Dynamics modelling of a multi body unicycle in three-dimensional space

by

Buddhika Lakmal Aththanayaka

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Alan Levin Department of Mechanical and Nuclear Engineering
Carl R. Ice College of Engineering

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Approved by:	Approved by:
Co-Major Professor Dr. Youqi Wang	Co-Major Professor Dr. Warren N. White

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Abstract

Self-balanced unicycle has received the attention of researchers for decades. Over the years, unicycle models with several different assemblies have been introduced by them. A thorough analysis of the dynamics of a unicycle with a frame and a rotating disk is discussed in this research. A torque applied to the rolling wheel maintains the longitudinal stability of the system by moving forward and backward. The rotating disk mounted on the top of the frame maintains the lateral stability of the system by providing a torque. Due to this torque the rolling wheel precess and change its yaw direction. The components of the unicycle assembly are addressed separately for the analysis of the dynamics. First, only the rolling wheel considered. Then, the rolling wheel and the frame are analyzed. Finally, the completed assembly with the rotating disk considered to build the dynamics model. In each of these cases both Newton-Euler and Lagrangian methods are used to obtain the dynamics equations for the unicycle.

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List of Nomenclature

Variable	Definition
a_{X}, a_{Y}, a_{Z}	Acceleration of the wheel center with respective to the global coordinate system
a_{dX}, a_{dY}, a_{dZ}	Acceleration of the rotating disk center with respective to the global coordinate system
a_{lX}, a_{lY}, a_{lZ}	Acceleration of the rod mass center with respective to the global coordinate system
$a_{X'}, a_{Y'}, a_{Z'}$	Acceleration of the wheel center with respective to the (X'Y'Z') coordinate system
$a_{dX'''}, a_{dY'''}, a_{dZ'''}$	Acceleration of the rotating disk center with respective to the (X'''Y'''Z''') coordinate system
$a_{lX''}, a_{lY''}, a_{lZ''}$	Acceleration of the rod mass center with respective to the (X"Y"Z") coordinate system
$\boldsymbol{DCF}$	The description of the center of mass of rotating disk, in terms of the global coordinate system
F_{X}, F_{Y}, N	Reactions on the rolling wheel at the contact point on the ground
$\boldsymbol{GFO}$	Global frame orientation in terms of the (X'Y'Z') frame
$\boldsymbol{GFO2}$	Global frame orientation in terms of the (X"Y"Z") frame
g	Gravitational acceleration constant
$\boldsymbol{H}$	Angular momentum of rolling wheel
$\boldsymbol{H_d}$	Angular momentum of the disk
$\boldsymbol{H_l}$	Angular momentum of the rod
$(\dot{\boldsymbol{H}})_{\boldsymbol{r}}$	The rate of change of angular momentum of rolling wheel with respect to the center of the wheel, as viewed by an observer on the moving (X'Y'Z") frame
$(\dot{\boldsymbol{H}})_{\boldsymbol{rd}}$	The rate of change of angular momentum of the disk, with respect to the center of the disk as viewed by an observer on the (X'''Y'''Z''') frame

$\left(\dot{\boldsymbol{H}}\right)_{rl}$	The rate of change of angular momentum of the rod, with respect to the center of mass of rod as viewed by an observer on the moving (X”Y”Z”) frame
$\boldsymbol{I_d}$	Inertia matrix of the rotating disk
$\boldsymbol{I_l}$	Inertia matrix of the rod / frame
$\boldsymbol{I_w}$	Inertia matrix of the wheel
$I_{d1,}\ I_{d2,}\ I_{d3}$	Moments of inertia of the rotating disk about three principle axes at the center of mass. I_{d1} is the inertia about the axel of the wheel
$I_{l1,}\ I_{l2,}\ I_{l3}$	Moments of inertia of the frame/ rod along three principle axes at the center of mass.
$I_{w1,}\ I_{w2,}\ I_{w3}$	Moments of inertia of the wheel about three principle axes at the center of mass. I_{w2} is the inertia about the axel of the wheel
$\boldsymbol{L}$	Lagrangian
$\boldsymbol{LCF}$	The description of the center of mass of rod, in terms of the global coordinate system
$\boldsymbol{LocalF}$	The local force vector in wheel centered frame
$\boldsymbol{LocalF2}$	The reaction force vector at the joint
l	Distance to the center of mass of rod from rolling wheel center, along Z” axis
l_d	The distance from rolling wheel center to rotating disk center, along Z” axis
l_X, l_Y, l_Z	Coordinates of the rod mass center with respective to the global coordinate system
l_{dX},l_{dY},l_{dZ}	Coordinates of the center of the rotating disk with respective to the global coordinate system
$\boldsymbol{M}$	External moment acting on the rolling wheel
$\boldsymbol{M_d}$	External moments acting on the disk
$\boldsymbol{M_l}$	External moment acting on the rod
m	Mass of the rolling wheel
m_2	Mass of the frame / rod

m_d	Mass of the rotating disk
$\boldsymbol{PoC}$	Matrix for transformation from global origin to point of contact
$\boldsymbol{Q_i}$	Generalized forces
$\boldsymbol{q_i}$	Generalized coordinates
R	Radius of the rolling wheel
$\boldsymbol{RotX}$	The rotation matrix for the rotation of θ angle about the X axis
$\boldsymbol{RotY}$	The rotation matrix for the rotation of β angle about the Y' axis
$\boldsymbol{RotZ}$	The rotation matrix for the rotation of ϕ angle about the Z axis
r	Radius of the rotating disk
r_X,r_Y,r_Z	Coordinates of the rolling wheel center of mass with respective to the global coordinate system
$S_{X'}, S_{Y'}, S_{Z'}$	Reactions at the joint where rolling wheel and frame connects
$S_{X'''}, S_{Y'''}, S_{Z'''}$	Reactions at the joint where rotating disk and frame connects
T	Total kinetic energy
$\boldsymbol{TR}$	Matrix for transformation from point of contact to wheel center
$\boldsymbol{Td}$	Matrix for transformation from rolling wheel center to the center of disk
$\boldsymbol{Tl}$	Matrix for transformation from wheel center to the center of mass of frame
V	Gravitational potential energy
v_X, v_Y, v_Z	Velocity components of the rolling wheel center with respective to the global coordinate system
v_{dX},v_{dY},v_{dZ}	Velocity components of the rotating disk center with respective to the global coordinate system
v_{lX}, v_{lY}, v_{lZ}	Velocity components of the rod mass center with respective to the global coordinate system
$\boldsymbol{WCF}$	Wheel centered frame (X'Y'Z'), in terms of the global coordinate system
$\boldsymbol{WFO}$	The description of the (X'Y'Z') frame in terms of the (X"Y"Z")
XYZ	Global coordinate system
X'Y'Z'	Wheel centered coordinate system
X"Y"Z"	Coordinate system attached to the frame/ Rod

X'''Y'''Z'''	Coordinate system attached to the rotating disk
x, y	Coordinates of the rolling wheel contact point on the ground
$\dot{x}, \dot{y}$	Velocity of rolling wheel at the contact point on the ground
$\boldsymbol{\alpha}$	Angular acceleration vector of the rolling wheel in the wheel centered local frame
$\boldsymbol{\alpha_d}$	Angular acceleration vector of the disk in (X'''Y'''Z''') frame
$\boldsymbol{\alpha_l}$	Angular acceleration vector of the rod in (X''Y''Z'') frame
$\alpha_{X'}, \alpha_{Y'}, \alpha_{Z'}$	Angular acceleration components of the rolling wheel in the wheel centered coordinate system
$\alpha_{dX'''}, \alpha_{dY'''}, \alpha_{dZ'''}$	Angular acceleration components of the disk in (X'''Y'''Z''') frame
$\alpha_{lX''}, \alpha_{lY''}, \alpha_{lZ''}$	Angular acceleration components of the rod in (X''Y''Z'') frame
β	Rod angle / Frame angle
$\dot{\beta}$	Time rate of change of rod angle
$\ddot{\beta}$	Rod angular acceleration
$\dot{\eta}$	Disk rotation velocity
$\ddot{\eta}$	Disk angular acceleration
θ	Pitch angle
$\dot{\theta}$	Time rate of change of pitch angle
$\ddot{\theta}$	Pitch angular acceleration
λ_k	Lagrange multipliers
τ_1	Torque applied to the rotating disk
τ_2	Torque applied to the rolling wheel
ϕ	Yaw angle
$\dot{\phi}$	Time rate of change of yaw angle
$\ddot{\phi}$	Yaw angular acceleration
ψ	Roll angle
$\dot{\psi}$	Time rate of change roll angle
$\ddot{\psi}$	Roll angular acceleration

$\boldsymbol{\Omega}$	Angular velocity vector of the rolling wheel in the wheel centered local frame
$\boldsymbol{\Omega_d}$	Angular velocity of the disk in (X'''Y'''Z''') frame
$\boldsymbol{\Omega_l}$	Angular velocity vector of the rod in (X''Y''Z'') frame
$\boldsymbol{\omega}$	Angular velocity vector of the frame fixed on the rolling wheel center
$\omega_{X'}, \omega_{Y'}, \omega_{Z'}$	Angular velocity components of the rolling wheel in the wheel centered coordinate frame
$\omega_{dX'''}, \omega_{dY'''}, \omega_{dZ'''}$	Angular velocity components of the disk in (X'''Y'''Z''') frame
$\omega_{lX''}, \omega_{lY''}, \omega_{lZ''}$	Angular velocity components of the rod in (X''Y''Z'') frame

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Last but not least, I'm forever in my debt to my parents who wished me all the success in the world. This work also a tribute to all my family members and friends who have supported me in every possible way.

Dedication

To my wife Kanchana, my son Thinuka and our parents….

Chapter 1 - Introduction

The unicycle has received attention for research for more than twenty years. Over the years, unicycle models with several different assemblies have been introduced by researchers. One of the earliest works found on unicycle, done by A.Schoonwinkle [1], mimics a human riding a unicycle. As illustrated in Figure 1.1, it consists of a wheel, the unicycle frame representing the lower part of the rider's body, and A rotary turntable modeling the rider's twisting torso and arms.

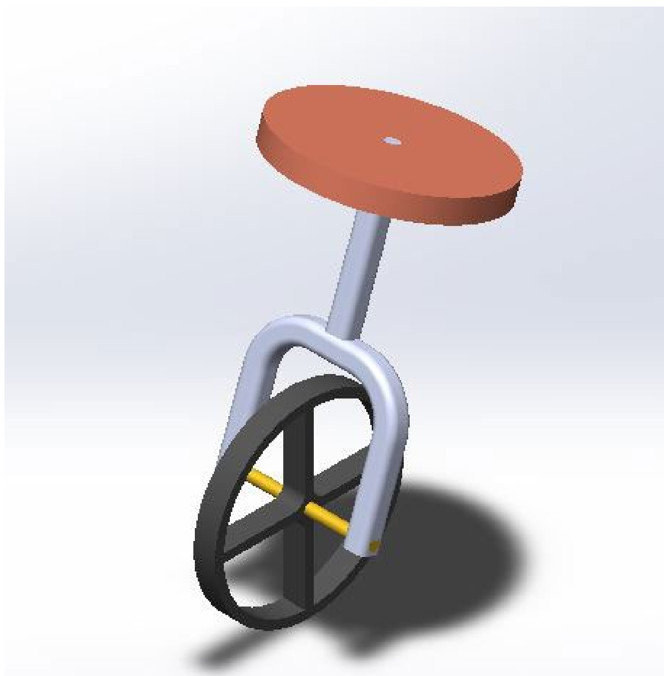

Figure 1.1. Computer Stabilized Unicycle

In this research [1], a linear dynamic model for the unicycle was found using Newton-Euler, Lagrange and D' Alembert's principle. In [2], there are few extra links compared to the configuration of [1] to represent the riders thighs, shanks and pedals as shown in Figure 1.2.

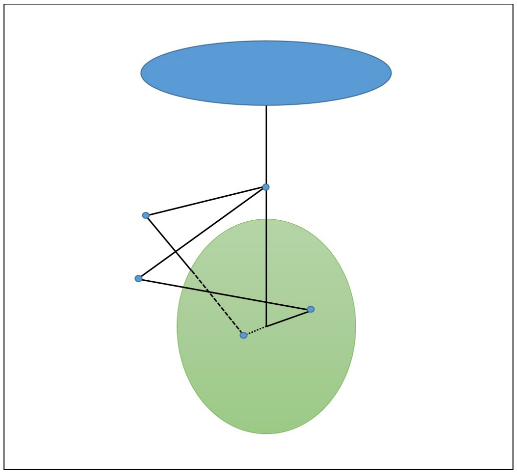

Figure 1.2. Model of a Human Riding Unicycle

These links provide the assistance to maintain the longitudinal stability of the unicycle while, lateral stability achieved by turning the unicycle in the direction it is falling. Another interesting model has been introduced in [3], by replacing the wheel with a rugby ball shaped wheel. Figure 1.3 shows this distinctive feature which helps to keep its rolling stability to a large extent.

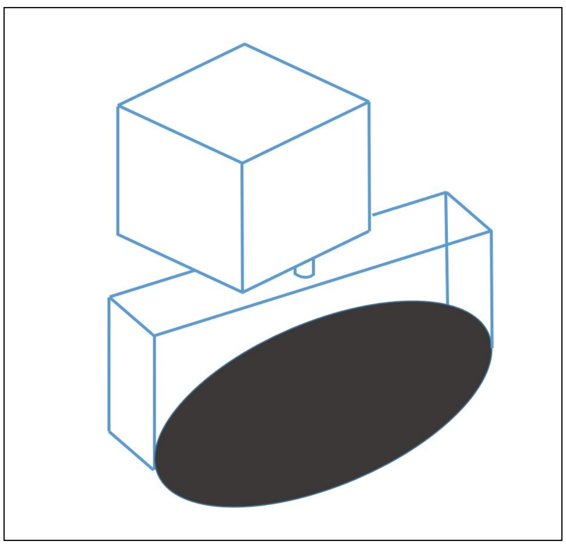

Figure 1.3. Unicycle with Rugby Ball Shaped Wheel

In [4], a single wheel, gyroscopically stabilized robot is developed using the principle of gyroscopic precession. Here, a gyroscope is installed to the wheel such a way to help stabilize the rolling motion and an inbuilt tilt mechanism enables the steering of the wheel. A sketch of this model is demonstrated in Figure 1.4.

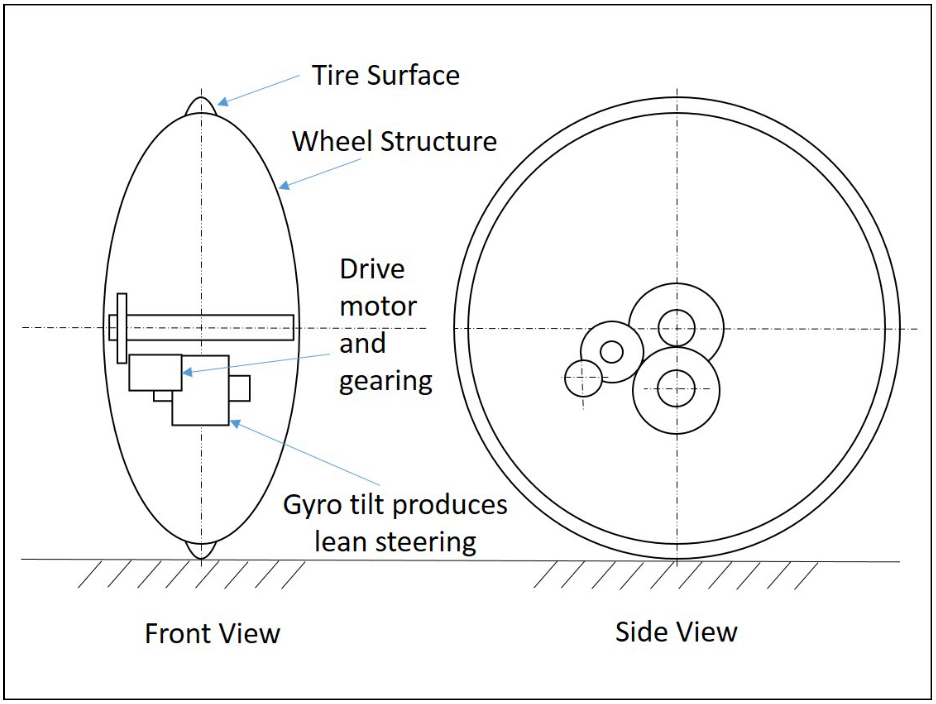

Figure 1.4. Gyroscopically Stabilized Unicycle Robot

The idea of using the precession to steer the wheel is again used in [5]. Figure 1.5 exhibits the unicycle they introduced, which equipped with a pendulum to control the lateral stability instead of a reaction wheel. It acts as the arms of a unicycle rider and maintains the stability by moving in left and right directions. Steering of the wheel is achieved by controlling the wheel and pendulum actuators at the same time. Once a constant speed has been gained by the wheel, pendulum provides a lateral torque for the wheel in order for it to precess. A detailed analysis on dynamics has been provided in a later article [6] by the same group with some additional work.

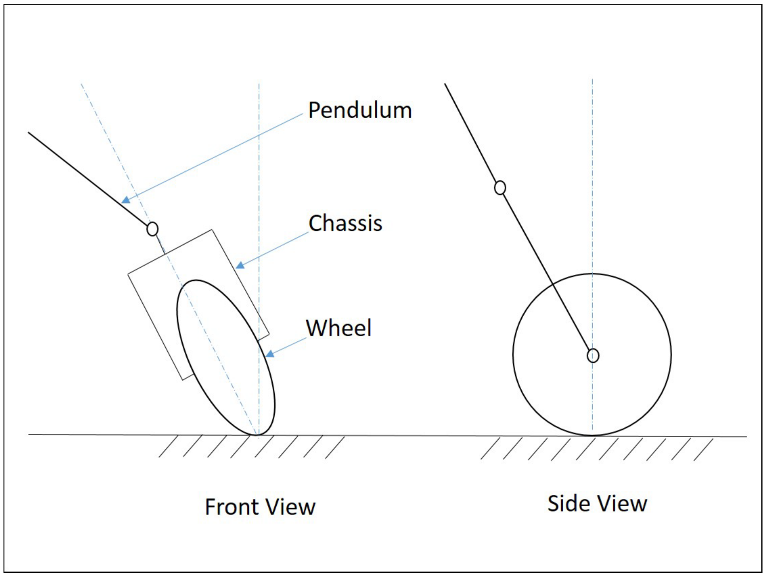

Figure 1.5. Pendulum-Balanced Autonomous Unicycle

The most common and vastly used method found in literature to stabilize a unicycle is the use of a reaction wheel or a flywheel, as shown in Figure 1.6. This wheel is attached to the unicycle frame such a way that its rotating axis is perpendicular to the driving wheel's spinning axis. In [7],[8] and [9] dynamic models were provided for a unicycle in 2D space, which have used this configuration. Several instances were found in literature [10],[11],[12],[13] and [14] where yaw angle control also taken in to consideration when building the dynamics model.

In the process of analyzing the dynamics of a unicycle with actuators for this study, the same model in Figure 1.6 is taken in to consideration. It is found much simpler to analyze the dynamics of the parts of the unicycle separately and add them together. Therefore, first the dynamics of the wheel has found in Chapter 2, following both Lagrangian and Newton- Euler methods. The results obtained in the second chapter for the dynamics of the rolling wheel, agrees with the examples found in [15] and [16], for the situation where, there aren't any actuators. Second, in Chapter 3, the dynamics of the frame is analyzed along with the rolling wheel. The frame is considered as a rod for the calculations, which is connected at the center of the wheel and it is free to rotate about the wheel axis.

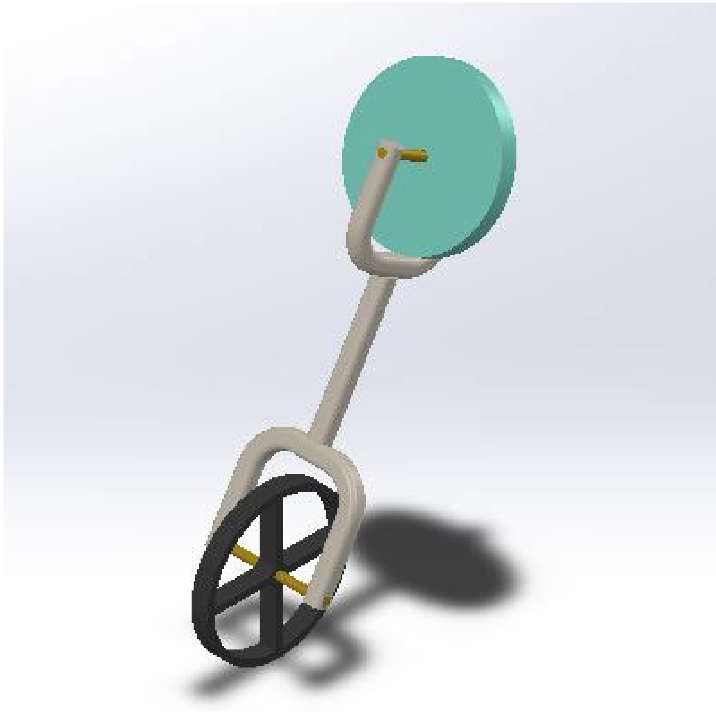

Figure 1.6. Unicycle with a Reaction Wheel

Again, both Newton-Euler and Lagrangian analysis carried out for this assembly. Finally, in Chapter 4, the dynamics of the complete unicycle assembly is analyzed. Here, a rotating disk is connected to the end of the rod such a way that, its axis is perpendicular to the rolling wheel axis. The calculations carried out related to these three chapters are included in Appendix A, B and C.

This thesis provides the dynamics equations for the entire system of unicycle in Figure 1.6, using the Lagrangian and Newton-Euler methods. It is for the first time, the Newton-Euler method is used to analyze this assembly. A similar system to this work is found in [11] but, it includes only the Lagrangian analysis. When comparing the final dynamics equations of [11] to this work, it is found that the generalized force vector has some dissimilarities. Since, the Newton-Euler method calculates the generalized force vector automatically, it can be considered as much reliable method, instead of a user defined vector as in the Lagrangian method.

Chapter 2 - Dynamics of a Rolling Wheel

Introduction

In this chapter, the dynamics of the rolling wheel of the unicycle with one actuator is found, following both Newton-Euler analysis and Lagrangian analysis. As mentioned earlier, due to the complexity of the system, only the wheel is considered first here. A matching set of equations of motion is found at the end of each method.

Dynamics of the Rolling Wheel - Newton Euler Analysis

Figure 2.1 illustrates the reference frames used to model the wheel orientation, geometry of the rolling wheel, and the external forces acting on it. The coordinate system (XYZ) represents the global coordinate system and (X'Y'Z') represents the local coordinate system fixed at the center of the rolling wheel. This frame is oriented with a yaw angle of ϕ about Z axis, and a pitch angle of θ about X' axis. The wheel roll about Y' axis and the roll angle is represented by ψ.

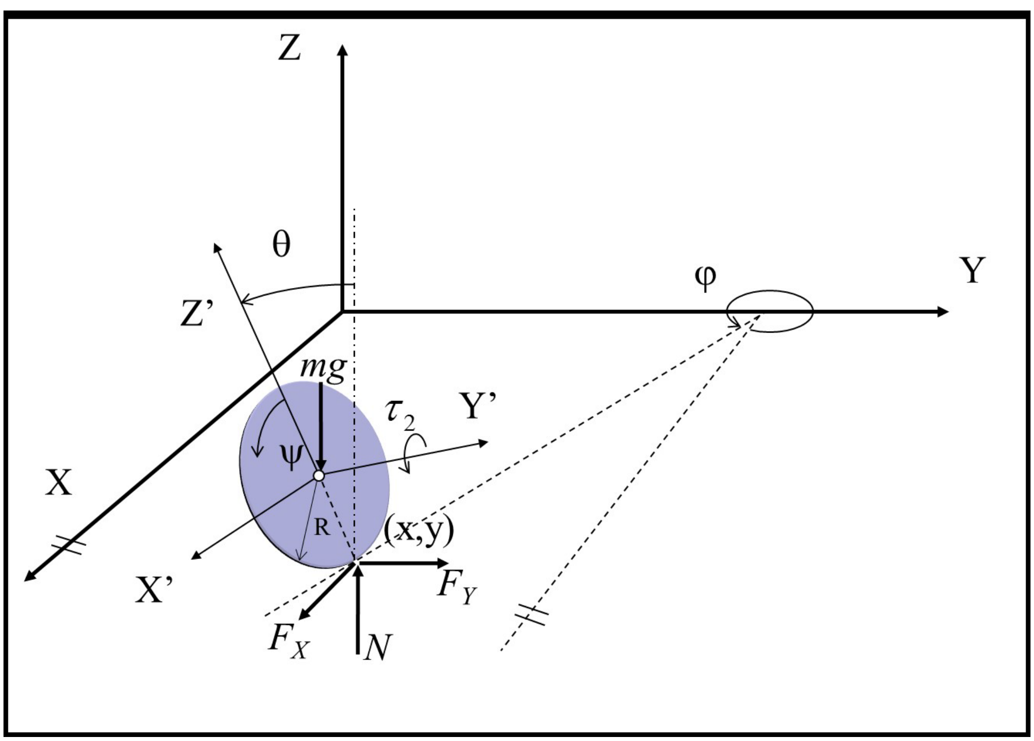

Figure 2.1. Geometry and Free Body Diagram of the Rolling Wheel

The wheel radius is R and it has a mass of m. The coordinates (x, y) represents the contact point of the wheel with the floor. The position coordinates (r_X,r_Y,r_Z) of the wheel center is defined with respective to the (XYZ) coordinate system as

$$r_X = x + R\sin(\phi)\sin(\theta), \quad \textbf{(2-1)}$$

$$r_Y = y - R\sin(\theta)\cos(\phi), \quad \textbf{(2-2)}$$

and

$$r_Z = R\cos(\theta). \quad \textbf{(2-3)}$$

The time derivatives of the position of the wheel center is calculated to obtain the velocity components

$$v_X = \dot{x} + R\cos(\phi)\sin(\theta)\,\dot{\phi} + R\sin(\phi)\sin(\theta)\dot{\theta}, \quad \textbf{(2-4)}$$

$$v_Y = \dot{y} + R\sin(\phi)\sin(\theta)\,\dot{\phi} - R\cos(\phi)\cos(\theta)\,\dot{\theta}, \quad \textbf{(2-5)}$$

and

$$v_Z = -R\sin(\theta)\dot{\theta}. \quad \textbf{(2-6)}$$

In above equations, $\dot{\phi}$ represents the time rate of change of yaw angle while, $\dot{\theta}$ represents the rate of change of pitch angle of the wheel. The wheel satisfies the non-holonomic constraints [15] defined by

$$\dot{x} = R\cos(\phi)\,\dot{\psi} \quad \textbf{(2-7)}$$

and

$$\dot{y} = R\sin(\phi)\,\dot{\psi}. \quad \textbf{(2-8)}$$

Here, $\dot{\psi}$ is the time rate of change of roll angle of the wheel. The acceleration components of the wheel center, relative to the global coordinate system, are found by taking the time derivative of the velocities of the wheel center defined by (2-4) through (2-8) to achieve

$$a_X = R\Big(2\cos(\phi)\cos(\theta)\,\dot{\phi}\dot{\theta} - \sin(\phi)\sin(\theta)\,\dot{\phi}^2 - \sin(\phi)\sin(\theta)\,\dot{\theta}^2 + \cos(\phi)\sin(\theta)\,\ddot{\phi} + \sin(\phi)\cos(\theta)\,\ddot{\theta} - \sin(\phi)\,\dot{\phi}\dot{\psi} + \cos(\phi)\,\ddot{\psi}\Big), \quad \textbf{(2-9)}$$

$$a_Y = R\left(\left(\left(-\dot{\phi}^2 - \dot{\theta}^2\right)\sin(\theta) + \cos(\theta)\,\ddot{\theta} - \dot{\phi}\dot{\psi}\right)\cos(\phi) - 2\sin(\phi)\left(\cos(\theta)\,\dot{\phi}\dot{\theta} + \frac{1}{2}\sin(\theta)\,\ddot{\phi} + \frac{\ddot{\psi}}{2}\right)\right), \quad \textbf{(2-10)}$$

and $$a_Z = -R\left(\cos(\theta)\,\dot{\theta}^2 + \sin(\theta)\,\ddot{\theta}\right). \quad \textbf{(2-11)}$$

The notations $\ddot{\psi}$, $\ddot{\theta}$ and $\ddot{\phi}$ represents the roll, pitch and yaw angular accelerations of the wheel respectively in (2-9), (2-10) and (2-11).

The angular velocity components of the wheel with respective to the wheel centered frame are

$$\omega_{X'} = \dot{\theta}, \quad \textbf{(2-12)}$$

$$\omega_{Y'} = \dot{\psi} + \sin(\theta)\,\dot{\phi}, \quad \textbf{(2-13)}$$

and $$\omega_{Z'} = \cos(\theta)\,\dot{\phi}. \quad \textbf{(2-14)}$$

The angular velocity vector of the wheel is

$$\mathbf{\Omega} = \begin{bmatrix} \omega_{X'} \\ \omega_{Y'} \\ \omega_{Z'} \end{bmatrix}. \quad \textbf{(2-15)}$$

The calculations carried out to obtain these velocity components are described in the Appendix A. Angular acceleration components of the wheel in the wheel centered coordinate system are

$$\alpha_{X'} = \ddot{\theta}, \quad \textbf{(2-16)}$$

$$\alpha_{Y'} = \ddot{\psi} + \ddot{\phi}\sin(\theta) + \dot{\theta}\dot{\phi}\cos(\theta), \quad \textbf{(2-17)}$$

and $$\alpha_{Z'} = \ddot{\phi}\cos(\theta) - \dot{\phi}\dot{\theta}\sin(\theta). \quad \textbf{(2-18)}$$

These are obtained by taking the time derivative of the angular velocity components of the wheel, defined by (2-12), (2-13) and (2-14). The angular acceleration vector of the wheel is

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{X'} \\ \alpha_{Y'} \\ \alpha_{Z'} \end{bmatrix}. \quad \textbf{(2-19)}$$

The inertia of the wheel along the principal axes in the (X’Y’Z’) frame are

$$\boldsymbol{I_w} = \begin{bmatrix} I_{X'X'} & 0 & 0 \\ 0 & I_{Y'Y'} & 0 \\ 0 & 0 & I_{Z'Z'} \end{bmatrix} = \begin{bmatrix} I_{w1} & 0 & 0 \\ 0 & I_{w2} & 0 \\ 0 & 0 & I_{w3} \end{bmatrix} \quad \textbf{(2-20)}$$

The unknown reactions F_X, F_Y and N at the contact point of the wheel with ground, are found by applying the Newton equation

$$F = ma, \tag{2-21}$$

to the wheel center in global frame. They are

$$F_X = ma_X, \tag{2-22}$$

$$F_Y = ma_Y, \tag{2-23}$$

and

$$N = ma_Z + mg. \tag{2-24}$$

These reaction forces are included in the Appendix A. The Euler equation of the rolling wheel is calculated in (X'Y'Z') frame, using

$$\boldsymbol{M} = \dot{\boldsymbol{H}}. \tag{2-25}$$

Here, $\boldsymbol{M}$ is the external moment acting on the rigid body and $\dot{\boldsymbol{H}}$ is the time rate of change of angular momentum about the center of mass of the wheel. The contact point reactions found by applying the Newton's equation, are transformed to the (X'Y'Z') frame in order to determine the external moments ($\boldsymbol{M}$) applied to the system. $\boldsymbol{M}$ is calculated by taking the cross product between, the wheel contact point vector in local frame and the contact point reaction forces vector in local frame. The calculations and vectors related to this can be found in Appendix A. The calculated external moment applied to the wheel in the (X'Y'Z') frame is given by the vector

$$\boldsymbol{M} = \begin{bmatrix} R(-F_X\cos(\theta)\sin(\phi) + F_Y\cos(\theta)\cos(\phi) + N\sin(\theta)) \\ -R(F_X\cos(\phi) + F_Y\sin(\phi)) \\ 0 \end{bmatrix}. \tag{2-26}$$

Using the angular accelerations, inertia and angular velocities defined previously, the rate of change of angular momentum is calculated for the wheel as

$$\dot{\boldsymbol{H}} = (\dot{\boldsymbol{H}})_{\boldsymbol{r}} + \boldsymbol{\omega} \times \boldsymbol{H}. \tag{2-27}$$

Here, $(\dot{\boldsymbol{H}})_{\boldsymbol{r}}$ is the rate of change of angular momentum with respect to the center of the wheel, as viewed by an observer on the moving (X'Y'Z') frame, which is calculated by multiplying the angular accelerations in (2-16) through (2-18) with inertia values in (2-20). The vector

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_{X'} \\ \omega_{Y'} - \dot{\psi} \\ \omega_{Z'} \end{bmatrix} \quad \textbf{(2-28)}$$

is the angular velocity vector of the frame (X'Y'Z') fixed at the wheel center. The angular momentum $(\boldsymbol{H})$ of the wheel in wheel centered frame is calculated by

$$\boldsymbol{H} = \boldsymbol{I_w} \times \boldsymbol{\Omega}. \quad \textbf{(2-29)}$$

The equation (2-25) was modified to include the known torque (τ_2) applied to the rolling wheel to show

$$\dot{\boldsymbol{H}} = \boldsymbol{M} + \begin{bmatrix} 0 \\ \tau_2 \\ 0 \end{bmatrix}. \quad \textbf{(2-30)}$$

The Euler equations calculated from (2-30) are

$$(R^2m + I_{w2})\sin(\theta)\,\ddot{\phi} + (R^2m + I_{w2})\,\ddot{\psi} + 2\left(R^2m + \frac{I_{w2}}{2}\right)\cos(\theta)\,\dot{\phi}\dot{\theta} - \tau_2 = 0, \quad \textbf{(2-31)}$$

$$(R^2m + I_{w1})\,\ddot{\theta} - (R^2m - I_{w1} + I_{w2})\cos(\theta)\sin(\theta)\,\dot{\phi}^2 - (R^2m + I_{w2})\dot{\phi}\dot{\psi}\cos(\theta) - mgR\sin(\theta) = 0, \quad \textbf{(2-32)}$$

and

$$I_{w1}\cos(\theta)\,\ddot{\phi} + \left(I_{w2}\,\dot{\psi} - 2\left(I_{w1} - \frac{I_{w2}}{2}\right)\sin(\theta)\,\dot{\phi}\right)\dot{\theta} = 0. \quad \textbf{(2-33)}$$

In order to obtain a symmetric mass matrix, a transformation was applied to the above equations. Therefore, (2-31) was multiplied by a $\sin(\theta)$ term and, (2-33) was multiplied by a $\cos(\theta)$ term and added them together to obtain the new equation

$$\begin{aligned} &((I_{w1} - I_{w2} - mR^2)\cos^2(\theta) + mR^2 + I_{w2})\ddot{\phi} + \sin(\theta)\,(I_{w2} + mR^2)\ddot{\psi} \\ &+ 2\cos(\theta)\,\dot{\theta}\left((mR^2 - I_{w1} + I_{w2})\sin(\theta)\,\dot{\phi} + \frac{I_{w2}\dot{\psi}}{2}\right) \\ &- \tau_2\sin(\theta) = 0. \end{aligned} \quad \textbf{(2-34)}$$

These dynamic equations are expressed in the matrix form at the end of this chapter.

Dynamics of the Rolling Wheel – Lagrangian Analysis

The Lagrangian (L) calculated for the same system illustrated in Figure 2.1 is

$$L = T - V, \tag{2-35}$$

where T is the total kinetic energy and V is the gravitational potential energy of the rolling wheel. Using the kinematic parameters found in previous section T and V are

$$\begin{aligned} T = \frac{1}{2}\, m\, &\Big(\big(\dot{x} + R\sin(\theta)\cos(\phi)\,\dot{\phi} + R\sin(\phi)\cos(\theta)\,\dot{\theta}\big)^2 \\ &+ \big(\dot{y} + R\sin(\phi)\sin(\theta)\,\dot{\phi} - R\cos(\theta)\cos(\phi)\,\dot{\theta}\big)^2 + R^2\sin(\theta)^2\,\dot{\theta}^2\Big) \\ &+ \frac{1}{2}\Big(I_{w1}\dot{\theta}^2 + I_{w2}\big(\dot{\psi} + \sin(\theta)\,\dot{\phi}\big)^2 + I_{w3}\cos(\theta)^2\,\dot{\phi}^2\Big) \end{aligned} \tag{2-36}$$

and

$$V = mgR\cos(\theta). \tag{2-37}$$

Since, the no slip condition is required for this problem the non-holonomic constraints for this system are

$$\dot{x} = R\cos(\phi)\,\dot{\psi} \tag{2-38}$$

and

$$\dot{y} = R\,\sin(\phi)\dot{\psi}. \tag{2-39}$$

The Lagrange dynamic equations for the wheel are obtained by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = Q_i + \sum_{k=1}^{n} \lambda_k a_{ki} \tag{2-40}$$

Here, Q_i are the generalized forces applied along the generalized coordinates denoted by q_i and λ_k are the Lagrange multipliers. The generalized forces obtained from Newton-Euler method are

$$Q_\psi = \tau_2, \tag{2-41}$$

$$Q_\theta = 0, \tag{2-42}$$

and

$$Q_\phi = \tau_2\sin(\theta). \tag{2-43}$$

After calculating the derivatives in (2-40), the first time derivatives of the non-holonomic constraints of (2-38) and (2-39) are substituted in order to find the Lagrange multipliers, λ_k. Finally, the Lagrange dynamics equations for the wheel are obtained and they exactly match the equations found in the Newton-Euler analysis. The calculations can be found in the Appendix A. The final dynamic equations are

$$(R^2m + I_{w2})\sin(\theta)\,\ddot{\phi} + (R^2m + I_{w2})\,\ddot{\psi} + 2\left(R^2m + \frac{I_{w2}}{2}\right)\cos(\theta)\,\dot{\phi}\dot{\theta} - \tau_2 = 0, \qquad \textbf{(2-44)}$$

$$(R^2m + I_{w1})\,\ddot{\theta} - (R^2m - I_{w1} + I_{w2})\cos(\theta)\sin(\theta)\;\dot{\phi}^2 - (R^2m + I_{w2})\dot{\phi}\dot{\psi}\cos(\theta) - mgR\sin(\theta) = 0, \qquad \textbf{(2-45)}$$

and

$$(I_{w2} + mR^2 + (I_{w3} - I_{w2} - mR^2)\cos(\theta)^2)\ddot{\phi} + (R^2m + I_{w2})\ddot{\psi}\sin(\theta) + 2\left(\frac{I_{w2}}{2}\dot{\psi} + (I_{w2} - I_{w3} + mR^2)\dot{\phi}\sin(\theta)\right)\cos(\theta)\,\dot{\theta} - \tau_2\sin(\theta) = 0. \qquad \textbf{(2-46)}$$

The equations (2-44) through (2-46) can be arranged in the matrix form as

$$\boldsymbol{Mass_w}(\ddot{q}_i) + \boldsymbol{C_w}(\dot{q}_i, q_i) + \boldsymbol{G_w}(q) - \boldsymbol{Q_i} = 0, \qquad \textbf{(2-47)}$$

where,

$$\boldsymbol{Mass_w} = \begin{bmatrix} mR^2 + I_{w2} & 0 & \left(mR^2 + I_{w2}\right)\sin(\theta) \\ 0 & mR^2 + I_{w1} & 0 \\ \left(mR^2 + I_{w2}\right)\sin(\theta) & 0 & mR^2 + I_{w2} + \left(I_{w3} - I_{w2} - mR^2\right)\cos^2(\theta) \end{bmatrix}, \qquad \textbf{(2-48)}$$

$$\boldsymbol{C_w} = \begin{bmatrix} \left(2mR^2 + I_{w2} + I_{w1} - I_{w3}\right)\cos(\theta)\,\dot{\phi}\dot{\theta} \\ -\left(\sin(\theta)\left(mR^2 - I_{w1} + I_{w2}\right)\dot{\phi} + \dot{\psi}\left(mR^2 + I_{w2}\right)\right)\cos(\theta)\,\dot{\phi} \\ 2\cos(\theta)\left(\left(mR^2 - I_{w1} + I_{w2}\right)\sin(\theta)\,\dot{\phi} + \dfrac{I_{w2}\dot{\psi}}{2}\right)\dot{\theta} \end{bmatrix}, \qquad \textbf{(2-49)}$$

$$\boldsymbol{G_w} = \begin{bmatrix} 0 \\ -mgR\sin(\theta) \\ 0 \end{bmatrix}, \qquad \textbf{(2-50)}$$

and

$$\boldsymbol{Q_i} = \begin{bmatrix} \tau_2 \\ 0 \\ \tau_2\sin(\theta) \end{bmatrix}. \qquad \textbf{(2-51)}$$

Analysis carried out to find the dynamic equations of the rolling wheel in this chapter, using both Newton-Euler and Lagrangian methods, confirms the accuracy of the final set of equations (2-44) through (2-46). Since, these two methods are independent from one another, it is fair to state that they are reliable.

Chapter 3 - Dynamics of the Rolling Wheel and Frame

Introduction

In this chapter, the dynamic equations for the rolling wheel and the frame are calculated. The frame is attached to the rolling wheel's axel, such a way that it is free to rotate independently about the Y' axis. Figure 3.1 illustrates this assembly and the coordinate systems used to describe the orientation. The frame is considered as a rod for the calculations. A new local frame (X"Y"Z") attached to the rod is introduced here. This frame rotates with the rod, about the Y' axis and it creates an angle of β with Z' axis.

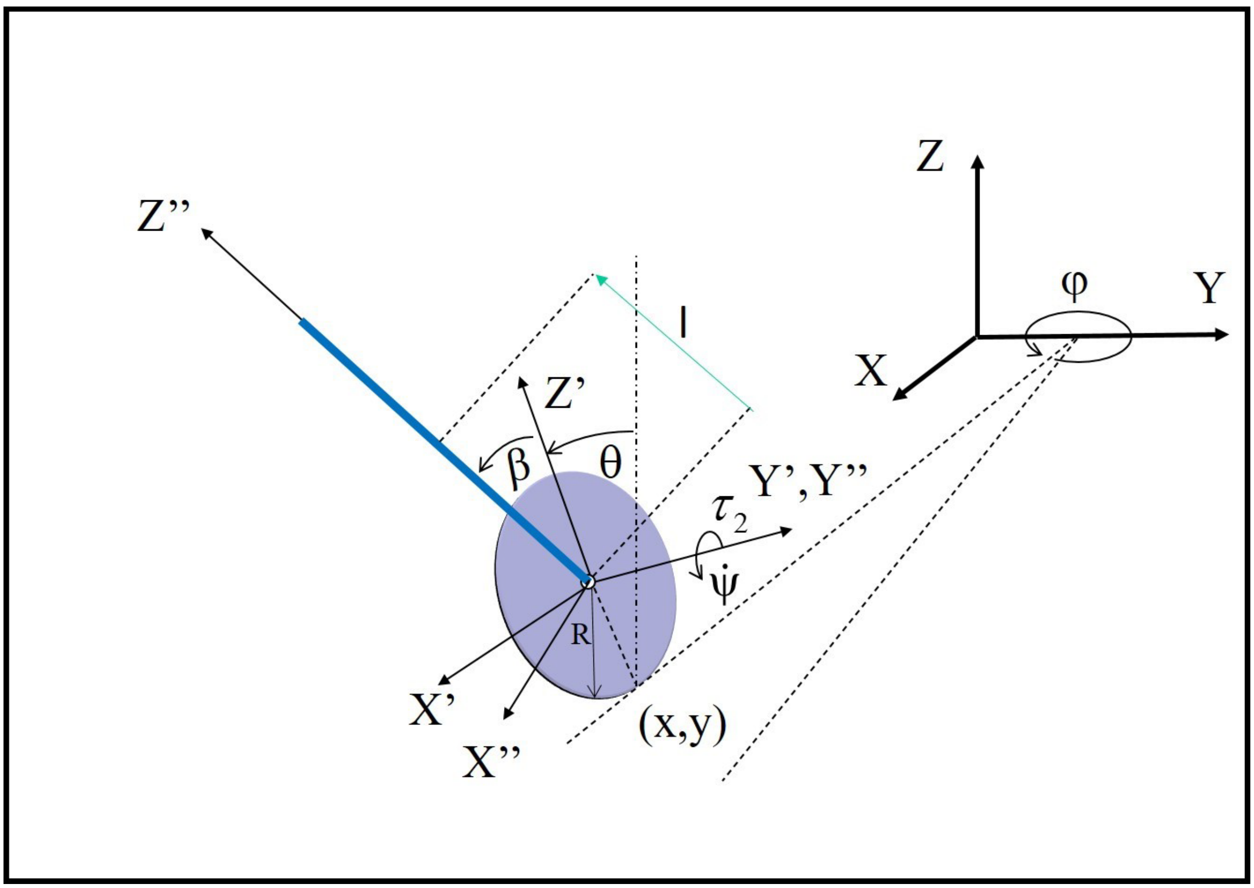

Figure 3.1. Orientation of the Rolling Wheel and Frame

As in Chapter 2, both Newton-Euler and Lagrangian methods were used to find the dynamic equations for this system. A matching set of equations are obtained at the end of the chapter confirming the accuracy of the calculations. Since, the kinematics related to the wheel are same as in Chapter 2, they are not repeated here.

Dynamics of the Wheel and Frame – Newton Euler Analysis

The free body diagram used for this analysis is shown in Figure 3.2. The frame is detached from the rolling wheel's axel in this figure. The forces acting on the rod and wheel at the joint marked here.

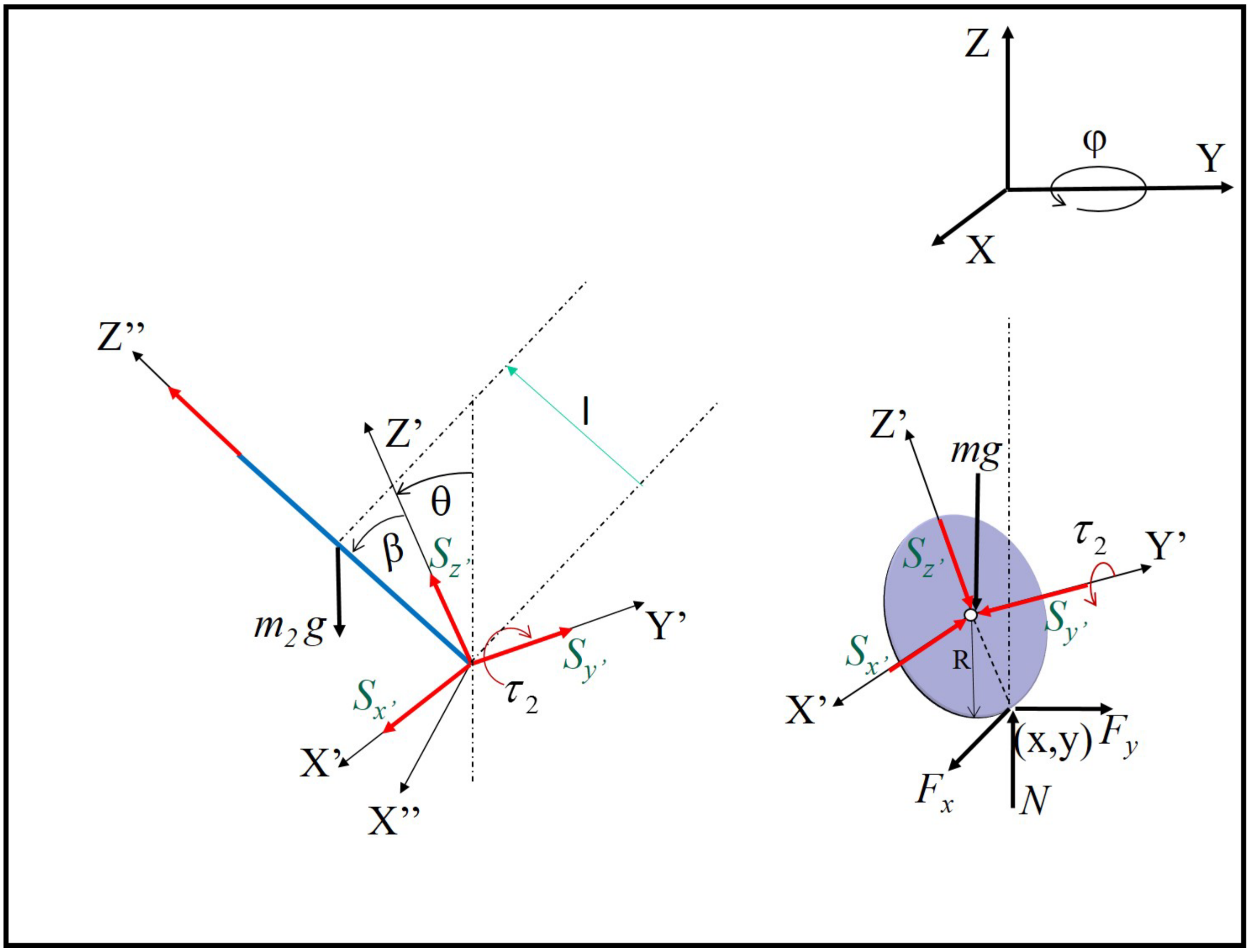

Figure 3.2. Free Body Diagram of Wheel and Rod

The rod has a mass of m_2 and the distance l to the center of mass from the joint is measured along the Z" axis. The coordinates of the rod mass center (l_X, l_Y, l_Z) with respective to the (XYZ) frame are

$$l_X = (\cos(\phi)\sin(\beta) + \sin(\phi)\sin(\theta)\cos(\beta))l + \sin(\phi)\sin(\theta)R + x, \tag{3-1}$$

$$l_Y = (\sin(\phi)\sin(\beta) - \cos(\phi)\sin(\theta)\cos(\beta))l - \cos(\phi)\sin(\theta)R + y, \tag{3-2}$$

and

$$l_Z = \cos(\theta)\cos(\beta)l + \cos(\theta)R. \tag{3-3}$$

The velocity of the rod mass center in reference to the (XYZ) frame obtained by taking the first time derivative of (3-1) through (3-3). The velocity components are

$$v_{lX} = \dot{x} + \big((-\sin(\phi)\sin(\beta) + \cos(\phi)\sin(\theta)\cos(\beta))l + \cos(\phi)\sin(\theta)R\big)\dot{\phi} + (\sin(\phi)\cos(\theta)\cos(\beta)l + \sin(\phi)\cos(\theta)R)\dot{\theta} + (\cos(\phi)\cos(\beta) - \sin(\phi)\sin(\theta)\sin(\beta))l\dot{\beta}, \tag{3-4}$$

$$v_{lY} = \dot{y} + \big((\cos(\phi)\sin(\beta) + \sin(\phi)\sin(\theta)\cos(\beta))l + \sin(\phi)\sin(\theta)R\big)\dot{\phi} + (-\cos(\phi)\cos(\theta)\cos(\beta)l - \cos(\phi)\cos(\theta)R)\dot{\theta} + (\sin(\phi)\cos(\beta) + \cos(\phi)\sin(\theta)\sin(\beta))l\dot{\beta}, \tag{3-5}$$

and

$$v_{lZ} = (-\sin(\theta)\cos(\beta)l - \sin(\theta)R)\dot{\theta} - \cos(\theta)\sin(\beta)l\dot{\beta}. \tag{3-6}$$

Then, to obtain the acceleration of the rod mass center in (XYZ) frame, the time derivatives of the velocity components in (3-4) through (3-6) calculated. They are

$$\begin{aligned} a_{lX} = \sin(\phi)\Bigg(&\sin(\theta)\Big(-l\big(\dot{\beta}^2 + \dot{\phi}^2 + \dot{\theta}^2\big)\cos(\beta) - \ddot{\beta}lsin(\beta) \\ &- R\big(\dot{\phi}^2 + \dot{\theta}^2\big)\Big) + l\big(\cos(\theta)\ddot{\theta} - 2\dot{\phi}\dot{\beta}\big)\cos(\beta) \\ &- 2l\left(\dot{\theta}\cos(\theta)\dot{\beta} + \frac{\ddot{\phi}}{2}\right)\sin(\beta) + R\big(\cos(\theta)\ddot{\theta} - \dot{\phi}\dot{\psi}\big)\Bigg) \\ &+ \cos(\phi)\Bigg(\big(-2\sin(\beta)l\dot{\phi}\dot{\beta} + l\ddot{\phi}\cos(\beta) + R\ddot{\phi}\big)\sin(\theta) \\ &+ 2l\left(\cos(\theta)\dot{\phi}\dot{\theta} + \frac{\ddot{\beta}}{2}\right)\cos(\beta) - l\big(\dot{\beta}^2 + \dot{\phi}^2\big)\sin(\beta) \\ &+ R\big(2\cos(\theta)\dot{\phi}\dot{\theta} + \ddot{\psi}\big)\Bigg), \end{aligned} \tag{3-7}$$

$$a_{lY} = \cos(\phi)\Bigg(\sin(\theta)\left(l\left(\dot{\beta}^2 + \dot{\phi}^2 + \dot{\theta}^2\right)\cos(\beta) + \ddot{\beta}lsin(\beta) + R\left(\dot{\phi}^2 + \dot{\theta}^2\right)\right) - l\left(\cos(\theta)\,\ddot{\theta} - 2\dot{\phi}\dot{\beta}\right)\cos(\beta) + 2l\left(\dot{\theta}\cos(\theta)\,\dot{\beta} + \frac{\ddot{\phi}}{2}\right)\sin(\beta) - R\left(\cos(\theta)\,\ddot{\theta} - \dot{\phi}\dot{\psi}\right)\Bigg) + \sin(\phi)\Bigg(\left(-2\sin(\beta)\,l\dot{\phi}\dot{\beta} + l\ddot{\phi}\cos(\beta) + R\ddot{\phi}\right)\sin(\theta) + 2l\left(\cos(\theta)\,\dot{\phi}\dot{\theta} + \frac{\ddot{\beta}}{2}\right)\cos(\beta) - l\left(\dot{\beta}^2 + \dot{\phi}^2\right)\sin(\beta) + R\left(2\cos(\theta)\,\dot{\phi}\dot{\theta} + \ddot{\psi}\right)\Bigg), \tag{3-8}$$

and

$$a_{lZ} = \left(-l\left(\dot{\beta}^2 + \dot{\theta}^2\right)\cos(\beta) - R\dot{\theta}^2 - \ddot{\beta}lsin(\beta)\right)\cos(\theta) - \sin(\theta)\left(-2\sin(\beta)\,l\dot{\beta}\dot{\theta} + l\ddot{\theta}\cos(\beta) + R\ddot{\theta}\right). \tag{3-9}$$

The angular velocity vector of the rod in (X”Y”Z”) frame is

$$\boldsymbol{\Omega_l} = \begin{bmatrix}\omega_{lX"}\\ \omega_{lY"}\\ \omega_{lZ"}\end{bmatrix}, \tag{3-10}$$

and the angular velocity components are

$$\omega_{lX"} = \dot{\theta}\cos(\beta) - \sin(\beta)\cos(\theta)\,\dot{\phi}, \tag{3-11}$$

$$\omega_{lY"} = \dot{\beta} + \sin(\theta)\,\dot{\phi}, \tag{3-12}$$

and

$$\omega_{lZ"} = \cos(\beta)\cos(\theta)\,\dot{\phi} + \dot{\theta}\sin(\beta). \tag{3-13}$$

The calculation carried out to obtain these angular velocity components are provided in Appendix B. The angular acceleration vector of the rod in (X”Y”Z”) frame,

$$\boldsymbol{\alpha_l} = \begin{bmatrix}\alpha_{lX"}\\ \alpha_{lY"}\\ \alpha_{lZ"}\end{bmatrix} \tag{3-14}$$

is obtained by taking the first time derivative of (3-11) through (3-13). The angular acceleration components are

$$\alpha_{lX"} = \cos(\beta)\,\ddot{\theta} + \sin(\beta)\sin(\theta)\,\dot{\phi}\dot{\theta} - \cos(\theta)\sin(\beta)\,\ddot{\phi} + \left(-\dot{\theta}\sin(\beta) - \cos(\beta)\cos(\theta)\,\dot{\phi}\right)\dot{\beta}, \tag{3-15}$$

$$\alpha_{lY"} = \ddot{\beta} + \cos(\theta)\,\dot{\phi}\dot{\theta} + \sin(\theta)\,\ddot{\phi}, \tag{3-16}$$

and

$$\alpha_{lZ"} = \sin(\beta)\,\ddot{\theta} - \cos(\beta)\sin(\theta)\,\dot{\phi}\dot{\theta} + \cos(\theta)\cos(\beta)\,\ddot{\phi} + \left(\dot{\theta}\cos(\beta) - \sin(\beta)\cos(\theta)\,\dot{\phi}\right)\dot{\beta}. \tag{3-17}$$

The inertia matrix of the rod in (X"Y"Z") frame along its principal axes is

$$\boldsymbol{I_l} = \begin{bmatrix} I_{X"X"} & 0 & 0 \\ 0 & I_{Y"Y"} & 0 \\ 0 & 0 & I_{Z"Z"} \end{bmatrix} = \begin{bmatrix} I_{l1} & 0 & 0 \\ 0 & I_{l2} & 0 \\ 0 & 0 & I_{l3} \end{bmatrix}. \tag{3-18}$$

The unknown reactions $S_{X'}, S_{Y'}, S_{Z'}$ at the joint and contact point reactions F_X,F_Y, and F_Z are calculated by applying (2-21) for the rolling wheel and the rod separately. It's applied in (X'Y'Z') frame for the wheel and in (X"Y"Z") frame for the rod. The equations achieved for the wheel are,

$$ma_{X'} - \cos(\phi)\,F_X - \sin(\phi)\,F_Y + S_{X'} = 0, \tag{3-19}$$

$$ma_{Y'} + \sin(\phi)\cos(\theta)\,F_X - \cos(\phi)\cos(\theta)\,F_Y - \sin(\theta)\,N + S_{Y'} + mgsin(\theta) = 0, \tag{3-20}$$

and

$$ma_{Z'} + mg\cos(\theta) - \sin(\phi)\sin(\theta)\,F_X + \cos(\phi)\sin(\theta)\,F_Y - \cos(\theta)\,N + S_{Z'} = 0. \tag{3-21}$$

The equations acquired for the rod are,

$$m_2 a_{lX"} - \cos(\beta)\,S_{X'} - m_2 g\cos(\theta)\sin(\beta) + \sin(\beta)\,S_{Z'} = 0, \tag{3-22}$$

$$m_2 a_{lY"} - S_{Y'} + m_2 g\sin(\theta) = 0, \tag{3-23}$$

and

$$m_2 a_{lZ"} - \cos(\beta)\,S_{Z'} - \sin(\beta)\,S_{X'} + m_2 g\cos(\theta)\cos(\beta) = 0. \tag{3-24}$$

The accelerations and the solved force components in (3-19) through (3-24) are provided in Appendix B. To obtain the Euler equations, (2-25) applied for both wheel and frame. The same procedure described in Chapter 2 is followed to obtain the external moment and the rate of change of angular momentum for the wheel. Since it is a repetition of the equations derived already, those equations are not described here. The external moment applied to the rod is calculated by finding the moments about the center of mass of the rod in (X"Y"Z") frame. The external moment vector of the rod is

$$\boldsymbol{M_l} = \begin{bmatrix} lS_{Y'} \\ -l(\cos(\beta)\, S_{X'} - \sin(\beta)\, S_{Z'} \\ 0 \end{bmatrix}. \quad \textbf{(3-25)}$$

The calculation carried out to obtain (3-25) is explained in Appendix B. To find the rate of change of angular momentum of the rod, (2-27) is modified as

$$\dot{\boldsymbol{H}}_{\boldsymbol{l}} = \left(\dot{\boldsymbol{H}}\right)_{\boldsymbol{rl}} + \boldsymbol{\Omega_l} \times \boldsymbol{H_l}, \quad \textbf{(3-26)}$$

where, $\left(\dot{\boldsymbol{H}}\right)_{\boldsymbol{rl}}$ is the rate of change of angular momentum of the rod, with respect to the center of mass of rod as viewed by an observer on the moving (X"Y"Z") frame. The calculation of $\left(\dot{\boldsymbol{H}}\right)_{\boldsymbol{rl}}$ is carried out as

$$\left(\dot{\boldsymbol{H}}\right)_{\boldsymbol{rl}} = \boldsymbol{\alpha_l} \times \boldsymbol{I_l}. \quad \textbf{(3-27)}$$

The angular momentum of the rod $\boldsymbol{H_l}$ is calculated by

$$\boldsymbol{H_l} = \boldsymbol{I_l} \times \boldsymbol{\Omega_l}. \quad \textbf{(3-28)}$$

Since, an equal and opposite torque applied to the rod, (2-25) can be applied to the rod as

$$\dot{\boldsymbol{H}}_{\boldsymbol{l}} = \boldsymbol{M_l} + \begin{bmatrix} 0 \\ -\tau_2 \\ 0 \end{bmatrix}. \quad \textbf{(3-29)}$$

The Euler equations obtained for the wheel are,

$$\begin{aligned} 2\left(R\cos(\beta) l m_2 + (m+m_2)R^2 + \frac{I_{w1}}{2} + \frac{I_{w2}}{2} - \frac{I_{w3}}{2}\right)\dot{\theta}\dot{\phi}\cos(\theta) \\ + \left(R\ddot{\phi} l m_2 \cos(\beta) - 2R\dot{\phi}\dot{\beta}\sin(\beta)\, l m_2 \right. \\ \left. + \left((m+m_2)R^2 + I_{w2}\right)\ddot{\phi}\right)\sin(\theta) + \cos(\beta)\, R l m_2 \ddot{\beta} \\ - R l m_2\left(\dot{\beta}^2 + \dot{\phi}^2\right)\sin(\beta) + \ddot{\psi}(m+m_2)R^2 + I_{w2}\ddot{\psi} - \tau_2 \\ = 0, \end{aligned} \quad \textbf{(3-30)}$$

$$\Big(-\dot{\phi}^2(R\cos(\beta)\,lm_2 + (m+m_2)R^2 + I_{w2} - I_{w3})\sin(\theta) \tag{3-31}$$
$$- 2R\dot{\phi}\dot{\beta}\cos(\beta)\,lm_2 - R\ddot{\phi}lm_2\sin(\beta)$$
$$- \dot{\phi}\dot{\psi}\big((m+m_2)R^2 + I_{w2}\big)\Big)\cos(\theta) - Rg(m+m_2)\sin(\theta)$$
$$+ \cos(\beta)\,Rlm_2\ddot{\theta} - 2\sin(\beta)\,R\dot{\beta}lm_2\dot{\theta}$$
$$+ \ddot{\theta}\big((m+m_2)R^2 + I_{w1}\big) = 0,$$

and

$$-\dot{\phi}\dot{\theta}(I_{w1} - I_{w2} + I_{w3})\sin(\theta) + I_{w2}\dot{\psi}\dot{\theta} + I_{w3}\cos(\theta)\,\ddot{\phi} = 0. \tag{3-32}$$

The Euler equations retrieved for the frame using (3-29) are,

$$-\sin(\beta)\,\dot{\phi}^2\big((l^2m_2 + I_{l1} - I_{l3})\cos(\beta) + Rlm_2\big)\cos(\theta)^2 \tag{3-33}$$
$$+ (2\dot{\phi}\dot{\theta}(l^2m_2 + I_{l1} - I_{l3})\cos(\beta)^2 + 2R\dot{\phi}\dot{\theta}\cos(\beta)\,lm_2$$
$$- \sin(\beta)\,glm_2 - \dot{\phi}\dot{\theta}(I_{l1} - I_{l2} - I_{l3}))\cos(\theta) + (\dot{\theta}^2(l^2m_2$$
$$+ I_{l1} - I_{l3})\sin(\beta) + Rlm_2(\sin(\theta)\,\ddot{\phi} + \ddot{\psi}))\cos(\beta)$$
$$+ Rlm_2(\sin(\theta)\,\dot{\phi}\dot{\psi} + \dot{\theta}^2 + \dot{\phi}^2)\sin(\beta) + \ddot{\phi}(l^2m_2$$
$$+ I_{l2})\sin(\theta) + l^2m\_2\ddot{\beta} + I_{l2}\ddot{\beta} + \tau_2 = 0,$$

$$(-\dot{\phi}(\dot{\phi}(l^2m_2 + I_{l2} - I_{l3})\sin(\theta) + 2\left(l^2m_2 + \frac{I_{l1}}{2} + \frac{I_{l2}}{2} - \frac{I_{l3}}{2}\right)\dot{\beta})\cos(\beta) \tag{3-34}$$
$$- \ddot{\phi}(l^2m_2 + I_{l1})\sin(\beta) - R\dot{\phi}lm_2(\sin(\theta)\,\dot{\phi} + \dot{\psi}))\cos(\theta)$$
$$+ \ddot{\theta}(l^2m_2 + I_{l1})\cos(\beta) + \dot{\theta}(\dot{\phi}(I_{l1} - I_{l2} + I_{l3})\sin(\theta)$$
$$- 2\left(l^2m_2 + \frac{I_{l1}}{2} + \frac{I_{l2}}{2} - \frac{I_{l3}}{2}\right)\dot{\beta})\sin(\beta) + lm\_2(R\ddot{\theta}$$
$$- g\sin(\theta)) = 0,$$

and

$$(\cos(\theta)\,I_{l3}\ddot{\phi} - \dot{\theta}(\dot{\phi}(I_{l1} - I_{l2} + I_{l3})\sin(\theta) + \dot{\beta}(I_{l1} - I_{l2} - I_{l3})))\cos(\beta) \tag{3-35}$$
$$+ (\dot{\phi}(\dot{\phi}(I_{l1} - I_{l2})\sin(\theta) + \dot{\beta}(I_{l1} - I_{l2} - I_{l3}))\cos(\theta)$$
$$+ \ddot{\theta}I_{l3})\sin(\beta) = 0.$$

To obtain a symmetric mass matrix two new equations obtained after few transformations were applied to the Euler equations found. First transformation applied is

$$(3-31) + ((3-34)\times\cos(\beta)) + ((3-35)\times\sin(\beta), \tag{3-36}$$

and the resulting equation is

$$
\begin{aligned}
&\Big(-\dot{\phi}(l^2m_2 + I_{l1} - I_{l3})\big(\sin(\theta)\,\dot{\phi} + 2\dot{\beta}\big)\cos(\beta)^2 \\
&+ \Big(-2R\dot{\phi}^2 \sin(\theta)\, lm_2 \;\; - \ddot{\phi}(l^2m_2 + I_{l1} - I_{l3})\sin(\beta) \\
&- R\dot{\phi}lm_2\big(\dot{\psi} + 2\dot{\beta}\big)\Big)\cos(\beta) \\
&- \dot{\phi}^2(R^2m + R^2m_2 - I_{l1} + I_{l2} + I_{w2} - I_{w3})\sin(\theta) \\
&- R\ddot{\phi}lm_2 \sin(\beta) \\
&- \big(R^2\dot{\psi}m + R^2\dot{\psi}m_2 + (-I_{l1} + I_{l2} + I_{l3})\dot{\beta} \\
&+ I_{w2}\dot{\psi})\dot{\phi}\big)\cos(\theta) + \ddot{\theta}(l^2m_2 + I_{l1} - I_{l3})\cos(\beta)^2 \\
&+ \big(-\sin(\theta)\, glm_2 - 2\dot{\beta}\dot{\theta}(l^2m_2 + I_{l1} - I_{l3})\sin(\beta) \\
&+ 2Rlm_2\ddot{\theta}\big)\cos(\beta) - Rg(m + \; m_2)\sin(\theta) \\
&- 2\sin(\beta)\, R\dot{\beta}lm_2\dot{\theta} + \ddot{\theta}(R^2m + R^2m_2 + I_{l3} + I_{w1}) = 0.
\end{aligned} \tag{3-37}
$$

The second transformation is

$$
\begin{aligned}
&((3-32)\times\cos(\theta)) + ((3-35)\times\cos(\beta)\cos(\theta)) \\
&- ((3-34)\times\sin(\beta)\cos(\theta)) + ((3-30)\times\sin(\theta)) \\
&+ \big((3-33)\times\sin(\theta)\big),
\end{aligned} \tag{3-38}
$$

And the resulting equation is

$$
\begin{aligned}
&\Big(-\ddot{\phi}(l^2m_2 + I_{l1} - I_{l3})\cos(\beta)^2 \\
&+ \big(2\dot{\phi}\dot{\beta}(l^2m_2 + I_{l1} - I_{l3})\sin(\beta) - 2R\ddot{\phi}lm_2\big)\cos(\beta) \\
&+ 2R\dot{\phi}\dot{\beta}\sin(\beta)\,lm_2 \\
&- \ddot{\phi}(R^2m + R^2m_2 - I_{l1} + I_{l2} + I_{w2} - I_{w3})\Big)\cos(\theta)^2 \\
&+ (2\dot{\theta}(l^2m_2 + I_{l1} - I_{l3})\big(\sin(\theta)\,\dot{\phi} - \dot{\beta}\big)\cos(\beta)^2 \\
&+ (4R\dot{\phi}\dot{\theta}lsin(\theta)m_2 - \ddot{\theta}\sin(\beta)(l^2m_2 + I_{l1} - I_{l3}))\cos(\beta) \\
&+ 2\dot{\phi}\dot{\theta}(R^2m + R^2m_2 - I_{l1} + I_{l2} + I_{w2} - I_{w3})\sin(\theta) \\
&- R\ddot{\theta}lsin(\beta)m_2 \\
&+ \big(2\beta l^2m_2 + (I_{l1} + I_{l2} - I_{l3})\dot{\beta} + I_{w2}\dot{\psi}\big)\dot{\theta})\cos(\theta) \\
&+ ((\dot{\theta}^2(l^2m_2 + I_{l1} - I_{l3})\sin(\beta) + Rlm\_2(\ddot{\psi} + \ddot{\beta}))\sin(\theta) \\
&+ 2R\ddot{\phi}lm_2)\cos(\beta) + (-Rlm_2(\dot{\beta} - \dot{\theta})(\dot{\beta} + \dot{\theta})\sin(\beta) \\
&+ \big(R^{2\ddot{\psi}} + l^2\ddot{\beta}\big)m_2 + \ddot{\psi}R^2m + I_{w2}\ddot{\psi} + I_{l2}\ddot{\beta})\sin(\theta) \\
&+ R\dot{\phi}lm_2(\dot{\psi} - 2\dot{\beta})\sin(\beta) \\
&+ \ddot{\phi}\big((R^2 + l^2)m_2 + R^2m + I_{l2} + I_{w2}\big) = 0.
\end{aligned}
\tag{3-39}
$$

The symmetric mass matrix then built using (3-30), (3-37), (3-39) and (3-33). This model built in matrix form is provided at the end of this chapter, since it is same for both Newton-Euler and Lagrange methods.

Dynamics of the Wheel and Frame – Lagrangian Analysis

The assembly displayed in Figure 3.1 is used to build the Lagrangian (L) utilizing (2-35). The total kinetic energy T for this assembly is given by

$$\begin{aligned}
T = \frac{m}{2}\Big\{&\big(\dot{x} + \cos(\phi)\sin(\theta)\,R\dot{\phi} + \sin(\phi)\cos(\theta)\,R\dot{\theta}\big)^2 \\
&+ \big(\dot{y} + \sin(\phi)\sin(\theta)\,R\dot{\phi} - \cos(\phi)\cos(\theta)\,R\dot{\theta}\big)^2 \\
&+ \sin(\theta)^2 R^2\dot{\theta}^2\Big\} + \frac{\dot{\theta}^2 I_{w1}}{2} + \frac{\big(\dot{\psi} + \sin(\theta)\,\dot{\phi}\big)^2 I_{w2}}{2} \\
&+ \frac{\cos(\theta)^2\,\dot{\phi}^2 I_{w3}}{2} \\
&+ \frac{m_2}{2}\Big(\big(\dot{x} \\
&+ \big((-\sin(\phi)\sin(\beta) + \cos(\phi)\sin(\theta)\cos(\beta))l \\
&+ \cos(\phi)\sin(\theta)\,R\big)\dot{\phi} \\
&+ (\sin(\phi)\cos(\theta)\cos(\beta)\,l + \sin(\phi)\cos(\theta)\,R)\dot{\theta} \\
&+ (\cos(\phi)\cos(\beta) - \sin(\phi)\sin(\theta)\sin(\beta))l\dot{\beta}\big)^2 \\
&+ \big(\dot{y} \\
&+ \big((\cos(\phi)\sin(\beta) + \sin(\phi)\sin(\theta)\cos(\beta))l \\
&+ \sin(\phi)\sin(\theta)\,R\big)\dot{\phi} \\
&+ (-\cos(\phi)\cos(\theta)\cos(\beta)\,l - \cos(\phi)\cos(\theta)\,R)\dot{\theta} \\
&+ (\sin(\phi)\cos(\beta) + \cos(\phi)\sin(\theta)\sin(\beta))l\dot{\beta}\big)^2 \\
&+ \big((-\sin(\theta)\cos(\beta)\,l - \sin(\theta)\,R)\dot{\theta} - \cos(\theta)\sin(\beta)\,l\dot{\beta}\big)^2\Big) \\
&+ \frac{\big(\dot{\theta}\cos(\beta) - \sin(\beta)\cos(\theta)\,\dot{\phi}\big)^2 I_{l1}}{2} + \frac{\big(\dot{\beta} + \sin(\theta)\,\dot{\phi}\big)^2 I_{l2}}{2} \\
&+ \frac{\big(\cos(\beta)\cos(\theta)\,\dot{\phi} + \dot{\theta}\sin(\beta)\big)^2 I_{l3}}{2}.
\end{aligned} \tag{3-40}$$

The kinematics defined in Chapter 2 and 3 used to calculate T. The total gravitational potential energy is

$$V = mgR\cos(\theta) + m_2 g(\cos(\theta)\cos(\beta)\, l + \cos(\theta)R). \tag{3-41}$$

The non-holonomic constraints defined in (2-38) and (2-39) applies to this assembly too.

The Lagrange dynamic equations are calculated using (2-40) again, and the generalized forces for this setup defined according to Newton-Euler method. They are,

$$Q_\psi = \tau_2, \tag{3-42}$$

$$Q_\theta = 0, \tag{3-43}$$

$$Q_\phi = 0, \tag{3-44}$$

and

$$Q_\beta = -\tau_2. \tag{3-45}$$

Following the same method in Chapter 2 the Lagrange dynamic equations are derived. They are

$$(\cos(\beta)\, R\ddot{\phi}lm_2 - 2\sin(\beta)\, R\dot{\beta}\dot{\phi}lm_2 + \big((m+m_2)R^2 + I_{w2}\big)\ddot{\phi})\sin(\theta) + 2\dot{\theta}\dot{\phi}(Rlm_2\cos(\beta) + (m+m_2)R^2 + \tfrac{I_{w2}}{2})\cos(\theta) + Rlm_2\ddot{\beta}\cos(\beta) - Rlm_2(\dot{\beta}^2 + \dot{\phi}^2)\sin(\beta) + \ddot{\psi}(m+m_2)R^2 + I_{w2}\ddot{\psi} - \tau_2 = 0, \tag{3-46}$$

$$(-\dot{\phi}\big(\sin(\theta)\,\dot{\phi} + 2\dot{\beta}\big)(l^2m_2 + I_{l1} - I_{l3})\cos(\beta)^2 + (-2R\dot{\phi}^2\sin(\theta)\, lm_2 - \ddot{\phi}(l^2m_2 + I_{l1} - I_{l3})\sin(\beta) - R\dot{\phi}lm_2(\dot{\psi} + 2\dot{\beta}))\cos(\beta) - \dot{\phi}^2(R^2m + R^2m_2 - I_{l1} + I_{l2} + I_{w2} - I_{w3})\sin(\theta) - R\ddot{\phi}\sin(\beta)\, lm_2 - \big(R^2\dot{\psi}m + R^2\dot{\psi}m_2 + (I_{l2} + I_{l3} - I_{l1})\dot{\beta} + I_{w2}\dot{\psi}\big)\dot{\phi})\cos(\theta) + \ddot{\theta}(l^2m_2 + I_{l1} - I_{l3})\cos(\beta)^2 + (-g\sin(\theta)\, lm_2 - 2\dot{\beta}\dot{\theta}(l^2m_2 + I_{l1} - I_{l3})\sin(\beta) + 2Rlm_2\ddot{\theta})\cos(\beta) - Rg(m+m_2)\sin(\theta) - 2\sin(\beta)\, R\dot{\beta}lm_2\dot{\theta} + \ddot{\theta}(R^2m + R^2m_2 + I_{l3} + I_{w1}) = 0, \tag{3-47}$$

$$\Big(-\ddot{\phi}(l^2m_2 + I_{l1} - I_{l3})\cos(\beta)^2 + \big(2\dot{\phi}\dot{\beta}(l^2m_2 + I_{l1} - I_{l3})\sin(\beta) - 2R\ddot{\phi}lm_2\big)\cos(\beta) + 2\sin(\beta)\, R\dot{\beta}\dot{\phi}lm_2 - \ddot{\phi}(R^2m + R^2m_2 - I_{l1} + I_{l2} + I_{w2} - I_{w3})\Big)\cos(\theta)^2 + (2\dot{\theta}(l^2m_2 + I_{l1} - I_{l3})\big(\sin(\theta)\,\dot{\phi} - \dot{\beta}\big)\cos(\beta)^2 + (4R\dot{\phi}\dot{\theta}l\sin(\theta)\, m_2 - \ddot{\theta}\sin(\beta)(l^2m_2 + I_{l1} - I_{l3}))\cos(\beta) + 2\dot{\phi}\dot{\theta}(R^2m + R^2m_2 - I_{l1} + I_{l2} + I_{w2} - I_{w3})\sin(\theta) - Rl\ddot{\theta}\sin(\beta)\, m_2 + \dot{\theta}(2\dot{\beta}l^2m_2 + (I_{l2} - I_{l3} + I_{l1})\dot{\beta} + I_{w2}\dot{\psi}))\cos(\theta) + ((\dot{\theta}^2(l^2m\_2 + I_{l1} - I_{l3})\sin(\beta) + \tag{3-48}$$

$$Rlm_2(\ddot{\psi}+\ddot{\beta}))\sin(\theta)+2R\ddot{\phi}lm_2)\cos(\beta)+(-Rlm_2(\dot{\beta}-\dot{\theta})(\dot{\beta}+\dot{\theta})\sin(\beta)+\left(R^2\ddot{\psi}+l^2\ddot{\beta}\right)m_2+R^2\ddot{\psi}m+I_{w2}\ddot{\psi}+I_{l2}\ddot{\beta})\sin(\theta)+R\dot{\phi}lm_2(\dot{\psi}-2\dot{\beta})\sin(\beta)+\left((R^2+l^2)m_2+R^2m+I_{l2}+I_{w2}\right)\ddot{\phi}=0,$$

and

$$-\dot{\phi}^2\left((l^2m_2+I_{l1}-I_{l3})\cos(\beta)+Rlm_2\right)\sin(\beta)\cos(\theta)^2+(2\dot{\phi}\dot{\theta}(l^2m_2+I_{l1}-I_{l3})\cos(\beta)^2+2R\dot{\phi}\dot{\theta}\cos(\beta)\,lm_2-g\sin(\beta)\,lm_2-\dot{\phi}\dot{\theta}(I_{l1}-I_{l2}-I_{l3}))\cos(\theta)+(\dot{\theta}^2(l^2m_2+I_{l1}-I_{l3})\sin(\beta)+Rlm_2(\ddot{\phi}\sin(\theta)+\ddot{\psi}))\cos(\beta)+Rlm_2(\dot{\phi}\dot{\psi}\sin(\theta)+\dot{\phi}^2+\dot{\theta}^2)\sin(\beta)+\ddot{\phi}(l^2m_2+I_{l2})\sin(\theta)+l^2m_2\ddot{\beta}+I_{l2}\ddot{\beta}+\tau_2=0. \quad \textbf{(3-49)}$$

The equations (3-46) through (3-49) arranged in matrix form according to (2-47) as,

$$\boldsymbol{Mass}=\begin{bmatrix} Mass_{11} & 0 & Mass_{13} & Mass_{14} \\ 0 & Mass_{22} & Mass_{23} & 0 \\ Mass_{31} & Mass_{32} & Mass_{33} & Mass_{34} \\ Mass_{41} & 0 & Mass_{43} & Mass_{44} \end{bmatrix} \quad \textbf{(3-50)}$$

where,

$$Mass_{11}=(m+m_2)R^2+I_{w2},$$

$$Mass_{13}=(R\cos(\beta)\,lm_2+(m+m_2)R^2+I_{w2})\sin(\theta)=Mass_{31},$$

$$Mass_{14}=R\cos(\beta)\,lm_2=Mass_{41},$$

$$Mass_{22}=(l^2m_2+I_{l1}-I_{l3})\cos(\beta)^2+2R\cos(\beta)\,lm_2+R^2m+R^2m_2+I_{l3}+I_{w1},$$

$$Mass_{23}=-\sin(\beta)\cos(\theta)\left((l^2m_2+I_{l1}-I_{l3})\cos(\beta)+Rlm_2\right)=Mass_{32},$$

$$Mass_{33}=\left((-l^2m_2-I_{l1}+I_{l3})\cos(\beta)^2-2Rcos(\beta)lm_2-R^2m-R^2m_2+I_{w3}+I_{l1}-I_{l2}-I_{w2}\right)\cos(\theta)^2+2R\cos(\beta)\,lm_2+(R^2+l^2)m_2+R^2m+I_{l2}+I_{w2},$$

$$Mass_{34}=\sin(\theta)\,(R\cos(\beta)\,lm_2+l^2m_2+I_{l2})=Mass_{43},$$

and

$$Mass_{44}=l^2m_2+I_{l2}.$$

The $\boldsymbol{C}$ vector is

$$\boldsymbol{C}=\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \quad \textbf{(3-51)}$$

where,

$$C_1 = 2\left(R\cos(\beta)\, lm_2 + (m+m_2)R^2 + \frac{I_{w1}}{2} + \frac{I_{w2}}{2} - \frac{I_{w3}}{2}\right)\dot{\theta}\dot{\phi}\cos(\theta)$$

$$- R\sin(\beta)\, lm_2\big(2\sin(\theta)\,\dot{\phi}\dot{\beta} + \dot{\phi}^2 + \dot{\beta}^2\big),$$

$$C_2 = m_2 g\big(\sin(\theta)\,(l\cos(\beta) + R)\big)$$

$$-\Bigg((l^2 m_2 + I_{l1} - I_{l3})\big(\sin(\theta)\,\dot{\phi} + 2\dot{\beta}\big)\cos(\beta)^2$$

$$+ 2lRm_2\left(\sin(\theta)\,\dot{\phi} + \frac{\dot{\psi}}{2} + \dot{\beta}\right)\cos(\beta)$$

$$+ \dot{\phi}(R^2 m + R^2 m_2 - I_{l1} + I_{l2} + I_{w2} - I_{w3})\sin(\theta) + R^2\dot{\psi}m_2 + (I_{l2} + I_{l3} - I_{l1})\dot{\beta}$$

$$+ \dot{\psi}(R^2 m + I_{w2})\Bigg)\dot{\phi}\cos(\theta)$$

$$+ \big(-g\sin(\theta)\, lm_2 - 2\dot{\beta}\dot{\theta}(l^2 m_2 + I_{l1} - I_{l3})\sin(\beta)\big)\cos(\beta)$$

$$- 2\left(\dot{\beta}\dot{\theta}\sin(\beta)\, l + \frac{g\sin(\theta)}{2}\right)Rm_{2,}$$

$$C_3 = 2\dot{\phi}\big((l^2 m_2 + I_{l1} - I_{l3})\cos(\beta) + Rlm_2\big)\sin(\beta)\,\dot{\beta}\cos(\theta)^2$$

$$+ 2\Big(\dot{\phi}\big((l^2 m_2 + I_{l1} - I_{l3})\cos(\beta)^2 + 2Rlm_2\cos(\beta) + R^2 m + R^2 m_2 + I_{l2} + I_{w2}$$

$$- I_{w3} - I_{l1}\big)\sin(\theta) - \dot{\beta}(l^2 m_2 + I_{l1} - I_{l3})\cos(\beta)^2 + \left(l^2 m_2 + \frac{I_{l1}}{2} + \frac{I_{l2}}{2} - \frac{I_{l3}}{2}\right)\dot{\beta}$$

$$+ \frac{I_{w2}\dot{\psi}}{2}\Big)\dot{\theta}\cos(\theta)$$

$$+ \sin(\beta)\Big(\Big(\dot{\theta}^2(l^2 m_2 + I_{l1} - I_{l3})\cos(\beta) - Rlm_2\big(\dot{\beta} - \dot{\theta}\big)\big(\dot{\beta} + \dot{\theta}\big)\Big)\sin(\theta)$$

$$+ R\dot{\phi}lm_2\big(\dot{\psi} - 2\dot{\beta}\big)\Big),$$

and

$$C_4 = -\big((l^2 m_2 + I_{l1} - I_{l3})\cos(\beta) + Rlm_2\big)\dot{\phi}^2 \sin(\beta)\cos(\theta)^2 \\ + 2\dot{\theta}\dot{\phi}\left((l^2 m_2 + I_{l1} - I_{l3})\cos(\beta)^2 + Rlm_2\cos(\beta) - \frac{I_{l1}}{2} + \frac{I_{l2}}{2} \\ + \frac{I_{l3}}{2}\right)\cos(\theta) \\ + \big(\dot{\theta}^2 (l^2 m_2 + I_{l1} - I_{l3})\cos(\beta) \\ + Rlm_2\big(\dot{\phi}\dot{\psi}\sin(\theta) + \dot{\phi}^2 + \dot{\theta}^2\big)\big)\sin(\beta).$$

$$\boldsymbol{G} = \begin{bmatrix} 0 \\ -mgR\sin(\theta) - m_2 g(\sin(\theta)\, R + \sin(\theta)\cos(\beta)\, l) \\ 0 \\ -m_2\, \mathrm{g}\cos(\theta)\sin(\beta)\, l \end{bmatrix} \tag{3-52}$$

and

$$\boldsymbol{Q_i} = \begin{bmatrix} \tau_2 \\ 0 \\ 0 \\ -\tau_2 \end{bmatrix}. \tag{3-53}$$

The equations (3-30), (3-37), (3-39) and (3-33) found by Newton -Euler method also provide the same matrices as (3-50) through (3-53).

Chapter 4 - Dynamics of the Unicycle with Rotating Disk

Introduction

The dynamic equations of the whole assembly of the unicycle are derived in this chapter. Figure 4.1 exhibits the model discussed here. A rotating disk is attached to the frame such a way that, it's rotating axis X''' is perpendicular to the wheel's rotating axis Y'. The orientation of this disk is described using the new local frame introduced as (X'''Y'''Z'''). Origin of this new frame is at the center of the disk which has a radius of r and a mass of m_d. The distance from wheel center to disk center is l_d. The disk rotates with an angular velocity of $\dot{\eta}$.

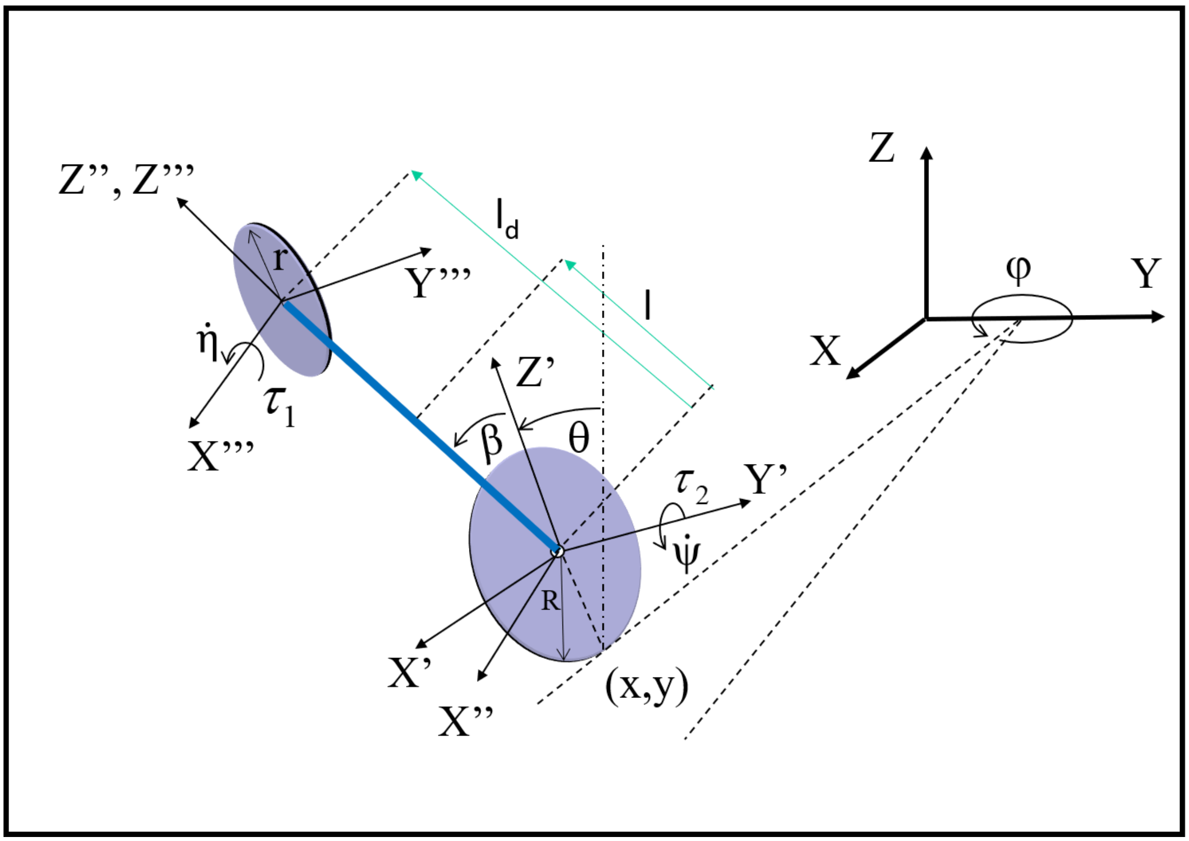

Figure 4.1. Unicycle with Rotating Disk Attached to the Frame

In addition to the equations derived in previous chapters, Newton- Euler and Lagrangian equations are derived for the unicycle with rotating disk here. A matching set of equations are obtained at the end of this chapter also, while confirming the accuracy of the analysis.

Dynamics of the Unicycle with Rotating Disk – Newton Euler Analysis

The free body diagram used for this analysis is shown in Figure 4.2. It illustrates the unicycle with its parts disassembled to mark the internal forces and the reactions.

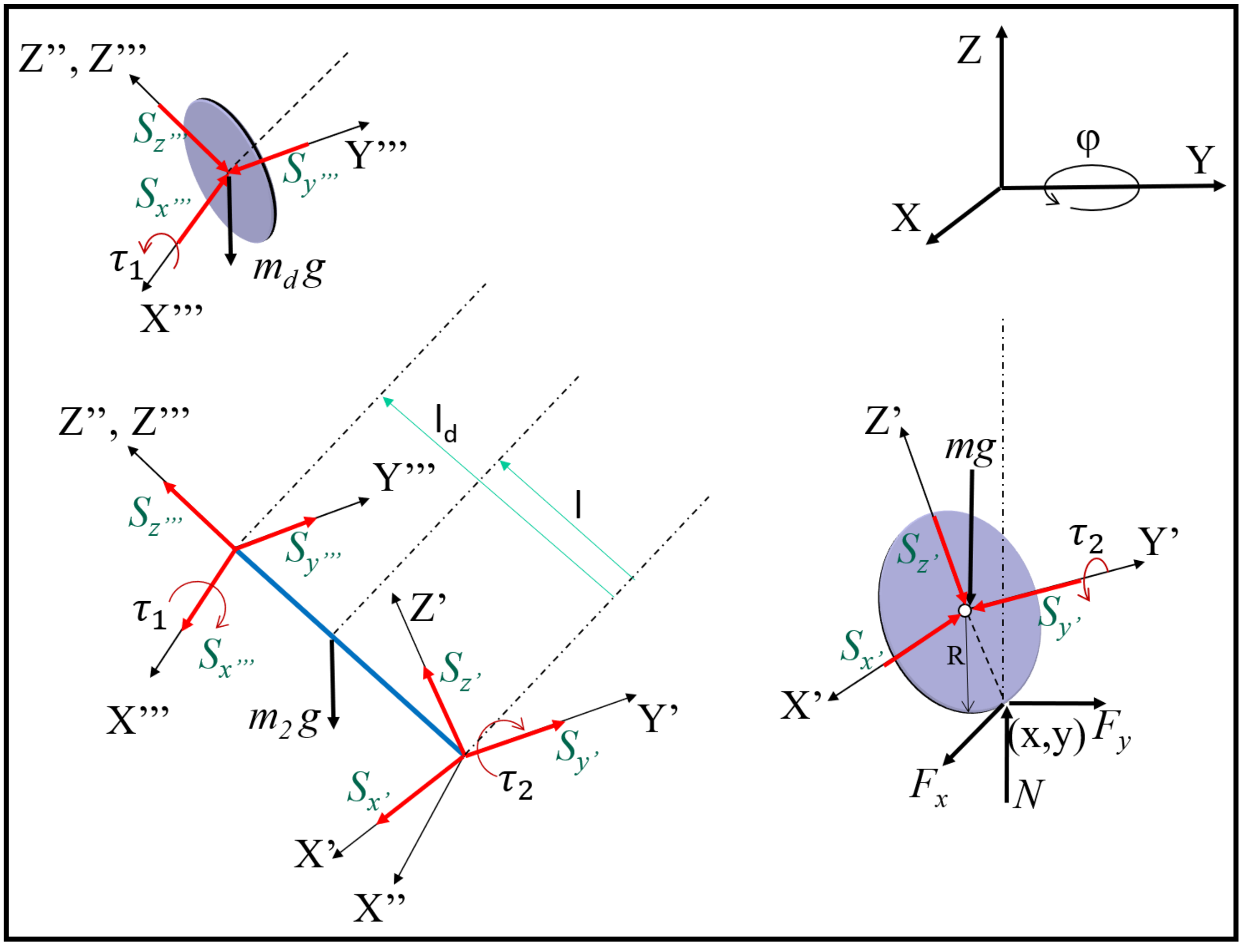

Figure 4.2. Free Body Diagram of Unicycle Assembly

The coordinates of the center of the disk (l_{dX},l_{dY},l_{dZ}) in (XYZ) frame are

$$l_{dX} = (\cos(\phi)\sin(\beta) + \sin(\phi)\sin(\theta)\cos(\beta))l_d + \sin(\phi)\sin(\theta)R + x, \quad \textbf{(4-1)}$$

$$l_{dY} = (\sin(\phi)\sin(\beta) - \cos(\phi)\sin(\theta)\cos(\beta))l_d - \cos(\phi)\sin(\theta)R + y, \quad \textbf{(4-2)}$$

and

$$l_{dZ} = \cos(\theta)\cos(\beta)\, l_d + \cos(\theta)R. \quad \textbf{(4-3)}$$

The velocity of the disk center obtained by taking the first time derivative of (4-1) through (4-3). Those velocity components are

$$v_{dx} = \dot{x} + \big((-\sin(\phi)\sin(\beta) + \cos(\phi)\sin(\theta)\cos(\beta))l_d + \cos(\phi)\sin(\theta)\,R\big)\dot{\phi} + (\sin(\phi)\cos(\theta)\cos(\beta)\,l_d + \sin(\phi)\cos(\theta)\,R)\dot{\theta} + (\cos(\phi)\cos(\beta) - \sin(\phi)\sin(\theta)\sin(\beta))l_d\dot{\beta}, \quad \textbf{(4-4)}$$

$$v_{dY} = \dot{y} + \big((\cos(\phi)\sin(\beta) + \sin(\phi)\sin(\theta)\cos(\beta))l_d + \sin(\phi)\sin(\theta)\,R\big)\dot{\phi} + (-\cos(\phi)\cos(\theta)\cos(\beta)\,l_d - \cos(\phi)\cos(\theta)\,R)\dot{\theta} + (\sin(\phi)\cos(\beta) + \cos(\phi)\sin(\theta)\sin(\beta))l_d\dot{\beta} \quad \textbf{(4-5)}$$

$$v_{dZ} = (-\sin(\theta)\cos(\beta)\,l_d - \sin(\theta)\,R)\dot{\theta} - \cos(\theta)\sin(\beta)\,l_d\dot{\beta} \quad \textbf{(4-6)}$$

The accelerations of the disk center with respective to (XYZ) frame calculated by taking the first time derivative of (4-4) through (4-6). The acceleration components are

$$a_{dX} = \sin(\phi)\left(\sin(\theta)\left(-l_d\left(\dot{\beta}^2 + \dot{\phi}^2 + \dot{\theta}^2\right)\cos(\beta) - \ddot{\beta}l_d sin(\beta) - R\left(\dot{\phi}^2 + \dot{\theta}^2\right)\right) + l_d\left(\cos(\theta)\,\ddot{\theta} - 2\dot{\phi}\dot{\beta}\right)\cos(\beta) - 2l_d\left(\dot{\theta}\cos(\theta)\,\dot{\beta} + \frac{\ddot{\phi}}{2}\right)\sin(\beta) + R\left(\cos(\theta)\,\ddot{\theta} - \dot{\phi}\dot{\psi}\right)\right) + \cos(\phi)\left(\left(-2\sin(\beta)\,l_d\dot{\phi}\dot{\beta} + l_d\ddot{\phi}\cos(\beta) + R\ddot{\phi}\right)\sin(\theta) + 2l_d\left(\cos(\theta)\,\dot{\phi}\dot{\theta} + \frac{\ddot{\beta}}{2}\right)\cos(\beta) - l_d\left(\dot{\beta}^2 + \dot{\phi}^2\right)\sin(\beta) + R\left(2\cos(\theta)\,\dot{\phi}\dot{\theta} + \ddot{\psi}\right)\right), \quad \textbf{(4-7)}$$

$$a_{dY} = \cos(\phi)\Bigg(\sin(\theta)\left(l_d(\dot{\beta}^2 + \dot{\phi}^2 + \dot{\theta}^2)\cos(\beta) + \ddot{\beta} l_d sin(\beta)\right.$$
$$\left. + R(\dot{\phi}^2 + \dot{\theta}^2)\right) - l_d(\cos(\theta)\ddot{\theta} - 2\dot{\phi}\dot{\beta})\cos(\beta)$$
$$+ 2l_d\left(\dot{\theta}\cos(\theta)\dot{\beta} + \frac{\ddot{\phi}}{2}\right)\sin(\beta) - R(\cos(\theta)\ddot{\theta} - \dot{\phi}\dot{\psi})\Bigg)$$
$$+ \sin(\phi)\Bigg((-2\sin(\beta) l_d\dot{\phi}\dot{\beta} + l_d\ddot{\phi}\cos(\beta) + R\ddot{\phi})\sin(\theta)$$
$$+ 2l_d\left(\cos(\theta)\dot{\phi}\dot{\theta} + \frac{\ddot{\beta}}{2}\right)\cos(\beta) - l_d(\dot{\beta}^2 + \dot{\phi}^2)\sin(\beta)$$
$$+ R(2\cos(\theta)\dot{\phi}\dot{\theta} + \ddot{\psi})\Bigg), \tag{4-8}$$

and

$$a_{dZ} = \left(-l_d(\dot{\beta}^2 + \dot{\theta}^2)\cos(\beta) - R\dot{\theta}^2 - \ddot{\beta} l_d sin(\beta)\right)\cos(\theta) - \sin(\theta)\left(-2\sin(\beta) l_d\dot{\beta}\dot{\theta} + l_d\ddot{\theta}\cos(\beta) + R\ddot{\theta}\right). \tag{4-9}$$

The angular velocity of the disk in (X'''Y'''Z''') frame is given by the vector

$$\boldsymbol{\Omega_d} = \begin{bmatrix}\omega_{dX'''}\\ \omega_{dY'''}\\ \omega_{dZ'''}\end{bmatrix}, \tag{4-10}$$

and the angular velocity components are

$$\omega_{dX'''} = \dot{\eta} + \dot{\theta}\cos(\beta) - \sin(\beta)\cos(\theta)\dot{\phi}, \tag{4-11}$$

$$\omega_{dY'''} = \dot{\beta} + \sin(\theta)\dot{\phi}, \tag{4-12}$$

and

$$\omega_{dZ'''} = \cos(\beta)\cos(\theta)\dot{\phi} + \dot{\theta}\sin(\beta). \tag{4-13}$$

The angular acceleration vector of the disk in (X'''Y'''Z''') frame is

$$\boldsymbol{\alpha_d} = \begin{bmatrix}\alpha_{dX'''}\\ \alpha_{dY'''}\\ \alpha_{dZ'''}\end{bmatrix}. \tag{4-14}$$

The acceleration components of (4-14), obtained by taking the first-time derivative of (4-11) through (4-13) are

$$\alpha_{dX'''} = \ddot{\eta} + \cos(\beta)\,\ddot{\theta} + \sin(\beta)\sin(\theta)\,\dot{\phi}\dot{\theta} - \cos(\theta)\sin(\beta)\,\ddot{\phi} + \left(-\dot{\theta}\sin(\beta) - \cos(\beta)\cos(\theta)\,\dot{\phi}\right)\dot{\beta}, \tag{4-15}$$

$$\alpha_{dY'''} = \ddot{\beta} + \cos(\theta)\,\dot{\phi}\dot{\theta} + \sin(\theta)\,\ddot{\phi}, \tag{4-16}$$

and

$$\alpha_{dZ'''} = \sin(\beta)\,\ddot{\theta} - \cos(\beta)\sin(\theta)\,\dot{\phi}\dot{\theta} + \cos(\theta)\cos(\beta)\,\ddot{\phi} + \left(\dot{\theta}\cos(\beta) - \sin(\beta)\cos(\theta)\,\dot{\phi}\right)\dot{\beta}. \tag{4-17}$$

The inertia matrix of the disk in (X'''Y'''Z''') frame along its principal axes is

$$\boldsymbol{I_d} = \begin{bmatrix} I_{X'''X'''} & 0 & 0 \\ 0 & I_{Y'''Y'''} & 0 \\ 0 & 0 & I_{Z'''Z'''} \end{bmatrix} = \begin{bmatrix} I_{d1} & 0 & 0 \\ 0 & I_{d2} & 0 \\ 0 & 0 & I_{d3} \end{bmatrix}. \tag{4-18}$$

Newton's equation in (2-21) is applied to the components shown in Figure 4.2 separately. For the rolling wheel, equations obtained in Chapter 3, (3-19) through (3-21) valid for this calculation also. Newton's equations derived for the rod in (X"Y"Z") frame are,

$$m_2 a_{lX"} - \cos(\beta)\,S_{X'} - m_2 g\cos(\theta)\sin(\beta) + \sin(\beta)\,S_{Z'} - S_{X'''} = 0, \tag{4-19}$$

$$m_2 a_{lY"} - S_{Y'} + m_2 g\sin(\theta) - S_{Y'''} = 0, \tag{4-20}$$

and

$$m_2 a_{lZ"} - \cos(\beta)\,S_{Z'} - \sin(\beta)\,S_{X'} + m_2 g\cos(\theta)\cos(\beta) - S_{Z'''} = 0. \tag{4-21}$$

The equations obtained for the rotating disk by applying (2-21) in (X'''Y'''Z''') frame are,

$$m_d a_{dX'''} - m_d g\cos(\theta)\sin(\beta) + S_{X'''} = 0, \tag{4-22}$$

$$m_d a_{dY'''} + m_d g\sin(\theta) + S_{Y'''} = 0, \tag{4-23}$$

and

$$m_d a_{dZ'''} + m_d g\cos(\theta)\cos(\beta) + S_{Z'''} = 0. \tag{4-24}$$

The results retrieved for unknown reactions, contact point reactions and the acceleration components are provided in Appendix C.

The Euler equations are derived for the separated components in Figure 4.2 by applying (2-25). For the rotating wheel it's the same procedure followed in Chapter 2 repeated. Therefore, only the final results are displayed in this chapter. When considering the rod, since there are new external forces and torques present, (3-29) is modified as

$$\dot{\boldsymbol{H}}_{\boldsymbol{l}} = \boldsymbol{M}_{\boldsymbol{l}} + \begin{bmatrix} -\tau_1 \\ -\tau_2 \\ 0 \end{bmatrix}, \tag{4-25}$$

where

$$\boldsymbol{M_l} = \begin{bmatrix} lS_{Y'} - (l_d - l)S_{Y'''} \\ -l(\cos(\beta)\, S_{X'} - \sin(\beta)\, S_{Z'} + (l_d - l)S_{X'''} \\ 0 \end{bmatrix}. \quad \textbf{(4-26)}$$

The same procedure in Chapter 3 is followed to calculate $\dot{\boldsymbol{H}}_{\boldsymbol{l}}$. In the calculation of Euler equations for the disk, external moments acting on the disk $\boldsymbol{M_d} = 0$, since the moments are taken about the center of the disk. The rate of change of angular momentum is calculated as

$$\dot{\boldsymbol{H}}_{\boldsymbol{d}} = (\dot{\boldsymbol{H}})_{\boldsymbol{rd}} + \boldsymbol{\Omega_l} \times \boldsymbol{H_d}, \quad \textbf{(4-27)}$$

where, $(\dot{\boldsymbol{H}})_{\boldsymbol{rd}}$ is the rate of change of angular momentum of the disk, with respect to the center of the disk as viewed by an observer on the (X'''Y'''Z''') frame. $(\dot{\boldsymbol{H}})_{\boldsymbol{rd}}$ is calculated as

$$(\dot{\boldsymbol{H}})_{\boldsymbol{rd}} = \boldsymbol{\alpha_d} \times \boldsymbol{I_d}. \quad \textbf{(4-28)}$$

In (4-27), $\boldsymbol{\Omega_l}$ is used since the frame (X'''Y'''Z''') also has the same angular velocity. The angular momentum of the disk $\boldsymbol{H_d}$, in (X'''Y'''Z''') frame is calculated by

$$\boldsymbol{H_d} = \boldsymbol{I_d} \times \boldsymbol{\Omega_d}. \quad \textbf{(4-29)}$$

Euler equations for the disk is obtained by solving

$$\dot{\boldsymbol{H}}_{\boldsymbol{d}} = \boldsymbol{M_d} + \begin{bmatrix} \tau_1 \\ 0 \\ 0 \end{bmatrix}. \quad \textbf{(4-30)}$$

The final Euler equations obtained for the rotating wheel are,

$$\begin{aligned} 2\dot{\theta}\dot{\phi}\Big(&R(lm_2 + l_d m_d)\cos(\beta) + (m_2 + m_d + m)R^2 - \frac{I_{w3}}{2} + \frac{I_{w1}}{2} \\ &+ \frac{I_{w2}}{2}\Big)\cos(\theta) \\ &+ \big(R\ddot{\phi}(lm_2 + l_d m_d)\cos(\beta) - 2R\dot{\phi}\dot{\beta}(lm_2 + l_d m_d)\sin(\beta) \\ &+ \big((m_2 + m_d + m)R^2 + I_{w2}\big)\ddot{\phi}\big)\sin(\theta) \\ &+ R\ddot{\beta}(lm_2 + l_d m_d)\cos(\beta) \\ &- R\big(\dot{\beta}^2 + \dot{\phi}^2\big)(lm_2 + l_d m_d)\sin(\beta) + \ddot{\psi}(m_2 + m_d + m)R^2 \\ &+ I_{w2}\ddot{\psi} - \tau_2 = 0, \end{aligned} \quad \textbf{(4-31)}$$

$$\left(-\dot{\phi}^2(R(lm_2 + l_d m_d)\cos(\beta) + (m_2 + m_d + m)R^2 + I_{w2} - I_{w3})\sin(\theta) - 2R\dot{\phi}\dot{\beta}(lm_2 + l_d m_d)\cos(\beta) - R\ddot{\phi}(lm_2 + l_d m_d)\sin(\beta) - \dot{\phi}\big((m_2 + m_d + m)R^2 + I_{w2}\big)\dot{\psi}\right)\cos(\theta) - Rg(m_2 + m_d + m)\sin(\theta) + R\ddot{\theta}(lm_2 + l_d m_d)\cos(\beta) - 2R\dot{\beta}\dot{\theta}(lm_2 + l_d m_d)\sin(\beta) + \ddot{\theta}\big((m_2 + m_d + m)R^2 + I_{w1}\big) = 0, \tag{4-32}$$

and

$$-\dot{\phi}\dot{\theta}(I_{w1} - I_{w2} + I_{w3})\sin(\theta) + I_{w2}\dot{\psi}\dot{\theta} + I_{w3}\cos(\theta)\ddot{\phi} = 0. \tag{4-33}$$

The Euler equations calculated for the rod from (4-25) are

$$\Big(\Big(\dot{\phi}(-l^2 m_2 - l_d^2 m_d - I_{l2} + I_{l3})\sin(\theta) + \dot{\beta}(-2l^2 m_2 - 2l_d^2 m_d - I_{l1} - I_{l2} + I_{l3})\Big)\dot{\phi}\cos(\beta) - R\dot{\phi}^2(lm_2 + l_d m_d)\sin(\theta) - \ddot{\phi}(l^2 m_2 + l_d^2 m_d + I_{l1})\sin(\beta) - R\dot{\phi}\dot{\psi}(lm_2 + l_d m_d)\Big)\cos(\theta) + \ddot{\theta}(l^2 m_2 + l_d^2 m_d + I_{l1})\cos(\beta) + \Big(\dot{\phi}\dot{\theta}(I_{l3} + I_{l1} - I_{l2})\sin(\beta) - g(lm_2 + l_d m_d)\Big)\sin(\theta) + \dot{\beta}\dot{\theta}(-2l^2 m_2 - 2l_d^2 m_d - I_{l1} - I_{l2} + I_{l3})\sin(\beta) + Rlm_2\ddot{\theta} + Rl_d m_d\ddot{\theta} + \tau_1 = 0, \tag{4-34}$$

$$-2\dot{\phi}\dot{\theta}\cos(\theta)(-l^2 m_2 - l_d^2 m_d - I_{l1} + I_{l3})\cos(\beta)^2 + \Bigg(\big(\cos(\theta)\dot{\phi} - \dot{\theta}\big)\big(\cos(\theta)\dot{\phi} + \dot{\theta}\big)(-l^2 m_2 - l_d^2 m_d - I_{l1} + I_{l3})\sin(\beta) + 2(lm_2 + l_d m_d)\left(\left(\cos(\theta)\dot{\phi}\dot{\theta} + \frac{\sin(\theta)\ddot{\phi}}{2} + \frac{\ddot{\psi}}{2}\right)R\right)\cos(\beta) - \Big(Rcos(\theta)^2\dot{\phi}^2 + g\cos(\theta) - R\big(\dot{\phi}\dot{\psi}\sin(\theta) + \dot{\theta}^2 + \dot{\phi}^2\big)\Big)(lm_2 + l_d m_d)\sin(\beta) + \dot{\phi}\dot{\theta}(I_{l3} - I_{l1} + I_{l2})\cos(\theta) + \ddot{\phi}(l^2 m_2 + l_d^2 m_d + I_{l2})\sin(\theta) + l^2 m_2\ddot{\beta} + l_d^2 m_d\ddot{\beta} + I_{l2}\ddot{\beta} + \tau_2 = 0, \tag{4-35}$$

and $$\left(\cos(\theta)\, I_{l3}\ddot{\phi} - \left(\dot{\phi}(I_{l3} + I_{l1} - I_{l2})\sin(\theta) + \dot{\beta}(I_{l1} - I_{l2} - I_{l3})\right)\dot{\theta}\right)\cos(\beta) + \left(\dot{\phi}\left(\dot{\phi}(I_{l1} - I_{l2})\sin(\theta) + \dot{\beta}(I_{l1} - I_{l2} - I_{l3})\right)\cos(\theta) + I_{l3}\ddot{\theta}\right)\sin(\beta) = 0. \quad \textbf{(4-36)}$$

Euler equations derived for the disk from (4-30) are

$$\Big(-\ddot{\phi}\cos(\theta)\, I_{d1} + \dot{\theta}\left(\dot{\phi}(I_{d1} - I_{d2} + I_{d3})\sin(\theta) - \dot{\beta}(I_{d1} + I_{d2} - I_{d3})\right)\Big)\sin(\beta) - \cos(\beta)\left(\dot{\phi}(I_{d2} - I_{d3})\sin(\theta) + \dot{\beta}(I_{d1} + I_{d2} - I_{d3})\right)\dot{\phi}\cos(\theta) + I_{d1}\cos(\beta)\,\ddot{\theta} + I_{d1}\ddot{\eta} - \tau_1 = 0, \quad \textbf{(4-37)}$$

$$-\cos(\beta)\sin(\beta)\,\dot{\phi}^2(I_{d1} - I_{d3})\cos(\theta)^2 + 2\dot{\phi}\left(\dot{\theta}(I_{d1} - I_{d3})\cos(\beta)^2 + \frac{I_{d1}\cos(\beta)\,\dot{\eta}}{2} - \frac{\dot{\theta}(I_{d1} - I_{d2} - I_{d3})}{2}\right)\cos(\theta) + \sin(\beta)\,\dot{\theta}^2(I_{d1} - I_{d3})\cos(\beta) + I_{d1}\sin(\beta)\,\dot{\eta}\dot{\theta} + I_{d2}\left(\sin(\theta)\,\ddot{\phi} + \ddot{\beta}\right) = 0, \quad \textbf{(4-38)}$$

and $$\left(I_{d3}\ddot{\phi}\cos(\theta) - \dot{\theta}\left(\dot{\phi}(I_{d1} - I_{d2} + I_{d3})\sin(\theta) + \dot{\beta}(I_{d1} - I_{d2} - I_{d3})\right)\right)\cos(\beta) + \dot{\phi}\sin(\beta)\left(\dot{\phi}(I_{d1} - I_{d2})\sin(\theta) + \dot{\beta}(I_{d1} - I_{d2} - I_{d3})\right)\cos(\theta) - I_{d1}\sin(\theta)\,\dot{\eta}\dot{\phi} + I_{d3}\sin(\beta)\,\ddot{\theta} - I_{d1}\dot{\beta}\dot{\eta} = 0. \quad \textbf{(4-39)}$$

Three new Euler equations were derived by applying few transformations to the Euler equations obtained above to build a symmetric mass matrix. One of the transformations applied is,

$$(4-32) + (4-34)\times\cos(\beta) + (4-36)\times\sin(\beta) + (4-37)\times\cos(\beta) + (4-39)\times\sin(\beta) = 0. \quad \textbf{(4-40)}$$

The results is

$$
\begin{aligned}
&\Big(\dot{\phi}\big(\sin(\theta)\,\dot{\phi} + 2\dot{\beta}\big)(-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\cos(\beta)^2 \\
&\quad + \Big(-2R\dot{\phi}^2(l m_2 + l_d m_d)\sin(\theta) \\
&\quad + \ddot{\phi}(-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta) \\
&\quad - R\dot{\phi}(l m_2 + l_d m_d)\big(\dot{\psi} + 2\dot{\beta}\big)\Big)\cos(\beta) \\
&\quad - \dot{\phi}^2\big((m_2 + m_d + m)R^2 + I_{d2} - I_{l1} + I_{l2} + I_{w2} - I_{w3} \\
&\quad - I_{d1}\big)\sin(\theta) - R\ddot{\phi}(l m_2 + l_d m_d)\sin(\beta) \\
&\quad - \dot{\phi}\big(\dot{\psi}(m_2 + m_d + m)R^2 \\
&\quad + (I_{d2} + I_{d3} - I_{l1} + I_{l2} + I_{l3} - I_{d1})\dot{\beta} + I_{w2}\dot{\psi}\big)\Big)\cos(\theta) \\
&\quad - \ddot{\theta}(-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\cos(\beta)^2 \\
&\quad + \big(-g(l m_2 + l_d m_d)\sin(\theta) \\
&\quad + 2\dot{\beta}\dot{\theta}(-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta) \\
&\quad + 2\ddot{\theta}(l m_2 + l_d m_d)R + I_{d1}\ddot{\eta}\big)\cos(\beta) \\
&\quad + \big(-\dot{\eta}\dot{\phi}\sin(\beta)\,I_{d1} - Rg(m_2 + m_d + m)\big)\sin(\theta) \\
&\quad - 2\dot{\beta}\left(\dot{\theta}(l m_2 + l_d m_d)R + \frac{\dot{\eta} I_{d1}}{2}\right)\sin(\beta) \\
&\quad + \big((m_2 + m_d + m)R^2 + I_{d3} + I_{l3} + I_{w1}\big)\ddot{\theta} = 0.
\end{aligned} \tag{4-41}
$$

Next transformation applied is

$$
\begin{aligned}
&(4-33) \times \cos(\theta) + (4-36) \times \cos(\beta)\cos(\theta) \\
&\quad + (4-39) \times \cos(\beta)\cos(\theta) - (4-34) \times \sin(\beta)\cos(\theta) \\
&\quad - (4-37) \times \sin(\beta)\cos(\theta) + (4-31) \times \sin(\theta) \\
&\quad + (4-35) \times \sin(\theta) + (4-38) \times \sin(\theta) = 0.
\end{aligned} \tag{4-42}
$$

The resulting Euler equation is

$$
\begin{aligned}
&\Big(\ddot{\phi}\,(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})(\cos(\beta))^2 \\
&+ \Big(-2\dot{\phi}\dot{\beta}(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta) \\
&- 2R\ddot{\phi}(lm_2 + l_dm_d)\Big)\cos(\beta) + 2R\dot{\phi}\dot{\beta}(lm_2 + l_dm_d)\sin(\beta) \\
&- \ddot{\phi}\,\Big((m_2 + m_d + m)R^2 + I_{d2} - I_{l1} + I_{l2} + I_{w2} - I_{w3} \\
&- I_{d1})\Big)(\cos(\theta))^2 \\
&+ \Big(-2\dot{\theta}\big(\sin(\theta)\,\dot{\phi} - \dot{\beta}\big)\big(-l^2m_2 - l_d{}_d^{2m} - I_{d1} + I_{d3} - I_{l1} \\
&+ I_{l3}\big)(\cos(\beta))^2 \\
&+ \big(4R\dot{\phi}\dot{\theta}(lm_2 + l_dm_d)\sin(\theta) \\
&+ \ddot{\theta}(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta) \\
&- I_{d1}\dot{\beta}\dot{\eta}\big)\cos(\beta) \\
&+ 2\dot{\theta}\dot{\phi}\big((m_2 + m_d + m)R^2 + I_{d2} - I_{l1} + I_{l2} + I_{w2} - I_{w3} \\
&- I_{d1}\big)\sin(\theta) + \big(-\ddot{\theta}(lm_2 + l_dm_d)R - I_{d1}\ddot{\eta}\big)\sin(\beta) \\
&- \dot{\theta}\,\Big((-2\,l^2m_2 - 2l_d^2m_d - I_{d1} - I_{d2} + I_{d3} - I_{l1} - I_{l2} + I_{l3})\dot{\beta} \\
&- I_{w2}\dot{\psi}\Big)\Big)\cos(\theta) \\
&+ \Big(\Big(-\dot{\theta}^2(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta) \\
&+ R\big(\ddot{\psi} + \ddot{\beta}\big)(lm_2 + l_dm_d)\Big)\sin(\theta) \\
&+ 2R\ddot{\phi}(lm_2 + l_dm_d)\Big)\cos(\beta) \\
&+ \Big(\Big(R(lm_2 + l_dm_d)\dot{\theta}^2 + I_{d1}\dot{\eta}\dot{\theta} - R\dot{\beta}^2(lm_2 + l_dm_d)\Big)\sin(\beta) \\
&+ \ddot{\psi}(m_2 + m_d + m)R^2 + l^2m_2\ddot{\beta} + l_d^2m_d\ddot{\beta} + (I_{l2} + I_{d2})\ddot{\beta} \\
&+ I_{w2}\ddot{\psi}\Big)\sin(\theta) + R\dot{\phi}(lm_2 + l_dm_d)\big(\dot{\psi} - 2\dot{\beta}\big)\sin(\beta) \\
&+ \ddot{\phi}\Big((m_2 + m_d + m)R^2 + l^2m_2 + l_d^2m_d + I_{l2} + I_{w2} + I_{d2}\Big) \\
&= 0.
\end{aligned}
\qquad \textbf{(4-43)}
$$

Final transformations applied is

$$(4-35)+(4-38)=0. \quad \textbf{(4-44)}$$

The result is

$$\begin{aligned}
&-\sin(\beta)\Big((l^2m_2+l_d^2m_d+I_{d1}-I_{d3}+I_{l1}-I_{l3})\cos(\beta) \qquad \textbf{(4-45)}\\
&\quad +R(lm_2+l_dm_d)\Big)\dot{\phi}^2\cos(\theta)^2\\
&\quad +\Big(-2\dot{\theta}\dot{\phi}(-l^2m_2-l_d^2m_d-I_{d1}+I_{d3}-I_{l1}+I_{l3})\cos(\beta)^2\\
&\quad +2\left(\dot{\theta}(lm_2+l_dm_d)R+\frac{I_{d1}\dot{\eta}}{2}\right)\dot{\phi}\cos(\beta)\\
&\quad -g(lm_2+l_dm_d)\sin(\beta)\\
&\quad +\dot{\theta}\dot{\phi}(I_{d2}+I_{d3}-I_{l1}+I_{l2}+I_{l3}-I_{d1})\Big)\cos(\theta)\\
&\quad +\Big(-\dot{\theta}^2(-l^2m_2-l_d^2m_d-I_{d1}+I_{d3}-I_{l1}+I_{l3})\sin(\beta)\\
&\quad +R(lm_2+l_dm_d)\big(\sin(\theta)\ddot{\phi}+\ddot{\psi}\big)\Big)\cos(\beta)\\
&\quad +\big(R\dot{\phi}\dot{\psi}(lm_2+l_dm_d)\sin(\theta)+R(lm_2+l_dm_d)\dot{\phi}^2\\
&\quad +\big(\dot{\theta}(lm_2+l_dm_d)R+I_{d1}\dot{\eta}\big)\dot{\theta}\big)\sin(\beta)\\
&\quad +\ddot{\phi}(l^2m_2+l_d^2m_d+I_{d2}+I_{l2})\sin(\theta)+l_d^2m_d\ddot{\beta}+l^2m_2\ddot{\beta}\\
&\quad +(I_{l2}+I_{d2})\ddot{\beta}+\tau_2=0.
\end{aligned}$$

Symmetric mass matrix of the unicycle model was built using (4-31), (4-41), (4-43), (4-45) and (4-37). These equations are expressed in matrix form at the end of this chapter.

Dynamics of the Unicycle with Rotating Disk – Lagrangian Analysis

The Lagrangian (L) is derived for the unicycle with the rotating disk shown in Figure 4.1 using (2-35). The total kinetic energy T for this assembly is given by

$$
\begin{aligned}
T = \frac{m}{2}\Big(&\big(\dot{x} + \cos(\phi)\sin(\theta)\,R\dot{\phi} + \sin(\phi)\cos(\theta)\,R\dot{\theta}\big)^2 \\
&+ \big(\dot{y} + \sin(\phi)\sin(\theta)\,R\dot{\phi} - \cos(\phi)\cos(\theta)\,R\dot{\theta}\big)^2 + \sin(\theta)^2 R^2\dot{\theta}^2\Big) \\
&+ \frac{\dot{\theta}^2 I_{w1}}{2} + \frac{\big(\dot{\psi} + \sin(\theta)\,\dot{\phi}\big)^2 I_{w2}}{2} + \frac{\cos(\theta)^2\,\dot{\phi}^2 I_{w3}}{2} \\
&+ \frac{m_2}{2}\Bigg(\bigg(\dot{x} + \left(\begin{matrix}(-\sin(\phi)\sin(\beta) + \cos(\phi)\sin(\theta)\cos(\beta))l + \\ \cos(\phi)\sin(\theta)\,R\end{matrix}\right)\dot{\phi} \\
&+ (\sin(\phi)\cos(\theta)\cos(\beta)\,l + \sin(\phi)\cos(\theta)\,R)\dot{\theta} \\
&+ (\cos(\phi)\cos(\beta) - \sin(\phi)\sin(\theta)\sin(\beta))l\dot{\beta}\bigg)^2 \\
&+ \bigg(\dot{y} + \left(\begin{matrix}(\cos(\phi)\sin(\beta) + \sin(\phi)\sin(\theta)\cos(\beta))l + \\ \sin(\phi)\sin(\theta)\,R\end{matrix}\right)\dot{\phi} \\
&+ (-\cos(\phi)\cos(\theta)\cos(\beta)\,l - \cos(\phi)\cos(\theta)\,R)\dot{\theta} \\
&+ (\sin(\phi)\cos(\beta) + \cos(\phi)\sin(\theta)\sin(\beta))l\dot{\beta}\bigg)^2 \\
&+ \Big((-\sin(\theta)\cos(\beta)\,l - \sin(\theta)\,R)\dot{\theta} - \cos(\theta)\sin(\beta)\,l\dot{\beta}\Big)^2\Bigg) \\
&+ \frac{\big(\dot{\theta}\cos(\beta) - \cos(\theta)\sin(\beta)\,\dot{\phi}\big)^2 I_{l1}}{2} + \frac{\big(\dot{\beta} + \sin(\theta)\,\dot{\phi}\,\big)^2 I_{l2}}{2} \\
&+ \frac{\big(\cos(\beta)\cos(\theta)\,\dot{\phi}\; + \dot{\theta}\sin(\beta)\big)^2 I_{l3}}{2} \\
&+ \frac{m_d}{2}\Bigg(\bigg(\dot{x} + \left(\begin{matrix}(-\sin(\phi)\sin(\beta) + \cos(\phi)\sin(\theta)\cos(\beta))l_d + \\ \cos(\phi)\sin(\theta)\,R\end{matrix}\right)\dot{\phi} \\
&+ (\sin(\phi)\cos(\theta)\cos(\beta)\,l_d + \sin(\phi)\cos(\theta)\,R)\dot{\theta} \\
&+ (\cos(\phi)\cos(\beta) - \sin(\phi)\sin(\theta)\sin(\beta))l_d\dot{\beta}\bigg)^2 \\
&+ \Big(\dot{y} \\
&+ \big((\cos(\phi)\sin(\beta) + \sin(\phi)\sin(\theta)\cos(\beta))l_d + \sin(\phi)\sin(\theta)\,R\big)\dot{\phi} \\
&+ (-\cos(\phi)\cos(\theta)\cos(\beta)\,l_d - \cos(\phi)\cos(\theta)\,R)\dot{\theta} \\
&+ (\sin(\phi)\cos(\beta) + \cos(\phi)\sin(\theta)\sin(\beta))l_d\dot{\beta}\Big)^2 \\
&+ \Big((-\sin(\theta)\cos(\beta)\,l_d - \sin(\theta)\,R)\dot{\theta} - \cos(\theta)\sin(\beta)\,l_d\dot{\beta}\Big)^2\Bigg) \\
&+ \frac{\big(\dot{\eta} + \dot{\theta}\cos(\beta) - \cos(\theta)\sin(\beta)\,\dot{\phi}\big)^2 I_{d1}}{2} + \frac{\big(\dot{\beta} + \sin(\theta)\,\dot{\phi}\big)^2 I_{d2}}{2} \\
&+ \frac{\big(\cos(\beta)\cos(\theta)\,\dot{\phi} + \dot{\theta}\sin(\beta)\big)^2 I_{d3}}{2}.
\end{aligned}
\tag{4-46}
$$

The total gravitational potential energy is

$$V = mgR\cos(\theta) + m_2 g(\cos(\theta)\cos(\beta)\, l + \cos(\theta)\, R) \qquad \textbf{(4-47)}$$
$$+ m_d g(\cos(\theta)\cos(\beta)\, l_d + \cos(\theta)\, R).$$

The non-holonomic constraints in this system are defined in (2-38) and (2-39). The generalized forces for the Lagrange equations calculation using (2-40) are obtained from the Newton-Euler method in Chapter 4. The generalized forces are

$$Q_\psi = \tau_2, \qquad \textbf{(4-48)}$$

$$Q_\theta = 0, \qquad \textbf{(4-49)}$$

$$Q_\phi = 0, \qquad \textbf{(4-50)}$$

$$Q_\beta = -\tau_2, \qquad \textbf{(4-51)}$$

and

$$Q_\eta = \tau_1 \qquad \textbf{(4-52)}$$

The final Lagrange equations for the unicycle are same as the (4-31), (4-41), (4-43), (4-45) and (4-37). These equations are arranged in a matrix form described in (2-47) as

$$\boldsymbol{Mass} = \begin{bmatrix} Mass_{11} & 0 & Mass_{13} & Mass_{14} & 0 \\ 0 & Mass_{22} & Mass_{23} & 0 & Mass_{25} \\ Mass_{31} & Mass_{32} & Mass_{33} & Mass_{34} & Mass_{35} \\ Mass_{41} & 0 & Mass_{43} & Mass_{44} & 0 \\ 0 & Mass_{52} & Mass_{53} & 0 & Mass_{55} \end{bmatrix}, \qquad \textbf{(4-53)}$$

where,

$$Mass_{11} = (m + m_2 + m_d)R^2 + I_{w2},$$

$$Mass_{13} = (R\cos(\beta)(\, lm_2 + l_d m_d) + (m + m_2 + m_d)R^2 + I_{w2})\sin(\theta) = Mass_{31},$$

$$Mass_{14} = R\cos(\beta)\,(lm_2 + l_d m_d) = Mass_{41},$$

$$Mass_{22} = -(-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\cos(\beta)^2 + 2R(lm_2 + l_d m_d)\cos(\beta) +$$
$$(m_2 + m_d + m)R^2 + I_{d3} + I_{l3} + I_{w1},$$

$$Mass_{23} = ((-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta)\cos(\beta) - R(lm_2 +$$
$$l_d m_d)\sin(\beta))\cos(\theta) = Mass_{32},$$

$$Mass_{25} = I_{d1}\cos(\beta) = Mass_{52},$$

$$Mass_{33} = \Big((-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\cos(\beta)^2 - 2R(lm_2 + l_d m_d)\cos(\beta) - (m_2 + m_d + m)R^2 - I_{d2} + I_{l1} - I_{l2} - I_{w2} + I_{w3} + I_{d1}\Big)\cos(\theta)^2 + 2R(lm_2 + l_d m_d)\cos(\beta) + (m_2 + m_d + m)R^2 + l^2 m_2 + l_d^2 m_d + I_{l2} + I_{w2} + I_{d2},$$

$$Mass_{34} = R(lm_2 + l_d m_d)\sin(\theta)\cos(\beta) + (l^2 m_2 + l_d^2 m_d + I_{d2} + I_{l2})\sin(\theta) = Mass_{43},$$

$$Mass_{35} = -I_{d1}\sin(\beta)\cos(\theta) = Mass_{53},$$

$$Mass_{44} = l^2 m_2 + l_d^2 m_d + I_{d2} + I_{l2},$$

and

$$Mass_{55} = I_{d1}.$$

The $\boldsymbol{C}$ vector is

$$\boldsymbol{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} \tag{4-54}$$

where,

$$C_1 = 2\left(R(lm_2 + l_d m_d)\cos(\beta) + (m_2 + m_d + m)R^2 - \frac{I_{w3}}{2} + \frac{I_{w1}}{2} + \frac{I_{w2}}{2}\right)\dot{\theta}\dot{\phi}\cos(\theta) - R * \sin(\beta)\,(lm_2 + l_d m_d)\big(2\sin(\theta)\,\dot{\phi}\dot{\beta} + \dot{\phi}^2 + \dot{\beta}^2\big),$$

$$C_2 = -\dot{\phi}\Bigg(-\big(\sin(\theta)\,\dot{\phi} + 2\dot{\beta}\big)(-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\cos(\beta)^2 + 2(lm_2 + l_d m_d)\left(\sin(\theta)\,\dot{\phi} + \frac{\dot{\psi}}{2} + \dot{\beta}\right) R * \cos(\beta) + \dot{\phi}\big((m_2 + m_d + m)R^2 + I_{d2} - I_{l1} + I_{l2} + I_{w2} - I_{w3} - I_{d1}\big)\sin(\theta) + (I_{d2} + I_{d3} - I_{l1} + I_{l2} + I_{l3} - I_{d1})\dot{\beta} + \big((m_2 + m_d + m)R^2 + I_{w2}\big)\dot{\psi}\Bigg)\cos(\theta) + \big(2\dot{\beta}\dot{\theta}(-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta)\big)\cos(\beta) + \big(-\dot{\eta}\dot{\phi}\sin(\beta)\,I_{d1}\big)\sin(\theta) - 2\dot{\beta}\left(\dot{\theta}(lm_2 + l_d m_d)R + \frac{\dot{\eta} I_{d1}}{2}\right)\sin(\beta),$$

$$
\begin{aligned}
C_3 = {} & 2\dot{\phi}\dot{\beta}\sin(\beta)\Big((l^2m_2 + l_d^2m_d + I_{d1} - I_{d3} + I_{l1} - I_{l3})\cos(\beta) + R(lm_2 + l_dm_d)\Big)\cos(\theta)^2 \\
& + \Big(2\dot{\phi}\Big((l^2m_2 + l_d^2m_d + I_{d1} - I_{d3} + I_{l1} - I_{l3})\cos(\beta)^2 \\
& + 2R(lm_2 + l_dm_d)\cos(\beta) + (m_2 + m_d + m)R^2 - I_{l1} + I_{l2} + I_{w2} - I_{w3} - I_{d1} \\
& + I_{d2}\Big)\dot{\theta}\sin(\theta) + 2\dot{\beta}\dot{\theta}(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\cos(\beta)^2 \\
& - \dot{\eta}\dot{\beta}\cos(\beta)\, I_{d1} \\
& - \Big((-2l^2m_2 - 2l_d^2m_d - I_{d1} - I_{d2} + I_{d3} - I_{l1} - I_{l2} + I_{l3})\dot{\beta} - I_{w2}\dot{\psi}\Big)\dot{\theta}\Big)\cos(\theta) \\
& + \sin(\beta)\Big(\Big(-\dot{\theta}^2(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\cos(\beta) \\
& + R(lm_2 + l_dm_d)\dot{\theta}^2 + \dot{\eta}I_{d1}\dot{\theta} - R\dot{\beta}^2(lm_2 + l_dm_d)\Big)\sin(\theta) \\
& + R\dot{\phi}(lm_2 + l_dm_d)(\dot{\psi} - 2\dot{\beta})\Big), \\
C_4 = {} & -\dot{\phi}^2\Big((l^2m_2 + l_d^2m_d + I_{d1} - I_{d3} + I_{l1} - I_{l3})\cos(\beta) + R(lm_2 + l_dm_d)\Big)\sin(\beta)\cos(\theta)^2 \\
& + 2\dot{\phi}\Bigg(-\dot{\theta}(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\cos(\beta)^2 \\
& + \left(\dot{\theta}(lm_2 + l_dm_d)R + \frac{\dot{\eta}I_{d1}}{2}\right)\cos(\beta) \\
& + \frac{\dot{\theta}(I_{d2} + I_{d3} - I_{l1} + I_{l2} + I_{l3} - I_{d1})}{2}\Bigg)\cos(\theta) \\
& + \big(-\dot{\theta}^2(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\cos(\beta) \\
& + R\dot{\phi}\dot{\psi}(lm_2 + l_dm_d)\sin(\theta) + R(lm_2 + l_dm_d)\dot{\phi}^2 \\
& + \big(\dot{\theta}(lm_2 + l_dm_d)R + \dot{\eta}I_{d1}\big)\dot{\theta}\big)\sin(\beta), \\
C_5 = {} & \Big(\dot{\phi}(I_{d1} - I_{d2} + I_{d3})\sin(\theta) - \dot{\beta}(I_{d1} + I_{d2} - I_{d3})\Big)\dot{\theta}\sin(\beta) \\
& - \Big(\dot{\phi}(I_{d2} - I_{d3})\sin(\theta) + \dot{\beta}(I_{d1} + I_{d2} - I_{d3})\Big)\dot{\phi}\cos(\beta)\cos(\theta),
\end{aligned}
$$

$$
\boldsymbol{G} = \begin{bmatrix} 0 \\ -mgR\sin(\theta) - m_2g\begin{pmatrix}\sin(\theta)\,R \\ +\sin(\theta)\cos(\beta)\,l\end{pmatrix} - m_dg\begin{pmatrix}\sin(\theta)\,R \\ +\sin(\theta)\cos(\beta)\,l_d\end{pmatrix} \\ 0 \\ -m_2\,g\cos(\theta)\sin(\beta)\,l - m_dg\cos(\theta)\sin(\beta)\,l_d \\ 0 \end{bmatrix}, \tag{4-55}
$$

and

$$
\boldsymbol{Q_i} = \begin{bmatrix} \tau_2 \\ 0 \\ 0 \\ -\tau_2 \\ \tau_1 \end{bmatrix} \tag{4-56}
$$

Conclusion

The main goal of this research was to build and check the accuracy of the equations of motion of the unicycle. In this study Newton Euler method and Lagrange method were used to accomplish the main goal. This work fills a gap in the literature available for the unicycle dynamics by providing the Newton Euler method. The unicycle was modeled in three-dimensional space considering the yaw angle which incorporates with the steering direction. It was separated in to three components namely, driving wheel, frame, and rotating disk, and analyzed separately using both methods. The dynamic equations obtained from the Lagrange method are exactly matches with the equations obtained from Newton Euler method with transformations. These equations are (4-31), (4-41), (4-43), (4-45) and (4-37). Therefore, it is reasonable to declare that the equations of motion developed for this unicycle model are accurate. Finally, for the easiness of comparison they are expressed in matrix form in (4-53) through (4-56).

The Newton Euler method provides 18 equations of motion before the transformations. For each component mentioned earlier 3 Newton equations and 3 Euler equations were derived. After careful comparison and using some mathematical transformations for Newton Euler method, a matching set of equations were obtained from both methods. These transformations can be found in (4-40), (4-42) and (4-44). Even though the Lagrange method is much simpler and faster in this calculation, the Newton Euler method provides much accurate results. The reason for this is, in Lagrange method the user must introduce the generalized force vector (4-56), by observing the free body diagram. This can be erroneous. But in Newton Euler method, after the transformations, this generalized force vector can be obtained automatically from the calculations.

The equations obtained from this work are highly coupled and non-linear. As for the next step of this project, the dynamics model can be simulated using MATLAB after linearizing the model. A state space system can be built for the linearized model. The states can be chosen considering the degrees of freedom and the actuators used in the unicycle. A numerical analysis can be carried out assigning some numerical values for the components, velocities, and the angles. Thus, the unicycle can be modeled and simulated in MATLAB. After modelling the unicycle, it is capable to build a controller for the latitude and longitude control of the unicycle. Finally, a unicycle can be built, and the optimized controllers can be utilized to control the stability of the unicycle.

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Appendix A - **Dynamic Calculations of Rolling Wheel**

Maple ® [17] software is used to do all the calculations in this research. Some complex derivations and long expressions obtained for results are presented here.

Newton-Euler Method

The free body diagram in Figure 2.1 is referred in order to do the calculations in this section. The variables used in the thesis are defined in the nomenclature section.

The transformation matrices used to do the kinematic calculations are defined in the immediate text to follow.

The transformation from global origin to point of contact is

$$\boldsymbol{PoC} = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad \textbf{(A-1)}$$

Transformation from point of contact to wheel center is given by

$$\boldsymbol{TR} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & R \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad \textbf{(A-2)}$$

The rotation matrix for the rotation of ϕ angle about the Z axis is

$$\boldsymbol{RotZ} = \begin{bmatrix} cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad \textbf{(A-3)}$$

The rotation matrix for the rotation of θ angle about the X axis is

$$\boldsymbol{RotX} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad \textbf{(A-4)}$$

The description of the wheel centered frame (X'Y'Z'), in terms of the global coordinate system is calculated as

$$\boldsymbol{WCF} = \boldsymbol{PoC} \times \boldsymbol{Rotz} \times \boldsymbol{Rotx} \times \boldsymbol{TR}, \quad \textbf{(A-5)}$$

and $\boldsymbol{WCF}$ **(A-6)**

$$= \begin{bmatrix} \cos(\phi) & -\sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta) & \sin(\phi)\sin(\theta)\,R + x \\ \sin(\phi) & \cos(\theta)\cos(\phi) & -\sin(\theta)\cos(\phi) & -\cos(\phi)\sin(\theta)\,R + y \\ 0 & \sin(\theta) & \cos(\theta) & \cos(\theta)\,R \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

is the output.

The description of the global frame orientation in terms of the (X'Y'Z') frame is obtained by

$$\boldsymbol{GFO} = (\boldsymbol{RotX})^T \times (\boldsymbol{RotZ})^T, \quad \textbf{(A-7)}$$

and **(A-8)**

$$\boldsymbol{GFO} = \begin{bmatrix} cos(\phi) & \sin(\phi) & 0 & 0 \\ -\sin(\phi)\cos(\theta) & \cos(\phi)\cos(\theta) & \sin(\theta) & 0 \\ \sin(\phi)\sin(\theta) & -\cos(\phi)\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here the superscript T denotes the transpose of that matrix. The wheel center kinematics are derived using the above matrices. The wheel center location coordinates in (XYZ) frame are obtained from WCF matrix and they are expressed in (2-1) through (2-3).

The angular velocity of the wheel is calculated in the wheel centered frame (X'Y'Z') for the easiness of the calculations. The velocity vector consists of three velocity components namely, wheel's rolling velocity $\dot{\psi}$, pitch angular velocity $\dot{\theta}$ and yaw angular velocity $\dot{\phi}$. While, roll and pitch angular velocities are expressed in the (X'Y'Z') frame, the yaw angular velocity is given in (XYZ) frame. Therefore, the $\dot{\phi}$ velocity component was transformed in to the (X'Y'Z') frame using the rotation matrix GFO. The translational part of this matrix is disregarded in this case. The angular velocity of the wheel calculated as

$$\boldsymbol{\Omega} = \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\psi} \\ 0 \end{bmatrix} + (GFO) \times \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}, \quad \textbf{(A-9)}$$

$$\boldsymbol{\Omega} = \begin{bmatrix} \dot{\theta} \\ \dot{\psi} + \dot{\phi}\sin(\theta) \\ \dot{\phi}\cos(\theta) \end{bmatrix}. \quad \textbf{(A-10)}$$

The reaction forces at the wheel contact point with the ground, calculated using (2-21) are

$$F_X = mR\left(\left(\left(-\dot{\phi}^2 - \dot{\theta}^2\right)\sin(\theta) + \ddot{\theta}\cos(\theta) - \dot{\phi}\dot{\psi}\right)\sin(\phi)\right.$$
$$\left. + 2\cos(\phi)\left(\dot{\phi}\dot{\theta}\cos(\theta) + \frac{\ddot{\phi}\sin(\theta)}{2} + \frac{\ddot{\psi}}{2}\right)\right), \quad \textbf{(A-11)}$$

$$F_Y = -(\left(\left(-\dot{\phi}^2 - \dot{\theta}^2\right)\sin(\theta) + \ddot{\theta}\cos(\theta) - \dot{\phi}\dot{\psi}\right)\cos(\phi)$$
$$\left. - 2\sin(\phi)\left(\dot{\phi}\dot{\theta}\cos(\theta) + \frac{\ddot{\phi}\sin(\theta)}{2} + \frac{\ddot{\psi}}{2}\right)\right), \quad \textbf{(A-12)}$$

and
$$N = mg - mR\dot{\theta}^2\cos(\theta) - mR\ddot{\theta}\sin(\theta). \quad \textbf{(A-13)}$$

In order to determine the external moments ($\boldsymbol{M}$) applied to the wheel, the contact point reactions are transformed in to the (X'Y'Z') frame. The local force vector in wheel centered frame is calculated by

$$\boldsymbol{LocalF} = \boldsymbol{GFO} \times \boldsymbol{RF}, \quad \textbf{(A-14)}$$

where
$$\boldsymbol{RF} = \begin{bmatrix} F_X \\ F_Y \\ N \\ 0 \end{bmatrix}, \quad \textbf{(A-15)}$$

and
$$\boldsymbol{LocalF} = \begin{bmatrix} \cos(\phi)\,F_X + \sin(\phi)\,F_Y \\ -\sin(\phi)\cos(\theta)\,F_X + \cos(\phi)\cos(\theta)\,F_Y + \sin(\theta)\,N \\ \sin(\phi)\cos(\theta)\,F_X - \cos(\phi)\sin(\theta)\,F_Y + \cos(\theta)\,N \\ 0 \end{bmatrix} \quad \textbf{(A-16)}$$
$$= \begin{bmatrix} LocalF_{X'} \\ LocalF_{Y'} \\ LocalF_{Z'} \\ 0 \end{bmatrix}.$$

$\boldsymbol{M}$ is calculated by taking the cross product between, the wheel contact point vector in local frame and the contact point reaction forces vector in local frame as

$$\boldsymbol{M} = \begin{bmatrix} 0 \\ 0 \\ -R \end{bmatrix} \times \begin{bmatrix} LocalF_{X'} \\ LocalF_{Y'} \\ LocalF_{Z'} \end{bmatrix} \quad \textbf{(A-17)}$$
$$= \begin{bmatrix} R(-F_X\cos(\theta)\sin(\phi) + F_Y\cos(\theta)\cos(\phi) + N\sin(\theta)) \\ -R(F_X\cos(\phi) + F_Y\sin(\phi)) \\ 0 \end{bmatrix}.$$

Then, the Euler's equations are calculated using (2-25) and (2-27), where

$$(\dot{\boldsymbol{H}})_{\boldsymbol{r}} = \boldsymbol{\alpha} \times \boldsymbol{I_w}. \quad \textbf{(A-18)}$$

The final set of dynamic equations for the wheel calculated using the Newton-Euler Method are expressed in (2-31), (2-32) and (2-34).

Lagrangian Method

The Lagrange dynamic equations for the wheel are obtained by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = Q_i + \sum_{k=1}^{n} \lambda_k a_{ki} \quad \textbf{(A-19)}$$

The variables in (A-19) are defined in Chapter 2 and the Nomenclature section. The results of this calculation are provided here. The equations obtained for $q_i = \psi, \theta, \phi$, x and y, respectively, are

$$(\ddot{\psi} + \dot{\theta}\cos(\theta)\dot{\phi} + \sin(\theta)\,\ddot{\phi})I_{w2} - \tau_2 - R\cos(\phi)\,\lambda_1 - R\sin(\phi)\lambda_2 = 0, \quad \textbf{(A-20)}$$

$$(R^2m + I_{w1})\ddot{\theta} + \ddot{x}\sin(\phi)\cos(\theta)\,Rm - \ddot{y}\cos(\phi)\cos(\theta)\,Rm - \quad \textbf{(A-21)}$$
$$\cos(\theta)\sin(\theta)\,(R^2m + I_{w2} - I_{w3})\dot{\phi}^2 - I_{w2}\dot{\phi}\cos(\theta)\,\dot{\psi} - mgR\sin(\theta) = 0,$$

$$\left((-R^2m - I_{w2} + I_{w3}) * \cos^2(\theta) + R^2m + I_{w2}\right)\ddot{\phi} + I_{w2}\ddot{\psi}\sin(\theta) \quad \textbf{(A-22)}$$
$$+ \ddot{x}\cos(\phi)\sin(\theta)\,Rm + \sin(\theta)\sin(\phi)\,\ddot{y}Rm$$
$$+ 2\left(\sin(\theta)\,(R^2m + I_{w2} - I_{w3})\dot{\phi} + \dot{\psi}\frac{I_{w2}}{2}\right)\cos(\theta)\,\dot{\theta}$$
$$- \sin(\theta)\;\tau_2 = 0,$$

$$2\left(\frac{\ddot{x}}{2} + R\left(\dot{\phi}\cos(\phi)\,\dot{\theta}\cos(\theta) - \dot{\phi}^2\sin(\phi)\frac{\sin(\theta)}{2} - \sin(\phi)\,\dot{\theta}^2\frac{\sin(\theta)}{2}\right.\right. \quad \textbf{(A-23)}$$
$$\left.\left. + \ddot{\phi}\cos(\phi)\frac{\sin(\theta)}{2} + \sin(\phi)\,\ddot{\theta}\frac{\cos(\theta)}{2}\right)\right)m + \lambda_1 = 0,$$

and

$$-\left(-\ddot{y} + R\left(-\cos(\phi)\sin(\theta)\,\dot{\phi}^2 - \cos(\phi)\sin(\theta)\,\dot{\theta}^2 - \right.\right. \quad \textbf{(A-24)}$$
$$\left.\left. 2\sin(\phi)\cos(\theta)\,\dot{\phi}\dot{\theta} + \cos(\phi)\cos(\theta)\,\ddot{\theta} - \sin(\phi)\sin(\theta)\,\ddot{\phi}\right)\right)m + \lambda_2 = 0.$$

The first time derivatives of the non-holonomic constraints of (2-38) and (2-39) are respectively

$$\ddot{x} = R\ddot{\psi}\cos(\phi) + R\dot{\psi}\dot{\phi}\sin(\phi), \quad \textbf{(A-25)}$$

and

$$\ddot{y} = -R\ddot{\psi}\sin(\phi) - R\dot{\psi}\dot{\phi}\cos(\phi). \quad \textbf{(A-26)}$$

These results in (A-25) and (A-26) then substituted in (A-23) and (A-24) to calculate λ_1 and λ_2. They are

$$\lambda_1 = -m(\ddot{x} + R\ddot{\phi}\cos(\phi)\sin(\theta) - R\dot{\phi}^2\sin(\phi)\sin(\theta) + 2R\dot{\phi}\cos(\phi)\,\dot{\theta}\cos(\theta) + R\sin(\phi)\,\ddot{\theta}\cos(\theta) - R\sin(\phi)\,\theta^2\sin(\theta), \quad \textbf{(A-27)}$$

and

$$\lambda_2 = -m\big(\ddot{y} - R\ddot{\theta}\cos(\theta)\cos(\phi) + R\theta^2\sin(\theta)\cos(\phi) + 2R\dot{\theta}\cos(\theta)\,\dot{\phi}\sin(\phi) + R\sin(\theta)\,\ddot{\phi}\sin(\phi) + R\sin(\theta)\,\phi^2\cos(\phi)\big). \quad \textbf{(A-28)}$$

Appendix B - **Dynamic Calculations of Rolling Wheel and Frame**

Newton-Euler Method

The calculations in this section is carried out in reference with Figure 3.2. The kinematics relationships related to wheel, explained in Appendix A are not repeated in here. Reader may refer to previous chapters to see those calculations.

The transformation from wheel center to the center of mass of frame is

$$\boldsymbol{Tl} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad \textbf{(B-1)}$$

The rotation matrix for the rotation of β angle about the Y' axis is

$$\boldsymbol{RotY} = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad \textbf{(B-2)}$$

The description of the center of mass of rod, in terms of the global coordinate system is calculated by

$$\boldsymbol{LCF} = \boldsymbol{WCF} \times \boldsymbol{RotY} \times \boldsymbol{Tl}. \quad \textbf{(B-3)}$$

The coordinates of the center of mass of the rod is provided in (3-1) through (3-3). The description of the (X'Y'Z') frame in terms of the (X"Y"Z") is obtained by

$$\boldsymbol{WFO} = (\boldsymbol{RotY})^T\ , \quad \textbf{(B-4)}$$

and

$$\boldsymbol{WFO} = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\beta) & 0 & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad \textbf{(B-5)}$$

The angular velocity vector $\boldsymbol{\Omega}_{lw}$ of the rod with respective to (X'Y'Z') frame is given by

$$\boldsymbol{\Omega}_{lw} = \begin{bmatrix} \dot{\theta} \\ \dot{\beta} + \dot{\phi}\sin(\theta) \\ \dot{\phi}\cos(\theta) \end{bmatrix}. \quad \textbf{(B-6)}$$

The angular velocity vector $\boldsymbol{\Omega}_{l}$ of the rod with respective to the (X"Y"Z") frame is calculated as

$$\boldsymbol{\Omega}_{l} = \boldsymbol{WFO} \times \boldsymbol{\Omega}_{lw}, \quad \textbf{(B-7)}$$

where,

$$\boldsymbol{\Omega_l} = \begin{bmatrix} \dot{\theta}\cos(\beta) - \sin(\beta)\cos(\theta)\,\dot{\phi} \\ \dot{\beta} + \sin(\theta)\,\dot{\phi} \\ \cos(\beta)\cos(\theta)\,\dot{\phi} + \dot{\theta}\sin(\beta) \end{bmatrix}. \quad \textbf{(B-8)}$$

The translational part of $\boldsymbol{WFO}$ matrix is disregarded in above calculation.

In order to apply the Newton's equation (2-21) for the rod in (X"Y"Z") frame, the reactions at the joint were transformed in to the (X"Y"Z") frame first. The reaction force vector at the joint is calculated as

$$\boldsymbol{LocalF2} = \boldsymbol{WFO} \times \boldsymbol{CF}, \quad \textbf{(B-9)}$$

where

$$\boldsymbol{CF} = \begin{bmatrix} S_{X'} \\ S_{Y'} \\ S_{Z'} \\ 0 \end{bmatrix}, \quad \textbf{(B-10)}$$

and

$$\boldsymbol{LocalF2} = \begin{bmatrix} \cos(\beta)\,S_{X'} - \sin(\beta)\,S_{Z'} \\ S_{Y'} \\ \sin(\beta)\,S_{X'} + \cos(\beta)\,S_{Z'} \\ 0 \end{bmatrix} = \begin{bmatrix} LocalF2_{X''} \\ LocalF2_{Y''} \\ LocalF2_{Z''} \\ 0 \end{bmatrix}. \quad \textbf{(B-11)}$$

Also, the wheel center accelerations calculated with respect to the (XYZ) frame in (2-9) through (2-11) transformed to the (X'Y'Z') frame as

$$\boldsymbol{a_{wcL}} = \boldsymbol{GFO} \times \boldsymbol{a_{wcG}}, \quad \textbf{(B-12)}$$

where

$$\boldsymbol{a_{wcG}} = \begin{bmatrix} a_X \\ a_Y \\ a_Z \\ 0 \end{bmatrix} \quad \textbf{(B-13)}$$

and

$$\boldsymbol{a_{wcL}} = \begin{bmatrix} a_{X'} \\ a_{Y'} \\ a_{Z'} \\ 0 \end{bmatrix} = \begin{bmatrix} R\big(2\cos(\theta)\,\dot{\phi}\dot{\theta} + \sin(\theta)\,\ddot{\phi} + \ddot{\psi}\big) \\ R\big(\cos(\theta)\,\dot{\phi}\big(\sin(\theta)\,\dot{\phi} + \dot{\psi}\big) - \ddot{\theta}\big) \\ R\big(\cos(\theta)^2\,\dot{\phi}^2 - \dot{\phi}\dot{\psi}\sin(\theta) - \dot{\phi}^2 - \dot{\theta}^2\big) \\ 0 \end{bmatrix}. \quad \textbf{(B-14)}$$

The description of the (XYZ) frame orientation in terms of the (X"Y"Z") is obtained by

$$\boldsymbol{GFO2} = (\boldsymbol{RotY})^T \times (\boldsymbol{RotX})^T \times (\boldsymbol{RotZ})^T \quad \textbf{(B-15)}$$

$$\boldsymbol{GFO2} = \begin{bmatrix} c(\phi)\,c(\beta) - s(\phi)s(\theta)s(\beta) & s(\phi)c(\beta) + c(\phi)s(\theta)\,s(\beta) & -c(\theta)s(\beta) & 0 \\ -s(\phi)c(\theta) & c(\phi)c(\theta) & s(\theta) & 0 \\ c(\phi)s(\beta) + s(\phi)s(\theta)c(\beta) & s(\phi)s(\beta) - c(\phi)s(\theta)c(\beta) & c(\theta)c(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The terms cos() and sin() represented by c() and s() respectively in (B-15).

The accelerations of the rod center calculated with respect to the (XYZ) frame in (3-7) through (3-9) transformed to the (X"Y"Z") frame as

$$\boldsymbol{a_{rcL}} = \boldsymbol{GFO2} \times \boldsymbol{a_{rcG}}, \tag{B-16}$$

where,

$$\boldsymbol{a_{rcG}} = \begin{bmatrix} a_{lX} \\ a_{lY} \\ a_{lZ} \\ 0 \end{bmatrix} \tag{B-17}$$

and

$$\boldsymbol{a_{rcL}} = \begin{bmatrix} a_{lX''} \\ a_{lY''} \\ a_{lZ''} \\ 0 \end{bmatrix}. \tag{B-18}$$

The acceleration components of (B-18) are

$$\begin{aligned} a_{lX''} = {} & 2\cos(\theta)\cos(\beta)^2\,\dot{\phi} l\dot{\theta} \\ & + \Bigg(\left(-\cos(\theta)^2\, l\dot{\phi}^2 + l\dot{\theta}^2\right)\sin(\beta) \\ & + 2R\left(\cos(\theta)\,\dot{\phi}\dot{\theta} + \frac{\sin(\theta)\,\ddot{\phi}}{2} + \frac{\ddot{\psi}}{2}\right)\Bigg)\cos(\beta) \\ & - R\left(\cos(\theta)^2\,\dot{\phi}^2 - \dot{\phi}\dot{\psi}\sin(\theta) - \dot{\phi}^2 - \dot{\theta}^2\right)\sin(\beta) + l\left(\sin(\theta)\,\ddot{\phi} + \ddot{\beta}\right), \end{aligned}$$

$$\begin{aligned} a_{lY''} = {} & \left(l\dot{\phi}\left(\sin(\theta)\,\dot{\phi} + 2\dot{\beta}\right)\cos(\beta) + R\sin(\theta)\dot{\phi}^2 + R\dot{\phi}\dot{\psi} + l\ddot{\phi}\sin(\beta)\right)\cos(\theta) \\ & + 2\sin(\beta)\, l\dot{\beta}\dot{\theta} - l\ddot{\theta}\cos(\beta) - R\ddot{\theta}, \end{aligned}$$

and

$$\begin{aligned} a_{lZ''} = {} & \left(\cos(\theta)^2\, l\dot{\phi}^2 - l\dot{\theta}^2\right)\cos(\beta)^2 \\ & + \Big(R\cos(\theta)^2\,\dot{\phi}^2 + 2\dot{\theta}\sin(\beta)\, l\dot{\phi}\cos(\theta) \\ & - R\left(\dot{\phi}\dot{\psi}\sin(\theta) + \dot{\theta}^2 + \dot{\phi}^2\right)\Big)\cos(\beta) + 2\cos(\theta)\sin(\beta)\, R\dot{\phi}\dot{\theta} \\ & + R\left(\sin(\theta)\,\ddot{\phi} + \ddot{\psi}\right)\sin(\beta) - 2l\left(\sin(\theta)\,\dot{\phi}\dot{\beta} + \frac{\dot{\phi}^2}{2} + \frac{\dot{\beta}^2}{2}\right). \end{aligned}$$

After solving (3-19) through (3-24), the contact point forces and the reaction force components at the wheel center are calculated. They are

$$F_X = ((-lm_2(\dot{\beta}^2 + \dot{\phi}^2 + \dot{\theta}^2)\cos(\beta) - \ddot{\beta} lm_2 \sin(\beta) - R(\dot{\phi}^2 + \dot{\theta}^2)(m + m_2))\sin(\theta) - 2m_2 l(\dot{\phi}\dot{\beta} - \frac{\cos(\theta)\ddot{\theta}}{2})\cos(\beta) + (-2\dot{\beta}\dot{\theta}\sin(\beta)\, lm_2 + \ddot{\theta}R(m + m_2))\cos(\theta) - \ddot{\phi} lm_2 \sin(\beta) - \dot{\phi}\dot{\psi}R(m + m_2))\sin(\phi) + ((\ddot{\phi} lm_2 \cos(\beta) - 2\dot{\phi}\dot{\beta}\sin(\beta)\, lm_2 + \ddot{\phi}R(m + m_2))\sin(\theta) + 2m_2(\cos(\theta)\,\dot{\phi}\dot{\theta} + \frac{\ddot{\beta}}{2}) l cos(\beta) + 2\dot{\phi}\dot{\theta}R(m + m_2)\cos(\theta) - lm_2(\dot{\beta}^2 + \dot{\phi}^2)\sin(\beta) + \ddot{\psi}R(m + m_2))\cos(\phi), \quad \textbf{(B-19)}$$

$$F_Y = ((lm_2(\dot{\beta}^2 + \dot{\phi}^2 + \dot{\theta}^2)\cos(\beta) + \ddot{\beta} lm_2 \sin(\beta) + R(\dot{\phi}^2 + \dot{\theta}^2)(m + m_2))\sin(\theta) + 2m_2 l(\dot{\phi}\dot{\beta} - \frac{\cos(\theta)\ddot{\theta}}{2})\cos(\beta) + (2\dot{\beta}\dot{\theta}\sin(\beta)\, lm_2 - \ddot{\theta}R(m + m_2))\cos(\theta) + \ddot{\phi} lm_2 \sin(\beta) + \dot{\phi}\dot{\psi}R(m + m_2))\cos(\phi) + ((\ddot{\phi} lm_2 \cos(\beta) - 2\dot{\phi}\dot{\beta}\sin(\beta)\, lm_2 + \ddot{\phi}R(m + m_2))\sin(\theta) + 2m_2\left(\cos(\theta)\,\dot{\phi}\dot{\theta} + \frac{\ddot{\beta}}{2}\right) l\cos(\beta) + 2\dot{\phi}\dot{\theta}R(m + m_2)\cos(\theta) - lm_2(\dot{\beta}^2 + \dot{\phi}^2)\sin(\beta) + \ddot{\psi}R(m + m_2))\sin(\phi), \quad \textbf{(B-20)}$$

$$N = \left(-lm_2(\dot{\beta}^2 + \dot{\theta}^2)\cos(\beta) - \ddot{\beta} lm_2 \sin(\beta) - R\dot{\theta}^2(m + m_2)\right)\cos(\theta) + (-\ddot{\theta} lm_2 \cos(\beta) + 2\dot{\beta}\dot{\theta}\sin(\beta)\, lm_2 - \ddot{\theta}R(m + m_2))\sin(\theta) + g(m + m_2), \quad \textbf{(B-21)}$$

$$S_{X'} = 2m_2(-(\sin(\theta)\,\dot{\phi}\dot{\beta} + \frac{\dot{\phi}^2}{2} + \frac{\dot{\beta}^2}{2}) l\sin(\beta) + l(\cos(\theta)\,\dot{\phi}\dot{\theta} + \frac{\sin(\theta)\,\ddot{\phi}}{2} + \frac{\ddot{\beta}}{2})\cos(\beta) + R(\cos(\theta)\,\dot{\phi}\dot{\theta} + \frac{\sin(\theta)\,\ddot{\phi}}{2} + \frac{\ddot{\psi}}{2})), \quad \textbf{(B-22)}$$

$$S_{Y'} = -m_2((-\dot{\phi}^2(l\cos(\beta) + R)\sin(\theta) - 2\cos(\beta)\, l\dot{\phi}\dot{\beta} - R\dot{\phi}\dot{\psi} - l\ddot{\phi}\sin(\beta))\cos(\theta) - 2\sin(\beta)\, l\dot{\beta}\dot{\theta} + l\ddot{\theta}\cos(\beta) + R\ddot{\theta} - \sin(\theta) g), \quad \textbf{(B-23)}$$

and $$S_{Z'} = -m\_2(l(-\cos(\theta)^2\,\dot{\phi}^2 + 2\sin(\theta)\,\dot{\phi}\dot{\beta} + \dot{\theta}^2 + \dot{\phi}^2 + \dot{\beta}^2)\cos(\beta) - Rcos(\theta)^2\dot{\phi}^2 - \cos(\theta)g + (R\dot{\phi}\dot{\psi} + l\ddot{\phi}\sin(\beta))\sin(\theta) + \ddot{\beta}l\sin(\beta) + R(\dot{\phi}^2 + \dot{\theta}^2)). \quad \textbf{(B-24)}$$

$\boldsymbol{M_l}$ is calculated about the rod center by taking the cross product between, the wheel center position vector in (X"Y"Z") frame and the wheel center reaction forces vector in same frame as

$$\boldsymbol{M_l} = \begin{bmatrix} 0 \\ 0 \\ -l \end{bmatrix} \times \begin{bmatrix} LocalF2_{X''} \\ LocalF2_{Y''} \\ LocalF2_{Z''} \end{bmatrix} = \begin{bmatrix} lS_{Y'} \\ -l(\cos(\beta)\,S_{X'} - \sin(\beta)\,S_{Z'}) \\ 0 \end{bmatrix} \quad \textbf{(B-25)}$$

Lagrangian Method

The Lagrange dynamic equations for the rolling wheel and frame are obtained by applying (A-19). The results of this calculation are provided here. The equations obtained for $q_i = \psi, \theta, \phi, \beta$, x and y, respectively, are

$$I_{w2}\ddot{\psi} + \cos(\theta)\,\dot{\phi}I_{w2}\dot{\theta} + \sin(\theta)\,I_{w2}\dot{\phi} - \tau_2 - R\cos(\phi)\,\lambda_1 - R\sin(\phi)\,\lambda_2 = 0, \quad \textbf{(B-26)}$$

$$(-\dot{\phi}(\sin(\theta)\,\dot{\phi} + 2\dot{\beta})(l^2m_2 + I_{l1} - I_{l3})\cos(\beta)^2 + (-2R\dot{\phi}^2\sin(\theta)\,lm_2 - \ddot{\phi}(l^2m_2 + I_{l1} - I_{l3})\sin(\beta) - 2\left(R\dot{\phi}\dot{\beta} + \frac{\cos(\phi)\ddot{y}}{2} - \frac{\sin(\phi)\ddot{x}}{2}\right)lm_2)\cos(\beta) - \dot{\phi}^2(R^2m + R^2m_2 - I_{l1} + I_{l2} + I_{w2} - I_{w3})\sin(\theta) - R\ddot{\phi}\sin(\beta)\,lm_2 - R\ddot{y}(m + m_2)\cos(\phi) + R\ddot{x}(m + m_2)\sin(\phi) + \dot{\phi}((I_{l1} - I_{l2} - I_{l3})\dot{\beta} - I_{w2}\dot{\psi}))\cos(\theta) + \ddot{\theta}(l^2m_2 + I_{l1} - I_{l3})\cos(\beta)^2 + (-g\sin(\theta)\,lm_2 - 2\dot{\beta}\dot{\theta}(l^2m_2 + I_{l1} - I_{l3})\sin(\beta) + 2Rlm_2\ddot{\theta})\cos(\beta) - Rg(m + m_2)\sin(\theta) - 2R\dot{\beta}lm_2\dot{\theta}\sin(\beta) + \ddot{\theta}(R^2m + R^2m_2 + I_{l3} + I_{w1}) = 0, \quad \textbf{(B-27)}$$

$$
\begin{aligned}
&\Big(\ddot{\phi}(-l^2 m_2 - I_{l1} + I_{l3})\cos(\beta)^2 \\
&\quad + \big(-2\dot{\phi}\dot{\beta}(-l^2 m_2 - I_{l1} + I_{l3})\sin(\beta) - 2R\ddot{\phi} l m_2\big)\cos(\beta) \\
&\quad + 2R\dot{\beta}\dot{\phi} l m_2 \sin(\beta) \\
&\quad - \ddot{\phi}(R^2 m + R^2 m_2 - I_{l1} + I_{l2} + I_{w2} - I_{w3})\Big)\cos(\theta)^2 \\
&\quad + \Big(2\big((l^2 m_2 + I_{l1} - I_{l3})\cos(\beta)^2 + 2Rlm_2\cos(\beta) + R^2 m \\
&\quad + R^2 m_2 + I_{l2} + I_{w2} - I_{w3} - I_{l1}\big)\dot{\theta}\dot{\phi}\sin(\theta) \\
&\quad + 2\dot{\beta}\dot{\theta}(-l^2 m_2 - I_{l1} + I_{l3})\cos(\beta)^2 \\
&\quad + \ddot{\theta}\sin(\beta)\,(-l^2 m_2 - I_{l1} + I_{l3})\cos(\beta) - Rl\ddot{\theta}\sin(\beta)\, m_2 \\
&\quad + \dot{\theta}\big(2\dot{\beta} l^2 m_2 + (I_{l2} - I_{l3} + I_{l1})\dot{\beta} + I_{w2}\dot{\psi}\big)\Big)\cos(\theta) \\
&\quad + \Big(\Big(-\dot{\theta}^2(-l^2 m_2 - I_{l1} + I_{l3})\sin(\beta) \\
&\quad + l m_2\big(R\ddot{\beta} + \ddot{x}\cos(\phi) + \sin(\phi)\,\ddot{y}\big)\Big)\cos(\beta) \\
&\quad - Rl\, m_{2(\dot{\beta}-\dot{\theta})(\dot{\beta}+\dot{\theta})\sin}(\beta) + R\ddot{x}(m + m_2)\cos(\phi) \\
&\quad + R\ddot{y}(m + m_2)\sin(\phi) + l^2 m_2\ddot{\beta} + I_{l2}\ddot{\beta} + I_{w2}\ddot{\psi}\Big)\sin(\theta) \\
&\quad + 2R\ddot{\phi} l m_2\cos(\beta) \\
&\quad - 2\left(R\dot{\phi}\dot{\beta} - \frac{\cos(\phi)\,\ddot{y}}{2} + \frac{\sin(\phi)\,\ddot{x}}{2}\right) l m_2 \sin(\beta) \\
&\quad + \ddot{\phi}\big((R^2 + l^2)m_2 + R^2 m + I_{l2} + I_{w2}\big),
\end{aligned} \tag{B-28}
$$

$$
\begin{aligned}
&-2\dot{\phi}\dot{\theta}\cos(\theta)\,(-l^2 m_2 - I_{l1} + I_{l3})\cos(\beta)^2 \\
&\quad + (\dot{\phi}^2\sin(\beta)\,(-l^2 m_2 - I_{l1} + I_{l3})\cos(\theta)^2 \\
&\quad + 2R\dot{\phi}\dot{\theta}\cos(\theta)\, l m_2 - \dot{\theta}^2(-l^2 m_2 - I_{l1} + I_{l3})\sin(\beta) \\
&\quad + l m_2(R\ddot{\phi}\sin(\theta) + \ddot{x}\cos(\phi) + \sin(\phi)\,\ddot{y}))\cos(\beta) \\
&\quad - \sin(\beta)\cos(\theta)^2 R\dot{\phi}^2 l m_2 + (-g\sin(\beta)\, l m_2 + \dot{\phi}\dot{\theta}(I_{l2} \\
&\quad + I_{l3} - I_{l1}))\cos(\theta) \\
&\quad + l\Big((\cos(\phi)\,\ddot{y} - \sin(\phi)\,\ddot{x})\sin(\theta) \\
&\quad + R\big(\dot{\phi}^2 + \dot{\theta}^2\big)\Big) m_2\sin(\beta) + \ddot{\phi}(l^2 m_2 + I_{l2})\sin(\theta) \\
&\quad + l^2 m_2\ddot{\beta} + I_{l2}\ddot{\beta} + \tau_2,
\end{aligned} \tag{B-29}
$$

$$((-lm_2(\dot{\beta}^2+\dot{\phi}^2+\dot{\theta}^2)\cos(\beta)-\ddot{\beta}lm_2\sin(\beta)-R(\dot{\phi}^2+\dot{\theta}^2)(m \tag{B-30}$$
$$+m_2))\sin(\theta)-2m_2l(\dot{\beta}\dot{\phi}-\frac{\ddot{\theta}\cos(\theta)}{2})\cos(\beta)$$
$$+(-2\dot{\beta}\dot{\theta}\sin(\beta)\,lm_2+R\ddot{\theta}(m+m_2))\cos(\theta)$$
$$-\ddot{\phi}\sin(\beta)\,lm_2)\sin(\phi)+(\ddot{\phi}lm_2\cos(\beta)-2\dot{\beta}\sin(\beta)\,lm_2\dot{\phi}$$
$$+R\ddot{\phi}(m+m_2))\cos(\phi)\sin(\theta)$$
$$+(2\left(\dot{\phi}\dot{\theta}\cos(\theta)+\frac{\ddot{\beta}}{2}\right)m_2l\cos(\beta)+2R\dot{\phi}\dot{\theta}(m$$
$$+m_2)\cos(\theta)-\sin(\beta)\,lm_2(\dot{\beta}^2+\dot{\phi}^2))\cos(\phi)+\ddot{x}m$$
$$+\ddot{x}m_2+\lambda_1,$$

and

$$((lm_2(\dot{\beta}^2+\dot{\phi}^2+\dot{\theta}^2)\cos(\beta)+\ddot{\beta}lm_2\sin(\beta)+R(\dot{\phi}^2+\dot{\theta}^2)(m \tag{B-31}$$
$$+m_2))\sin(\theta)+2m_2l\left(\frac{\dot{\beta}\dot{\phi}-\ddot{\theta}\cos(\theta)}{2}\right)\cos(\beta)$$
$$+(2\dot{\beta}\dot{\theta}\sin(\beta)\,lm_2-R\ddot{\theta}(m+m_2))\cos(\theta)$$
$$+\ddot{\phi}\sin(\beta)\,lm_2)\cos(\phi)+\sin(\phi)(\ddot{\phi}lm_2\cos(\beta)$$
$$-2\dot{\beta}\sin(\beta)\,lm_2\dot{\phi}+R\ddot{\phi}(m+m_2))\sin(\theta)$$
$$+(2\left(\dot{\phi}\dot{\theta}\cos(\theta)+\frac{\ddot{\beta}}{2}\right)m_2l\cos(\beta)+2R\dot{\phi}\dot{\theta}(m$$
$$+m_2)\cos(\theta)-\sin(\beta)\,lm_2(\dot{\beta}^2+\dot{\phi}^2))\sin(\phi)+\ddot{y}m$$
$$+\ddot{y}m_2+\lambda_2.$$

The results in (A-25) and (A-26) then substituted in (B-30) and (B-31) to calculate λ_1 and λ_2. They ar

$$\lambda_1 = ((lm_2(\dot{\beta}^2 + \dot{\phi}^2 + \dot{\theta}^2)\cos(\beta) + \ddot{\beta} lm_2 \sin(\beta) + R(\dot{\phi}^2 + \dot{\theta}^2)(m + m_2))\sin(\theta) + 2l\left(\dot{\beta}\dot{\phi} - \frac{\ddot{\theta}\cos(\theta)}{2}\right) m_2 \cos(\beta) + (2\dot{\beta}\dot{\theta}\sin(\beta)\, lm_2 - R\ddot{\theta}(m + m_2))\cos(\theta) + \ddot{\phi}\sin(\beta)\, lm_2 + R\dot{\phi}\dot{\psi}(m + m_2))\sin(\phi) - \cos(\phi)((\ddot{\phi} lm_2 \cos(\beta) - 2\dot{\beta}\sin(\beta)\, lm_2 \dot{\phi} + R\ddot{\phi}(m + m_2))\sin(\theta) + 2lm_2(\dot{\theta}\cos(\theta)\,\dot{\phi} + \frac{\ddot{\beta}}{2})\cos(\beta) + 2R\dot{\theta}\dot{\phi}(m + m_2)\cos(\theta) - \sin(\beta)\, lm_2(\dot{\beta}^2 + \dot{\phi}^2) + R\ddot{\psi}(m + m_2)), \tag{B-32}$$

$$\lambda_2 = ((-lm_2(\dot{\beta}^2 + \dot{\phi}^2 + \dot{\theta}^2)\cos(\beta) - \ddot{\beta} lm_2 \sin(\beta) - R(\dot{\phi}^2 + \dot{\theta}^2)(m + m_2))\sin(\theta) - 2l\left(\dot{\beta}\dot{\phi} - \frac{\ddot{\theta}\cos(\theta)}{2}\right) m_2 \cos(\beta) + (-2\dot{\beta}\dot{\theta}\sin(\beta)\, lm_2 + R\ddot{\theta}(m + m_2))\cos(\theta) - \ddot{\phi}\sin(\beta)\, lm_2 - R\dot{\phi}\dot{\psi}(m + m_2))\cos(\phi) - ((\ddot{\phi} lm_2 \cos(\beta) - 2\dot{\beta}\sin(\beta)\, lm_2 \dot{\phi} + R\ddot{\phi}(m + m_2))\sin(\theta) + 2lm_2(\dot{\theta}\cos(\theta)\,\dot{\phi} + \frac{\ddot{\beta}}{2})\cos(\beta) + 2R\dot{\theta}\dot{\phi}(m + m_2)\cos(\theta) - \sin(\beta)\, lm_2(\beta^2 + \dot{\phi}^2) + R\ddot{\psi}(m + m_2))\sin(\phi). \tag{B-33}$$

Appendix C - **Dynamic Calculation of Unicycle with Rotating Disk**

Newton-Euler Method

The free body diagram used for the calculations in this section is shown in Figure 4.2.

The transformation from rolling wheel center to center of disk is

$$\boldsymbol{Td} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_d \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad \textbf{(C-1)}$$

The description of the center of mass of rotating disk, in terms of the global coordinate system is calculated by

$$\boldsymbol{DCF} = \boldsymbol{WCF} \times \boldsymbol{RotY} \times \boldsymbol{Td}. \quad \textbf{(C-2)}$$

The coordinates of the center of the disk obtained from (C-2) is given in (4-1) through (4-3).

The accelerations of the rotating disk center calculated with respect to the (XYZ) frame in (4-7) through (4-9) transformed to the (X'''Y'''Z''') frame as

$$\boldsymbol{a_{dcL}} = \boldsymbol{GFO2} \times \boldsymbol{a_{dcG}}, \quad \textbf{(C-3)}$$

where,

$$\boldsymbol{a_{dcG}} = \begin{bmatrix} a_{dX} \\ a_{dY} \\ a_{dZ} \\ 0 \end{bmatrix}, \quad \textbf{(C-4)}$$

and

$$\boldsymbol{a_{dcL}} = \begin{bmatrix} a_{dX'''} \\ a_{dY'''} \\ a_{dZ'''} \\ 0 \end{bmatrix}. \quad \textbf{(C-5)}$$

The acceleration components of (C-5) are

$$\begin{aligned} a_{dX'''} = {} & 2\cos(\theta)\cos(\beta)^2\,\dot{\phi} l_d \dot{\theta} + ((-\cos(\theta)^2\, l_d \dot{\phi}^2 + l_d \dot{\theta}^2)\sin(\beta) \\ & + 2R(\cos(\theta)\,\dot{\phi}\dot{\theta} + \frac{\sin(\theta)\,\ddot{\phi}}{2} + \frac{\ddot{\psi}}{2}))\cos(\beta) - R(\cos(\theta)^2\,\dot{\phi}^2 \\ & - \dot{\phi}\dot{\psi}\sin(\theta) - \dot{\phi}^2 - \dot{\theta}^2)\sin(\beta) + l_d(\sin(\theta)\,\ddot{\phi} + \ddot{\beta}), \end{aligned} \quad \textbf{(C-6)}$$

$$\begin{aligned} a_{dY'''} = {} & (l_d \dot{\phi}(\sin(\theta)\,\dot{\phi} + 2\dot{\beta})\cos(\beta) + R\sin(\theta)\,\dot{\phi}^2 + R\dot{\phi}\dot{\psi} \\ & + l_d \ddot{\phi}\sin(\beta))\cos(\theta) + 2\sin(\beta)\, l_d \dot{\beta}\dot{\theta} - l_d \ddot{\theta}\cos(\beta) - R\ddot{\theta}, \end{aligned} \quad \textbf{(C-7)}$$

and $$a_{dZ'''} = \left(\cos(\theta)^2 l_d\dot{\phi}^2 - l_d\dot{\theta}^2\right)\cos(\beta)^2 + \left(R\cos(\theta)^2\dot{\phi}^2 + 2\dot{\theta}\sin(\beta) l_d\dot{\phi}\cos(\theta) - R\left(\dot{\phi}\dot{\psi}\sin(\theta) + \dot{\theta}^2 + \dot{\phi}^2\right)\right)\cos(\beta) + 2\cos(\theta)\sin(\beta) R\dot{\phi}\dot{\theta} + R\left(\sin(\theta)\ddot{\phi} + \ddot{\psi}\right)\sin(\beta) - 2\left(\sin(\theta)\dot{\phi}\dot{\beta} + \frac{\dot{\phi}^2}{2} + \frac{\dot{\beta}^2}{2}\right) l_d. \quad \textbf{(C-8)}$$

The contact point forces, reaction force components at the joint where wheel and frame connects and the reaction force components at the rotating disk joint are calculated, by solving (3-19) through (3-21) and (4-19) through (4-24). They are,

$$F_X = ((-(\dot{\beta}^2 + \dot{\phi}^2 + \dot{\theta}^2)(lm_2 + l_d m_d)\cos(\beta) - \ddot{\beta}(lm_2 + l_d m_d)\sin(\beta) - R(\dot{\phi}^2 + \dot{\theta}^2)(m_2 + m_d + m))\sin(\theta) - 2(lm_2 + l_d m_d)(\dot{\phi}\dot{\beta} - \frac{\cos(\theta)\ddot{\theta}}{2})\cos(\beta) + (-2\dot{\beta}\dot{\theta}\,(lm_2 + l_d m_d)\sin(\beta) + \ddot{\theta}R(m_2 + m_d + m))\cos(\theta) - \ddot{\phi}(lm_2 + l_d m_d)\sin(\beta) - \dot{\phi}\dot{\psi}R(m_2 + m_d + m))\sin(\phi) + 2\cos(\phi)((\frac{\ddot{\phi}(lm_2 + l_d m_d)\cos(\beta)}{2} - \dot{\phi}\dot{\beta}\,(lm_2 + l_d m_d)\sin(\beta) + \frac{\ddot{\phi}R(m_2 + m_d + m)}{2})\sin(\theta) + (lm_2 + l_d m_d)(\cos(\theta)\dot{\phi}\dot{\theta} + \frac{\ddot{\beta}}{2})\cos(\beta) + \dot{\phi}\dot{\theta}R(m_2 + m_d + m)\cos(\theta) - \frac{\left(\dot{\beta}^2 + \dot{\phi}^2\right)(lm_2 + l_d m_d)\sin(\beta)}{2} + \frac{\ddot{\psi}R(m_2 + m_d + m)}{2}), \quad \textbf{(C-9)}$$

$$F_Y = (\Big(\big(\dot{\beta}^2 + \dot{\phi}^2 + \dot{\theta}^2\big)(lm_2 + l_d m_d)\cos(\beta) + \ddot{\beta}(lm_2 + l_d m_d)\sin(\beta) + R\big(\dot{\phi}^2 + \dot{\theta}^2\big)(m_2 + m_d + m)\Big)\sin(\theta) + 2(lm_2 + l_d m_d)\left(\dot{\phi}\dot{\beta} - \frac{\cos(\theta)\,\ddot{\theta}}{2}\right)\cos(\beta) + \Big(2\dot{\beta}\dot{\theta}(lm_2 + l_d m_d)\sin(\beta) - \ddot{\theta}R(m_2 + m_d + m)\Big)\cos(\theta) + \ddot{\phi}(lm_2 + l_d m_d)\sin(\beta) + \dot{\phi}\dot{\psi}R(m_2 + m_d + m))\cos(\phi) + 2\sin(\phi)((\frac{\ddot{\phi}(lm_2 + l_d m_d)\cos(\beta)}{2} - \dot{\phi}\dot{\beta}\,(lm_2 + l_d m_d)\sin(\beta) + \frac{\ddot{\phi}R(m_2 + m_d + m)}{2})\sin(\theta) + (lm_2 + l_d m_d)(\cos(\theta)\,\dot{\phi}\dot{\theta} + \frac{\ddot{\beta}}{2})\cos(\beta) + \dot{\phi}\dot{\theta}R(m_2 + m_d + m)\cos(\theta) - \frac{\big(\dot{\beta}^2 + \dot{\phi}^2\big)(lm_2 + l_d m_d)\sin(\beta)}{2} + \frac{\ddot{\psi}R(m_2 + m_d + m)}{2}), \tag{C-10}$$

$$N = (-(\dot{\beta}^2 + \dot{\theta}^2)(lm_2 + l_d m_d)\cos(\beta) - \ddot{\beta}(lm_2 + l_d m_d)\sin(\beta) - R\dot{\theta}^2(m_2 + m_d + m))\cos(\theta) + (-\ddot{\theta}(lm_2 + l_d m_d)\cos(\beta) + 2\dot{\beta}\dot{\theta}(lm_2 + l_d m_d)\sin(\beta) - \ddot{\theta}R(m_2 + m_d + m))\sin(\theta) + g(m_2 + m_d + m), \tag{C-11}$$

$$S_{X'} = (\ddot{\phi}(lm_2 + l_d m_d)\cos(\beta) - 2\dot{\phi}\dot{\beta}(lm_2 + l_d m_d)\sin(\beta) + R\ddot{\phi}(m_2 + m_d))\sin(\theta) + 2(lm_2 + l_d m_d)(\cos(\theta)\,\dot{\phi}\dot{\theta} + \frac{\ddot{\beta}}{2})\cos(\beta) - (\dot{\beta}^2 + \dot{\phi}^2)(lm_2 + l_d m_d)\sin(\beta) + R(m_2 + m_d)(2\cos(\theta)\,\dot{\phi}\dot{\theta} + \ddot{\psi}), \tag{C-12}$$

$$S_{Y'} = ((\big((lm_2 + l_d m_d)\cos(\beta) + R(m_2 + m_d)\big)\dot{\phi}^2\sin(\theta) + 2\dot{\beta}\dot{\phi}(lm_2 + l_d m_d)\cos(\beta) + \ddot{\phi}(lm_2 + l_d m_d)\sin(\beta) + R\dot{\psi}\dot{\phi}(m_2 + m_d))\cos(\theta) + g(m_2 + m_d)\sin(\theta) - \ddot{\theta}(lm_2 + l_d m_d)\cos(\beta) + 2\dot{\beta}\dot{\theta}(lm_2 + l_d m_d)\sin(\beta) - R\ddot{\theta}(m_2 + m_d), \tag{C-13}$$

$$S_{Z'} = -(lm_2 + l_d m_d)(-\cos(\theta)^2\,\dot{\phi}^2 + 2\sin(\theta)\,\dot{\phi}\dot{\beta} + \dot{\beta}^2 + \dot{\theta}^2 + \dot{\phi}^2)\cos(\beta) + R\dot{\phi}^2(m_2 + m_d)\cos(\theta)^2 + g(m_2 + m_d)\cos(\theta) + (-\ddot{\phi}(lm_2 + l_d m_d)\sin(\beta) - R\dot{\psi}\dot{\phi}(m_2 + m_d))\sin(\theta) - \ddot{\beta}(lm_2 + l_d m_d)\sin(\beta) - R(\dot{\phi}\hat{\ }2 + \dot{\theta}^2)(m_2 + m_d), \tag{C-14}$$

$$S_{X'''} = -m_d(((-\cos(\theta)^2\, l_d\dot{\phi}^2 + l_d\dot{\theta}^2)\cos(\beta) - R * \cos(\theta)^2\,\dot{\phi}^2 - g\cos(\theta) + R(\dot{\phi}\dot{\psi}\sin(\theta) + \dot{\theta}^2 + \dot{\phi}^2))\sin(\beta) + 2\cos(\theta)\cos(\beta)^2\,\dot{\phi} l_d\dot{\theta} + R(2\cos(\theta)\,\dot{\phi}\dot{\theta} + \sin(\theta)\,\ddot{\phi} + \ddot{\psi})\cos(\beta) + l_d(\sin(\theta)\,\ddot{\phi} + \ddot{\beta})), \tag{C-15}$$

$$S_{Y'''} = \Big(\big(-\dot{\phi}^2(l_d\cos(\beta) + R)\sin(\theta) - 2\cos(\beta)\, l_d\dot{\phi}\dot{\beta} - R\dot{\phi}\dot{\psi} - l_d\ddot{\phi}\sin(\beta)\big)\cos(\theta) - 2\sin(\beta)\, l_d\dot{\beta}\dot{\theta} + l_d\ddot{\theta}\cos(\beta) + R\ddot{\theta} - \sin(\theta)\, g\Big) m_d, \tag{C-16}$$

and

$$S_{Z'''} = -m_d(\big(\cos(\theta)^2\, l_d\dot{\phi}^2 - l_d\dot{\theta}^2\big)\cos(\beta)^2 + (R\cos(\theta)^2\,\dot{\phi}^2 + (2\dot{\theta}\sin(\beta)\, l_d\dot{\phi} + g)\cos(\theta) - R(\dot{\phi}\dot{\psi}\sin(\theta) + \dot{\theta}^2 + \dot{\phi}^2))\cos(\beta) + 2\cos(\theta)\sin(\beta)\, R\dot{\phi}\dot{\theta} + R(\sin(\theta)\,\ddot{\phi} + \ddot{\psi})\sin(\beta) - 2\left(\sin(\theta)\,\dot{\phi}\dot{\beta} + \frac{\dot{\phi}^2}{2} + \frac{\dot{\beta}^2}{2}\right) l_d). \tag{C-17}$$

Lagrangian Method

The Lagrange dynamic equations for the unicycle with rotating disk are obtained by applying (A-19). The results of this calculation are provided here. The equations obtained for $q_i = \psi, \theta$,ϕ,β, η,x and y, respectively, are

$$I_{w2}\ddot{\psi} + \cos(\theta)\,\dot{\phi} I_{w2}\dot{\theta} + \sin(\theta)\, I_{w2}\dot{\phi} - \tau_2 - R\cos(\phi)\,\lambda_1 - R\sin(\phi)\,\lambda_2 = 0, \tag{C-18}$$

(C-19)

$$\Bigg(\dot{\phi}\big(\sin(\theta)\,\dot{\phi} + 2\dot{\beta}\big)(-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\cos(\beta)^2$$

$$+ \Bigg(-2R\dot{\phi}^2(l m_2 + l_d m_d)\sin(\theta)$$

$$+ \ddot{\phi}(-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta) - 2(R\dot{\phi}\dot{\beta}$$

$$+ \frac{\cos(\phi)\,\ddot{y}}{2} - \frac{\sin(\phi)\,\ddot{x}}{2})(l m_2 + l_d m_d)\Bigg)\cos(\beta)$$

$$- \dot{\phi}^2\big((m_2 + m_d + m)R^2 + I_{d2} - I_{l1} + I_{l2} + I_{w2} - I_{w3}$$

$$- I_{d1}\big)\sin(\theta)$$

$$- R\ddot{\phi}(l m_2 + l_d m_d)\sin(\beta) - R\ddot{y}(m_2 + m_d + m)\cos(\phi)$$

$$+ R\ddot{x}(m_2 + m_d + m)\sin(\phi)$$

$$- \dot{\phi}\left((I_{d2} + I_{d3} - I_{l1} + I_{l2} + I_{l3} - I_{d1})\dot{\beta} + I_{w2}\dot{\psi}\right)\Bigg)\cos(\theta)$$

$$- \ddot{\theta}(-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\cos(\beta)^2$$

$$+ \big(-g(l m_2 + l_d m_d)\sin(\theta)$$

$$+ 2\dot{\beta}\dot{\theta}(-l^2 m_2 - l_d^2 m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta)$$

$$+ 2\ddot{\theta}(l m_2 + l_d m_d)R + I_{d1}\ddot{\eta}\big)\cos(\beta)$$

$$+ \big(-\dot{\eta}\dot{\phi}\sin(\beta)\,I_{d1} - Rg(m_2 + m_d + m)\big)\sin(\theta)$$

$$- 2\dot{\beta}\left(\dot{\theta}(l m_2 + l_d m_d)R + \frac{\dot{\eta} I_{d1}}{2}\right)\sin(\beta)$$

$$+ \big((m_2 + m_d + m)R^2 + I_{d3} + I_{l3} + I_{w1}\big)\ddot{\theta} = 0.$$

$$\Big(\ddot{\phi}\,(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})(\cos(\beta))^2 \tag{C-20}$$

$$+ \Big(-2\dot{\phi}\dot{\beta}(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta)$$

$$- 2R\ddot{\phi}(lm_2 + l_dm_d)\Big)\cos(\beta) + 2R\dot{\phi}\dot{\beta}(lm_2 + l_dm_d)\sin(\beta)$$

$$- \ddot{\phi}\,\Big((m_2 + m_d + m)R^2 + I_{d2} - I_{l1} + I_{l2} + I_{w2} - I_{w3}$$

$$- I_{d1})\Big)(\cos(\theta))^2$$

$$+ \Big(-2\dot{\theta}\big(\sin(\theta)\,\dot{\phi} - \dot{\beta}\big)\big(-l^2m_2 - l_{d_d}^{\;2m} - I_{d1} + I_{d3} - I_{l1}$$

$$+ I_{l3}\big)(\cos(\beta))^2$$

$$+ \Big(4R\dot{\phi}\dot{\theta}(lm_2 + l_dm_d)\sin(\theta)$$

$$+ \ddot{\theta}(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta) - I_{d1}\dot{\beta}\dot{\eta}\Big)\cos(\beta)$$

$$+ 2\dot{\theta}\dot{\phi}\big((m_2 + m_d + m)R^2 + I_{d2} - I_{l1} + I_{l2} + I_{w2} - I_{w3}$$

$$- I_{d1}\big)\sin(\theta) + \big(-\ddot{\theta}(lm_2 + l_dm_d)R - I_{d1}\ddot{\eta}\big)\sin(\beta)$$

$$- \dot{\theta}\,\Big((-2\,l^2m_2 - 2l_d^2m_d - I_{d1} - I_{d2} + I_{d3} - I_{l1} - I_{l2} + I_{l3})\dot{\beta}$$

$$- I_{w2}\dot{\psi})\Big)\cos(\theta)$$

$$+ \Big(\big(-\dot{\theta}^2(-l^2m_2 - l_d^2m_d - I_{d1} + I_{d3} - I_{l1} + I_{l3})\sin(\beta) + (lm_2$$

$$+ l_dm_d)(R\ddot{\beta} + \ddot{x}\cos(\phi) + \sin(\phi)\,\ddot{y})\big)\sin(\theta)$$

$$+ 2R\ddot{\phi}(lm_2 + l_dm_d)\Big)\cos(\beta)$$

$$+ \Big(\Big(R(lm_2 + l_dm_d)\dot{\theta}^2 + I_{d1}\dot{\eta}\dot{\theta} - R\dot{\beta}^2(lm_2 + l_dm_d)\Big)\sin(\beta)$$

$$+ R\ddot{x}(m_2 + m_d + m)\cos(\phi) \; + \; R\ddot{y}(m_2 + m_d + m)\sin(\phi) + l^2m_2\ddot{\beta}$$

$$+ l_d^2m_d\ddot{\beta} + (I_{l2} + I_{d2})\ddot{\beta} \; + I_{w2}\ddot{\psi}\Big)\sin(\theta)$$

$$- 2\left(R\dot{\phi}\dot{\beta} - \frac{\cos(\phi)\,\ddot{y}}{2} + \frac{\sin(\phi)\,\ddot{x}}{2}\right)(lm_2 + l_dm_d)\sin(\beta)$$

$$+ \ddot{\phi}\Big((m_2 + m_d + m)R^2 \; + l^2m_2 + l_d^2m_d + I_{l2} + I_{w2} + I_{d2}\Big) = 0,$$

$$
\begin{aligned}
&2\dot{\phi}\dot{\theta}\cos(\theta)\,(l^2 m_2 + l_d^2 m_d + I_{d1} - I_{d3} + I_{l1} - I_{l3})\cos(\beta)^2 \\
&\quad + \Big(-\dot{\phi}^2 \sin(\beta)\,(l^2 m_2 + l_d^2 m_d + I_{d1} - I_{d3} + I_{l1} - I_{l3})\cos(\theta)^2 \\
&\quad + 2\left(\dot{\theta}(l m_2 + l_d m_d)R + \frac{\dot{\eta} I_{d1}}{2}\right)\dot{\phi}\cos(\theta) \\
&\quad + \dot{\theta}^2 (l^2 m_2 + l_d^2 m_d + I_{d1} - I_{d3} + I_{l1} - I_{l3})\sin(\beta) \\
&\quad + (l m_2 + l_d m_d)\big(R\ddot{\phi}\sin(\theta) + \ddot{x}\cos(\phi) + \sin(\phi)\,\ddot{y}\big)\Big)\cos(\beta) \\
&\quad - R\dot{\phi}^2 \sin(\beta)\,(l m_2 + l_d m_d)\cos(\theta)^2 \\
&\quad + \Big(-g(l m_2 + l_d m_d)\sin(\beta) \\
&\quad - \dot{\phi}\dot{\theta}(I_{d1} - I_{d2} - I_{d3} + I_{l1} - I_{l2} - I_{l3})\Big)\cos(\theta) \\
&\quad + \Big((\cos(\phi)\,\ddot{y} - \sin(\phi)\,\ddot{x})(l m_2 + l_d m_d)\sin(\theta) \\
&\quad + R(l m_2 + l_d m_d)\dot{\phi}^2 + \dot{\theta}\big(\dot{\theta}(l m_2 + l_d m_d)R + \dot{\eta} I_{d1}\big)\Big)\sin(\beta) \\
&\quad + \ddot{\phi}(l^2 m_2 + l_d^2 m_d + I_{d2} + I_{l2})\sin(\theta) + \ddot{\beta} m_d l_d^2 + \ddot{\beta} m_2 l^2 \\
&\quad + (I_{d2} + I_{l2})\ddot{\beta} + \tau_2 = 0,
\end{aligned}
\tag{C-21}
$$

$$
\begin{aligned}
&\Big(-\Big((l^2m_2 + l_d^2m_d + I_{d1} - I_{d3} + I_{l1} - I_{l3})\cos(\beta) \\
&\quad + R(lm_2 + l_dm_d)\Big)\dot{\phi}^2\cos(\theta)^2 \\
&\quad + \left(-\dot{\theta}\dot{y}(lm_2 + l_dm_d)\cos(\phi) + \dot{\theta}\dot{x}(lm_2 + l_dm_d)\sin(\phi)\right. \\
&\quad \left. - m_dgl_d - m_2gl - I_{d1}\ddot{\phi}\right)\cos(\theta) \\
&\quad + \dot{\theta}^2(l^2m_2 + l_d^2m_d + I_{d1} - I_{d3} + I_{l1} - I_{l3})\cos(\beta) \\
&\quad + \Big(\dot{x}(lm_2 + l_dm_d)\cos(\phi) + \dot{y}(lm_2 + l_dm_d)\sin(\phi) + I_{d1}\dot{\theta} \\
&\quad + R\dot{\beta}(lm_2 + l_dm_d)\Big)\dot{\phi}\sin(\theta) + \dot{\beta}\dot{x}(lm_2 + l_dm_d)\cos(\phi) \\
&\quad + \dot{\beta}\dot{y}(lm_2 + l_dm_d)\sin(\phi) + R(lm_2 + l_dm_d)\dot{\phi}^2 \\
&\quad + \Big(\dot{\theta}(lm_2 + l_dm_d)R + I_{d1}(\dot{\eta} - \dot{\beta})\Big)\dot{\theta}\Big)\sin(\beta) \\
&\quad + \Big(2\dot{\theta}(l^2m_2 + l_d^2m_d + I_{d1} - I_{d3} + I_{l1} - I_{l3})\cos(\beta)^2 \\
&\quad + \Big(\dot{\theta}(lm_2 + l_dm_d)R + I_{d1}(\dot{\eta} - \dot{\beta})\Big)\cos(\beta) \\
&\quad - \dot{\theta}(l^2m_2 + l_d^2m_d + I_{d1} - I_{d3} + I_{l1} - I_{l3})\Big)\dot{\phi}\cos(\theta) \\
&\quad + \left(-\dot{\beta}(\cos(\phi)\dot{y} - \sin(\phi)\dot{x})(lm_2 + l_dm_d)\sin(\theta)\right. \\
&\quad - \dot{\phi}\dot{y}(lm_2 + l_dm_d)\cos(\phi) + \dot{\phi}\dot{x}(lm_2 + l_dm_d)\sin(\phi) \\
&\quad \left. + I_{d1}\ddot{\theta}\right)\cos(\beta) + \ddot{\eta}I_{d1} - \tau_1 = 0,
\end{aligned}
\tag{C-22}
$$

$$
\begin{aligned}
&((-(\dot{\beta}^2 + \dot{\phi}^2 + \dot{\theta}^2)(lm_2 + l_dm_d)\cos(\beta) - \ddot{\beta}(lm_2 + l_dm_d)\sin(\beta) - R(\dot{\phi}^2 \\
&\quad + \dot{\theta}^2)(m_2 + m_d + m))\sin(\theta) - 2(\dot{\phi}\dot{\beta} - \frac{\ddot{\theta}\cos(\theta)}{2})(lm_2 \\
&\quad + l_dm_d)\cos(\beta) + (-2\dot{\beta}\dot{\theta}(lm_2 + l_dm_d)\sin(\beta) + R\ddot{\theta}(m_2 + m_d \\
&\quad + m))\cos(\theta) - \ddot{\phi}\sin(\beta)(lm_2 + l_dm_d))\sin(\phi) + (\ddot{\phi}(lm_2 \\
&\quad + l_dm_d)\cos(\beta) - 2\dot{\phi}\dot{\beta}(lm_2 + l_dm_d)\sin(\beta) + R\ddot{\phi}(m_2 + m_d \\
&\quad + m))\cos(\phi)\sin(\theta) + (2(\dot{\phi}\dot{\theta}\cos(\theta) + \frac{\ddot{\beta}}{2})(lm_2 + l_dm_d)\cos(\beta) \\
&\quad + 2R\dot{\phi}\dot{\theta}(m_2 + m_d + m)\cos(\theta) - \sin(\beta)(\dot{\beta}^2 + \dot{\phi}^2)(lm_2 \\
&\quad + l_dm_d))\cos(\phi) + \ddot{x}m + \ddot{x}m_2 + \ddot{x}m_d + \lambda_1 = 0,
\end{aligned}
\tag{C-23}
$$

and $$(((\dot\beta^2+\dot\phi^2+\dot\theta^2)(lm_2+l_dm_d)\cos(\beta)+\ddot\beta(lm_2+l_dm_d)\sin(\beta)$$
$$+R(\dot\phi^2+\dot\theta^2)(m_2+m_d+m))\sin(\theta)+2(\dot\phi\dot\beta$$
$$-\frac{\ddot\theta\cos(\theta)}{2})(lm_2+l_dm_d)\cos(\beta)+(2\dot\beta\dot\theta(lm_2$$
$$+l_dm_d)\sin(\beta)-R\ddot\theta(m_2+m_d+m))\cos(\theta)$$
$$+\ddot\phi\sin(\beta)(lm_2+l_dm_d))\cos(\phi)+(\ddot\phi(lm_2$$
$$+l_dm_d)\cos(\beta)-2\dot\phi\dot\beta(lm_2+l_dm_d)\sin(\beta)$$
$$+R\ddot\phi(m_2+m_d+m))\sin(\phi)\sin(\theta)+(2(\dot\phi\dot\theta\cos(\theta)$$
$$+\frac{\ddot\beta}{2})(lm_2+l_dm_d)\cos(\beta)$$
$$+2R\dot\phi\dot\theta(m_2+m_d+m)\cos(\theta)-\sin(\beta)(\dot\beta^2$$
$$+\dot\phi^2)(lm_2+l_dm_d))\sin(\phi)+\ddot ym+\ddot ym_2+\ddot ym_d$$
$$+\lambda_2=0 \qquad \textbf{(C-24)}$$

The results in (A-25) and (A-26) then substituted in (C-23) and (C-24) to calculate λ_1 and λ_2. They are

$$\lambda_1=(((\dot\beta^2+\dot\phi^2+\dot\theta^2)(lm_2+l_dm_d)\cos(\beta)+\ddot\beta(lm_2+l_dm_d)\sin(\beta) \qquad \textbf{(C-25)}$$
$$+R(\dot\phi^2+\dot\theta^2)(m_2+m_d+m))\sin(\theta)+2(\dot\phi\dot\beta$$
$$-\frac{\ddot\theta\cos(\theta)}{2})(lm_2+l_dm_d)\cos(\beta)+(2\dot\beta\dot\theta(lm_2$$
$$+l_dm_d)\sin(\beta)-R\ddot\theta(m_2+m_d+m))\cos(\theta)$$
$$+\ddot\phi\sin(\beta)(lm_2+l_dm_d)+R\dot\phi\dot\psi(m_2+m_d+m))\sin(\phi)$$
$$-2\cos(\phi)(\left(\frac{\ddot\phi(lm_2+l_dm_d)\cos(\beta)}{2}\right.$$
$$\left.-\dot\phi\dot\beta(lm_2+l_dm_d)\sin(\beta)+\frac{R\ddot\phi(m_2+m_d+m)}{2}\right)\sin(\theta)$$
$$+(lm_2+l_dm_d)(\dot\phi\dot\theta\cos(\theta)+\frac{\ddot\beta}{2})\cos(\beta)+R\dot\phi\dot\theta(m_2+m_d$$
$$+m)\cos(\theta)-\frac{\sin(\beta)\left(\dot\beta^2+\dot\phi^2\right)(lm_2+l_dm_d)}{2}$$
$$+\frac{R\ddot\psi(m_2+m_d+m)}{2}),$$

$$
\begin{aligned}
\lambda_2 = \; & ((-(\dot{\beta}^2+\dot{\phi}^2+\dot{\theta}^2)(lm_2+l_d m_d)\cos(\beta) - \ddot{\beta}(lm_2+l_d m_d)\sin(\beta) \\
& - R(\dot{\phi}^2+\dot{\theta}^2)(m_2+m_d+m))\sin(\theta) - 2(\dot{\phi}\dot{\beta} \\
& - \frac{\ddot{\theta}\cos(\theta)}{2})(lm_2+l_d m_d)\cos(\beta) + (-2\dot{\beta}\dot{\theta}(lm_2 \\
& + l_d m_d)\sin(\beta) + R\ddot{\theta}(m_2+m_d+m))\cos(\theta) \\
& - \ddot{\phi}\sin(\beta)(lm_2+l_d m_d) - R\dot{\phi}\dot{\psi}(m_2+m_d+m))\cos(\phi) \\
& - 2((\frac{\ddot{\phi}(lm_2+l_d m_d)\cos(\beta)}{2} - \dot{\phi}\dot{\beta}(lm_2+l_d m_d)\sin(\beta) \\
& + \frac{R\ddot{\phi}(m_2+m_d+m)}{2})\sin(\theta) + (lm_2+l_d m_d)(\dot{\phi}\dot{\theta}\cos(\theta) \\
& + \frac{\ddot{\beta}}{2})\cos(\beta) + R\dot{\phi}\dot{\theta}(m_2+m_d+m)\cos(\theta) \\
& - \frac{\sin(\beta)\left(\dot{\beta}^2+\dot{\phi}^2\right)(lm_2+l_d m_d)}{2} \\
& + \frac{R\ddot{\psi}(m_2+m_d+m)}{2})\sin(\phi)
\end{aligned}
\qquad \textbf{(C-26)}
$$