\mathbb{H}^n}(1)}{J_{\beta,\mathbb{H}^n}(1)} \ge 0$, then there exists $r_0 > 0$ such that $J_{\beta,\mathbb{H}^n}$ is monotone increasing in $(0, r_0)$.

In this statement, $d\sigma_{\mathbb{H}^n}$ denotes the surface element in the Heisenberg group $\mathbb{H}^n$, see Section 2.4 for a short introduction and [52], [45].

Proof. It follows from Lemma 2.7 that $J_{\beta,\mathbb{H}^n}$ is well defined in $(0, 1)$. Differentiating with respect to r and recalling the co-area formula in the Heisenberg group, we obtain

$$
\begin{aligned}
J'_{\beta,\mathbb{H}^n}(r) &= -\beta r^{-\beta-1}\int_{B_r^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_1|^2}{|\xi|_{\mathbb{H}^n}^{Q-2}}d\xi\int_{B_r^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_2|^2}{|\xi|_{\mathbb{H}^n}^{Q-2}}d\xi\\
&+r^{-\beta}\int_{\partial B_r^{\mathbb{H}^n}(0)}|\xi|_{\mathbb{H}^n}\frac{|\nabla_{\mathbb{H}^n}u_1|^2}{r^{Q-2}\sqrt{|x|^2+|y|^2}}d\sigma_{\mathbb{H}^n}(\xi)\int_{B_r^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_2|^2}{|\xi|_{\mathbb{H}^n}^{Q-2}}d\xi\\
&+r^{-\beta}\int_{B_r^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_1|^2}{|\xi|_{\mathbb{H}^n}^{Q-2}}d\xi\int_{\partial B_r^{\mathbb{H}^n}(0)}|\xi|_{\mathbb{H}^n}\frac{|\nabla_{\mathbb{H}^n}u_2|^2}{r^{Q-2}\sqrt{|x|^2+|y|^2}}d\sigma_{\mathbb{H}^n}(\xi)\\
&=-\beta r^{-\beta-1}\int_{B_r^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_1|^2}{|\xi|_{\mathbb{H}^n}^{Q-2}}d\xi\int_{B_r^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_2|^2}{|\xi|_{\mathbb{H}^n}^{Q-2}}d\xi\\
&+r^{-\beta}\int_{\partial B_r^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_1|^2}{r^{Q-3}\sqrt{|x|^2+|y|^2}}d\sigma_{\mathbb{H}^n}(\xi)\int_{B_r^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_2|^2}{|\xi|_{\mathbb{H}^n}^{Q-2}}d\xi\\
&+r^{-\beta}\int_{B_r^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_1|^2}{|\xi|_{\mathbb{H}^n}^{Q-2}}d\xi\int_{\partial B_r^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_2|^2}{r^{Q-3}\sqrt{|x|^2+|y|^2}}d\sigma_{\mathbb{H}^n}(\xi).
\end{aligned}
$$

Notice that by a change of variables, denoting $\kappa \in \partial B_1^{\mathbb{H}^n}(0)$ as $\kappa=(\kappa_x,\kappa_y,k_t)$, with $\kappa_x,\ \kappa_y\in\mathbb{R}^n$ and $\kappa_t\in\mathbb{R}$, we have thus

$$
\begin{aligned}
&J'_{\beta,\mathbb{H}^n}(r) = -\beta r^{-\beta-1}r^{2Q}\int_{B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_1(\delta_r(\kappa))|^2}{|\delta_r(\kappa)|_{\mathbb{H}^n}^{Q-2}}d\kappa\int_{B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_2(\delta_r(\kappa))|^2}{|\delta_r(\kappa)|_{\mathbb{H}^n}^{Q-2}}d\kappa\\
&+r^{-\beta}r^{2Q-1}\int_{\partial B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_1(\delta_r(\kappa))|^2}{r^{Q-3}\sqrt{|r\kappa_x|^2+|r\kappa_y|^2}}d\sigma_{\mathbb{H}^n}(\kappa)\int_{B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_2(\delta_r(\kappa))|^2}{|\delta_r(\kappa)|_{\mathbb{H}^n}^{Q-2}}d\kappa\\
&+r^{-\beta}r^{2Q-1}\int_{B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_1(\delta_r(\kappa))|^2}{|\delta_r(\kappa)|_{\mathbb{H}^n}^{Q-2}}d\kappa\int_{\partial B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_2(\delta_r(\kappa))|^2}{r^{Q-3}\sqrt{|r\kappa_x|^2+|r\kappa_y|^2}}d\sigma_{\mathbb{H}^n}(\kappa)\\
&=-\beta r^{-\beta-1}r^{Q}\int_{B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_1(\delta_r(\kappa))|^2}{r^{Q-2}|\kappa|_{\mathbb{H}^n}^{Q-2}}d\kappa\quad r^{Q}\int_{B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_2(\delta_r(\kappa))|^2}{r^{Q-2}|\kappa|_{\mathbb{H}^n}^{Q-2}}d\kappa\\
&+r^{-\beta}r^{2}\int_{\partial B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_1(\delta_r(\kappa))|^2}{r\sqrt{|\kappa_x|^2+|\kappa_y|^2}}d\sigma_{\mathbb{H}^n}(\kappa)\quad r^{Q}\int_{B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_2(\delta_r(\kappa))|^2}{r^{Q-2}|\kappa|_{\mathbb{H}^n}^{Q-2}}d\kappa
\end{aligned}
$$

$$
+r^{-\beta}r^{Q}\int_{B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_1(\delta_r(\kappa))|^2}{r^{Q-2}|\kappa|_{\mathbb{H}^n}^{Q-2}}d\kappa\quad r^{2}\int_{\partial B_1^{\mathbb{H}^n}(0)}\frac{|\nabla_{\mathbb{H}^n}u_2(\delta_r(\kappa))|^2}{r\sqrt{|\kappa_x|^2+|\kappa_y|^2}}d\sigma_{\mathbb{H}^n}(\kappa),
$$

which gives

$$J'_{\beta,\mathbb{H}^n}(r) = r^{-\beta-1}r^4\Big(-\beta \int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_1(\delta_r(\kappa))|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa \int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_2(\delta_r(\kappa))|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa$$
$$+ \int_{\partial B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_1(\delta_r(\kappa))|^2}{\sqrt{|\kappa_x|^2 + |\kappa_y|^2}} d\sigma_{\mathbb{H}^n}(\kappa) \int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_2(\delta_r(\kappa))|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa$$
$$+ \int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_1(\delta_r(\kappa))|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa \int_{\partial B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} u_2(\delta_r(\kappa))|^2}{\sqrt{|\kappa_x|^2 + |\kappa_y|^2}} d\sigma_{\mathbb{H}^n}(\kappa)\Big). \tag{2.28}$$

Let now $v_i(\kappa) := \frac{u_i(\delta_r(\kappa))}{r}$, $i = 1, 2$. Then $\nabla_{\mathbb{H}^n} v_i(\kappa) = (\nabla_{\mathbb{H}^n} u_i)(\delta_r(\kappa))$. Hence, by (2.28), it holds

$$J'_{\beta,\mathbb{H}^n}(r) = r^{-\beta+3}\Big(-\beta \int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} v_1|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa \int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} v_2|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa$$
$$+ \int_{\partial B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} v_1|^2}{\sqrt{|\kappa_x|^2 + |\kappa_y|^2}} d\sigma_{\mathbb{H}^n}(\kappa) \int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} v_2|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa \tag{2.29}$$
$$+ \int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} v_1|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa \int_{\partial B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} v_2|^2}{\sqrt{|\kappa_x|^2 + |\kappa_y|^2}} d\sigma_{\mathbb{H}^n}(\kappa)\Big).$$

In particular, using this last expression and (2.4), we have

$$\frac{J'_{\beta,\mathbb{H}^n}(1)}{J_{\beta,\mathbb{H}^n}(1)} = \frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} v_1|^2}{\sqrt{|\kappa_x|^2 + |\kappa_y|^2}} d\sigma_{\mathbb{H}^n}(\kappa)}{\displaystyle\int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} v_1|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa}
+ \frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} v_2|^2}{\sqrt{|\kappa_x|^2 + |\kappa_y|^2}} d\sigma_{\mathbb{H}^n}(\kappa)}{\displaystyle\int_{B_1^{\mathbb{H}^n}(0)} \frac{|\nabla_{\mathbb{H}^n} v_2|^2}{|\kappa|_{\mathbb{H}^n}^{Q-2}} d\kappa} - \beta.$$

The fact that it is sufficient to show that $\frac{J'_{\beta,\mathbb{H}^n}(1)}{J_{\beta,\mathbb{H}^n}(1)} \geq 0$ to get that $J_{\beta,\mathbb{H}^n}$ is monotone increasing follows again from (2.29) and (2.4), since by (2.29) it results $J'_{\beta,\mathbb{H}^n}(r)r^{\beta-3} = J'_{\beta,\mathbb{H}^n}(1)$ and by (2.4) $J_{\beta,\mathbb{H}^n}$ is always nonnegative respectively. □

In the next section, we reduce ourselves to the simplest case given by $\mathbb{H}^1$.

2.6 Kohn-Laplace operator in spherical coordinates in $\mathbb{H}^1$

This section is devoted to represent the Kohn-Laplace operator $\Delta_{\mathbb{H}^n}$ in spherical coordinates in the Heisenberg group. As a matter of fact, in order to obtain some estimates of $\frac{J'_{\beta,\mathbb{H}^n}(1)}{J_{\beta,\mathbb{H}^n}(1)}$, we need to write $\Delta_{\mathbb{H}^n}$ in terms of radial coordinates. This issue has been faced in [58] by using an abstract and more elegant approach, even if very theoretical, see also [56] and [8]. Here, we describe in details this problem in $\mathbb{H}^1$ with explicit computations.

Specifically, we consider the following coordinates in $\mathbb{H}^1$

$$\begin{cases} x = \rho\sqrt{\sin(\varphi)}\cos(\theta) \\ y = \rho\sqrt{\sin(\varphi)}\sin\theta \\ t = \rho^2\cos\varphi. \end{cases} \tag{2.30}$$

They mimic the classical polar coordinates in $\mathbb{R}^3$. From (2.30), we obtain the expressions of ρ, φ and θ with respect to the cartesian coordinates x, y and t, that is

$$\begin{cases} \rho = ((x^2+y^2)^2+t^2)^{1/4} \\ \theta = \arctan\left(\frac{y}{x}\right) \\ \varphi = \arccos\left(\frac{t}{\rho^2}\right). \end{cases} \tag{2.31}$$

Recalling now from Section 2.4 the vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \tag{2.32}$$

and the operators

$$\nabla_{\mathbb{H}^1} \equiv (X, Y), \quad \Delta_{\mathbb{H}^1} = X^2 + Y^2, \tag{2.33}$$

we want to determine $\nabla_{\mathbb{H}^1}\rho$, $\nabla_{\mathbb{H}^1}\theta$, $\nabla_{\mathbb{H}^1}\varphi$, using (2.32), (2.31) and (2.33).

Lemma 2.9. *We have the equalities*

$$\nabla_{\mathbb{H}^1}\varphi = \frac{2}{\rho(x^2+y^2)}\left(t\nabla_{\mathbb{H}^1}\rho + \rho(-y,x)\right),$$
$$\nabla_{\mathbb{H}^1}\rho = \rho^{-3}((x^2+y^2)x + ty, (x^2+y^2)y - tx),$$
$$\nabla_{\mathbb{H}^1}\theta = \frac{1}{x^2+y^2}(-y,x).$$

Proof. Let us begin by calculating

$$X\varphi = -\frac{1}{\sqrt{1-\left(\frac{t}{\rho^2}\right)^2}}X\left(\frac{t}{\rho^2}\right) = -\frac{1}{\sqrt{1-\frac{t^2}{\rho^4}}}\left(\frac{2y}{\rho^2} - 2\rho^{-3}tX\rho\right) = -\frac{2}{\sqrt{\frac{\rho^4-t^2}{\rho^4}}}$$
$$\frac{1}{\rho^2}\left(y - \frac{tX\rho}{\rho}\right) = -\frac{2\rho^2}{\sqrt{\rho^4-t^2}}\frac{1}{\rho^2}\left(\frac{\rho y - tX\rho}{\rho}\right) = \frac{2}{\rho\sqrt{\rho^4-t^2}}\left(tX\rho - \rho y\right),$$

and

$$Y\varphi = -\frac{1}{\sqrt{1-\left(\frac{t}{\rho^2}\right)^2}}Y\left(\frac{t}{\rho^2}\right) = -\frac{\rho^2}{\sqrt{\rho^4-t^2}}\left(-\frac{2x}{\rho^2} - 2\rho^{-3}tY\rho\right)$$
$$= -\frac{2}{\sqrt{\rho^4-t^2}}\left(\frac{-x\rho - tY\rho}{\rho}\right) = \frac{2}{\rho\sqrt{\rho^4-t^2}}\left(x\rho + tY\rho\right),$$

which gives

$$\nabla_{\mathbb{H}^1}\varphi = \left(\frac{2}{\rho\sqrt{\rho^4-t^2}}\left(tX\rho - \rho y\right), \frac{2}{\rho\sqrt{\rho^4-t^2}}\left(x\rho + tY\rho\right)\right) = \frac{2}{\rho\sqrt{\rho^4-t^2}}$$
$$(tX\rho - \rho y, x\rho + tY\rho) = \frac{2}{\rho(x^2+y^2)}\left(t\nabla_{\mathbb{H}^1}\rho + \rho(-y,x)\right). \tag{2.34}$$

Let us calculate now $\nabla_{\mathbb{H}^1}\theta$. For this purpose, we have

$$X\theta = \frac{1}{1+\left(\frac{y}{x}\right)^2}X\left(\frac{y}{x}\right) = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

and

$$Y\theta = \frac{1}{1+\left(\frac{y}{x}\right)^2}Y\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2},$$

which imply

$$\nabla_{\mathbb{H}^1}\theta = \frac{1}{x^2+y^2}(-y,x). \tag{2.35}$$

About $\nabla_{\mathbb{H}^1}\rho$, instead, we exploit (2.21) in the particular case of $\mathbb{H}^1$ and it holds

$$\nabla_{\mathbb{H}^1}\rho = \rho^{-3}((x^2+y^2)x + yt, (x^2+y^2)y - xt). \tag{2.36}$$

Using (2.34), (2.35) and (2.36), we achieve

$$\begin{aligned}
\nabla_{\mathbb{H}^1}\varphi &= \frac{2}{\rho(x^2+y^2)}\left(t\nabla_{\mathbb{H}^1}\rho + \rho(-y,x)\right), \\
\nabla_{\mathbb{H}^1}\rho &= \rho^{-3}((x^2+y^2)x + ty, (x^2+y^2)y - tx), \\
\nabla_{\mathbb{H}^1}\theta &= \frac{1}{x^2+y^2}(-y,x).
\end{aligned}$$

□

Let us establish, at this point, also the values $|\nabla_{\mathbb{H}^1}\varphi|^2$, $|\nabla_{\mathbb{H}^1}\rho|^2$ and $|\nabla_{\mathbb{H}^1}\theta|^2$.

Lemma 2.10. *The following relationships hold:*

$$|\nabla_{\mathbb{H}^1}\varphi|^2 = \frac{4(x^2+y^2)}{\rho^4}, \quad |\nabla_{\mathbb{H}^1}\rho|^2 = \frac{x^2+y^2}{\rho^2}, \quad |\nabla_{\mathbb{H}^1}\theta|^2 = \frac{1}{x^2+y^2}.$$

Proof. According to Lemma 2.9, we obtain first

$$|\nabla_{\mathbb{H}^1}\theta|^2 = \frac{1}{(x^2+y^2)^2}(y^2+x^2) = \frac{1}{x^2+y^2}. \tag{2.37}$$

Concerning $|\nabla_{\mathbb{H}^1}\rho|^2$, it comes directly from (2.23) and it holds

$$|\nabla_{\mathbb{H}^1}\rho|^2 = \frac{x^2+y^2}{\rho^2}. \tag{2.38}$$

About $|\nabla_{\mathbb{H}^1}\varphi|^2$, instead, we calculate explicitly and we get, using (2.38) and Lemma 2.9,

$$\begin{aligned}
|\nabla_{\mathbb{H}^1}\varphi|^2 &= \frac{4}{\rho^2(x^2+y^2)^2}(t^2\,|\nabla_{\mathbb{H}^1}\rho|^2 + \rho^2(y^2+x^2) + 2t\rho\langle\nabla_{\mathbb{H}^1}\rho, (-y,x)\rangle) \\
&= \frac{4}{\rho^2(x^2+y^2)^2}\left(t^2\frac{x^2+y^2}{\rho^2} + \rho^2(y^2+x^2) + 2t\rho\rho^{-3}(-(x^2+y^2)yx - y^2t + (x^2\right. \\
&\left. + y^2)yx - x^2t)\right) = \frac{4}{\rho^2(x^2+y^2)^2}\left(t^2\frac{x^2+y^2}{\rho^2} + \rho^2(y^2+x^2) - 2t^2\frac{x^2+y^2}{\rho^2}\right) \\
&= \frac{4}{\rho^2(x^2+y^2)^2}\left(\rho^2(x^2+y^2) - t^2\frac{x^2+y^2}{\rho^2}\right) = \frac{4}{\rho^2(x^2+y^2)}\left(\frac{\rho^4-t^2}{\rho^2}\right)
\end{aligned}$$

namely

$$|\nabla_{\mathbb{H}^1}\varphi|^2 = \frac{4(x^2+y^2)}{\rho^4}. \tag{2.39}$$

Putting together (2.37), (2.37) and (2.39), we then achieve the expected results. □

To conclude the computations on $\nabla_{\mathbb{H}^1}\rho$, $\nabla_{\mathbb{H}^1}\theta$, $\nabla_{\mathbb{H}^1}\varphi$, we calculate $\langle\nabla_{\mathbb{H}^1}\varphi, \nabla_{\mathbb{H}^1}\rho\rangle$, $\langle\nabla_{\mathbb{H}^1}\rho, \nabla_{\mathbb{H}^1}\theta\rangle$ and $\langle\nabla_{\mathbb{H}^1}\varphi, \nabla_{\mathbb{H}^1}\theta\rangle$ as well.

Lemma 2.11. *The following equalities hold:*

$$\langle\nabla_{\mathbb{H}^1}\varphi, \nabla_{\mathbb{H}^1}\rho\rangle = 0, \quad \langle\nabla_{\mathbb{H}^1}\rho, \nabla_{\mathbb{H}^1}\theta\rangle = -\frac{\cos(\varphi)}{\rho}, \quad \langle\nabla_{\mathbb{H}^1}\varphi, \nabla_{\mathbb{H}^1}\theta\rangle = \frac{2(x^2+y^2)}{\rho^4}.$$

Proof. Let us compute $\langle\nabla_{\mathbb{H}^1}\varphi, \nabla_{\mathbb{H}^1}\rho\rangle$, $\langle\nabla_{\mathbb{H}^1}\varphi, \nabla_{\mathbb{H}^1}\theta\rangle$ and $\langle\nabla_{\mathbb{H}^1}\rho, \nabla_{\mathbb{H}^1}\theta\rangle$, using Lemma 2.9 and Lemma 2.10. Specifically, we have first

$$\begin{aligned}
&\langle\nabla_{\mathbb{H}^1}\varphi, \nabla_{\mathbb{H}^1}\rho\rangle = \frac{2}{\rho(x^2+y^2)}(t\,|\nabla_{\mathbb{H}^1}\rho|^2 + \rho\langle(-y,x), \nabla_{\mathbb{H}^1}\rho\rangle) \\
&= \frac{2}{\rho(x^2+y^2)}\left(t\frac{x^2+y^2}{\rho^2} + \rho^{-2}(-(x^2+y^2)yx - y^2t + (x^2+y^2)xy - x^2t)\right) \\
&= \frac{2}{\rho(x^2+y^2)}\left(t\frac{x^2+y^2}{\rho^2} + \rho^{-2}(-t(x^2+y^2))\right) = 0.
\end{aligned} \tag{2.40}$$

Next, we calculate

$$\begin{aligned}
&\langle\nabla_{\mathbb{H}^1}\rho, \nabla_{\mathbb{H}^1}\theta\rangle = \frac{\rho^{-3}}{x^2+y^2}(-(x^2+y^2)xy - y^2t + (x^2+y^2)xy - x^2t) = -\frac{t}{\rho^3} \\
&= -\frac{\cos(\varphi)}{\rho}.
\end{aligned} \tag{2.41}$$

Concerning $\langle\nabla_{\mathbb{H}^1}\varphi, \nabla_{\mathbb{H}^1}\theta\rangle$, lastly, using (2.41), it results

$$\begin{aligned}
&\langle\nabla_{\mathbb{H}^1}\varphi, \nabla_{\mathbb{H}^1}\theta\rangle = \frac{2}{\rho(x^2+y^2)^2}(t\rho^{-3}(-(x^2+y^2)xy - y^2t + (x^2+y^2)xy - x^2t) \\
&+ \rho(y^2+x^2)) = \frac{2}{\rho(x^2+y^2)^2}(-t^2\rho^{-3}(x^2+y^2) + \rho(y^2+x^2)) = \frac{2}{\rho(x^2+y^2)} \\
&(-t^2\rho^{-3}+\rho) = \frac{2}{\rho(x^2+y^2)}\left(\frac{-t^2+\rho^4}{\rho^3}\right) = \frac{2(x^2+y^2)}{\rho^4}.
\end{aligned}$$

Considering this together with (2.40) and (2.41), we obtain the desired relationships. □

We are in position now to compute $\Delta_{\mathbb{H}^1}\varphi$, $\Delta_{\mathbb{H}^1}\rho$ and $\Delta_{\mathbb{H}^1}\theta$, using Lemma 2.9. In particular, we have the following results.

Lemma 2.12. *We have the relationships*

$$\Delta_{\mathbb{H}^1}\theta = 0, \quad \Delta_{\mathbb{H}^1}\rho = \frac{3(x^2+y^2)}{\rho^3}, \quad \Delta_{\mathbb{H}^1}\varphi = \frac{4\cos\varphi}{\rho^2}.$$

Proof. We first remark that by Lemma 2.9

$$\Delta_{\mathbb{H}^1}\theta = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0. \tag{2.42}$$

About $\Delta_{\mathbb{H}^1}\rho$, instead, we can employ the formula, see [9],

$$\Delta_{\mathbb{H}^1} f(\rho) = |\nabla_{\mathbb{H}^1}\rho|^2 \left(f'' + \frac{Q-1}{\rho} f' \right), \tag{2.43}$$

in the particular case of $f(\rho) = \rho$ and $Q = 4$, and we achieve, in view of Lemma 2.10,

$$\Delta_{\mathbb{H}^1}\rho = \frac{x^2+y^2}{\rho^2}\frac{3}{\rho} = \frac{3(x^2+y^2)}{\rho^3}. \tag{2.44}$$

About $\Delta_{\mathbb{H}^1}\varphi$, we obtain, from Lemma 2.9, and by virtue of (2.44) and Lemma 2.10,

$$\begin{aligned}
\Delta_{\mathbb{H}^1}\varphi &= X\left(\frac{2}{\rho(x^2+y^2)}\right)(tX\rho - \rho y) + \frac{2}{\rho(x^2+y^2)}(XtX\rho + tX^2\rho - yX\rho \\
&- \rho Xy) + Y\left(\frac{2}{\rho(x^2+y^2)}\right)(tY\rho + \rho x) + \frac{2}{\rho(x^2+y^2)}(YtY\rho + tY^2\rho + xY\rho \\
&+ \rho Yx) = (tX\rho - \rho y)\left(-\frac{2}{\rho^2(x^2+y^2)^2}\right)((x^2+y^2)X\rho + \rho 2x) + (tY\rho + \rho x) \\
&\left(-\frac{2}{\rho^2(x^2+y^2)^2}\right)((x^2+y^2)Y\rho + \rho 2y) + \frac{2}{\rho(x^2+y^2)}((Xt-y)X\rho - \rho Xy \\
&+ (Yt+x)Y\rho + \rho Yx) + \frac{2t}{\rho(x^2+y^2)}\Delta_{\mathbb{H}^1}\rho = -\frac{2}{\rho^2(x^2+y^2)^2}(t(x^2+y^2)(X\rho)^2 \\
&- \rho y(x^2+y^2)X\rho + 2\rho xtX\rho - 2\rho^2 xy + t(x^2+y^2)(Y\rho)^2 + \rho x(x^2+y^2)Y\rho \\
&2\rho ytY\rho + 2\rho^2 xy) + \frac{6t}{\rho^4} + \frac{2y}{\rho(x^2+y^2)}X\rho - \frac{2x}{\rho(x^2+y^2)}Y\rho
\end{aligned}$$

so that

$$\Delta_{\mathbb{H}^1}\varphi = -\frac{2}{\rho^2(x^2+y^2)^2}(t(x^2+y^2)((X\rho)^2+(Y\rho)^2)+\rho(x^2+y^2)(xY\rho-yX\rho)$$
$$+2t\rho(xX\rho+yY\rho))+\frac{6t}{\rho^4}+\frac{2y}{\rho(x^2+y^2)}X\rho-\frac{2x}{\rho(x^2+y^2)}Y\rho=-\frac{2t}{\rho^2(x^2+y^2)}$$
$$|\nabla_{\mathbb{H}^1}\rho|^2-\frac{2x}{\rho(x^2+y^2)}Y\rho+\frac{2y}{\rho(x^2+y^2)}X\rho-\frac{4t}{\rho(x^2+y^2)^2}(xX\rho+yY\rho)+\frac{6t}{\rho^4}$$
$$+\frac{2y}{\rho(x^2+y^2)}X\rho-\frac{2x}{\rho(x^2+y^2)}Y\rho=-\frac{2t}{\rho^4}+\frac{4}{\rho(x^2+y^2)}(yX\rho-xY\rho)$$
$$-\frac{4t}{\rho(x^2+y^2)^2}(x\rho^{-3}((x^2+y^2)x+yt)+y\rho^{-3}((x^2+y^2)y-xt))+\frac{6t}{\rho^4}$$
$$=\frac{4t}{\rho^4}+\frac{4}{\rho(x^2+y^2)}(y\rho^{-3}((x^2+y^2)x+ty)-x\rho^{-3}((x^2+y^2)y-xt))$$
$$-\frac{4t}{\rho^4(x^2+y^2)^2}((x^2+y^2)x^2+xyt+(x^2+y^2)y^2-yxt)=\frac{4t}{\rho^4}+\frac{4}{\rho^4(x^2+y^2)}$$
$$((x^2+y^2)yx+ty^2-(x^2+y^2)xy+x^2t)-\frac{4t}{\rho^4}=\frac{4t}{\rho^4}=\frac{4\cos(\varphi)}{\rho^2}$$

i.e.

$$\Delta_{\mathbb{H}^1}\varphi = \frac{4\cos\varphi}{\rho^2}.$$

Putting together this with (2.42) and (2.44), we have the expected relationships. □

We complete this section providing the expression of $\Delta_{\mathbb{H}^1}u$, assuming that $u=\rho^\alpha f(\theta,\varphi)$.

Lemma 2.13. *Let $u=\rho^\alpha f(\theta,\varphi)$. Then*

$$\Delta_{\mathbb{H}^1}u=\rho^{\alpha-2}\Bigg(\alpha(\alpha+2)\sin(\varphi)f-2\alpha(\cos(\varphi))\frac{\partial f}{\partial\theta}+\frac{1}{\sin(\varphi)}\frac{\partial^2 f}{\partial\theta^2}$$
$$+4\sin(\varphi)\frac{\partial^2 f}{\partial\varphi\partial\theta}+4\sin(\varphi)\frac{\partial^2 f}{\partial\varphi^2}+4\cos(\varphi)\frac{\partial f}{\partial\varphi}\Bigg).$$

Proof. Let us start the proof by computing Xu and Yu. We have

$$Xu=\alpha\rho^{\alpha-1}(X\rho)f+\rho^\alpha\left(\frac{\partial f}{\partial\theta}X\theta+\frac{\partial f}{\partial\varphi}X\varphi\right),$$
$$Yu=\alpha\rho^{\alpha-1}(Y\rho)f+\rho^\alpha\left(\frac{\partial f}{\partial\theta}Y\theta+\frac{\partial f}{\partial\varphi}Y\varphi\right).$$

We are ready now to calculate $\Delta_{\mathbb{H}^1} u$. Precisely, we achieve

$$\Delta_{\mathbb{H}^1} u = \alpha(\alpha-1)\rho^{\alpha-2}((X\rho)^2+(Y\rho)^2)f + \alpha\rho^{\alpha-1}(X^2\rho+Y^2\rho)f + 2\alpha\rho^{\alpha-1}$$
$$(X\rho)\left(\frac{\partial f}{\partial\theta}X\theta+\frac{\partial f}{\partial\varphi}X\varphi\right) + 2\alpha\rho^{\alpha-1}(Y\rho)\left(\frac{\partial f}{\partial\theta}Y\theta+\frac{\partial f}{\partial\varphi}Y\varphi\right)$$
$$+\rho^\alpha\left(X\left(\frac{\partial f}{\partial\theta}X\theta+\frac{\partial f}{\partial\varphi}X\varphi\right)+Y\left(\frac{\partial f}{\partial\theta}Y\theta+\frac{\partial f}{\partial\varphi}Y\varphi\right)\right)$$
$$=\alpha(\alpha-1)\rho^{\alpha-2}\left|\nabla_{\mathbb{H}^1}\rho\right|^2 f + \alpha\rho^{\alpha-1}(\Delta_{\mathbb{H}^1}\rho)f + 2\alpha\rho^{\alpha-1}\left(\frac{\partial f}{\partial\theta}X\rho X\theta + \frac{\partial f}{\partial\varphi}X\rho\right.$$
$$\left.X\varphi + \frac{\partial f}{\partial\theta}Y\rho Y\theta + \frac{\partial f}{\partial\varphi}Y\rho Y\varphi\right) + \rho^\alpha\left(X\left(\frac{\partial f}{\partial\theta}\right)X\theta + \frac{\partial f}{\partial\theta}X^2\theta + X\left(\frac{\partial f}{\partial\varphi}\right)X\varphi\right.$$
$$\left.+\frac{\partial f}{\partial\varphi}X^2\varphi + Y\left(\frac{\partial f}{\partial\theta}\right)Y\theta + \frac{\partial f}{\partial\theta}Y^2\theta + Y\left(\frac{\partial f}{\partial\varphi}\right)Y\varphi + \frac{\partial f}{\partial\varphi}Y^2\varphi\right)$$
$$=(\alpha(\alpha-1)\rho^{\alpha-2}\left|\nabla_{\mathbb{H}^1}\rho\right|^2 + \alpha\rho^{\alpha-1}\Delta_{\mathbb{H}^1}\rho)f + 2\alpha\rho^{\alpha-1}\left(\frac{\partial f}{\partial\theta}\langle\nabla_{\mathbb{H}^1}\rho,\nabla_{\mathbb{H}^1}\theta\rangle + \frac{\partial f}{\partial\varphi}\right.$$
$$\left.\langle\nabla_{\mathbb{H}^1}\rho,\nabla_{\mathbb{H}^1}\varphi\rangle\right) + \rho^\alpha\left(\left(\frac{\partial^2 f}{\partial\theta^2}X\theta + \frac{\partial^2 f}{\partial\varphi\partial\theta}X\varphi\right)X\theta + \left(\frac{\partial^2 f}{\partial\theta\partial\varphi}X\theta + \frac{\partial^2 f}{\partial\varphi^2}X\varphi\right)\right.$$
$$X\varphi + \left(\frac{\partial^2 f}{\partial\theta^2}Y\theta + \frac{\partial^2 f}{\partial\varphi\partial\theta}Y\varphi\right)Y\theta + \left(\frac{\partial^2 f}{\partial\theta\partial\varphi}Y\theta + \frac{\partial^2 f}{\partial\varphi^2}Y\varphi\right)Y\varphi + \frac{\partial f}{\partial\theta}(X^2\theta$$
$$\left.+Y^2\theta) + \frac{\partial f}{\partial\varphi}(X^2\varphi+Y^2\varphi)\right) = (\alpha(\alpha-1)\rho^{\alpha-2}\left|\nabla_{\mathbb{H}^1}\rho\right|^2 + \alpha\rho^{\alpha-1}\Delta_{\mathbb{H}^1}\rho)f$$
$$+2\alpha\rho^{\alpha-1}\left(\frac{\partial f}{\partial\theta}\langle\nabla_{\mathbb{H}^1}\rho,\nabla_{\mathbb{H}^1}\theta\rangle + \frac{\partial f}{\partial\varphi}\langle\nabla_{\mathbb{H}^1}\rho,\nabla_{\mathbb{H}^1}\varphi\rangle\right) + \rho^\alpha\left(\frac{\partial^2 f}{\partial\theta^2}\left|\nabla_{\mathbb{H}^1}\theta\right|^2\right.$$
$$\left.+2\frac{\partial^2 f}{\partial\varphi\partial\theta}\langle\nabla_{\mathbb{H}^1}\varphi,\nabla_{\mathbb{H}^1}\theta\rangle + \frac{\partial^2 f}{\partial\varphi^2}\left|\nabla_{\mathbb{H}^1}\varphi\right|^2 + \frac{\partial f}{\partial\theta}\Delta_{\mathbb{H}^1}\theta + \frac{\partial f}{\partial\varphi}\Delta_{\mathbb{H}^1}\varphi\right),$$

which yields

$$\Delta_{\mathbb{H}^1} u = (\alpha(\alpha-1)\rho^{\alpha-2}\left|\nabla_{\mathbb{H}^1}\rho\right|^2 + \alpha\rho^{\alpha-1}\Delta_{\mathbb{H}^1}\rho)f + 2\alpha\rho^{\alpha-1}\left(\frac{\partial f}{\partial\theta}\langle\nabla_{\mathbb{H}^1}\rho,\nabla_{\mathbb{H}^1}\theta\rangle\right.$$
$$\left.+\frac{\partial f}{\partial\varphi}\langle\nabla_{\mathbb{H}^1}\rho,\nabla_{\mathbb{H}^1}\varphi\rangle\right) + \rho^\alpha\left(\frac{\partial^2 f}{\partial\theta^2}\left|\nabla_{\mathbb{H}^1}\theta\right|^2 + 2\frac{\partial^2 f}{\partial\varphi\partial\theta}(\nabla_{\mathbb{H}^1}\varphi\cdot\nabla_{\mathbb{H}^1}\theta) + \frac{\partial^2 f}{\partial\varphi^2}\right.$$
$$\left.\left|\nabla_{\mathbb{H}^1}\varphi\right|^2 + \frac{\partial f}{\partial\theta}\Delta_{\mathbb{H}^1}\theta + \frac{\partial f}{\partial\varphi}\Delta_{\mathbb{H}^1}\varphi\right).$$

In particular, using Lemma 2.10, 2.11 and 2.12, we get

$$\Delta_{\mathbb{H}^1} u = \left(\alpha(\alpha-1)\rho^{\alpha-2}\frac{x^2+y^2}{\rho^2} + \alpha\rho^{\alpha-1}\frac{3(x^2+y^2)}{\rho^3}\right)f - 2\alpha\rho^{\alpha-1}\frac{\partial f}{\partial\theta}\frac{\cos\varphi}{\rho}$$

$$+\rho^{\alpha}\bigg(\frac{\partial^2 f}{\partial\theta^2}\frac{1}{x^2+y^2}+4\frac{\partial^2 f}{\partial\varphi\partial\theta}\frac{x^2+y^2}{\rho^4}+\frac{\partial^2 f}{\partial\varphi^2}\frac{4(x^2+y^2)}{\rho^4}+\frac{\partial f}{\partial\varphi}\frac{4\cos(\varphi)}{\rho^2}\bigg)$$
$$=\alpha(\alpha+2)\bigg(\frac{x^2+y^2}{\rho^2}\bigg)\rho^{\alpha-2}f-2\alpha(\cos(\varphi))\rho^{\alpha-2}\frac{\partial f}{\partial\theta}+\rho^{\alpha-2}\bigg(\frac{\rho^2}{x^2+y^2}\frac{\partial^2 f}{\partial\theta^2}$$
$$+4\frac{x^2+y^2}{\rho^2}\frac{\partial^2 f}{\partial\varphi\partial\theta}+4\frac{x^2+y^2}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2}+4\cos(\varphi)\frac{\partial f}{\partial\varphi}\bigg).$$

Thus, we lastly obtain the desired expression for $\Delta_{\mathbb{H}^1}u$, since $\frac{x^2+y^2}{\rho^2}=\sin(\varphi)$ from (2.30). □

2.7 e_ρ, e_φ as orthonormal basis

In this section, we look for the points $p=(x,y,t)\in\mathbb{H}^1$ where $\{e_\rho,e_\varphi\}$ is an orthonormal basis of $H\mathbb{H}^1_p$, with $H\mathbb{H}^1_p$ denoting the horizontal vector space at $p\in\mathbb{H}^1$, see Section 2.4.
Let $p\in\mathbb{H}^1$. Let us define

$$e_\rho(p):=\bigg(\frac{\nabla_{\mathbb{H}^1}\rho}{|\nabla_{\mathbb{H}^1}\rho|}\bigg)(p),\quad e_\varphi(p):=\bigg(\frac{\nabla_{\mathbb{H}^1}\varphi}{|\nabla_{\mathbb{H}^1}\varphi|}\bigg)(p).$$

We recall, according to Lemma 2.11, that $\langle e_\rho(p),e_\varphi(p)\rangle_{\mathbb{R}^2}=0$. Then, whenever $e_\rho(p),e_\varphi(p)$ exist, we have

$$\operatorname{span}\{e_\rho(p),e_\varphi(p)\}=H\mathbb{H}^1_p.$$

As a consequence, in these cases, since $\{e_\rho(p),e_\varphi(p)\}$ is an orthonormal basis, if $u:\mathbb{H}^1\to\mathbb{R}$ is sufficiently smooth, then

$$\nabla_{\mathbb{H}^1}u(p)=\langle\nabla_{\mathbb{H}^1}u(p),e_\rho(p)\rangle e_\rho(p)+\langle\nabla_{\mathbb{H}^1}u(p),e_\varphi(p)\rangle e_\varphi(p),$$

since $\nabla_{\mathbb{H}^1}u(p)\in H\mathbb{H}^1_p$, and denoting

$$\nabla^\rho_{\mathbb{H}^1}u(p)=\langle\nabla_{\mathbb{H}^1}u(p),e_\rho(p)\rangle e_\rho(p),\quad \nabla^\varphi_{\mathbb{H}^1}u(p)=\langle\nabla_{\mathbb{H}^1}u(p),e_\varphi(p)\rangle e_\varphi(p),\tag{2.45}$$

we have

$$|\nabla_{\mathbb{H}^1}u(p)|^2=\langle\nabla_{\mathbb{H}^1}u(p),e_\rho(p)\rangle^2+\langle\nabla_{\mathbb{H}^1}u(p),e_\varphi(p)\rangle^2,$$

and

$$|\nabla_{\mathbb{H}^1} u(p)|^2 = |\nabla^\rho_{\mathbb{H}^1} u(p)|^2 + |\nabla^\varphi_{\mathbb{H}^1} u(p)|^2. \tag{2.46}$$

Lemma 2.14. *The couple $(\nabla_{\mathbb{H}^1}\rho)(p)$, $(\nabla_{\mathbb{H}^1}\varphi)(p)$ determines a basis of $H\mathbb{H}^1_p$, for every $p = (x, y, t)$, such that $x^2 + y^2 \neq 0$.*

Proof. We look for the points where $\nabla_{\mathbb{H}^1}\rho$ and $\nabla_{\mathbb{H}^1}\varphi$ vanish. We have that $\nabla_{\mathbb{H}^1}\rho = 0$ if

$$\begin{cases} \rho^{-3}((x^2+y^2)x + yt) = 0 \\ \rho^{-3}((x^2+y^2)y - xt) = 0 \end{cases} \iff \begin{cases} (x^2+y^2)x + yt = 0 \\ (x^2+y^2)y - xt = 0, \end{cases} \tag{2.47}$$

which gives, multiplying the first row by $y \neq 0$ and the second one by $x \neq 0$,

$$\begin{cases} (x^2+y^2)yx + y^2t = 0 \\ (x^2+y^2)yx - x^2t = 0. \end{cases} \tag{2.48}$$

Subtracting the second row to the first one in (2.48), we get

$$0 = y^2t + x^2t = (x^2+y^2)t,$$

which implies $t = 0$, because $x \neq 0$ and $y \neq 0$. Now, if $t = 0$, we obtain, from the first row in (2.47), $(x^2+y^2)x = 0$, which is a contradiction, recalling that we have supposed that $x \neq 0$.

Therefore, suppose that $y = 0$, and in view of the first row in (2.47), we have $x = 0$. Analogously, if we assume $x = 0$, we achieve, by the second row in (2.47), $y = 0$. To sum up, we have $\nabla_{\mathbb{H}^1}\rho = 0$ in points $p = (x, y, t)$, with $x = 0$ and $y = 0$.

Concerning $\nabla_{\mathbb{H}^1}\varphi$, we have $\nabla_{\mathbb{H}^1}\varphi = 0$ if

$$\frac{2}{\rho(x^2+y^2)}\left(\rho(-y, x) + t\nabla_{\mathbb{H}^1}\rho\right) = 0,$$

which immediately yields that x and y can not be equal to 0 at the same time, so it is equivalent to

$$\rho(-y, x) + t\nabla_{\mathbb{H}^1}\rho = 0,$$

that is

$$\begin{cases} -\rho y + tX\rho = 0 \\ \rho x + tY\rho = 0 \end{cases} \iff \begin{cases} -\rho y + t\rho^{-3}((x^2+y^2)x + yt) = 0 \\ \rho x + t\rho^{-3}((x^2+y^2)y - xt) = 0 \end{cases}$$

$$\iff \begin{cases} t((x^2+y^2)x + yt) = \rho^4 y \\ t((x^2+y^2)y - xt) = -\rho^4 x, \end{cases}$$

thus, we have to solve

$$\begin{cases} t((x^2+y^2)x + yt) = \rho^4 y \\ t((x^2+y^2)y - xt) = -\rho^4 x. \end{cases} \tag{2.49}$$

Specifically, multiplying the first row in (2.49) by $y \neq 0$ and the second one by $x \neq 0$, we get

$$\begin{cases} t(x^2+y^2)xy + t^2y^2 = \rho^4 y^2 \\ t(x^2+y^2)yx - t^2x^2 = -\rho^4 x^2. \end{cases} \tag{2.50}$$

Subtracting the second row in (2.50) to the first one,

$$(x^2+y^2)t^2 = (x^2+y^2)\rho^4,$$

and dividing by $(x^2+y^2) \neq 0$, recalling that x and y can not be equal to 0 at the same time,

$$t^2 = \rho^4,$$

which implies

$$|t| = \rho^2,$$

and hence $t = \pm\rho^2$. Substituting $t = \rho^2$ in the first row of (2.50), we achieve

$$\rho^2(x^2+y^2)yx + \rho^4y^2 = \rho^4y^2,$$

which gives

$$\rho^2(x^2+y^2)yx = 0,$$

which is a contradiction, since $x \neq 0$, $y \neq 0$ and $\rho \neq 0$. Analogously, if we take $t = -\rho^2$, we have, always from the first row in (2.50),

$$-\rho^2(x^2+y^2)yx + \rho^4y^2 = \rho^4y^2,$$

i.e.

$$-\rho^2(x^2+y^2)yx = 0,$$

which is a contradiction, again because $x \neq 0$, $y \neq 0$ and $\rho \neq 0$.
Suppose now that $y = 0$, and we have, according to the first row in (2.49), $tx^3 = 0$, which entails $t = 0$, since x and y can not be equal to 0 at the same time. At this point, if $y = t = 0$, we have, by the second row in (2.49), $\rho^4 x = 0$, in other words $x = 0$, recalling that $\rho \neq 0$, which is impossible, since $y = 0$. Analogously, if we assume $x = 0$, we have from the second row in (2.49) that the only possibility is $t = 0$, but this condition yields, by virtue of the first row in (2.49), $y = 0$, which is impossible, because $x = 0$. To recap, $\nabla_{\mathbb{H}^1}\varphi \neq 0$, $\forall p \in \mathbb{H}^1$, where it is well defined, i.e. in points $p = (x, y, t)$ such that $x^2 + y^2 \neq 0$.

This fact, together with $\nabla_{\mathbb{H}^1}\rho = 0$ if $x = y = 0$, gives that $(\nabla_{\mathbb{H}^1}\rho)(p)$ and $(\nabla_{\mathbb{H}^1}\varphi)(p)$ are a basis of $H\mathbb{H}^1_p$ in points $p = (x, y, t)$ with $x^2 + y^2 \neq 0$. □

2.8 Some crucial estimates in $\mathbb{H}^1$

In this section, we show a crucial lower bound for

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi},$$

with $u \in C(B_1^{\mathbb{H}^1}(0)) \cap H^1_{\mathbb{H}^1}(B_1^{\mathbb{H}^1}(0))$ nonnegative, and such that $u(0) = 0$, $\Delta_{\mathbb{H}^1} u \geq 0$.
Let us introduce the following notation:

$$A_\rho := \int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla^\rho_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi), \quad A_\varphi := \int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla^\varphi_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi),$$
$$A_u := \int_{\partial B_1^{\mathbb{H}^1}(0)} u^2\sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi). \tag{2.51}$$

As in the Euclidean setting, we obtain the following lower bound.

Lemma 2.15. *Let $u \in C(B_1^{\mathbb{H}^1}(0)) \cap H^1_{\mathbb{H}^1}(B_1^{\mathbb{H}^1}(0))$ be nonnegative, and such that $u(0) = 0$, $\Delta_{\mathbb{H}^1} u \geq 0$. Then, the following lower bound holds:*

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi} \geq \frac{A_\rho + A_\varphi}{A_u + A_u^{1/2} A_\rho^{1/2}}. \tag{2.52}$$

Proof. Using (2.46), we get

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi} = \frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{\left|\nabla^\rho_{\mathbb{H}^1} u\right|^2 + \left|\nabla^\varphi_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi}$$

$$= \frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{\left|\nabla^\rho_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi) + \int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{\left|\nabla^\varphi_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi}. \tag{2.53}$$

We now look for an upper bound of

$$\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi.$$

Specifically, as for the classical Laplacian, we have

$$\Delta_{\mathbb{H}^1} u^2 = 2\left|\nabla_{\mathbb{H}^1} u\right|^2 + 2u\Delta_{\mathbb{H}^1} u,$$

which implies, if u satisfies $u\Delta_{\mathbb{H}^1} u \geq 0$,

$$2\left|\nabla_{\mathbb{H}^1} u\right|^2 \leq \Delta_{\mathbb{H}^1} u^2.$$

In view of this, we then achieve

$$\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi \leq \frac{1}{2}\int_{B_1^{\mathbb{H}^1}(0)} \frac{\Delta_{\mathbb{H}^1} u^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi. \tag{2.54}$$

At this point, recalling the definition of $\operatorname{div}_{\mathbb{H}^1}$ from Section 2.4, it holds

$$|\xi|^{-2}_{\mathbb{H}^1} \Delta_{\mathbb{H}^1} u^2 = \operatorname*{div}_{\mathbb{H}^1}\left(|\xi|^{-2}_{\mathbb{H}^1} \nabla_{\mathbb{H}^1} u^2\right) - \langle \nabla_{\mathbb{H}^1} |\xi|^{-2}_{\mathbb{H}^1}, \nabla_{\mathbb{H}^1} u^2\rangle,$$

which entails, by virtue of (2.54),

$$\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi \le \frac{1}{2}\int_{B_1^{\mathbb{H}^1}(0)} \operatorname*{div}_{\mathbb{H}^1}\left(|\xi|_{\mathbb{H}^1}^{-2}\nabla_{\mathbb{H}^1} u^2\right) d\xi - \frac{1}{2}\int_{B_1^{\mathbb{H}^1}(0)} \langle \nabla_{\mathbb{H}^1} |\xi|_{\mathbb{H}^1}^{-2}, \nabla_{\mathbb{H}^1} u^2\rangle\, d\xi. \tag{2.55}$$

Now, we have, again as for the classical Laplacian,

$$\nabla_{\mathbb{H}^1} u^2 = 2u\nabla_{\mathbb{H}^1} u.$$

Consequently, by the analogue of the divergence theorem in $\mathbb{H}^1$, see Section 2.4, we get

$$\int_{B_1^{\mathbb{H}^1}(0)} \operatorname*{div}_{\mathbb{H}^1}\left(|\xi|_{\mathbb{H}^1}^{-2}\nabla_{\mathbb{H}^1} u^2\right) d\xi = \int_{\partial B_1^{\mathbb{H}^1}(0)} 2\,|\xi|_{\mathbb{H}^1}^{-2}\, u\langle \nabla_{\mathbb{H}^1} u, \nu_{\mathbb{H}^1}\rangle\, d\sigma_{\mathbb{H}^1}(\xi), \tag{2.56}$$

where

$$\nu_{\mathbb{H}^1} = \frac{\nabla_{\mathbb{H}^1}\rho}{|\nabla_{\mathbb{H}^1}\rho|}, \tag{2.57}$$

with

$$\rho = |\xi|_{\mathbb{H}^1}. \tag{2.58}$$

In particular, on $\partial B_1^{\mathbb{H}^1}(0)$, we have $|\xi|_{\mathbb{H}^1} = 1$, therefore, in view of (2.56), we obtain

$$\int_{B_1^{\mathbb{H}^1}(0)} \operatorname*{div}_{\mathbb{H}^1}\left(|\xi|_{\mathbb{H}^1}^{-2}\nabla_{\mathbb{H}^1} u^2\right) d\xi = \int_{\partial B_1^{\mathbb{H}^1}(0)} 2u\langle \nabla_{\mathbb{H}^1} u, \nu_{\mathbb{H}^1}\rangle\, d\sigma_{\mathbb{H}^1}(\xi). \tag{2.59}$$

In addition, it results

$$\langle \nabla_{\mathbb{H}^1} u^2, \nabla_{\mathbb{H}^1} |\xi|_{\mathbb{H}^1}^{-2}\rangle = \operatorname*{div}_{\mathbb{H}^1}\left(u^2\nabla_{\mathbb{H}^1} |\xi|_{\mathbb{H}^1}^{-2}\right) - u^2 \operatorname*{div}_{\mathbb{H}^1}\left(\nabla_{\mathbb{H}^1} |\xi|_{\mathbb{H}^1}^{-2}\right). \tag{2.60}$$

Hence, by the analogue of the divergence theorem in $\mathbb{H}^1$, we have

$$\int_{B_1^{\mathbb{H}^1}(0)} \langle \nabla_{\mathbb{H}^1} u^2, \nabla_{\mathbb{H}^1} |\xi|_{\mathbb{H}^1}^{-2}\rangle\, d\xi = \int_{\partial B_1^{\mathbb{H}^1}(0)} u^2 \langle \nabla_{\mathbb{H}^1} |\xi|_{\mathbb{H}^1}^{-2}, \nu_{\mathbb{H}^1}\rangle\, d\sigma_{\mathbb{H}^1}(\xi) - \int_{B_1^{\mathbb{H}^1}(0)} u^2 \operatorname*{div}_{\mathbb{H}^1}\left(\nabla_{\mathbb{H}^1} |\xi|_{\mathbb{H}^1}^{-2}\right) d\xi. \tag{2.61}$$

As a consequence, by virtue of (2.59) and (2.61), we achieve from (2.55)

$$\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi \le \int_{\partial B_1^{\mathbb{H}^1}(0)} u \langle \nabla_{\mathbb{H}^1} u, \nu_{\mathbb{H}^1}\rangle \, d\sigma_{\mathbb{H}^1}(\xi)$$
$$+ \frac{1}{2}\int_{B_1^{\mathbb{H}^1}(0)} u^2 \operatorname*{div}_{\mathbb{H}^1}\left(\nabla_{\mathbb{H}^1} |\xi|^{-2}_{\mathbb{H}^1}\right) d\xi - \frac{1}{2}\int_{\partial B_1^{\mathbb{H}^1}(0)} u^2 \langle \nabla_{\mathbb{H}^1} |\xi|^{-2}_{\mathbb{H}^1}, \nu_{\mathbb{H}^1}\rangle \, d\sigma_{\mathbb{H}^1}(\xi). \tag{2.62}$$

At this point, it holds, according to (2.58),

$$\nabla_{\mathbb{H}^1} |\xi|^{-2}_{\mathbb{H}^1} = -2\, |\xi|^{-3}_{\mathbb{H}^1} \nabla_{\mathbb{H}^1} |\xi|_{\mathbb{H}^1} = -2\rho^{-3} \nabla_{\mathbb{H}^1}\rho.$$

which yields, using (2.57),

$$\langle \nabla_{\mathbb{H}^1} |\xi|^{-2}_{\mathbb{H}^1}, \nu_{\mathbb{H}^1}\rangle = -2\rho^{-3} \left|\nabla_{\mathbb{H}^1}\rho\right|. \tag{2.63}$$

Notice, in particular, that by (2.57) we have

$$\nu_{\mathbb{H}^1} = e_\rho. \tag{2.64}$$

Thus, by virtue of this and (2.63), we obtain from (2.62), also exploiting (2.45),

$$\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi \le \int_{\partial B_1^{\mathbb{H}^1}(0)} u \langle \nabla_{\mathbb{H}^1} u, e_\rho\rangle \, d\sigma_{\mathbb{H}^1}(\xi)$$
$$+ \frac{1}{2}\int_{B_1^{\mathbb{H}^1}(0)} u^2 \operatorname*{div}_{\mathbb{H}^1}\left(\nabla_{\mathbb{H}^1} |\xi|^{-2}_{\mathbb{H}^1}\right) d\xi + \int_{\partial B_1^{\mathbb{H}^1}(0)} u^2 \left|\nabla_{\mathbb{H}^1}\rho\right| \, d\sigma_{\mathbb{H}^1}(\xi),$$

which implies

$$\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi \le \int_{\partial B_1^{\mathbb{H}^1}(0)} \left(u \langle \nabla_{\mathbb{H}^1} u, e_\rho\rangle + u^2 \left|\nabla_{\mathbb{H}^1}\rho\right|\right) d\sigma_{\mathbb{H}^1}(\xi)$$
$$+ \frac{1}{2}\int_{B_1^{\mathbb{H}^1}(0)} u^2 \operatorname*{div}_{\mathbb{H}^1}\left(\nabla_{\mathbb{H}^1} |\xi|^{-2}_{\mathbb{H}^1}\right) d\xi. \tag{2.65}$$

At this point, we know that $|\xi|^{-2}_{\mathbb{H}^1}$ is, up to a multiplicative constant, the fundamental solution of $\Delta_{\mathbb{H}^1}$, and in addition

$$\Delta_{\mathbb{H}^1} = \operatorname*{div}_{\mathbb{H}^1} \nabla_{\mathbb{H}^1},$$

thus

$$\operatorname*{div}_{\mathbb{H}^1}\left(\nabla_{\mathbb{H}^1} |\xi|_{\mathbb{H}^1}^{-2}\right) = \Delta_{\mathbb{H}^1} |\xi|_{\mathbb{H}^1}^{-2} = \delta_0,$$

with δ_0 the Dirac delta centered at 0.

Consequently, recalling that $u(0) = 0$, and therefore also $u^2(0) = 0$, we achieve

$$u^2 \operatorname*{div}_{\mathbb{H}^1}\left(\nabla_{\mathbb{H}^1} |\xi|_{\mathbb{H}^1}^{-2}\right) = u^2\delta_0 = 0 \quad \text{in } B_1^{\mathbb{H}^1}(0),$$

which entails, in view of (2.65),

$$\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi \le \int_{\partial B_1^{\mathbb{H}^1}(0)} \left(u\langle \nabla_{\mathbb{H}^1} u, e_\rho\rangle + u^2 |\nabla_{\mathbb{H}^1}\rho|\right)\, d\sigma_{\mathbb{H}^1}(\xi). \tag{2.66}$$

In particular, we have from Lemma 2.10

$$|\nabla_{\mathbb{H}^1}\rho| = \frac{\sqrt{x^2+y^2}}{\rho},$$

which gives

$$|\nabla_{\mathbb{H}^1}\rho| = \sqrt{x^2+y^2} \quad \text{on } \partial B_1^{\mathbb{H}^1}(0).$$

Substituting this in (2.66), we then get

$$\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi \le \int_{\partial B_1^{\mathbb{H}^1}(0)} \left(u\langle \nabla_{\mathbb{H}^1} u, e_\rho\rangle + u^2 \sqrt{x^2+y^2}\right)\, d\sigma_{\mathbb{H}^1}(\xi). \tag{2.67}$$

Specifically, we can rewrite the right term in (2.67) as

$$\int_{\partial B_1^{\mathbb{H}^1}(0)} u\langle \nabla_{\mathbb{H}^1} u, e_\rho\rangle \frac{\sqrt[4]{x^2+y^2}}{\sqrt[4]{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi) + \int_{\partial B_1^{\mathbb{H}^1}(0)} u^2\sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi),$$

which gives, by Hölder inequality,

$$\begin{aligned}\int_{\partial B_1^{\mathbb{H}^1}(0)} \left(u\langle \nabla_{\mathbb{H}^1} u, e_\rho\rangle + u^2 \sqrt{x^2+y^2}\right)\, d\sigma_{\mathbb{H}^1}(\xi) &\le \int_{\partial B_1^{\mathbb{H}^1}(0)} u^2\sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi) \\ + \left(\int_{\partial B_1^{\mathbb{H}^1}(0)} u^2\sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi)\right)^{1/2} &\left(\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{\left|\nabla_{\mathbb{H}^1}^\rho u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)\right)^{1/2}.\end{aligned}$$

As a consequence, we have, from (2.67),

$$\begin{aligned}\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi &\le \int_{\partial B_1^{\mathbb{H}^1}(0)} u^2\sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi) \\ &+ \left(\int_{\partial B_1^{\mathbb{H}^1}(0)} u^2\sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi)\right)^{1/2} \left(\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{\left|\nabla_{\mathbb{H}^1}^\rho u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)\right)^{1/2}.\end{aligned} \tag{2.68}$$

To recap, we have increased the denominator of (2.53) with (2.68) and using this, we then obtain from (2.53) the desired lower bound. □

Let us introduce now the notation

$$\lambda_{\varphi(\Sigma)} := \inf_{v \in H_0^1(\Sigma)} \frac{\displaystyle\int_\Sigma \frac{\left|\nabla^{\varphi}_{\mathbb{H}^1} v\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_\Sigma v^2 \sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi)}, \tag{2.69}$$

where, in general, $\Sigma \subset \partial B_1^{\mathbb{H}^1}(0)$ is a rectifiable set. In particular, for our purposes, we consider

$$\Sigma := \partial B_1^{\mathbb{H}^1}(0) \cap \{u > 0\}, \tag{2.70}$$

for u as in Lemma 2.52.

Theorem 2.16. *Let $u \in C(B_1^{\mathbb{H}^1}(0)) \cap H^1_{\mathbb{H}^1}(B_1^{\mathbb{H}^1}(0))$ be nonnegative, and such that $u(0) = 0$, $\Delta_{\mathbb{H}^1} u \geq 0$. Then*

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi} \geq 2\left(\sqrt{1+\lambda_{\varphi(\Sigma)}} - 1\right). \tag{2.71}$$

Proof. First of all, we remark that $A_u \neq 0$, hence the right term in (2.52) becomes

$$\frac{A_\rho + A_\varphi}{A_u + A_u^{1/2} A_\rho^{1/2}} = \frac{\dfrac{A_\rho}{A_u} + \dfrac{A_\varphi}{A_u}}{1 + \left(\dfrac{A_\rho}{A_u}\right)^{1/2}}. \tag{2.72}$$

Substituting it in (2.52) we then achieve

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi} \geq \frac{\dfrac{A_\rho}{A_u} + \dfrac{A_\varphi}{A_u}}{1 + \left(\dfrac{A_\rho}{A_u}\right)^{1/2}}. \tag{2.73}$$

Furthermore, recalling that $u \in H_0^1(\Sigma)$ and (2.70), we have

$$\frac{A_\varphi}{A_u} \geq \lambda_{\varphi(\Sigma)},$$

which entails, in view of (2.73),

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi} \geq \frac{\dfrac{A_\rho}{A_u} + \lambda_{\varphi(\Sigma)}}{1 + \left(\dfrac{A_\rho}{A_u}\right)^{1/2}}. \tag{2.74}$$

At this point, if we call

$$s = \frac{A_\rho}{A_u}, \tag{2.75}$$

we can rewrite the right term in (2.74) as a function depending on s, precisely as

$$F(s) = \frac{s + \lambda_{\varphi(\Sigma)}}{1+\sqrt{s}}, \quad s \in \mathbb{R},\ s > 0.$$

Our idea is to find the minimum of F to get a lower bound of

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi}.$$

Specifically, we have

$$\begin{aligned} F'(s) &= \frac{1+\sqrt{s} - (s+\lambda_{\varphi(\Sigma)})\frac{1}{2\sqrt{s}}}{(1+\sqrt{s})^2} = \frac{(1+\sqrt{s})2\sqrt{s} - s - \lambda_{\varphi(\Sigma)}}{2\sqrt{s}(1+\sqrt{s})^2} \\ &= \frac{2\sqrt{s} + s - \lambda_{\varphi(\Sigma)}}{2\sqrt{s}(1+\sqrt{s})^2}. \end{aligned}$$

At this point, we notice that the denominator in the expression of F' is always positive, so we have to study the numerator to find the minimum.
In particular, it results

$$2\sqrt{s} + s - \lambda_{\varphi(\Sigma)} \geq 0 \quad \overset{z=\sqrt{s}}{\Longleftrightarrow} \quad 2z + z^2 - \lambda_{\varphi(\Sigma)} \geq 0.$$

Now, the roots of $z^2 + 2z - \lambda_{\varphi(\Sigma)}$ are

$$z_\pm = -1 \pm \sqrt{1 + \lambda_{\varphi(\Sigma)}},$$

but because $z > 0$, we obtain

$$z^2 + 2z - \lambda_{\varphi(\Sigma)} \geq 0 \iff z \geq -1 + \sqrt{1 + \lambda_{\varphi(\Sigma)}},$$

which implies that

$$s = z^2 = \left(-1 + \sqrt{1 + \lambda_{\varphi(\Sigma)}}\right)^2$$

is the minimum point of F.
Consequently, we achieve

$$\begin{aligned} F(s) \geq F\left(\left(-1 + \sqrt{1 + \lambda_{\varphi(\Sigma)}}\right)^2\right) &= \frac{1 + 1 + \lambda_{\varphi(\Sigma)} - 2\sqrt{1 + \lambda_{\varphi(\Sigma)}} + \lambda_{\varphi(\Sigma)}}{1 - 1 + \sqrt{1 + \lambda_{\varphi(\Sigma)}}} \\ = 2\frac{1 + \lambda_{\varphi(\Sigma)} - \sqrt{1 + \lambda_{\varphi(\Sigma)}}}{\sqrt{1 + \lambda_{\varphi(\Sigma)}}} &= 2\left(\sqrt{1 + \lambda_{\varphi(\Sigma)}} - 1\right), \end{aligned}$$

which yields

$$\frac{s + \lambda_{\varphi(\Sigma)}}{1 + \sqrt{s}} \geq 2\left(\sqrt{1 + \lambda_{\varphi(\Sigma)}} - 1\right),$$

and thus, using (2.75),

$$\frac{\dfrac{A_\rho}{A_u} + \lambda_{\varphi(\Sigma)}}{1 + \left(\dfrac{A_\rho}{A_u}\right)^{1/2}} \geq 2\left(\sqrt{1 + \lambda_{\varphi(\Sigma)}} - 1\right).$$

This fact, together with (2.74), then entails the expected lower bound.

□

We now show an alternative proof of Theorem 2.16, which follows the approach developed in [15] about the classical Alt-Caffarelli-Friedman monotonicity formula.
We first consider for every $\beta \in (0, 1)$ the following lower bound, recalling the

definition of $\lambda_{\varphi(\Sigma)}$ in (2.69), (2.46) and employing the Cauchy inequality:

$$\begin{aligned}
&\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi) \geq \int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{\left|\nabla^\rho_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi) + \lambda_{\varphi(\Sigma)} \\
&\int_{\partial B_1^{\mathbb{H}^1}(0)} u^2 \sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi) = \int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{\left|\nabla^\rho_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi) + \lambda_{\varphi(\Sigma)}\beta \\
&\int_{\partial B_1^{\mathbb{H}^1}(0)} u^2 \sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi) + \lambda_{\varphi(\Sigma)}(1-\beta) \int_{\partial B_1^{\mathbb{H}^1}(0)} u^2 \sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi) \\
&\geq 2 \left(\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{\left|\nabla^\rho_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi) \right)^{\frac{1}{2}} \left(\lambda_{\varphi(\Sigma)}\beta \int_{\partial B_1^{\mathbb{H}^1}(0)} u^2 \sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi) \right)^{\frac{1}{2}} \\
&+ \lambda_{\varphi(\Sigma)}(1-\beta) \int_{\partial B_1^{\mathbb{H}^1}(0)} u^2 \sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi).
\end{aligned} \tag{2.76}$$

In view of (2.76) and (2.68), it then follows, since $A_u \neq 0$,

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi} \geq \frac{2(\lambda_{\varphi(\Sigma)}\beta)^{1/2} A_\rho^{1/2} A_u^{1/2} + (1-\beta)\lambda_{\varphi(\Sigma)} A_u}{A_u + A_u^{1/2} A_\rho^{1/2}}$$

$$= \frac{2(\lambda_{\varphi(\Sigma)}\beta)^{1/2} \frac{A_\rho^{1/2}}{A_u^{1/2}} + (1-\beta)\lambda_{\varphi(\Sigma)}}{1 + \frac{A_\rho^{1/2}}{A_u^{1/2}}} \geq \min\{(1-\beta)\lambda_{\varphi(\Sigma)}, 2(\lambda_{\varphi(\Sigma)}\beta)^{1/2}\}.$$

At this point, let $\beta \in (0,1)$ be such that

$$(1-\beta)\lambda_{\varphi(\Sigma)} = 2(\lambda_{\varphi(\Sigma)}\beta)^{1/2}. \tag{2.77}$$

Then, denoting $\alpha := (\lambda_{\varphi(\Sigma)}\beta)^{1/2}$, we obtain that the previous relationship is satisfied whenever

$$\alpha^2 + 2\alpha - \lambda_{\varphi(\Sigma)} = 0. \tag{2.78}$$

We point out that, from (2.77), it follows

$$(1-\beta)\sqrt{\lambda_{\varphi(\Sigma)}} = 2\sqrt{\beta},$$

as well, so that, denoting $\gamma = \sqrt{\beta}$, we get

$$\sqrt{\lambda_{\varphi(\Sigma)}}\gamma^2 + 2\gamma - \sqrt{\lambda_{\varphi(\Sigma)}} = 0.$$

This yields

$$\gamma = \frac{-1 \pm \sqrt{1+\lambda_{\varphi(\Sigma)}}}{\sqrt{\lambda_{\varphi(\Sigma)}}},$$

but, since $\beta > 0$, it results

$$\gamma = \frac{-1 + \sqrt{1+\lambda_{\varphi(\Sigma)}}}{\sqrt{\lambda_{\varphi(\Sigma)}}} = -\frac{1}{\sqrt{\lambda_{\varphi(\Sigma)}}} + \sqrt{1+\frac{1}{\lambda_{\varphi(\Sigma)}}}.$$

Now, the function $r \to -r + \sqrt{1+r^2}$ is positive in $[0, +\infty)$ and since $(-r + \sqrt{1+r^2})' = -1 + \frac{r}{\sqrt{1+r^2}} < 0$ for every $r > 0$, this function is monotone decreasing, so that $0 < -r + \sqrt{1+r^2} \le 1$.

As a consequence, there exists $\beta \in (0,1)$ given by

$$\beta = \left(\frac{-1 + \sqrt{1+\lambda_{\varphi(\Sigma)}}}{\sqrt{\lambda_{\varphi(\Sigma)}}}\right)^2$$

such that, when (2.77) is realized, it holds

$$\min\{(1-\beta)\lambda_{\varphi(\Sigma)}, 2(\lambda_{\varphi(\Sigma)}\beta)^{1/2}\} = 2\left(\sqrt{1+\lambda_{\varphi(\Sigma)}} - 1\right),$$

as stated in Theorem 2.16.

2.9 Straightforward computation of the two basic cases

In this section, we want to state and prove two further lemmas and next give a generalization of one of them, which will be useful in the proof of Theorem 2.1.

Lemma 2.17. *If* $u = x^+$, *then*

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi} = 2.$$

Proof. First of all, we note that

$$\nabla_{\mathbb{H}^1} x^+ = \chi_{\{x>0\}},$$

hence

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi} = \frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{x>0\}} \frac{1}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)\cap\{x>0\}} \frac{1}{|\xi|^2_{\mathbb{H}^1}}\, d\xi}. \tag{2.79}$$

Let us calculate now

$$\int_{B_1^{\mathbb{H}^1}(0)\cap\{x>0\}} \frac{1}{|\xi|^2_{\mathbb{H}^1}}\, d\xi.$$

To this end, we apply the change of variables in spherical coordinates, that is, denoting $\xi := (x, y, t)$,

$$\xi = T(\rho, \varphi, \theta) := (\rho\sqrt{\sin(\varphi)}\cos(\theta), \rho\sqrt{\sin(\varphi)}\sin(\theta), \rho^2\cos(\varphi)), \tag{2.80}$$

and, since $|\det J_T| = \rho^3$ and

$$x = \rho\sqrt{\sin(\varphi)}\cos(\theta) > 0 \Longleftrightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \tag{2.81}$$

we get

$$\int_{B_1^{\mathbb{H}^1}(0)\cap\{x>0\}} \frac{1}{|\xi|^2_{\mathbb{H}^1}}\, d\xi = \int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 \frac{\rho^3}{\rho^2}\, d\rho\, d\varphi\, d\theta = \pi^2\left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=1} = \frac{\pi^2}{2}. \tag{2.82}$$

Regarding

$$\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{x>0\}} \frac{1}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi),$$

instead, we recall first that, by definition of $d\sigma_{\mathbb{H}^1}$, see Section 2.4, it holds

$$d\sigma_{\mathbb{H}^1}(\xi) = \frac{\sqrt{x^2+y^2}}{|\nabla\rho|}\, d\sigma(\xi), \tag{2.83}$$

hence we achieve

$$\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{x>0\}} \frac{1}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi) = \int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{x>0\}} \frac{1}{|\nabla\rho|}\, d\sigma(\xi). \tag{2.84}$$

At this point, we consider the following parametrization of $\partial B_1^{\mathbb{H}^1}(0)$:

$$\xi = K(\theta,\varphi) \coloneqq (\sqrt{\sin(\varphi)}\cos(\theta), \sqrt{\sin(\varphi)}\sin(\theta), \cos(\varphi)). \tag{2.85}$$

Then, we obtain

$$d\sigma(\xi) = \left|\frac{\partial K}{\partial\theta}\wedge\frac{\partial K}{\partial\varphi}\right|\, d\theta\, d\varphi,$$

which yields

$$\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{x>0\}} \frac{1}{|\nabla\rho|}\, d\sigma(\xi) = \int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{|\nabla\rho|}\left|\frac{\partial K}{\partial\theta}\wedge\frac{\partial K}{\partial\varphi}\right|\, d\theta\, d\varphi, \tag{2.86}$$

using (2.81).
In particular, we have

$$\begin{aligned}|\nabla\rho| &= \left|\frac{1}{4}((x^2+y^2)^2+t^2)^{-3/4}(2(x^2+y^2)2x, 2(x^2+y^2)2y, 2t)\right|\\ &= \frac{1}{2}\rho^{-3}\sqrt{4x^2(x^2+y^2)^2+4y^2(x^2+y^2)^2+t^2} = \frac{1}{2}\sqrt{4(x^2+y^2)^3+t^2},\end{aligned}$$

that is, according to (2.85),

$$|\nabla\rho|\,|_{\partial B_1^{\mathbb{H}^1}(0)} = \frac{1}{2}\sqrt{4\sin^3(\varphi)+\cos^2(\varphi)} \quad \text{in } \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times[0,2\pi).$$

On the other hand, in view of (2.85), it results

$$\begin{aligned}\left|\frac{\partial K}{\partial\theta}\wedge\frac{\partial K}{\partial\varphi}\right| &= \left|\left(-\sin^{3/2}(\varphi)\cos(\theta), -\sin^{3/2}(\varphi)\sin(\theta), -\frac{\cos(\varphi)}{2}\right)\right|\\ &= \sqrt{\sin^3\varphi+\frac{\cos^2(\varphi)}{4}} = \frac{1}{2}\sqrt{4\sin^3\varphi+\cos^2(\varphi)},\end{aligned}$$

which thus implies

$$\frac{\left|\frac{\partial K}{\partial\theta}\wedge\frac{\partial K}{\partial\varphi}\right|}{|\nabla\rho|} = 1 \quad \text{in } \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times[0,2\pi). \tag{2.87}$$

Consequently, we have

$$\int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{|\nabla\rho|}\left|\frac{\partial K}{\partial\theta}\wedge\frac{\partial K}{\partial\varphi}\right| d\theta\, d\varphi = \pi^2,$$

hence, from (2.84) and (2.86), we get

$$\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{x>0\}} \frac{1}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi) = \pi^2. \tag{2.88}$$

Putting together (2.82) and (2.88), we lastly achieve by (2.79) the expected result. □

Lemma 2.18. *If* $u = t^+$, *then*

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi} = 4.$$

Proof. We point out first that, in this case, we have

$$\nabla_{\mathbb{H}^1} t^+ = 2(y, -x)\chi_{\{t>0\}}.$$

Therefore, it holds

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1} u|^2}{|\xi|^2_{\mathbb{H}^1}}\, d\xi} = \frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{t>0\}} 4\sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)\cap\{t>0\}} \frac{4(x^2+y^2)}{|\xi|^2_{\mathbb{H}^1}}\, d\xi}. \tag{2.89}$$

Now, let us compute

$$\int_{B_1^{\mathbb{H}^1}(0)\cap\{t>0\}} \frac{4(x^2+y^2)}{|\xi|^2_{\mathbb{H}^1}}\, d\xi.$$

Using the change of variables in spherical coordinates (2.80) and noting that

$$t = \rho^2\cos(\varphi) > 0 \Longleftrightarrow 0 < \varphi < \frac{\pi}{2}, \tag{2.90}$$

we then obtain

$$\int_{B_1^{\mathbb{H}^1}(0)\cap\{t>0\}} \frac{4(x^2+y^2)}{|\xi|^2_{\mathbb{H}^1}}\,d\xi = \int_{-\pi}^{\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \frac{4\rho^2\sin(\varphi)}{\rho^2}\rho^3\,d\rho d\varphi d\theta = 2\pi. \tag{2.91}$$

In parallel, in view of (2.83), (2.87) and (2.90), we get

$$\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{t>0\}} 4\sqrt{x^2+y^2}\,d\sigma_{\mathbb{H}^1}(\xi) = 4\int_{-\pi}^{\pi}\int_0^{\frac{\pi}{2}} \sin(\varphi)\,d\varphi\,d\theta = 8\pi \tag{2.92}$$

as well. So, from (2.89), (2.91) and (2.92), we lastly have the desired equality. □

Lemma 2.19. *For every $a, b \in \mathbb{R}$ such that $a^2+b^2 \neq 0$, let $u = (ax+by)^+$ be defined in $B_1^{\mathbb{H}^1}(0)$. Then*

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1}u|^2}{\sqrt{x^2+y^2}}\,d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1}u|^2}{|\xi|^2_{\mathbb{H}^1}}\,d\xi} = 2.$$

2.10 Proof of Theorem 2.1

In this section, we prove the main result Theorem 2.1, as a consequence of Lemma 2.8 and Lemma 2.19.

Proof of Theorem 2.1. Let $u_1 = (ax+by)^+$ and $u_2 = (ax+by)^-$, with $a, b \in \mathbb{R}$ such that $a^2+b^2 \neq 0$. Then, we employ Lemma 2.17, which holds in the same way for u_2, concluding that

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1}u_1|^2}{\sqrt{x^2+|y|^2}}d\sigma_{\mathbb{H}^1}(\kappa)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1}u_1|^2}{|\kappa|^{Q-2}_{\mathbb{H}^1}}d\kappa} + \frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1}u_2|^2}{\sqrt{x^2+|y|^2}}d\sigma_{\mathbb{H}^1}(\kappa)}{\displaystyle\int_{B_1^{\mathbb{H}^1}(0)} \frac{|\nabla_{\mathbb{H}^1}u_2|^2}{|\kappa|^{Q-2}_{\mathbb{H}^1}}d\kappa} = 4.$$

Thus, if $\beta > 4$, then $u_1 = (ax+by)^+$ and $u_2 = (ax+by)^-$ satisfy the hypotheses of Lemma 2.8, but $\frac{J'_{\beta,\mathbb{H}^1}(1)}{J_{\beta,\mathbb{H}^1}(1)} < 0$ when tested on these choices of u_1 and u_2. Hence, in order to preserve the increasing monotonicity of $J_{\beta,\mathbb{H}^1}$, we must assume that $\beta \leq 4$. □

2.11 The case f independent of θ

In this section, we analyze the case of $\mathbb{H}^1$-harmonic functions in the form $u = \rho^\alpha f(\varphi)$, with f a given function depending on φ.
First, we recall that in Lemma 2.13 we proved that if $u = \rho^\alpha f(\theta, \varphi)$, then

$$\Delta_{\mathbb{H}^1} u = \rho^{\alpha-2}\Bigg(\alpha(\alpha+2)\sin(\varphi) f(\theta,\varphi) - 2\alpha(\cos(\varphi))\frac{\partial f}{\partial \theta} + \frac{1}{\sin(\varphi)}\frac{\partial^2 f}{\partial \theta^2} + 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi \partial \theta} + 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi^2} + 4\cos(\varphi)\frac{\partial f}{\partial \varphi}\Bigg). \tag{2.93}$$

Now, if we evaluate this expression on $\partial B_1^{\mathbb{H}^1}(0)$, we get

$$\Delta_{\mathbb{H}^1} u\big|_{\partial B_1^{\mathbb{H}^1}(0)} = \alpha(\alpha+2)\sin(\varphi) f(\theta,\varphi) - 2\alpha\cos(\varphi)\frac{\partial f}{\partial \theta} + \frac{1}{\sin(\varphi)}\frac{\partial^2 f}{\partial \theta^2} + 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi \partial \theta} + 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi^2} + 4\cos(\varphi)\frac{\partial f}{\partial \varphi}. \tag{2.94}$$

Corollary 2.20. *If $u = \rho^\alpha f(\varphi)$, then*

$$\Delta_{\mathbb{H}^1} u\big|_{\partial B_1^{\mathbb{H}^1}(0)} = \alpha(\alpha+2)\sin(\varphi) f + 4\frac{\partial}{\partial \varphi}\left(\sin(\varphi)\frac{\partial f}{\partial \varphi}\right).$$

Proof. For the sake of simplicity, in the following we will denote $\Delta_{\mathbb{H}^1} u\big|_{\partial B_1^{\mathbb{H}^1}(0)}$ with $\Delta_{\mathbb{H}^1} u$.

In particular, if $f(\theta,\varphi)$ does not depend on θ, i.e. $f = f(\varphi)$, we obtain, in view of (2.94),

$$\Delta_{\mathbb{H}^1} u = \alpha(\alpha+2)\sin(\varphi) f + 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi^2} + 4\cos(\varphi)\frac{\partial f}{\partial \varphi}. \tag{2.95}$$

At this point, we note that

$$4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi^2} + 4\cos(\varphi)\frac{\partial f}{\partial \varphi} = 4\frac{\partial}{\partial \varphi}\left(\sin(\varphi)\frac{\partial f}{\partial \varphi}\right),$$

which implies, from (2.95), our thesis. □

Corollary 2.20 yields the following lemma as well.

Lemma 2.21. *If $\alpha = 2$ and we take $u = \rho^2 \cos(\varphi)$, we have that u is $\Delta_{\mathbb{H}^1}$ -harmonic, that is $\Delta_{\mathbb{H}^1} u = 0$, and*

$$8 = \frac{4 \displaystyle\int_0^{\frac{\pi}{2}} \sin(\varphi)((\cos(\varphi))')^2 \, d\varphi}{\displaystyle\int_0^{\frac{\pi}{2}} \sin(\varphi) \cos^2(\varphi) \, d\varphi}.$$

Proof. This result can be found in [56]. However, to help the reader, we give a straightforward proof of this fact.

First of all, from (2.93), we achieve that if $u = \rho^\alpha f(\varphi)$,

$$\Delta_{\mathbb{H}^1} u = \rho^{\alpha-2}\left(\alpha(\alpha+2)\sin(\varphi) f + 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi^2} + 4\cos(\varphi)\frac{\partial f}{\partial \varphi}\right).$$

As a consequence, if $u = \rho^2 \cos(\varphi)$, we have, because $\alpha = 2$ and $f(\varphi) = \cos(\varphi)$,

$$\Delta_{\mathbb{H}^1} u = 8\sin(\varphi)\cos(\varphi) - 4\sin(\varphi)\cos(\varphi) - 4\cos(\varphi)\sin(\varphi) = 0.$$

Now, if $u = \rho^\alpha f(\varphi)$ satisfies $\Delta_{\mathbb{H}^1} u|_{\partial B_1^{\mathbb{H}^1}(0)} = 0$, we have, according to Corollary 2.20,

$$-4(\sin(\varphi) f')' = \alpha(\alpha+2)\sin(\varphi) f,$$

writing $\frac{\partial}{\partial \varphi}\left(\sin(\varphi)\frac{\partial f}{\partial \varphi}\right) = (\sin(\varphi) f')'$, since f is a function depending only on φ, and multiplying both the terms of the equality by η sufficiently smooth with $\eta\left(\frac{\pi}{2}\right) = 0$, it holds

$$\alpha(\alpha+2)\sin(\varphi) f\eta = -4(\sin(\varphi) f')'\eta.$$

Integrating over $\left[0, \dfrac{\pi}{2}\right]$ the previous equality, we then obtain

$$\alpha(\alpha+2)\int_0^{\frac{\pi}{2}} \sin(\varphi) f\eta \, d\varphi = -4\int_0^{\frac{\pi}{2}} (\sin(\varphi) f')'\eta \, d\varphi.$$

In particular, if we choose $\eta = f$, we get, because $f\left(\frac{\pi}{2}\right) = 0$ by virtue of the choice of f,

$$\alpha(\alpha+2)\int_0^{\frac{\pi}{2}} \sin(\varphi) f^2 \, d\varphi = -4\left(\left[(\sin(\varphi) f') f\right]_{\varphi=0}^{\varphi=\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin(\varphi) f' f' \, d\varphi\right)$$
$$= 4\int_0^{\frac{\pi}{2}} \sin(\varphi)(f')^2 \, d\varphi.$$

In addition, in view of this, we also have

$$\alpha(\alpha+2) = \frac{4\displaystyle\int_0^{\frac{\pi}{2}} \sin(\varphi)(f')^2\, d\varphi}{\displaystyle\int_0^{\frac{\pi}{2}} \sin(\varphi) f^2\, d\varphi}. \tag{2.96}$$

At this point, we recall that, from Lemma 2.21, $\rho^2\cos(\varphi)$ is $\mathbb{H}^1$-harmonic, where $\alpha = 2$ and $f(\varphi) = \cos(\varphi)$, with $\cos\left(\dfrac{\pi}{2}\right) = 0$, hence, repeating the same argument used to achieve the last equality, it results

$$8 = \frac{4\displaystyle\int_0^{\frac{\pi}{2}} \sin(\varphi)(\cos'(\varphi))^2\, d\varphi}{\displaystyle\int_0^{\frac{\pi}{2}} \sin(\varphi)\cos^2(\varphi)\, d\varphi}.$$

□

Now, denoting by $\mathcal{L}f := 4(\sin(\varphi)f')'$, and considering the following eigenvalues problem

$$\begin{cases} \mathcal{L}f + \lambda\sin(\varphi)f = 0, & \varphi_1 < \varphi < \varphi_2 \\ f(\varphi_1) = 0, \\ f(\varphi_2) = 0, \end{cases} \tag{2.97}$$

it results that α has to satisfy the relationship

$$\alpha(\alpha+2) = \lambda, \tag{2.98}$$

which is exactly the same one obtained in (2.78). Furthermore, we know, from the proof of Lemma 2.21, that the identity below is valid:

$$\alpha(\alpha+2) = \frac{4\displaystyle\int_0^{\varphi_0} \sin(\varphi)(f')^2\, d\varphi}{\displaystyle\int_0^{\varphi_0} \sin(\varphi) f^2\, d\varphi}.$$

On the other hand, performing a change of variables $\tau = \pi - \varphi$, it holds

$$\alpha(\alpha+2) = \frac{-4\displaystyle\int_{\pi}^{\pi-\varphi_0} \sin(\pi-\tau)(f')^2(\pi-\tau)\,d\tau}{-\displaystyle\int_{\pi}^{\pi-\varphi_0} \sin(\pi-\tau)f^2(\pi-\tau)\,d\tau} = \frac{4\displaystyle\int_{\pi-\varphi_0}^{\pi} \sin(\tau)(f')^2(\pi-\tau)\,d\tau}{\displaystyle\int_{\pi-\varphi_0}^{\pi} \sin(\tau)f^2(\pi-\tau)\,d\tau}.$$

To sum up, we have achieved

$$\begin{aligned} \alpha_1(\varphi_0)(\alpha_1(\varphi_0)+2) &= \frac{4\displaystyle\int_0^{\varphi_0} \sin(\varphi)(f')^2\,d\varphi}{\displaystyle\int_0^{\varphi_0} \sin(\varphi)f^2\,d\varphi} \\ \alpha_1(\eta_0)(\alpha_1(\eta_0)+2) &= \frac{4\displaystyle\int_{\eta_0}^{\pi} \sin(\tau)(f')^2(\pi-\tau)\,d\tau}{\displaystyle\int_{\eta_0}^{\pi} \sin(\tau)f^2(\pi-\tau)\,d\tau} \end{aligned} \tag{2.99}$$

where $\varphi_0 + \eta_0 = \pi$.

Lemma 2.22. *The function*

$$G(\varphi) = \alpha_1(\varphi)(\alpha_1(\varphi)+2) + \alpha_1(\pi-\varphi)(\alpha_1(\pi-\varphi)+2), \quad \varphi \in [0,\pi],$$

is symmetric with respect to $\frac{\pi}{2}$.

Proof. For every $\varphi_0 \in [0, \frac{\pi}{2}]$ we have

$$\begin{aligned} &G(\varphi_0) = \alpha_1(\pi-(\pi-\varphi_0))(\alpha_1(\pi-(\pi-\varphi_0))+2) + \alpha_1(\pi-\varphi_0) \\ &(\alpha_1(\pi-\varphi_0)+2) = G(\pi-\varphi_0). \end{aligned}$$

□

In particular $u = \rho^2\cos(\varphi)$ is $\mathbb{H}^1-$harmonic in $\{(x,y,t) \in \mathbb{H}^1 : \ t \geq 0\}$.

Let us denote, at this point, by $\lambda_{\varphi_1}(\varphi_2)$ the eigenvalue of the problem (2.97). We note that, in particular, the first eigenvalue $\lambda_{\varphi_0,\varphi_1}$ is determined by the Rayleigh quotient given, in this case, by

$$\lambda_{\varphi_1}(\varphi_2) := \inf_{f\in H_0^1((\varphi_1,\varphi_2))} \frac{4\displaystyle\int_{\varphi_0}^{\varphi_1} \sin(\varphi) f'(\varphi)^2 d\varphi}{\displaystyle\int_{\varphi_0}^{\varphi_1} \sin(\varphi) f(\varphi)^2 d\varphi}.$$

Moreover, we denote with $G(\varphi)$ and $h(\varphi)$ the functions

$$G(\varphi) := \lambda_0(\varphi) + \lambda_\varphi(\pi), \quad \varphi \in [0,\pi].$$

and

$$h(\varphi) := 2(\sqrt{1+\lambda_0(\varphi)} - 1) + 2(\sqrt{1+\lambda_0(\pi-\varphi)} - 1) \tag{2.100}$$

respectively. Then, we get the following result.

Lemma 2.23. *If the minimum value of h corresponds to the configuration in which the Koranyi ball is split in two parts by the plane $t = 0$, then $G(\varphi) \geq 16$ and $h(\varphi) \geq 4$. In general, assuming only that $G(\varphi) \geq q > 0$, we obtain $h(\varphi) \geq 2(\sqrt{2+q} - 2)$.*

Proof. Let us recall first that, in general, it holds $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \leq \sqrt{2}\sqrt{a+b}$, so from (2.100) it follows

$$h(\varphi) \geq 2(\sqrt{2+\lambda_0(\varphi)+\lambda_0(\pi-\varphi)} - 2) = 2(\sqrt{2+G(\varphi)} - 2). \tag{2.101}$$

Now, we distinguish two cases, i.e. whether the minimum value of h corresponds to the configuration in which the Koranyi ball is split in two parts by the plane $t = 0$ or not. Let us treat the first case. Then, in view of (2.98) (2.99) and Lemma 2.21, we achieve by definition of G

$$G(\varphi) = \lambda_0(\varphi) + \lambda_\varphi(\pi) = \lambda_0(\varphi) + \lambda_0(\pi-\varphi) \geq 2\lambda_0(\frac{\pi}{2}) = 16 \tag{2.102}$$

which yields, according to (2.101),

$$h(\varphi) \geq 2(\sqrt{18} - 2) > 2(4-2) = 4.$$

In case, instead, we only know that $G(\varphi) \geq q > 0$, again by (2.102) we reach $h(\varphi) \geq 2(\sqrt{2+q} - 2)$. □

2.12 The case f depending on φ and θ

In this section, we deal with the case of $\mathbb{H}^1$-harmonic functions whose expression is $u = \rho^\alpha f(\theta, \varphi)$, with f a given function depending on θ, φ.
We begin recalling that, according to Lemma 2.13, it holds if $u = \rho^\alpha f$,

$$\Delta_{\mathbb{H}^1} u = \rho^{\alpha-2}\Bigg(\alpha(\alpha+2)\sin(\varphi) f - 2\alpha(\cos(\varphi))\frac{\partial f}{\partial \theta} + \frac{1}{\sin(\varphi)}\frac{\partial f^2}{\partial \theta^2} + 4\sin(\varphi)\frac{\partial^2 f}{\partial\varphi\partial\theta} + 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi^2} + 4\cos(\varphi)\frac{\partial f}{\partial\varphi}\Bigg),$$

which also implies

$$\Delta_{\mathbb{H}^1} u|_{\partial B_1^{\mathbb{H}^1}(0)} = \alpha(\alpha+2)\sin(\varphi) f - 2\alpha(\cos(\varphi))\frac{\partial f}{\partial \theta} + \frac{1}{\sin(\varphi)}\frac{\partial f^2}{\partial \theta^2} + 4\sin(\varphi)\frac{\partial^2 f}{\partial\varphi\partial\theta} + 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi^2} + 4\cos(\varphi)\frac{\partial f}{\partial\varphi}. \tag{2.103}$$

Let now

$$A(\theta,\varphi) := \begin{bmatrix} \dfrac{1}{\sin(\varphi)} & (4+2\alpha)\sin(\varphi) \\ -2\alpha\sin(\varphi) & 4\sin(\varphi) \end{bmatrix}. \tag{2.104}$$

and define

$$\mathcal{L}_{\theta,\varphi} := \operatorname*{div}_{\theta,\varphi}\left(A(\theta,\varphi)\nabla_{\theta,\varphi}\right), \tag{2.105}$$

where

$$\nabla_{\theta,\varphi} g = \left(\frac{\partial g}{\partial\theta}, \frac{\partial g}{\partial\varphi}\right), \quad \operatorname*{div}_{\theta,\varphi} G = \frac{\partial G_1}{\partial\theta} + \frac{\partial G_2}{\partial\varphi}, \tag{2.106}$$

with $g : [0, 2\pi] \times [0, \pi] \to \mathbb{R}$ and $G : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^2$ smooth. Then, the following lemma is valid.

Lemma 2.24. *Let $\Omega_{\theta,\varphi} \subseteq [0, 2\pi] \times [0, \pi]$ and $T(\Omega_{\theta,\varphi}) \subset \partial B_1^{\mathbb{H}^1}(0)$. If $u = \rho^\alpha f(\theta, \varphi)$ is solution of $\Delta_{\mathbb{H}^1} u = 0$ in $\delta_R(T(\Omega_{\theta,\varphi}))$, $R > 0$, then*

$$\mathcal{L}_{\theta,\varphi} f = -\alpha(\alpha+2)\sin(\varphi) f \quad \text{in } \Omega_{\theta,\varphi},$$

with

$$\alpha(\alpha+2) = \frac{\displaystyle\int_{\Omega_{\theta,\varphi}} \left(\frac{1}{\sin(\varphi)} \left(\frac{\partial f}{\partial \theta} \right)^2 + 4\sin(\varphi) \frac{\partial f}{\partial \theta}\frac{\partial f}{\partial \varphi} + 4\sin(\varphi) \left(\frac{\partial f}{\partial \varphi} \right)^2 \right) d\theta d\varphi}{\displaystyle\int_{\Omega_{\theta,\varphi}} \sin(\varphi) f^2 \, d\theta d\varphi}. \tag{2.107}$$

Proof. First of all, we point out that (2.103) yields if $\Delta_{\mathbb{H}^1} u|_{\partial B_1^{\mathbb{H}^1}(0)} = 0$, with $u = \rho^\alpha f(\theta, \varphi)$,

$$\begin{aligned} &\alpha(\alpha+2)\sin(\varphi) f - 2\alpha\cos(\varphi)\frac{\partial f}{\partial \theta} + \frac{1}{\sin(\varphi)}\frac{\partial^2 f}{\partial \theta^2} + 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi \partial \theta} \\ &+ 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi^2} + 4\cos(\varphi)\frac{\partial f}{\partial \varphi} = 0. \end{aligned}$$

Now, we claim that this equation also implies

$$\alpha(\alpha+2)\sin(\varphi) f + \mathcal{L}_{\theta,\varphi} f = 0, \tag{2.108}$$

with $\mathcal{L}_{\theta,\varphi}$ defined as in (2.105). Let us check it. We have by (2.105)

$$\begin{aligned} \mathcal{L}_{\theta,\varphi} &= \operatorname*{div}_{\theta,\varphi} \left(\frac{1}{\sin(\varphi)}\frac{\partial}{\partial \theta} + (4+2\alpha)\sin(\varphi)\frac{\partial}{\partial \varphi}, -2\alpha\sin(\varphi)\frac{\partial}{\partial \theta} + 4\sin(\varphi)\frac{\partial}{\partial \varphi} \right) \\ &= \frac{1}{\sin(\varphi)}\frac{\partial^2}{\partial \theta^2} + (4+2\alpha)\sin(\varphi)\frac{\partial^2}{\partial \theta \partial \varphi} - 2\alpha\cos(\varphi)\frac{\partial}{\partial \theta} - 2\alpha\sin(\varphi)\frac{\partial^2}{\partial \varphi \partial \theta} \\ &+ 4\cos(\varphi)\frac{\partial}{\partial \varphi} + 4\sin(\varphi)\frac{\partial^2}{\partial \varphi^2}, \end{aligned}$$

in other words

$$\mathcal{L}_{\theta,\varphi} = \frac{1}{\sin(\varphi)}\frac{\partial^2}{\partial \theta^2} + 4\sin(\varphi)\frac{\partial^2}{\partial \theta \partial \varphi} - 2\alpha\cos(\varphi)\frac{\partial}{\partial \theta} + 4\cos(\varphi)\frac{\partial}{\partial \varphi} + 4\sin(\varphi)\frac{\partial^2}{\partial \varphi^2},$$

hence (2.108) indeed holds.

Therefore, if we consider $T(\Omega_{\theta,\varphi}) \subseteq \partial B_1^{\mathbb{H}^1}(0)$, (2.108) in $\Omega_{\theta,\varphi}$ can be read as

$$\operatorname*{div}_{\theta,\varphi} (A_{\theta,\varphi} \nabla_{\theta,\varphi} f) = -\alpha(\alpha+2)\sin(\varphi) f \quad \text{in } \Omega_{\theta,\varphi},$$

and, multiplying both the terms of the equality by η sufficiently smooth, with compact support in $\Omega_{\theta,\varphi}$, we get

$$\operatorname*{div}_{\theta,\varphi} (A_{\theta,\varphi} \nabla_{\theta,\varphi} f)\, \eta = -\alpha(\alpha+2)\sin(\varphi) f \eta \quad \text{in } \Omega_{\theta,\varphi}.$$

Integrating this equality over $\Omega_{\theta,\varphi}$, we then achieve

$$\int_{\Omega_{\theta,\varphi}} \operatorname*{div}_{\theta,\varphi}(A_{\theta,\varphi}\nabla_{\theta,\varphi}f)\,\eta\,d\theta d\varphi = -\alpha(\alpha+2)\int_{\Omega_{\theta,\varphi}} \sin(\varphi) f\eta\,d\theta d\varphi.$$

In particular, if we choose $\eta = f$ in the previous equality, we have

$$\int_{\Omega_{\theta,\varphi}} \operatorname*{div}_{\theta,\varphi}(A_{\theta,\varphi}\nabla_{\theta,\varphi}f)\,f\,d\theta d\varphi = -\alpha(\alpha+2)\int_{\Omega_{\theta,\varphi}} \sin(\varphi) f^2\,d\theta d\varphi. \tag{2.109}$$

In addition, by the divergence theorem, we obtain

$$\int_{\Omega_{\theta,\varphi}} \operatorname*{div}_{\theta,\varphi}(A_{\theta,\varphi}\nabla_{\theta,\varphi}f)\,f\,d\theta d\varphi = \int_{\partial\Omega_{\theta,\varphi}} \langle f A_{\theta,\varphi}\nabla_{\theta,\varphi}f, \nu\rangle\,d\sigma(\theta,\varphi)$$
$$-\int_{\Omega_{\theta,\varphi}} \langle A_{\theta,\varphi}\nabla_{\theta,\varphi}f, \nabla_{\theta,\varphi}f\rangle\,d\theta d\varphi,$$

which implies

$$\int_{\Omega_{\theta,\varphi}} \operatorname*{div}_{\theta,\varphi}(A_{\theta,\varphi}\nabla_{\theta,\varphi}f)\,f\,d\theta d\varphi = -\int_{\Omega_{\theta,\varphi}} \langle A_{\theta,\varphi}\nabla_{\theta,\varphi}f, \nabla_{\theta,\varphi}f\rangle\,d\theta d\varphi,$$

because f has compact support in $\Omega_{\theta,\varphi}$ by the choice of f.
As a consequence, substituting this in (2.109), we then have

$$\int_{\Omega_{\theta,\varphi}} \langle A_{\theta,\varphi}\nabla_{\theta,\varphi}f, \nabla_{\theta,\varphi}f\rangle\,d\theta d\varphi = \alpha(\alpha+2)\int_{\Omega_{\theta,\varphi}} \sin(\varphi) f^2\,d\theta d\varphi. \tag{2.110}$$

Now, let us note that with $4+2\alpha \neq -2\alpha$, namely $\alpha \neq -1$, from (2.104), $A_{\theta,\varphi}$ is not symmetric.
Thus, we can consider the symmetrized form of $A_{\theta,\varphi}$,

$$A^S_{\theta,\varphi} := \frac{A_{\theta,\varphi} + A^T_{\theta,\varphi}}{2},$$

and we observe that

$$\langle A^S_{\theta,\varphi}v, v\rangle = \frac{1}{2}(\langle A_{\theta,\varphi}v, v\rangle + \langle A^T_{\theta,\varphi}v, v\rangle) = \frac{1}{2}(\langle A_{\theta,\varphi}v, v\rangle + \langle v, A_{\theta,\varphi}v\rangle), \quad v \in \mathbb{R}^2,$$

i.e.

$$\langle A^S_{\theta,\varphi}v, v\rangle = \langle A_{\theta,\varphi}v, v\rangle, \quad v \in \mathbb{R}^2.$$

Using it, we then achieve from (2.110)

$$\int_{\Omega_{\theta,\varphi}} \langle A^S_{\theta,\varphi}\nabla_{\theta,\varphi} f, \nabla_{\theta,\varphi} f\rangle \, d\theta d\varphi = \alpha(\alpha+2)\int_{\Omega_{\theta,\varphi}} \sin(\varphi) f^2 \, d\theta d\varphi. \qquad (2.111)$$

At this point, we look for an explicit expression of $A^S_{\theta,\varphi}$.
Specifically, we have, using (2.104),

$$A^S_{\theta,\varphi} = \frac{1}{2}\left(\begin{bmatrix} \dfrac{1}{\sin(\varphi)} & (4+2\alpha)\sin(\varphi) \\ \\ -2\alpha\sin(\varphi) & 4\sin(\varphi)\end{bmatrix} + \begin{bmatrix} \dfrac{1}{\sin(\varphi)} & -2\alpha\sin(\varphi) \\ \\ (4+2\alpha)\sin(\varphi) & 4\sin(\varphi)\end{bmatrix}\right)$$
$$= \frac{1}{2}\begin{bmatrix} \dfrac{2}{\sin(\varphi)} & 4\sin(\varphi) \\ \\ 4\sin(\varphi) & 8\sin(\varphi)\end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sin(\varphi)} & 2\sin(\varphi) \\ \\ 2\sin(\varphi) & 4\sin(\varphi)\end{bmatrix}.$$

Consequently, we get

$$\langle A^S_{\theta,\varphi}\nabla_{\theta,\varphi} f, \nabla_{\theta,\varphi} f\rangle = \langle \begin{bmatrix} \dfrac{1}{\sin(\varphi)}\dfrac{\partial f}{\partial \theta} + 2\sin(\varphi)\dfrac{\partial f}{\partial \varphi} \\ \\ 2\sin(\varphi)\dfrac{\partial f}{\partial \theta} + 4\sin(\varphi)\dfrac{\partial f}{\partial \varphi}\end{bmatrix}, \begin{bmatrix} \dfrac{\partial f}{\partial \theta} \\ \\ \dfrac{\partial f}{\partial \varphi}\end{bmatrix}\rangle$$
$$= \frac{1}{\sin(\varphi)}\left(\frac{\partial f}{\partial \theta}\right)^2 + 2\sin(\varphi)\frac{\partial f}{\partial \varphi}\frac{\partial f}{\partial \theta} + 2\sin(\varphi)\frac{\partial f}{\partial \theta}\frac{\partial f}{\partial \varphi} + 4\sin(\varphi)\left(\frac{\partial f}{\partial \varphi}\right)^2$$
$$= \frac{1}{\sin(\varphi)}\left(\frac{\partial f}{\partial \theta}\right)^2 + 4\sin(\varphi)\frac{\partial f}{\partial \theta}\frac{\partial f}{\partial \varphi} + 4\sin(\varphi)\left(\frac{\partial f}{\partial \varphi}\right)^2.$$

Substituting this expression in (2.111), it then results the desired equality. □

At this point, we remark that if $\alpha = 1$ and $f = \sqrt{\sin(\varphi)}\cos(\theta)$, we get, in view of (2.103),

$$\Delta_{\mathbb{H}^1}\left(\rho\sqrt{\sin(\varphi)}\cos(\theta)\right)|_{\partial B^{\mathbb{H}^1}_1(0)} = 3\sin^{3/2}(\varphi)\cos(\theta) + 2\cos(\varphi)\sqrt{\sin(\varphi)}\sin(\theta)$$
$$-\frac{\sqrt{\sin(\varphi)}\cos(\theta)}{\sin(\varphi)} + 4\sin(\varphi)\frac{\partial}{\partial\varphi}\left(-\sqrt{\sin(\varphi)}\sin(\theta)\right) + 4\sin(\varphi)\cos(\theta)$$

$$\frac{\partial}{\partial\varphi}\left(\frac{\cos(\varphi)}{2\sqrt{\sin(\varphi)}}\right)+4\cos(\varphi)\frac{\cos(\varphi)}{2\sqrt{\sin(\varphi)}}\cos(\theta)=3\sin^{3/2}(\varphi)\cos(\theta)+2\sqrt{\sin(\varphi)}$$
$$\cos(\varphi)\sin(\theta)-\frac{\sqrt{\sin(\varphi)}\cos(\theta)}{\sin(\varphi)}+4\sin(\varphi)\left(-\frac{\cos(\varphi)}{2\sqrt{\sin(\varphi)}}\sin(\theta)\right)+4\sin(\varphi)$$
$$\frac{(-\sin(\varphi))\,2\sqrt{\sin(\varphi)}-\cos(\varphi)\left(\dfrac{\cos(\varphi)}{\sqrt{\sin(\varphi)}}\right)}{4\sin(\varphi)}\cos(\theta)+2\,\frac{\cos^2(\varphi)\cos(\theta)}{\sqrt{\sin(\varphi)}}$$
$$=3\sin^{3/2}(\varphi)\cos(\theta)+2\sqrt{\sin(\varphi)}\cos(\varphi)\sin(\theta)-\frac{\cos(\theta)}{\sqrt{\sin(\varphi)}}-2\,\frac{\sin(\varphi)\cos(\varphi)}{\sqrt{\sin(\varphi)}}$$
$$\sin(\theta)-2\sin(\varphi)^{3/2}\cos(\theta)-\frac{\cos^2(\varphi)\cos(\theta)}{\sqrt{\sin(\varphi)}}+2\,\frac{\cos^2(\varphi)\cos(\theta)}{\sqrt{\sin(\varphi)}}=\frac{\sin^2\varphi\cos(\theta)}{\sqrt{\sin(\varphi)}}$$
$$+\frac{2\sin(\varphi)\cos(\varphi)\sin(\theta)-\cos(\theta)-2\sin(\varphi)\cos(\varphi)\sin(\theta)+\cos^2(\varphi)\cos(\theta)}{\sqrt{\sin(\varphi)}},$$

namely

$$\Delta_{\mathbb{H}^1}\left(\rho\sqrt{\sin(\varphi)}\cos(\theta)\right)|_{\partial B_1^{\mathbb{H}^1}(0)}=0.$$

Hence, in view of (2.107), with $\Omega_{\theta,\varphi}=(0,\pi)\times(0,\pi)$, we should have

$$3=\frac{\displaystyle\int_{\Omega_{\theta,\varphi}}\left(\frac{1}{\sin(\varphi)}\left(\frac{\partial f}{\partial\theta}\right)^2+4\sin(\varphi)\frac{\partial f}{\partial\theta}\frac{\partial f}{\partial\varphi}+4\sin(\varphi)\left(\frac{\partial f}{\partial\varphi}\right)^2\right)d\theta d\varphi}{\displaystyle\int_{\Omega_{\theta,\varphi}}\sin(\varphi)f^2\,d\theta d\varphi},\tag{2.112}$$

with $f=\sqrt{\sin(\varphi)}\cos(\theta)$.

Let us check now that this equality holds. Precisely, we have first

$$\int_{(0,\pi)\times(0,\pi)}\sin(\varphi)(\sqrt{\sin(\varphi)}\cos(\theta))^2\,d\theta d\varphi=\int_0^\pi\sin^2\varphi\,d\varphi\int_0^\pi\cos^2(\theta)\,d\theta,$$

which implies

$$\int_{(0,\pi)\times(0,\pi)}\sin(\varphi)(\sqrt{\sin(\varphi)}\cos(\theta))^2\,d\theta d\varphi=\left(\frac{\pi}{2}-\left[\frac{\sin(2\varphi)}{4}\right]_{\varphi=0}^{\varphi=\pi}\right)\left(\frac{\pi}{2}+\left[\frac{\sin(2\theta)}{4}\right]_{\theta=0}^{\theta=\pi}\right)=\frac{\pi^2}{4},\tag{2.113}$$

since

$$\begin{aligned} \cos^2\tau &= \frac{1+\cos(2\tau)}{2}, \\ \sin^2\tau &= \frac{1-\cos(2\tau)}{2}. \end{aligned} \tag{2.114}$$

In parallel, it is also valid

$$\int_{(0,\pi)\times(0,\pi)} \left(\frac{1}{\sin(\varphi)} \left(\frac{\partial}{\partial\theta}(\sqrt{\sin(\varphi)}\cos(\theta)) \right)^2 + 4\sin(\varphi) \frac{\partial}{\partial\theta}(\sqrt{\sin(\varphi)}\cos(\theta)) \right.$$
$$\left. \frac{\partial}{\partial\varphi}(\sqrt{\sin(\varphi)}\cos(\theta))\, d\theta d\varphi + 4\sin(\varphi) \left(\frac{\partial}{\partial\varphi}(\sqrt{\sin(\varphi)}\cos(\theta)) \right)^2 \right) d\theta d\varphi$$
$$= \int_{(0,\pi)\times(0,\pi)} \left(\sin^2(\theta) - 2\sin(\varphi)\cos(\varphi)\cos(\theta)\sin(\theta) + \cos^2(\varphi)\cos^2(\theta) \right) d\theta d\varphi$$
$$= \int_{(0,\pi)\times(0,\pi)} \sin^2(\theta)\, d\theta d\varphi - \int_{(0,\pi)\times(0,\pi)} \sin(2\varphi)\cos(\theta)\sin(\theta)\, d\theta d\varphi$$
$$+ \int_{(0,\pi)\times(0,\pi)} \cos^2(\varphi)\cos^2(\theta)\, d\theta d\varphi,$$

and thus, from (2.114),

$$\int_{(0,\pi)\times(0,\pi)} \left(\frac{1}{\sin(\varphi)} \left(\frac{\partial}{\partial\theta}(\sqrt{\sin(\varphi)}\cos(\theta)) \right)^2 + 4\sin(\varphi) \frac{\partial}{\partial\theta}(\sqrt{\sin(\varphi)}\cos(\theta)) \right.$$
$$\left. \frac{\partial}{\partial\varphi}(\sqrt{\sin(\varphi)}\cos(\theta))\, d\theta d\varphi + 4\sin(\varphi) \left(\frac{\partial}{\partial\varphi}(\sqrt{\sin(\varphi)}\cos(\theta)) \right)^2 \right) d\theta d\varphi$$
$$= \pi \int_0^\pi \sin^2(\theta)\, d\theta - \int_0^\pi \sin(2\varphi)\, d\varphi \int_0^\pi \cos(\theta)\sin(\theta)\, d\theta + \int_0^\pi \cos^2(\varphi)\, d\varphi$$
$$\int_0^\pi \cos^2(\theta)\, d\theta = \pi \int_0^\pi \left(\frac{1-\cos(2\theta)}{2} \right) d\theta - \left[-\frac{\cos(2\varphi)}{2} \right]_{\varphi=0}^{\varphi=\pi} \int_0^\pi \cos(\theta)\sin(\theta)\, d\theta$$
$$+ \int_0^\pi \left(\frac{1+\cos(2\varphi)}{2} \right) d\varphi \int_0^\pi \left(\frac{1+\cos(2\theta)}{2} \right) d\theta = \pi \left(\frac{\pi}{2} - \left[\frac{\sin(2\theta)}{4} \right]_{\theta=0}^{\theta=\pi} \right)$$
$$+ \left(\frac{\pi}{2} + \left[\frac{\sin(2\varphi)}{4} \right]_{\varphi=0}^{\varphi=\pi} \right) \left(\frac{\pi}{2} + \left[\frac{\sin(2\theta)}{4} \right]_{\theta=0}^{\theta=\pi} \right) = \frac{\pi^2}{2} + \frac{\pi^2}{4} = \frac{3}{4}\pi^2,$$

that is

$$\int_{(0,\pi)\times(0,\pi)} \left(\frac{1}{\sin(\varphi)} \left(\frac{\partial}{\partial\theta}(\sqrt{\sin(\varphi)}\cos(\theta)) \right)^2 + 4\sin(\varphi) \frac{\partial}{\partial\theta}(\sqrt{\sin(\varphi)}\cos(\theta)) \right.$$

$$\frac{\partial}{\partial\varphi}(\sqrt{\sin(\varphi)}\cos(\theta))\,d\theta d\varphi + 4\sin(\varphi)\left(\frac{\partial}{\partial\varphi}(\sqrt{\sin(\varphi)}\cos(\theta))\right)^2\Bigg)\,d\theta d\varphi = \frac{3}{4}\pi^2. \tag{2.115}$$

As a consequence, putting together (2.113) and (2.115), we indeed get (2.112).

2.13 Evaluation of $\lambda_{\varphi_{(\Sigma)}}$ for symmetric caps

In this section, we analyze the behavior of $\lambda_{\varphi_{(\Sigma)}}$ for symmetric caps. We observe, at this point, that a symmetric cap with respect to the $t-$axis may be described by only using the variable φ in the change of variables T, see (2.80). It is worth to recall now that the Koranyi ball is not symmetric along all the directions like the Euclidean ball. Precisely, the following results hold.

Lemma 2.25. *If $u = x^+$, then*

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} \frac{\left|\nabla^{\varphi}_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\,d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} u^2\sqrt{x^2+y^2}\,d\sigma_{\mathbb{H}^1}(\xi)} = 2.$$

If $u = t^+$, then

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} \frac{\left|\nabla^{\varphi}_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\,d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} u^2\sqrt{x^2+y^2}\,d\sigma_{\mathbb{H}^1}(\xi)} = 8.$$

As a consequence, it results

$$\lambda_{\varphi_{(\Sigma)}} \leq 2,$$

with $\Sigma = \partial B_1^{\mathbb{H}^1}(0) \cap \{u > 0\}$.

Proof. We start with $u = x^+$. In particular, we want to compute

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} \frac{\left|\nabla^{\varphi}_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\,d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} u^2\sqrt{x^2+y^2}\,d\sigma_{\mathbb{H}^1}(\xi)}.$$

Using the parametrization in spherical coordinates of the boundary of unitary Koranyi ball in (2.85), $u = x^+$ reads on $\partial B_1^{\mathbb{H}^1}(0)$

$$u = (\sqrt{\sin(\varphi)}\cos(\theta))^+, \tag{2.116}$$

which is positive if $-\pi/2 < \theta < \pi/2$ by (2.81). At this point, we want to express $\left|\nabla^{\varphi}_{\mathbb{H}^1} u\right|^2$ according to (2.116). Let us recall first that if $v = \rho^\alpha f(\theta, \varphi)$, we have

$$\nabla_{\mathbb{H}^1} v = \alpha\rho^{\alpha-1}\nabla_{\mathbb{H}^1}\rho f + \rho^\alpha\left(\frac{\partial f}{\partial\theta}\nabla_{\mathbb{H}^1}\theta + \frac{\partial f}{\partial\varphi}\nabla_{\mathbb{H}^1}\varphi\right). \tag{2.117}$$

Moreover, we know by (2.45) that

$$\nabla^{\varphi}_{\mathbb{H}^1} v = \langle\nabla_{\mathbb{H}^1} v, e_\varphi\rangle e_\varphi,$$

where $e_\varphi = \frac{\nabla_{\mathbb{H}^1}\varphi}{|\nabla_{\mathbb{H}^1}\varphi|}$, thus, in view of (2.117), Lemma 2.10 and 2.11, we achieve

$$\nabla^{\varphi}_{\mathbb{H}^1} v = \langle\alpha\rho^{\alpha-1}\nabla_{\mathbb{H}^1}\rho f + \rho^\alpha\left(\frac{\partial f}{\partial\theta}\nabla_{\mathbb{H}^1}\theta + \frac{\partial f}{\partial\varphi}\nabla_{\mathbb{H}^1}\varphi\right), \frac{\nabla_{\mathbb{H}^1}\varphi}{|\nabla_{\mathbb{H}^1}\varphi|}\rangle e_\varphi$$
$$= \frac{\rho^\alpha}{|\nabla_{\mathbb{H}^1}\varphi|}\left(\frac{\partial f}{\partial\theta}\langle\nabla_{\mathbb{H}^1}\theta, \nabla_{\mathbb{H}^1}\varphi\rangle + \frac{\partial f}{\partial\varphi}\left|\nabla_{\mathbb{H}^1}\varphi\right|^2\right) e_\varphi = \frac{\rho^{\alpha+2}}{2\sqrt{x^2+y^2}}\left(\frac{\partial f}{\partial\theta}\frac{2(x^2+y^2)}{\rho^4}\right.$$
$$\left. + \frac{\partial f}{\partial\varphi}\frac{4(x^2+y^2)}{\rho^4}\right) e_\varphi,$$

namely

$$\nabla^{\varphi}_{\mathbb{H}^1} v = \rho^{\alpha-2}\sqrt{x^2+y^2}\left(\frac{\partial f}{\partial\theta} + 2\frac{\partial f}{\partial\varphi}\right) e_\varphi,$$

which implies

$$\left|\nabla^{\varphi}_{\mathbb{H}^1} v\right|^2 = \rho^{2(\alpha-2)}(x^2+y^2)\left(\frac{\partial f}{\partial\theta} + 2\frac{\partial f}{\partial\varphi}\right)^2.$$

In particular, on $\partial B_1^{\mathbb{H}^1}(0)$ this reads, by virtue of (2.85),

$$\left|\nabla^{\varphi}_{\mathbb{H}^1} v\right|^2|_{\partial B_1^{\mathbb{H}^1}(0)} = \sin(\varphi)\left(\frac{\partial f}{\partial\theta} + 2\frac{\partial f}{\partial\varphi}\right)^2. \tag{2.118}$$

Now, the function $u = x^+$ in the spherical coordinates (2.80) has the expression $u = (\rho\sqrt{\sin(\varphi)}\cos(\theta))^+$, which corresponds to the form $\rho^\alpha f(\theta, \varphi)$, with

$\alpha = 1$ and $f = \sqrt{\sin(\varphi)}\cos(\theta)$. Thus, we obtain, according to (2.118),

$$\left|\nabla^{\varphi}_{\mathbb{H}^1} u\right|^2 |_{\partial B_1^{\mathbb{H}^1}(0)} = \sin(\varphi)\left(- \sqrt{\sin(\varphi)}\sin(\theta) + 2\frac{\cos(\varphi)\cos(\theta)}{2\sqrt{\sin(\varphi)}}\right)^2$$
$$= \sin^2(\varphi)\sin^2(\theta) + \cos^2(\varphi)\cos^2(\theta) - 2\cos(\varphi)\sin(\varphi)\cos(\theta)\sin(\theta)$$
$$= (\sin(\varphi)\sin(\theta) - \cos(\varphi)\cos(\theta))^2 = \cos^2(\theta+\varphi),$$

that is

$$\left|\nabla^{\varphi}_{\mathbb{H}^1} u\right|^2 |_{\partial B_1^{\mathbb{H}^1}(0)} = \cos^2(\theta+\varphi). \tag{2.119}$$

Let us recall, moreover, that, from (2.83) and (2.87), it results

$$d\sigma_{\mathbb{H}^1}(\xi) = \frac{|\nabla_{\mathbb{H}^1}\rho|}{|\nabla\rho|}\, d\sigma(\xi) = \sqrt{\sin(\varphi)}\, d\theta\, d\varphi, \tag{2.120}$$

therefore, exploiting (2.85), (2.116) and (2.119), we get

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} \frac{\left|\nabla^{\varphi}_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} u^2\sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi)} = \frac{\displaystyle\int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta+\varphi)\, d\theta\, d\varphi}{\displaystyle\int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(\varphi)\cos^2(\theta)\, d\theta\, d\varphi}. \tag{2.121}$$

Let us compute, at this point, the numerator and the denominator of the right hand side in (2.121) separately. In both cases, we use the duplication formulas recalled in (2.114).

About the numerator, in particular, we have

$$\int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta+\varphi)\, d\theta\, d\varphi = \frac{\pi^2}{2}. \tag{2.122}$$

Concerning the denominator, it holds

$$\int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(\varphi)\cos^2(\theta)\, d\theta\, d\varphi = \frac{\pi^2}{4}. \tag{2.123}$$

Consequently, from (2.121), (2.122) and (2.123), we lastly obtain

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} \frac{\left|\nabla^{\varphi}_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} u^2\sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi)} = 2.$$

Let us deal now with $u = t^+$. In view of (2.85), we achieve $u^+ = \cos(\varphi)^+$, which is positive if $0 \leq \varphi < \pi/2$. Then, keeping in mind (2.118) and (2.120), we get

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} \frac{\left|\nabla^{\varphi}_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} u^2\sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi)} = \frac{8\pi \displaystyle\int_0^{\frac{\pi}{2}} \sin(\varphi)\sin^2(\varphi) d\varphi}{2\pi \displaystyle\int_0^{\frac{\pi}{2}} \cos^2(\varphi)\sin(\varphi) d\varphi}$$

$$= \frac{8\pi \displaystyle\int_0^{\frac{\pi}{2}} \sin(\varphi)(1-\cos^2(\varphi)) d\varphi}{2\pi \left[-\frac{1}{3}\cos^3\varphi\right]_{\varphi=0}^{\varphi=\frac{\pi}{2}}} = 8.$$

□

In particular, from the proof of Lemma 2.25 we derive the following corollary as well.

Corollary 2.26. *It holds*

$$\frac{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} \frac{\left|\nabla^{\varphi}_{\mathbb{H}^1} u\right|^2}{\sqrt{x^2+y^2}}\, d\sigma_{\mathbb{H}^1}(\xi)}{\displaystyle\int_{\partial B_1^{\mathbb{H}^1}(0)\cap\{u>0\}} u^2\sqrt{x^2+y^2}\, d\sigma_{\mathbb{H}^1}(\xi)} = \frac{\displaystyle\int_{\Omega_{\theta,\varphi}} \sin(\varphi)\left(\frac{\partial f}{\partial \theta} + 2\frac{\partial f}{\partial \varphi}\right)^2 d\theta d\varphi}{\displaystyle\int_{\Omega_{\theta,\varphi}} \sin(\varphi) f^2 d\theta d\varphi},$$

where $T(\Omega_{\theta,\varphi}) = \partial B_1^{\mathbb{H}^1}(0) \cap \{u > 0\}$ *and* $u = \rho^\alpha f(\theta, \varphi)$.

2.14 Last considerations

In this section, we expose our last considerations about the question of the existence of an Alt-Caffarelli-Friedman monotonicity formula in the Heisenberg group $\mathbb{H}^1$.
In view of Sections 2.12 and 2.11, whenever f satisfies

$$\alpha(\alpha+2)\sin(\varphi) f(\theta,\varphi) - 2\alpha\cos(\varphi)\frac{\partial f}{\partial \theta} + \frac{1}{\sin(\varphi)}\frac{\partial^2 f}{\partial \theta^2} + 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi \partial \theta}$$
$$+ 4\sin(\varphi)\frac{\partial^2 f}{\partial \varphi^2} + 4\cos(\varphi)\frac{\partial f}{\partial \varphi} = 0, \tag{2.124}$$

on $\Gamma \subset \partial B_1^{\mathbb{H}^1}(0)$ or

$$\alpha(\alpha+2)(\sin\varphi)f(\varphi)+4\frac{\partial}{\partial\varphi}\left(\sin(\varphi)\frac{\partial f}{\partial\varphi}\right)=0$$

on $\Gamma \subset \partial B_1^{\mathbb{H}^1}(0)$ for f depending only on φ, then $u=\rho^\alpha f(\theta,\varphi)$ is $\mathbb{H}^1$-harmonic in the set

$$\mathcal{P}_\Gamma := \{(x,y,t)\in\mathbb{H}^1 : \quad (x,y,t)=\delta_\lambda(\xi,\eta,\tau),\ \lambda>0,\ (\xi,\eta,\tau)\in\Gamma\},$$

where $\delta_\lambda(\xi,\eta,\tau) := (\lambda\xi,\lambda\eta,\lambda^2\tau)$, $\lambda>0$, is the dilation semigroup in the smallest Heisenberg group $\mathbb{H}^1$, see Section 2.4. This follows directly by Lemma 2.13. For instance, if $\Gamma=\{(x,y,t)\in\partial B_1^{\mathbb{H}^1}(0) : \quad x^2+y^2<Mt\}$, where $M>0$ is a constant, then

$$\mathcal{P}_\Gamma = \{(x,y,t)\in\mathbb{H}^1 : \quad x^2+y^2<Mt\}.$$

Moreover, if we add a boundary condition to the equation (2.124) by requiring that $f=0$ on $\partial\Gamma$, then $u=\rho^\alpha f$ satisfies

$$\begin{cases}\Delta_{\mathbb{H}^1}u=0, & (x,y,t)\in\mathcal{P}_\Gamma,\\ u=0, & (x,y,t)\in\partial\mathcal{P}_\Gamma.\end{cases} \tag{2.125}$$

Of course, if we fix Γ and we assume that $f=0$ on $\partial\Gamma$ as well, then the equation (2.124) has a solution only for some particular values of α. This type of problem, in particular, has been faced (the authors having in mind different applications respect to those in the work of Ferrari and myself) in [58] and [8], without entering into the details as done in [38], but considering all the Heisenberg groups $\mathbb{H}^n$.
Now, see Section 2.12, (2.125) can be rewritten as the eigenvalues problem

$$\begin{cases}\mathcal{L}_{\theta,\varphi}f=-\lambda(\Gamma)f & \text{in } \Omega\subset\mathbb{R}^2,\\ f=0 & \text{on } \partial\Omega,\end{cases} \tag{2.126}$$

with $T(\Omega) = \Gamma$ and

$$\lambda(\Gamma) := \inf_{v \in H_0^1(\Omega_{\theta,\varphi})} \frac{\displaystyle\int_{\Omega_{\theta,\varphi}} G_f(\theta,\varphi)\, d\theta d\varphi}{\displaystyle\int_{\Omega_{\theta,\varphi}} \sin(\varphi) f^2\, d\theta d\varphi}, \tag{2.127}$$

where

$$G_f(\theta,\varphi) := \frac{1}{\sin(\varphi)}\left(\frac{\partial f}{\partial \theta}\right)^2 + 4\sin(\varphi)\frac{\partial f}{\partial \theta}\frac{\partial f}{\partial \varphi} + 4\sin(\varphi)\left(\frac{\partial f}{\partial \varphi}\right)^2.$$

As a consequence, as in the Euclidean framework, we have reduced ourselves to study Rayleigh quotients to understand if a monotonicity formula can hold in $\mathbb{H}^1$ and, in particular, to deal with the correspondent of the characteristic number, see Section 2.3 for this notion, in such a noncommutative framework. Specifically, it would be fundamental to know if the result by [48] recalled in Section 2.3, that is the cap on ∂B_1 having the same $\mathcal{H}^{n-1}$ measure of some sets Σ on ∂B_1 has the smallest Rayleigh quotient, is true even in the Heisenberg case. Let us say that we would like to know if there exists a set $\Gamma^* \subset \partial B_1^{\mathbb{H}^1}(0)$ such that for every $\Gamma \subset \partial B_1^{\mathbb{H}^1}(0)$,

$$P_{\mathbb{H}^1}^{B_1^{\mathbb{H}^1}(0)}(\Gamma) = P_{\mathbb{H}^1}^{B_1^{\mathbb{H}^1}(0)}(\Gamma^*),$$

it results

$$\alpha_{\mathbb{H}^1}(\Gamma) \geq \alpha_{\mathbb{H}^1}(\Gamma^*),$$

where $\alpha_{\mathbb{H}^1}(\Gamma)$ denotes the unique positive solution to the equation

$$\alpha(\alpha+2) = \lambda(\Gamma),$$

and $\lambda(\Gamma)$ is the first eigenvalue of the problem (2.126) defined as in (2.127). The existence in the Heisenberg group of the properties of the characteristic number associated with the set Γ, as far as we know, is still unknown. This part corresponds to the topic discussed in [74] in the Euclidean setting. Precisely, just for having an idea about the difficulty in solving the problem, we remark that

$$P_{\mathbb{H}^1}^{B_1^{\mathbb{H}^1}(0)}(\Gamma) = \int_{\Omega} \sqrt{\sin(\varphi)} d\theta d\varphi,$$

where $\Gamma = T(\{1\} \times \Omega)$. At this point, we may decide to symmetrize the set Ω in many ways. For instance, for every φ, we might define Ω^*_φ in such a way that

$$\mathcal{H}^1(\Omega^*_\varphi) = 2\theta_\varphi = \mathcal{H}^1(\Omega_\varphi),$$

and consider $\Omega^* \coloneqq \cup_{\varphi \in \Pi_2(\Omega)} \Omega^*_\varphi$, where $\Pi_2(\Omega) \coloneqq \{\varphi : \ \Omega_\varphi \neq \emptyset\}$. Nevertheless, the lack of an isoperimetric result in the framework of $\mathbb{H}^1$ does not permit to conclude anything.

Trying to recap the situation in $\mathbb{H}^1$, first, by Lemma 2.25, we know that, in general, $\lambda_{\varphi(\Sigma)} \leq 2$, where $\lambda_{\varphi(\Sigma)}$ is the Rayleigh quotient in this context. In the particular case of functions defined on caps depending only on φ, i.e. in the form $u = \rho^\alpha f(\varphi)$ in what exposed in this chapter, the function h defined in (2.100) is symmetric with respect to $\frac{\pi}{2}$ in $[0, \pi]$. However, we do not know if the minimum of the function h is realized when $\varphi = \frac{\pi}{2}$, even if this fact would seem natural. In any case, if it were true that the Koranyi ball is split in two half parts by the plane $t = 0$ when h realizes the minimum, which is unfortunately still unknown, then $\min_{\varphi \in [0,\pi]} h(\varphi) = 8$, since from Lemma 2.21 $\lambda_0(\frac{\pi}{2}) = 8$, see Section 2.11 for all the notation mentioned. As a consequence, choosing $\beta = 8$ in the expression of $J_{\beta,\mathbb{H}^1}$, we would achieve, according to Lemma 2.8 and Theorem 2.16, that $J_{8,\mathbb{H}^1}$ is increasing monotone only considering functions defined on caps depending just on φ, since $\lambda_{\varphi(\Sigma)} = \lambda_0(\frac{\pi}{2})$, by virtue of (2.78) and (2.98). In the more general case of functions defined on caps depending on both φ and θ, the question is more delicate because in this case $\lambda_{\varphi(\Sigma)} \neq \lambda(\Gamma)$, as we can see from Corollary 2.26 and (2.127).

Chapter 3

Regularity of almost minimizers for the p-Laplacian

In this chapter, the aim is to investigate some extensions of the results in [30] to the functional

$$J_p(u,\Omega) := \int_\Omega (|\nabla u|^p + \chi_{\{u>0\}})\,dx, \quad p>1, \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $u \geq 0$. This functional is, precisely, a generalization of the classical one-phase (Bernoulli) energy functional

$$J(u,\Omega) := \int_\Omega (|\nabla u|^2 + \chi_{\{u>0\}})\,dx,$$

studied in [30], to each $p > 1$. We note that J corresponds to J_2 in (3.1), so in the following we will refer to J in this way to exploit just a single definition.

3.1 State of the art for almost minimizers of J_2

In this section, we present the state of the play concerning almost minimizers for J_2.
In [30], specifically, the two main theorems concern the optimal Lipschitz

regularity of almost minimizers for J_2 and the $C^{1,\alpha}$ regularity of their free boundary outside a closed singular set of Hausdorff dimension $n-5$, together with finite $n-1$ dimension. In particular, this last result relies on an improvement of flatness theorem, in the spirit of [26], because the authors show that almost minimizers are "viscosity solutions" in the following more general sense. Roughly speaking, in this case viscosity solutions satisfy a comparison principle in a neighborhood of a touching point whose size depends on the properties of the test functions. This strategy is inspired by [31].
About further literature on almost minimizers for J_2, we quote the recent works [24, 23]. In [24], the authors achieved local Lipschitz continuity of almost minimizers in the more general case of a two-phase energy functional. Later, in [23] the authors showed uniform rectifiability of the free boundary, and in the purely one-phase case they obtained that the free boundary is $C^{1,\alpha}$ almost-everywhere.

3.2 Regularity issues for almost minimizers for the p-Laplacian

In this section, we introduce the regularity issues for almost minimizers of J_p on which I have been working together with Serena Dipierro, my advisor Fausto Ferrari, and Enrico Valdinoci. Precisely, the optimal Lipschitz regularity of almost minimizers for J_p, $p > 2$, has been proved. The statement of this result is the following.

Theorem 3.1. *Let u be an almost minimizer for J_p in B_1 (with constant κ and exponent β) with $p > 2$. Then, it holds*

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C(\|u\|_{W^{1,p}(B_1)} + 1)$$

with C some constant depending on κ, β and n. In addition, u is uniformly Lipschitz continuous in a neighborhood of $\{u = 0\}$, in other words if $u(0) = 0$ then

$$|\nabla u| \leq C(n) \quad \text{in } B_{r_0},$$

for some r_0 *depending on* κ, β, n *and* $\|\nabla u\|_{L^p}$.

About, instead, the free boundary regularity for an almost minimizer to (3.1), a regularity result has not been proved yet. The trickiest point is related to a nondegeneracy condition on almost minimizers. Another aspect to carefully investigate is the possible existence of a Weiss type monotonicity formula, always for almost minimizers. We refer, for instance, to [62, 63, 61] for the theory about free boundary problems for the p-Laplace operator. Furthermore, we cite [82] as a possible reference work in the study of the existence of the Weiss type monotonicity formula, and we mention [67], which is related to the behavior of p-harmonic functions of two variables. Concerning, instead, the other aspects in [30] about the regularity of the free boundary, most of them are valid also for almost minimizers of (3.1), $p > 2$.
We provide in the following the details of the study I have been doing with S. Dipierro, F. Ferrari and Enrico Valdinoci. Let us begin by recalling in Section 3.3 a few of general things we exploit hereinafter. In Section 3.4, we show the optimal Lipschitz regularity of almost minimizers for J_p, $p > 2$. In the next section, we deal with nondegeneracy properties and in the final Section 3.6 we focus on the partial regularity of the free boundary.

3.3 Some general facts

In this section, we give some general definitions and results which will be useful in this chapter.
First, for the sake of completeness, let us provide the definition of the p-Laplacian.

Definition 3.2. Given a function $u : \mathbb{R}^n \to \mathbb{R} \in C^1$, we define the p-Laplacian of u as

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

We point out that the p-Laplacian can be rewritten as

$$\Delta_p u = |\nabla u|^{p-2} \Big(\Delta u + (p-2) \langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \rangle \Big).$$

Now, we recall the definition of the p-harmonic replacement and next, we state and prove a technical lemma on it.

Definition 3.3. Let $u \in W^{1,p}(B_r(x))$ be given. We say that $v \in W^{1,p}(B_r(x))$ is the p-harmonic replacement of u in $B_r(x)$, if

$$\int_{B_r(x)} |\nabla v|^p \, dx = \min_{u-w \in W_0^{1,p}(B_r(x))} \int_{B_r(x)} |\nabla w|^p \, dx.$$

Lemma 3.4. *Let $B_r = B_r(x_0) \subset \mathbb{R}^n$ and let $u \in W^{1,p}(B_r)$. Then, if v is the p-harmonic replacement of u in B_r, we have the following inequalities:*

(i) if $1 < p < 2$, then

$$\int_{B_r} |\nabla u - \nabla v|^p \, dx \leq C \left(\int_{B_r} (|\nabla u|^p - |\nabla v|^p) \, dx \right)^{\frac{p}{2}} \left(\int_{B_r} (|\nabla u| + |\nabla v|)^p dx \right)^{1-\frac{p}{2}}, \quad C = C(n,p) > 0; \tag{3.2}$$

(ii) if $p \geq 2$, then

$$\int_{B_r} |\nabla u - \nabla v|^p \, dx \leq C \int_{B_r} (|\nabla u|^p - |\nabla v|^p) \, dx, \tag{3.3}$$

with $C = C(n,p) > 0$.

Proof. By definition, v satisfies

$$\int_{B_r} |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi \, dx = 0 \tag{3.4}$$

for all $\varphi \in W_0^{1,p}(B_r)$, that is v is a weak solution of $\Delta_p(v) = 0$.
Now, we want to get an upper bound of

$$\int_{B_r} |\nabla u - \nabla v|^p \, dx.$$

To this end, let us consider the following family of functions

$$u^s(x) = su(x) + (1-s)v(x), \quad 0 \leq s \leq 1, \tag{3.5}$$

in such a way that $u^0 = v$ and $u^1 = u$, and we compute

$$\int_{B_r} (|\nabla u|^p - |\nabla v|^p)\,dx = \int_{B_r} \left(\left|\nabla u^1\right|^p - \left|\nabla u^0\right|^p\right) dx$$
$$= \int_{B_r} \left(\int_0^1 \frac{d}{ds} |\nabla u^s|^p \,ds\right) dx$$
$$= \int_{B_r} \left(\int_0^1 p\,|\nabla u^s|^{p-2}\, \nabla u^s \cdot \nabla(u-v)\,ds\right) dx,$$

which gives using (3.4), since $u - v \in W_0^{1,p}(B_r)$,

$$\int_{B_r} (|\nabla u|^p - |\nabla v|^p)\,dx = p\Bigg(\int_{B_r} \left(\int_0^1 |\nabla u^s|^{p-2}\, \nabla u^s \cdot \nabla(u-v)\,ds\right) dx$$
$$- \int_0^1 \left(\int_{B_r} |\nabla v|^{p-2}\, \nabla v \cdot \nabla(u-v)\,dx\right) ds\Bigg)$$
$$= p\int_{B_r} \left(\int_0^1 \left(|\nabla u^s|^{p-2}\, \nabla u^s - |\nabla v|^{p-2}\, \nabla v\right) \cdot \nabla(u-v)\,ds\right) dx.$$

Let us remark that

$$u^s - v = su(x) - sv(x) = s(u(x) - v(x)), \tag{3.6}$$

thus, from the previous equality, we achieve

$$\int_{B_r} (|\nabla u|^p - |\nabla v|^p)\,dx$$
$$= p\int_0^1 \frac{1}{s}\left(\int_{B_r} \left(|\nabla u^s|^{p-2}\, \nabla u^s - |\nabla v|^{p-2}\, \nabla v\right) \cdot \nabla(u^s - v)\,dx\right) ds. \tag{3.7}$$

At this point, we want to apply to (3.7) the well-known inequality

$$(|\xi|^{p-2}\,\xi - |\zeta|^{p-2}\,\zeta)\cdot(\xi-\zeta) \geq \gamma \begin{cases} |\xi-\zeta|^2\,(|\xi|+|\zeta|)^{p-2} & \text{if } 1<p<2, \\ |\xi-\zeta|^p & \text{if } p \geq 2, \end{cases}$$

for any nonzero ξ, $\zeta \in \mathbb{R}^n$ and a constant $\gamma = \gamma(n,p) > 0$.

Precisely, if we choose $\xi = \nabla u^s$ and $\zeta = \nabla v$ in the inequality above, we obtain, by (3.7),

$$\int_{B_r} (|\nabla u|^p - |\nabla v|^p)\, dx$$
$$\geq p\gamma \begin{cases} \displaystyle\int_0^1 \frac{1}{s}\left(\int_{B_r} |\nabla u^s - \nabla v|^2 \,(|\nabla u^s| + |\nabla v|)^{p-2} dx\right) ds, \\ \text{if } 1 < p < 2, \\ \displaystyle\int_0^1 \frac{1}{s}\left(\int_{B_r} |\nabla u^s - \nabla v|^p \, dx\right) ds, \\ \text{if } p \geq 2. \end{cases} \tag{3.8}$$

Let us analyze the two cases above separately.

First, let us consider the case $p \geq 2$. Specifically, from (3.6) and (3.8), we get

$$\int_{B_r} (|\nabla u|^p - |\nabla v|^p) dx \geq p\gamma \int_0^1 s^{p-1}\left(\int_{B_r} |\nabla u - \nabla v|^p \, dx\right) ds$$
$$= \gamma \Big[s^p\Big]_{s=0}^{s=1} \int_{B_r} |\nabla u - \nabla v|^p \, dx,$$

namely

$$\int_{B_r} (|\nabla u|^p - |\nabla v|^p) dx \geq \gamma \int_{B_r} |\nabla u - \nabla v|^p \, dx. \tag{3.9}$$

About the case $1 < p < 2$, instead, we observe that

$$|\nabla u^s| + |\nabla v| \leq s\,|\nabla u| + (1-s)\,|\nabla v| + |\nabla v| = s\,|\nabla u| + (2-s)\,|\nabla v|$$
$$\overset{0\leq s\leq 1}{\leq} 2(|\nabla u| + |\nabla v|),$$

which yields by (3.6) and (3.8), because $p - 2 < 0$ if $1 < p < 2$,

$$\int_{B_r} (|\nabla u|^p - |\nabla v|^p)\, dx$$
$$\geq p\gamma \int_0^1 \frac{1}{s} \int_{B_r} s^2 \,|\nabla u - \nabla v|^2\, 2^{p-2}\,(|\nabla u| + |\nabla v|)^{p-2}\, dx$$
$$= C(n,p) \left[\frac{s^2}{2}\right]_{s=0}^{s=1} \int_{B_r} |\nabla u - \nabla v|^2 \,(|\nabla u| + |\nabla v|)^{p-2}\, dx$$

i.e.

$$\int_{B_r} (|\nabla u|^p - |\nabla v|^p)\, dx \geq C(n,p) \int_{B_r} |\nabla u - \nabla v|^2 \,(|\nabla u| + |\nabla v|)^{p-2}\, dx. \tag{3.10}$$

Now, let us consider

$$\int_{B_r} |\nabla u - \nabla v|^p \, dx,$$

and by Hölder's inequality with Hölder exponent $2/p$ and conjugate exponent

$$\left(\frac{2}{p}\right)' = \frac{2/p}{2/p - 1} = \frac{2}{2-p},$$

we have

$$\begin{aligned}
&\int_{B_r} |\nabla u - \nabla v|^p \, dx \\
&= \int_{B_r} |\nabla u - \nabla v|^p \left(|\nabla u| + |\nabla v|\right)^{\frac{p(p-2)}{2}} \left(|\nabla u| + |\nabla v|\right)^{-\frac{p(p-2)}{2}} dx \\
&\leq \left(\int_{B_r} |\nabla u - \nabla v|^2 \left(|\nabla u| + |\nabla v|\right)^{p-2} dx\right)^{\frac{p}{2}} \left(\int_{B_r} (|\nabla u| + |\nabla v|)^p dx\right)^{1-\frac{p}{2}},
\end{aligned}$$

which implies, from (3.10),

$$\begin{aligned}
&\int_{B_r} |\nabla u - \nabla v|^p \, dx \\
&\leq c(n,p) \left(\int_{B_r} \left(|\nabla u|^p - |\nabla v|^p\right) dx\right)^{\frac{p}{2}} \left(\int_{B_r} (|\nabla u| + |\nabla v|)^p dx\right)^{1-\frac{p}{2}}.
\end{aligned}$$

□

Let us go on recalling the definition of Campanato spaces and a result which we will use in the proof of the corollary 3.12, see [55].

Definition 3.5 (Campanato spaces)**.** Let Ω be a bounded open set in $\mathbb{R}^n$, and let $1 \leq p < +\infty$ and $\lambda \geq 0$. We denote by $\mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^n)$ the space of functions $u \in L^p(\Omega, \mathbb{R}^n)$ such that

$$[u]^p_{\mathcal{L}^{p,\lambda}} := \sup_{\substack{x_0 \in \Omega \\ \rho > 0}} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p \, dx < +\infty, \tag{3.11}$$

where

$$\Omega_{x_0,\rho} := \Omega \cap B(x_0, \rho), \quad u_{x_0,\rho} := \fint_{\Omega_{x_0,\rho}} u \, dx.$$

Remark 3.6. The quantity $[u]_{\mathcal{L}^{p,\lambda}}$ is a seminorm in $\mathcal{L}^{p,\lambda}$ and it is equivalent to

$$\left(\sup_{\substack{x_0\in\Omega \\ \rho>0}} \rho^{-\lambda} \inf_{\xi\in\mathbb{R}^n} \int_{\Omega(x_0,\rho)} |u-\xi|^p \, dx \right)^{1/p}.$$

We define then the norm in $\mathcal{L}^{p,\lambda}$ as

$$\|u\|_{\mathcal{L}^{p,\lambda}} := \|u\|_{L^p} + [u]_{\mathcal{L}^{p,\lambda}}. \tag{3.12}$$

For simplicity, let us assume now that $\Omega = B_r(x)$.

Theorem 3.7. *Let $\Omega = B_r(x)$ and let $n < \lambda \le n+p$. The space $\mathcal{L}^{p,\lambda}(B_r(x))$ is isomorphic to $C^{0,\alpha}(\overline{B_r(x)})$, with $\alpha = \frac{\lambda-n}{p}$.*

3.4 Lipschitz continuity of almost minimizers for J_p

In this section, we prove the main result Theorem 3.1 about the Lipschitz continuity of almost minimizers of J_p, $p > 2$. Let us start by recalling the definition of almost minimizer for J_p.

Definition 3.8. We say that u is an almost minimizer for J_p in Ω (with constant κ and exponent β) if $u \in W^{1,p}(\Omega)$, $u \ge 0$ a.e. in Ω and

$$J_p(u, B_r(x)) \le (1+\kappa r^\beta) J_p(v, B_r(x)). \tag{3.13}$$

for every ball $B_r(x) \subset \Omega$ such that $\overline{B_r(x)} \subset \Omega$ and every $v \in W^{1,p}(\Omega)$ such that $v = u$ on $\partial B_r(x)$ in the trace sense.

Throughout the chapter, constants depending only on n and p are called universal and moreover, when u is an almost minimizer, these constants may depend on κ and β as well. It may happen that we denote with the same symbol universal constants changing from line to line.
Proceeding in parallel to [30], the first result is the subsequent dichotomy.

Proposition 3.9. *Let $u \in W^{1,p}(B_1)$ and assume that*

$$J_p(u, B_1) \le (1+\sigma) J_p(v, B_1) \tag{3.14}$$

for all $v \in W^{1,p}(B_1)$ such that $v = u$ on ∂B_1. Denote by

$$a := \left(\fint_{B_1} |\nabla u|^p \, dx \right)^{1/p}. \tag{3.15}$$

For every $\varepsilon > 0$ small, there exist constants η, M, σ_0 (depending on ε) such that if $\sigma \le \sigma_0$ and $a \ge M$ then the following dichotomy holds. Either

$$\left(\fint_{B_\eta} |\nabla u|^p \, dx \right)^{1/p} \le \frac{a}{2}, \tag{3.16}$$

or

$$\left(\fint_{B_\eta} |\nabla u - q|^p \, dx \right)^{1/p} \le \varepsilon a, \tag{3.17}$$

with $q \in \mathbb{R}^n$ such that

$$\frac{a}{2^{(2p+1)/p}} < |q| \le C_0 a, \tag{3.18}$$

and $C_0 > 0$ universal.

Proof. Let v be the p-harmonic replacement of u in B_1. We want to get an estimate of

$$\int_{B_1} |\nabla u - \nabla v|^p \, dx$$

in terms of a. According to Lemma 3.4, we have to distinguish two cases. Let us suppose first that $p \ge 2$. Then, from (3.3), we achieve, using (3.14),

$$\int_{B_1} |\nabla u - \nabla v|^p \, dx \le C\bigg(J_p(u, B_1) - \int_{B_1} |\nabla v|^p \, dx \bigg) \le C\bigg((1+\sigma) J_p(v, B_1)$$
$$- \int_{B_1} |\nabla v|^p \, dx \bigg) = C\bigg(\sigma \bigg(\int_{B_1} |\nabla v|^p \, dx + |\{v > 0\} \cap B_1| \bigg) + |\{v > 0\} \cap B_1| \bigg)$$
$$\le C \left(\sigma \int_{B_1} |\nabla v|^p \, dx + 1 \right),$$

which gives, since v is the p-harmonic replacement of u in B_1,

$$\int_{B_1} |\nabla u - \nabla v|^p \, dx \le C \left(\sigma \int_{B_1} |\nabla u|^p \, dx + 1 \right).$$

Taking the average integral of the previous inequality, we then obtain

$$\fint_{B_1} |\nabla u - \nabla v|^p \, dx \le C \left(\sigma \fint_{B_1} |\nabla u|^p \, dx + 1 \right) = C(\sigma a^p + 1), \qquad (3.19)$$

for $C > 0$ universal.

Let us assume now that $1 < p < 2$. By virtue of (3.2), repeating the same argument used to get (3.19), we have

$$\int_{B_1} |\nabla u - \nabla v|^p \, dx \le C \left(\sigma \int_{B_1} |\nabla u|^p \, dx + \tilde{C} \right)^{\frac{p}{2}} \left(\int_{B_1} (|\nabla u| + |\nabla v|)^p dx \right)^{1-\frac{p}{2}}$$
$$\le C 2^{\frac{p}{2}} \left(\sigma^{\frac{p}{2}} \left(\int_{B_1} |\nabla u|^p \, dx \right)^{\frac{p}{2}} + \tilde{C}^{\frac{p}{2}} \right) \left(\int_{B_1} 2^p (|\nabla u|^p + |\nabla v|^p) \, dx \right)^{1-\frac{p}{2}}$$
$$\le C(n,p) \left(\sigma^{\frac{p}{2}} \left(\int_{B_1} |\nabla u|^p \, dx \right)^{\frac{p}{2}} + 1 \right) \left(\int_{B_1} |\nabla u|^p \, dx + \int_{B_1} |\nabla v|^p \, dx \right)^{1-\frac{p}{2}},$$

which implies, because v is the p-harmonic replacement of u in B_1,

$$\int_{B_1} |\nabla u - \nabla v|^p \, dx \le C(n,p) \left(\sigma^{\frac{p}{2}} \left(\int_{B_1} |\nabla u|^p \, dx \right)^{\frac{p}{2}} + 1 \right)$$
$$\left(2 \int_{B_1} |\nabla u|^p \, dx \right)^{1-\frac{p}{2}} = C(n,p) \left(\sigma^{\frac{p}{2}} \int_{B_1} |\nabla u|^p \, dx + \left(\int_{B_1} |\nabla u|^p \, dx \right)^{1-\frac{p}{2}} \right).$$

Taking the average integral in the previous inequality, we then have

$$\fint_{B_1} |\nabla u - \nabla v|^p \, dx \le C(n,p)(\sigma^{\frac{p}{2}} a^p + (\omega_n)^{-\frac{p}{2}} (a^p)^{1-\frac{p}{2}})$$
$$\le C(n,p)(\sigma^{\frac{p}{2}} a^p + (a^p)^{1-\frac{p}{2}}). \qquad (3.20)$$

At this point, for every p, we know by Theorem 3.19 in [66] that, since v is the p-harmonic replacement of u in B_1, if we fix $x \in B_{1/2}$, then, because $B_{1/2}(x) \subset B_1$,

$$|\nabla v|^p (x) \le \sup_{B_{1/4}(x)} |\nabla v|^p \le C \fint_{B_{1/2}(x)} |\nabla v|^p \, dy \le C \fint_{B_1} |\nabla u|^p \, dy,$$

thus we achieve

$$|\nabla v| (x) \le C_0 a, \quad x \in B_{1/2}. \qquad (3.21)$$

Denoting by $q := \nabla v(0)$, from (3.21), we hence obtain $|q| \leq C_0 a$, and by Theorem 2 in [68] we have

$$\fint_{B_\eta} |\nabla v - q|^p \, dx \leq \fint_{B_\eta} \left(C \left(\frac{\eta}{1/2} \right)^\alpha \|\nabla v\|_{L^\infty(B_{1/2})} \right)^p dx \leq C\eta^{\alpha p} \|\nabla v\|^p_{L^\infty(B_{1/2})}$$

i.e., together with (3.21),

$$\fint_{B_\eta} |\nabla v - q|^p \, dx \leq C_1 a^p \eta^{\alpha p}, \quad \forall \eta \leq 1/2, \tag{3.22}$$

for some $0 < \alpha \leq 1$ and C_1 universal.

Now, let us distinguish two cases again. If $p \geq 2$, using (3.19) and (3.22), we get

$$\begin{aligned} \fint_{B_\eta} |\nabla u - q|^p \, dx &\leq \fint_{B_\eta} 2^p (|\nabla u - \nabla v|^p + |\nabla v - q|^p) dx \\ &\leq 2^p (\eta^{-n} C(\sigma a^p + 1) + C_1 a^p \eta^{\alpha p}), \end{aligned}$$

namely

$$\fint_{B_\eta} |\nabla u - q|^p \, dx \leq 2^p C \eta^{-n} \sigma a^p + 2^p C \eta^{-n} + 2^p C_1 a^p \eta^{\alpha p}, \tag{3.23}$$

which yields

$$\fint_{B_\eta} |\nabla u|^p \, dx \leq 4^p C \eta^{-n} \sigma a^p + 4^p C \eta^{-n} + 4^p C_1 a^p \eta^{\alpha p} + 2^p |q|^p \tag{3.24}$$

as well.

Otherwise, if $1 < p < 2$, we have, by virtue of (3.20) and (3.22), and repeating the same argument to achieve (3.23),

$$\fint_{B_\eta} |\nabla u - q|^p \, dx \leq 2^p C \eta^{-n} \sigma^{\frac{p}{2}} a^p + 2^p C \eta^{-n} (a^p)^{1-\frac{p}{2}} + 2^p C_1 a^p \eta^{\alpha p}, \tag{3.25}$$

which also gives

$$\fint_{B_\eta} |\nabla u|^p \, dx \leq 4^p C \eta^{-n} \sigma^{\frac{p}{2}} a^p + 4^p C \eta^{-n} (a^p)^{1-\frac{p}{2}} + 4^p C_1 a^p \eta^{\alpha p} + 2^p |q|^p . \tag{3.26}$$

At this point, given $\varepsilon > 0$, we want to show that we can choose η small (depending on ε) and then σ small and a large depending on η, such that

$$\begin{cases} 4^pC\eta^{-n}\sigma a^p + 4^pC\eta^{-n} + 4^pC_1a^p\eta^{\alpha p} \leq 2^p\varepsilon^pa^p \leq \frac{a^p}{2^{p+1}} & \text{if } p \geq 2, \\ 4^pC\eta^{-n}\sigma^{\frac{p}{2}}a^p + 4^pC\eta^{-n}(a^p)^{1-\frac{p}{2}} + 4^pC_1a^p\eta^{\alpha p} \leq 2^p\varepsilon^pa^p \leq \frac{a^p}{2^{p+1}} \\ \text{if } 1 < p < 2. \end{cases} \tag{3.27}$$

Precisely, first we fix η small in both cases. Then, if $p \geq 2$, we can choose $\sigma = \eta^{n+1}$ and thus

$$4^pC\eta^{-n}\sigma a^p + 4^pC\eta^{-n} + 4^pC_1a^p\eta^{\alpha p} = 4^pC\eta a^p + 4^pC\eta^{-n} + 4^pC_1a^p\eta^{\alpha p},$$

so we want to choose a such that

$$4^pC\eta a^p + 4^pC\eta^{-n} + 4^pC_1a^p\eta^{\alpha p} \leq 2^p\varepsilon^pa^p \iff 2^pC\eta^{-n} \leq a^p(\varepsilon^p - 2^pC\eta - 2^pC_1\eta^{\alpha p}) \iff a \geq \left(\frac{2^pC\eta^{-n}}{\varepsilon^p - 2^pC\eta - 2^pC_1\eta^{\alpha p}}\right)^{1/p}.$$

In parallel, if $1 < p < 2$, we can choose $\sigma = \eta^{(n+1)\frac{2}{p}}$, which entails

$$4^pC\eta^{-n}\sigma^{\frac{p}{2}}a^p + 4^pC\eta^{-n}(a^p)^{1-\frac{p}{2}} + 4^pC_1a^p\eta^{\alpha p} = 4^pC\eta a^p + 4^pC\eta^{-n}(a^p)^{1-\frac{p}{2}} + 4^pC_1a^p\eta^{\alpha p},$$

hence we want to choose a so that

$$\begin{aligned} &4^pC\eta a^p + 4^pC\eta^{-n}(a^p)^{1-\frac{p}{2}} + 4^pC_1a^p\eta^{\alpha p} \leq 2^p\varepsilon^pa^p \iff a^p(\varepsilon^p - 2^pC\eta \\ &- 2^pC\eta^{-n}a^{-\frac{p^2}{2}} - 2^pC_1\eta^{\alpha p}) \geq 0 \iff \varepsilon^p - 2^pC\eta - 2^pC\eta^{-n}a^{-\frac{p^2}{2}} - 2^pC_1\eta^{\alpha p} \\ &\geq 0 \iff \varepsilon^p - 2^pC\eta - 2^pC_1\eta^{\alpha p} \geq 2^pC\eta^{-n}a^{-\frac{p^2}{2}} \\ &\iff a^{\frac{p^2}{2}} \geq \frac{2^pC\eta^{-n}}{\varepsilon^p - 2^pC\eta - 2^pC_1\eta^{\alpha p}} \iff a \geq \left(\frac{2^pC\eta^{-n}}{\varepsilon^p - 2^pC\eta - 2^pC_1\eta^{\alpha p}}\right)^{2/p^2}. \end{aligned}$$

Now, we distinguish two cases according to the size of $|q|$. In particular, if

$$|q| \leq \frac{a}{2^{(2p+1)/p}},$$

we obtain, in view of (3.24) and (3.27),

$$\fint_{B_\eta} |\nabla u|^p \, dx \leq \frac{a^p}{2^{p+1}} + 2^p \frac{a^p}{2^{2p+1}} = \frac{a^p}{2^{p+1}} + \frac{a^p}{2^{p+1}} = \frac{a^p}{2^p},$$

and thus

$$\left(\fint_{B_\eta} |\nabla u|^p \, dx \right)^{1/p} \leq \frac{a}{2}.$$

Analogously, repeating the same computation, we achieve the same conclusion from (3.26) and (3.27). Otherwise, we have

$$\frac{a}{2^{(2p+1)/p}} < |q| \leq C_0 a,$$

and, by (3.23) and (3.27), we get

$$\fint_{B_\eta} |\nabla u - q|^p \, dx \leq \varepsilon^p a^p,$$

i.e.

$$\left(\fint_{B_\eta} |\nabla u - q|^p \, dx \right)^{1/p} \leq \varepsilon a.$$

Analogously, repeating the same argument, according to (3.25) and (3.27), we obtain same sentence as before. □

We want to show now that alternative (3.17) can be "improved" when ε and σ are sufficiently small, again taking inspiration from [30]. The next result expresses this fact.

Lemma 3.10. *Let u be as in Proposition 3.9 with $p \geq 2$ and $a \geq a_0 > 0$. Assume also that*

$$\left(\fint_{B_1} |\nabla u - q|^p \, dx \right)^{1/p} \leq \varepsilon a, \tag{3.28}$$

for some $\varepsilon > 0$ and $q \in \mathbb{R}^n$ such that

$$\frac{a}{2^{(3p+1)/p}} < |q| \leq 2C_0 a, \tag{3.29}$$

with $C_0 > 0$ the universal constant in Proposition 3.9.
There exist $0 < \alpha < 1$, and corresponding $\rho = \rho(\alpha) > 0$, $\varepsilon_0 = \varepsilon_0(\alpha, a_0)$, $c_0 = c_0(\alpha, a_0)$, such that if

$$\varepsilon \leq \varepsilon_0 \quad \text{and} \quad \sigma \leq c_0 \varepsilon^p,$$

then

$$\left(\fint_{B_\rho} |\nabla u - \tilde{q}|^p \, dx\right)^{1/p} \le \varepsilon \rho^\alpha a \quad \textit{Poincaré-Sobolev}$$

with $\tilde{q} \in \mathbb{R}^n$ *such that*

$$|q - \tilde{q}| \le \tilde{C}\varepsilon a,$$

for some $\tilde{C} > 0$ *universal.*

Proof. Let $\bar{v}$ denote the p-harmonic replacement of u in $B_{1/2}$ and define by v the competitor

$$\begin{cases} v = \bar{v} & \text{in } B_{1/2}, \\ v = u & \text{outside } B_{1/2}. \end{cases} \tag{3.30}$$

Then, since $v = u$ on ∂B_1 and $v \in W^{1,p}(B_1)$ by (3.30), using the hypotheses on u, we have

$$J_p(u, B_1) \le (1+\sigma) J_p(v, B_1),$$

which gives, because $v = u$ outside $B_{1/2}$,

$$\begin{aligned} J_p(u, B_{1/2}) + J_p(u, B_1 \setminus B_{1/2}) \le J_p(v, B_1) + \sigma J_p(v, B_1) = J_p(v, B_{1/2}) \\ + J_p(u, B_1 \setminus B_{1/2}) + \sigma J_p(v, B_1), \end{aligned}$$

that is

$$J_p(u, B_{1/2}) \le J_p(v, B_{1/2}) + \sigma J_p(v, B_1). \tag{3.31}$$

Now, rewriting (3.31) in view of definition of J_p, we obtain

$$\begin{aligned} \int_{B_{1/2}} |\nabla u|^p \, dx + \left|\{u > 0\} \cap B_{1/2}\right| \le \int_{B_{1/2}} |\nabla v|^p \, dx + \left|\{v > 0\} \cap B_{1/2}\right| \\ + \sigma J_p(v, B_1) \le \int_{B_{1/2}} |\nabla v|^p \, dx + \left|B_{1/2}\right| + \sigma J_p(v, B_1), \end{aligned}$$

which yields

$$\int_{B_{1/2}} (|\nabla u|^p - |\nabla v|^p) \, dx \le \left|\{u = 0\} \cap B_{1/2}\right| + \sigma J_p(v, B_1).$$

This inequality, by virtue of (3.3), also implies

$$\int_{B_{1/2}} |\nabla u - \nabla v|^p \, dx \leq C \left|\{u = 0\} \cap B_{1/2}\right| + C\sigma J_p(v, B_1), \tag{3.32}$$

with $C = C(n, p) > 0$.
Lastly, because v is the p-harmonic replacement of u in $B_{1/2}$ and is equal to u outside $B_{1/2}$, it holds

$$\int_{B_1} |\nabla v|^p \, dx + |\{v > 0\} \cap B_1| \leq \int_{B_1} |\nabla u|^p \, dx + |B_1| \leq a^p + C,$$

so, after relabeling C, we conclude, from (3.32),

$$\int_{B_{1/2}} |\nabla u - \nabla v|^p \, dx \leq C \left|\{u = 0\} \cap B_{1/2}\right| + C\sigma(a^p + 1). \tag{3.33}$$

Moreover, we claim that

$$\left|\{u = 0\} \cap B_{1/2}\right| \leq C_1 \varepsilon^{p+\delta}, \tag{3.34}$$

with C_1, δ universal, which gives, by (3.33), after renaming C_1,

$$\int_{B_{1/2}} |\nabla u - \nabla v|^p \, dx \leq C_1 \varepsilon^{p+\delta} + C\sigma(a^p + 1). \tag{3.35}$$

At this point, we want to show that, even if $v - q \cdot x$ is not the p-harmonic replacement of $u - q \cdot x$, differently from the classical case (see [30]), it still satisfies a uniformly elliptic equation in $B_{1/2}$ with C^γ coefficients, $0 < \gamma < 1$. Precisely, using (3.28) and (3.35), we get (universal constants can change from line to line)

$$\begin{aligned} \int_{B_{1/2}} |\nabla v - q|^p \, dx &\leq 2^p \left(\int_{B_{1/2}} |\nabla u - q|^p \, dx + \int_{B_{1/2}} |\nabla v - \nabla u|^p \, dx \right) \\ &\leq 2^p (\varepsilon^p a^p + C_1 \varepsilon^{p+\delta} + C\sigma(a^p + 1)), \end{aligned}$$

in other words

$$\int_{B_{1/2}} |\nabla v - q|^p \, dx \leq 2^p \varepsilon^p a^p + C_1 \varepsilon^{p+\delta} + C\sigma(a^p + 1).$$

Let us assume, at this moment, that $\sigma \leq c_0 \varepsilon^p$, with c_0 to be made precise later. As a consequence, from the previous inequality, we have

$$\int_{B_{1/2}} |\nabla v - q|^p \, dx \leq 2^p \varepsilon^p a^p + C_1 \varepsilon^{p+\delta} + C c_0 \varepsilon^p (a^p + 1),$$

which implies, since $a \geq a_0 > 0$ and $\varepsilon^{p+\delta} \leq \varepsilon^p$,

$$\int_{B_{1/2}} |\nabla v - q|^p \, dx \leq C \varepsilon^p a^p. \tag{3.36}$$

Specifically, if $a_0 \geq 1$ this fact easily follows. Otherwise, we can get an upper bound of $C c_0 \varepsilon^p$ multiplying and dividing it by a^p and then using that $1/a^p \leq 1/a_0^p$.

Furthermore, (3.36) also yields

$$|\nabla v - q| \leq C(\varepsilon a)^\nu, \tag{3.37}$$

where ν is a small exponent, non necessarily 1. This condition however, together with (3.18) and the fact that $a \geq a_0 > 0$, guarantees that $v - q \cdot x$ satisfies a uniformly elliptic equation with C^γ coefficients.

Precisely, let us remark first that $q \cdot x$ is p-harmonic as v, since $\nabla(q \cdot x) = q$, which is a constant. Let us define now the function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$F(z) = |z|^{p-2} z, \tag{3.38}$$

and we consider

$$F(\nabla v) - F(q) = \int_0^1 \frac{d}{dt} F(t\nabla v + (1-t)q) \, dt,$$

that is

$$|\nabla v|^{p-2} \nabla v - |q|^{p-2} q = \int_0^1 \frac{d}{dt} F(t\nabla v + (1-t)q) \, dt. \tag{3.39}$$

Let us compute explicitly $\frac{d}{dt} F(t\nabla v + (1-t)q)$. We have

$$\frac{d}{dt} F(t\nabla v + (1-t)q) = DF(t\nabla v + (1-t)q)(\nabla v - q),$$

which implies, from (3.39),

$$|\nabla v|^{p-2} \nabla v - |q|^{p-2} q = \int_0^1 DF(t\nabla v + (1-t)q)(\nabla v - q) \, dt,$$

namely

$$|\nabla v|^{p-2}\nabla v - |q|^{p-2} q = \left(\int_0^1 DF(t\nabla v + (1-t)q)\,dt\right)(\nabla v - q). \qquad (3.40)$$

At this point, applying the divergence to both sides in (3.40), because v and $q \cdot x$ are both p-harmonic in $B_{1/2}$, we achieve

$$\operatorname{div}(A(x)(\nabla v - q)) = 0 \quad \text{in } B_{1/2},$$

with

$$A(x) := \int_0^1 DF(t\nabla v + (1-t)q)\,dt.$$

In particular, in view of (3.29), the fact that $a \geq a_0 > 0$ and (3.37), $v - q \cdot x$ satisfies a uniformly elliptic equation in $B_{1/2}$ with C^γ coefficients, $0 < \gamma < 1$. Therefore, applying again Theorem 3.19 in [66], we obtain fixing $x \in B_{1/4}$, since $B_{1/4}(x) \subset B_{1/2}$ and following the argument used to have (3.21),

$$|\nabla v - q|^p(x) \leq C \fint_{B_{1/2}(x)} |\nabla v - q|^p\,dy,$$

which yields, using (3.36),

$$|\nabla v - q|(x) \leq \tilde{C}\varepsilon a, \quad x \in B_{1/4}. \qquad (3.41)$$

By virtue of this, denoting $\bar{q}$ the gradient of $v - q \cdot x$ at 0, it holds, again from Theorem 2 in [68], always because $v - q \cdot x$ solves a uniformly elliptic equation in $B_{1/2}$ with C^γ coefficients, recalling the steps done to have (3.22),

$$\fint_{B_\rho} |\nabla(v - q\cdot x) - \bar{q}|^p\,dx \leq C_2 \rho^{\beta p}\varepsilon^p a^p, \quad \forall \rho \leq 1/4,$$

with C_2 universal and $0 < \beta \leq 1$, namely, denoting $\tilde{q} := q + \bar{q}$,

$$\fint_{B_\rho} |\nabla v - \tilde{q}|^p\,dx \leq C_2 \rho^{\beta p}\varepsilon^p a^p. \qquad (3.42)$$

We point out that, by (3.41) and the definitions of $\bar{q}$ and $\tilde{q}$,

$$|\tilde{q} - q| = |\bar{q}| \leq \tilde{C}\varepsilon a.$$

Putting together now (3.35) and (3.42), we then achieve

$$\fint_{B_\rho} |\nabla u - \tilde{q}|^p \, dx \leq 2^p \left(\frac{1}{\omega_n \rho^n} (C_1 \varepsilon^{p+\delta} + C\sigma(a^p + 1)) + C_2 \rho^{\beta p} \varepsilon^p a^p \right),$$

that is

$$\fint_{B_\rho} |\nabla u - \tilde{q}|^p \, dx \leq 2^p C_1 \varepsilon^{p+\delta} \rho^{-n} + \bar{C}\sigma(a^p + 1)\rho^{-n} + 2^p C_2 \rho^{\beta p} \varepsilon^p a^p. \quad (3.43)$$

At this point, we want to choose ρ so that

$$2^p C_2 \rho^{\beta p} \varepsilon^p a^p \leq \frac{1}{4} \rho^{\alpha p} \varepsilon^p a^p, \quad (3.44)$$

and this is possible only if we take $0 < \alpha < \beta$. Precisely, we have to take ρ depending on α in such a way that

$$\rho^{(\beta - \alpha)p} \leq \frac{1}{2^{p+2} C_2},$$

in other words

$$\rho \leq (2^{p+2} C_2)^{\frac{1}{(\beta - \alpha)p}}.$$

Moreover, we want to choose ε small depending on ρ (and thus α) and a_0 such that

$$2^p C_1 \varepsilon^{p+\delta} \rho^{-n} \leq \frac{1}{4} \rho^{\alpha p} \varepsilon^p a_0^p, \quad (3.45)$$

and thus we take ε which satisfies

$$\varepsilon^\delta \leq \frac{1}{2^{p+2} C_1} \rho^{\alpha p + n} a_0^p,$$

i.e.

$$\varepsilon \leq \left(\frac{1}{2^{p+2} C_1} \rho^{\alpha p + n} a_0^p \right)^{1/\delta}.$$

Lastly, we choose σ small depending on ρ (and hence α) and a_0 in such a way that

$$\bar{C}\sigma(a^p + 1)\rho^{-n} \leq \frac{1}{4} \rho^{\alpha p} \varepsilon^p (a^p + a_0^p), \quad (3.46)$$

which means that we take σ so that

$$\sigma \leq \frac{\rho^{\alpha p + n} \varepsilon^p (a^p + a_0^p)}{\bar{C}(a^p + 1)},$$

which is coherent with our hypothesis that $\sigma \le c_0\varepsilon^p$. Putting together the choices (3.44), (3.45) and (3.46), we then get, in view of (3.43), since $a \ge a_0$,

$$\begin{aligned}&\fint_{B_\rho} |\nabla u - \tilde q|^p\,dx \le \frac14 \rho^{\alpha p}\varepsilon^p a_0^p + \frac14 \rho^{\alpha p}\varepsilon^p (a^p + a_0^p) + \frac14 \rho^{\alpha p}\varepsilon^p a^p \\ &\le \frac14 \rho^{\alpha p}\varepsilon^p a^p + \frac12 \rho^{\alpha p}\varepsilon^p a^p + \frac14 \rho^{\alpha p}\varepsilon^p a^p = \rho^{\alpha p}\varepsilon^p a^p,\end{aligned}$$

which gives

$$\left(\fint_{B_\rho} |\nabla u - \tilde q|^p\,dx\right)^{1/p} \le \rho^\alpha \varepsilon a,$$

as desired.

It remains to show that (3.34) is true.

To this end, we consider the linear function

$$l(x) := b + q\cdot x, \quad b := \fint_{B_1} u, \tag{3.47}$$

and we observe that

$$\fint_{B_1} (u-l)\,dx = 0.$$

Indeed, since $q \cdot x$ is harmonic,

$$\begin{aligned}&\fint_{B_1} (u-l)\,dx = \fint_{B_1} u\,dx + \fint_{B_1} q\cdot x\,dx - \fint_{B_1} b\,dx = b + \fint_{B_1} q\cdot x\,dx - b \\ &= (q\cdot x)(0) = 0.\end{aligned}$$

As a consequence, by Poincaré inequality, denoting

$$(u-l)_{B_1} = \fint_{B_1} (u-l)\,dx,$$

we achieve

$$\|u - l - (u-l)_{B_1}\|_{L^p(B_1)} = \|u-l\|_{L^p(B_1)} \le C\,\|\nabla(u-l)\|_{L^p(B_1)},$$

with C universal, which entails, from (3.28), because $\nabla l = q$, using (3.47),

$$\fint_{B_1} |u-l|^p\,dx \le C \fint_{B_1} |\nabla u - q|^p\,dx \le C\varepsilon^p a^p. \tag{3.48}$$

Now, since $u \geq 0$, $l^- \leq |u-l|$. Precisely, when $l^- = 0$ it follows immediately, whereas when $l^- > 0$,

$$l^- = -l \leq u - l = |u-l|\,.$$

Hence, by virtue of (3.48), we have

$$\fint_{B_1} (l^-)^p\,dx \leq C\varepsilon^p a^p. \tag{3.49}$$

This fact, together with (3.29), yields

$$l \geq c_1 a \quad \text{in } B_{1/2}, \tag{3.50}$$

with c_1 universal and ε small.
Specifically, we first note that l^- is subharmonic, because l is harmonic. So, by Hölder's inequality with exponent p, we obtain, according to (3.49),

$$l^-(x) \leq \fint_{B_{1/4}(x)} l^-(y)\,dy \leq \frac{1}{\left|B_{1/4}(x)\right|}\left(\int_{B_{1/4}(x)} (l^-(y))^p\,dy\right)^{1/p} \left|B_{1/4}(x)\right|^{1-1/p}$$
$$= \left(\fint_{B_{1/4}(x)} (l^-(y))^p\,dy\right)^{1/p} \leq C\varepsilon a, \quad x \in B_{3/4}.$$

In particular, this condition implies

$$-l(x) \leq l^-(x) \leq C\varepsilon a, \quad x \in B_{3/4},$$

in other words

$$l(x) \geq -C\varepsilon a, \quad x \in B_{3/4}. \tag{3.51}$$

At this point, let us fix $x \in B_{1/2}$ and we consider

$$y \coloneqq x - \frac{q}{4\,|q|}.$$

We remark that

$$|y| \leq |x| + \left|\frac{q}{4\,|q|}\right| < \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$

i.e. $y \in B_{3/4}$, and thus, by (3.51), $l(y) \geq -C\varepsilon a$. Developing this last inequality, in view of (3.47), we then have

$$q \cdot y + b = q \cdot \left(x - \frac{q}{4\,|q|}\right) + b = q \cdot x - \frac{|q|^2}{4\,|q|} + b = q \cdot x - \frac{|q|}{4} + b \geq -C\varepsilon a,$$

which gives, using (3.29),

$$q \cdot x + b \geq \frac{|q|}{4} - C\varepsilon a \geq \frac{a}{2^{(3p+1)/p+2}} - C\varepsilon a = c_1 a,$$

with c_1 universal, if

$$\frac{1}{2^{(3p+1)/p+2}} - C\varepsilon > 0,$$

namely ε sufficiently small. To recap, we have shown that

$$q \cdot x + b \geq c_1 a, \quad x \in B_{1/2},$$

and so, by the arbitrariness of x and definition (3.47), we get (3.50).
Now, dividing u and l by a, we can assume without loss of generality that $a = 1$ in (3.28) and (3.50). Moreover, from the Poincaré-Sobolev inequality, we obtain

$$\begin{cases} \left(\int_{B_1} |u - l|^{p^*} \, dx \right)^{1/p^*} \leq C \left(\int_{B_1} |\nabla(u - l)|^p \, dx \right)^{1/p} & \text{if } p < n, \\ \sup\limits_{B_1} |u - l| \leq C \left(\int_{B_1} |\nabla(u - l)|^p \, dx \right)^{1/p} & \text{if } p > n, \end{cases} \tag{3.52}$$

with $C = C(n, p)$ universal.
Let us treat the two cases above separately. Let us suppose first that $p < n$. Then, recalling (3.52), definition (3.47), and (3.28), we get

$$\left(\int_{B_1} |u - l|^{p^*} \, dx \right)^{1/p^*} \leq C\varepsilon.$$

Furthermore, exploiting (3.50), we achieve

$$\begin{aligned} C\varepsilon &\geq \left(\int_{B_1} |u - l|^{p^*} \, dx \right)^{1/p^*} \geq \left(\int_{B_{1/2} \cap \{u=0\}} |u - l|^{p^*} \, dx \right)^{1/p^*} \\ &= \left(\int_{B_{1/2} \cap \{u=0\}} |l|^{p^*} \, dx \right)^{1/p^*} \geq c_1 \left| B_{1/2} \cap \{u = 0\} \right|^{1/p^*}, \end{aligned}$$

which yields

$$\left| B_{1/2} \cap \{u = 0\} \right| \leq C\varepsilon^{p^*}. \tag{3.53}$$

At this point, we recall that

$$p^* := \frac{np}{n-p},$$

and hence we can rewrite p^* as

$$p^* = \frac{np + p^2 - p^2}{n-p} = p + \frac{p^2}{n-p}.$$

Therefore, calling $\delta = p^2/(n-p)$, which satisfies $\delta > 0$, since $p < n$, we lastly get, according to (3.53),

$$\left|B_{1/2} \cap \{u = 0\}\right| \leq C\varepsilon^{p+\delta},$$

as desired.

Let us assume now that $p > n$. We note that in this case, in view of (3.52), we also have, with $\delta > 0$,

$$\left(\int_{B_1} |u-l|^{p+\delta}\, dx\right)^{1/(p+\delta)} \leq \sup_{B_1} |u-l| \, |B_1|^{1/(p+\delta)} \leq C\left(\int_{B_1} |\nabla(u-l)|^p\, dx\right)^{1/p}.$$

Therefore, we can repeat exactly the same reasoning done in case of $p < n$ and we obtain the desired result.

It remains to analyze the case $p = n$. For this purpose, we know by Theorem 7.15 in [54] that since $u - l \in W^{1,n}(B_1)$, then

$$\int_{B_1} \exp\left(\frac{|u-l|}{c_1 \|\nabla(u-l)\|_{L^n(B_1)}}\right)^{n/(n-1)} dx \leq c_2 |B_1| \tag{3.54}$$

with $c_1 = c_1(n)$ and $c_2 = c_2(n)$. As a consequence, because we know in general that

$$e^x \geq cx^\mu, \quad x, \mu > 0,$$

we also achieve from (3.54), by virtue of (3.47),

$$\int_{B_1} c\left(\frac{|u-l|}{c_1 \|\nabla u - q\|_{L^n(B_1)}}\right)^{n\mu/(n-1)} dx \leq c_2 |B_1|,$$

which implies, according to (3.28),

$$\int_{B_1} |u-l|^{n\mu/(n-1)}\, dx \leq C \|\nabla u - q\|^{n\mu/(n-1)}_{L^n(B_1)} |B_1| \leq C\varepsilon^{n\mu/(n-1)}, \tag{3.55}$$

with C universal. In particular, we can choose $\mu = n$ and we remark that

$$\frac{n^2}{n-1} = \frac{n^2 - n + n}{n-1} = n + \frac{n}{n-1} = n + \delta, \quad \delta > 0,$$

hence, from (3.55),

$$\int_{B_1} |u - l|^{n\mu/(n-1)}\, dx \leq C\varepsilon^{n+\delta}.$$

Arguing as in case $p < n$, we eventually get

$$\left|\{u = 0\} \cap B_{1/2}\right| \leq C_1 \varepsilon^{n+\delta},$$

with C_1, δ universal. □

Remark 3.11. We remark that Lemma 3.10 still holds if we replace the lower bound in (3.29) with the assumption

$$\fint_{B_1} u\, dx \geq C_1 a, \tag{3.56}$$

for C_1 universal large enough (depending on C_0). Indeed, in the proof of Lemma 3.10, the lower bound in (3.29) is only used to get (3.50). Therefore, it is enough to show that (3.50) still holds under the assumption (3.56). To this end, we first note that, by definition of b in (3.47), the assumption (3.56) reads

$$b \geq C_1 a.$$

Hence, using (3.47) and (3.29), we achieve

$$l(x) = q \cdot x + b \geq -|q|\,|x| + C_1 a \geq -2C_0 a \frac{1}{2} + C_1 a = a\left(C_1 - C_0\right) = c_1 a, \quad x \in B_{1/2},$$

in other words

$$l \geq c_1 a \quad \text{in } B_{1/2},$$

if $C_1 > C_0$ (from which the dependence of C_1 on C_0), which is exactly (3.50).

At this point, a corollary of Lemma 3.10 holds.

Corollary 3.12. *Let u be an almost minimizer for J_p in B_1 (with constant κ and exponent β), $p > 2$ and suppose that u satisfies* (3.28)-(3.18)*, together with $a \geq a_0 > 0$. Then, there exist ε_0, κ_0 depending on β, n and a_0, so that if $\varepsilon \leq \varepsilon_0$, $\kappa \leq \kappa_0 \varepsilon^2$ then*

$$\|u - l\|_{C^{1,\beta/p}(B_{1/2})} \leq C\varepsilon a, \tag{3.57}$$

where C is a universal constant and l is a linear function of slope q. Furthermore, it also holds

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq \tilde{C} a, \tag{3.58}$$

with $\tilde{C}$ universal.

Remark 3.13. In view of (3.57), we have $\nabla u \neq 0$, which implies that $u > 0$ in $B_{1/2}$.
Specifically, by (3.57), using the definition of $\|u - l\|_{C^{1,\beta/p}(B_{1/2})}$ and the fact that l is a linear function with slope q, we obtain

$$\|\nabla(u - l)\|_{L^\infty(B_{1/2})} = \|\nabla u - q\|_{L^\infty(B_{1/2})} \leq C\varepsilon a,$$

which gives

$$|\nabla u - q|\,(x) \leq C\varepsilon a, \quad x \in B_{1/2},$$

and thus also

$$|\nabla u|\,(x) \geq |q| - C\varepsilon a, \quad x \in B_{1/2}.$$

As a consequence, since (3.18) holds, we get

$$|\nabla u|\,(x) \geq \frac{a}{2^{(2p+1)/p}} - C\varepsilon a > 0, \quad x \in B_{1/2},$$

if ε is sufficiently small, which indeed yields $\nabla u \neq 0$ in $B_{1/2}$.
From this fact, since u is nonnegative, we get that $u > 0$ in $B_{1/2}$. Precisely, because $u \geq 0$, if $u > 0$ in $B_{1/2}$ was not true, it would mean that there exists a point $x_0 \in B_{1/2}$ such that $u(x_0) = 0$. Since $u \in C^{1,\beta/p}(B_{1/2})$ by (3.57), we have now two alternatives. Either $u \equiv 0$ in $B_{1/2}$ or x_0 is a minimum point for u. In both cases, however, $\nabla u(x_0) = 0$, and this contradicts the fact that $\nabla u \neq 0$ in $B_{1/2}$.

Proof of Corollary 3.12. We show first that we can iterate Lemma 3.10 indefinitely with $\alpha = \beta/p$. Indeed, if ($q_0 := q$)

$$\left(\fint_{B_r} |\nabla u - q_k|^p \, dx \right)^{1/p} \leq \varepsilon r^{\beta/p} a, \quad \text{with } r = \rho^k, \tag{3.59}$$

holds, then the rescaling

$$u_r(x) := \frac{u(rx)}{r} \tag{3.60}$$

satisfies the hypotheses of Lemma 3.10 with

$$\sigma_r := \kappa r^{\beta}, \quad \varepsilon_r := \varepsilon r^{\beta/p}. \tag{3.61}$$

Precisely, from (3.60) and (3.59), we have

$$\left(\fint_{B_r} |\nabla u - q_k|^p \, dx \right)^{1/p} \overset{x=ry}{=} \left(\frac{1}{\omega_n r^n} \int_{B_1} |(\nabla u)(ry) - q_k|^p \, r^n dy \right)^{1/p}$$
$$\left(\frac{1}{\omega_n} \int_{B_1} |\nabla u_r(y) - q_k|^p \, dy \right)^{1/p} \leq \varepsilon r^{\beta/p} a,$$

in other words

$$\left(\fint_{B_1} |\nabla u_r - q_k|^p \, dy \right)^{1/p} \leq \varepsilon r^{\beta/p} a,$$

and so u_r defined as in (3.60) satisfies (3.28) with ε_r defined in (3.61). For the almost minimality condition, see instead Remark 3.14.
Moreover, the conclusion of Lemma 3.10 implies that

$$|q_{i+1} - q_i| \leq C\varepsilon \rho^{i\beta/p} a, \quad i \leq k-1, \tag{3.62}$$

which, together with (3.18), gives that (3.29) is true for any q_k, provided that ε_0 is sufficiently small, see Remark 3.15. As a consequence, the rescaling u_r and q_k satisfy all the hypotheses of Lemma 3.10 and so we can apply it with $\alpha = \beta/p$ to get

$$\left(\fint_{B_\rho} |\nabla u_r - q_{k+1}|^p \, dx \right)^{1/p} \leq \varepsilon \rho^{\beta/p} r^{\beta/p} a = \varepsilon \rho^{\beta/p} \rho^{k\beta/p} a = \varepsilon \rho^{(k+1)\beta/p} a. \tag{3.63}$$

In particular, in view of (3.60), we can rewrite the left hand side of (3.63) as

$$\left(\fint_{B_\rho} |\nabla u_r - q_{k+1}|^p \, dx\right)^{1/p} = \left(\fint_{B_\rho} |(\nabla u)(rx) - q_{k+1}|^p \, dx\right)^{1/p}$$
$$\overset{y=rx}{=} \left(\frac{1}{\omega_n \rho^n} \int_{B_{r\rho}} |\nabla u(y) - q_{k+1}|^p \, r^{-n} dy\right)^{1/p} \overset{r=\rho^k}{=} \left(\fint_{B_{\rho^{k+1}}} |\nabla u - q_{k+1}|^p \, dy\right)^{1/p},$$

which yields, from (3.63),

$$\left(\fint_{B_{\rho^{k+1}}} |\nabla u - q_{k+1}|^p \, dy\right)^{1/p} \leq \varepsilon \rho^{(k+1)\beta/p} a.$$

Therefore, (3.59) holds for $r = \rho^{k+1}$ as well and thus it is satisfied for any k. Now, we want to show that the same conclusion is true for all balls $B_r(x) \subset B_{3/4}$, after relabeling ε by $C\varepsilon$ if necessary.
Precisely, we fix $x \in B_{3/4}$ and we take $B_{1/4}(x)$. Since $x \in B_{3/4}$, $B_{1/4}(x) \subset B_1$, hence, according to (3.28), we achieve, because $q_0 := q$,

$$\left(\fint_{B_{1/4}(x)} |\nabla u - q_0|^p \, dy\right)^{1/p} \leq C\varepsilon a, \quad x \in B_{3/4},$$

with C universal. Repeating the reasoning used to obtain (3.59), we then have

$$\left(\fint_{B_r(x)} |\nabla u - q_k|^p \, dy\right)^{1/p} \leq C\varepsilon r^{\beta/p} a, \quad \text{with } r = \frac{1}{4}\rho^k, \quad x \in B_{3/4}. \tag{3.64}$$

At this point, it remains to show that (3.64) holds for any r such that $B_r(x) \subset B_{3/4}$. To this end, we distinguish two cases, i.e. either $r > 1/4$ or $r \leq 1/4$. Let us assume first that $r > 1/4$. Then, always from (3.28), we get

$$\left(\fint_{B_r(x)} |\nabla u - q_0|^p \, dy\right)^{1/p} \leq \left(\frac{1}{\omega_n (1/4)^n} \int_{B_1} |\nabla u - q_0|^p \, dy\right)^{1/p} \leq C\varepsilon a. \tag{3.65}$$

If instead $r \leq 1/4$, then $\exists k$ so that

$$\frac{1}{4}\rho^{k+1} \leq r \leq \frac{1}{4}\rho^k,$$

thus, using (3.64), we have

$$\left(\fint_{B_r(x)} |\nabla u - q_k|^p \, dy\right)^{1/p} \le \left(\frac{1}{\omega_n(\frac{1}{4}\rho^{(k+1)})^n} \int_{B_{\frac{1}{4}\rho^k}} |\nabla u - q_k|^p \, dy\right)^{1/p}$$

$$= \left(\frac{1}{\omega_n(\frac{1}{4}\rho^k)^n \rho^n} \int_{B_{\frac{1}{4}\rho^k}} |\nabla u - q_k|^p \, dy\right)^{1/p} \le C\varepsilon r^{\beta/p} a,$$

which gives

$$\left(\fint_{B_r(x)} |\nabla u - q_k|^p \, dy\right)^{1/p} \le C\varepsilon r^{\beta/p} a. \tag{3.66}$$

Consequently, putting together (3.64), (3.65) and (3.66), we lastly achieve

$$\left(\fint_{B_r(x)} |\nabla u - q_k|^p \, dy\right)^{1/p} \le C\varepsilon r^{\beta/p} a, \quad B_r(x) \subset B_{3/4}. \tag{3.67}$$

From this fact, by virtue of standard Campanato estimates, we then obtain

$$\|\nabla u - q_0\|_{C^{0,\beta/p}(B_{1/2})} \le C\varepsilon a, \tag{3.68}$$

from which our claims follow.
We show first this fact and then how we use Campanato estimates to get (3.68).
About the claim (3.58), from (3.68) we also have

$$\|\nabla u - q_0\|_{L^\infty(B_{1/2})} \le C\varepsilon a,$$

which gives, in view of (3.18), since $q_0 \coloneqq q$,

$$|\nabla u|(x) \le |\nabla u - q_0|(x) + |q_0| \le C\varepsilon a + C_0 a = \tilde{C} a, \quad x \in B_{1/2},$$

if ε is small enough, and thus

$$\|\nabla u\|_{L^\infty(B_{1/2})} \le \tilde{C} a.$$

Concerning claim (3.57), instead, we consider a linear function

$$l(x) \coloneqq q_0 \cdot x + b, \tag{3.69}$$

in such a way that

$$b = u(0). \tag{3.70}$$

Therefore, since $u \in C^{1,\beta/p}(B_{1/2})$ by (3.68), we can apply the mean value theorem to $u - l$ in the segment $[0, x]$, with $x \in B_{1/2}$, and we have

$$(u - l)(x) = (u - l)(0) + \nabla(u - l)(z) \cdot x, \quad z \in [0, x]. \tag{3.71}$$

In particular, using (3.69) and (3.70), (3.71) can be rewritten as

$$(u - l)(x) = (\nabla u - q_0)(z) \cdot x,$$

which entails from (3.68), because $x, z \in B_{1/2}$,

$$|u - l|\,(x) \leq |\nabla u - q_0|\,(z)\,|x| \leq C\varepsilon a \frac{1}{2} = C\varepsilon a,$$

and hence

$$\|u - l\|_{L^\infty(B_{1/2})} \leq C\varepsilon a.$$

This fact, together with (3.68), then yields

$$\|u - l\|_{C^{1,\beta/p}(B_{1/2})} \leq C\varepsilon a,$$

with l a linear function of slope q_0 and so of slope q.
We are left with the proof of (3.68). By Theorem 3.7, it is equivalent to show

$$\|\nabla u - q_0\|_{\mathcal{L}^{p,n+\beta}(B_{1/2})} \leq C\varepsilon a,$$

and from Remark 3.6 this means

$$\|\nabla u - q_0\|_{L^p(B_{1/2})} + [\nabla u - q_0]_{\mathcal{L}^{p,n+\beta}(B_{1/2})} \leq C\varepsilon a.$$

In view of (3.28), since $q_0 \coloneqq q$, we have

$$\|\nabla u - q_0\|_{L^p(B_{1/2})} \leq C\varepsilon a. \tag{3.72}$$

Therefore, it remains to show that

$$[\nabla u - q_0]_{\mathcal{L}^{p,n+\beta}(B_{1/2})} \leq C\varepsilon a.$$

Specifically, by virtue of Remark 3.6, it suffices to prove that

$$\left(\sup_{\substack{x_0 \in B_{1/2} \\ r>0}} r^{-(n+\beta)} \inf_{\xi \in \mathbb{R}^n} \int_{B_{1/2}(x_0,r)} |\nabla u - q_0 - \xi|^p \, dx \right)^{1/p} \leq C\varepsilon a.$$

For this purpose, we fix $x_0 \in B_{1/2}$ and by (3.67) we also obtain, for any $B_r(x_0) \subset B_{1/2}$,

$$\frac{1}{r^{\beta/p}} \left(\fint_{B_r(x_0)} |\nabla u - q_k|^p \, dx \right)^{1/p} = \left(\frac{1}{\omega_n r^{\beta+n}} \int_{B_r(x_0)} |\nabla u - q_k|^p \, dx \right)^{1/p} \leq C\varepsilon a$$

which yields

$$\left(\frac{1}{r^{\beta+n}} \int_{B_r(x_0)} |\nabla u - q_k|^p \, dx \right)^{1/p} \leq C\varepsilon a,$$

with C universal. Moreover, we can rewrite this as

$$\left(\frac{1}{r^{\beta+n}} \int_{B_r(x_0)} |\nabla u - q_0 + (q_0 - q_k)|^p \, dx \right)^{1/p} \leq C\varepsilon a,$$

so we get in addition

$$\left(\frac{1}{r^{\beta+n}} \inf_{\xi \in \mathbb{R}^n} \int_{B_r(x_0)} |\nabla u - q_0 + \xi|^p \, dx \right)^{1/p} \leq C\varepsilon a, \tag{3.73}$$

for any $B_r(x_0) \subset B_{1/2}$.

Now, we note that if r is such that $B_r(x_0) \subset B_{1/2}$, then by Definition 3.5 $B_{1/2}(x_0, r) = B_r(x_0)$, hence to achieve

$$\left(\sup_{\substack{x_0 \in B_{1/2} \\ r>0}} r^{-(n+\beta)} \inf_{\xi \in \mathbb{R}^n} \int_{B_{1/2}(x_0,r)} |\nabla u - q_0 - \xi|^p \, dx \right)^{1/p} \leq C\varepsilon a,$$

we also need to know

$$\left(\frac{1}{r^{\beta+n}} \inf_{\xi \in \mathbb{R}^n} \int_{B_{1/2}(x_0,r)} |\nabla u - q_0 + \xi|^p \, dx \right)^{1/p} \leq C\varepsilon a,$$

for any $B_r(x_0) \not\subseteq B_{1/2}$. To this end, we distinguish two cases. If $B_r(x_0)$ $\not\subseteq B_{3/4}$ this means that $r \geq 1/4$, since $x_0 \in B_{1/2}$. Thus, in view of (3.28), we have, because $B_{1/2}(x_0,r) \subset B_{1/2}$ from Definition 3.5,

$$\begin{aligned}
&\left(\frac{1}{r^{\beta+n}} \inf_{\xi\in\mathbb{R}^n} \int_{B_{1/2}(x_0,r)} |\nabla u - q_0 + \xi|^p \, dx\right)^{1/p} \\
&\leq \left(\frac{1}{(1/4)^{\beta+n}} \inf_{\xi\in\mathbb{R}^n} \int_{B_{1/2}(x_0,r)} |\nabla u - q_0 + \xi|^p \, dx\right)^{1/p} \\
&\leq \left(\frac{\omega_n}{(1/4)^{\beta+n}\omega_n} \int_{B_1} |\nabla u - q_0|^p \, dx\right)^{1/p} \leq C\varepsilon a,
\end{aligned}$$

that is

$$\left(\frac{1}{r^{\beta+n}} \inf_{\xi\in\mathbb{R}^n} \int_{B_{1/2}(x_0,r)} |\nabla u - q_0 + \xi|^p \, dx\right)^{1/p} \leq C\varepsilon a, \tag{3.74}$$

with C universal. If instead $B_r(x_0) \subset B_{3/4}$, by (3.67), repeating the same reasoning to obtain (3.73), we get

$$\left(\frac{1}{r^{\beta+n}} \inf_{\xi\in\mathbb{R}^n} \int_{B_r(x_0)} |\nabla u - q_0 + \xi|^p \, dx\right)^{1/p} \leq C\varepsilon a, \tag{3.75}$$

with C universal, for these balls as well. Consequently, considering together (3.73), (3.74) and (3.75), we obtain

$$\left(\frac{1}{r^{\beta+n}} \inf_{\xi\in\mathbb{R}^n} \int_{B_r(x_0)} |\nabla u - q_0 + \xi|^p \, dx\right)^{1/p} \leq C\varepsilon a,$$

with C universal, for every $x_0 \in B_{1/2}$ and for any r, which also gives

$$\frac{1}{r^{\beta+n}} \inf_{\xi\in\mathbb{R}^n} \int_{B_r(x_0)} |\nabla u - q_0 + \xi|^p \, dx \leq C\varepsilon^p a^p,$$

for every $x_0 \in B_{1/2}$ and for every r, and thus

$$[\nabla u - q_0]^p_{\mathcal{L}^{p,n+\beta}(B_{1/2})} = \sup_{\substack{x_0\in B_{1/2}\\ r>0}} \frac{1}{r^{\beta+n}} \inf_{\xi\in\mathbb{R}^n} \int_{B_r(x_0)} |\nabla u - q_0 + \xi|^p \, dx \leq C\varepsilon^p a^p,$$

i.e.

$$[\nabla u - q_0]_{\mathcal{L}^{p,n+\beta}(B_{1/2})} \leq C\varepsilon a.$$

□

Remark 3.14. We remark that if u is an almost minimizer for J_p in B_1 (with constant κ and exponent β), then the rescaling defined as in (3.60) satisfies (3.14) with $\sigma_r \coloneqq \kappa r^\beta$.
Indeed, by definition of almost minimizer for J_p in B_1 (with constant κ and exponent β), we know that

$$J_p(u, B_r) \leq (1 + \kappa r^\beta) J_p(v, B_r) \tag{3.76}$$

for every ball $B_r \subset B_1$ and for every function $v \in W^{1,p}(B_1)$ so that $v = u$ on ∂B_r in the trace sense. Now, we take $v \in W^{1,p}(B_1)$ such that $v = u_r$ on ∂B_1 in the trace sense. Then, using (3.60), we have (the equalities are always in the trace sense)

$$rv(x) = rv\left(\frac{rx}{r}\right) = u(rx), \quad x \in \partial B_1,$$

which means

$$v_r(x) = u(x), \quad x \in \partial B_r, \tag{3.77}$$

if we call

$$v_r(x) \coloneqq rv\left(\frac{x}{r}\right). \tag{3.78}$$

As a consequence, from (3.76), we get

$$\int_{B_r} (|\nabla u|^p + \chi_{\{u>0\}}) dx \leq (1 + \kappa r^\beta) \int_{B_r} (|\nabla v_r|^p + \chi_{\{v_r>0\}}) dx. \tag{3.79}$$

We want to rewrite (3.79) in terms of u_r and v. We have, according to (3.60) and (3.78),

$$
\begin{aligned}
&\int_{B_r} (|\nabla u|^p + \chi_{\{u>0\}})dx \overset{x=ry}{=} \int_{B_1} (|\nabla u(ry)|^p + \chi_{\{ry|\, u(ry)>0\}})\, r^n dy \\
&= r^n \int_{B_1} (|\nabla u_r(y)|^p + \chi_{\{ry|\, ru_r(y)>0\}})\, dy \overset{r\geq 0}{=} r^n \int_{B_1} (|\nabla u_r|^p + \chi_{\{ry|\, u_r(y)>0\}})dy \\
&= r^n \int_{B_1} (|\nabla u_r|^p + \chi_{\{y|\, u_r(y)>0\}})dy = r^n J_p(u_r, B_1) \\
&\leq (1+\kappa r^\beta) \int_{B_r} (|\nabla v_r|^p + \chi_{\{v_r>0\}})dx \\
&= (1+\kappa r^\beta) \int_{B_r} \left(\left|\nabla v\left(\frac{x}{r}\right)\right|^p + \chi_{\{x|\, v_r(x)>0\}} \right) dx \\
&\overset{x=ry}{=} (1+\kappa r^\beta) \int_{B_1} \left(\left|\nabla v\left(\frac{yr}{r}\right)\right|^p + \chi_{\{yr|\, v_r(yr)>0\}} \right) r^n dy \\
&= (1+\kappa r^\beta) r^n \int_{B_1} (|\nabla v(y)|^p + \chi_{\{yr|\, rv(y)>0\}})dy \\
&\overset{r\geq 0}{=} (1+\kappa r^\beta) r^n \int_{B_1} (|\nabla v|^p + \chi_{\{y|\, v(y)>0\}})dy = r^n(1+\kappa r^\beta) J_p(v, B_1),
\end{aligned}
$$

which yields

$$
r^n J_p(u_r, B_1) \leq r^n (1+\kappa r^\beta) J_p(v, B_1), \tag{3.80}
$$

and hence, recalling that $\sigma_r := \kappa r^\beta$,

$$
J_p(u_r, B_1) \leq (1+\sigma_r) J_p(v, B_1),
$$

for all $v \in W^{1,p}(B_1)$ such that $v = u_r$ on ∂B_1 in the trace sense.

Remark 3.15. (3.29) holds for any q_k by induction on $k \geq 1$, provided that ε_0 is sufficiently small. Specifically, according to (3.62), we have with $k = 1$

$$
|q_1 - q_0| \leq C\varepsilon a, \tag{3.81}
$$

which yields, using (3.18),

$$
|q_1| \leq |q_1 - q_0| + |q_0| \leq C\varepsilon a + C_0 a \leq 2C_0 a, \tag{3.82}
$$

if ε and hence ε_0 is sufficiently small. On the other hand, from (3.81), we also achieve, by virtue of (3.18),

$$
|q_1| \geq |q_0| - C\varepsilon a \geq \frac{a}{2^{(2p+1)/p}} - C\varepsilon a \geq \frac{a}{2^{(3p+1)/p}}, \tag{3.83}
$$

if ε and thus ε_0 is sufficiently small. Therefore, putting together (3.82) and (3.83), we obtain that (3.29) holds for q_1.
At this point, we suppose that (3.29) is true for q_k. By (3.62), we have

$$\begin{aligned}|q_{k+1} - q_0| &\leq |q_{k+1} - q_k| + |q_k - q_{k-1}| + \ldots + |q_1 - q_0| \\ &\leq C\varepsilon\rho^{k\beta/p}a + C\varepsilon\rho^{(k-1)\beta/p}a + \ldots + C\varepsilon a \leq C\varepsilon a,\end{aligned}$$

So, we can repeat the argument used in case of $k = 1$, and we get that (3.29) holds for $k + 1$ as well. Consequently, (3.29) is satisfied by any q_k.

Having proved the previous results, we are now able to show the proof of main Theorem 3.1.

Proof of Theorem 3.1. First of all, we note that without loss of generality we can assume that u is an almost minimizer with constant $\tilde{\kappa} = \kappa s^\beta$, which can be made arbitrary small. Indeed, we can consider the rescaling

$$u_s(x) \coloneqq \frac{u(sx)}{s},$$

and the fact follows arguing as in Remark 3.14.
Now, let us choose $\alpha = \beta/p$, and $a_0 = 1$ and let $\varepsilon_0 = \varepsilon_0(\beta, 1)$, $c_0 = c_0(\beta, 1)$ be given from Lemma 3.10. Moreover, let η small, $M \geq 1$ and σ_0 be the constants from Proposition 3.9 depending on $\varepsilon = \varepsilon_0$. Let us define at this point

$$a(\tau) \coloneqq \left(\fint_{B_\tau} |\nabla u|^p \, dx \right)^{1/p} \tag{3.84}$$

and we consider the integers $k \geq 0$ such that

$$a(\eta^k) \leq C(\eta)M + 2^{-k}a(1), \tag{3.85}$$

for $C(\eta)$ a large constant.
For $k = 0$, (3.85) is clearly true. Let us also suppose that (3.85) holds for all k's. Then, it follows that

$$a(r) \leq C(M, \eta)(1 + a(1)), \quad \forall r < 1. \tag{3.86}$$

Precisely, if $r < 1$, there exists $k_0 \geq 0$ so that $\eta^{k_0+1} < r \leq \eta^{k_0}$. Therefore, by (3.84) and (3.85), we get

$$a(r) \leq \left(\frac{1}{\omega_n \eta^{(k_0+1)n}} \int_{B_{\eta^{k_0}}} |\nabla u|^p \, dx\right)^{1/p} = \eta^{-n/p} a(\eta^{k_0}) \leq \eta^{-n/p}(C(\eta)M + 2^{-k_0}$$
$$a(1)) \leq \eta^{-n/p} \max(C(\eta)M, 1)(1 + a(1)) = C(M,\eta)(1 + a(1)).$$

Let us assume otherwise that (3.85) is not true for all k's and let $k_0 + 1$ be the first integer for which (3.85) fails. We distinguish then two cases. If

$$a(\eta^{k_0}) \leq M,$$

then (3.85) holds for $k_0 + 1$ as well, because, exploiting the steps to obtain (3.86), we have

$$a(\eta^{k_0+1}) \leq \eta^{-n/p} a(\eta^{k_0}) \leq C(\eta)M + 2^{-(k_0+1)} a(1),$$

which is a contradiction with the choice of k_0. As a consequence,

$$a(\eta^{k_0}) > M,$$

and, in view of Proposition 3.9 (rescaled) it holds that either, from (3.85),

$$a(\eta^{k_0+1}) \leq \frac{1}{2} a(\eta^{k_0}) \leq C(\eta)M + 2^{-(k_0+1)} a(1),$$

which yields again a contradiction, or

$$\left(\fint_{B_{\eta^{k_0+1}}} |\nabla u - q|^p \, dx\right)^{1/p} \leq \varepsilon a(\eta^{k_0}), \tag{3.87}$$

with

$$\frac{a(\eta^{k_0})}{2^{(2p+1)/p}} < |q| \leq C_0 a(\eta^{k_0}).$$

Hence, in this case we can apply Corollary 3.12 (rescaled) and we achieve, since (3.85) is true for k_0,

$$a(r) \leq \bar{C} a(\eta^{k_0}) \leq C(M,\eta)(1 + a(1)), \quad r \leq \eta^{k_0}. \tag{3.88}$$

Indeed, by Corollary 3.12 (rescaled), we have

$$\|\nabla u\|_{L^\infty(B_{\eta^{k_0}/2})} \le \bar{C} a(\eta^{k_0}).$$

Therefore, if $r \le \eta^{k_0}/2$, we get, because (3.85) is true for k_0,

$$a(r) \le \left(\fint_{B_r} \|\nabla u\|^p_{L^\infty(B_{\eta^{k_0}/2})}\, dx \right)^{1/p} = \|\nabla u\|_{L^\infty(B_{\eta^{k_0}/2})} \le C(M,\eta)(1 + a(1)),$$

which gives

$$a(r) \le C(M,\eta)(1 + a(1)), \quad r \le \eta^{k_0}/2.$$

Now, this fact is then true for all r's such that $r \le \eta^{k_0}$, since if $r > \eta^{k_0}/2$ we can repeat the same argument used to achieve (3.86).
On the other hand, if $r > \eta^{k_0}$, we can repeat again the argument to get (3.86) and it holds

$$a(r) \le C(M,\eta)(1 + a(1)), \quad r > \eta^{k_0},$$

which, together with (3.88), implies again (3.86).
In particular, repeating the same reasoning done to obtain (3.67), we have in addition that (3.86) holds for all balls with center in $B_{1/2}$ which are contained in B_1, namely, if we denote

$$a(r)(x) := \left(\fint_{B_r(x)} |\nabla u|^p\, dy \right)^{1/p}, \tag{3.89}$$

we have

$$a(r)(x) \le C(M,\eta)(1 + a(1)(0)), \quad x \in B_{1/2},\ B_r(x) \subset B_1.$$

Consequently, we get from (3.89), by virtue of Lebesgue Differentiation Theorem, since $u \in W^{1,p}(B_1)$,

$$|\nabla u(x)| = (|\nabla u(x)|^p)^{1/p} = \lim_{r\to 0} a(r)(x) \le C(M,\eta)(1 + a(1)(0)) = C(1 + a(1)(0))$$
$$= C(1 + \omega_n^{-1/p} \|\nabla u\|_{L^p(B_1)}) \le C(1 + \|u\|_{W^{1,p}(B_1)}),$$

which yields

$$\|\nabla u\|_{L^\infty(B_{1/2})} \le C(1 + \|u\|_{W^{1,p}(B_1)}). \tag{3.90}$$

So, the first claim is proved.
We prove at this point the second claim. First of all, we observe that if $u(0) = 0$, we can never end up in the alternative (3.87), otherwise Corollary 3.12 applies to u and by Remark 3.13 $u > 0$ in $B_{1/2}$, which is a contradiction. Therefore, arguing as above, this implies that (3.85) is true for all $k \geq 0$, and thus

$$a(\eta^k) \leq C,$$

with C universal depending on $\|\nabla u\|_{L^p(B_1)}$, i.e. $a(\eta^k)$ is uniformly bounded. Indeed, since

$$2^{-k}a(1) \overset{k\to\infty}{\to} 0,$$

$2^{-k}a(1)$ is bounded by a universal constant depending on $\|\nabla u\|_{L^p(B_1)}$, which gives in view of (3.85) the desired fact.
As a consequence, repeating the same argument used above to achieve (3.90), we get

$$|\nabla u(x)| \leq C(1 + a(\eta^k)(0)) \leq C, \quad x \in B_{\eta^k/2},$$

i.e

$$|\nabla u| \leq C \quad \text{in } B_{\eta^k/2},$$

with C universal depending on $\|\nabla u\|_{L^p(B_1)}$, which is the second claim with $r_0 = \eta^k/2$.

□

3.5 Nondegeneracy

In this section, our goal is twofold. First, we show that almost minimizers for J_p are well approximated by p-harmonic functions (in their positivity set). Next, we deal with nondegeneracy properties of almost minimizers, which are a crucial ingredient to use compactness arguments. Concerning the last topic, we have already mentioned that it is a tricky point. In particular, a strong nondegeneracy property in the spirit of [30] has not been proved for almost

minimizers to (3.1) yet.
Throughout the section, we assume that

$$\|\nabla u\|_{L^\infty(B_1)} \le K, \quad \text{and} \quad J_p(u, B_1) \le J_p(v, B_1) + \sigma. \tag{3.91}$$

We note that the second inequality in (3.91) directly comes from the condition of almost minimality. Indeed, by (3.91), we have

$$J_p(u, B_1) \le K^p |B_1| + |B_1| = C,$$

and so the energy inequality

$$J_p(u, B_1) \le (1+\sigma) J_p(v, B_1)$$

for any $v \in W^{1,p}(B_1)$ which agrees with u on ∂B_1, can be read as

$$J_p(u, B_1) \le J_p(v, B_1) + \tilde{C}\sigma,$$

with $\tilde{C}$ sufficiently large, i.e., relabeling σ, (3.91) holds. The advantage of (3.91), rather than (3.14), is that the energies cancel in a region where $u = v$ and (3.91) rescales better, see Remark 3.16 below.

Remark 3.16. The rescaling (3.60) satisfies (3.91) with $\sigma_r := r^{-n}\sigma$. Precisely, let us take a function $v \in W^{1,p}(B_1)$ such that $v = u_r$ on ∂B_1. Then, exploiting the same argument used to have (3.77), it holds $u = v_r$ on ∂B_r, with v_r defined as in (3.78). As a consequence, because $u = v_r$ on ∂B_r, we obtain

$$J_p(u, B_r) \le J_p(v_r, B_r) + \sigma,$$

which entails, repeating the same computations done to achieve (3.80),

$$r^n J_p(u_r, B_1) \le r^n J_p(v, B_1) + \sigma,$$

namely

$$J_p(u_r, B_1) \le J_p(v, B_1) + \sigma_r$$

for any $v \in W^{1,p}(B_1)$ such that $v = u_r$ on ∂B_1.

First, we prove the following lemma, which provides a comparison between the almost minimizer u and its p-harmonic replacement in the positivity set of u.

Lemma 3.17. *Assume that u satisfies* (3.91) *and $B_1 \subset \{u > 0\}$. Let v be the p-harmonic replacement of u in B_1. Then*

$$\|u - v\|_{L^\infty(B_{1/2})} \leq c(\sigma), \quad c(\sigma) \to 0 \quad \text{as } \sigma \to 0. \tag{3.92}$$

Proof. Since v is the p-harmonic replacement of u in B_1, we can use (3.91) to get

$$J_p(u, B_1) \leq J_p(v, B_1) + \sigma,$$

which gives, because $B_1 \subset \{u > 0\}$,

$$\int_{B_1} |\nabla u|^p \, dx + |B_1| \leq \int_{B_1} |\nabla v|^p \, dx + |B_1| + \sigma$$

and thus

$$\int_{B_1} |\nabla u|^p \, dx \leq \int_{B_1} |\nabla v|^p \, dx + \sigma.$$

From the last inequality, in view of Lemma 3.4, we then have

$$\int_{B_1} |\nabla u - \nabla v|^p \, dx \leq C\sigma, \tag{3.93}$$

with C universal.

Moreover, we can apply Poincaré inequality to $u - v \in W_0^{1,p}(B_1)$ and we obtain

$$\int_{B_{3/4}} |u - v|^p \, dx \leq C\sigma, \tag{3.94}$$

with $u - v$ uniformly Lipschitz in $B_{3/4}$, because both u and v are uniformly Lipschitz in $B_{3/4}$.

Let us assume now that (3.92) fails. This means that there exists $x_0 \in B_{1/2}$ such that $|u - v|(x_0) \geq \mu$ and in particular, without loss of generality, we can suppose (changing $u - v$ in $v - u$ otherwise) $(u - v)(x_0) \geq \mu$, with $\mu > 0$ independent of σ. Thus, by the uniform Lipschitz continuity of $u - v$, we get

$$(u - v)(x) \geq (u - v)(x_0) - C\,|x - x_0| \geq c_0\mu \quad \text{in } B_{c\mu}(x_0),$$

provided c is small enough. From this and (3.94), we then have

$$\mu^{p+n} \leq C\sigma,$$

which contradicts the hypothesis on μ and so (3.92) holds.

□

Lemma 3.18. *Let u be a function that satisfies* (3.91) *and suppose $B_1 \subset \{u > 0\}$. Let w be a $p-$harmonic function such that $u \geq w$ in B_1, $\nabla w \neq 0$ and $u - w \geq \mu > 0$ at 0 for some $\mu \leq \mu_0$, μ_0 small depending on K. Then $u - w \geq c\mu$ in $B_{1/2}$ for some c universal, provided that $\sigma \leq \mu^{n+p+1}$.*

Proof. Let v be the $p-$harmonic replacement of u in B_1. Then recalling Lemma 3.17, we deduce that in $B_{1/2}$

$$-c\sigma^{\frac{1}{n+p}} + v \leq u \leq v + c\sigma^{\frac{1}{n+p}}. \tag{3.95}$$

As a consequence, we obtain that in $B_{1/2}$

$$w \leq u \leq v + c\sigma^{\frac{1}{n+p}},$$

so that in $B_{1/2}$

$$(v + c\sigma^{\frac{1}{n+p}}) - w \geq 0 \tag{3.96}$$

and, in addition,

$$(v(0) + c\sigma^{\frac{1}{n+p}}) - w(0) \geq u(0) - w(0) \geq \mu,$$

namely

$$(v(0) + c\sigma^{\frac{1}{n+p}}) - w(0) \geq \mu. \tag{3.97}$$

Let $v_1 := v + c\sigma^{\frac{1}{n+p}}$. Then v_1 is $p-$harmonic as well. Thus

$$\int_{B_{1/2}} \langle A(\nabla v_1) - A(\nabla w), \nabla\varphi\rangle dx = \int_{B_{1/2}} \langle A(\nabla v) - A(\nabla w), \nabla\varphi\rangle dx = 0,$$

where $A(h) := |h|^{p-2}h$. On the other hand

$$A(\nabla v) - A(\nabla w) = \int_0^1 \frac{d}{dt} A(\nabla(tv + (1-t)w))dt,$$

so that by writing

$$b_{ij}(x) := \int_0^1 \frac{\partial}{\partial h_j} A^i(\nabla(tv + (1-t)w))dt, \quad B(x) := (b_{ij}(x))_{i,j},$$

we obtain

$$\int_{B_{1/2}} \langle B(x)\nabla(v_1 - w), \nabla\varphi\rangle dx = 0,$$

where

$$\lambda I \le B \le \Lambda I$$

and

$$\min\{1, p-1\} \int_0^1 |\nabla(tv_1 + (1-t)w)|^{p-2} dt =: \lambda(x) \le \Lambda(x) := \max\{1, p-1\} \int_0^1 |\nabla(tv_1 + (1-t)w)|^{p-2} dt.$$

Now, recalling that $\nabla w \neq 0$ it results that B is elliptic and $v_1 - w \ge 0$, see (3.96), satisfies $\mathrm{div}(B(x)\nabla(v_1 - w)) = 0$ in $B_{1/2}$. Then by Harnack inequality, see [64], using (3.97), it follows that in $B_{1/4}$

$$C_H(v_1 - w) \ge (v_1 - w)(0) \ge \mu.$$

Then, in view of the definition of v_1, we have

$$v + c\sigma^{\frac{1}{n+p}} - w \ge \frac{\mu}{C_H} \quad \text{in } B_{1/4}$$

which gives, from (3.95), if $\sigma \le \mu^{n+p+1}$,

$$u - w \ge v - c\sigma^{\frac{1}{n+p}} - w \ge \frac{\mu}{C_H} - 2\sigma^{\frac{1}{n+p}} \ge c\mu \quad \text{in } B_{1/4}.$$

Finally, to recap, it results

$$u - w \ge c\mu \quad \text{in } B_{1/4}.$$

□

Remark 3.19. This theorem is different with respect to the harmonic case, see [30], because the $p-$Laplace operator is not linear and we need to know that after the linearization of the operator, the matrix B has to be elliptic. This hypothesis is satisfied whenever we consider a $p-$harmonic function whose gradient does not vanish. In the application, this fact is satisfied for every polynomial of degree 1.

At this point, we are able to state and prove the weak nondegeneracy lemma.

Lemma 3.20 (Weak nondegeneracy)**.** *Suppose that u satisfies* (3.91) *for σ small and $B_1 \subset \{u > 0\}$. Then $u(0) \geq c$ with $c = c(K) > 0$.*

Proof. Denote by v the p-harmonic replacement of u in B_1. Then, in view of Lemma 3.17, it suffices to show that the statement holds for v. Indeed, if $v(0) \geq c$, $c = c(K) > 0$, by Lemma 3.17 we have $u(0) - v(0) \geq -C\sigma^{1/(n+p)}$, which gives

$$u(0) \geq v(0) - C\sigma^{1/(n+p)} \geq c - C\sigma^{1/(n+p)} = c$$

if σ is small enough.
Now, let us take $\varphi \in C_0^\infty(B_{1/2})$ such that $\varphi \equiv 1$ in $B_{1/4}$ and $0 \leq \varphi \leq 1$. Then, by definition of the p-harmonic replacement, since $\varphi \in C^\infty(B_{1/2})$, $v(1-\varphi) = v = u$ on ∂B_1, thus, using again the definition of p-harmonic replacement, (3.91) and the fact that $B_1 \subset \{u > 0\}$, we get

$$J_p(v, B_1) \leq \int_{B_1} |\nabla u|^p \, dx + |B_1| = J_p(u, B_1) \leq J_p(v(1-\varphi), B_1) + \sigma. \quad (3.98)$$

On the other hand, because v is p-harmonic in B_1 and $v = u \geq 0$ on ∂B_1, we can apply Comparison Principle, see Theorem 2.15 in [64], and we have $v \geq 0$ in B_1. Therefore, we can use Harnack inequality for v and we have

$$\|v\|_{L^\infty(B_{1/2})} \leq Cv(0), \quad (3.99)$$

with $C = C(n,p)$. Moreover, the fact that $v \geq 0$ in B_1 also yields by the Strong Maximum Principle, see Corollary 2.21 in [64], that $v > 0$ in B_1.

Precisely, by the Strong Maximum Principle, if $\exists\, x_0 \in B_1$ such that $v(x_0) = 0$, then, since $\min_{B_1} v \geq 0$, it means that $v \equiv 0$ in $\overline{B}_1$. Hence, because $u = v$ on ∂B_1, this entails $u \equiv 0$ on ∂B_1 and so we can choose the zero function as a test function in (3.91), which gives

$$J_p(u, B_1) \leq \sigma,$$

which is an absurd with σ small, because $J_p(u, B_1) \geq |B_1|$, since $u > 0$ in B_1. As a consequence, $v > 0$ in B_1.
In parallel, by Theorem 2 in [78], we know that, dividing v by $\|v\|_{L^\infty(B_{1/2})}$, which is positive because $v > 0$ in B_1, and noting that $v/\|v\|_{L^\infty(B_{1/2})}$ is still p-harmonic,

$$|\nabla v(x)| \leq C\|v\|_{L^\infty(B_{1/2})} \leq Cv(0), \quad x \in B_{1/2},$$

where C depends only on n, p and some other a priori constants, see [78]. Thus, we obtain

$$\|\nabla v\|_{L^\infty(B_{1/2})} \leq Cv(0),$$

which implies from (3.99), after relabeling C if necessary,

$$\|v\|_{L^\infty(B_{1/2})}, \|\nabla v\|_{L^\infty(B_{1/2})} \leq Cv(0). \tag{3.100}$$

We consider, at this point,

$$\begin{aligned}
&\int_{B_1} |\nabla v|^p\, dx = \int_{B_1} |\nabla(v(1-\varphi)) + \nabla(v\varphi)|^p\, dx \\
&= \int_{B_1} \left(|\nabla(v(1-\varphi))|^2 + 2\langle \nabla(v(1-\varphi)), \nabla(v\varphi)\rangle + |\nabla(v\varphi)|^2\right)^{p/2} dx \\
&= \int_{B_1 \setminus B_{1/2}} |\nabla(v(1-\varphi))|^p\, dx \\
&+ \int_{B_{1/2} \setminus B_{1/4}} \left(|\nabla(v(1-\varphi))|^2 + 2\langle \nabla(v(1-\varphi)), \nabla(v\varphi)\rangle + |\nabla(v\varphi)|^2\right)^{p/2} dx \\
&+ \int_{B_{1/4}} |\nabla v|^p\, dx,
\end{aligned}$$

so that by the Cauchy-Schwarz inequality we achieve

$$\begin{aligned}\int_{B_1} |\nabla v|^p\,dx \geq \int_{B_1\setminus B_{1/2}} |\nabla(v(1-\varphi))|^p\,dx + \int_{B_{1/4}} |\nabla v|^p\,dx \\ + \int_{B_{1/2}\setminus B_{1/4}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx.\end{aligned} \tag{3.101}$$

Now, let us analyze the term

$$\int_{B_{1/2}\setminus B_{1/4}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx.$$

First, we rewrite it as

$$\begin{aligned}&\int_{B_{1/2}\setminus B_{1/4}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx \\ &= \int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|\geq|\nabla(v\varphi)|\}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx \\ &+ \int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx\end{aligned} \tag{3.102}$$

and we treat the two terms of the right hand side separately. About

$$\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx,$$

since $|\nabla(v(1-\varphi))| < |\nabla(v\varphi)|$, it also holds $|\nabla(v(1-\varphi))|^p < |\nabla(v\varphi)|^p$, thus we have

$$\begin{aligned}&\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx \\ &\geq \int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} (|\nabla(v(1-\varphi))|^p - |\nabla(v\varphi)|^p)\,dx \\ &= \int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} |\nabla(v(1-\varphi))|^p\,dx \\ &- \int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} |(\nabla v)\varphi + v\nabla\varphi|^p\,dx,\end{aligned}$$

which yields, from triangle inequality of $|\cdot|$ and keeping in mind that $(a+b)^p$

$\leq 2^{p-1}(a^p + b^p)$ for every $a, b \geq 0$ and for every $p \geq 1$,

$$\begin{aligned}&\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx\\&\geq \int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} |\nabla(v(1-\varphi))|^p\,dx\\&\quad - 2^{p-1}\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} (|(\nabla v)\varphi|^p + |v\nabla\varphi|^p)\,dx\end{aligned}$$

that is

$$\begin{aligned}&\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx\\&\geq \int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} |\nabla(v(1-\varphi))|^p\,dx\\&\quad - 2^{p-1}\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} (|\nabla v|^p|\varphi|^p + |v|^p|\nabla\varphi|^p)\,dx.\end{aligned}$$

By this, together with the fact that $0 \leq \varphi \leq 1$ and $\varphi \in C_0^\infty(B_{1/2})$, we then get

$$\begin{aligned}&\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx\\&\geq \int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} |\nabla(v(1-\varphi))|^p\,dx\\&\quad - C\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} (|\nabla v|^p + |v|^p)\,dx,\end{aligned}$$

which gives, according to (3.100),

$$\begin{aligned}&\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx\\&\geq \int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|<|\nabla(v\varphi)|\}} |\nabla(v(1-\varphi))|^p\,dx - Cv(0)^p.\end{aligned} \tag{3.103}$$

Concerning

$$\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|\geq|\nabla(v\varphi)|\}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx,$$

instead, by Bernoulli's inequality, because $\frac{|\nabla(v\varphi)|}{|\nabla(v(1-\varphi))|} \leq 1$, we have

$$||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p = |\nabla(v(1-\varphi))|^p \left(1 - \frac{|\nabla(v\varphi)|}{|\nabla(v(1-\varphi))|}\right)^p$$
$$\geq |\nabla(v(1-\varphi))|^p \left(1 - p\frac{|\nabla(v\varphi)|}{|\nabla(v(1-\varphi))|}\right) = |\nabla(v(1-\varphi))|^p$$
$$- p\,|\nabla(v(1-\varphi))|^{p-1}\,|\nabla(v\varphi)|\,.$$

In addition, we can apply Young's inequality for products

$$ab \leq \frac{a^m}{m} + \frac{b^n}{n}, \quad a, b \geq 0, \quad m, n > 1, \quad \frac{1}{m} + \frac{1}{n} = 1,$$

with

$$a = |\nabla(v(1-\varphi))|)^{p-1},\ b = |\nabla(v\varphi)|\,,\ m = \frac{p}{p-1},$$

and the choice of m entails that

$$\frac{1}{n} = 1 - \frac{p-1}{p} = \frac{1}{p},$$

i.e.

$$n = p.$$

As a consequence, we obtain, repeating the same considerations done to achieve (3.103),

$$\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|\geq|\nabla(v\varphi)|\}} ||\nabla(v(1-\varphi))| - |\nabla(v\varphi)||^p\,dx$$
$$\geq \int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|\geq|\nabla(v\varphi)|\}} |\nabla(v(1-\varphi))|^p\,dx$$
$$- p\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|\geq|\nabla(v\varphi)|\}} \left(|\nabla(v(1-\varphi))|^p\,\frac{p-1}{p} + |\nabla(v\varphi)|^p\,\frac{1}{p}\right)dx$$
$$\geq \int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|\geq|\nabla(v\varphi)|\}} |\nabla(v(1-\varphi))|^p\,dx$$
$$- (p-1)\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|\geq|\nabla(v\varphi)|\}} |\nabla(v(1-\varphi))|^p\,dx - Cv(0)^p. \tag{3.104}$$

In particular, since

$$\nabla(v(1-\varphi)) = (\nabla v)(1-\varphi) - v\nabla\varphi,$$

we can repeat again the considerations used to get (3.103) and we have

$$-(p-1)\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|\geq|\nabla(v\varphi)|\}}|\nabla(v(1-\varphi))|^p\,dx\geq -Cv(0)^p,$$

which implies, from (3.104),

$$\begin{aligned}&\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|\geq|\nabla(v\varphi)|\}}||\nabla(v(1-\varphi))|-|\nabla(v\varphi)||^p\,dx\\&\geq\int_{(B_{1/2}\setminus B_{1/4})\cap\{|\nabla(v(1-\varphi))|\geq|\nabla(v\varphi)|\}}|\nabla(v(1-\varphi))|^p\,dx-Cv(0)^p.\end{aligned}\tag{3.105}$$

Putting together (3.103) and (3.105), we then obtain, by virtue of (3.101) and (3.102), since $1-\varphi\equiv 0$ in $B_{1/4}$,

$$\begin{aligned}&\int_{B_1}|\nabla v|^p\,dx\geq\int_{B_1\setminus B_{1/2}}|\nabla(v(1-\varphi))|^p\,dx+\int_{B_{1/4}}|\nabla v|^p\,dx\\&+\int_{B_{1/2}\setminus B_{1/4}}|\nabla(v(1-\varphi))|^p\,dx-Cv(0)^p\geq\int_{B_1}|\nabla(v(1-\varphi))|^p\,dx-Cv(0)^p.\end{aligned}$$

Now, combining the last inequalities with (3.98) we achieve

$$\int_{B_1}|\nabla v|^p dx+\int_{B_1}\chi_{\{v>0\}}\,dx\leq\int_{B_1}|\nabla v|^p\,dx+Cv(0)^p+\int_{B_1}\chi_{\{v(1-\varphi)>0\}}\,dx+\sigma,$$

and, because $v>0$ in B_1 and again $1-\varphi\equiv 0$ in $B_{1/4}$, it results

$$\int_{B_1}|\nabla v|^p dx+|B_1|\leq\int_{B_1}|\nabla v|^p\,dx+Cv(0)^p+|B_1|-\left|B_{1/4}\right|+\sigma$$

Therefore, we conclude

$$\left|B_{1/4}\right|\leq Cv(0)^p+\sigma,$$

which gives the thesis for σ sufficiently small. □

3.6 Partial regularity of the free boundary

In this section, we deal with the regularity of the free boundary for almost minimizers of J_p, $p>2$.

3.6.1 Almost minimizers as viscosity solutions

This subsection is devoted to show that almost minimizers satisfy a comparison principle with suitable families of sub and supersolutions of the one-phase free boundary problem

$$\begin{cases} \Delta_p u = 0 & \text{in } \{u > 0\}, \\ |\nabla u| = 1 & \text{on } F(u) := \partial\{u > 0\}. \end{cases} \tag{3.106}$$

As in [30], the difference with the infinitesimal case is that in order to obtain a contradiction in the proof of the comparison principle, we have to make clear the size of the neighborhood around the contact point between the solution and an explicit barrier.
Let us start by providing a viscosity supersolution lemma. Actually, we have not been able to show this result yet. Specifically, in the proof there are two tricky points we are still facing and which we will point out for clarity in the remainder of the proof. Hereinafter, we label the still unproven results with "Expected". In particular, these are both direct and indirect consequences of the following viscosity supersolution lemma (Lemma 3.21) and the corresponding viscosity subsolution lemma (Lemma 3.23).

Expected Lemma 3.21 (Supersolution)**.** *Let u satisfy* (3.91) *and let P be a quadratic polynomial such that*

$$\left\|D^2 P\right\| \leq 1, \quad \Delta_p P \geq \mu,$$

for some $0 < \mu \leq \mu_0$ small. Suppose also that

$$\textit{either } u > 0 \textit{ or } |\nabla P| \geq 1 + \mu \textit{ in } B_1.$$

Then P cannot stay below u in B_1 and touch u by below at a point in $B_{1/2}$ if $\sigma \leq \mu^{n+p+1}$.

We point out that the supersolution Lemma 3.21 is used to get a comparison principles for a function u satisfying (3.91). In particular, one way to apply Lemma 3.21 can be found in the following version of the comparison principle for almost minimizers.

Expected Corollary 3.22 (Comparison principle)**.** *Assume that* u *satisfies* (3.91) *and*

$$u \geq P \text{ in a } \delta\text{-neighborhood of } \partial\mathcal{U} \text{ of some domain } \mathcal{U} \subset B_1,$$

for some quadratic polynomial P *such that* $\|D^2P\| \leq \delta^{-1}$, $\Delta_p P \geq \mu$. *Let us suppose also that either* $u > 0$ *or* $|\nabla P| \geq 1 + \mu$ *in* $\mathcal{U}$. *If*

$$\mu^{n+p+1} \geq C(K,\delta)\sigma,$$

then $u \geq P$ *in* $\mathcal{U}$.

Proof. Suppose by contradiction that the thesis is not true. This means that

$$\exists\, \bar{x} : u(\bar{x}) < P(\bar{x}). \tag{3.107}$$

Let us define then

$$M := \min_{\overline{\mathcal{U}}}(u - P) = (u - P)(x_0), \tag{3.108}$$

where x_0 exists because u is Lipschitz from (3.91). We remark first that M is negative by (3.107). Thus, since $u \geq P$ in a δ-neighborhood of $\partial\mathcal{U}$, $x_0 \in \mathcal{U}$ at distance greater than δ from $\partial\mathcal{U}$. Moreover, in view of (3.108), we have

$$P + M \leq P + u - P = u \quad \text{in } \overline{\mathcal{U}},$$

and

$$(P + M)(x_0) = P(x_0) + u(x_0) - P(x_0) = u(x_0).$$

Therefore, $P+M$ touches u by below at x_0. Without loss of generality, $P+M$ is denoted by P. Being $\operatorname{dist}(x_0, \partial\mathcal{U}) > \delta$, $B_\delta(x_0) \subset \mathcal{U}$, so we can rescale the situation from $B_\delta(x_0)$ to B_1 and contradict Lemma 3.21.
Precisely, we define

$$\tilde{u}(x) := \frac{u(x_0 + \delta x)}{\delta}, \quad \tilde{P}(x) := \frac{P(x_0 + \delta x)}{\delta}, \quad x \in B_1. \tag{3.109}$$

Then, because P touches u by below at x_0, $\tilde{P}$ touches $\tilde{u}$ at 0. Also, by (3.109), we achieve

$$\nabla \tilde{P}(x) = (\nabla P)(x_0 + \delta x), \quad D^2\tilde{P}(x) = \delta(D^2P)(x_0 + \delta x),$$

which give

$$\Delta_p \tilde{P}(x) = |(\nabla P)(x_0 + \delta x)|^{p-2} \, \delta(\Delta P)(x_0 + \delta x) + (p-2) \, |(\nabla P)(x_0 + \delta x)|^{p-4}$$
$$\delta \langle (D^2 P)(x_0 + \delta x)(\nabla P)(x_0 + \delta x), (\nabla P)(x_0 + \delta x) \rangle = \delta(\Delta_p P)(x_0 + \delta x),$$

To recap, we have for $\tilde{P}$

$$\begin{aligned} &\nabla \tilde{P}(x) = (\nabla P)(x_0 + \delta x), \quad D^2 \tilde{P}(x) = \delta(D^2 P)(x_0 + \delta x), \\ &\Delta_p \tilde{P}(x) = \delta(\Delta_p P)(x_0 + \delta x). \end{aligned} \tag{3.110}$$

Hence, using the hypotheses on P, we get, according to (3.110),

$$\begin{aligned} \|D^2 \tilde{P}\| &= \delta \left\| D^2 P \right\| \leq \delta \delta^{-1} = 1, \\ \Delta_p \tilde{P}(x) &= \delta(\Delta_p P)(x_0 + \delta x) \geq \delta \mu, \end{aligned}$$

namely

$$\|D^2 \tilde{P}\| \leq 1, \quad \Delta_p \tilde{P} \geq \delta \mu,$$

and if $|\nabla P| \geq 1 + \mu$,

$$|\nabla \tilde{P}| = |\nabla P(x_0 + \delta x)| \geq 1 + \mu \geq 1 + \delta \mu,$$

thus from (3.109), either $\tilde{u} > 0$ or $|\nabla \tilde{P}| \geq 1 + \delta \mu$ in B_1. Finally, by Remark (3.16), $\tilde{u}$ satisfies (3.91) with $\tilde{\sigma} \coloneqq \delta^{-n} \sigma$.
To sum up, all the assumptions of Lemma 3.21 are satisfied by $\tilde{u}$ and $\tilde{P}$ with $\tilde{\sigma}$ and $\tilde{\mu} \coloneqq \delta \mu$. As a consequence, if $\tilde{\sigma} \leq \tilde{\mu}^{n+p+1}$, we can apply Lemma 3.21 and the fact that $\tilde{P} \leq \tilde{u}$ in B_1 with $\tilde{P}$ touching u by below in 0 gives a contradiction. It remains to show that $\tilde{\sigma} \leq \tilde{\mu}^{n+p+1}$ holds. Precisely, we obtain

$$\tilde{\sigma} = \delta^{-n} \sigma \leq \tilde{\mu}^{n+p+1} = \mu^{n+p+1} \delta^{n+p+1}$$

if we suppose $\mu^{n+p+1} \geq \delta^{-2n-p-1} \sigma = C(K, \delta) \sigma$. □

In a similar way, we also have a viscosity subsolution lemma, from which we can achieve a version of Corollary 3.22 for polynomials P lying above u.

Expected Lemma 3.23 (Subsolution)**.** *Let u satisfy* (3.91) *and let P be a quadratic polynomial such that*

$$\left\|D^2P\right\| \leq 1, \quad \Delta_p P \leq -\mu,$$

for some $\mu > 0$ small. Suppose also that

$$\text{either } u > 0 \text{ or } |\nabla P| \leq 1-\mu \text{ in } B_1.$$

Then P^+ cannot stay above u in B_1 and touch u by above at a point in $B_{1/2} \cap \overline{\{P>0\}}$ if $\sigma \leq \mu^{n+p+1}$.

Before showing the proofs of Lemma 3.21 and 3.23, we state and prove two auxiliary lemmas about perturbations of quadratic polynomials as classical subsolutions to (3.106). In particular, we will use the second one of them in the proof of Lemma 3.21.

Lemma 3.24. *Let $Q \coloneqq c|x-x_0|^{-\gamma}+m$, where $\gamma > 0$ and $m \in \mathbb{R}$. For every quadratic polynomial P such that*

$$\left\|D^2P\right\| \leq 1, \quad \Delta_p P \geq \mu, \quad |\nabla P| \geq 1+\mu,$$

for some $0 < \mu \leq \mu_0$ small, there exist positive constants c_μ and C_p such that if $|c| \leq c_\mu$, then

$$\Delta_p \bar{P} \geq \frac{\mu}{C_p} \quad \text{and} \quad |\nabla \bar{P}| > 1,$$

where $\bar{P}(x) \coloneqq P(x) + Q(x)$.

Proof. We want to show that we can choose x_0 sufficiently large and c small enough in such a way that $\Delta_p \bar{P} \geq \frac{\mu}{C_p}$. To this end, we compute

$$\begin{aligned}
&\nabla Q = -c\gamma|x-x_0|^{-\gamma-1}\frac{x-x_0}{|x-x_0|}\\
&D^2Q = -c\gamma\bigg(-(\gamma+1)|x-x_0|^{-\gamma-2}\frac{x-x_0}{|x-x_0|}\otimes\frac{x-x_0}{|x-x_0|} + |x-x_0|^{-\gamma-2}I\\
&-|x-x_0|^{-\gamma-1}\frac{1}{|x-x_0|^2}(x-x_0)\otimes\frac{x-x_0}{|x-x_0|}\bigg) = -c\gamma\bigg(|x-x_0|^{-\gamma-2}I\\
&-(\gamma+2)|x-x_0|^{-\gamma-2}\frac{x-x_0}{|x-x_0|}\otimes\frac{x-x_0}{|x-x_0|}\bigg),
\end{aligned}$$

that is

$$\nabla Q = -c\gamma|x-x_0|^{-\gamma-1}\frac{x-x_0}{|x-x_0|},$$
$$D^2Q = c\gamma|x-x_0|^{-\gamma-2}\left((\gamma+2)\frac{x-x_0}{|x-x_0|}\otimes\frac{x-x_0}{|x-x_0|}-I\right),$$

which also give

$$\Delta Q = c\gamma|x-x_0|^{-\gamma-2}(\gamma+2-n).$$

To sum up, we have

$$\begin{aligned}&\nabla Q = -c\gamma|x-x_0|^{-\gamma-1}\frac{x-x_0}{|x-x_0|}, \quad \Delta Q = c\gamma|x-x_0|^{-\gamma-2}(\gamma+2-n),\\&D^2Q = c\gamma|x-x_0|^{-\gamma-2}\left((\gamma+2)\frac{x-x_0}{|x-x_0|}\otimes\frac{x-x_0}{|x-x_0|}-I\right).\end{aligned} \tag{3.111}$$

Now, calculating $\Delta_p\bar{P}$, we get

$$\Delta_p\bar{P} = |\nabla(P+Q)|^{p-2}\Big(\Delta P+\Delta Q+(p-2)\langle D^2Q\,v,v\rangle+(p-2)\langle D^2P\,v,v\rangle\Big),$$

denoting

$$v := \frac{\nabla(P+Q)}{|\nabla(P+Q)|}. \tag{3.112}$$

In particular, we can rewrite the expression of $\Delta_p\bar{P}$ as

$$\begin{aligned}\Delta_p\bar{P} &= |\nabla(P+Q)|^{p-2}\Big(\Delta P\pm(p-2)\langle D^2P\frac{\nabla P}{|\nabla P|},\frac{\nabla P}{|\nabla P|}\rangle+\Delta Q+(p-2)\langle D^2Q\,v,\\&v\rangle+(p-2)\langle D^2P\,v,v\rangle\Big) = |\nabla P|^{p-2}\Big(\Delta P+(p-2)\langle D^2P\frac{\nabla P}{|\nabla P|},\frac{\nabla P}{|\nabla P|}\rangle\Big)\\&+(|\nabla(P+Q)|^{p-2}-|\nabla P|^{p-2})\Big(\Delta P+(p-2)\langle D^2P\frac{\nabla P}{|\nabla P|},\frac{\nabla P}{|\nabla P|}\rangle\Big)\\&+|\nabla(P+Q)|^{p-2}(p-2)\Big(\langle D^2P\,v,v\rangle-\langle D^2P\frac{\nabla P}{|\nabla P|},\frac{\nabla P}{|\nabla P|}\rangle\Big)+|\nabla(P+Q)|^{p-2}\\&(\Delta Q+(p-2)\langle D^2Q\,v,v\rangle),\end{aligned}$$

i.e.

$$\Delta_p \bar{P} = \Delta_p P + (|\nabla(P+Q)|^{p-2} - |\nabla P|^{p-2})\Big(\Delta P + (p-2)\langle D^2 P \frac{\nabla P}{|\nabla P|}, \frac{\nabla P}{|\nabla P|}\rangle\Big)$$
$$+ |\nabla(P+Q)|^{p-2}(p-2)\Big(\langle D^2 P v, v\rangle - \langle D^2 P \frac{\nabla P}{|\nabla P|}, \frac{\nabla P}{|\nabla P|}\rangle\Big) + |\nabla(P+Q)|^{p-2}$$
$$(\Delta Q + (p-2)\langle D^2 Q v, v\rangle). \tag{3.113}$$

We treat the terms in (3.113) separately. Let us analyze first

$$|\nabla(P+Q)|^{p-2}(p-2)\Big(\langle D^2 P v, v\rangle - \langle D^2 P \frac{\nabla P}{|\nabla P|}, \frac{\nabla P}{|\nabla P|}\rangle\Big).$$

Specifically, according to the mean value theorem for the function $|\nabla P + x|^{p-2}$, we achieve

$$|\nabla P + x|^{p-2} = |\nabla P|^{p-2} + (p-2)|\nabla P + \xi|^{p-3}\langle \frac{\nabla P + \xi}{|\nabla P + \xi|}, x\rangle, \quad \xi \in [0, x],$$

where $[0, x]$ represents the segment which connects 0 and x. In particular, with $x = \nabla Q$, we get

$$|\nabla P + \nabla Q|^{p-2} = |\nabla P|^{p-2} + (p-2)|\nabla P + \xi|^{p-3}\langle \frac{\nabla P + \xi}{|\nabla P + \xi|}, \nabla Q\rangle, \quad \xi \in [0, \nabla Q]. \tag{3.114}$$

Exploiting the same argument, but for the function

$$\frac{1}{\left|\frac{\nabla P}{|\nabla P|} + x\right|},$$

we also obtain

$$\frac{1}{|\nabla P + \nabla Q|} = \frac{1}{|\nabla P|\left|\frac{\nabla P}{|\nabla P|} + \frac{\nabla Q}{|\nabla P|}\right|} = \frac{1}{|\nabla P|}\Bigg(\frac{1}{\left|\frac{\nabla P}{|\nabla P|}\right|} - \left|\frac{\nabla P}{|\nabla P|} + \eta\right|^{-2}$$
$$\langle \frac{\frac{\nabla P}{|\nabla P|} + \eta}{\left|\frac{\nabla P}{|\nabla P|} + \eta\right|}, \frac{\nabla Q}{|\nabla P|}\rangle\Bigg) = \frac{1}{|\nabla P|}\Bigg(1 - \left|\frac{\nabla P}{|\nabla P|} + \eta\right|^{-2}\langle \frac{\frac{\nabla P}{|\nabla P|} + \eta}{\left|\frac{\nabla P}{|\nabla P|} + \eta\right|}, \frac{\nabla Q}{|\nabla P|}\rangle\Bigg),$$

$\eta \in \left[0, \frac{\nabla Q}{|\nabla P|}\right]$, which yields

$$\frac{\nabla P + \nabla Q}{|\nabla P + \nabla Q|} = \frac{\nabla P + \nabla Q}{|\nabla P|}\left(1 - \left|\frac{\nabla P}{|\nabla P|} + \eta\right|^{-2} \langle \frac{\frac{\nabla P}{|\nabla P|} + \eta}{\left|\frac{\nabla P}{|\nabla P|} + \eta\right|}, \frac{\nabla Q}{|\nabla P|}\rangle\right)$$
$$= \frac{\nabla P}{|\nabla P|} + \mathcal{O}(|\nabla Q|),$$

since our idea is to let $|\nabla Q|$ go to 0. As a consequence, putting this condition together with (3.114), we achieve, by virtue of (3.112),

$$\left||\nabla(P+Q)|^{p-2}(p-2)\left(\langle D^2P v, v\rangle - \langle D^2P\frac{\nabla P}{|\nabla P|}, \frac{\nabla P}{|\nabla P|}\rangle\right)\right| = \Big(|\nabla P|^{p-2}$$
$$+ (p-2)|\nabla P + \xi|^{p-3}\langle\frac{\nabla P+\xi}{|\nabla P+\xi|}, \nabla Q\rangle\Big)(p-2)\left|\langle D^2P\left(\frac{\nabla P}{|\nabla P|} + \mathcal{O}(|\nabla Q|)\right),\right.$$
$$\left.\frac{\nabla P}{|\nabla P|} + \mathcal{O}(|\nabla Q|)\rangle - \langle D^2P\frac{\nabla P}{|\nabla P|}, \frac{\nabla P}{|\nabla P|}\rangle\right| \le (p-2)\Big(|\nabla P|^{p-2} + (p-2)|\nabla P$$
$$+ \xi|^{p-3}|\nabla Q|\Big)\mathcal{O}(|\nabla Q|) \le C_1|\nabla P|^{p-2}|\nabla Q| + C_2|\nabla P|^{p-3}|\nabla Q|^2,$$

in other words

$$\left||\nabla(P+Q)|^{p-2}(p-2)\left(\langle (D^2P)v, v\rangle - \langle D^2P\frac{\nabla P}{|\nabla P|}, \frac{\nabla P}{|\nabla P|}\rangle\right)\right|$$
$$\le |\nabla P|^{p-2}|\nabla Q|(C_1 + C_2|\nabla Q||\nabla P|^{-1}), \tag{3.115}$$

because by hypothesis $\|D^2P\| \le 1$ and $|\xi| \to 0$ if $|\nabla Q| \to 0$. Concerning the term

$$(|\nabla(P+Q)|^{p-2} - |\nabla P|^{p-2})\left(\Delta P + (p-2)\langle D^2P\frac{\nabla P}{|\nabla P|}, \frac{\nabla P}{|\nabla P|}\rangle\right),$$

instead, we get, from (3.114),

$$\left|(|\nabla(P+Q)|^{p-2} - |\nabla P|^{p-2})\left(\Delta P + (p-2)\langle D^2P\frac{\nabla P}{|\nabla P|}, \frac{\nabla P}{|\nabla P|}\rangle\right)\right|$$
$$\le \left|(p-2)|\nabla P+\xi|^{p-3}\langle\frac{\nabla P+\xi}{|\nabla P+\xi|}, \nabla Q\rangle\right|\left(|\Delta P| + (p-2)\left|D^2P\frac{\nabla P}{|\nabla P|}\right|\left|\frac{\nabla P}{|\nabla P|}\right|\right)$$

which entails

$$\left|(|\nabla(P+Q)|^{p-2} - |\nabla P|^{p-2})\left(\Delta P + (p-2)\langle D^2P\frac{\nabla P}{|\nabla P|}, \frac{\nabla P}{|\nabla P|}\rangle\right)\right|$$
$$\le (p-2)2|\nabla P|^{p-3}|\nabla Q|(n+p-2), \tag{3.116}$$

since $\|D^2P\| \leq 1$ implies $|\Delta P| \leq n$ as well and again $|\xi| \to 0$ if $|\nabla Q| \to 0$. Lastly, we have to estimate

$$|\nabla(P+Q)|^{p-2}(\Delta Q + (p-2)\langle D^2Q\,v, v\rangle).$$

For this purpose, we recall (3.111) and we obtain, in view of (3.112),

$$|\nabla(P+Q)|^{p-2}(\Delta Q + (p-2)\langle D^2Q\,v, v\rangle) = |\nabla(P+Q)|^{p-2}(\mathrm{tr}(D^2Q) + (p-2)$$
$$\mathrm{tr}(D^2Q(v\otimes v))) = |\nabla(P+Q)|^{p-2}\,\mathrm{tr}(D^2Q(I+(p-2)(v\otimes v)))$$
$$\geq |\nabla(P+Q)|^{p-2}\mathcal{P}^-_{1,p-1}(D^2Q) \geq |\nabla(P+Q)|^{p-2}c\gamma|x-x_0|^{-\gamma-2}(\gamma+1-(p-1)$$
$$(n-1)) \geq \frac{1}{2}|\nabla P|^{p-2}c\gamma|x-x_0|^{-\gamma-2}(\gamma+1-(p-1)(n-1)),$$

namely

$$|\nabla(P+Q)|^{p-2}(\Delta Q + (p-2)\langle D^2Qv, v\rangle) \geq \frac{1}{2}|\nabla P|^{p-2}c\gamma|x-x_0|^{-\gamma-2}(\gamma+1$$
$$-(p-1)(n-1)), \tag{3.117}$$

because $|\nabla Q| \to 0$.

Therefore, considering together (3.115), (3.116) and (3.117), from (3.113), (3.111) and the hypothesis $\Delta_p P \geq \mu$, it holds

$$\Delta_p\bar{P} \geq \mu + |\nabla P|^{p-2}|\nabla Q|\bigg(-(p-2)2|\nabla P|^{-1}(n+p-2) - C_1 - C_2c\gamma$$
$$|x-x_0|^{-\gamma-1}|\nabla P|^{-1} + \frac{1}{2}|x-x_0|^{-1}(\gamma+1-(p-1)(n-1))\bigg),$$

which gives, if $M-2<|x-x_0|<M$, with M very large, and recalling that $|\nabla P| \geq 1+\mu$,

$$\Delta_p\bar{P} \geq \mu + |\nabla P|^{p-2}c\gamma|x-x_0|^{-\gamma-1}\bigg(-(p-2)2(n+p-2)(1+\mu)^{-1} - C_1$$
$$-C_2c\gamma(M-2)^{-\gamma-1}(1+\mu)^{-1} + \frac{1}{2}M^{-1}(\gamma+1-(p-1)(n-1))\bigg) \geq \frac{\mu}{C_p},$$

if $M \to \infty$, i.e. x_0 is extremely large, and c is sufficiently small as well.
The condition $|\nabla\bar{P}| > 1$ now easily follows by the facts that $|\nabla P| \geq 1+\mu$ and $|\nabla Q| \to 0$. □

Lemma 3.25. *Let* $Q = c_q|x|^q + m$*, where* $q \geq 2$ *and* $m \in \mathbb{R}$*. For every quadratic polynomial* P *such that*

$$\|D^2 P\| \leq 1, \quad \Delta_p P \geq \mu, \quad |\nabla P| \geq 1 + \mu,$$

for some $0 < \mu \leq \mu_0$ *small, there exist positive constants* c_μ *and* C_p *such that if* $|c_q| \leq c_\mu$*, then*

$$\Delta_p \bar{P} \geq \frac{\mu}{C_p} \quad \text{and} \quad |\nabla \bar{P}| > 1,$$

where $\bar{P}(x) \coloneqq P(x) + Q(x)$*.*

Proof. Let

$$\bar{P}(x) \coloneqq P(x) + Q(x), \tag{3.118}$$

where

$$Q(x) \coloneqq c_q|x|^q + m, \quad q \geq 2. \tag{3.119}$$

We want to prove that we can choose c_q sufficiently small in such a way that $\Delta_p \bar{P} \geq \frac{\mu}{C_p}$. For this purpose, first, repeating the same computations done to achieve (3.120), we have

$$\begin{aligned} \nabla Q &= c_q q|x|^{q-1}\frac{x}{|x|}, \qquad \Delta Q = c_q q|x|^{q-2}(q-2+n), \\ D^2 Q &= c_q q|x|^{q-2}\Big((q-2)\frac{x}{|x|} \otimes \frac{x}{|x|} + I\Big). \end{aligned} \tag{3.120}$$

Next, since (3.113) is not depending on the particular expression of Q, it is valid in the same way in this case as well, together with (3.112).
We focus then, as in the proof of Lemma 3.24, on the terms in (3.113). Again, (3.115) and (3.116) do not rely on the particular expression of Q, so they hold also in this case. Let us analyze then

$$|\nabla(P+Q)|^{p-2}(\Delta Q + (p-2)\langle D^2 Q v, v\rangle),$$

and we point out that, repeating the same argument to have (3.117), but with (3.119), we get, because $p > 2$,

$$|\nabla(P+Q)|^{p-2}(\Delta Q + (p-2)\langle D^2 Q v, v\rangle) \geq \frac{1}{2}|\nabla P|^{p-2} c_q q|x|^{q-2}(q+n-2), \tag{3.121}$$

because again we want to let $|\nabla Q|$ go to 0. Consequently, according to (3.115), (3.116) and (3.121), we obtain this time from (3.113), (3.120), the hypotheses $\Delta_p P \geq \mu$, $|\nabla P| \geq 1+\mu$, $\mu > 0$ and the fact that we ask $1 \geq |x| \geq \frac{1}{2}$,

$$\Delta_p \bar{P} \geq \mu + |\nabla P|^{p-2}|\nabla Q|\bigg(-(p-2)2|\nabla P|^{-1}(n+p-2) - C_1 - C_2c_q q|x|^{q-1}$$
$$|\nabla P|^{-1} + \frac{1}{2}|x|^{-1}(q+n-2)\bigg) \geq \mu + |\nabla P|^{p-2}|\nabla Q|\bigg(-(p-2)2(1+\mu)^{-1}(n$$
$$+p-2) - C_1 - C_2c_q q(1+\mu)^{-1} + \frac{1}{2}(q+n-2)\bigg) \geq \mu + |\nabla P|^{p-2}|\nabla Q|$$
$$(-C(n,p) - C_1 - C_2c_q q + C(q,n)),$$

i.e.

$$\Delta_p \bar{P} \geq \mu + |\nabla P|^{p-2}|\nabla Q|(-C(n,p) - C_1 - C_2c_q q + C(q,n)). \tag{3.122}$$

Now, we distinguish two cases, depending on the sign of

$$-C(n,p) - C_1 - C_2c_q q + C(q,n).$$

Precisely, if this is nonnegative then, by (3.122), it directly follows that $\Delta_p \bar{P} \geq \frac{\mu}{C_p}$, and provided c_q is chosen sufficiently small, also $|\nabla \bar{P}| > 1$ is satisfied. Hence, it remains to consider the case when

$$-C(n,p) - C_1 - C_2c_q q + C(q,n) < 0.$$

First, by this we can get a lower bound of (3.122) as

$$\Delta_p \bar{P} \geq \mu + C(P)c_q q(-C(n,p) - C_1 - C_2c_q q + C(q,n)), \tag{3.123}$$

where $C(P) := \max_{\overline{B}_1} |\nabla P|^{p-2}$ and again because $1 \geq |x| \geq \frac{1}{2}$. Let us focus, at this point, on the term

$$C(P)c_q q(-C(n,p) - C_1 - C_2c_q q + C(q,n)).$$

Developing it, we have

$$-C_2C(P)(c_q q)^2 + C(P)c_q q(-C(n,p) - C_1 + C(q,n)),$$

and we ask this to be greater or equal to $-\frac{\mu}{k(p,q)}$, where $k(p,q)$ satisfies

$$1 - \frac{1}{k(p,q)} \geq \frac{1}{C_p},$$

with C_p to be made precise later. Denoting for the sake of simplicity

$$a \coloneqq c_q q, \quad b \coloneqq C(P)(-C(n,p) - C_1 + C(q,n)),$$

we need then

$$-C_2 C(P) a^2 + ab + \frac{\mu}{k(p,q)} \geq 0,$$

which is satisfied if

$$\frac{b - \sqrt{b^2 + 4C_2 C(P)\frac{\mu}{k(p,q)}}}{2C_2 C(P)} \leq a \leq \frac{b + \sqrt{b^2 + 4C_2 C(P)\frac{\mu}{k(p,q)}}}{2C_2 C(P)}.$$

In particular, since $a > 0$, the last condition reads

$$0 < a \leq \frac{b + \sqrt{b^2 + 4C_2 C(P)\frac{\mu}{k(p,q)}}}{2C_2 C(P)}.$$

Therefore, with this choice of $a \coloneqq c(q)q$, we achieve from (3.123) $\Delta_p \bar{P} \geq \frac{\mu}{C_p}$, and provided c_q is chosen enough small, $|\nabla \bar{P}| > 1$ is satisfied as well.
To recap, in both cases, we have $\Delta_p \bar{P} \geq \frac{\mu}{C_p}$ and $|\nabla \bar{P}| > 1$. □

As already remarked, we will exploit Lemma 3.25 in the proof of Lemma 3.21. Specifically, we will see that the proofs of Lemma 3.21 and 3.23 follow the same structure. So, we state the correspondent lemma to Lemma 3.25 which will be employed in the proof of Lemma 3.23.

Lemma 3.26. *Let $Q = c_q|x|^q + m$, where $q \geq 2$ and $m \in \mathbb{R}$. For every quadratic polynomial P such that*

$$\left\|D^2 P\right\| \leq 1, \quad \Delta_p P \leq -\mu, \quad |\nabla P| \leq 1 - \mu$$

for some $0 < \mu \leq \mu_0$ small, there exist positive constants c_μ and C_p such that if $|c_q| \leq c_\mu$, then

$$\Delta_p \bar{P} \leq -\frac{\mu}{C_p} \quad \text{and} \quad |\nabla \bar{P}| < 1,$$

where $\bar{P}(x) \coloneqq P(x) + Q(x)$.

We now give the proofs of Lemma 3.21 and 3.23. Actually, in these proofs there is a tricky point we are still facing and which we will point out for clarity in the proof of Lemma 3.21.

Proof of Lemma 3.21. Let us assume first that $u > 0$ in B_1. Then, the conclusion follows by Lemma 3.17. Indeed, let us assume that the thesis of Lemma 3.21 is not true. This means that $P \leq u$ in B_1 and P touches u by below at x_0 in $B_{1/2}$. According to Lemma 3.17, we know that

$$\|u - v\|_{L^\infty(B_{1/2})} \leq c(\sigma), \quad c(\sigma) \to 0 \quad \text{as } \sigma \to 0, \tag{3.124}$$

with v the p-harmonic replacement of u in B_1. In particular from (3.124), since $u(x_0) = P(x_0)$, $x_0 \in B_{1/2}$, we get

$$v(x_0) - c(\sigma) \leq u(x_0) = P(x_0), \quad c(\sigma) \to 0 \quad \text{as } \sigma \to 0,$$

i.e.

$$v(x_0) - c(\sigma) \leq P(x_0), \quad c(\sigma) \to 0 \quad \text{as } \sigma \to 0. \tag{3.125}$$

Moreover, because v is the p-harmonic replacement of u in B_1, we have that $v = u$ on ∂B_1. Therefore, because $P \leq u$ in B_1 implies (by continuity) $P \leq u$ on ∂B_1 as well, we obtain $P \leq v$ on ∂B_1. Now, using the comparison principle for the $p-$Laplace operator, see [64, Remark at p. 11], we conclude that $P \leq v$ in all of B_1, because by hypothesis the polynomial P is a $p-$subsolution. Nevertheless, from (3.125) we deduce that it is not possible to have $P < v$ in whole $B_{1/2}$, so it has to occur that either $P \equiv v$ in $B_{1/2}$ or there exists a sequence of points $\{x_j\}_{j\in\mathbb{N}} \subset B_{1/2}$ such that $P(x_j) = v(x_j)$, but not $P \equiv v$ in $B_{1/2}$. The first alternative can not happen, since v is p-harmonic, whereas P is strictly p-subharmonic (a strict subsolution of the $p-$Laplace equation). It remains to analyze the second case. This second alternative is a little bit tricky and it should follow in view of a compactness argument we are dealing with. However, let us suppose, for the moment, that even this second possibility is valid. Therefore, to recap, it can not hold that $P < v$ in whole $B_{1/2}$.

Now we assume by contradiction that there exists a quadratic polynomial such that $u \geq P$, $u(x_0) = P(x_0)$ for some $x_0 \in B_{1/2}$ and $|\nabla P| \geq 1 + \mu$. We know that $D^2 P$ is bounded, u is Lipschitz and P touches u by below at x_0. As a consequence P is uniformly Lipschitz continuous in B_1, with Lipschitz constant independent from P. Indeed, from the fact that P touches u by below at x_0, we achieve

$$\nabla(u - P)(x_0) = 0,$$

which yields, in view of (3.91),

$$|\nabla P(x_0)| = |\nabla u(x_0)| \leq K. \tag{3.126}$$

Moreover, since $\|D^2 P\| \leq 1$, we get

$$|\nabla P(x) - \nabla P(x_0)| \leq \left\|D^2 P\right\| |x - x_0| \leq |x| + |x_0| \leq 2, \quad x \in B_1,$$

namely

$$|\nabla P(x) - \nabla P(x_0)| \leq 2, \quad x \in B_1.$$

Putting together this condition with (3.126), we then have

$$|\nabla P(x)| \leq |\nabla P(x_0)| + |\nabla P(x) - \nabla P(x_0)| \leq K + 2, \quad x \in B_1,$$

which lastly implies that P is uniformly Lipschitz continuous in B_1, with Lipschitz constant independent of P.

Now, instead of perturbing the polynomial with a function w satisfying $\Delta_p w = -1$, see for instance [20] for the proof of $\Delta_p |x|^{\frac{p}{p-1}} = c_{p,n}$, we exploit the result proved in Lemma 3.25. Indeed we define, similarly to the classical case,

$$\bar{P} := P + \mu c_2 (1 - |x|^2), \tag{3.127}$$

so that we can apply Lemma 3.25 for $q = 2$. In particular, we have $\Delta_p \bar{P} \geq \frac{\mu}{C_p}$ and $|\nabla \bar{P}| > 1$. Being $\bar{P}$ Lipschitz continuous in B_1, we achieve that $\bar{P}^+$ is Lipschitz continuous in B_1 as well. Indeed, if either $\bar{P}^+ = \bar{P}$ or $\bar{P} = 0$ in a couple of points, the Lipschitz continuity is clear. It remains to check the fact when $\bar{P}^+(x) = \bar{P}(x)$ and $\bar{P}^+(y) = 0$, $x, y \in B_1$. In this case, we have

$$|\bar{P}^+(x) - \bar{P}^+(y)| = |\bar{P}^+(x)| = \bar{P}^+(x) = \bar{P}(x) \leq \bar{P}(x) - \bar{P}(y) \leq L|x-y|,$$

namely

$$|\bar{P}^+(x) - \bar{P}^+(y)| \leq L|x-y|, \quad x, y \in B_1,$$

since, because $\bar{P}^+(y) = 0$, $\bar{P}(y) \leq 0$, and thus $-\bar{P}(y) \geq 0$. As a consequence, $\bar{P}^+$ is actually Lipschitz continuous.
By the Lipschitz continuity of $\bar{P}^+$, together with that of u, we then obtain that

$$|(\bar{P}^+ - u)(x) - (\bar{P}^+ - u)(x_0)| \leq L|x - x_0| \quad \text{in } B_{\bar{c}\mu}(x_0),$$

which gives, in view of (3.127), since $u(x_0) = P(x_0)$, $x_0 \in B_{1/2}$,

$$(\bar{P}^+ - u)(x) \geq (\bar{P}^+ - u)(x_0) - L|x - x_0| = \mu c_2(1 - |x_0|^2) - L|x - x_0|$$

in other words

$$\bar{P}^+ - u \geq \frac{\mu}{C_1} \quad \text{in } B_{\bar{c}\mu}(x_0), \tag{3.128}$$

with $\bar{c}$ sufficiently small.
At this point, let us set

$$u_{\max} := \max\{u, \bar{P}^+\}, \quad u_{\min} := \min\{u, \bar{P}^+\}, \tag{3.129}$$

and we remark that

$$u_{\max} = u, \quad u_{\min} = \bar{P}^+ \quad \text{on } \partial B_1, \tag{3.130}$$

because $P \leq u$ in B_1 implies by continuity $P \leq u$ on ∂B_1 as well and

$$\bar{P} = P + \mu c_2(1 - |x|^2) = P \leq u \quad \text{on } \partial B_1,$$

which yields $\bar{P}^+ \leq u$ on ∂B_1, since $u \geq 0$ everywhere.
Then, using (3.130), we get, according to (3.91),

$$J_p(u, B_1) \leq J_p(u_{\max}, B_1) + \sigma,$$

which can be rewritten as

$$J_p(u_{\min}, B_1) - J_p(\bar{P}^+, B_1) \le \sigma. \tag{3.131}$$

Precisely, by virtue of (3.129), we have

$$\begin{aligned}
J_p(u, B_1) - J_p(u_{\max}, B_1) &= \int_{B_1 \cap \{u \ge \bar{P}^+\}} (|\nabla u|^p + \chi_{\{u>0\}})\, dx \\
&+ \int_{B_1 \cap \{u < \bar{P}^+\}} (|\nabla u|^p + \chi_{\{u>0\}})\, dx - \int_{B_1 \cap \{u \ge \bar{P}^+\}} (|\nabla u|^p + \chi_{\{u>0\}})\, dx \\
&- \int_{B_1 \cap \{u < \bar{P}^+\}} (|\nabla \bar{P}^+|^p + \chi_{\{\bar{P}^+>0\}})\, dx = \int_{B_1 \cap \{u < \bar{P}^+\}} (|\nabla u|^p + \chi_{\{u>0\}})\, dx \\
&- \int_{B_1 \cap \{u < \bar{P}^+\}} (|\nabla \bar{P}^+|^p + \chi_{\{\bar{P}^+>0\}})\, dx = \int_{B_1 \cap \{u < \bar{P}^+\}} (|\nabla u|^p + \chi_{\{u>0\}})\, dx \\
&- \int_{B_1 \cap \{u < \bar{P}^+\}} (|\nabla \bar{P}^+|^p + \chi_{\{\bar{P}^+>0\}})\, dx \pm \int_{B_1 \cap \{u \ge \bar{P}^+\}} (|\nabla \bar{P}^+|^p + \chi_{\{\bar{P}^+>0\}})\, dx \\
&= \int_{B_1} (|\nabla u_{\min}|^p + \chi_{\{u_{\min}>0\}})\, dx - \int_{B_1} (|\nabla \bar{P}^+|^p + \chi_{\{\bar{P}^+>0\}})\, dx,
\end{aligned}$$

which hence entails

$$J_p(u, B_1) - J_p(u_{\max}, B_1) = J_p(u_{\min}, B_1) - J_p(\bar{P}^+, B_1),$$

and so (3.131) holds.

We claim, at this point, that

$$J_p(u_{\min}, B_1) - J_p(\bar{P}^+, B_1) \ge \frac{\mu}{C_p} \int_{B_1} (\bar{P}^+ - u_{\min})\, dx.$$

Then this claim, together with (3.128) and (3.131), lastly implies that

$$\begin{aligned}
\sigma &\ge J_p(u_{\min}, B_1) - J_p(\bar{P}^+, B_1) \ge \frac{\mu}{C_p} \int_{B_1} (\bar{P}^+ - u_{\min})\, dx \ge \frac{\mu}{C_p} \int_{B_{\bar{c}\mu}} (\bar{P}^+ \\
&- u_{\min})\, dx \ge \frac{\mu}{C_p} \int_{B_{\bar{c}\mu}(x_0)} (\bar{P}^+ - u)\, dx = \frac{\mu}{C_p} \frac{\mu}{C_1} |B_{\bar{c}\mu}(x_0)| = c\mu^{n+2},
\end{aligned}$$

that is $\sigma \ge c\mu^{n+2}$, which gives a contradiction with the hypothesis $\sigma \le \mu^{n+p+1}$ and μ small. As a consequence, Lemma 3.21 holds.

It remains to show that the claim is true. For this purpose, we minimize the functional

$$F_p(v) := \int_{B_1} \left(|\nabla v|^p + \chi_{\{v>0\}} + \frac{\mu}{C_p}(v - \bar{P}^+) \right) dx \tag{3.132}$$

among all functions $0 \leq v \leq \bar{P}^+$ which are equal to $\bar{P}^+$ on ∂B_1, and we argue that $\bar{P}^+$ is the minimizer. This fact will yield the claim since $u_{\min}$ is an admissible competitor. Indeed, by (3.132), if $\bar{P}^+$ is the minimizer and $u_{\min}$ is a competitor, we have

$$F_p(\bar{P}^+) = J_p(\bar{P}^+, B_1) \leq F_p(u_{\min}) = \int_{B_1} \Big(|\nabla u_{\min}|^p + \chi_{\{u_{\min}>0\}} + \frac{\mu}{C_p}(u_{\min} - \bar{P}^+) \Big) dx = J_p(u_{\min}, B_1) + \frac{\mu}{C_p} \int_{B_1} (u_{\min} - \bar{P}^+)\, dx,$$

namely the claim holds.
To prove that $\bar{P}^+$ is actually the minimizer, we first remark that in the region where the minimizer v is strictly below $\bar{P}^+$, v satisfies in a variational sense the following problem

$$\begin{cases} \Delta_p v = \frac{\mu}{pC_p} & \text{in } \{v > 0\}, \\ |\nabla v|^p = \frac{1}{p-1} & \text{on } F(v), \end{cases} \tag{3.133}$$

see [61] or [33]. About, instead, the fact that v is also solution of (3.133) in the viscosity sense, it is a more delicate argument, that we are still investigating. Let us assume, for the moment, that it is true, as we would expect.

This facts means that v fulfills the comparison principle with the continuous family of functions $(\bar{P}+t)^+$, which are classical subsolutions to (3.133), because $\Delta_p \bar{P} \geq \frac{\mu}{C_p}$, so that $\Delta_p \bar{P} \geq \frac{\mu}{pC_p}$, and $|\nabla \bar{P}| > 1$, see [61] for the details. In particular, we can increase t from a large negative constant up to $t = 0$ and thus we achieve that $v \equiv \bar{P}^+$.

□

Proof of Lemma 3.23. The proof follows the same scheme of the previous one, so we only sketch it. Exploiting the same argument used in the proof of Lemma 3.21, but with opposite inequalities, we get the conclusion in case $u > 0$. Therefore, it suffices to assume that $|\nabla P| \leq 1 - \mu$ and we suppose that the conclusion does not hold, i.e. $u \leq P^+$ in B_1 and $u(x_0) = P^+(x_0)$ with $x_0 \in B_{1/2} \cap \overline{\{P > 0\}}$. We define, at this point, as in the proof of Lemma 3.21,

$$\bar{P} \coloneqq P - \mu c_2(1 - |x|^2), \quad c_2 > 0, \tag{3.134}$$

so that in this case Lemma 3.26 applies since $\|D^2P\| \le 1$ and $\Delta_p P \le -\mu$, and thus we have $\Delta_p \bar{P} \le -\frac{\mu}{C_p}$ and $|\nabla \bar{P}| < 1$. Setting again

$$u_{\max} := \max\{u, \bar{P}^+\}, \quad u_{\min} := \min\{u, \bar{P}^+\}, \tag{3.135}$$

we note this time that $u_{\min} = u$ on ∂B_1, because by continuity $u \le P^+$ in B_1 implies $u \le P^+$ on ∂B_1 as well and in view of (3.134), $P^+ = \bar{P}^+$ on ∂B_1. As a consequence, (3.91) yields then

$$J_p(u, B_1) \le J_p(u_{\min}, B_1) + \sigma,$$

which can be rewritten as

$$J_p(u_{\max}, B_1) - J_p(\bar{P}^+, B_1) \le \sigma, \tag{3.136}$$

using the same computation exploited to obtain (3.131).
Furthermore, as before, it turns out that $\bar{P}^+$ is the minimizer of the functional

$$F_p(v) := \int_{B_1} \left(|\nabla v|^p + \chi_{\{v>0\}} + \frac{\mu}{C_p}(\bar{P}^+ - v) \right) dx \tag{3.137}$$

among all competitors $v \ge \bar{P}^+$ which are equal to $\bar{P}^+$ on ∂B_1. The argument used is again that the minimizer v satisfies the comparison principle with the continuous family of functions $(\bar{P}+t)^+$, $t \ge 0$, which are in this case classical supersolutions to

$$\begin{cases} \Delta_p v = -\frac{\mu}{pC_p} & \text{in } \{v > 0\}, \\ |\nabla v|^p = \frac{1}{p-1} & \text{on } F(v), \end{cases} \tag{3.138}$$

because $\Delta_p \bar{P} \le -\frac{\mu}{C_p}$ and so also $\Delta_p \bar{P} \le -\frac{\mu}{pC_p}$, and $|\nabla \bar{P}| < 1$. We point out that v is a viscosity solution to (3.138).
The fact that $\bar{P}^+$ is the minimizer of (3.137) then entails that

$$\sigma \ge J_p(u_{\max}, B_1) - J_p(\bar{P}^+, B_1) \ge \frac{\mu}{C_p} \int_{B_1} (u_{\max} - \bar{P}^+)\, dx, \tag{3.139}$$

hence we are left with the proof that the right hand side is greater than $c\mu^{n+2}$ to conclude as above. To this end, we have to distinguish two cases.

If $u(x_0) < \frac{c_2\mu}{8}$, we can deduce first from $u(x_0) = P^+(x_0)$ that $P(x_0) < \frac{c_2\mu}{8}$. In addition, by the mean value theorem, together with the assumption $|\nabla P| \leq 1 - \mu < 1$, we have

$$|P(x) - P(x_0)| \leq |x - x_0| < \frac{c_2\mu}{8} \quad \text{in } B_{\frac{c_2\mu}{8}}(x_0),$$

which gives, using $P(x_0) < \frac{c_2\mu}{8}$,

$$P(x) < P(x_0) + \frac{c_2\mu}{8} < \frac{c_2\mu}{4} \quad \text{in } B_{\frac{c_2\mu}{8}}(x_0),$$

namely $P < \frac{c_2\mu}{4}$ in $B_{\frac{c_2\mu}{8}}(x_0)$. In particular, according to (3.134), this fact then implies

$$\bar{P} = P - \mu c_2(1-|x|^2) < \frac{c_2\mu}{4} - \mu c_2\left(1-\frac{1}{4}\right) = \frac{c_2\mu}{4} - \mu c_2\frac{3}{4} = -\frac{c_2\mu}{2} \quad \text{in } B_{\frac{c_2\mu}{8}}(x_0)$$

i.e. $\bar{P} < 0$ in $B_{\frac{c_2\mu}{8}}(x_0)$, since we can suppose that $B_{\frac{c_2\mu}{8}}(x_0) \subset B_{1/2}$ provided μ is small enough. As a consequence, we also obtain

$$\bar{P}^+ \equiv 0 \quad \text{in } B_{\frac{c_2\mu}{8}}(x_0). \tag{3.140}$$

Now, we distinguish two cases, that is whether $u(x_0) > 0$ or not. Let us begin with the case $u(x_0) > 0$. Then, provided μ is sufficiently small, we can assume that $u > 0$ in whole $B_{\frac{c_2\mu}{8}}(x_0)$. Therefore, by virtue of Lemma 3.20, Lipschitz continuity of u, (3.140) and (3.135), since $u \geq 0$ everywhere, we achieve

$$\begin{aligned} &u_{\max} - \bar{P}^+ = u \geq u(x_0) - L|x - x_0| \geq c(K) - Lc'\mu = c(K)\mu \\ &\text{in } B \subset B_{\frac{c_2\mu}{8}}(x_0), \quad |B| \sim \mu^n, \end{aligned}$$

if c' is small enough, which yields that the right hand side in (3.139) is greater than $c\mu^{n+2}$.

If instead $x_0 \in \partial\{u > 0\}$, the desired conclusion would follow from the strong nondegeneracy of u, nevertheless we recall that do not have the strong nondegeneracy for u yet. However, for the moment, we assume that such property holds . Consequently, we get

$$\max_{B_{\frac{c_2\mu}{8}}(x_0)} u = u(x_M) \geq c(K)\frac{c_2\mu}{8}.$$

Now, if we take $B = B_{c'\mu}(x_M) \subset B_{\frac{c_2\mu}{8}}(x_0)$ this condition entails, together with the Lipschitz continuity of u,

$$u(x) \geq u(x_M) - L|x - x_M| \geq c(K)\frac{c_2\mu}{8} - Lc'\mu = c(K)\mu \quad \text{in } B_{c'\mu}(x_M)$$

if c' is sufficiently small, and hence, from (3.140), since $u \geq 0$ everywhere,

$$u_{\max} - \bar{P}^+ = u \geq c(K)\mu \quad \text{in } B_{c'\mu}(x_M),$$

which gives that the right hand side in (3.139) is greater than $c\mu^{n+2}$.
If $u(x_0) \geq \frac{c_2\mu}{8}$, then, according to the fact that $u(x_0) = P^+(x_0)$ and (3.134), we achieve $u(x_0) \geq \bar{P}^+(x_0)$, which gives by continuity $u \geq \bar{P}^+$ and so $u = u_{\max}$ in $B_{c'\mu}(x_0)$. This, together with the Lipschitz continuity of u and $\bar{P}^+$, lastly yields

$$(u_{\max} - \bar{P}^+)(x) \geq (u - \bar{P}^+)(x_0) - L|x - x_0| \geq \mu c_2(1 - |x_0|^2) - Lc'\mu \geq c\mu$$

in $B_{c'\mu}(x_0)$, in other words

$$u_{\max} - \bar{P}^+ \geq c\mu \quad \text{in } B_{c'\mu}(x_0),$$

which implies that the right hand side in (3.139) is greater than $c\mu^{n+2}$. □

3.6.2 Partial regularity of the free boundary

As in the classical case, see [30], this subsection provides the proof of the main regularity result for the free boundary of almost minimizers. Our proof is inspired by the techniques developed in [61], which have been largely inspired by the techniques in [26]. The structure of the subsection will follow that of the corresponding subsection in [30]. All the results in the current subsection are expected ones, but not completely proved, since they depend on both Lemma 3.21 and 3.23. Therefore, as previously anticipated, they are labeled with “Expected”.
First of all, we present the statement of the main theorem.

Expected Theorem 3.27 (Flatness implies regularity)**.** *Let u be an almost minimizer for J_p in B_1 (with constant κ and exponent β) and assume that $|\nabla u| \leq K$. Suppose also that $|u - x_n^+| \leq \varepsilon_0$ in B_1 and $0 \in F(u) := \partial\{u > 0\} \cap B_1$. Then, if ε_0 and κ are small enough depending on β and K, it holds that $F(u)$ is $C^{1,\alpha}$ in a neighborhood of 0, for some $\alpha \leq \beta/(n+p+2)$.*

As in the classical case [30], it turns out that Theorem 3.27 is a direct consequence of an improvement of flatness lemma. The precise statement of the improvement of flatness lemma is the following.

Expected Lemma 3.28 (Improvement of flatness)**.** *Let u satisfy* (3.91). *Suppose also that $|u - x_n^+| \leq \varepsilon$ in B_1, $0 \in F(u)$ and σ in* (3.91) *satisfies $\sigma \leq \varepsilon^{n+p+2}$. Then, given $\alpha \in (0,1)$, there exists η depending on α such that*

$$|u - (x \cdot \nu)^+| \leq \varepsilon \eta^{1+\alpha} \quad \text{in } B_\eta \tag{3.141}$$

for some unit direction ν, provided that $\varepsilon \leq \varepsilon_0(K,\alpha)$ is small enough.

About the proof of Lemma 3.28, instead, we do not provide it immediately. Before giving it, we point out indeed that due to results in subsection 3.6.1, the proof of Lemma 3.28 follows the scheme of the case of minimizers as in [61]. In particular, we sketch the details in the following two subsections.

3.6.3 Two properties

Let us define first the ε-scaled function

$$\bar{u}_\varepsilon := \frac{u - x_n}{\varepsilon} \quad \text{in the set } \{u > 0\} \cap B_1, \tag{3.142}$$

and we introduce the notation

$$\begin{aligned} B_\rho^+ &:= B_\rho \cap \{x_n > 0\} \\ \Gamma_\rho &:= B_\rho \cap \{x_n = 0\} \end{aligned} \tag{3.143}$$

We introduce now two properties (P1) and (P2) for the function u which result to be sufficient for obtaining the approximation of $\bar{u}_\varepsilon$ as in (3.142) with solutions of the linearized Neumann problem

$$\begin{cases} \mathcal{L}_p \bar{u}_0 = 0 & \text{in } B^+_{1/2}, \\ \partial_\nu \bar{u}_0 = 0 & \text{on } \Gamma_{1/2}, \end{cases} \tag{3.144}$$

with $\nu := e_n$ and $\mathcal{L}_p := \Delta + (p-2)\partial_{nn}$, and for achieving the improvement of flatness Lemma 3.28. These properties are written in terms of two small parameters δ and $\varepsilon > 0$.

(P1) *Harnack inequality*, (see Theorem 3.2 in [61]).
Given $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\delta)$ so that if $\varepsilon \leq \varepsilon_0$,

$$u \geq l^+ := (x_n + a)^+ \quad \text{in } B_r(x_0) \subset B_1,$$

with a constant, $r \geq \delta$, $|a| \leq \varepsilon$ and

$$u(y) \geq l^+(y) + \gamma\varepsilon \text{ for some } y \text{ for which } B_{r/2}(y) \subset \{l^+ > 0\} \cap B_r(x_0),$$

for some $\gamma \in [\delta, 1]$, then

$$u \geq (x_n + a + c\gamma\varepsilon)^+ \quad \text{in } B_{r/2}(x_0),$$

with $c > 0$ universal.
In a similar way, the above is true when we replace $\geq$ by $\leq$ and γ by $-\gamma$.

(P2) *Viscosity property.* Given $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\delta)$ so that if $\varepsilon \leq \varepsilon_0$ we cannot have $u(x_0) = P(x_0)$ and $u \geq P$ in $B_\delta(x_0) \subset B_1$, where P is a quadratic polynomial with $\|D^2 P\| \leq \delta^{-1}\varepsilon$, $\Delta_p P \geq \delta\varepsilon$, and in the ball $B_\delta(x_0)$ either $u > 0$ or $|\nabla P| > 1 + \delta\varepsilon$.
In a similar way, the above is true when $u \leq P$, $\Delta_p P \leq -\delta\varepsilon$ and $|\nabla P| < 1 - \delta\varepsilon$.

At this point, we show that (P1) and (P2) are enough to get the improvement of flatness property as in [61].
Precisely, we first state and prove a lemma which roughly says that a function u which satisfies (P1) and (P2), satisfies the improvement of flatness

property as well. Secondly, we show that if u verifies the hypotheses of the improvement of flatness Lemma 3.28, then it satisfies (P1) and (P2).

Expected Lemma 3.29. *Suppose a family of functions u satisfy properties (P1) and (P2). If $|u - x_n^+| \le \varepsilon$ in B_1 and $0 \in F(u)$, then*

$$|u - (x \cdot \nu)^+| \le \varepsilon \eta^{1+\alpha} \quad \text{in } B_\eta$$

with ν a unit direction, provided that $\varepsilon \le \varepsilon_1$ with ε_1 depending on n, α and the dependence $\delta \mapsto \varepsilon_0(\delta)$ of properties (P1) and (P2).

Proof. The proof is achieved by compactness. Precisely, we argue by contradiction and we find sequences $\varepsilon_k \to 0$ and a sequence of functions u_k satisfying the assumptions but not the conclusion of Lemma 3.29. Let, at this point,

$$\bar{u}_k := \frac{u_k - x_n}{\varepsilon_k} \quad \text{in the set } \{u_k > 0\} \cap B_1. \tag{3.145}$$

Then, taking $\varepsilon_k \le \varepsilon_0(\delta_k)$ with $\delta_k = 2^{-k}$, property (P1) together with the Ascoli-Arzelà theorem produce that (up to a subsequence) the graphs of $\bar{u}_k$ converge in the Hausdorff distance to the graph of a Hölder function $\bar{u}_0$ defined in the half ball $B^+_{1/2} \cup \Gamma_{1/2}$, see [61].

Now, we want to show that the property (P2) entails that the function $\bar{u}_0$ satisfies (3.144) in the viscosity sense. To this end, we take a quadratic polynomial Q which touches $\bar{u}_0$ from below at some point $x_0 \in B^+_{1/2} \cup \Gamma_{1/2}$. By the convergence of the graphs of $\bar{u}_k$ to the graph of $\bar{u}_0$, this means that there exist constants $c_k \to 0$ such that $Q + c_k$ touches $\bar{u}_k$ from below at $x_k \to x_0$ in a fixed neighborhood of x_0. In particular, using the definition of $\bar{u}_k$ as in (3.145), this also implies that

$$P_k := x_n + \varepsilon_k(Q + c_k) \tag{3.146}$$

touches u_k by below at x_k. At this point, we distinguish two cases.

(i) If $x_0 \in B^+_{1/2}$, we have to prove that $(\Delta Q + (p-2)Q_{nn})(x_0) \le 0$. For this purpose, let us assume by contradiction that $\Delta Q + (p-2)Q_{nn} > 0$. Then, choosing $\delta > 0$ sufficiently small, we can assume that $\|D^2Q\| \le \delta^{-1}$, $\Delta Q + (p-2)Q_{nn} \ge C_1\delta$, with C_1 to be made precise later, and that P_k as in (3.146) touches u_k from below at x_k in a δ-neighborhood $\mathcal{O}_k$ of x_k. Without loss of generality, we can assume that $\mathcal{O}_k = B_\delta(x_k)$. The strategy now is to contradict property (P2) for u_k by means of P_k. We note first that since $(x_0)_n > 0$, for k large also $(x_k)_n > 0$. Thus, in view of (3.146), taking δ and ε_k sufficiently small, we get that $P_k > 0$ in $B_\delta(x_k)$. Precisely, because $x_n > (x_k)_n - \delta$ and $Q \ge \min\limits_{B_\delta(x_k)} Q = C(Q)$ in $B_\delta(x_k)$, we have from (3.146)

$$P_k > (x_k)_n - \delta + \varepsilon_k(C(Q) + c_k) \ge \frac{(x_0)_n}{2} - \delta + \varepsilon_k(C(Q) + c_k) > 0$$

in $B_\delta(x_k)$, if indeed δ and ε_k are small enough. Moreover, $\|D^2Q\| \le \delta^{-1}$ immediately gives by (3.146) $\|D^2P_k\| \le \delta^{-1}\varepsilon_k$. So, it remains to check that $\Delta_p P_k \ge \delta\varepsilon_k$. For this, we compute

$$\begin{aligned}\Delta_p P_k = |\nabla P_k|^{p-2}(\varepsilon_k \Delta Q + (p-2)|\nabla P_k|^{-2}\langle \varepsilon_k D^2Q(e_n + \varepsilon_k \nabla Q), e_n + \varepsilon_k \\ \nabla Q\rangle) = |\nabla P_k|^{p-2}(\varepsilon_k \Delta Q + (p-2)|\nabla P_k|^{-2}\varepsilon_k\langle D^2Q e_n, e_n\rangle + (p-2) \\ |\nabla P_k|^{-2}(2\varepsilon_k^2\langle D^2Q e_n, \nabla Q\rangle + \varepsilon_k^3\langle D^2Q\nabla Q, \nabla Q\rangle))\end{aligned}$$

which yields

$$\begin{aligned}\Delta_p P_k = |\nabla P_k|^{p-2}(\varepsilon_k(\Delta Q + |\nabla P_k|^{-2}(p-2)Q_{nn}) + (p-2)|\nabla P_k|^{-2}(2\varepsilon_k^2 \\ \langle D^2Q e_n, \nabla Q\rangle + \varepsilon_k^3\langle D^2Q\nabla Q, \nabla Q\rangle)).\end{aligned} \tag{3.147}$$

Let us analyze the terms in (3.147) separately. About

$$|\nabla P_k|^{p-2}\varepsilon_k(\Delta Q + |\nabla P_k|^{-2}(p-2)Q_{nn}),$$

we remark first that $|\nabla P_k| \to 1$ as $\varepsilon_k \to 0$. As a consequence, because $\Delta Q + (p-2)Q_{nn} > 0$ by the assumption on Q, we obtain

$$\Delta Q + |\nabla P_k|^{-2}(p-2)Q_{nn} \to \Delta Q + (p-2)Q_{nn} > 0 \quad \text{as } \varepsilon_k \to 0,$$

which means by the definition of limit that

$$\Delta Q + |\nabla P_k|^{-2}(p-2)Q_{nn} \geq \frac{\Delta Q + (p-2)Q_{nn}}{2} \geq \frac{C_1\delta}{2} \quad \text{for } k \text{ large,}$$

that is

$$\Delta Q + |\nabla P_k|^{-2}(p-2)Q_{nn} \geq \frac{C_1\delta}{2} \quad \text{for } k \text{ large.} \tag{3.148}$$

In addition, always since $|\nabla P_k| \to 1$ as $\varepsilon_k \to 0$, we can assume for instance that $|\nabla P_k| \geq 1/2$ for these k's. Hence, using (3.148), we achieve, because $p > 2$,

$$|\nabla P_k|^{p-2}\varepsilon_k(\Delta Q + |\nabla P_k|^{-2}(p-2)Q_{nn}) \geq \varepsilon_k \left(\frac{1}{2}\right)^{p-2}\frac{C_1\delta}{2} \quad \text{for } k \text{ large.} \tag{3.149}$$

Concerning the term

$$(p-2)|\nabla P_k|^p(2\varepsilon_k^2\langle D^2Qe_n, \nabla Q\rangle + \varepsilon_k^3\langle D^2Q\nabla Q, \nabla Q\rangle),$$

instead, we first note that for previous k's large for which $|\nabla P_k| \geq 1/2$, it also holds $|\nabla P_k| \leq 3/2$, thus we get for these k's, because $\|D^2Q\| \leq \delta^{-1}$ and $|\nabla Q| \leq C(Q) = \max_{\overline{B}_\delta(x_k)} |\nabla Q|$,

$$(p-2)|\nabla P_k|^p(2\varepsilon_k^2\langle D^2Qe_n, \nabla Q\rangle + \varepsilon_k^3\langle D^2Q\nabla Q, \nabla Q\rangle) \geq (p-2)|\nabla P_k|^p$$
$$(-2\varepsilon_k^2\|D^2Q\||e_n||\nabla Q| - \varepsilon_k^3\|D^2Q\||\nabla Q||\nabla Q|) \geq -(p-2)\left(\frac{3}{2}\right)^p(2\varepsilon_k^2$$
$$\delta^{-1}C(Q) + \varepsilon_k^3\delta^{-1}C(Q)^2)$$

and we ask

$$-(p-2)\left(\frac{3}{2}\right)^p(2\varepsilon_k^2\delta^{-1}C(Q)+\varepsilon_k^3\delta^{-1}C(Q)^2) \geq -c_1\varepsilon_k\delta, \quad \text{with } c_1 \text{ universal.}$$

In other words, we want

$$(p-2)\left(\frac{3}{2}\right)^p(2\varepsilon_k^2\delta^{-1}C(Q) + \varepsilon_k^3\delta^{-1}C(Q)^2) \leq c_1\varepsilon_k\delta, \quad \text{with } c_1 \text{ universal,}$$

namely

$$(p-2)\left(\frac{3}{2}\right)^p (2\varepsilon_k\delta^{-1}C(Q)+\varepsilon_k^2\delta^{-1}C(Q)^2)\le c_1\delta, \quad \text{with } c_1 \text{ universal,}$$

and this is true if ε_k is sufficiently small, i.e. if k is large enough. To recap, we have obtained

$$(p-2)|\nabla P_k|^p(2\varepsilon_k^2\langle D^2Qe_n,\nabla Q\rangle+\varepsilon_k^3\langle D^2Q\nabla Q,\nabla Q\rangle)\ge -c_1\varepsilon_k\delta,$$

with c_1 universal and k large.
Putting together this condition with (3.149), we then achieve, by virtue of (3.147),

$$\Delta_p P_k \ge \varepsilon_k\left(\frac{1}{2}\right)^{p-2}\frac{C_1\delta}{2}-c_1\varepsilon_k\delta\ge\varepsilon_k\delta,$$

if C_1 is sufficiently large, which gives $\Delta_p P_k \ge \delta\varepsilon_k$.
Therefore, to sum up, we have that P_k touches u_k from below in $B_\delta(x_k)$ at x_k, with $\|D^2P_k\|\le\delta^{-1}\varepsilon_k$, $\Delta_pP_k\ge\delta\varepsilon_k$ and $u_k>0$ in $B_\delta(x_k)$. This fact contradicts property (P2) for u_k and hence $\Delta Q+(p-2)Q_nn>0$ can not happen. As a consequence, $\Delta Q+(p-2)Q_nn(x_0)\le 0$.

(ii) If $x_0\in\Gamma_{1/2}$, we want to show that $Q_n(x_0)\le 0$. Suppose by contradiction that $Q_n(x_0)>0$. Then, proceeding as above, we find that P_k as in (3.146) touches u_k from below in $\overline{\{u_k>0\}}\cap B_{2\delta}(x_0)$ at $x_k\to x_0$. Provided δ is sufficiently small, we can assume that $Q_n(x_0)>\delta$. At this point, exploiting (3.146), we notice that $\partial_nP_k=1+\varepsilon_kQ_n\ge 1+\delta\varepsilon_k$ if again δ is small enough, which thus implies that $Q_n\ge\delta$ in $\overline{\{u_k>0\}}$ $\cap\, B_{2\delta}(x_0)$. This remark yields that both P_k is increasing in the x_n direction and $|\nabla P_k|\ge\partial_nP_k\ge 1+\delta\varepsilon_k$ in $\overline{\{u_k>0\}}\cap B_{2\delta}(x_0)$, namely $|\nabla P_k|\ge 1+\delta\varepsilon_k$ in $\overline{\{u_k>0\}}\cap B_{2\delta}(x_0)$. In particular, the fact that P_k is increasing in the x_n direction entails that P_k is below u_k in a whole δ-neighborhood of x_k, which we can assume to be $B_\delta(x_k)$. Indeed, if $x_k\in\{u_k>0\}\cap B_{2\delta}(x_0)$, it immediately follows from the remark that $\{u_k>0\}\cap B_{2\delta}(x_0)$ is open, whereas we need the x_n-monotonicity of P_k

if $x_k \in (\overline{\{u_k > 0\}} \setminus \{u_k > 0\}) \cap B_{2\delta}(x_0)$, because we only have $P_k \leq u_k$ in $B_\delta(x_k) \cap \overline{\{u_k > 0\}}$. Nevertheless, by the x_n-monotonicity of P_k and the fact that $|u - x_n^+| \leq \varepsilon_k$, $P_n \leq 0$ in $\overline{\{u_k > 0\}}^c \cap B_\delta(x_k)$. Consequently, P_k is actually below u_k in $B_\delta(x_k)$.

To recap, we have that P_k touches u_k from below in $B_\delta(x_k)$ at x_k, with P_k satisfying $\|D^2 P_k\| \leq \delta^{-1}\varepsilon_k$ and $\Delta_p P_k \geq \varepsilon_k \delta$ by (i). Hence, since in addition $|\nabla P_k| \geq 1 + \delta\varepsilon_k$ in $\overline{\{u_k > 0\}} \cap B_{2\delta}(x_0)$, and thus $|\nabla P_k| \geq 1 + \delta\varepsilon_k$ in $B_\delta(x_k)$ as well, we contradict property (P2) for u_k again and this implies that $Q_n(x_0) \leq 0$.

A similar argument can be used if a quadratic polynomial Q touches $\bar{u}_0$ from above at some point $x_0 \in B^+_{1/2} \cup \Gamma_{1/2}$, exploiting the version of property (P2) for quadratic polynomials touching from above.

As a consequence, $\bar{u}_0$ is actually a viscosity solution to (3.144).

Now, we sketch the conclusion of the lemma, see [61] for the details. Precisely, from Lemma 2.8 in [61], we know that

$$|\bar{u}_0(x) - \bar{u}_0(0) - \nabla \bar{u}_0(0) \cdot x| \leq C_0 \eta^2 \quad \text{in } B^+_\eta \cup \Gamma_\eta, \tag{3.150}$$

where, specifically, $\bar{u}_0(0) = 0$, since $\bar{u}_k(0) = 0$ for every k and $\bar{u}_k$ converge uniformly to $\bar{u}_0$. Therefore, choosing $\eta > 0$ sufficiently small depending only on α and n, we also get by (3.150)

$$|\bar{u}_0 - l| \leq \frac{1}{2}\eta^{1+\alpha} \quad \text{in } B^+_\eta \cup \Gamma_\eta.$$

At this point, we argue as in [61] to obtain that each u_k satisfies the conclusion of Lemma 3.29, which is a contradiction with our assumptions on u_k's. □

3.6.4 Properties (P1) and (P2) are satisfied

To get Lemma 3.28 it is enough to show that if $\sigma \leq \varepsilon^{n+p+2}$ then properties (P1) and (P2) are satisfied by u. About (P1), in view of Remark 3.16 it is sufficient to analyze the case $B_r(x_0) = B_1$ and replace σ by

$$\bar{\sigma} := r^{-n}\sigma. \tag{3.151}$$

Furthermore, we assume $a = 0$ for simplicity. At this point, we want to apply Lemma 3.18 to u and x_n. Indeed, we know by assumptions of property (P1) that $u \geq x_n^+ \geq x_n$ in B_1, with x_n $p-$harmonic and $\nabla x_n \neq 0$. In addition, we also have

$$u(y) \geq y_n^+ + \gamma\varepsilon \text{ for some } y \text{ for which } B_{1/2}(y) \subset \{x_n^+ > 0\} \cap B_1. \quad (3.152)$$

Hence, we can consider $u \geq x_n$ in $B_{1/2}(y)$ and since $B_{1/2}(y) \subset \{x_n^+ > 0\} \cap B_1$, it holds, from $u \geq x_n$ in $B_{1/2}(y)$, that $B_{1/2}(y) \subset \{u > 0\}$ as well. As a consequence, according to (3.152), we can apply Lemma 3.18 to u and x_n with $\mu = \gamma\varepsilon$ provided that $\bar{\sigma} \leq \mu^{n+p+1}$. This condition, in particular, follows by (3.151) and the fact that $\sigma \leq \varepsilon^{n+p+2}$. Precisely, using (3.151) and $\sigma \leq \varepsilon^{n+p+2}$, we achieve

$$\bar{\sigma} = r^{-n}\sigma \leq \delta^{-n}\varepsilon^{n+p+2} \leq (\gamma\varepsilon)^{n+p+1} = \mu^{n+p+1},$$

where $\gamma \geq \delta$ in the hypotheses of property (P2) and taking ε small depending on δ, for instance $\varepsilon \leq \delta^{2n+p+1}$. We can then apply Lemma 3.18 to obtain

$$u \geq x_n + c\gamma\varepsilon \quad \text{in } B_{1/4}(y). \quad (3.153)$$

We note now that $B_{1/2}(y) \subset B_1$ means that $|y| \leq 1/2$, so $B_{1/4}(y) \subset B_{3/4}$. Moreover, the fact that $B_{1/2}(y) \subset \{x_n > 0\}$ implies $y_n = 1/2$, from which we deduce that $B_{1/4}(y) \subset \{x_n \geq 1/4\}$. To recap, we have $B_{1/4}(y) \subset B_{3/4} \cap \{x_n \geq 1/4\}$. Consequently, we can actually extend (3.153) to the whole $B_{3/4} \cap \{x_n \geq c_0(n)\}$, with $c_0(n)$ a universal constant to be fixed, provided changing c and exploiting that $B_{3/4} \cap \{x_n \geq c_0(n)\}$ is bounded. Therefore, we get, from (3.153),

$$u \geq x_n + c\gamma\varepsilon \quad \text{in } B_{3/4} \cap \{x_n \geq c_0(n)\},$$

which also gives, because $x_n \geq c_0(n) > 0$,

$$u \geq x_n^+ + c\gamma\varepsilon \quad \text{in } B_{3/4} \cap \{x_n \geq c_0(n)\}. \quad (3.154)$$

At this point, we want to apply Corollary 3.22 to show that u must be greater than

$$P \coloneqq x_n + \frac{c}{2}\gamma\varepsilon(c_0 + x_n + 2nx_n^2 - |x'|^2) \tag{3.155}$$

in the cylinder

$$\mathcal{C} \coloneqq \{|x_n| \leq 2c_0, \quad |x'| \leq 1/4\}.$$

To do this, we need that $u \geq P$ in a δ-neighborhood of $\partial\mathcal{C}$, with $\|D^2P\| \leq \delta^{-1}$, $\Delta_p P \geq \mu$, μ small, and either $u > 0$ or $|\nabla P| \geq 1+\mu$ in $\mathcal{C}$. Let us start checking the differential conditions on P. Precisely, by virtue of (3.155), we have

$$\nabla P = e_n + \frac{c}{2}\gamma\varepsilon(e_n + 4nx_n e_n - 2x') = \left(1 + \frac{c}{2}\gamma\varepsilon(1 + 4nx_n)\right)e_n - c\gamma\varepsilon x',$$

$$D^2P = c\gamma\varepsilon \begin{bmatrix} -2 & 0 & \cdots & \cdots & 0 \\ 0 & -2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 4n \end{bmatrix},$$

$$\Delta P = \frac{c}{2}\gamma\varepsilon(4n - 2(n-1)) = \frac{c}{2}\gamma\varepsilon(2n+2) = c\gamma\varepsilon(n+1).$$

Then, using the expression of D^2P, we get

$$\|D^2P\| = c\gamma\varepsilon 4n \leq \delta^{-1}, \tag{3.156}$$

with δ small to be chosen later, provided c and ε are sufficiently small.
In parallel, exploiting the equality for ∇P, we obtain

$$|\nabla P|^2 = \left(1 + \frac{c}{2}\gamma\varepsilon(1 + 4nx_n)\right)^2 + (c\gamma\varepsilon)^2|x'|^2 \geq \left(1 + \frac{c}{2}\gamma\varepsilon(1 - 8nc_0)\right)^2$$
$$\geq \left(1 + \frac{c}{4}\gamma\varepsilon\right)^2 = (1+\mu)^2 \quad \text{in } \mathcal{C},$$

which yields

$$|\nabla P| \geq 1 + \mu \quad \text{in } \mathcal{C}, \tag{3.157}$$

taking $\mu = c\gamma\varepsilon/4$ and c_0 such that

$$1 - 8nc_0 \geq \frac{1}{2} \iff 8nc_0 \leq \frac{1}{2} \iff c_0 \leq \frac{1}{16n}.$$

The conditions above let us to compute $\Delta_p P$ as well. Indeed, we have

$$\Delta_p P = |\nabla P|^{p-2}(\Delta P + (p-2)|\nabla P|^{-2}\langle D^2 P \nabla P, \nabla P\rangle) = |\nabla P|^{p-2}(c\gamma\varepsilon(n+1)$$
$$+ (p-2)|\nabla P|^{-2}\langle(-2P_1, \dots, -2P_{n-1}, 4nP_n), \nabla P\rangle) = |\nabla P|^{p-2}(c\gamma\varepsilon(n+1)$$
$$+ (p-2)|\nabla P|^{-2}(-2P_1^2 - \dots - 2P_{n-1}^2 + 4nP_n^2)) = |\nabla P|^{p-2}(c\gamma\varepsilon(n+1)$$
$$+ (p-2)|\nabla P|^{-2}(-2|\nabla P|^2 + (4n+2)P_n^2)) = |\nabla P|^{p-2}(c\gamma\varepsilon(n+1) - 2$$
$$(p-2) + (p-2)|\nabla P|^{-2}(4n+2)P_n^2),$$

namely

$$\Delta_p P = |\nabla P|^{p-2}(c\gamma\varepsilon(n+1) - 2(p-2) + (p-2)|\nabla P|^{-2}(4n+2)P_n^2). \quad (3.158)$$

Let us analyze now the term $|\nabla P|^{-2}P_n^2$. Specifically, using the expression of ∇P, we achieve

$$\frac{P_n^2}{|\nabla P|^2} = \frac{\left(1 + \frac{c}{2}\gamma\varepsilon(1 + 4nx_n)\right)^2}{\left(1 + \frac{c}{2}\gamma\varepsilon(1 + 4nx_n)\right)^2 + (c\gamma\varepsilon)^2|x'|^2} \geq \frac{1}{2} \quad \text{in } \mathcal{C},$$

choosing c and ε small enough, since $|x_n| \leq 2c_0$ and $|x'| \leq 1/4$ in $\mathcal{C}$. Exploiting this inequality, we then obtain, in view of (3.158) and (3.157),

$$\Delta_p P \geq |\nabla P|^{p-2}\left(c\gamma\varepsilon(n+1) - 2(p-2) + (p-2)(4n+2)\frac{1}{2}\right)$$
$$\geq (1+\mu)^{p-2}(c\gamma\varepsilon(n+1) + (p-2)(2n-1)) \geq \mu,$$

i.e.

$$\Delta_p P \geq \mu,$$

if μ is sufficiently small. To recap, putting together this last condition with (3.156) and (3.157), we have $\|D^2 P\| \leq \delta^{-1}$, $|\nabla P| \geq 1 + \mu$ in $\mathcal{C}$ and $\Delta_p P \geq \mu$. So, it remains to show that $u \geq P$ in a δ-neighborhood of $\partial\mathcal{C}$ to can apply Corollary 3.22 to u and P. To this end, we distinguish two cases. If $c_0 \leq x_n \leq 2c_0$, this fact follows directly from (3.154) if c_0 is sufficiently small depending on n. Precisely, by (3.154) we get, always using (3.155),

$$P \leq x_n + \frac{c}{2}\gamma\varepsilon(c_0 + 2c_0 + 8nc_0^2) \leq x_n^+ + c\gamma\varepsilon \leq u \quad \text{in } B_{3/4} \cap \{c_0 \leq x_n \leq 2c_0\},$$

which implies, if we call $\mathcal{O}$ a δ-neighborhood of $\partial\mathcal{C}$,

$$P \le u \quad \text{in } \mathcal{O} \cap \{c_0 \le x_n \le 2c_0\}, \tag{3.159}$$

provided that c_0 is small enough depending on n.

About the complementary case, what we want to prove is a consequence of the hypothesis

$$u \ge x_n^+ \quad \text{in } B_1$$

of Property (P1). Actually, if $-2c_0 \le x_n \le -c_0$, provided c and ε are sufficiently small, we have $P \le 0$ from (3.155) and thus, because $u \ge 0$, $u \ge P$ in $\{-2c_0 \le x_n \le -c_0\}$. Hence, it remains to consider the case $-c_0 < x_n < c_0$. Here, a δ-neighborhood $\mathcal{O}$ of $\partial\mathcal{C}$ is guaranteed by the fact that $|x'| > 1/4 - \delta$, so we obtain, according to (3.155),

$$P \le x_n + \frac{c}{2}\gamma\varepsilon\left(c_0 + c_0 + 2nc_0^2 - \left(\frac{1}{4} - \delta\right)^2\right) \le x_n^+ \le u,$$

i.e.

$$P \le u \quad \text{in } \mathcal{O} \cap \{-c_0 < x_n < c_0\},$$

which yields, together with the considerations above,

$$P \le u \quad \text{in } \mathcal{O} \cap \{-2c_0 \le x_n < c_0\}.$$

This condition and (3.159) then give that P is below u in a δ-neighborhood of $\partial\mathcal{C}$. As a consequence, we can apply Corollary 3.22 to u and P and we achieve $u \ge P$ in $\mathcal{C}$.

Next, we show that this fact entails the conclusion of property (P1). The strategy is to use $u \ge P$ in $\mathcal{C}$ to get the result on $\{x' = 0\}$ and then to slide it to each x' in $B'_{1/2}$.

Specifically, on $x' = 0$, by virtue of (3.155) we have

$$P = x_n + \frac{c}{2}\gamma\varepsilon(c_0 + x_n + 2nx_n^2) \quad \text{on } \{x' = 0\},$$

thus, from $u \ge P$ in $\mathcal{C}$ we obtain

$$u \ge x_n + \frac{c}{2}\gamma\varepsilon(c_0 + x_n + 2nx_n^2) \quad \text{on } \{x' = 0\} \cap \mathcal{C}. \tag{3.160}$$

We remark first that, independently from (3.160), when $x_n \leq -c\gamma\varepsilon$ the fact that $u \geq (x_n + c\gamma\varepsilon)^+$ is a direct consequence of $u \geq 0$. So, we have to analyze the case when $x_n > -c\gamma\varepsilon$. For this purpose, we need (3.160). Indeed, in view of (3.160), we have with $x_n > -c\gamma\varepsilon$

$$u \geq x_n + \frac{c}{2}\gamma\varepsilon(c_0 - c\gamma\varepsilon) = x_n + c\gamma\varepsilon \quad \text{on } \{x' = 0,\ x_n > -c\gamma\varepsilon\} \cap \mathcal{C},$$

in other words, since $u \geq 0$,

$$u \geq (x_n + c\gamma\varepsilon)^+ \quad \text{on } \{x' = 0,\ x_n > -c\gamma\varepsilon\} \cap \mathcal{C},$$

taking c and ε sufficiently small. Putting together this with the remark above, we then achieve

$$u \geq (x_n + c\gamma\varepsilon)^+ \quad \text{on } \{x' = 0\} \cap (\mathcal{C} \cup \{x_n \leq -c\gamma\varepsilon\}). \tag{3.161}$$

In addition, from (3.154), since $x_n \geq c_0 > 0$ and $c\gamma\varepsilon > 0$, we get

$$u \geq x_n^+ + c\gamma\varepsilon = (x_n + c\gamma\varepsilon)^+ \quad \text{in } B_{3/4} \cap \{x_n \geq c_0(n)\},$$

i.e. also

$$u \geq (x_n + c\gamma\varepsilon)^+ \quad \text{in } B_{1/2} \cap \{x_n \geq c_0(n)\},$$

which implies, together with (3.161) and the definition of $\mathcal{C}$, that

$$u \geq (x_n + c\gamma\varepsilon)^+ \quad \text{on } \{x' = 0\} \cap B_{1/2}. \tag{3.162}$$

Now, we want to extend the result in (3.162) to the whole $B_{1/2}$. To this end, we fix $x_0' \in B_{1/2}'$ and we consider the translation of P as in (3.155)

$$\tilde{P}(x) := P(x' - x_0', x_n).$$

We note that $\tilde{P}(x_0', x_n) = P(0, x_n)$, hence we can repeat the reasoning used before to obtain (3.162) with $\tilde{P}$ instead of P and we have

$$u \geq (x_n + c\gamma\varepsilon)^+ \quad \text{on } \{x' = x_0'\} \cap B_{1/2}.$$

Varying x_0' in all of $B_{1/2}'$, we lastly get the conclusion of property (P1) in the whole $B_{1/2}$.

Concerning property (P2), instead, we exploit Lemma 3.21. Precisely, we first apply a rescaling of factor δ^{-1}, in other words considering

$$\tilde{u}(x) \coloneqq \frac{u(x_0+\delta x)}{\delta}, \quad \tilde{P}(x) \coloneqq \frac{P(x_0+\delta x)}{\delta}, \quad x \in B_1,$$

and thus, by virtue of the hypotheses of property (P2), the rescaled polynomial $\tilde{P}$ satisfies, according to (3.110),

$$\|D^2\tilde{P}\| \le \varepsilon \le 1, \quad \Delta_p \tilde{P} \ge \delta^2\varepsilon.$$

Also, always using the hypotheses of property (P2) and (3.110), it holds either $\tilde{u} > 0$ or $|\nabla\tilde{P}| \ge 1+\delta\varepsilon \ge 1+\delta^2\varepsilon \ge$ in B_1. As a consequence, we can apply Lemma 3.21 to $\tilde{u}$ and $\tilde{P}$, with $\mu = \delta^2\varepsilon$, if $\bar{\sigma} \le \mu^{n+p+1}$ and this fact, recalling (3.151), is true if

$$\bar{\sigma} = r^{-n}\sigma \le \delta^{-n}\sigma \le \delta^{-n}\varepsilon^{n+p+2} \le (\delta^2\varepsilon)^{n+p+1} = \mu^{n+p+1},$$

which means $\varepsilon \le \delta^{2(n+p+1)+n}$, that is ε small enough depending on δ. Therefore, property (P2) holds as well.
To recap, both property (P1) and (P2) are satisfied if $\sigma \le \varepsilon^{n+p+2}$, and so Lemma 3.28 holds.
Now, we sketch the proof of Theorem 3.27.

Proof of Theorem 3.27. We first point out that because u is an almost minimizer, it satisfies (3.91) with $\sigma = \kappa$. At this point, given $\alpha \in (0, \beta/(n+p+2)]$, we take ε_0 depending on K, α as determined by Lemma 3.28 and $\kappa \le \varepsilon_0^{n+p+2}$. Then, in view of Lemma 3.28 and the hypotheses of Theorem 3.27, u satisfies (3.141) with ε_0, in other words

$$|u-(x\cdot\nu)^+| \le \varepsilon_0\eta^{1+\alpha} \quad \text{in } B_\eta. \tag{3.163}$$

Let us consider now the rescaling

$$u_\eta(x) \coloneqq \frac{u(\eta x)}{\eta}, \quad x \in B_1,$$

and, by (3.163), we achieve, writing $x = \eta y$, $y \in B_1$,

$$\frac{1}{\eta}|u(\eta y)-(\eta y\cdot\nu)^+| \le \varepsilon_0\eta^\alpha \quad \text{in } B_1 \iff |u_\eta(y)-(y\cdot\nu)^+| \le \varepsilon_0\eta^\alpha \quad \text{in } B_1,$$

i.e., relabeling $y = x$,

$$|u_\eta - (x \cdot \nu)^+| \leq \varepsilon_0 \eta^\alpha \quad \text{in } B_1. \tag{3.164}$$

Furthermore, using Remark 3.14, (3.91) still holds for u_η, but with $\bar{\sigma} \coloneqq \kappa\eta^\beta$. Consequently, we can apply Lemma 3.28 again, but this time for u_η, if

$$\bar{\sigma} \coloneqq \kappa\eta^\beta \leq (\varepsilon_0 \eta^\alpha)^{n+p+2},$$

that is, because $\kappa \leq \varepsilon_0^{n+p+2}$, if

$$\eta^\beta \leq \eta^{\alpha(n+p+2)},$$

which is true for η sufficiently small for our choice of α. Therefore, repeating the same argument, we can apply Lemma 3.28 indefinitely and the theorem follows in a standard way. □

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