THE EXTREMAL FUNCTION FOR K_{10} MINORS

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To my parents.

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SUMMARY

For two graphs G and H, G has H as a minor if a graph isomorphic to H can be obtained from a subgraph of G by repeatedly contracting edges. The Four Color Theorem (4CT) says that every planar graph is 4-colorable. Due to the Kuratowski-Wagner theorem, the 4CT can be restated that every graph with no K_5 minor and $K_{3,3}$ minor is 4-colorable. The famous Hadwiger Conjecture is a generalization of the 4CT, which says that every graph with no K_{t+1} minor for integers $t \geq 1$ is t-colorable. The Hadwiger's Conjecture is true for all $t \leq 5$ and remains widely open for $t \geq 6$.

To make progress on Hadwiger's Conjecture for $t \geq 6$, one major line of work has focused on giving an upper bound on the number of edges for graphs without a K_t minor. The maximum number of edges of an n-vertex graph with no K_t minor is known as the extremal function for K_t minors. This dissertation focuses on the extremal function for K_{10} minors.

Our main theorem says that every graph on $n \geq 8$ vertices and at least $8n - 35$ edges either has a K_{10} minor or falls into a few families of exceptional graphs. In Chapter 1, we discuss more into the motivation and related results on the K_{10} minor work. In Chapter 2, we present necessary graph theoretical background and a series of observations of the exceptional graphs in the main theorem. We study structural properties of possible minimal counter-examples to the main theorem in Chapter 3 and later dive into proving a main technical lemma in Chapter 4. Finally, we conclude our main theorem in Chapter 5.

We note that the proof for the main technical lemma (Lemma 4.1.1) in our proof for the main theorem is computer-assisted. We do not yet have a computer-free proof for Lemma 4.1.1.

CHAPTER 1
INTRODUCTION

We discuss motivation, related results, and further impact of my work in this chapter.

1.1 Motivation

My work is motivated by Hadwiger's conjecture, which is a longstanding open problem that generalizes the Four Color Theorem. The Four Color Theorem (4CT) states that every planar graph is 4-colorable. By the Kuratowski-Wagner theorem [20, 39], a graph is planar if and only if it has no K_5 or $K_{3,3}$ minor, which allows us to restate the 4CT that every graph with no K_5 minor and no $K_{3,3}$ minor is 4-colorable. As $K_{3,3}$ can in fact be colored by four colors, one might wonder if every graph with no K_5 minor is 4-colorable. This leads us to Hadwiger's famous conjecture [7].

Conjecture 1.1.1 (Hagwidger's Conjecture)**.** *For every integer $t \geq 0$, every graph with no K_{t+1} minor is t-colorable.*

For $t \leq 3$, Hadwiger's conjecture is reasonably easy, as shown by Hadwiger [7] and Dirac [5]. Wagner [39] proved the case $t = 4$ is equivalent to the 4CT in 1937, so the case $t = 4$ was eventually proved in 1976 when Appel and Haken [2, 3] proved the 4CT. In 1993, Robertson, Seymour, and Thomas [23] showed the case $t = 5$ is also equivalent to the 4CT. Hadwiger's conjecture remains open for $t \geq 6$.

One major line of work on Hadwiger's conjecture has focused on giving an upper bound on the number of edges for graphs that lack a K_t minor. For positive integers t and n, the maximum number of edges that an n-vertex graph with no K_t minor can have is known as the *extremal function for K_t minors*. My work in this dissertation focuses on the case $t = 10$. Mader [21] proved the following theorem in 1968.

Theorem 1.1.2. *For every integer $t = 1, 2, ..., 7$, a graph on $n \geq t$ vertices and at least $(t-2)n - \binom{t-1}{2} + 1$ edges has a K_t minor.*

Mader also pointed out $K_{2,2,2,2,2}$ is a counter-example for the case $t = 8$. One can construct further counter-examples by repeatedly identifying cliques of size 5.

In general, for graphs H_1 and H_2 and an integer k, we define an (H_1, H_2, k)-cockade recursively as follows: Every graph isomorphic to H_1 or H_2 is an (H_1, H_2, k)-cockade; If G_1, G_2 are both (H_1, H_2, k)-cockades, then the graph obtained from the disjoint union of G_1 and G_2 by identifying a clique of size k in G_1 with a clique of the same size in G_2 is also an (H_1, H_2, k)-cockade; Every (H_1, H_2, k)-cockade can be constructed this way. If $H_1 = H_2 = H$, then an (H_1, H_2, k)-cockade is also called an (H, k)-cockade. A graph G is a *trivial* (H, k)-cockade if $G \cong H$, and otherwise a *non-trivial* (H, k)-cockade. For a (H, k)-cockade G, the multiplicity of G is defined recursively as follows: G has multiplicity 1 if it is a trivial (H, k)-cockade; G has multiplicity $m = m_1 + m_2$ for some $m_1, m_2 \geq 1$ if there exist induced subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$, $G_1 \cap G_2 \cong K_k$, and G_i is an (H, k)-cockade of multiplicity m_i for $i = 1, 2$.

Jørgensen [10] and Song and Thomas [32] generalized Theorem 1.1.2 for K_8 minors and K_9 minors, respectively, as follows.

Theorem 1.1.3. *Every graph on $n \geq 8$ vertices and at least $6n - 20$ edges either has a K_8 minor or is a $(K_{2,2,2,2,2}, 5)$-cockade.*

Theorem 1.1.4. *Every graph on $n \geq 9$ vertices and at least $7n - 27$ edges either has a K_9 minor or is a $(K_{1,2,2,2,2,2}, 6)$-cockade, or is isomorphic to $K_{2,2,2,3,3}$.*

It is then natural to ask if every graph on $n \geq 10$ vertices and $8n - 35$ edges with no K_{10} minor falls into a few families of graphs. We prove Theorem 1.1.5 for K_{10} minors, which is the main theorem of this dissertation.

For graphs H and G, let $H + G$ denote the graph obtained from the disjoint union of H and G by adding edges xy for all $x \in V(H)$ and $y \in V(G)$.

Theorem 1.1.5. *Every graph on $n \geq 8$ vertices and at least $8n - 35$ edges either has a K_{10} minor or is isomorphic to one of the following graphs:*

(1) a $(K_{1,1,2,2,2,2,2}, 7)$-cockade;

(2) $K_{1,2,2,2,3,3}$;

(3) $K_{2,2,2,2} + C_5$;

(4) $K_{2,2,3,3,4}$;

(5) $K_{3,3,3} + C_5$;

(6) $K_{2,2,2,2,2,3}$;

(7) $G_1 = K_{2,2,2,2,2,3} - e$ where $e \in E(K_{2,2,2,2,2,3})$ and both ends of e have degree 11;

(8) $G_2 = K_{2,2,2,2,2,3} - e$ where $e \in E(K_{2,2,2,2,2,3})$ and the ends of e have degree 10 and 11;

(9) $K_{2,3,3,3,3}$

(10) $G_3 = K_{2,3,3,3,3} - e$ where $e \in E(K_{2,3,3,3,3})$ and both ends of e have degree 11;

(11) $G_4 = K_{2,3,3,3,3} - e$ where $e \in E(K_{2,3,3,3,3})$ and the ends of e have degree 11 and 12;

(12) a $(K_{2,2,2,2,2,3}, 6)$-cockade of multiplicity 2.

We note that in the proof for Theorem 1.1.5 that we will present later in this dissertation, the proof for the main technical lemma (Lemma 4.1.1) is computer-assisted, and we do not yet have a computer-free proof for it.

1.2 Related Work and Impact of the Main Theorem

We now discuss related work to our problem and impact of Theorem 1.1.5 to give a more general context.

1.2.1 The extremal function for K_t minors

We first point out that the linear edge bound given by Mader in Theorem 1.1.2 actually is incorrect for large t. Kostochka [17, 18] and de la Vega [38] proved that for large t, a graph on n vertices must have at least $\Omega(t\sqrt{\log t}n)$ edges to guarantee a K_t minor by showing that a random graph with no K_t minor may have average degree of order $t\sqrt{\log t}$. Kostochaka

[17, 18] and Thomason [35] proved the order of $t\sqrt{\log t}n$ is also an upper bound, and later Thomason [36] was able to determine the constant of proportionality exactly. Although it may now seem unnecessary to study the extremal function for specific small values of t, the random graph examples only provide finitely many counter-examples. Of course, for any given value of t, more counter-examples can be made by taking disjoint unions or gluing counter-examples along small cut sets, but we know of no construction of highly connected infinite families of counter-examples. More specifically, Seymour and Thomas conjecture the following.

Conjecture 1.2.1. *For every integer $t \geq 1$, there exists a constant $N = N(t)$ such that every $(t-2)$-connected graph on $n \geq N$ vertices and at least $(t-2)n - \binom{t-1}{2} + 1$ edges has a K_t minor.*

Note that Theorems 1.1.2, 1.1.3, and 1.1.4 imply that Conjecture 1.2.1 is true for $t \leq 9$. Since every K_{10}-minor-free graph in Theorem 1.1.5 is not 8-connected, Theorem 1.1.5 implies that Conjecture 1.2.1 is also true for $t = 10$. In particular, we have the following corollary of Theorem 1.1.5.

Corollary 1.2.2 (Corollary of Theorem 1.1.5)**.** *Every 8-connected graph on $n \geq 8$ vertices and at least $8n - 35$ edges has a K_{10} minor.*

1.2.2 Relating to Hadwiger's Conjecture

The proof of the case $t = 5$ of Hadwiger's conjecture [23] uses the case $t = 6$ of Theorem 1.1.2 to get an upper bound on the number of edges for K_6 minor-free graphs. As the case $t = 6$ of Hadwiger's conjecture remains open, Kawarabayashi and Toft [15] proved that every graph with no K_7 minor is either 6-colorable or has a $K_{4,4}$ minor. It is not known yet if every K_7 minor-free graph is 7-colorable. Albar and Gonçalves [1] and Rolek and Song [28] proved that for $t = 7, 8, 9$, a graph with no K_t minor is $(2t-6)$-colorable. Their proofs use the extremal function results for $t \leq 9$ to find a vertex of degree of at most $2t - 5$ in every graph with no K_t minor. Now, Corollary 1.2.2 and Theorem 5.2 in [28] immediately imply the following corollary that every graph with no K_{10} minor is 14-colorable.

Corollary 1.2.3 (Corollary of Theorem 1.1.5)**.** *Every graph with no K_{10} minor is 14-colorable.*

Another weaker version of Hadwiger's conjecture is the doubly-critical conjecture by Kawarabayashi, Pedersen, and Toft [13]. A connected t-chromatic graph G is called *doubly-critical* if $G - \{u, v\}$ is $(t - 2)$-colorable for every edge $uv \in E(G)$. The doubly-critical conjecture states that every doubly-critical t-chromatic graph contains a K_t minor. Rolek and Song showed in [27] that the doubly-critical conjecture is true for all $t \leq 9$, and their proof again uses the extremal function for $t \leq 9$. According to Song (private communication), by following the ideas in [27] and the ideas proving Theorem 1.1.5 in this dissertation, one can prove, with effort, that every double-critical 10-chromatic graph has a K_{10} minor, which then resolves the the doubly-critical conjecture for $t = 10$.

Another way of weakening Hadwiger's conjecture is to only consider t-chromatic graphs with a unique t-coloring. A recent work by Kriesell [19] shows that for $t \leq 10$, every graph of chromatic number t with a unique t-coloring has a K_{10} minor. Following the ideas in [19] and the ideas proving Theorem 1.1.5 in this dissertation, we can then extend to obtain the following corollary.

Corollary 1.2.4 (Corollary of Theorem 1.1.5)**.** *Every 11-chromatic graph with a unique 11-coloring has a K_{11} minor.*

The last line of work related to Hadwiger's conjecture we want to mention here is the Erdös-Lovász Tihany Conjecture.

Conjecture 1.2.5 (Erdös-Lovász Tihany Conjecture)**.** *For any pair of integers $s, t \geq 2$ and any graph G with $\omega(w) < \chi(G) = s + t - 1$, there are two disjoint subgraphs G_1 and G_2 of G such that $\chi(G_1) \geq s$ and $\chi(G_2) \geq t$.*

For integers $s, t \geq 2$ with $s \leq t$, say a graph an (s,t)-*graph* if it is a connected $(s+t-1)$-chromatic graph and does not contain two disjoint subgraphs with chromatic numbers s and t, respectively. Then following the ideas in [16] by Kawarabayashi, Pedersen, and Toft and the ideas of proving Theorem 1.1.5 in this dissertation, we can conclude the following corollary, settling Conjecture 1.4 in the same paper [16] for $s = 2$ and $2 \leq t \leq 9$.

Corollary 1.2.6 (Corollary of Theorem 1.1.5)**.** *For* $t = 2, 3, ..., 9$*, every* $(2,t)$*-graph with clique number at most t has a* $K_2 \cup K_t$ *minor.*

1.2.3 Variants of the extremal function for K_t minors

Thomas and Yoo [34] studied the extremal function for K_t minors for triangle-free graphs. They proved a theorem that for $t = 2, 3, ..., 9$, a triangle-free graph on $n \geq 2t - 5$ vertices and at least $(t-2)n - (t-2)^2 + 1$ edges has a K_t minor. Now by Theorem 1.1.5 in this dissertation and Theorem 3.2 in [34], we can extend the triangle-free theorem to the case $t = 10$ and conclude the following corollary immediately.

Corollary 1.2.7 (Corollary of Theorem 1.1.5)**.** *Every triangle-free graph on* $n \geq 15$ *vertices and at least* $8n - 63$ *edges has a* K_{10} *minor.*

The extremal functions for K_t^- minors have also been studied, where K_t^- denotes the graph obtained from K_t by deleting one edge. Jakobsen [8, 9] proved that for $t \leq 7$, every graph on $n \geq t$ vertices and at least $(t - \frac{5}{2})n - \frac{1}{2}(t-3)(t-1)$ edges has a K_t^- minor, or is a $(K_{t-1}, t-3)$-cockade, or G is a $(K_{2,2,2,2}, K_6, 4)$-cockade in the case $t = 7$. Song [31] later showed that every graph on $n \geq 8$ vertices and at least $\frac{11n-35}{2}$ edges either has a K_8^- minor or is a $(K_{1,2,2,2,2}, K_7, 5)$-cockade. Moreover, Song pointed out (private communication) it is promising that the way of using the 3-linkage theorem by Thomas and Wollan [33] in our proof for Theorem 1.1.5 in this dissertation can be applied to prove an analogous theorem for K_9^- minor-free graphs.

CHAPTER 2

PRELIMINARIES

2.1 Graph Basics

All graphs are simple in this dissertation. For a graph G, $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. If two vertices x, y are adjacent in G, we say they are *neighbors* in G and write xy or yx to denote the edge between them. x and y are called the *end vertices*, or simply *ends*, of the edge xy. We use $|G| = |V(G)|$ to denote the number of vertices in G.

If a graph G' satisfies that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$, then G' is a *subgraph* of G, denoted by $G' \subseteq G$. For a subset $A \subseteq V(G)$ of vertices, $G[A]$ denotes the subgraph of G that has set of vertices equal to A and set of edges containing every edge of G with both end vertices in A. Say $G[A]$ is *induced* by the subset A. An *induced subgrpah* of G is a subgraph of G equal to $G[A]$ for some $A \subseteq V(G)$.

For a vertex $x \in V(G)$, $N(x)$ is the set of neighbors of x in G, and $N[x] = N(x) \cup \{x\}$. We also use $N(x)$ and $N[x]$ denote the induced subgraphs of G on the subsets of vertices $N(x)$ and $N[x]$, respectively. The degree of x is the size of $N(x)$, denoted by $d(x)$. The minimum degree over all vertices in G is denoted by $\delta(G)$. A subset $A \subseteq V(G)$ of vertices is a *clique* if every pair of vertices in A is an edge in G. The *clique number* of G is maximum size of a clique in G, denoted by $\omega(G)$. For a subset $A \subseteq V(G)$ of vertices and a subset $F \subseteq E(G)$ of edges, $G - A$ denotes the graph obtained from G by deleting all vertices in A, and $G - F$ denotes the graph obtained from G by deleting all edges in F. In the case $A = \{v\}$, $G - v = G - A$. Let $e \in E(G)$. $G - e$ is the graph obtained from G by deleting e.

We now define edge contraction and a more general notion of graph containment called minor. Let $e = xy \in E(G)$. The graph obtained from G by *contracting* e, denoted as $G\backslash e$, is the graph obtained from G by deleting both x and y and adding a new vertex whose neighborhood in the new graph is equal to $N_G(x) \cup N_G(y) - \{x, y\}$. For some graph H, say G contains H as a *minor*, or simply G has an H *minor*, if a graph isomorphic to H can be obtained from a subgraph of G by repeatedly contracting edges, denoted as $G > H$. Equivalently, G has an H minor if there exist pairwise disjoint subsets $S_1, ..., S_{|H|} \subseteq V(G)$ of vertices such that and a bijective function $\phi : V(H) \rightarrow \{S_1, ..., S_{|H|}\}$ such that $G[S_i]$ is a connected subgraph of G for all $i = 1, ..., |H|$, and that for every $xy \in E(H)$, there is an edge in G with one end in $\phi(x)$ and one end in $\phi(y)$. H is called a *proper minor* of G if $G > H$ and $G \not\cong H$.

A graph P is a *path* if we can label the vertices of P as $v_1, ..., v_k$ such that $E(P) = \{v_1v_2, ..., v_{k-1}v_k\}$. We denote $P = v_1v_2...v_k$. The *length* of a path is the number of edges on it. v_0 and v_k are the *end vertices* or simply *ends* of P. Say P *links* v_0 and v_k or *joins* v_0 and v_k, and say P is a v_0-v_k *path*. Say P is an A-B path if $A, B \subseteq V(G)$ such that $v_0 \in A$ and $v_k \in B$. The vertices in $V(P) - \{v_0, v_k\}$ are the *internal vertices* of P. We also consider the graph on a single vertex as a path. A path is *trivial* if it has length zero, and otherwise *non-trivial*. A subgraph P' of P is a *subpath* if it is also a path. For $v_i, v_j \in V(P)$, v_iPv_j denotes the subpath of P linking v_i and v_j. A graph C is a *cycle* if we can label the vertices of C as $v_1, ..., v_k$ such that $E(C) = \{v_1v_2, ..., v_{k-1}v_k, v_kv_1\}$. We write $C = v_1v_2...v_kv_1$. The *length* of a cycle is the number of edges on it. A cycle of length k is called a k-cycle, denoted C_k.

In a graph G, two paths $P, Q \subseteq G$ are *disjoint* if $V(P) \cap V(Q) = \emptyset$, and they are *internally disjoint* if the sets of internal vertices of P, Q are *disjoint*. For $A, B \subseteq V(G)$, say a set of paths $P_1, ..., P_t \subseteq G$ *links* or *joins* A, B, if every P_i for all $i = 1, ..., t$ has one end in A, one end in B, and is otherwise disjoint from $A \cup B$.

A graph G is *connected* if for every $x, y \in V(G)$, there exists some path in G linking x and y; it is *disconnected* otherwise. A *connected component* or simply a *component* of

G is a maximal connected subgraph of G. G is *k-connected* for some integer k if for every $X \subseteq V(G)$ such that $|X| < k$, $G - X$ is connected. A subset $X \subseteq V(G)$ is a *separating set* of G if $G - X$ is disconnected. For vertices $a, b \in V(G) - X$, say X separates a from b if there is no path linking a, b in $G - X$. For $a \in V(G) - X$ and $B \subseteq V(G)$, say X *separates* a from B, or X *separates* B from a, if there is no path linking a and some vertex in B in $G - X$.

A *separation* of G is a pair (A, B) of subsets of vertices of G such that $A \cup B = V(G)$ and $ab \notin E(G)$ for all $a \in A - B$ and $b \in B - A$. For $X \subseteq V(G)$, the pair (A, B) is a *separation* of (G, X) if (A, B) is a separation of G such that $X \subseteq A$. For a separation (A, B) of G, or (G, X) for some $X \subseteq V(G)$, the *order* of the separation is the size of $A \cap B$. It is called a *k-separation* (or *$\leq k$-separation*), if $|A \cap B| = k$ (or $|A \cap B| \leq k$, respectively). A separation (A, B) is *trivial* if $A \subseteq B$ or $B \subseteq A$; it is *non-trivial* otherwise. It is an easy exercise that a graph is k-connected if and only if it has no non-trivial separation of order at most $k - 1$.

2.2 Rooted K_3 and Rooted K_4 Minors

Rooted minor is a special type of minor. Let G, H be graphs, and let $X \subseteq V(G)$ such that $|X| = |H|$. Say G has an *H minor rooted at X* if there is a function ϕ mapping vertices in H to disjoint connected subgraphs of G such that $|V(\phi(u)) \cap X| = 1$ for all $u \in V(H)$, and that if $uv \in E(H)$ then there exists an edge in G joining a vertex in $\phi(u)$ to a vertex in $\phi(v)$.

In this section, we will present our own result on rooted K_3 minors, which is relatively straight-forward, followed by a result on rooted K_4 minors due to Robertson, Seymour, and Thomas.

2.2.1 Rooted K_3 minors

Lemma 2.2.1. *Let G be a connected graph, and let $X = \{x_1, x_2, x_3\}$ be a subset of three distinct vertices in G. Then, G has a K_3 minor rooted at X if and only if there does not exist a cut vertex w of G such that every component of $G - \{w\}$ contains at most one vertex from X.*

Proof. Since G connected, we assume without loss of generality that there is a path P in G linking x_1, x_2 such that $x_3 \notin V(P)$.

First assume that G has a K_3 minor rooted at X. Without loss of generality, we can then assume that there exist two paths Q_1, Q_2 such that Q_1, Q_2 each link x_3 and some vertex on P and are otherwise disjoint from P, and that $V(Q_1) \cap V(Q_2) = \{x_3\}$. It follows that there is no cut vertex separating x_3 from P in G, and therefore there is no cut vertex w of G such that every component of $G - \{w\}$ contains at most one vertex in X.

Now, assume that G does not have a K_3 minor rooted at X. We will prove that there exists a cut vertex w of G such that every component of $G - \{w\}$ contains at most one vertex in X. Since G is connected, there exists a path Q linking x_3 and some vertex $w \in V(P)$ that is otherwise disjoint from P. Since G does not have a K_3 minor rooted at X, there do not exist two paths Q_1, Q_2 each joining x_3 and some vertex on P such that Q_1, Q_2 are disjoint except for x_3. Therefore, w is a cut vertex of G, and there is a component J_3 of $G - \{w\}$ such that $x_3 \in V(J_3)$ and $V(P) \cap V(J_3) = \emptyset$.

If $w \in \{x_1, x_2\}$, without loss of generality, assume that $w = x_2$. Then, the component of $G - \{w\}$ that contains x_1, say J_1, satisfies that $V(J_1) \cap X = \{x_1\}$. It follows that w satisfies that every component of $G - \{w\}$ contains at most one vertex in X. If $w \notin \{x_1, x_2\}$, we consider the block decomposition of $G - V(J_3)$. Notice that $P \subseteq G - V(J_3)$. Choose blocks $B_1, ..., B_k$ of $G - V(J_3)$ and vertices $v_1, v_2, ..., v_{k+1} \in V(P)$ such that $v_1 = x_1$, $v_{k+1} = x_2$, and $v_i, v_{i+1} \in V(B_i)$ for all $i = 1, .., k$. Assume for a moment that $w \in V(B_i) - \{v_i, v_{i+1}\}$ for some $i \in \{1, ..., k\}$. Since B_i is 2-connected, it follows that B_i

has K_3 minor rooted at $\{w, v_i, v_{i+1}\}$. It follows that $B_i \cup P \cup Q = B_i \cup v_1 P v_i \cup v_{i+1} P v_{k+1} \cup Q$ has a K_3 minor rooted at X, a contradiction. Hence, $w = v_i$ for some $i \in \{2, ..., k\}$. Let J_1, J_2 be the components of $G - \{w\} \cup V(J_3)$ such that $V(v_1 P v_i) - \{v_i\} \subseteq V(J_1)$ and $V(v_i P v_{k+1}) - \{v_i\} \subseteq V(J_2)$. It follows that J_1, J_2, J_3 are distinct components of $G - \{w\}$ such that $x_i \in V(J_i)$ for $i = 1, 2, 3$. □

2.2.2 Rooted K_4 minors

The following theorem on rooted K_4-minors was proven by Robertson, Seymour, and Thomas in their proof of Hadwiger's Conjecture for graphs with no K_6-minor [23], in which a K_4-minor rooted at x_1, x_2, x_3, x_4 is called a *cluster traversing* $\{x_1, x_2, x_3, x_4\}$. A *trisection* of a graph G is a triple (A, B, C) of subsets of $V(G)$ such that $A \cap B = A \cap C = B \cap C$ and $G[A] \cup G[B] \cup G[C] = G$; the *order* of the trisection (A, B, C) is $|A \cap B \cap C|$.

Theorem 2.2.2 (Rooted K_4-minor Theorem)**.** *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Then*

(i) G has a K_4 minor rooted at X, or

(ii) there is a trisection (A_1, A_2, B) of order 2 such that $|Z \cap (A_i - B)| = 1$ for $i \in \{1, 2\}$, or

(iii) there is a (≤ 3)-separation (A, B) with $Z \subseteq A$ and $|B - A| \geq 2$ and $|Z \cap B| \leq 2$, or

(iv) G can be drawn in the plane so that every vertex in Z is incident with the infinite region.

We note that Fabila-Monroy and Wood [6] proved a stronger theorem than Theorem 2.2.2 by giving a complete characterization of graphs that have a K_4 minor rooted at four nominated vertices.

2.3 Disjoint Paths

Let G be a graph. For a path P in G and vertices $s, t \in V(G)$, say P *links* s, t if s, t are the end vertices of P. For an integer k and distinct vertices $s_1, ..., s_k, t_1, ..., t_k \in V(G)$, the ***k***

disjoint paths problem asks whether there exist k disjoint paths $P_1, ..., P_k$ in G such that P_i link s_i and t_i for all $i = 1, ..., k$. Robertson and Seymour [26] showed that there is a polynomial time algorithm for deciding whether such disjoint paths exist. We now discuss results on the k disjoint paths problem for $k = 2, 3$.

2.3.1 Two Disjoint Paths

In fact, the two disjoint path problem is a special case for another. Let C be a cycle in a graph G. A *C-cross* is a pair of disjoint paths P_1, P_2 with ends x_1, y_1 and x_2, y_2, respectively, such that x_1, x_2, y_1, y_2 occur on C in the order listed, and the paths are otherwise disjoint from C. The feasibility problem for a C-cross generalizes the feasibility problem for the two disjoint paths problem: Notice that for distinct vertices $s_1, s_2, t_1, t_2 \in V(G)$ in a graph G, there exist two disjoint paths P_1, P_2 in G such that P_i for $i = 1, 2$ links s_i and t_i if and only if the graph $G' = G \cup \{s_1s_2, s_2t_1, t_1t_2, t_2s_1\}$ has a C-cross, where $C = s_1s_2t_1t_2s_1$ is the cycle that goes through the four vertices s_1, s_2, t_1, t_2 in order. It follows that, to study the feasibility of the two disjoint path problem, it suffices to study the characterization of C-crosses in a graph.

Theorem 2.3.1 gives a characterization of graphs containing C-crosses. This exact version of the theorem is Theorem 1.3 in [14], obtained in varous forms by Jung [11], Robertson and Seymour [24], Seymour [29], Shiloach [30], and Thomassen [37]. Let G be a graph, and let $X \subseteq V(G)$. Let (A, B) be a ≤ 3-separation of (G, X) such that there exist $|A \cap B|$ paths from some vertex $v \in B - A$ to X that are disjoint except for v. Let H be the graph obtained from $G[A]$ by adding an edge joining every pair of distinct vertices in $A \cap B$. We say that H is an *elementary X-reduction of G (determined by (A, B))*. We say that a graph J is an *X-reduction* of G if it can be obtained from G by a series of elementary X-reductions. If C is a subgraph of G, then by an *(elementary) C-reduction* we mean an *(elementary) $V(C)$-reduction*.

Theorem 2.3.1 (Jung; Robertson and Seymour; Seymour; Shiloach; Thomassen)**.** *Let G be* *a graph, and let C be a cycle in G. Then, G has no C-cross if and only if some C-reduction of G can be drawn in the plane with C bounding a face.*

2.3.2 Three Disjoint Paths

We introduce a result on 3-linkage in Theorem 2.3.2 due to Thomas and Wollan [33]. Theorem 2.3.2 is used later in this dissertation to find three disjoint paths. Let G be a graph. For $X \subseteq V(G)$ and an integer t, the pair (G, X) is *t-linked* if for all $k \leq t$ and distinct vertices $s_1, ..., s_k, t_1, ..., t_k \in X$, there exist k disjoint paths $P_1, ..., P_k$ in G such that P_i links s_i, t_i for all $i = 1, ..., k$. The pair (G, X) is *linked* if it is $\lfloor |X|/2 \rfloor$-linked. A separation (A, B) of G is *t-linked* if $(G[B], A \cap B)$ is t-linked. Use $\rho_G(X)$ to denote the number of edges of G that have at least one end in $V(G) - X$, i.e. $\rho_G(X) = |E(G)| - |E(G[X])|$.

Theorem 2.3.2 (Wollan and Thomas)**.** *Let G be a graph. Let $X \subseteq V(G)$ be a subset of vertices such that $|X| = 6$. Then, (G, X) is linked if $\rho(V(G) - X) \geq 5|V(G) - X| + 4$ and $\rho(B - A) \leq 5|B - A|$ for every ≤ 5-separation (A, B) of (G, X).*

2.3.3 Menger's Theorem and Perfect's Theorem

A classic theorem of Menger states that if a graph G is k-connected, then for any two disjoint subsets $A, B \subseteq V(G)$ such that $|A| = |B| = k$, there exist k disjoint paths $P_1, ..., P_k$ such that each P_i links one vertex in A and one vertex in B.

Theorem 2.3.3 is a stronger version of Menger's theorem due to Perfect [22], which is also stated in Section 3.3 in Diestel's text [4].

Theorem 2.3.3 (Perfect)**.** *Let G be a graph with $A, B \subseteq V(G)$. Let k be the minimum number of vertices separating A from B in G. If $\mathcal{P}$ is any set of fewer than k disjoint $A - B$ paths in G, then G contains a set $\mathcal{Q}$ of disjoint $A - B$ paths such that $|\mathcal{Q}| = |\mathcal{P}| + 1$, and that the set of vertices in A that lie on a path in $\mathcal{P}$ is a proper subset of the vertices in A that lie on a path in $\mathcal{Q}$, and likewise for B.*

Corollary 2.3.4 (Corollary of Theorem 2.3.3)**.** *Let G be a graph with $a \in V(G)$ and $B \subseteq V(G) - \{a\}$. Let k be the minimum number of vertices separating a from B in G. If $\mathcal{P}$ is any set of fewer than k paths from a to B that are disjoint except for a, then G contains a set $\mathcal{Q}$ of paths from a to B that are disjoint except for a such that $|\mathcal{Q}| = |\mathcal{P}| + 1$, and that the set of vertices in B that lie on a path in $\mathcal{P}$ is a proper subset of the vertices in B that lie on a path in $\mathcal{Q}$.*

Let S_k denote the permutation group on k elements.

Corollary 2.3.5 (Corollary of Theorem 2.3.3)**.** *Let G be a graph. Let $A, B \subseteq V(G)$ with a vertex $a_1 \in A - B$. Let $k = |A|$. Suppose there exist k disjoint paths $P_1, ..., P_k$ such that $a_i \in A$ and $b_i \in B$ are the ends of P_i for $i = 1, ..., k$. If there does not exist a $\leq k$-separation (X, Y) of G such that $A \subseteq X$, $B \subseteq Y$, and $X - Y \neq \emptyset$, then there exists some $b_{k+1} \in B - \{b_1, ..., b_k\}$, a permutation $\theta \in S_{k+1}$, and $k + 1$ internally disjoint paths $Q_1, ..., Q_{k+1}$ linking A and B such that Q_i links a_i and $b_{\theta(i)}$ for all $i = 1, ..., k$, and that Q_{k+1} links a_1 and $b_{\theta(k+1)}$.*

Proof. Let G' be the graph obtained from G by adding a new vertex a'_1 and an edge between a'_1 and every neighbor of a_1 in G. Let $A' = A \cup \{a'_1\}$. Note that $P_1, ..., P_k$ are k disjoint paths linking A' and B in G'.

Assume for a moment that there exists a k-separation (X', Y') such that $A' \subseteq X'$ and $B \subseteq Y'$. Since there is no $\leq k$-separation (X, Y) of G such that $A \subseteq X$, $B \subseteq Y$, and $X - Y \neq \emptyset$, it follows that $(X' - Y') \cap V(G) = \emptyset$, and therefore $X' - Y' = \{a'_1\}$ and $X' \cap Y' = \{a_1, ..., a_k\}$. Since $a_1 \notin B$, P_1 contains at least 2 vertices. Let u be the unique vertex in P_1 that is adjacent to a_1, and note that $u \in Y' - X'$. However, since $a_1 u \in E(G)$, we know $a'_1 u \in E(G')$, which is a contradiction to that (X', Y') is a separation of G' and there is no edge between $X' - Y'$ and $Y' - X'$.

Hence, there is no k-cut separating A' and B in G'. By Theorem 2.3.3, there exist some $b_{k+1} \in B - \{b_1, ..., b_k\}$, a permutation $\theta \in S_{k+1}$, and $k + 1$ disjoint paths $Q'_1, ..., Q'_{k+1}$

linking A' and B in G' such that Q'_i links a_i and $b_{\theta(i)}$ for $i = 1, ..., k$ and Q'_{k+1} links a'_1 and $b_{\theta(k+1)}$. Back in the graph G, let $Q_i = Q'_i$ for all $i = 1, ..., k$, and let Q_{k+1} be the path obtained from Q'_{k+1} by replacing a'_1 with a_1. It follows that the vertex b_{k+1}, the permutation $\theta \in S_{k+1}$, and the paths $Q_1, ..., Q_{k+1}$ are as desired. □

2.4 Bridges and Tripods

2.4.1 Bridges

Let G be a graph, and let S be a subgraph of G. An S*-bridge* in G is a connected subgraph B of G such that $E(B) \cap E(S) = \emptyset$ and either $E(B)$ consists of a unique edge with both ends in S, or for some component C of $G \backslash V(S)$ the set $E(B)$ consists of all edges of G with at least one end in $V(C)$. The vertices in $V(B) \cap V(S)$ are called the *attachments* of B on S. We say an S-bridge B *attaches to* a subgraph H of S if $V(H) \cap V(B) \neq \emptyset$, and in the case $H = \{v\}$ for some $v \in V(S)$, we say B *attaches to* v. An S-bridge B is called *trivial* if it consists of a unique edge with both ends in S, and it is called *non-trivial* otherwise.

Let S be a subgraph of G, and let $W \subseteq V(S)$. A W*-segment* of S is a subpath P of S of length at least one such that both end vertices of P are contained in W, and that every internal vertex v of P is not in W and has degree two in S. Say W is a *segmenting set* of S if S is equal to the union of all W-segments of S. It is an easy exercise to check that a segmenting set of S includes all vertices of degree not equal to two in S. Note that if W is a segmenting set of S, every edge of S is contained in a unique W-segment of S. Say an S-bridge B is W*-unstable* if all attachments of B on S belong to some W-segment of S, and otherwise B is W*-stable*.

The next lemma, Lemma 2.4.1, says that it is possible to make every S-bridge W-stable by making the following "local" changes. For a segmenting set $W \subseteq V(S)$ of S, let P be a W-segment of S of length at least two, and let Q be a path in G linking x, y that is otherwise disjoint from S. Let S' be obtained from S by replacing the path xPy by Q. Then we say that S' is obtained from S by *rerouting P along Q*, or simply that S' is obtained from S by *rerouting*. Please note that P is required to have length at least two, and hence this relation is not symmetric. Also note that W is a segmenting set of S', as S' is equal to the union of all W-segments of S'. We say the rerouting is *proper with respect to W* if all attachments of the S-bridge that contains Q belong to P. Lemma 2.4.1 is a generalization of Lemma 2.1 in [12] and is essentially due to Tutte.

Lemma 2.4.1 (Rerouting Lemma)**.** *Let G be a graph. Let S be a subgraph of G, and $W \subseteq V(S)$ be a segmenting set of S. Then, there exists a subgraph S' of G obtained from S by a sequence of proper reroutings with respect to W such that if all attachments of an S'-bridge B belong to some W-segment P of S', then there exist vertices $x, y \in V(P)$ such that some component of $G \backslash \{x, y\}$ includes a vertex of B and is disjoint from $S' \backslash V(P)$.*

Proof. Choose a subgraph $S' \subseteq G$ that can be obtained from S by a sequence of proper reroutings with respect to W such that the number of vertices in $G - V(S')$ belonging to W-stable S'-bridges is maximum, and subject to this, $|V(S')|$ is minimum. We will prove that S' is as desired. Let B be a W-unstable S'-bridge, and say all attachments of B on S' belong to a W-segment P of S'.

Let $v_0, v_1, ..., v_k$ be distinct vertices of P, listed in order of occurrence on P such that v_0 and v_k are the ends of P and $\{v_1, ..., v_{k-1}\}$ is the set of all internal vertices of P that are attachments of W-stable S'-bridges.

Assume for a moment that for some $i \in \{1, ..., k-1\}$, there exist two attachments u, v of B such that v_i is an internal vertex of uPv. Let S'' be a subgraph of G obtained from G by replacing uPv by an induced subpath of B linking u, v that is otherwise disjoint from S'. It follows that every vertex in $V(G) - V(S')$ belonging to a W-stable S'-bridge is in $V(G) - V(S'')$ and belongs to a W-stable S''-bridge, and that $v_i \in V(S')$ is in $V(G) - V(S'')$ and belongs to a W-stable S''-bridge, a contradiction to the choice of S'. Therefore, all attachments of B are on v_iPv_{i+1} for some $i \in \{0, 1, ..., k-1\}$.

Since B is arbitrary, it follows that for every W-unstable S'-bridge B' that attaches to some vertex in the interior of v_iPv_{i+1}, all attachments of B' are on v_iPv_{i+1}. Hence, there is a component K of $G\backslash\{v_i, v_{i+1}\}$ such that $V(B) - \{v_i, v_{i+1}\} \subseteq V(K)$ and $V(K) \cap (V(S')\backslash V(P)) = \emptyset$. This means that if $V(B) - \{v_i, v_{i+1}\} \neq \emptyset$, then v_i and v_{i+1} give the desired x and y. Therefore, we may assume that B is simply an edge joining v_i and v_{i+1}. Note this implies that v_iPv_{i+1} has length at least two. Let S'' be the subgraph of G obtained from G by replacing v_iPv_{i+1} by the edge v_iv_{i+1}. It follows that $|V(S'')| < |V(S')|$ and every vertex in $V(G) - V(S')$ belonging to a W-stable S'-bridge is in $V(G) - V(S'')$ and belongs to a W-stable S''-bridge, a contradiction to the choice of S'. □

2.4.2 Tripods

We now introduce the tripod structure, which is originally due to Robertson and Seymour [25].

Definition 2.4.2. In a graph G with a subset of three distinct vertices $X = \{x_1, x_2, x_3\} \subseteq V(G)$, a subgraph T of G is called an X-*tripod* if T can be written as a union of internally disjoint subpaths $P_1, P_2, P_3, Q_1, Q_2, Q_3, L_1, L_2, L_3$ of G satisfying the following: For some distinct vertices $z_1, z_2, z_3 \in V(G)$, L_i links z_i and x_i for $i = 1, 2, 3$; and for distinct vertices $p, q \in V(G) - \bigcup_{i=1}^{3} V(L_i)$, P_i links p, z_i and Q_i links q, z_i for $i = 1, 2, 3$. Each path L_i for $i = 1, 2, 3$ is called a *leg* of T. A leg is *trivial* if it is a single vertex; it is *non-trivial* otherwise.

Lemma 2.4.3. *Let G be a graph and $X = \{x_1, x_2, x_3\} \subseteq V(G)$ be a subset of three distinct vertices such that G cannot be drawn in the plane with x_1, x_2, x_3 incident to the infinite face. If there is no non-trivial ≤ 2-separation of (G, X), then G has an X-tripod.*

Proof. We will prove the lemma by inducting on $|V(G)|$.

First consider the case that $|V(G)| \leq 5$. Notice that the lemma is true if $G \cong K_5$, as the complete graph K_5 is non-planar, has no non-trivial ≤ 2-separation, and contains an X'-tripod as a subgraph for every subset X' of three distinct vertices in it. Also notice that the

Lemma is trivially true if $|V(G)| \leq 4$, since the complete graph K_4 is planar, meaning that, if $|V(G)| \leq 4$, G can always be drawn in the plane with x_1, x_2, x_3 incident to the infinite face . We may then assume $|V(G)| = 5$. If every vertex in X and every vertex in $G - X$ are adjacent, then G has an X-tripod. So assume that there is some non-edge between a vertex in X and a vertex in $V(G) - X$. This then implies that $G \cup \{x_1x_2, x_1x_3, x_2x_3\}$ is isomorphic to some proper subgraph of K_5 and therefore is planar. It follows that $G \cup \{x_1x_2, x_1x_3, x_2x_3\}$ can be drawn in the plane with $x_1x_2x_3x_1$ bounding the infinite face, and therefore G can be drawn in the plane with x_1, x_2, x_3 incident to the infinite face.

From now on, we can assume $|V(G)| \geq 6$ and the assertion holds for all graphs on fewer than $|V(G)|$ vertices.

Observe that x_2 does not separate x_1 from x_3 in G: For the sake of a contradiction, assume that x_2 is a cut vertex of G, and there exist distinct components K_1, K_3 of $G - \{x_2\}$ such that $x_i \in V(K_i)$ for $i = 1, 3$. Note that if $K_3 - \{x_3\}$ is non-empty, then $\{x_2, x_3\}$ separates the non-trivial subgraphs K_1 and $K_3 - \{x_3\}$, a contradiction to the fact that (G, X) does not have a non-trivial ≤ 2-separation. By symmetry, it follows that $V(K_i) = \{x_i\}$ for $i = 1, 3$, meaning that x_2 is the only neighbor for x_1 and x_3 in G. Since $|V(G)| \geq 6$, we know that $|V(G) - X| > 0$. It follows that x_2 separates $\{x_1, x_3\}$ from $G - X$, again a contradiction to the fact that (G, X) does not have a non-trivial ≤ 2-separation.

Next, observe that we may assume $E(G[X]) = \{x_1x_2, x_2x_3\}$: Let $G' = (G \cup \{x_1x_2, x_2x_3\}) - \{x_1x_3\}$. Note that since G cannot be drawn in the plane with x_1, x_2, x_3 incident to the infinite plane, neither can G'; and since (G, X) does not have a non-trivial ≤ 2-separation, neither does (G', X). Also note that G has an X-tripod if and only if G' has one. Hence, it suffices to consider G' instead of G, so we may assume that $E(G[X]) = \{x_1x_2, x_2x_3\}$.

Let P_X be the path $x_1x_2x_3$ in G. We next prove that there exists some cycle C such that $P_X \subseteq C$ and every C-bridge attaches to x_2.

Claim 1. There exists a cycle C such that $P_X \subseteq C$ and every C-bridge attaches to x_2.

Proof of Claim 1. First observe that X is a segmenting set for every cycle in G that includes all three vertices in X. Since x_2 does not separate x_1 from x_3 in G, there exists some cycle in G that includes all three vertices in X. By Lemma 2.4.1, since (G, X) does not have a non-trivial ≤ 2-separation, there exists a cycle $C' \subseteq G$ such that $X \subseteq V(C')$ and every C'-bridge in G is X-stable. For $i = 1, 2$, let P_i be the subpath of C' linking x_i and x_{i+1} such that $V(P_i) \cap X = \{x_i, x_{i+1}\}$.

Let C be the cycle obtained from C' by replacing P_1 with the edge x_1x_2 and replacing P_2 with the edge x_2x_3. Let $B \subseteq G$ be any C-bridge. Since $V(C) \subseteq V(C')$, B is contained in a C'-bridge B'. Since every C'-bridge is X-stable, we know that B' is an X-stable C'-bridge and therefore attaches to some vertex on $P_1 \cup P_2 - \{x_1, x_3\}$. By the construction of C, it follows that B attaches to x_2. Since B is arbitrary, it follows that the cycle C is as desired. $\dashv$

By Claim 1, let $C \subseteq G$ be a cycle such that $P_X \subseteq C$ and every C-bridge attaches to x_2. We next show that we may assume G has a C-cross.

Claim 2. If G has no C-cross, then it has an X-tripod.

Proof of Claim 2. Assume that G has no C-cross. Recall that G cannot be drawn in the plane with x_1, x_2, x_3 incident to the infinite face. Since x_1, x_2, x_3 are all contained in C, it follows that G cannot be drawn in the plane with C bounding a face. By Theorem 2.3.1, some non-trivial C-reduction of G can be drawn in the plane with C bounding a face. This means that there exists a non-trivial ≤ 3-separation (A, B) of $(G, V(C))$ such that $G_1 = G[A] \cup \{uv : u, v \in A \cap B\}$ can be drawn in the plane with C bounding a face. Since there is no non-trivial ≤ 2-separation of (G, X), every non-trivial separation of $(G, V(C))$ has order at least three. It follows that $|A \cap B| = 3$, and there exist three disjoint paths L_1, L_2, L_3 linking X and $A \cap B$ in $G[A]$. It also follows that there is no non-trivial ≤ 2-separation of $(G[B], A \cap B)$. Due to the disjoint paths L_1, L_2, L_3, since G does not have an X-tripod, we know that $G[B]$ does not have an $(A \cap B)$-tripod. Since $|B| < |V(G)|$, by induction, $G[B]$ can be drawn in the plane with every vertex in $A \cap B$ incident to the infinite face. This means that the graph $G_2 = G[B] \cup \{uv : u, v \in A \cap B\}$ can be drawn in the plane with the cycle on $A \cap B$ bounding the infinite face. Now, the drawings of G_1 and G_2 can be combined to form a planar drawing of $G_1 \cup G_2 = G \cup \{uv : u, v \in A \cap B\}$ with C bounding a face, a contradiction. $\dashv$

By Claim 2, we may assume that G has a C-cross. This means that there exist four distinct vertices s_1, s_2, t_1, t_2 in order on C and two disjoint paths R_1, R_2 in G such that R_i links s_i, t_i for $i = 1, 2$ and is otherwise disjoint from C. Let $P = C - \{x_2\}$.

First observe that we may assume $x_2 \in \{s_1, t_1, s_2, t_2\}$. To see this is true, assume that $x_2 \notin \{s_1, t_1, s_2, t_2\}$, which means that $\{s_1, s_2, t_1, t_2\} \subseteq V(P)$. Without loss of generality, assume that P goes through $x_1, s_1, s_2, t_1, t_2, x_3$ in order, where $\{x_1, x_3\}$ may or may not be disjoint from $\{s_1, t_2\}$. Since every C-bridge attaches to x_2, it follows that there is a path R_0 linking x_2 and some $r \in V(R_1 \cup R_2) - V(P)$ such that R_0 is otherwise disjoint from $C \cup R_1 \cup R_2$. Without loss of generality, assume that r is an internal vertex of R_2. By replacing R_2 with the path $R_0 \cup rR_2s_2$, we would then have x_2 as an end of R_2, as desired.

Now, without loss of generality, say s_1, t_1 are distinct vertices on P such that $V(x_1Ps_1) \cap V(x_3Pt_1) = \emptyset$, s_2 is an internal vertex of s_1Pt_1, and that $x_2 = t_2$. Since every C-bridge attaches to x_2, the path R_1 has length at least two and is contained in some C-bridge attaching to x_2. It follows that there exists a path W linking an internal vertex of R_1 and some vertex on $R_2 - \{s_2\}$ such that W is otherwise disjoint from $C \cup R_1 \cup R_2$. Let $T = P \cup R_1 \cup R_2 \cup W$. It follows that T is an X-tripod as desired. $\square$

Here we introduce more notations and definitions related to tripods.

Let G be a graph and $X = \{x_1, x_2, x_3\} \subseteq V(G)$ be a subset of three distinct vertices in G. Let $T \subseteq G$ be an X-tripod. Let vertices $z_1, z_2, z_3, p, q \in V(T)$ and paths L_1, L_2, L_3, P_1, P_2, P_3, Q_1, Q_2, Q_3 be labeled as in Definition 2.4.2 for the X-tripod T.

Let $\mathcal{L}(T) = V(L_1 \cup L_2 \cup L_3)$, $\mathcal{Z}(T) = \{z_1, z_2, z_3\}$, and $\mathcal{R}(T) = V(P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3)$. For two subsets of vertices $A, B \subseteq V(G)$, say the ordered pair (A, B) *splits* T if (A, B) is a 3-separation of G such that $\mathcal{L}(T) \subseteq A$, $\mathcal{R}(T) \subseteq B$, $\mathcal{Z}(T) = A \cap B$.

Equivalently, we also say T is *split* by (A, B) or simply that T is *split* in G if some ordered pair (A, B) *splits* T in G. Let $\mathcal{W}(T) = \mathcal{Z}(T) \cup X \cup \{p, q\}$. Observe that $\mathcal{W}(T)$ is a segmenting set of T. For convenience, we say a T-bridge is *stable* if it is $\mathcal{W}(T)$-stable and otherwise *unstable*, and we say a subpath of T is a *segment* of T if it is a $\mathcal{W}(T)$-segment of T. Also, we call a (proper) rerouting with respect to $\mathcal{W}(T)$ simply a *(proper) rerouting*.

We next define three types of "local" changes, called tripod-transformations, that could be made on some X-tripods.

Let R be a path in G linking some $r_1 \in \mathcal{L}(T) - \mathcal{Z}(T)$ and $r_2 \in \mathcal{R}(T) - \mathcal{Z}(T)$. Without loss of generality (up to relabeling the vertices and segments in T), assume that $r_1 \in V(L_1) - \{z_1\}$ and either $r_2 \in V(P_1) - \{z_1\}$ or $r_2 \in V(P_2) - \{z_2\}$. If $r_2 \in V(P_1) - \{z_1\}$, let T' be obtained from T by including R and deleting internal vertices of $z_1 P_1 r_2$. If $r_2 \in V(P_2) - \{z_2\}$, let T' be obtained from T by including R and deleting internal vertices of P_1. Then, T' is an X-tripod in both cases. We say T' is obtained from T by a *tripod-transformation of Type I*, or simply that T' is obtained from T by a *tripod-transformation*.

For some distinct indices $i, j \in \{1, 2, 3\}$, let $S_1, S_2 \subseteq G$ be two disjoint paths such that S_1 links x_i and a vertex on $L_j - \{x_j\}$, S_2 links x_j and a vertex on $L_i - \{x_i\}$, and that S_1, S_2 each are internally disjoint from T. Without loss of generality, say $i = 1$ and $j = 2$. Let T' be obtained from T by including $S_1 \cup S_2$ and deleting the internal vertices of P_2 and Q_1. Then, T' is an X-tripod. We say T' is obtained from T by a *tripod-transformation of Type II*, or simply that T' is obtained from T by a *tripod-transformation*.

Let $r \in V(G) - V(T)$ and R_1, R_2, R_3 be three paths in G such that R_i for $i = 1, 2, 3$ links r and some $u_i \in V(L_i)$ and is otherwise disjoint from T, $u_i \neq z_i$ for some $i \in \{1, 2, 3\}$, and that R_1, R_2, R_3 are pairwise disjoint except for r. Let T' be obtained from T by including $R_1 \cup R_2 \cup R_3$ and deleting $V(P_1 \cup P_2 \cup P_3) \backslash \mathcal{Z}(T)$, and let T'' be obtained from T by including $R_1 \cup R_2 \cup R_3$ and deleting $V(Q_1 \cup Q_2 \cup Q_3) \backslash \mathcal{Z}(T)$. Then, we say T' and T'' are obtained from T by a *tripod-transformation of Type III*, or simply that T' is obtained from T by a *tripod-transformation*.

In the next lemma, we observe a series of properties of tripod-transformations and reroutings on tripods.

Lemma 2.4.4. *Let G be a graph and $X = \{x_1, x_2, x_3\} \subseteq V(G)$ be a subset of three distinct vertices in G. Let $T \subseteq G$ be an X-tripod. Then, the following statements are true.*

(1) There is a 3-separation of G splitting T if and only if there is no X-tripod in G that can be obtained from T by a tripod-transformation of Type I.

(2) If at least two legs of T are trivial, then there is no X-tripod in G that can be obtained from T by a tripod-transformation of Type II.

(3) If T' is an X-tripod in G obtained from T by a tripod-transformation of Type II, then T' has at least two trivial legs.

(4) There exists an X-tripod $T' \subseteq G$ that can be obtained from T by a sequence of tripod-transformations of Type I such that some 3-separation of G splits T'.

(5) If there is no non-trivial ≤ 2-separation of (G, X), then there exists an X-tripod $T' \subseteq G$ obtained from T by a sequence of proper reroutings such that every T'-bridge is stable. Furthermore, if some 3-separation (A, B) of G splits T, then (A, B) also splits T'.

Proof. (1)-(3) are simply true due to the definitions of tripod-transformations of Type I and Type II.

To prove (4), we may assume that T is not split by any 3-separation of G, since otherwise we could just let $T' = T$. Let $T_0 = T$. By (1), we can recursively find a sequence of X-tripods $T_1, T_2, ...$ such that for every $i >= 0$, if T_i is not split by a 3-separation of G, then T_{i+1} is an X-tripod obtained by a tripod-transformation of Type I. Notice that $3 \leq |\mathcal{L}(T_{i+1})| < |\mathcal{L}(T_i)|$ for all i by the definition of tripod-transformations of Type I. It follows that the sequence of X-tripods $T_0, T_1, T_2, ...$ must have finite length. Let T' be the last X-tripod in this sequence. By (1), it follows that T' is split by some 3-separation of G.

It remains to prove (5). Let $W = \mathcal{W}(T)$, and recall that W is a segmenting set of T. By Lemma 2.4.1, there exists an X-tripod $T' \subseteq G$ obtained from T by a sequence of proper reroutings such that if all attachments of an unstable T'-bridge B_0 belong to some segment P of T', then there exist vertices $x, y \in V(P)$ such that some component K of $G \backslash \{x, y\}$ includes a vertex of B_0 and is disjoint from $T' \backslash P$. Note that if such an unstable T-bridge B_0 exists, let the segment P of T', vertices $x, y \in V(P)$, and the component K of $G \backslash \{x, y\}$ be labeled as in the description above. Then, $V(K) \cap V(T)$ is a subset of the set of internal vertices of P, and therefore $\{x, y\}$ separates K from X in G, a contradiction to the fact that there is no non-trivial ≤ 2-separation of (G, X). Therefore, every T'-bridge is stable. Now, assume that (A, B) is a 3-separation of G that splits T. Then, since T' is obtained from T by a sequence of proper reroutings, which contains $A \cap B$ as a subset, every proper rerouting in the sequence is completely includes in either A or B. It follows that (A, B) also splits T' as well. □

Lemma 2.4.5. *Let G be a graph and $X = \{x_1, x_2, x_3\} \subseteq V(G)$ be a subset of three distinct vertices such that there exists some X-tripod in $G - N[x]$. If there is no non-trivial ≤ 2-separation of (G, X), there exist an X-tripod T satisfying the following properties:*

(i) Some 3-separation of G splits T.

(ii) Every T-bridge in G is stable.

(iii) There is no X-tripod in G that can be obtained from T by a tripod-transformation.

Proof. By (4) and (5) of Lemma 2.4.4, there exists an X-tripod T_1 in G such that every T_1-bridge in G is stable and some 3-separation of G splits T_1. By (1) of Lemma 2.4.4, there is no X-tripod in G that can be obtained from T_1 by a tripod-transformation of Type I. If there is no X-tripod in G that can be obtained from T_1 by a tripod-transformation of Type II, then T_1 would be as desired. So we may assume that there exists an X-tripod T_2 in G that can be obtained from T_1 by a tripod-transformation of Type II. By (3) of Lemma 2.4.4, at least two legs of T_2 are trivial.

By (4) and (5) of Lemma 2.4.4 again, we can obtain an X-tripod T_3 from T_2 by a sequence of tripod-transformations of Type I, followed by a sequence of proper reroutings, such that every T_3-bridge is stable and some 3-separation of G splits T_3. Note that T_2

has at least two trivial legs, and so does T_3, due to the construction of T_3. By (1) and (2) of Lemma 2.4.4, there is no X-tripod in G that can be obtained from T by a tripod-transformation. Hence, T_3 is an X-tripod in G satisfying the desired properties. □

2.5 Exceptional graphs

Say a graph is an *exceptional graph* if it is isomorphic to one of the K_{10} minor-free graphs stated in Theorem 1.1.5. In a graph G, a subset $U \subseteq V(G)$ of vertices is called an *island* of G if it is a minimal subset of vertices such that $G = G[U] + G[U']$ where $U' = V(G) \backslash U$. An island of size k is called a *k-island*. A partition $\mathcal{P} = (V_1, ..., V_t)$ of $V(G)$ is called an *island partition* of G if every V_i is an island of G.

Lemma 2.5.1. *Every graph G has a unique island partition.*

Proof. If there exist two non-empty graphs K and L such that $G \cong K + L$, then G has an island partition of size at least 2; otherwise $V(G)$ itself is an island in G. This shows the existence of an island partition of G.

For the sake of a contradiction, suppose $\mathcal{P} = (V_1, ..., V_t)$ and $\mathcal{Q} = (U_1, ..., U_r)$ are two distinct island partitions of G. Without loss of generality, assume $V_1 \cap U_1 \neq \emptyset$ and $V_1 - U_1 \neq \emptyset$. Since $V_1 - U_1$ is not included in U_1 in the partition $\mathcal{Q}$, for every $a \in V_1 - U_1$ and every $b \in V_1 \cap U_1$, $ab \in E(G)$. This shows $G[V_1] = G[V_1 - U_1] + G[V_1 \cap U_1]$ where both $V_1 - U_1$ and $V_1 \cap U_1$ are non-empty, a contradiction to the fact that V_1 is an island. □

Let G be a graph. For any edge $xy \in E(G)$, the vertex $w \in V(G/xy)$ obtained from the contraction of xy in G is called *the new vertex of G/xy*. For a subset $U \subseteq V(G)$ of vertices, denote $q_G(U) = |U| - \omega(G[U])$, i.e. the minimum number of vertices to delete from U such that the remaining vertices induce a complete subgraph. For a partition $\mathcal{P} = (V_1, ..., V_t)$ of $V(G)$, denote $q_G(\mathcal{P}) = \sum_{V_i \in \mathcal{P}} q_G(V_i)$ and $l_G(\mathcal{P}) = \lceil q_G(\mathcal{P})/2 \rceil$.

Lemma 2.5.2. *Let G be a graph on n vertices. Let $\mathcal{P} = (V_1, ..., V_t)$ be a partition of $V(G)$ such that $G = G[V_1] + ... + G[V_t]$. If $\max\{q_G(V_i) : 1 \le i \le t\} \le \frac{1}{2}q_G(\mathcal{P})$, then $G > K_{n-l}$ where $l = l_G(\mathcal{P})$.*

Proof. For convenience, let $q_i = q_G(V_i)$ for all i. If $q_i = 0$ for some i, then every vertex in V_i is adjacent to all other vertices in G. We can then just delete V_i from G since $G > K_{n-l}$ if and only if $G - V_i > K_{n-|V_i|-l}$. This means we may assume $q_i \ge 1$ for all i. Since $q_i \le |V_i| - 1$ for every i, it follows that $|V_i| \ge 2$ for all i. Also, since $\max\{q_G(V_i) : 1 \le i \le t\} \le \frac{1}{2}q_G(\mathcal{P})$ and $q_i \ge 1$ for all i, it follows that $t \ge 2$. By relabeling the subsets in $\mathcal{P}$, we may assume V_i are sorted in the decreasing order of q_i, meaning that $q_1 \le \frac{1}{2}\sum_{i=1}^{t} q_i$.

We are going to prove the lemma by inducting on $\sum_{i=1}^{t} q_i$. The base case is $\sum_{i=1}^{t} q_i = 2$, which happens precisely when $t = 2$ and $q_1 = q_2 = 1$. In this case, $l = \lceil (1+1)/2 \rceil = 1$. Let $u_i \in V_i$ for each $i \in \{1, 2\}$ such that $G[V_i - \{u_i\}]$ is complete. By contracting the edge u_1u_2 we can get a K_{n-1} minor.

So assume $\sum_{i=1}^{t} q_i \ge 3$. Choose $u_i \in V_i$ for each $i \in \{1, 2\}$ such that $\omega(G[V_i - \{u_i\}]) = \omega(G[V_i])$. Let w be the new vertex of G/u_1u_2. Let $H = G/u_1u_2 - \{w\}$. Note $|H| = n - 2$. To show $G > K_{n-l}$, we will show $H > K_{n-l-1} = K_{|H|-l'}$ where $l' = l - 1$ by proving that a complete minor can be obtained within $l' = l - 1$ contractions from H.

Let $W_i = V_i - \{u_i\}$ for $i \in \{1, 2\}$ and let $W_i = V_i$ for all $3 \le i \le t$. For each i, since $|V_i| \ge 2$, $|W_i| \ge 1$, and therefore $\mathcal{P}' = (W_1, ..., W_t)$ is a partition of $V(H)$ where each W_i is non-empty. Let $q'_i = q_H(W_i)$. Note $q'_i = q_i$ for $i \ge 3$ and $q'_i = q_i - 1$ for $i = 1, 2$. This implies

$$l' = l - 1 = \lceil (\sum_{i=1}^{t} q_i)/2 \rceil - 1 = \lceil \frac{1}{2}\big((q_1 - 1) + (q_2 - 1)\big) + \frac{1}{2}(\sum_{i=3}^{t} q_i) \rceil = \lceil (\sum_{i=1}^{t} q'_i)/2 \rceil = l_H(\mathcal{P}').$$

By induction, it suffices to prove $\max\{q'_i : 1 \le i \le t\} \le \frac{1}{2}\sum_{i=1}^{t} q'_i$. Since q_i are in the decreasing order, either $\max\{q'_i : 1 \le i \le t\} = q'_1$ or $\max\{q'_i : 1 \le i \le t\} = q'_3$ in the case

$t \geq 3$. If $\max\{q'_i : 1 \leq i \leq t\} = q'_1$, then

$$q'_1 = q_1 - 1 \leq \sum_{i=2}^{t} q_i - 1 = \sum_{i=2}^{t} q'_i,$$

meaning that $q'_1 \leq \frac{1}{2}\sum_{i=1}^{t} q'_i$. We may then assume $t \geq 3$ and $q'_3 > q'_1 \geq q'_2$. Since $q_1 \geq q_2 \geq q_3$, it follows that $q_1 = q_2 = q_3$, and we let this value be q for convenience. If $q \geq 2$, then

$$q'_3 = q \leq 2q - 2 = (q-1) + (q-1) = q'_1 + q'_2 \leq \sum_{i=1}^{t} q'_i - q'_3,$$

meaning $q'_3 \leq \frac{1}{2}\sum_{i=1}^{t} q'_i$. We may then assume $q = 1$, meaning $q_i = 1$ for all i. If there exists $q_4 \geq 1$, then we get $q'_3 \leq \frac{1}{2}\sum_{i=1}^{t} q'_i$ again. If $t = 3$, then $l' = \lceil (0+0+1)/2 \rceil = 1$. Recall $|W_1| \geq 1$. Let $u'_1 \in W_1$ be arbitrary and let $u_3 \in W_3$ such that $H[W_3 - \{u_3\}]$ is complete. By contracting the edge $u'_1 u_3$ in H, we can get a complete minor. □

Lemma 2.5.3. *Let G be an exceptional graph that is not isomorphic to a non-trivial $(K_{1,1,2,2,2,2,2}, 7)$-cockade or a non-trivial $(K_{2,2,2,2,2,3}, 6)$-cockade. Let $\mathcal{P}$ be the island partition of G. Then, $q_G(\mathcal{P}) = 2(|G| - 10) + 1$ and $q_G(C) \leq |G| - 10$ for every $C \in \mathcal{P}$.*

Proof. Suppose $G \ncong G_i$ for any i. Let C be an island of G. Note that C is either an independent set or induces some 5-cycle. If C is an independent set, then $q_G(C) = |C| - 1$; and if $G[C] \cong C_5$, then $q_G(C) = 3$. One can then check the proposition holds by simply counting.

Suppose $G = G_i$ for some $i \in \{1, 2, 3, 4\}$ and $e \notin E(G)$ such that $G + e \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$. Let C be an island of G. Note if C is an independent set, then again $q_G(C) = |C| - 1 = q_{G+e}(C)$. Otherwise, we can write $C = C_1 \cup C_2$ where C_1, C_2 are two distinct islands of $G + e$, and e has one end in C_1 and the other one in C_2. Observe that, in any case, we always have $q_G(C) = q_G(C_1) + q_G(C_2) = q_{G+e}(C_1) + q_{G+e}(C_1)$. It follows that $q_G(\mathcal{P}) = q_{G+e}(\mathcal{P}')$ where $\mathcal{P}'$ is the island partition for $G + e$. If $G + e \cong K_{2,2,2,2,2,3}$,

$|G| = 13$ and $\max\{q_G(C) : C \in \mathcal{P}\} = 3$; and if $G + e \cong K_{2,3,3,3,3}$, $|G| = 14$ and $\max\{q_G(C) : C \in \mathcal{P}\} = 4$. Therefore, the proposition holds for $G \cong G_i$ where $i \in \{1, 2, 3, 4\}$, too. □

Lemma 2.5.4. *Let G be an exceptional graph. If $G \ncong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$, then $e(G) = 8|G| - 35$. If $G \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$, then $e(G) = 8|G| - 34$.*

Proof. If G is not a non-trivial $(K_{1,1,2,2,2,2,2}, 7)$-cockade or a $(K_{2,2,2,2,2,3}, 6, 2)$-cockade of multiplicity 2, it suffices to check the following:

$$e(K_{1,1,2,2,2,2,2}) = 61 = 8 \cdot 12 - 35,$$
$$e(K_{1,2,2,2,3,3}) = 69 = 8 \cdot 13 - 35,$$
$$e(K_{2,2,2,2} + C_5) = 69 = 8 \cdot 13 - 35,$$
$$e(K_{2,2,3,3,4}) = 77 = 8 \cdot 14 - 35,$$
$$e(K_{3,3,3} + C_5) = 77 = 8 \cdot 14 - 35,$$
$$e(K_{2,2,2,2,2,3}) = 70 = (8 \cdot 13 - 35) + 1, \text{ and}$$
$$e(K_{2,3,3,3,3}) = 78 = (8 \cdot 14 - 35) + 1.$$

If G is a $(K_{2,2,2,2,2,3}, 6)$-cockade of multiplicity 2, then $e(G) = 2e(K_{2,2,2,2,2,3}) - \binom{6}{2} = 8 \cdot 20 - 35$.

If G is a non-trivial $(K_{1,1,2,2,2,2,2}, 7)$-cockade, we prove $e(G) = 8|G| - 35$ by inducting on $|G|$. The base case is $|G| = 12$ and was just shown above. Write $G = G_1 \cup G_2$ such that G_1, G_2 are both $(K_{1,1,2,2,2,2,2}, 7)$-cockades, $G_1 \cap G_2 \cong K_7$, and there is no edge between $G_1 - G_2$ and $G_2 - G_1$. By induction, $e(G_i) = 8|G_i| - 35$ for $i \in \{1, 2\}$. It follows that

$$e(G) = e(G_1) + e(G_2) - e(K_7) = 8(|G_1| + |G_2|) - 70 - 21 = 8(|G| + 7) - 91 = 8|G| - 35.$$

□

Lemma 2.5.5. *Let H be a t-connected graph for integer $t > 0$. Let G be an (H, t)-cockade. Then, G is t-connected.*

Proof. We prove this lemma by inducting on $|G|$. The base case $G \cong H$ is trivially true, and we may assume that $G = G_1 \cup G_2$ such that G_1, G_2 are both (H, t)-cockades, $G_1 \cap G_2 \cong K_t$, and there is no edge between $G_1 - G_2$ and $G_2 - G_1$. By induction, both G_1, G_2 are t-connected. Let $A \subseteq V(G)$ such that $|A| = t - 1$. Let $A_i = A \cap V(G_i)$ for $i = 1, 2$. Note that $|A_i| \leq t - 1$ for $i = 1, 2$ and $V(G_1 \cap G_2) - A \neq \emptyset$. For $i = 1, 2$, since G_i is t-connected, $G_i - A_i$ is connected. Since $V(G_1 \cap G_2) - A \neq \emptyset$, we know that $G - A = (G_1 - A_1) \cup (G_2 - A_2)$is connected. It follows that G is t-connected, as A was chosen arbitrarily. □

Lemma 2.5.6. *Let H be a graph, and let $\omega(H) = t$. Let G be an (H, t)-cockade.Then, G does not contain a subgraph isomorphic to K_{t+1}.*

Proof. We prove this lemma by inducting on $|G|$. The base case $G \cong H$ is trivially true, as $\omega(H) = t$. We may then assume that $G = G_1 \cup G_2$ such that G_1, G_2 are both (H, t)-cockades, $G_1 \cap G_2 \cong K_t$, and there is no edge between $G_1 - G_2$ and $G_2 - G_1$. By induction, neither G_1 nor G_2 contains a subgraph isomorphic to K_{t+1}. Since there is no edge between $V(G_1 - G_2)$ and $V(G_2 - G_1)$, every subset $A \subseteq V(G)$ of vertices such that $G[A]$ is a clique is completely contained in either G_1 or G_2. It follows that G does not have a subgraph isomorphic to K_{t+1}. □

2.5.1 Deletion Lemma

We will next prove that adding an edge to an exceptional graph would make it have a K_{10} minor, unless the new graph is isomorphic to another exceptional graph. We call this lemma the deletion lemma, which will later be used to show that a minimum counter-example graph to our main theorem on n vertices has exactly $8n - 35$ edges.

Lemma 2.5.7 (Deletion Lemma)**.** *Let G be an exceptional graph. Let x and y be two non-adjacent vertices in G. Then, either $G + xy$ is an exceptional graph, or $G + xy > K_{10}$.*

Proof. We will consider the two cases whether or not G is a non-trivial (H,t)-cockade, where $(H,t) = (K_{2,2,2,2,2,3}, 2)$ or $(K_{1,1,2,2,2,2,2}, 7)$, separately.

Case 1: G is NOT a non-trivial (H,t)-cockade, where $(H,t) = (K_{2,2,2,2,2,3}, 2)$ or $(K_{1,1,2,2,2,2,2}, 7)$.

Let $\mathcal{P}$ be the island partition of G. By Proposition 2.5.3, $q_G(\mathcal{P}) = 2(|G| - 10) + 1$. Note that x, y are in the same island $C \in \mathcal{P}$ as x, y are non-adjacent. We will first show $q_{G+xy}(\mathcal{P}) = 2(|G| - 10)$.

If $G \ncong G_i$ for some $i \in \{1, 2, 3, 4\}$, then every island of G is either an independent set or induces a 5-cycle. It follows that $q_{G+xy}(C) = q_G(C) - 1$ and therefore $q_{G+xy}(\mathcal{P}) = q_G(\mathcal{P}) - 1 = 2(|G| - 10)$.

If $G \cong G_i$ for some $i \in \{1, 2, 3, 4\}$, let $e \notin E(G)$ such that $G + e \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$. Notice that since $e \notin E(G)$, C includes either zero or two ends of e. If C does not include any end of e, then C is just an independent set and we have $q_{G+xy}(C) = q_G(C) - 1$ again. It follows that $q_{G+xy}(\mathcal{P}) = 2(|G| - 10)$. We may then assume C includes both ends of e. This means that C is the disjoint union of islands C_1, C_2 of $G + e$, where $(G + e)[C_i] \cong \overline{K_2}$ or $\overline{K_3}$ for both $i = 1, 2$. Note that if $e = xy$, then $G + xy \cong G_i$ for some $i \in \{1, 2, 3, 4\}$, and we would then be done with the proof. So we may assume that $e \neq xy$, meaning that x, y are both contained in C_i for some $i \in \{1, 2\}$. One can then check that we have $q_{G+xy}(C) = q_G(C) - 1$ in all cases. It follows that $q_{G+xy}(\mathcal{P}) = q_G(\mathcal{P}) - 1 = 2(|G| - 10)$.

Now in both cases, we have $q_{G+xy}(\mathcal{P}) = 2(|G| - 10)$. By Lemma 2.5.3, $q_G(C') \leq |G| - 10 = \frac{1}{2} q_{G+xy}(\mathcal{P})$ for every island C' of G. It follows that $q_{G+xy}(C') \leq q_G(C') \leq \frac{1}{2} q_{G+xy}(\mathcal{P})$. By Lemma 2.5.2, we conclude that $G + xy > K_{10}$.

Case 2: G is a non-trivial (H,t)-cockade, where $(H,t) = (K_{2,2,2,2,2,3}, 2)$ or $(K_{1,1,2,2,2,2,2}, 7)$.

Note that in the case $(H,t) = (K_{2,2,2,2,2,3}, 2)$, G is exactly a $(K_{2,2,2,2,2,3}, 2)$-cockade of multiplicity 2. Write $G = G_1 \cup G_2$ such that G_1, G_2 are both (H,t)-cockades, $G_1 \cap G_2 \cong K_t$, and there is no edge between $G_1 - G_2$ and $G_2 - G_1$. Due to Case 1 and Lemma 2.5.4, we may assume that for both $i = 1, 2$, $G_i + zw > K_{10}$ for every pair of non-adjacent vertices $z, w \in V(G_i)$. This means that we may assume that x, y not both contained in one of G_1 and G_2.

Without loss of generality, say $x \in V(G_1) - V(G_2)$ and $y \in V(G_2) - V(G_1)$. Since G_1, G_2 are both (H, t)-cockades, where $(H, t) = (K_{2,2,2,2,2,3}, 2)$ or $(K_{1,1,2,2,2,2,2}, 7)$, we know that there exists some $x' \in V(G_1 \cap G_2)$ such that $xx' \notin E(G)$, and that $G[V(G_2 - G_1) \cup \{x'\}]$ is connected due to Lemma 2.5.5. It follows that there exists a path $Q \subseteq G[V(G_2 - G_1) \cup \{x'\}]$ linking x' and y. Notice that $G_1 + xx' > K_{10}$, as $xx' \notin E(G_1)$. It follows that by contracting edges on Q to one single vertex in $G + xy$, we would then obtain resulting graph that contains a subgraph isomorphic to $G_1 + xx'$, meaning that $G + xy > K_{10}$. □

2.5.2 Contraction Lemmas

The goal of this subsection is to prove that if a graph G has $\delta(G) \geq 8$ and two adjacent vertices x, y such that G/xy is isomorphic to an exceptional graph, and that x, y share exactly $8|G| - e(G/xy) - 36$, then either $G > K_{10}$ or G is isomorphic to some other exceptional graphs. We call lemmas in this form contraction lemmas. We will prove 5 contraction lemmas in this subsection, namely Lemma 2.5.10, Lemma 2.5.11, Lemma 2.5.12, Lemma 2.5.13, and Lemma 2.5.14. These contraction lemmas will be later used to show that in a minimum counter-example graph to our main theorem, every edge is contained in at least eight triangles.

Lemma 2.5.8. *Let G be a graph, and let $x, y \in V(G)$ be two distinct vertices. Let $N = N_G(x) \cup N_G(y) - \{x, y\}$. Let $N' \subseteq N$ and $\alpha, \beta \in \mathbb{Z}^+$ be such that x, y each have at least α neighbors in N', and that they have at least β common neighbors in N'. Let $\mathcal{P} = (C_1, ..., C_t)$ be the island partition of $G[N']$, and let d be the number of 1-islands of $G[N']$.*

Suppose the triple $(G[N'], \alpha, \beta)$ satisfies the following properties: (i) $t \geq 2$, (ii) $\alpha \geq \beta + 1$, *(iii)* $\beta \geq \sum_{i=1}^{t} \omega(G[C_i])$*, and (iv)* $\beta \geq \max_i\{|C_i|\} + d + 1$*. Then, there exist two distinct islands* C_i*,* C_j *of* $G[N']$ *such that* C_i *contains non-adjacent vertices* $w_1, w_2 \in N(x)$*, and* C_j *contains non-adjacent vertices* $w_3, w_4 \in N(y)$*.*

Proof. Let $Z = N(x) \cap N(y) \cap N'$, and let $Z_i = Z \cap C_i$ for all $i = 1, ..., t$. For convenience, let $\omega_i = \omega(G[C_i])$ for all $i = 1, ..., t$. By (iii), $\sum_{i=1}^{t} \omega_i \leq \beta = \sum_{i=1}^{t} |Z_i|$. Note that $w_i \geq 1$ for all i.

Note that $|Z| = \beta \geq \alpha + 1 \geq 2$. We now observe that we may assume $G[Z]$ is a clique. To see this is true, assume that there exist non-adjacent vertices $w_1, w_2 \in Z_i \subseteq C_i \in \mathcal{P}$ for some island C_i. By (iv), there exists some C_j in $\mathcal{P} - \{C_i\}$ such that $|C_j| \geq 2$ and $Z_j \neq \emptyset$. Let $w_3 \in Z_j$ and $w_4 \in C_j - \{w_3\}$. Note that w_4 is adjacent to at least one of x and y. It follows that C_i, C_j and w_1, w_2, w_3, w_4 are as desired.

Therefore, we now assume that $G[Z_i]$ is a clique for all $i = 1, ..., t$, and hence $\omega_i \geq |Z_i|$. Since $\sum_{i=1}^{t} \omega_i \leq \sum_{i=1}^{t} |Z_i|$, we know that $|Z_i| = \omega_i$ and that $G[Z_i]$ is a maximum clique in $G[C_i]$ for all $i = 1, ..., t$. By (ii), there exists an island $C_i \in \mathcal{P}$ such that x has at least $\omega_i + 1$ neighbors in C_i, meaning that there exists some vertex of C_i that is adjacent to x but not to y. For the sake of a contradiction, we may assume that for every island $C_j \in \mathcal{P} - \{C_i\}$, every vertex in $C_j - Z_j$ is adjacent to x but not to y.

Assume for a moment that $C_j - Z_j = \emptyset$ for all $j \neq i$. Since $G[Z_j]$ is a clique and $C_j = Z_j$ is an island of $G[N']$, it follows that $|Z_j| = |C_j| = 1$ for all $j \neq i$. Since $|C_i| \geq 2$, we have $d = t - 1$ and thus $\beta = |Z_i| + t - 1 = |Z_i| + d < |C_i| + d$, a contradiction to (iv). It follows that there exists some island $C_j \in \mathcal{P} - \{C_i\}$ such that $C_j - Z_j \neq \emptyset$. Since every vertex in $C_j - Z_j$ is adjacent to x but not to y, we may then assume every vertex in $C_i - Z_i$ is adjacent to x but not to y. It follows that every vertex in $N' - Z$ is adjacent to x but not to y, meaning that y has exactly $|Z| = \beta$ neighbors in N', a contradiction to (ii). □

Lemma 2.5.9. *Let* G *be a graph with* $\delta(G) \geq 8$*. Suppose there is an edge* $xy \in E(G)$ *such that* x *and* y *share at least 6 common neighbors in* G*. Suppose* G/xy *is isomorphic to an exceptional graph that is neither a non-trivial* $(K_{1,1,2,2,2,2,2}, 7)$*-cockade nor non-trivial*

$(K_{2,2,2,2,2,3},6)$*-cockade. If* $G/xy \cong G_i$ *for some* $i \in \{1,2,3,4\}$*, let* $e \notin E(G/xy)$ *be the unique non-edge of* G/xy *such that* $G/xy+e \cong K_{2,2,2,2,2,3}$ *or* $K_{2,3,3,3,3}$*, and let* $H = G/xy+e$*; otherwise, let* $H = G/xy$*. Let* w *be the new vertex of* G/xy*. Let* C_w *be the island of* H *that contains* w*. Then, there are two distinct islands* C_1, C_2 *of* $H - C_w$ *such that*

(1) C_1 *contains two non-adjacent vertices* $w_1, w_2 \in N_G(x)$*,*

(2) C_2 *contains two non-adjacent vertices* $w_3, w_4 \in N_G(y)$*,*

(3) for every $u \in V(G/xy) - N_{G/xy}[w]$*,* u *is adjacent to* w_i *for every* $i \in \{1,2,3,4\}$ *in* G/xy*,*

(4) if $G/xy \cong K_{2,2,2,2}+C_5$ *and* $G/xy[C_w] \cong \overline{K_2}$*, then* $G/xy[C_i]$ *is isomorphic to a 5-cycle for some* $i \in \{1,2\}$*.*

Proof. Let $N = N_{G/xy}(w) = N_G(x) \cup N_G(y) - \{x,y\} \subseteq V(G)$. Note that $H = G/xy$ if and only if $H \not\cong G_i$ for any $i \in \{1,2,3,4\}$. We will prove this lemma by considering the following two cases: C_w is an island of G/xy, or $G/xy \cong G_i$ for some $i \in \{1,2,3,4\}$, and C_w is not an island of G/xy.

Case 1: C_w is an island of H as well as G/xy.

Note now it is not the case that $G/xy \cong G_i$ for some $i \in \{1,2,3,4\}$ and an end of e is contained in C_w. Since C_w is an island of G/xy, every vertex $u \in V(G/xy) - N_{G/xy}[w]$ must be in C_w and thus is adjacent to every vertex in $V(G/xy) - C_w$ in the graph G/xy. This means that it suffices to find two distinct islands satisfying (1), (2), and (4).

First assume that C_w is an independent set of H first, namely $H \cong K_{2,2,3,3,4}$, $K_{2,2,2,2,2,3}$, $K_{2,3,3,3,3}$, $K_{1,2,2,2,3,3}$, $K_{1,1,2,2,2,2,2}$, or $H \cong K_{3,3,3}+C_5$ or $K_{2,2,2,2}+C_5$ with $H[C_w]$ not isomorphic to a 5-cycle. In this case, it suffices to find desired islands satisfying (1) and (2), and we will use Lemma 2.5.8 to find them. Note that $H[N] = H - C_w$, and every island of $H[N]$ is equal to a unique island of $H - C_w$. Also note that x and y have at least $\beta = 6$ common neighbors in N, and since $\delta(G) \geq 8$ they each have at least $\alpha = 7$ neighbors in N. Let $\mathcal{P} = (C'_1, ..., C'_t)$ be the island partition of $G[N]$, and let d be the number of

1-islands in it. Notice that for every $C_i' \in \mathcal{P}$, $\omega(C_i') = 1$ if $H[C_i']$ is an independent set, and $\omega(C_i') = 2$ if $H[C_i']$ is isomorphic to a 5-cycle. Since $H[N]$ contains at most one island isomorphic to a 5-cycle, $\sum_{i=1}^{t} \omega(H[C_i']) = t + 1$ if $H[N]$ contains a 5-cycle, and $\sum_{i=1}^{t} \omega(H[C_i']) = t$ otherwise. One can then check that the triple $(H[N], \alpha, \beta)$ satisfies all (i)-(iv) in Lemma 2.5.8, and it follows that the desired islands satisfying (1) and (2) can be found.

We may then assume that $H \cong K_{3,3,3} + C_5$ or $K_{2,2,2,2} + C_5$, and $H[C_w]$ is isomorphic to a 5-cycle. We will again use Lemma 2.5.8 to find islands C_1, C_2 that satisfy (1) and (2) first. Let $N' = V(H) - C_w \subseteq N$, so every island of $H[N']$ is equal to a unique island of $H - C_w$. Since in either case w has exactly two neighbors in C_w in G/xy, we know $|N - N'| = 2$. It follows that x and y each have at least $\alpha' = 7 - 2 = 5$ neighbors in N', and they have at least $\beta' = 6 - 2 = 4$ common neighbors in N'. One can then check the triple $(H[N'], \alpha', \beta')$ satisfies all (i)-(iv) in Lemma 2.5.8, and therefore there exist distinct islands C_1, C_2 satisfying (1) and (2).

It remains to show (4) in the case $H = G/xy \cong K_{2,2,2,2} + C_5$ and $G/xy[C_w] \cong \overline{K_2}$. Note that we may assume the two islands C_1, C_2 of $H - C_w$ that we just found using Lemma 2.5.8 are both 2-islands. Let C denote the 5-island of H. Since every vertex in C is adjacent to w in G/xy, it must be adjacent to at least one of x and y in G. Since $G/xy[C]$ is a 5-cycle, there exist two non-adjacent vertices $w_1', w_2' \in C$ such that they are both adjacent to x or both adjacent to y in the graph G. Without loss of generality, assume w_1' and w_2' are both adjacent to x. We can then use C to replace C_1, use w_1' and w_2' to replace w_1 and w_2, and keep $C_2 = \{w_3, w_4\}$ the same. The modified islands C_1 and C_2 are as desired.

Case 2: C_w is not an island of H as well as G/xy.

In this case, $G/xy \cong G_i$ for some $i \in \{1, 2, 3, 4\}$, and C_w is not an island of G/xy. It suffices to find distinct islands C_1, C_2 that satisfy (1)-(3) now. Recall that e is the unique edge in $E(H) - E(G/xy)$, and note in this case C_w must contain an end of e. Let $N' = \{v \in N : v \text{ is adjacent to } u \text{ for every } u \in V(G/xy) - N_{G/xy}[w]\} \subseteq N$. Note that since every island of H is an independent set, every island C' of $H[N']$ is a subset of a unique island of $H - C_w$, say $h(C')$. Assume that there exist distinct islands C_i', C_j' of $H[N']$ such that C_i' contains two non-adjacent vertices both adjacent to x in G, and that C_j' contains two non-adjacent vertices both adjacent to y in G. Then by the definition of N', $C_1 = h(C_i')$ and $C_2 = h(C_j')$ are distinct islands of $H - C_w$ satisfying (1), (2), and (3). Note that x and y each have at least $\alpha' = 7 - |N - N'|$ neighbors in N', and they have at least $\beta' = 6 - |N - N'|$ common neighbors in N'. It follows that to find the desired islands, it suffices to show $(H[N'], \alpha', \beta')$ satisfies (i)-(iv) in Lemma 2.5.8. Again let $\mathcal{P} = (C_1', ..., C_t')$ be the island partition of $H[N']$, and let d be the number of 1-islands in it. We consider all cases in the following table.

Table 2.1

G/xy	$\lvert C_w \rvert$	if w is an end of e	$G[N]$	$H[N'] = G[N']$	$\lvert N - N' \rvert$	(α', β', t, d)
G_1	2	yes	$K_{1,2,2,2,3}$	$K_{2,2,2,3}$	1	$(6,5,4,0)$
	2	no	$K_{2,2,2,2,3}$	$K_{1,2,2,2,3}$	1	$(6,5,5,1)$
G_2	2	yes	$K_{2,2,2,2,2}$	$K_{2,2,2,2}$	2	$(5,4,4,0)$
	3	yes	$K_{1,2,2,2,2}$	$K_{2,2,2,2}$	1	$(6,5,4,0)$
	2	no	$K_{2,2,2,2,3}$	$K_{2,2,2,2,2}$	1	$(6,5,5,0)$
	3	no	$K_{2,2,2,2,2}$	$K_{1,2,2,2,2}$	1	$(6,5,5,1)$
G_3	3	yes	$K_{2,2,3,3}$	$K_{2,3,3}$	2	$(5,4,3,0)$
	3	no	$K_{2,3,3,3}$	$K_{2,2,3,3}$	1	$(6,5,4,0)$
G_4	2	yes	$K_{2,3,3,3}$	$K_{3,3,3}$	2	$(5,4,3,0)$
	3	yes	$K_{1,3,3,3}$	$K_{3,3,3}$	1	$(6,5,3,0)$
	2	no	$K_{3,3,3,3}$	$K_{2,3,3,3}$	1	$(6,5,4,0)$
	3	no	$K_{2,3,3,3}$	$K_{1,3,3,3}$	1	$(6,5,4,1)$

For each case in this table, one can check that the triple $(H[N'], \alpha', \beta')$ satisfies (i)-(iv) in Lemma 2.5.8. Therefore, the desired islands can be found. □

Lemma 2.5.10 (Contraction Lemma 1)**.** *Let G be a graph with $\delta(G) \geq 8$. Suppose there is an edge $xy \in E(G)$ such that x and y share at least 6 common neighbors in G. Suppose G/xy is isomorphic to an exceptional graph that is neither a non-trivial $(K_{1,1,2,2,2,2,2}, 7)$-cockade nor a $(K_{2,2,2,2,2,3}, 6)$-cockade of multiplicity 2. If the new vertex of G/xy is not adjacent to all other vertices, then $G > K_{10}$.*

Proof. We continue using the same definitions and notations used in Lemma 2.5.9: If $G/xy \cong G_i$ for some $i \in \{1,2,3,4\}$, let $e \notin E(G/xy)$ be the unique non-edge of $E(G/xy)$ such that $G/xy + e \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$, and let $H = G/xy + e$; otherwise, let $H = G/xy$. Let w be the new vertex of G/xy and C_w be the island of H containing w.

By Lemma 2.5.9, we can choose distinct islands C_1, C_2 of $H - C_w$ and vertices w_1, w_2, w_3, w_4 such that (1) $w_1w_2 \notin E(G)$ and $\{w_1, w_2\} \subseteq C_1 \cap N_G(x)$, (2) $w_3w_4 \notin E(G)$ and $\{w_3, w_4\} \subseteq C_2 \cap N_G(y)$, (3) for every $u \in V(G/xy) - N_{G/xy}[w]$, u is adjacent to w_i for every $i \in \{1,2,3,4\}$ in G/xy, and (4) if $G/xy \cong K_{2,2,2,2} + C_5$ and $G/xy[C_w] \cong \overline{K_2}$, then $G/xy[C_i]$ is isomorphic to a 5-cycle for some $i \in \{1,2\}$.

Define $H' = (H - \{w\}) + w_1w_2 + w_3w_4$. We first prove the following claim that $H' > K_{10}$.

Claim 1. $H' > K_{10}$.

Proof of Claim 1. Let $\mathcal{P}$ be the island partition of H. Note $\mathcal{P}$ contains C_w, C_1, and C_2. Let $\mathcal{P}_1 = \{C_w, C_1, C_2\}$ and $\mathcal{P}_2 = \mathcal{P} - \mathcal{P}_1$. Let $\mathcal{P}'$ be the partition of $V(H')$ obtained from $\mathcal{P}$ by replacing C_w with $C_w - \{w\}$. Let $\mathcal{P}'_1 = \{C_w - \{w\}, C_1, C_2\}$ and $\mathcal{P}'_2 = \mathcal{P}' - \mathcal{P}'_1$. Observe that for every $C \in \mathcal{P}_2$, $H[C] = H'[C]$ and thus $q_{H'}(C) = q_H(C)$. Also observe that every island in $\mathcal{P}$ either is an independent set of size at least 2 or induces a 5-cycle. This implies that for $i \in \{1,2\}$, $q_{H'}(C_1) = q_{H+w_1w_2}(C_1) = q_H(C_1) - 1$ and $q_{H'}(C_2) = q_{H+w_3w_4}(C_2) = q_H(C_2) - 1$, and furthermore that $q_{H'}(C_w - \{w\}) = q_H(C_w) - 1$. Hence, we can write

$$q_{H'}(\mathcal{P}') = \sum_{C' \in \mathcal{P}'_1} q_{H'}(C') + \sum_{C' \in \mathcal{P}'_2} q_{H'}(C')$$

$$= (q_H(C_w) - 1) + \sum_{i=1,2} (q_H(C_i) - 1) + \sum_{C \in \mathcal{P}_2} q_H(C)$$

$$= \sum_{C \in \mathcal{P}_1} q_H(C) - 3 + \sum_{C \in \mathcal{P}_2} q_H(C)$$

$$= q_H(\mathcal{P}) - 3.$$

Since H is an exceptional graph which is neither a non-trivial $(K_{1,1,2,2,2,2,2,2}, 7)$-cockade nor a $(K_{2,2,2,2,2,3}, 6)$-cockade of multiplicity 2, by Lemma 2.5.3, $q_H(\mathcal{P}) = 2(|H| - 10) + 1$. Since $|H'| = |H| - 1$, it follows that

$$l_{H'}(\mathcal{P}') = \lceil \frac{1}{2} q_{H'}(\mathcal{P}') \rceil = \lceil \frac{1}{2} \cdot (2(|H| - 10) + 1 - 3) \rceil = \lceil |H| - 11 \rceil = |H'| - 10.$$

By Lemma 2.5.2, to show $H' > K_{10}$ it now suffices to prove $q_{H'}(C') \leq \frac{1}{2} q_{H'}(\mathcal{P}')$ for every island $C' \in \mathcal{P}'$. Note that $\frac{1}{2} q_{H'}(\mathcal{P}') = |H| - 11$ as shown above. Let $C' \in \mathcal{P}'$ be arbitrary. If $C' = C_w - \{w\}$, let $C = C_w$; otherwise let $C = C'$. Recall that if $C \in \mathcal{P}_1$, then $q_{H'}(C') = q_H(C) - 1$; and if $C \in \mathcal{P}_2$, then $q_{H'}(C') = q_H(C)$. Note H is isomorphic to one of the following graphs: $K_{1,1,2,2,2,2,2,2}$, $K_{1,2,2,2,2,3,3}$, $K_{2,2,2,2} + C_5$, $K_{2,2,3,3,4}$, $K_{3,3,3} + C_5$, $K_{2,2,2,2,2,2,3}$, and $K_{2,3,3,3,3,3}$. By checking through every one of these exceptional graphs, we observe that $q_H(C) \leq |H| - 11$ unless $H \cong K_{2,2,2,2} + C_5$ and $H[C] \cong C_5$. If $q_H(C) \leq |H| - 11$, then we have $q_{H'}(C') \leq q_H(C) \leq |H| - 11$. If $H \cong K_{2,2,2,2} + C_5$ and $H[C] \cong C_5$, $q_H(C) = 3 = |K_{2,2,2,2} + C_5| - 11 + 1$. We may then assume $q_{H'}(C') = q_H(C)$, meaning that $C \in \mathcal{P}_2$ and thus $|C_w| = |C_1| = |C_2| = 2$, a contradiction property (4) of our choice of C_1 and C_2. □

Let L be the graph obtained from G by contracting the edges xw_1 and yw_3. To prove $G > K_{10}$, we just need to show $L > K_{10}$. Note if $G/xy = H$, then $H' = (H - \{w\}) + w_1w_2 + w_3w_4 \subseteq L$ by properties (1)-(3). By Claim 1, it follows that $L > K_{10}$ if $G/xy = H$. We can then assume $G/xy \neq H$, meaning that $G/xy \cong G_i$ for some $i \in \{1, 2, 3, 4\}$. Recall that in this case $H \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3,3}$, and e is the unique edge in $H - G/xy$. Let a, b be the ends of e, and note a and b are in two distinct islands of H. Let C_a and C_b be the islands of H containing a and b, respectively. Observe that if w is an end of e or $e = w_1w_3$, then $L \supseteq H' > K_{10}$. This means we may assume w is not an end of e and $e \neq w_1w_3$. More generally, we can assume e is not an edge between $\{w_1, w_2\}$ and $\{w_3, w_4\}$, since otherwise we can relabel the vertices w_1, w_2, w_3, w_4 and use the previous argument to show $L > K_{10}$.

Let $G' = (G/xy - \{w\}) + w_1w_2 + w_3w_4$. Observe that $G' \subseteq L$, so it suffices to prove $G' > K_{10}$. The rest of the proof now falls into two cases: $\{C_a, C_b\} \subseteq \{C_w, C_1, C_2\}$ or $\{C_a, C_b\} \nsubseteq \{C_w, C_1, C_2\}$.

Case 1: $\{C_a, C_b\} \subseteq \{C_w, C_1, C_2\}$.

Note $\{C_a, C_b\} \subseteq \{C_w, C_1, C_2\}$ means that $e = ab$ crosses two distinct islands of H among C_w, C_1, and C_2. Let $\mathcal{Q}$ be the island partition of G/xy. Let D_w be the island in $\mathcal{Q}$ that contains w. Let $\mathcal{Q}'$ be the partition of $V(G')$ obtained from $\mathcal{Q}$ by replacing D_w with $D'_w = D_w - \{w\}$. We make the following claim.

Claim 1. (i) $q_{G'}(\mathcal{Q}') = q_{G/xy}(\mathcal{Q}) - 3$, and (ii) $q_{G'}(C') \leq |G'| - 10$ for every $C' \in \mathcal{Q}'$.

Before proving Claim 1, we first show it implies $G' > K_{10}$. Assume Claim 1 is true. By Lemma 2.5.3, since G/xy is an exceptional graph that neither a non-trivial $(K_{1,1,2,2,2,2,2,2}, 7)$-cockade nor a $(K_{2,2,2,2,2,2,3}, 6)$-cockade of multiplicity 2, $q_{G/xy}(\mathcal{Q}) = 2(|G/xy| - 10) + 1$. Since $|G'| = |G/xy| - 1$, by (i) in Claim 1,

$$q_{G'}(\mathcal{Q}') = q_{G/xy}(\mathcal{Q}) - 3 = 2(|G/xy| - 10) + 1 - 3 = 2(|G'| - 10),$$

meaning that $l_{G'}(\mathcal{Q}') = \lceil \frac{1}{2} q_{G'}(\mathcal{Q}') \rceil = |G'| - 10$. By (ii) and Lemma 2.5.2, it follows that $G' > K_{10}$.

Proof of Claim 1. Note that $D_e = C_a \cup C_b$ is one single island in $\mathcal{Q}$. D_e and D_w may or may not be distinct islands, but it does not matter. Let $\mathcal{Q}_1$ be the minimal subset of $\mathcal{Q}$ that covers vertices in C_1, C_2, and C_w, and let $\mathcal{Q}'_1$ be the minimal subset of $\mathcal{Q}'$ that covers vertices in C_1, C_2, and $C_w - \{w\}$. Observe that $D_w \in \mathcal{Q}_1$ and $D'_w \in \mathcal{Q}'_1$. Let $\mathcal{Q}_2 = \mathcal{Q} - \mathcal{Q}_1$ and $\mathcal{Q}'_2 = \mathcal{Q}' - \mathcal{Q}'_1$. Since $\{C_a, C_b\} \subseteq \{C_w, C_1, C_2\}$, we know $\mathcal{Q}_2 = \mathcal{Q}'_2$ and $E_{G/xy}(C) = E_{G'}(C)$ for every $C \in \mathcal{Q}_2 = \mathcal{Q}'_2$. It follows that $q_{G/xy}(C) = q_{G'}(C)$ for every $C \in \mathcal{Q}_2 = \mathcal{Q}'_2$.

We now prove (i) first. If $\{C_a, C_b\} = \{C_1, C_2\}$, then $D_e = C_a \cup C_b = C_1 \cup C_2 \in \mathcal{Q}$

and $D_e \neq D_w$. It follows that $\mathcal{Q}_1 = \{D_e, D_w\}$ and $\mathcal{Q}'_1 = \{D_e, D'_w\}$. Since $e = ab$ is not between $\{w_1, w_2\}$ and $\{w_3, w_4\}$, one of C_1 and C_2 contains at least 3 vertices. Without loss of generality, assume $|C_1| = 3$, $C_1 = \{w_1, w_2, a\}$, and $b \in C_2$. Note $E(G'[D_e]) - E(G/xy[D_e]) = \{w_1w_2, w_3w_4\}$. One can check that if $|C_2| = 2$, $q_{G/xy}(D_e) = 3$ and $q_{G'}(D_e) = 1$; and if $|C_2| = 3$, $q_{G/xy}(D_e) = 4$ and $q_{G'}(D_e) = 2$. Therefore, in any case, we have $q_{G'}(D_e) = q_{G/xy}(D_e) - 2$ and $q_{G'}(D_e) \leq 2$. Since D_w is an independent set in G/xy and $|D'_w| = |D_w| - 1$, it follows that $q_{G'}(D'_w) = q_{G/xy}(D_w) - 1$. Therefore,

$$q_{G'}(\mathcal{Q}') = \sum_{C' \in \mathcal{Q}'_1} q_{G'}(C') + \sum_{C' \in \mathcal{Q}'_2} q_{G'}(C')$$

$$= q_{G'}(D_e) + q_{G'}(D'_w) + \sum_{C' \in \mathcal{Q}'_2} q_{G'}(C')$$

$$= (q_{G/xy}(D_e) - 2) + (q_{G/xy}(D_w) - 1) + \sum_{C \in \mathcal{Q}_2} q_{G/xy}(C)$$

$$= q_{G/xy}(\mathcal{Q}) - 3.$$

This proves (i) for the case $\{C_a, C_b\} = \{C_1, C_2\}$. To finish proving (i), we may assume $\{C_a, C_b\} = \{C_w, C_i\}$ for some $i \in \{1, 2\}$. Without loss of generality, assume $i = 1$, $C_w = C_a$, and $C_1 = C_b$. It follows that $D_e = C_w \cup C_1 = D_w$, $\mathcal{Q}_1 = \{D_w, C_2\}$, and $\mathcal{Q}'_1 = \{D'_w, C_2\}$. It is easy to see that $q_{G'}(C_2) = q_{G/xy}(C_2) - 1$ since $G/xy[C_2] - G'[C_2] = \{w_3w_4\}$. Since w is not an end of e, $w \in C_a - \{a\}$ and is not adjacent to a in G/xy. By property (3), both w_1 and w_2 are adjacent to every vertex that is not adjacent to w in G/xy. It follows that $\{w_1, w_2\} \subseteq N_{G/xy}(a)$, and thus $b \neq w_1$ or w_2 and $C_1 = \{b, w_1, w_2\}$ is a 3-island. One can then check that if $|C_w| = 2$, $q_{G/xy}(D_w) = 3$ and $q_{G'}(D'_w) = 1$; and if $|C_w| = 3$, $q_{G/xy}(D_w) = 4$ and $q_{G'}(D'_w) = 2$. In either case, we have $q_{G'}(D_w) = q_{G/xy}(D'_w) - 2$ and $q_{G'}(D'_w) \leq 2$. Using the same argument as above, we can show $q_{G'}(\mathcal{Q}') = q_{G/xy}(\mathcal{Q}) - 3$ again. This finishes proving (i) in Claim 1.

To prove (ii), note when we were proving (i) we also showed that $q_{G'}(D_e) \leq 2$ if

$\{C_a, C_b\} = \{C_1, C_2\}$, and that $q_{G'}(D'_w) = q_{G'}(D_e - \{w\}) \leq 2$ if $\{C_a, C_b\} = \{C_w, C_i\}$ for some $i \in \{1, 2\}$. For every island $C' \in \mathcal{Q}'$ that is not D_e or $D_e - \{w\}$, $|C'| \leq 3$ and therefore $q_{G'}(C') \leq 2$. It follows that for every $C' \in \mathcal{Q}'$, $q_{G'}(C') \leq 2$. Since $G/xy \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$, $|G'| - 10 = |G/xy| - 1 - 10 \geq 13 - 11 = 2$. It follows that $q_{G'}(C') \leq 2 \leq |G'| - 10$ for every $C' \in \mathcal{Q}'$, which proves (ii). □

Case 2: $\{C_a, C_b\} \not\subseteq \{C_w, C_1, C_2\}$.

Without loss of generality, assume $C_a \notin \{C_w, C_1, C_2\}$. In the case $\{C_a, C_b\} \cap \{C_w, C_1, C_2\} = \emptyset$, we choose C_a to be a 3-island if possible. For the rest of the proof, the goal is to find a vertex $a' \in N_{G'}(a)$ such that $G'/aa' > K_{10}$. Note this then implies $G' > K_{10}$.

Case 2.1: $H \cong K_{2,2,2,2,2,3}$

Note $H \cong K_{2,2,2,2,2,3}$ now has exactly 6 islands, and recall that $H' = (H - \{w\}) + w_1w_2 + w_3w_4$. Let $H'' = H' - C_1 \cup C_2 \cup C_w$, and note H'' contains exactly 3 islands of H and that C_a is one of them. Also note this implies there exists an island $C_{a'}$ of H'' such that $C_{a'} \notin \{C_a, C_b\}$. Let $a' \in C_{a'}$. Since a' and b are adjacent in G', it follows that the new vertex of G'/aa' is adjacent to b in G'/aa', and therefore $G'/aa' = H'/aa'$. This means that to prove $G' > K_{10}$, we just need to choose an island $C_{a'}$ of H'' such that $C_{a'} \notin \{C_a, C_b\}$ and $H'/aa' > K_{10}$ for some $a' \in C_{a'}$. In the following table, we list all possible cases; and for each case, we show our choice of $C_{a'}$ by giving the size of it as well as the graph H/aa' where a' is any vertex in the chosen $C_{a'}$.

Table 2.2

$\lvert C_w \rvert$	C_1 and C_2	$H'' = H' - C_1 \cup C_2 \cup C_w$	$\lvert C_a \rvert$	$\lvert C_{a'} \rvert$	H'/aa'
2	$\lvert C_1 \rvert = \lvert C_2 \rvert = 2$	$K_{2,2,3}$	3	2	$K_7 + K_{2,2}$
2	$\lvert C_1 \rvert = \lvert C_2 \rvert = 2$	$K_{2,2,3}$	2	3	$K_7 + K_{2,2}$
2	$\lvert C_1 \rvert \neq \lvert C_2 \rvert$	$K_{2,2,2}$	2	2	$K_6 + \overline{P_3} + \overline{P_2}$
3	$\lvert C_1 \rvert = \lvert C_2 \rvert = 2$	$K_{2,2,2}$	2	2	$K_7 + K_{2,2}$

We note that in the case $|C_w| = |C_1| = |C_2| = 2$ and $|C_a| = 2$ (second row in the table), C_b must not be the 3-island of H'' by the choice of C_a, which allows us to choose the 3-island of H'' to be $C_{a'}$. In other cases, it is easy to see that it is possible to choose the island $C_{a'}$ of H'' of the size listed in the table such that $C_{a'} \notin \{C_a, C_b\}$. Since each case in the table has $H'/aa' \cong K_7 + K_{2,2}$ or $K_6 + \overline{P_3} + \overline{P_2}$, and both of these two graphs have a K_{10} minor, it follows that $H'/aa' > K_{10}$. It follows that $G'/aa' = H'/aa' > K_{10}$.

Case 2.2: $H \cong K_{2,3,3,3,3}$ In this case, there are fewer islands in H, and we will have to choose the vertex a' more carefully. Observe that at least one of C_1 and C_2 is not equal to C_b, so without loss of generality we can assume $C_1 \neq C_b$. In the table below, we list all possible cases with the range to choose a' from and the corresponding resulting graph H'/aa' in each case.

Table 2.3

$\lvert C_w\rvert$	C_1 and C_2	$\lvert C_a\rvert$	choose a' in	H'/aa'
2	$\lvert C_1\rvert = \lvert C_2\rvert = 3$	3	$C_1 - \{w_1, w_2\}$	$K_4 + K_{2,3} + \overline{P_3}$
3	$\lvert C_1\rvert = \lvert C_2\rvert = 3$	2	$C_1 - \{w_1, w_2\}$	$K_4 + K_{2,3} + \overline{P_3}$
3	$\lvert C_1\rvert = \lvert C_2\rvert = 3$	3	$C_1 - \{w_1, w_2\}$	$K_3 + K_{2,2,2} + \overline{P_3}$
3	$\lvert C_1\rvert = 3, \lvert C_2\rvert = 2$	3	$C_1 - \{w_1, w_2\}$	$K_5 + K_{2,2,3}$
3	$\lvert C_1\rvert = 2, \lvert C_2\rvert = 3, C_2 \neq C_b$	3	$C_2 - \{w_3, w_4\}$	$K_5 + K_{2,2,3}$
3	$\lvert C_1\rvert = 2, \lvert C_2\rvert = 3, C_2 = C_b$	3	$H - C_w \cup C_1 \cup C_2 \cup C_a$	$K_3 + K_{2,2,2} + \overline{P_3}$

Observe that for every case in this table, the graph H'/aa' always has a K_{10} minor. Since a' is always chosen from an island of H that is not C_a or C_b, a' is adjacent to b in G', and therefore the new vertex of G'/aa' is adjacent to b in G'/aa'. It follows that $G'/aa' = H'/aa' > K_{10}$.

□

Lemma 2.5.11 (Contraction Lemma 2)**.** *Let G be a graph with $\delta(G) \geq 8$. Suppose there is an edge $xy \in E(G)$ such that x and y share exactly 7 common neighbors in G. If $G/xy \cong K_{1,2,2,2,3,3}$, then either $G > K_{10}$ or G is isomorphic to $K_{2,2,3,3,4}$, G_3, or G_4.*
In particular, (1) if $G \cong K_{2,2,3,3,4}$, then one of x and y has degree 10 and the other one has degree 11, and (2) if $G \cong G_3$ or G_4, let e be the unique non-edge such that $G + e \cong K_{2,3,3,3,3}$: if $G \cong G_3$, one of x and y is an end of e, and the other one is not an end of e in a 3-island of $G + e$; if $G \cong G_4$, one of x and y is an end of e in a 3-island, and the other one is not an end of e in a 3-island of $G + e$.

Proof. Let w be the new vertex of G/xy. By Lemma 2.5.10, we may assume w is the vertex in G/xy that is adjacent to all other vertices. Let $N = N_{G/xy}(w) = N_G(x) \cup N_G(y) - \{x, y\}$. Note $G[N] \cong K_{2,2,2,3,3}$. It follows that

$$(d_G(x) - 1) + (d_G(y) - 1) - 7 = d_{G/xy}(w) = 12,$$

meaning that $d_G(x) + d_G(y) = 21$. Without loss of generality, assume $d_G(x) \leq d_G(y)$. In the rest of the proof, we consider the following two cases: $d_G(x) \leq 9$ and $d_G(y) \geq 12$, or $d_G(x) = 10$ and $d_G(y) = 11$.

Case 1: $d_G(x) \leq 9$ and $d_G(y) \geq 12$. Note that $d_G(y) \geq 12$ means that there is at most one vertex in N that is not adjacent to y. Choose $y' \in N$ such that y is adjacent to every vertex in $N - \{y'\}$. Let $C_{y'}$ be the island of $G[N]$ that contains y'. Since $d_G(x) \geq 8$, x has at least 6 neighbors in $N - \{y'\}$. Since $G[N] \cong K_{2,2,2,3,3}$, there exist two non-adjacent vertices $w_1, w_2 \in N - \{y'\}$ that are both adjacent to x. Let C_x be the island of $G[N]$ that contains w_1 and w_2. Note C_x and $C_{y'}$ may or may not be the same island, but it does not matter. Let C' be an island of $G[N] - C_x \cup C_{y'}$, and let $w_3 \in C'$. By contracting $y'w_3$ and xw_1, we can obtain a resulting graph that has a subgraph isomorphic to $G' = (G[N \cup \{y\}]/y'w_3) + w_1w_2$. Let $H = G[N \cup \{y\}]/y'w_3$, and note it is now enough to prove $G' = H + w_1w_2$ has a K_{10} minor. Since y is adjacent to every vertex in $N - \{y'\}$, it is adjacent to w_3 in G and thus adjacent to the new vertex of H. It follows that $H \cong (G[N \cup \{y\}] + yy')/y'w_3$. Since $G[N \cup \{y\}] + yy' \cong K_{1,2,2,2,3,3}$ and y', w_3 are in two distinct islands of $G[N \cup \{y\}] + yy'$ of size at least 2, it follows that $H \cong (G[N \cup \{y\}] + yy')/y'w_3$ is isomorphic to $K_4 + K_{2,3,3}$, $K_3 + K_{2,2,2,3}$, or $K_{1,1,2,2,2,2,2}$. Note that in the graph $H = (G[N \cup \{y\}]/y'w_3)$, w_1 and w_2 are not adjacent. This means that to prove $G' = H + w_1w_2$ has a K_{10} minor, it suffices to show for every $f \notin E(H)$, $H + f > K_{10}$. Let $f \notin E(H)$ be arbitrary. If $H \cong K_4 + K_{2,3,3}$, then $H + f \cong K_6 + K_{3,3}$ or $K_4 + K_{2,3} + \overline{P_3}$; if $H \cong K_3 + K_{2,2,2,3}$, then $H + f \cong K_5 + K_{2,2,3}$ or $K_3 + K_{2,2,2} + \overline{P_3}$; and if $H \cong K_{1,1,2,2,2,2,2}$, then $H + f \cong K_4 + K_{2,2,2,2}$. In any case,

$H + f$ is isomorphic to a graph that has a K_{10} minor, and it follows that $H + f > K_{10}$.

Case 2: $d_G(x) = 10$ and $d_G(y) = 11$. In this case, x has exactly 3 non-neighbors and y and exactly 2 non-neighbors in N. Let x_1, x_2, x_3 and y_1, y_2 be the non-neighbors of x and y, respectively. Note that $\{x_1, x_2, x_3\} \cap \{y_1, y_1\} = \emptyset$, and $\{x_1, x_2, x_3\} \subseteq N_G(y)$ and $\{y_1, y_2\} \subseteq N_G(x)$. Let $N' = \{x_1, x_2, x_3, y_1, y_2\} \subseteq N$.

Assume y_1 and y_2 are in two distinct islands of $G[N]$. Let these two islands be C_y^1 and C_y^2. Note there are exactly three islands of $G[N] - C_y^1 \cup C_y^2$. If some island C' of $G[N] - C_y^1 \cup C_y^2$ is such that every vertex in C' is adjacent to x, let the two islands in $G[N] - C_y^1 \cup C_y^2 \cup C'$ be C_x^1 and C_x^2. Let $w_1 \in C_x^1$, $w_2 \in C_x^2$, and $w_3 \in C'$. Contract y_1w_1, y_2w_2, and xw_3 in G, and we can get a resulting graph isomorphic to $K_7 + K_{2,2} > K_{10}$. We may then assume that the three islands in $G[N] - C_y^1 \cup C_y^2$ are C_x^1, C_x^2, and C_x^3, and that $x_i \in C_x^i$ for each $i \in \{1, 2, 3\}$. Now, each one of the five islands of $G[N]$ contains exactly one vertex in N. Observe that we can choose a vertex $w_0 \in N - N'$ that is in some 3-island of $G[N] - C_x^3$. Contract edges x_1y_1, x_2y_2, and x_3w_0, and we can then obtain a graph isomorphic to $K_{11}^- > K_{10}$.

Now we may assume y_1 and y_2 are in the same island C_y of $G[N]$, meaning that N' can cover up to 4 islands of $G[N]$. This means there exists some island C_0 of $G[N]$ such that $N' \cap C_0 = \emptyset$. Choose C_0 to be a 3-island if possible, and let w_0 be a vertex in C_0.

If N' covers exactly four islands of $G[N]$, then x_1, x_2, x_3 are in three distinct islands that are distinct from C_y. By contracting x_1y_1, x_2y_2, and x_3w_0, we can obtain a resulting graph isomorphic to $K_7 + K_{2,2}$ if $|C_y| = 2$ and K_{11}^- if $|C_y| = 3$. In either case, the resulting graph has a K_{10} minor and hence $G > K_{10}$.

Assume N' covers exactly three islands of $G[N]$, say C_y, C_1, and C_2. Without loss of generality, assume $x_i \in C_i$ for $i = 1, 2$. If $x_3 \in C_y$, then $C_y = \{y_1, y_2, x_3\}$. By contracting x_1y_1, x_2y_2, and xw_0, we can obtain a resulting graph isomorphic to $K_7 + K_{2,2} > K_{10}$ since we chose C_0 to be a 3-island if possible. Without loss of generality, we may then assume $x_3 \in C_1$. If $|C_y| = |C_1| = 2$, then $G \cong G_4$ if $|C_2| = 2$ and $G \cong G_3$ if $|C_2| = 3$. In particular, let e be the unique non-edge of G such that $G + e \cong K_{2,3,3,3,3,3}$, then if $|C_2| = 2$ and $G \cong G_4$, x is an end of e in a 3-island of $G + e$ and y is not an end of e in a 3-island $G+e$; if $|C_2| = 3$ and $G \cong G_3$, x is an end of e and y is not an end of e in a 3-island of $G+e$. We may then assume at least one of C_y and C_1 contains 3 vertices. By contracting x_1y_1, x_2y_2, and x_3w_0, we can get a resulting graph isomorphic to $K_{11}^- > K_{10}$ if $|C_y| = |C_1| = 3$ or $K_7 + K_{2,2} > K_{10}$ if exactly one of C_y and C_1 is a 3-island.

Finally, consider that N' covers exactly two islands of $G[N]$. If $|C_y| = 2$, then x_1, x_2, x_3 form a 3-island of $G[N]$, meaning that $G \cong K_{2,3,3,3,3,4}$ with $d(x) = 10$ and $d(y) = 11$. If $|C_y| = 3$, first assume $C_1 = \{x_1, x_2, x_3\}$. By contracting x_1y_1, x_2y_2, and x_3w_0, we can obtain a resulting graph isomorphic to $K_7 + K_{2,2} > K_{10}$. We can then assume, without loss of generality, that $C_y = \{y_1, y_2, x_3\}$ and $\{x_1, x_2\} \subseteq C_1$. If $|C_1| = 3$, contract x_1y_1, x_2y_2, and xw_0, and we can obtain a graph isomorphic to $K_7 + K_{2,2} > K_{10}$. If $|C_1| = 2$, then by the choice of C_0 we have $|C_0| = 3$. Let w_1, w_2 be two vertices from distinct islands in $G[N] - C_y \cup C_1 \cup C_0$. By contracting y_1w_1, y_2w_2, and xw_0, we will obtain a graph isomorphic to $K_7 + K_{2,2} > K_{10}$. □

Lemma 2.5.12 (Contraction Lemma 3)**.** *Let G be a graph with $\delta(G) \geq 8$. Suppose there is an edge $xy \in E(G)$ such that x and y share exactly 7 common neighbors in G. If $G/xy \cong K_{1,1,2,2,2,2,2,2}$, then either $G > K_{10}$ or G is isomorphic to $K_{1,2,2,2,3,3}$, G_1, or G_2. In particular, (1) if $G \cong K_{1,2,2,2,3,3}$, then $d_G(x) = d_G(y) = 10$, and (2) if $G \cong G_1$ or G_2, let e be the unique non-edge of G such that $G + e \cong K_{2,2,2,2,2,3}$: if $G \cong G_1$, one of x and y is in the 3-island in $G + e$ and the other one is an end of e; if $G \cong G_2$, one of x and y is the end of e in the 3-island of $G + e$, and that the other one is not an end of e and in a 2-island of G.*

Proof. Let w be the vertex in G/xy obtained by the contraction of xy in G. Let C_w be the island of G/xy that contains w. By Lemma 2.5.10, we may assume w is adjacent to all other vertices in G/xy, meaning that $|C_w| = 1$. Let $N = N_{G/xy}(w) = N_G(x) \cup N_G(y) - \{x, y\}$.

Note $G[N] \cong K_{1,2,2,2,2,2}$. It follows that

$$(d_G(x) - 1) + (d_G(y) - 1) - 7 = d_{G/xy}(w) = 11,$$

meaning that $d_G(x) + d_G(y) = 20$. Without loss of generality, assume $d_G(x) \leq d_G(y)$. Note that $\delta(G) \geq 8$. We will proceed the rest of the proof following these three cases: $d_G(x) = 8$ and $d_G(y) = 12$, $d_G(x) = 9$ and $d_G(y) = 11$, and $d_G(x) = d_G(y) = 10$.

Case 1: $d_G(x) = 8$ and $d_G(y) = 12$. Since $d_G(y) = 12$, y is adjacent to every vertex in N and x has 7 neighbors in N. Since $G[N] \cong K_{1,2,2,2,2,2}$ has exactly 6 islands, there exists a 2-island $\{w_1, w_2\}$ of it such that both w_1 and w_2 are adjacent to x. By contracting xw_1, we get a resulting graph isomorphic to $K_4 + K_{2,2,2,2} > K_{10}$, and therefore $G > K_{10}$.

Case 2: $d_G(x) = 9$ and $d_G(y) = 11$. In this case, x has exactly 8 neighbors in N, and that there is a unique $y' \in N$ that is not adjacent to y. Let $C_{y'}$ be the island of $G[N]$ that contains y'. If $|C_{y'}| = 2$, then x has at least 6 neighbors in $N - C_{y'}$. Since $G[N] - C_{y'} \cong K_{1,2,2,2,2}$, there exists a 2-island $C_1 = \{w_1, w_2\}$ of it such that both w_1 and w_2 are adjacent to x. Let $C_2 = \{w_3, w_4\}$ be any 2-island of $G[N] - C_{y'} \cup C_1$. Note that both w_3 and w_4 are adjacent to y and y'. Contracting xw_1 and $y'w_3$ in G, and we get a resulting graph isomorphic to $K_7 + K_{2,2} > K_{10}$, implying that $G > K_{10}$. We may then assume $|C_{y'}| = 1$. Note x has exactly three non-neighbors in N. Call them x_1, x_2, and x_3. If x_1, x_2, x_3 are in three distinct islands in $G[N - y'] \cong K_{2,2,2,2,2}$, then there remain two 2-islands $C_1 = \{w_1, w_2\}$ and $C_2 = \{w_3, w_4\}$ that do not contain any non-neighbor of x. By contracting x_1w_1, x_2w_3, and x_3y', we obtain a resulting graph isomorphic to K_{10}. Without loss of generality, we may then assume $\{x_1, x_2\}$ is a 2-island of $G[N]$. It follows that $G \cong G_2$. In particular, let $e \notin E(G)$ be the unique non-edge $G + e \cong K_{2,2,2,2,2,3}$. It follows that x is the end of e in the 3-island of $G + e$ and y is not an end of e in a 2-island of G.

Case 3: $d_G(x) = d_G(y) = 10$. In this case, x and y each have exactly two non-neighbors in N. Let the non-neighbors for x be x_1 and x_2 and the non-neighbors for y be y_1 and y_2. Note that x_1, x_2, y_1, y_2 are four distinct vertices. Assume $\{x_1, x_2\}$ is a 2-island of $G[N]$ first. If $\{y_1, y_2\}$ is another 2-island of $G[N]$, then $G \cong K_{1,2,2,2,3,3}$ with $d_G(x) = d_G(y) = 10$. If one of y_1 and y_2 is in the unique 1-island of $G[N]$, then $G \cong G_1$. In particular, let $e \notin E(G)$ be the unique non-edge such that $G + e \cong K_{2,2,2,2,2,3}$, and then x is in the 3-island of $G + e$ and y is an end of e. If y_1 and y_2 are in two distinct 2-islands in $G[N] - \{x_1, x_2\}$, by contracting x_1y_1 and x_2y_2 we get a resulting graph isomorphic to $K_7 + K_{2,2} > K_{10}$. We may then assume x_1, x_2 are in distinct islands of $G[N]$, and by symmetry y_1 and y_2 are in distinct islands of $G[N]$ too. At most one vertex among x_1, x_2, y_1, y_2 is in the 1-island, so without loss of generality assume that x_1 and x_2 are in two distinct 2-islands. Note that there exists a 2-island C_0 of $G[N]$ such that both vertices in C_0 are common neighbors for x and y. Let $w_0 \in C_0$. This implies that the new vertex of G/yw_0 is adjacent to both y_1 and y_2. It follows that $G/yw_0 \cong K_3 + K_{2,2} + \overline{P_5}$. It is easy to observe that $K_3 + K_{2,2} + \overline{P_5} > K_{10}$, implying that $G > K_{10}$.

□

Lemma 2.5.13 (Contraction Lemma 4)**.** *Let G be a graph with $\delta(G) \geq 8$. Suppose there is an edge $xy \in E(G)$ such that x and y share exactly 7 common neighbors in G. If G/xy is a $(K_{1,1,2,2,2,2,2}, 7)$-cockade, then either $G > K_{10}$ or G is isomorphic to $K_{1,2,2,2,3,3}$, G_1, or G_2.*

In particular, (1) if $G \cong K_{1,2,2,2,3,3}$, then $d_G(x) = d_G(y) = 10$, and (2) if $G \cong G_1$ or G_2, let e be the unique non-edge of G such that $G + e \cong K_{2,2,2,2,2,3}$: if $G \cong G_1$, one of x and y is in the 3-island in $G + e$ and the other one is an end of e; if $G \cong G_2$, one of x and y is the end of e in the 3-island of $G + e$, and that the other one is not an end of e and in a 2-island of G.

Proof. We will prove the lemma by inducting on $|G/xy|$. The base case is $G/xy \cong K_{1,1,2,2,2,2,2}$, which is proved in Lemma 2.5.12. Assume that G/xy is a non-trivial $(K_{1,1,2,2,2,2,2}, 7)$-cockade. Choose subgraphs H_1, H_2 of G/xy such that H_1, H_2 are both $(K_{1,1,2,2,2,2,2}, 7)$-

cockades, $H_1 \cup H_2 \cong G/xy$, and $H_1 \cap H_2 \cong K_7$. Let w be the new vertex of G/xy. For each $i \in \{1, 2\}$, let $H_i^* = G[(V(H_i) - \{w\}) \cup \{x, y\}]$. Observe that since each H_i is a $(K_{1,1,2,2,2,2,2}, 7)$-cockade, $\delta(H_i) \geq 10$. It follows that for every vertex $v \in V(H_i^*) - \{x, y\}$, $d_{H_i^*}(v) \geq d_{H_i}(v) \geq 10$. Suppose $w \in V(H_1) - V(H_2)$. Then x, y have exactly 7 common neighbors in H_1^*, implying that $d_{H_1^*}(x) \geq 8$ and $d_{H_1^*}(y) \geq 8$. It follows that $\delta(H_1^*) \geq 8$, and by induction we may assume H_1^* is isomorphic to $K_{1,2,2,2,3,3}$, G_1, or G_2. This is then a contradiction, since H_1^* contains a subgraph isomorphic to K_7 but any one of these three exceptional graphs does not.

By symmetry, we may assume $w \in V(H_1 \cap H_2)$. Let $S = V(H_1 \cap H_2) - \{w\}$, and note that $G[S] \cong K_6$. Let $Z \subseteq V(G)$ be the set of the 7 common neighbors of x and y. Assume $Z \subseteq V(H_1^*)$ for a moment. This again implies that both x and y have at least 8 neighbors in H_1^* and therefore $\delta(H_1^*) \geq 8$. By induction, H_1^* is isomorphic to $K_{1,2,2,2,3,3}$, G_1, or G_2. If $H_1^* \cong G_1$ or G_2, let $f \notin E(H_1^*)$ be the unique non-edge such that $H_1^* + f \cong K_{2,2,2,2,2,3}$ and let $L = H_1^* + f$; if $H_1^* \cong K_{1,2,2,2,3,3}$, let $L = H_1^*$. Note that in either case, L has exactly 6 islands. Since $H_1^*[S] \cong K_6$, every vertex in S is in a distinct island in L. Since x and y are adjacent in L, they must be in different islands. It follows that there exist unique vertices $x', y' \in S$ such that x' and x are in the same island, and that y' and y are in the same island of L. Let K be a component of $H_2^* - H_1^*$. Note that $N_G(K) \subseteq S \cup \{x, y\}$. By Lemma 2.5.5, since $K_{1,1,2,2,2,2,2}$ is 7-connected and H_2 is a $(K_{1,1,2,2,2,2,2}, 7)$-cockade, we know that H_2 is 7-connected. It follows that $|N_{H_2}(K)| \geq 7$ and therefore $|N_G(K)| = |N_{H_2^*}(K)| \geq 7$. Since $|S \cup \{x, y\}| = 8$, without loss of generality we can assume both x and x' are contained in $N_G(K)$. By contracting all vertices in K to x, we can get a resulting graph on $V(H_1^*)$ that contains $H_1^* + xx'$ as a subgraph. Since x and x' are in the same island of L, $xx' \neq f$ in the case $H_1^* \cong G_1$ or G_2. By Lemma 2.5.7, $H_1^* + xx' \geq K_{10}$ and therefore $G > K_{10}$.

Now, we may assume $Z \not\subseteq V(H_1^*)$, and by symmetry we may also assume $Z \not\subseteq V(H_2^*)$.

For each $i \in \{1, 2\}$, let $Z_i = Z \cap V(H_i^* - H_{3-i}^*)$, and note that $|Z_i| \geq 1$. It follows that

$$|S - Z| = 6 - |Z \cap S| = 6 - (7 - |Z_1| - |Z_2|) = |Z_1| + |Z_2| - 1 \geq |Z_i|$$

for every $i \in \{1, 2\}$. Again by Lemma 2.5.5, H_1, H_2 are both 7-connected. This implies that $H_i - \{w\} - S \cap Z$ is $(6 - |Z \cap S|)$-connected, and equivalently $|S - Z|$-connected, for each $i \in \{1, 2\}$. Since $|Z_i| \leq |S - Z|$, there exist $|Z_i|$ disjoint paths from Z_i to $S - Z$ in $H_i - \{w\} - S \cap Z$ for each $i \in \{1, 2\}$. For each $i \in \{1, 2\}$, choose disjoint paths $P_1^i, ..., P_{|Z_i|}^i$ between Z_i and $S - Z$ in $H_i - \{w\} - S \cap Z$, and let $Z_i' \subseteq S - Z$ be the set of endpoints of these paths in $S - Z$. Contract $P_1^i, ..., P_{|Z_i|}^i$, and let H_i' be the induced subgraph on $V(H_i^*)$ of the resulting graph. Note now in H_i', x and y have exactly 7 common neighbors, implying that $\delta(H_1') \geq 8$. By induction, for each $i \in \{1, 2\}$, H_i' is isomorphic to $K_{1,2,2,2,3,3}$, G_1, or G_2, with detailed positions of x and y in H_i' described in the statement of the lemma.

For each $i \in \{1, 2\}$, if $H_i' \cong G_1$ or G_2, let $e_i \notin E(H_i')$ be the unique non-edge such that $H_i' + e_i \cong K_{2,2,2,2,2,3}$ and let $L_i = H_i' + e_i$; and if $H_i' \cong K_{1,2,2,2,3,3}$, let $L_i = H_i'$. In either case, L_i contains exactly 6 islands, implying that every vertex in S is in a distinct island of L_i. By induction, for each $i \in \{1, 2\}$, if $H_i' \cong K_{1,2,2,2,3,3}$, then $d_{H_i'}(x) = d_{H_i'}(y) = 10$, if $H_i' \cong G_1$, one of x and y is in the 3-island in L_i and the other one is an end of e_i; if $H_i' \cong G_2$, one of x and y is the end of e_i in the 3-island of L_i, and that the other one is not an end of e_i and in a 2-island of H_i'. Observe that in any of the three cases, there exist exactly three islands, say C_i^1, C_i^2, and C_i^3, that are all 2-islands of L_i such that both vertices in C_i^j are common neighbors for x and y for $j = 1, 2, 3$. For every $i \in \{1, 2\}$ and every $j \in \{1, 2, 3\}$, since exactly one vertex in C_i^j is contained in S, the other one must be in $V(H_i^* - H_{3-i}^*)$. Note the vertex that is in $C_i^j \cap V(H_i^* - H_{3-i}^*)$ must be adjacent to both x and y in the graph H_i^*, meaning that this vertex is contained in Z_i. It follows that $|Z_i| \geq 3$ for both $i = 1, 2$. Since $|Z_1| + |Z_2| \leq |Z| = 7$, without loss of generality we can assume $|Z_2| = 3$, and that the three vertices in Z_2 are precisely one from each C_2^1, C_2^2, and C_2^3. For each $v \in Z_2$, note that $N_{H_2^*}(v) = N_{H_2'}(v)$. It follows that in the graph H_2^*, each $v \in Z_2$ has exactly one non-neighbor in S. Recall P_1, P_2, P_3 are three disjoint paths linking $S - Z$ and Z_2 in $H_2 - \{w\} - S \cap Z$. Therefore, we may choose each path P_j to have length exactly 1 for $j = 1, 2, 3$.

Now, since x and y are adjacent, they are in distinct islands of L_1. Since each vertex in S is in a distinct island of L_1, there exist unique vertices $x', y' \in S$ such that x' and x are in the same island of L_1, and that y' and y are in the same island of L_1. Note if $H_1' \cong G_1$ or G_2, $xx' \neq e_1$ and $yy' \neq e_1$. By Lemma 2.5.7, $H_1' + xx' > K_{10}$ and $H_1' + yy' > K_{10}$. Therefore, it suffices to show that there exists a path Q in H_2^* that is internally contained in $H_2^* - H_1^* - Z_2$ and links x and x' or links y and y'.

Now, since $H_2' \cong K_{1,2,2,2,3,3}$, G_1, or G_2, we know that $|H_2^*| = |H_2'| = 13$ and therefore $|V(H_2^* - H_1^*) - Z_2| = 2$, meaning that there are exactly two vertices, say u_1, u_2, in $H_2^* - H_1^*$ that are not common neighbors of x and y in G. Observe that $N_G(u_i) = N_{H_2^*}(u_i) = N_{H_2'}(u_i)$ for $i = 1, 2$. Also observe that regardless of which graph H_2' is isomorphic to, any vertex in it that is not x or y has degree at least 10. It follows that u_1, u_2 each has at most two non-neighbors in H_2^*. We may then assume, in the graph H_2', u_1, u_2 each has exactly two non-neighbors among $\{x, x', y, y'\}$ and that u_1, u_2 are adjacent to each other.

Note that in any case, at least one of x and y is contained in a 3-island of L_2. Without loss of generality, assume that y is contained in a 3-island of L_2. Call this island C_y, and note that exactly one of u_1, u_2 is contained in C_y. Without loss of generality, say $u_1 \in C_y$. Then, observe that in any case we have u_1 adjacent to y in H_2'. This means we may assume $N_{H_2'}(u_1) \cap \{x, x', y, y'\} = \{x, y'\}$. Since u_1, u_2 are adjacent to each other, we may further assume that $N_{H_2'}(u_2) \cap \{x, x', y, y'\} = \{x, y'\}$ too. It follows that $H_2'[\{x', y, u_1, u_2\}]$ has a supergraph isomorphic to $\overline{K_{2,2}}$, which is a contradiction, since none of $K_{1,2,2,2,3,3}$, G_1, and G_2 contains a subgraph isomorphic to $\overline{K_{2,2}}$. □

Lemma 2.5.14 (Contraction Lemma 5)**.** *Let G be a graph with $\delta(G) \geq 8$. Suppose there is an edge $xy \in E(G)$ such that x and y share exactly 7 common neighbors in G. Suppose*

G/xy is isomorphic to a $(K_{2,2,2,2,2,3}, 6)$-cockade of multiplicity 2. Then, $G > K_{10}$.

Proof. Let $H = G/xy$. Let H_1, H_2 be induced subgraphs of H such that $H = H_1 \cup H_2$, $H_1 \cap H_2 \cong K_6$, and $H_i \cong K_{2,2,2,2,2,3}$ for $i = 1, 2$. Let w be the new vertex of $H = G/xy$. If $w \in V(H_i - H_{3-i})$ for some $i \in \{1, 2\}$, then we can apply Lemma 2.5.10 to $G[V(H_i - \{w\}) \cup \{x, y\}]$ and show that $G > K_{10}$. Therefore, we may assume that $w \in V(H_1 \cap H_2)$. For $i = 1, 2$, let $H_i^* = G[V(H_i - \{w\}) \cup \{x, y\}]$.

Observe that for $i = 1, 2$, since $H_i \cong K_{2,2,2,2,2,3}$, $d_G(v) \geq d_{H_i}(v) \geq 10$ for every $v \in V(G) - \{x, y\}$. Let $Z \subseteq V(G) - \{x, y\}$ be the subset of 7 vertices that are common neighbors of x and y in G. For $i = 1, 2$, let $Z_i = Z \cap V(H_i^* - H_{3-i}^*)$. If $|Z_1| = 0$, then x and y have exactly 7 common neighbors in H_2^*, which implies that $d_{H_2^*}(x), d_{H_2^*}(y) \geq 8$. It follows that $\delta(H_2^*) \geq 8$. Since $H_2^*/xy = H_2 \cong K_{2,2,2,2,2,3}$, by Lemma 2.5.10 it follows that $H_2^* > K_{10}$ and therefore $G > K_{10}$. By symmetry, we may then assume $|Z_i| \geq 1$ for $i = 1, 2$. Let $S = V(H_1 \cap H_2) - \{w\}$, and it follows that $|Z_i| + |S \cap Z| \leq 6$ for $i = 1, 2$. Since $|S| = 5$, we have $|Z_i| + 5 - |S - Z| \leq 6$ and thus $|S - Z| \geq |Z_i| - 1$ for $i = 1, 2$.

Assume for a moment that $|S - Z| = |Z_1| - 1 = |Z_2| - 1$. Then,

$$7 = |Z| = |Z_1| + |Z_2| + |S \cap Z| = 2|S - Z| + 2 + |S \cap Z| = 7 + |S - Z|,$$

and therefore $|S - Z| = 0$. It follows that $S \subseteq Z$ and $|Z_1| = |Z_2| = 1$. Let z_i be the unique vertex in Z_i for $i = 1, 2$. Note now x and y have exactly 6 common neighbors in H_1^*, meaning that $d_{H_1^*}(x), d_{H_1^*}(y) \geq 7$. Note that if $N(x) \cap V(H_1^* - H_2^*) - \{z_1\} \neq \emptyset$ and $N(x) \cap V(H_1^* - H_2^*) - \{z_1\} \neq \emptyset$, then we would have $d_{H_1^*}(x), d_{H_1^*}(y) \geq 8$ and therefore $\delta(H_1^*) \geq 8$. By Lemma 2.5.10, it follows that $H_1^* > K_{10}$ and thus $G > K_{10}$. Without loss of generality, we may assume that in the graph G, every vertex in $N_{G/xy}(w) \cap V(H_1) - \{z_1\}$ is adjacent to x only but not adjacent to y. Note this means that $H_1^* - \{y\} \cong K_{2,2,2,2,2,3}$. Now, note that there exists a unique vertex $s_1 \in S$ that is not adjacent to z_1 in H_1^*. Since $S \subseteq Z$, s_1 is adjacent to y. It follows that $(H_1^* - \{y\}) \cup \{z_1 s_1\} \subseteq H_1^*/ys_1$. Since

$H_1^* - \{y\} \cong K_{2,2,2,2,2,2,3}$, by Lemma 2.5.7 we know that $(H_1^* - \{y\}) \cup \{z_1 s_1\} > K_{10}$. It follows that $H_1^* > K_{10}$ and thus $G > K_{10}$.

Hence, we may assume there exists some $i \in \{1, 2\}$ such that $|S - Z| \geq |Z_i|$. Without loss of generality, assume that $|S - Z| \geq |Z_1|$. Observe that if there exist $|Z_1|$ disjoint paths linking Z_1 and $S - Z$ in $H_1^* - \{x, y\} \cup (S \cap Z)$, then by contracting each of these paths to its end in $S - Z$, we would obtain a resulting graph H_2' on $V(H_2^*)$ such that $\delta(H_2') \geq 8$ and x, y have 7 common neighbors in H_2'. By Lemma 2.5.10, it follows that $H_2' > K_{10}$ and thus $G > K_{10}$. Therefore, it suffices to prove such disjoint paths exist.

Note that $H_1 \cong K_{2,2,2,2,2,2,3}$ and $G[S] \cong K_5$. If $|Z_1| \geq 3$, then there exists a complete matching from Z_1 to $S - Z$, and therefore the desired disjoint paths exist. If $|Z_1| = 2$, we may assume $|Z_1| = |S - Z| = 2$ and the two vertices $u_1, u_2 \in Z_1$ and one vertex $v_1 \in S - Z$ form a 3-island in H_1. Let v_2 be the vertex in $S - Z$ that is not equal to u_3. Let $w_1 \in V(H_1^* - H_2^*) - Z_1$ be the unique vertex that is not adjacent to v_2, and let w_2 be any vertex in $V(H_1^* - H_2^*) - Z_1 \cup \{w_1\}$. We can then observe that w_i is adjacent to both u_i and v_i for $i = 1, 2$, and therefore paths going through u_i, w_i, v_i in order for both $i = 1, 2$ are as desired. If $|Z_1| = 1$, then we may assume $|S - Z| = 1$ as well and the vertex $u \in Z_1$ is not adjacent to the vertex $v \in S - Z$. Again since $H_1 \cong K_{2,2,2,2,2,2,3}$, there exists some $w \in V(H_1^* - H_2^*) - \{u\}$ that is a common neighbor for u and v. The path going through u, w, v in order is then as desired. □

CHAPTER 3

STRUCTURE OF POSSIBLE MINIMAL COUNTER-EXAMPLES

In this chapter, we study the structure of possible minimal counter-examples to Theorem 1.1.5. We will prove a series of lemmas on the number of edges, minimum degree, connectivity, and separations of possible minimal counter-examples to Theorem 1.1.5.

In particular, we say a graph G on $n \geq 8$ vertices is a *minimal counter-example to Theorem 1.1.5* if the following statements hold:

(1) $e(G) \geq 8n - 35$,

(2) $G \not> K_{10}$,

(3) G is not isomorphic to any exceptional graph,

(4) For every graph G' such that $8 \leq |G'| \leq n-1$ and $e(G') \geq 8|G'| - 35$, either $G' > K_{10}$ or G' is isomorphic to an exceptional graph, and

(5) Subject to (1)-(4), $e(G)$ is minimum.

To prove Theorem 1.1.5, for the sake of a contradiction, we assume that a minimal counter-example to Theorem 1.1.5 exists. For convenience, we will use G to denote a fixed minimal counter-example to Theorem 1.1.5 in the rest of Chapter 3, Chapter 4, and Chapter 5.

3.1 Basic Properties

Lemma 3.1.1. *G has the following properties:*

(1) $|V(G)| = n \geq 11$, $e(G) = 8n - 35$.

(2) $\delta(G) \geq 10$, and $\delta(N(x)) \geq 8$ for every $x \in V(G)$.

(3) If G' is a proper minor of G such that $|G'| \geq 8$, then $e(G') \leq 8|G'| - 34$ and the equality holds if and only if $G' \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$.

Proof. To see (1) is true, first observe that there is no graph on at least $8n - 35$ edges for $n = 8$ or 9, and that the only graph on $n = 10$ vertices with at least $8n - 35$ edges is K_{10}. It follows that $n \geq 11$. If $e(G) > 8n - 35$, then by the definition of a minimal counter-example to Theorem 1.1.5, $G \backslash e$ must be an exceptional graph for every $e \in E(G)$, which is a contradiction to Lemma 2.5.7. Hence, $e(G) = 8n - 35$.

To show (2), we first prove $\delta(G) \geq 8$ and $\delta(N(x)) \geq 6$ for every $x \in V(G)$. Suppose there exists an edge $xy \in E(G)$ such that x and y share at most 5 common neighbors. This means that $e(G/xy) \geq 8n - 35 - 6 = 8(n-1) - 35 + 2$. By Lemma 2.5.4, G/xy is not an exceptional graph, a contradiction to the fact that G is a minimal counter-example to Theorem 1.1.5. It follows that every pair of adjacent vertices in G share at least 6 common neighbors, meaning that $\delta(G) \geq 7$ and $\delta(N(x)) \geq 6$ for every $x \in V(G)$. Suppose there exists some $x \in V(G)$ such that $d(x) = 7$. This implies that $N(x) \cong K_7$, which is a subgraph of $G \backslash x$. Note $e(G \backslash x) = 8(n-1) - 35 + 1$. Since G is a minimal counter-example to Theorem 1.1.5, $G \backslash x$ must be an exceptional graph. By Lemma 2.5.4, $G \backslash x \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$, which is a contradiction since neither one of these two exceptional graphs contains a subgraph isomorphic to K_7. We conclude that $\delta(G) \geq 8$ and $\delta(N(x)) \geq 6$ for every $x \in V(G)$.

To continue proving (2), for the sake of a contradiction, assume that there exists some $xy \in E(G)$ such that x and y have k common neighbors where $k = 6$ or 7. It follows that $e(G/xy) = 8n - 35 - (k+1) = 8(n-1) - 35 + (7-k) \geq 8|G/xy| - 35$. Again since G is a minimal counter-example to Theorem 1.1.5, G/xy must be isomorphic an exceptional graph. If $k = 6$, then $e(G/xy) = 8|G/xy| - 34$, meaning that $G/xy \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$ by Lemma 2.5.4. Since $\delta(G) \geq 8$, Lemma 2.5.10 implies that $G > K_{10}$, a contradiction. We may then assume $k = 7$ and $e(G/xy) = 8|G/xy| - 35$, so again G/xy is isomorphic to an exceptional graph. Since $\delta(G) \geq 8$, by Lemma 2.5.10-Lemma 2.5.14, it follows that either $G > K_{10}$ or G is isomorphic to some exceptional graph, again a contradiction to the fact that G is a minimal counter-example to Theorem 1.1.5. Therefore, we have so far proved that $\delta(N(x)) \geq 8$ for every $x \in V(G)$, which then implies $\delta(G) \geq 9$. Notice that if $d(x) = 9$ for some $x \in V(G)$, then we would immediately have $N[x] > K_{10}$ as $\delta(N(x)) \geq 8$, a contradiction to the fact that $G \not> K_{10}$. This completes the proof for (2).

To show (3), assume that G' is a proper minor of G with $|G'| \geq 8$ and $e(G') \geq 8|G'| - 34$. It suffices to prove that $G' \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$ and $e(G') = 8|G'| - 34$. Since G is a minimal counter-example to Theorem 1.1.5, we know that G' is isomorphic to some exceptional graph. By Lemma 2.5.4, it follows that $G' \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$, and $e(G') = 8|G'| - 34$. □

3.2 Separations and Connectivity

The major goal of this section is to prove Lemma 3.2.9 that G is 7-connected. To prove it, we will first need to prove s series of lemmas on separations of G. Some of these lemmas before Lemma 3.2.9 will be used later in this thesis as well.

Lemma 3.2.1. *Let (A_1, A_2) be a non-trivial separation of G. For $i = 1, 2$, let $G_i = G[A_i]$, and let $\mathcal{G}_i$ be a non-empty subset of minors of G on $V(G_i)$, i.e. every graph in $\mathcal{G}_i$ has its set of vertices equal to $V(G_i)$ and can be obtained from G by deleting or contracting edges that have at least one end in $V(G_{3-i} - G_i)$. For $i = 1, 2$, define $d(\mathcal{G}_i)$ and $r(\mathcal{G}_i)$ as follows: $d(\mathcal{G}_i) = \max_{H_i \in \mathcal{G}_i}\{e(H_i) - e(G_i)\}$; $r(\mathcal{G}_i) = 1$ if there exists a graph in $\mathcal{G}_i$ isomorphic to $K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$, and $r(\mathcal{G}_i) = 0$ otherwise. Let $S = A_1 \cap A_2$. Then,*

$$8|S| \geq 35 + d(\mathcal{G}_1) + d(\mathcal{G}_2) + e(G[S]) - r(\mathcal{G}_1) - r(\mathcal{G}_2) \geq 33 + d(\mathcal{G}_1) + d(\mathcal{G}_2) + e(G[S]).$$

Proof. For convenience, let $d_i = d(\mathcal{G}_i)$ and $r_i = r(\mathcal{G}_i)$ for $i = 1, 2$. Choose $H_i \in \mathcal{G}_i$ with $e(H_i) - e(G_i) = d_i$ such that $H_i \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$ if possible, for both $i = 1, 2$. It follows that $e(G_1)+e(G_2) = e(H_1)+e(H_2)-d_1-d_2$. By (2) of Lemma 3.1.1, $\delta(G) \geq 10$. Since (A_1, A_2) is a non-trivial separation of G, $|H_i| = |G_i| \geq 11$ for both $i = 1, 2$. Since H_1, H_2 are both proper minors of G with $|H_i| \geq 11$ for $i = 1, 2$, by (3) of Lemma 3.1.1

it follows that $e(H_i) \leq 8|H_i| - 35 + r_i$ for $i = 1, 2$. Since $e(G) = 8n - 35$ by (1) of Lemma 3.1.1, it follows that

$$\begin{aligned} 8n - 35 &= e(G_1) + e(G_2) - e(G[S]) \\ &= e(H_1) + e(H_2) - d_1 - d_2 - e(G[S]) \\ &\leq 8(|H_1| + |H_2|) - 70 + r_1 + r_2 - d_1 - d_2 - e(G[S]) \\ &= 8n + 8|S| - 70 + r_1 + r_2 - d_1 - d_2 - e(G[S]). \end{aligned}$$

Therefore,

$$8|S| \geq 35 + d_1 + d_2 + e(G[S]) - r_1 - r_2.$$

Since $r_i = r(\mathcal{G}_i) \leq 1$ for both $i = 1, 2$, it follows that

$$8|S| \geq 35 + d_1 + d_2 + e(G[S]) - r_1 - r_2 \geq 33 + d_1 + d_2 + e(G[S]).$$

□

Lemma 3.2.2. *G is 6-connected.*

Proof. Let (A_1, A_2) be a non-trivial separation of G of minimum order. Let $S = A_1 \cap A_2$. For $i = 1, 2$, let $G_i = G[A_i]$ and let d_i be the maximum number of edges that can be added to S by contracting edges that have at least one end in $G_{3-i} - G_i$. By Lemma 3.2.1, $8|S| \geq 33 + d_1 + d_2 + e(G[S])$, which implies $|S| \geq 5$. For the sake of a contradiction, assume $|S| = 5$. It follows that $d_1 + d_2 + e(G[S]) \leq 7$. Let $\delta = \delta(G[S])$ for convenience. Since S is a minimum separating set of G, every vertex in S has some neighbor in $G_i - G_{3-i}$ for both $i = 1, 2$. It follows that $d_i \geq |S| - 1 - \delta = 4 - \delta$ for $i = 1, 2$, as we can contract all of $G_{3-i} - G_i$ to some vertex $v \in S$ with $d_{G[S]}(v) = \delta$ to make v adjacent to all other vertices in S. Since $e(G[S]) \geq \frac{1}{2}|S|\delta = \frac{5}{2}\delta$, it follows that

$$7 \geq d_1 + d_2 + e(G[S]) \geq 2(4 - \delta) + \frac{5}{2}\delta = 8 + \frac{1}{2}\delta,$$

which means that $\delta \leq -2$, a contradiction. □

Lemma 3.2.3. *Let $U \subseteq V(G)$ such that $U \neq \emptyset$ and $|N(U)| \leq \delta(G) - 1$. If there is no non-trivial separation of $(G[U \cup N(U)], N(U))$ of order at most $|N(U)| - 1$, then for every $Z \subseteq N(U)$ such that $|Z| = 4$, the following statements are true:*
(1) If $|N(U)| \leq \delta(G) - 2$, then $G[U \cup Z]$ has a K_4 minor rooted at Z.
(2) If $|N(U)| = \delta(G) - 1$, then one of the following two statements is true: (2a) For every two vertices $y_1, y_2 \in Z$, $G[U \cup Z]$ has a minor L such that $V(L) = Z$ and $L \cup \{y_1 y_2\} \cong K_4$.
(2b) If $\delta(G) = 11$, then for every $Z' \subseteq N(U) - Z$ such that $|Z'| = 4$, $G[U \cup Z']$ has a K_4 minor rooted at Z'.

Proof. Let $H = G[U \cup Z]$. Note we may assume that H does not have a K_4 minor rooted at Z. We will show either a contradiction, or that one of (2a) and (2b) holds true and $|N(U)| = \delta(G) - 1$.

Since there is no non-trivial separation of $(G[U \cup N(U)], N(U))$ of order at most $|N(U)| - 1$, there is no non-trivial $\leq$ 3-separation of (H, Z). Choose (X, Y) to be a 4-separation of (H, Z) such that $Y - X \neq \emptyset$, and subject to that $|Y|$ is minimum. Note that such a separation exists due to the trivial 4-separation $(Z, Z \cup U)$ of (H, Z). Let $Z^* = X \cap Y$. Then note that the minimality of $|Y|$ implies that there is no non-trivial $\leq$ 4-separation $(G[Y], Z^*)$. Since there is no non-trivial $\leq$ 3-separation of (H, Z), there exist four disjoint paths linking Z and Z^* in $G[X]$. Since H does not have a K_4 minor rooted at Z, it follows that $G[Y]$ does not have a K_4 minor rooted at Z^*.

By Theorem 2.2.2, one of (ii)-(iv) is true for H and Z. Since there is no non-trivial $\leq$ 3-separation of (H, Z), we know (iii) is not true. Therefore, one of (ii) and (iv) is true.

Case 1: (ii) is true, i.e. there is a trisection (A_1, A_2, B) of $G[Y]$ of order 2 such that $|Z^* \cap (A_i - B)| = 1$ for $i = 1, 2$.

Let z_i be the unique vertex in Z^* that is in $A_i - B$ for $i = 1, 2$, and let a, b be the two vertices in $A_1 \cap A_2 \cap B$. Notice that if $A_1 - B - \{z_1\} \neq \emptyset$, then $(A_2 \cup B \cup \{z_1\}, A_1)$ would be a non-trivial 3-separation of $(G[Y], Z^*)$, a contradiction to the fact that there is no non-trivial ≤ 4-separation $(G[Y], Z^*)$. By symmetry, it follows that $A_i - B = \{z_i\}$ for $i = 1, 2$. If $B - A_1 \cup A_2 \cup Z^* \neq \emptyset$, then similarly we would have a non-trivial ≤ 4-separation $(A_1 \cup A_2 \cup Z^*, B)$ of $(G[Y], Z^*)$, again a contradiction. It follows that $B - A_1 \cup A_2 \cup Z^* = \emptyset$, and therefore $Y - Z^* \subseteq \{a, b\}$. Notice that vertices in $Y - Z^* = Y - X$ have no neighbor in $X - Y$, meaning that $N_G(Y - Z^*) \subseteq Z^* \cup (N_G(U) - Z)$. It follows that

$$|N_G(Y - Z^*)| \leq |Z^* \cup (N_G(U) - Z)| = |Z^*| + |N_G(U) - Z| = |N_G(U)|.$$

If $|N_G(U)| \leq \delta(G) - 2$, then $|N_G(Y - Z^*)| \leq \delta(G) - 2$. This means that every vertex in $Y - Z^*$ has at least two neighbors in $Y - Z^*$ and thus $|Y - Z^*| \geq 3$, a contradiction to the fact that $Y - Z^* \subseteq \{a, b\}$.

If $|N_G(U)| = \delta(G) - 1$, then $|N_G(Y - Z^*)| \leq \delta(G) - 1$. We will prove that (2a) is true. Note that $|N_G(Y - Z^*)| \leq \delta(G) - 1$ means that every vertex in $Y - Z^*$ has at least one neighbor in $Y - Z^*$. Since $Y - Z^* \subseteq \{a, b\}$, it follows that $Y - Z^* = \{a, b\}$, $ab \in E(G)$, and that a, b each are adjacent to all vertices in $N_G(Y - Z^*) = Z^* \cup (N_G(U) - Z)$. This means that for every pair of vertices $y'_1, y'_2 \in Z^*$, $G[Y] = G[Z^* \cup \{a, b\}]$ has a minor L' on Z^* such that $L' \cup \{y'_1 y'_2\} \cong K_4$, as we can simply contract the edges ay'_3 and by'_4, where $Z^* - \{y'_1, y'_2\} = \{y'_3, y'_4\}$. Recall that there exist four disjoint paths linking Z and Z^* in $H[X]$. It follows that for every pair of vertices $y_1, y_2 \in Z$, H has a minor L such that $V(L) = Z$ and $L \cup \{y_1 y_2\} \cong K_4$. Therefore, (2a) is true.

Case 2: (iv) is true, i.e. $G[Y]$ can be drawn in the plane so that every vertex in Z^* is incident with the infinite region.

Since $G[Y]$ can be drawn in the plane so that every vertex in Z^* is incident with the infinite region, there exists a planar graph J that can be obtained from $G[Y]$ by making $J[Z^*]$ isomorphic to K_4^-. Note that

$$e(J) = e(J[Z^*]) + e(Z^*, Y - Z^*) + e(G[Y - Z^*]) = 5 + e(Z^*, Y - Z^*) + e(G[Y - Z^*]).$$

Since J is planar, $e(J) \leq 3|J| - 6$. It follows that

$$5+e(Z^*, Y-Z^*)+e(G[Y-Z^*]) = e(J) \leq 3|J|-6 = 3(|Y-Z^*|+4)-6 = 3|Y-Z^*|+6.$$

Therefore,

$$e(Z^*, Y - Z^*) + e(G[Y - Z^*]) \leq 3|Y - Z^*| + 1.$$

If $|N(U)| \leq \delta(G) - 2$, then $\delta(G) - |N(U)| \geq 2$. It follows that

$$\begin{aligned}
e(Z^*, Y - Z^*) + e(G[Y - Z^*]) &= \frac{1}{2}e(Z^*, Y - Z^*) + \frac{1}{2}\sum_{v \in Y - Z^*} d_J(v) \\
&\geq \frac{1}{2}e(Z^*, Y - Z^*) + \frac{1}{2}(\delta(G) - |N(U) - Z|) \cdot |Y - Z^*| \\
&= \frac{1}{2}e(Z^*, Y - Z^*) + \frac{1}{2}(\delta(G) - |N(U)| + 4) \cdot |Y - Z^*| \\
&\geq \frac{1}{2}e(Z^*, Y - Z^*) + \frac{1}{2}(2 + 4)|Y - Z^*| \\
&= \frac{1}{2}e(Z^*, Y - Z^*) + 3|Y - Z^*|.
\end{aligned}$$

Since $e(Z^*, Y - Z^*) + e(G[Y - Z^*]) \leq 3|Y - Z^*| + 1$, it follows that

$$\frac{1}{2}e(Z^*, Y - Z^*) + 3|Y - Z^*| \leq e(Z^*, Y - Z^*) + e(G[Y - Z^*]) \leq 3|Y - Z^*| + 1,$$

meaning that $e(Z^*, Y - Z^*) \leq 2$. By the minimality of $|Y|$ when choosing (X, Y), every vertex in Z^* has at least one neighbor in $Y - Z^*$, meaning $e(Z^*, Y - Z^*) \geq 4$, a contradiction.

It remains to consider $|N(U)| = \delta(G) - 1$, and we will prove that (2b) is true. To prove (2b), assume that $\delta(G) = 11$. Since J is planar, $G[Y - Z^*]$ which is a subgraph of J is also planar, and therefore $e(G[Y - Z^*]) \leq 3|Y - Z^*| - 6$. Recall that $e(Z^*, Y - Z^*) + e(G[Y - Z^*]) \leq 3|Y - Z^*| + 1$. It follows that

$$e(Z^*, Y - Z^*) + 2e(G[Y - Z^*]) \leq 6|Y - Z^*| - 5.$$

Since $\delta(G) = 11$, we now have

$$e(N(Y - Z^*), Y - Z^*) + 2e(G[Y - Z^*]) = \sum_{v \in Y - Z^*} d_G(v) \geq 11|Y - Z^*|.$$

Notice that

$$\begin{aligned} e(N(U) - Z, Y - Z^*) =& \Big(e(N(Y - Z^*), Y - Z^*) + 2e(G[Y - Z^*])\Big) - \\ & \Big(e(Z^*, Y - Z^*) + 2e(G[Y - Z^*])\Big) \\ \geq& 11|Y - Z^*| - (6|Y - Z^*| - 5) \\ =& 5|Y - Z^*| + 5. \end{aligned}$$

Since $N(U) = \delta(G) - 1 = 10$, we have $|N(U) - Z| = |N(U)| - 4 = 6$. It follows that there exist five distinct vertices $v_1, v_2, v_3, v_4, v_5 \in Y - Z^*$ such that each of them is adjacent all 6 vertices in $N(U) - Z$. Let $Z' \subseteq N(U) - Z$ such that $|Z'| = 4$ be arbitrary, and let $Z' = \{z'_1, z'_2, z'_3, z'_4\}$. By contracting edges $z'_i v_i$ for $i = 1, 2, 3$, we would then obtain a K_4 minor of $G[(Y - Z^*) \cup Z']$ rooted at Z'. Since $G[(Y - Z^*) \cup Z'] \subseteq G[U \cup Z']$, it follows that $G[U \cup Z']$ has a K_4 minor rooted at Z'. $\square$

Lemma 3.2.4. *Let S be a separating set of G. The following statements are true:*

(1) There is no $w \in S$ such that $G[S - \{w\}]$ is complete.

(2) If $|S| \leq \delta(G) - 2$ and is minimum over all separating sets of G, then there is no $Z \subseteq S$ with $|Z| = 4$ such that the graph obtained from $G[S]$ by making Z a clique is complete.

(3) $G[S]$ contains an independent set of size 3 or two disjoint non-edges.

Proof. Let (A_1, A_2) be a non-trivial separation of G. Let $S = A_1 \cap A_2$ and $G_i = G[A_i]$ for $i = 1, 2$.

We first prove that (1) implies (3). Assume (1) is true. Since $G[S - \{w\}]$ is not a complete graph for all $w \in S$, for every non-adjacent vertices $x, y \in S$, neither $G[S - \{x\}]$ nor $G[S - \{y\}]$ is a complete graph. It follows that there exists some $z \in S - \{x, y\}$ such that $xz, yz \notin E(G)$, or that there exist some $z_1, z_2 \in S - \{x, y\}$ such that $z_1z_2 \notin E(G)$. The former case implies that $G[S]$ contains an independent set of size 3 on $\{x, y, z\}$, and the latter case implies that $G[S]$ has two disjoint non-edges, namely xy and z_1z_2.

It now suffices to prove (1) and (2) in the rest of this proof. Let $w \in S$ and $Z \subseteq S$ such that $|Z| = 4$. For $i = 1, 2$, let H_i^w be the graph obtained from G_i by making w adjacent to all other vertices in S, and let H_i^Z be the graph obtained from G_i by making Z a clique. If S is a minimum separating set of G and $|S| \leq \delta(G) - 2$, let (H_1, H_2) be equal to one of (H_1^w, H_2^w) and (H_1^Z, H_2^Z). Otherwise, let $(H_1, H_2) = (H_1^w, H_2^w)$. Let $H = H_1 \cup H_2$. It suffices to prove that $H[S]$ is not a complete graph. For the sake of a contradiction, assume that it is not.

Observe that we can choose S such that it is a minimal separating set of G. This is because, if S is not minimal, we would have $(H_1, H_2) = (H_1^w, H_2^w)$, and we can then replace S with a minimal subset $S' \subseteq S$ such that S' separates G, and that there exists some $w' \in S'$ where $G[S' - \{w'\}]$ is complete. It follows that $G > H_i$ for $i = 1, 2$ in both cases: If $(H_1, H_2) = (H_1^w, H_2^w)$, since S is a minimal separating set, we can contract all of $V(G_{3-i} - G_i)$ to w for $i = 1, 2$; and if $(H_1, H_2) = (H_1^Z, H_2^Z)$, we know $|S| \leq \delta(G) - 2$ and S is a minimum separating set of G, which then allows us to apply Lemma 3.2.3 to obtain that $G > H_i$ for $i = 1, 2$. Finally, notice that $|S| \leq 8$, since otherwise we could contract all of $V(H_1) - S$ to one single vertex and obtain a K_{10} minor of H_1, meaning $G > K_{10}$, a contradiction. By Lemma 3.2.2, it follows that $6 \leq |S| \leq 8$.

Claim 1. $H_i \ncong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$ for $i = 1, 2$.

Proof of Claim 1. For the sake of a contradiction, assume $H_1 \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$. Note that $K_{2,3,3,3,3}$ contains no clique of size greater than 5, and $K_{2,2,2,2,2,3}$ contains no clique of size greater than 6. Since $|S| \geq 6$, it follows that $H_1 \cong K_{2,2,2,2,2,3}$, $|S| = 6$, and $H[S] \cong K_6$. Let $d = e(H[S]) - e(G[S])$. Note $e(G_i) = e(H_i) - d$ for $i = 1, 2$, and that

$e(G[S]) + d = e(H[S]) = e(K_6) = 15$. Since $e(G) = 8n - 35$, we have

$$8n-35 = e(G_1)+e(G_2)-e(G[S]) = e(H_1)-d+e(G_2)-e(G[S]) = e(H_1)+e(G_2)-15.$$

Since $H_1 \cong K_{2,2,2,2,2,3}$, $e(H_1) = 70$ and $|G_2| = n - (|K_{2,2,2,2,2,3}| - |S|) = n - 7$. It follows that

$$e(G_2) = 8n - 35 - 70 + 15 = 8(n - 7) - 34 = 8|G_2| - 34.$$

Note that $|S| \leq 8$, so $|G_2| > 8$. Notice that G_2, H_2 are both proper minors of G such that $e(H_2) \geq e(G_2)$. By (3) of Lemma 3.1.1, it follows that $G_2 = H_2 \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$. Since $H_2[S]$ is a clique on $|S| \geq 6$ vertices, it follows that $G_2 = H_2 \cong K_{2,2,2,2,2,3}$. This means that G is isomorphic to a $(K_{2,2,2,2,2,3}, 6)$-cockade of multiplicity 2, which is an exceptional graph, a contradiction. ⊣

Now, let d_i be the maximum number of edges that can be added to S by contracting edges that have at least one end in $G_{3-i} - G_i$ for $i = 1, 2$. Since $H[S]$ is complete, $d_1 = d_2 = e(H[S]) - e(G[S])$. Since $H_i \not\cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$ for $i = 1, 2$, by Lemma 3.2.1 it follows that

$$8|S| \geq 35 + d_1 + d_2 + e(G[S]).$$

Claim 2. $G[S] \cong K_7, K_8$, or K_8^-

Proof of Claim 2. We consider the case $(H_1, H_2) = (H_1^w, H_2^w)$ and the case $(H_1, H_2) = (H_1^Z, H_2^Z)$ separately.

Case 1: $(H_1, H_2) = (H_1^w, H_2^w)$. Let $\delta = d_{G[S]}(w)$. Then $d_i = |S| - 1 - \delta$ for $i = 1, 2$ and $e(G[S]) = \binom{|S|-1}{2} + \delta$. It follows that $8|S| \geq 35 + 2(|S| - 1 - \delta) + \binom{|S|-1}{2} + \delta$, meaning that $\delta \geq 33 + \binom{|S|-1}{2} - 6|S|$. Since $\delta \leq |S| - 1$ and $6 \leq |S| \leq 8$, it follows that $\delta \geq 6$ and $|S| = 7$ or 8. Therefore, $G[S] \cong K_7, K_8$, or K_8^-.

Case 2: $(H_1, H_2) = (H_1^Z, H_2^Z)$. Let $z = e(G[Z])$. Then $d_i = 6 - z$ for $i = 1, 2$ and $e(G[S]) = \binom{|S|}{2} - (6 - z)$. It follows that $8|S| \geq 35 + 2(6 - z) + \binom{|S|}{2} - (6 - z)$, meaning that $z \geq 41 + \binom{|S|}{2} - 8|S|$. Since $z \leq 6$ and $6 \leq |S| \leq 8$,we have either $|S| = 7$ and $z = 6$, or that $|S| = 8$ and $z \geq 5$. Again, it follows that $G[S] \cong K_7, K_8$, or K_8^-. $\dashv$

Consider the case $G[S] \cong K_7$ first, which implies that $H_i = G_i$ for $i = 1, 2$. By Claim 1, $G_i \ncong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$ for $i = 1, 2$. By Lemma 3.1.1, $|G_i| \geq 11$ and therefore $e(G_i) \leq 8|G_i| - 35$ for both $i = 1, 2$. Therefore, $e(G_1) + e(G_2) \leq 8(|G| + 7) - 70 = 8n - 14$. On the other hand, $e(G_1) + e(G_2) = e(G) + \binom{7}{2} = 8n - 35 + 21 = 8n - 14$. It follows that for both $i = 1, 2$, $e(G_i) = 8|G_i| - 35$ and hence G_i is isomorphic to some exceptional graph, due Lemma 3.1.1 as G_i is a proper minor of G on at least 11 vertices. Note that G_i contains a clique of size 7 for both $i = 1, 2$. It follows that each G_i for $i = 1, 2$ is isomorphic to a $(K_{1,1,2,2,2,2,2}, 7)$-cockade and therefore $G = G_1 \cup G_2$ is also isomorphic to a $(K_{1,1,2,2,2,2,2}, 7)$-cockade, a contradiction.

We may now assume $G[S] \cong K_8$ or K_8^-. Note that H_i for $i = 1, 2$ is a proper minor of G on at least 11 vertices that contains a clique of size 8. Also note that no exceptional graph contains a clique of size 8. It follows that $e(H_i) \leq 8|H_i| - 36$ for both $i = 1, 2$. Let $t = e(H_i) - e(G_i) = \binom{8}{2} - e(G[S]) = 28 - e(G[S])$, and note that $t = 0$ or 1. Then,

$$\begin{aligned} e(G) =& e(G_1) + e(G_2) - e(G[S]) \\ \leq& (8|H_1| - 36 - t) + (8|H_2| - 36 - t) - (28 - t) \\ =& 8(|H_1| + |H_2|) - 100 - t \\ =& 8(n + 8) - 100 - t \\ =& 8n - 36 - t. \end{aligned}$$

Since $t = 0$ or 1, $e(G) \leq 8n - 36$, a contradiction to the fact that $e(G) = 8n - 35$. $\square$

Lemma 3.2.5. *Let (A_1, A_2) be a non-trivial separation of G such that $|A_1 \cap A_2| \leq \delta(G) - 2$ and $|A_1 \cap A_2|$ is minimum over all non-trivial separations of G. Let $w \in A_1 \cap A_2$, and let H_i for $i = 1, 2$ be the graph obtained from $G[A_i]$ by making w adjacent to all other vertices in $A_1 \cap A_2$. Then, $H_i \ncong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$ for $i = 1, 2$.*

Proof. Let $S = A_1 \cap A_2$ and $G_i = G[A_i]$ for $i = 1, 2$. By Lemma 3.2.2 and Lemma 3.2.4, $|S| \geq 6$ and $H_1[S]$ is not a clique.

For the sake of a contradiction, assume $H_1 \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$. If $H_1 \cong K_{2,2,2,2,2,3}$, let $\{x_i, y_i\}$ for $i = 1, 2, 3, 4, 5$ be the five 2-islands and $\{r_1, r_2, r_3\}$ be the 3-island of H_1; if $H_1 \cong K_{2,3,3,3,3}$, let $\{x_i, y_i, z_i\}$ for $i = 1, 2, 3, 4$ be the four 3-islands and $\{r_1, r_2\}$ be the 2-island of H_1. Observe that up to isomorphism, we may assume either $w = x_1$ or $w = r_1$ in both cases. Let C_w be the island of H_1 that contains w. Since w is adjacent to every vertex in S in H_1, it follows that $C_w - \{w\} \subseteq A_1 - S$.

Claim 1. There do not exist four distinct vertices $a_1, b_1, a_2, b_2 \in S - \{w\}$ such that $a_i b_i \notin E(H_1)$ for $i = 1, 2$.

Proof of Claim 1. For the sake of a contradiction, assume that $a_1, b_1, a_2, b_2 \in S - \{w\}$ are distinct vertices such that $a_i b_i \notin E(H_1)$ for $i = 1, 2$. Let $Z_1 = \{a_1, b_1, a_2, b_2\}$, and let R_1 be the graph obtained from G_1 by making Z_1 a clique. By Lemma 3.2.3, we know that $G > R_1$. We then consider all cases, up to isomorphism, in the following table. Notice that $R_1 - \{w\} > K_{10}$ in every case in the table. It follows that $G > R_1 > K_{10}$, a contradiction.

Table 3.1

H_1	w	Z_1	$R_1 - \{w\}$
$K_{2,2,2,2,2,3}$	x_1	$\{x_2, y_2, x_3, y_3\}$	$K_5 + K_{2,2,3}$
$K_{2,2,2,2,2,3}$	x_1	$\{x_2, y_2, r_1, r_2\}$	$K_3 + K_{2,2,2} + \overline{P_3}$
$K_{2,2,2,2,2,3}$	r_1	$\{x_1, y_1, x_2, y_2\}$	$K_4 + K_{2,2,2,2}$
$K_{2,3,3,3,3}$	x_1	$\{x_2, y_2, x_3, y_3\}$	$K_{2,2,3} + \overline{P_3} + \overline{P_3}$
$K_{2,3,3,3,3}$	x_1	$\{x_2, y_2, r_1, r_2\}$	$K_{1,1,2,3,3} + \overline{P_3}$
$K_{2,3,3,3,3}$	r_1	$\{x_2, y_2, x_3, y_3\}$	$K_{1,3,3} + \overline{P_3} + \overline{P_3}$

⊣

Note that $H_1[S]$ is not a clique, and w is adjacent to all other vertices in $S - \{w\}$. It follows that there exists some non-edge in $H_1[S - \{w\}]$. By Claim 1, there is a unique island C_0 of H_1 such that $|C_0 \cap S| \geq 2$. Choose distinct vertices $p_1, p_2 \in C_0 \cap S$. Notice that $|S - C_0 \cup \{w\}| \geq 6 - 3 - 1 = 2$, meaning that at least two islands if H_1 have exactly one vertex in $S - \{w\}$. Let C_2 be the island of H_1 that contains x_2. Without loss of generality, in both cases we can then assume that $C_2 \cap S = \{x_2\}$. Observe that that $Z_2 = \{w, x_2, p_1, p_2\}$ is a set of four distinct vertices by construction. Let R_2 be the graph obtained from G_1 by making Z_2 a clique. Again by Lemma 3.2.3, we have $G > R_2$.

Now, note that $y_2 \in C_2 - \{x_2\}$ and thus $y_2 \in V(H_1) - S$, which then implies that w, y_2 are adjacent in G and therefore in R_2. Furthermore, observe that the new vertex in R_2/wy_2 is adjacent to all other vertices in it. In the following table, we consider all cases up to isomorphism and show that $R_2/wy_2 > K_{10}$ in every case. It follows that $G > R_2 > K_{10}$, a contradiction.

Table 3.2

H_1	w	$\{p_1, p_2\}$	R_2/wy_2
$K_{2,2,2,2,2,3}$	x_1	$\{x_3, y_3\}$	$K_5 + K_{2,2,3}$
$K_{2,2,2,2,2,3}$	x_1	$\{r_1, r_2\}$	$K_3 + K_{2,2,2} + \overline{P_3}$
$K_{2,2,2,2,2,3}$	r_1	$\{x_3, y_3\}$	$K_4 + K_{2,2,2,2}$
$K_{2,3,3,3,3}$	x_1	$\{x_3, y_3\}$	$K_{1,2,2,2,3} + \overline{P_3}$
$K_{2,3,3,3,3}$	x_1	$\{r_1, r_2\}$	$K_3 + K_{2,2,3,3}$
$K_{2,3,3,3,3}$	r_1	$\{x_3, y_3\}$	$K_{1,1,2,3,3} + \overline{P_3}$

□

Lemma 3.2.6. *Let (A_1, A_2) be a non-trivial separation of G such that $|A_1 \cap A_2| \leq \delta(G) - 2$ and $|A_1 \cap A_2|$ is minimum over all non-trivial separations of G. Let $Z \subseteq S$ such that $|Z| = 4$, and let $H_i = G[A_i] \cup \{z_1 z_2 : z_1, z_2 \in Z\}$ for $i = 1, 2$. Then, $H_i \ncong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$ for $i = 1, 2$.*

Proof. Let $S = A_1 \cap A_2$ and $G_i = G[A_i]$ for $i = 1, 2$. By Lemma 3.2.2 and Lemma 3.2.4, $|S| \geq 6$ and $H_1[S]$ is not a clique. For the sake of a contradiction, assume $H_1 \cong K_{2,2,2,2,2,3}$

or $K_{2,3,3,3,3}$. Label the vertices in H as in the proof for Lemma 3.2.5. Note that $H_1[Z] \cong K_4$. Without loss of generality, assume that either $Z = \{x_1, x_2, x_3, x_4\}$ or $Z = \{x_1, x_2, x_3, r_1\}$ in both cases.

Claim 1. For every $z \in Z$ and $z' \in S - Z$, $zz' \in E(G)$.

Proof of Claim 1. For the sake of a contradiction, assume that there are vertices $z \in Z$ and $z' \in S - Z$ such that $zz' \notin E(G)$, and note that this means that $zz' \notin E(H_1)$ either. Without loss of generality, assume that either $z = x_1$ and $z' = y_1$, or that $z = r_1$ and $z' = r_2$. Let $Z_1 = \{z, z', x_2, x_3\}$ and R_1 be the graph obtained from G_1 by making Z_1 a clique. By Lemma 3.2.3, $G > R_1$. Note that $|Z \cap Z_1| = 3$ in all cases. Let z_0 be the unique vertex in $Z - Z_1$.

Let $z_0' \in V(H_1) - Z$ such that z_0' is adjacent to every vertex in Z in H_1. Then, by the construction of H_1, observe that z_0' is also adjacent to every vertex in Z in G and therefore in R_1 as well. It follows that the new vertex in R_1/z_0z_0' is adjacent to all vertices in $Z \cap Z_1$. In the following table, we consider all cases up to isomorphism, and in each case we show that there exists some $z_0' \in V(H_1) - Z$ such that z_0' is adjacent to every vertex in Z in H_1 and $R_1/z_0z_0' > K_{10}$. It follows that $G > R_1 > K_{10}$, a contradiction.

Table 3.3

H_1	Z	(z, z')	(z_0, z_0')	R_1/z_0z_0'
$K_{2,2,2,2,2,3}$	$\{x_1, x_2, x_3, x_4\}$ or $\{x_1, x_2, x_3, r_1\}$	(x_1, y_1)	(x_4, r_1) or (r_1, x_4)	$K_4 + K_{2,2,2,2}$
$K_{2,2,2,2,2,3}$	$\{x_1, x_2, x_3, r_1\}$	(r_1, r_2)	(x_1, x_4)	$K_3 + K_{2,2,2} + \overline{P_3}$
$K_{2,3,3,3,3}$	$\{x_1, x_2, x_3, x_4\}$ or $\{x_1, x_2, x_3, r_1\}$	(x_1, y_1)	(x_4, r_1) or (r_1, x_4)	$K_{1,1,2,3,3} + \overline{P_3}$
$K_{2,3,3,3,3}$	$\{x_1, x_2, x_3, r_1\}$	(r_1, r_2)	(x_1, x_4)	$K_3 + K_{2,2,3,3}$

$\dashv$

Recall that $H_1[S]$ is not a clique and $H_1[Z] \cong K_4$. By Claim 1, there exist vertices $p_1, p_2 \in S - Z$ such that $p_1p_2 \notin E(H_1)$. Let $Z_2 = \{x_1, x_2, p_1, p_2\}$, and let R_2 be the graph obtained from G_1 by making Z_2 a clique. Again we have $G > R_2$ by Lemma 3.2.3, and therefore it suffices to prove $R_2 > K_{10}$. We will consider the case $H_1 \cong K_{2,2,2,2,2,3}$ and the case $H_1 \cong K_{2,3,3,3,3}$ separately in the remaining proof.

First assume that $H_1 \cong K_{2,2,2,2,2,3}$. In this case, $Z \cap Z_2 = \{x_1, x_2\}$. Let u_1, u_2 be the two distinct vertices in $Z - Z_2$, and note that $N_{G_1}(v) = N_{H_1}(v)$ for every $v \in V(G_1) - Z$. Therefore, if there exist vertices $u_1', u_2' \in V(H_1) - Z \cup Z_2$ both contained in some island of H_1 that is disjoint from $Z \cup Z_2$, then by contracting edges u_1u_1' and u_2u_2' in R_2, we can then have the two new vertices w_1, w_2 in the resulting graph satisfying that w_1, w_2 are adjacent to each other and w_i for $i = 1, 2$ is adjacent to both x_1 and x_2. It follows that $(R_2/u_1u_1')/u_2u_2' = \big((H_1/u_1u_1')/u_2u_2'\big) \cup \{vv' : v, v' \in Z_2\}$. In the table below, we show that in each case we can always find some $u_1', u_2' \in V(H_1) - Z \cup Z_2$ contained in some island of H_1 disjoint from $Z \cup Z_2$ such that the corresponding graph $(R_2/u_1u_1')/u_2u_2' = \big((H_1/u_1u_1')/u_2u_2'\big) \cup \{vv' : v, v' \in Z_2\}$ has a K_{10} minor. It follows that $G > R_2 > K_{10}$, a contradiction.

Table 3.4

H_1	Z	$\{p_1, p_2\}$	$\{u_1, u_2\}$	$\{u_1', u_2'\}$	$(R_2/u_1u_1')/u_2u_2'$
$K_{2,2,2,2,2,3}$	$\{x_1, x_2, x_3, x_4\}$	$\{x_5, y_5\}$	$\{x_3, x_4\}$	$\{r_1, r_2\}$	$K_7 + K_{2,2}$
$K_{2,2,2,2,2,3}$	$\{x_1, x_2, x_3, x_4\}$	$\{r_1, r_2\}$	$\{x_3, y_4\}$	$\{x_5, y_5\}$	$K_7 + K_{2,2}$
$K_{2,2,2,2,2,3}$	$\{x_1, x_2, x_3, r_1\}$	$\{x_4, y_4\}$	$\{x_3, r_1\}$	$\{x_5, y_5\}$	$K_7 + K_{2,2}$

Now, assume that $H_2 \cong K_{2,3,3,3,3}$. If $Z = \{x_1, x_2, x_3, x_4\}$, then $\{p_1, p_2\} = \{r_1, r_2\}$. Observe that x_3, x_4 each are adjacent to all of y_1, y_2, z_1, z_2 in G_1. It follows that $(R_2/x_3y_1)/x_4y_2$ contains $H_1[V(G_1) - \{x_3, x_4\}] \cup \{y_1z_1, y_2z_2, r_1r_2\} \cong K_{1,1,2,2} + \overline{P_3} + \overline{P_3}$ as a subgraph. Since $K_{1,1,2,2} + \overline{P_3} + \overline{P_3} > K_{10}$, it follows that $G > R_2 > K_{10}$, a contradiction. If $Z = \{x_1, x_2, x_3, r_1\}$, then without loss of generality assume that $\{p_1, p_2\} = \{x_4, y_4\}$. Now, x_3, r_1 each are adjacent to all of y_1, y_2, z_1, z_2 in G_1. It follows that $(R_1/x_3y_1)/r_1y_2$ contains $H_1[V(G_1) - \{x_3, r_1\}] \cup \{y_1z_1, y_2z_2, x_4y_4\} \cong K_{1,2} + \overline{P_3} + \overline{P_3} + \overline{P_3}$ as a subgraph. Since $K_{1,2} + \overline{P_3} + \overline{P_3} + \overline{P_3} K_{10}$, it follows that $G > R_2 > K_{10}$, again a contradiction. $\square$

We can now combine Lemma 3.2.5 and Lemma 3.2.6 to form the next lemma.

Lemma 3.2.7. *Let (A_1, A_2) be a non-trivial separation of G. Let $S = A_1 \cap A_2$. If $|S| \leq \delta(G) - 2$ and $|S|$ is minimum over all non-trivial separations of G, then the following* *statements are true for both* $i = 1, 2$.

(1) For every $w \in S$, $G[A_i] \cup \{wr : r \in A_1 \cap A_2 - \{w\}\} \ncong K_{2,2,2,2,2,3}$ *or* $K_{2,3,3,3,3}$.

(2) For every $Z \subseteq S$ *of size 4,* $G[A_i] \cup \{z_1 z_2 : z_1, z_2 \in Z\} \ncong K_{2,2,2,2,2,3}$ *or* $K_{2,3,3,3,3}$.

The next lemma is an immediate consequence of Lemma 3.2.1 and Lemma 3.2.7.

Lemma 3.2.8. *Let* (A_1, A_2) *be a non-trivial separation of* G *and* $S = A_1 \cap A_2$ *such that* $|S| \leq \delta(G) - 2$ *and* $|S|$ *is minimum over all non-trivial separations of* G*. Let* $S = A_1 \cap A_2$*. Let* $d' = \max_{Z: Z \subseteq S, |Z|=4}\{6 - e(G[Z])\}$ *and* $d = \max\{d', |S| - 1 - \delta(G[S])\}$*. Then,* $8|S| \geq 35 + 2d + e(G[S])$.

Proof. For $i = 1, 2$, let $\mathcal{G}_i'$ be the set of graphs obtained from G_i by making some $w \in S$ adjacent to all other vertices in S, and let $\mathcal{G}_i''$ be the set of graphs obtained from G_i by making some subset $Z \subseteq S$ of size 4 a clique. Let $\mathcal{G}_i = \mathcal{G}_i' \cup \mathcal{G}_i''$ for $i = 1, 2$. By Lemma 3.2.3, every graph in $\mathcal{G}_i$ for $i = 1, 2$ is a minor of G with vertex set $V(G_i)$.

Note that $|S| - 1 - \delta(G[S]) = \max_{H \in \mathcal{G}_i'}\{e(H) - e(G_i)\}$ and $d' = \max_{H \in \mathcal{G}_i''}\{e(H) - e(G_i)\}$ for $i = 1, 2$. It follows that for both $i = 1, 2$,

$$d = \max\{d', |S| - 1 - \delta(G[S])\} = \max_{H \in \mathcal{G}_i}\{e(H) - e(G_i)\}.$$

By Lemma 3.2.7, no graph in $\mathcal{G}_i$ is isomorphic to $K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$ for $i = 1, 2$. Therefore, by Lemma 3.2.1, we have that $8|S| \geq 35 + 2d + e(G[S])$. □

Lemma 3.2.9. *G is 7-connected.*

Proof. Let (A_1, A_2) be a non-trivial separation of G of minimum order. Let $S = A_1 \cap A_2$ and $G_i = G[A_i]$ for $i = 1, 2$. By Lemma 3.2.2, $|S| \geq 6$, so we may assume $|S| = 6$ for the sake of a contradiction. Note that $\delta(G) \geq 10$, so $|S| < \delta(G) - 2$, which allows us to apply Lemma 3.2.8. Let $\delta = \delta(G[S])$. Let $d = \max\{5 - \delta, \max_{Z: Z \subseteq S, |Z|=4}\{6 - e(G[Z])\}\}$. By Lemma 3.2.8, $8|S| \geq 35 + 2d + e(G[S])$. With $|S| = 6$, it follows that

$$2d \leq 13 - e(G[S]).$$

Since $d \geq 5 - \delta$, we have $e(G[S]) \leq 13 - 2d \leq 3 + 2\delta$. Since $e(G[S]) \geq \frac{1}{2}\delta|S| = 3\delta$, it follows that $3\delta \leq e(G[S]) \leq 3 + 2\delta$ and hence $\delta \leq 3$.

Claim 1. Either $e(G[S]) \leq 1$ or $E(G[S])$ is precisely a perfect matching of size 3.

Proof of Claim 1. By the definition of d, $e(G[Z]) \geq 6 - d$ for every $Z \subseteq S$ with $|Z| = 4$. Note that each pair of vertices is contained in exactly $\binom{4}{2}$ subsets of size 4 of S. It follows that

$$e(G[S]) = \frac{\sum_{Z:Z\subseteq S,|Z|=4} e(G[Z])}{\binom{4}{2}} \geq \frac{\binom{6}{4}(6-d)}{\binom{4}{2}} = 15 - \frac{5}{2}d.$$

Since $2d \leq 13 - e(G[S])$, we know $15 - \frac{5}{2}d \leq e(G[S]) \leq 13 - 2d$ and therefore $d \geq 4$. Note that $e(G[S]) \geq \frac{1}{2}|S|\delta = 3\delta$. It follows that $3\delta \leq e(G[S]) \leq 13 - 2d \leq 13 - 8 = 5$, which shows $\delta \leq \frac{5}{3}$ and therefore $\delta \leq 1$.

If $d = 4$, observe that $5 = 15 - \frac{5}{2} \cdot 4 \leq e(G[S]) \leq 13 - 2 \cdot 4 = 5$. It follows that (i) $e(G[S]) = 5$ and (ii) $e(G[Z]) = 2$ for every $Z \subseteq S$ with $|Z| = 4$. Note that (i) implies $\Delta(G) \geq 2$ and (ii) implies $\Delta(G) \leq 2$, so we can choose $v \in S$ such that $d_{G[S]}(v) = 2$. Let $N_{G[S]}(v) = \{u_1, u_2\}$ and $S - N_{G[S]}[v] = \{w_1, w_2, w_3\}$. Let $Z_1 = \{v, u_1, u_2, w_1\}$. Since $e(G[Z_1]) = 2$, w_1 is adjacent to neither u_1 nor u_2. By symmetry, it follows that w_i is adjacent to neither u_1 nor u_2 for $i = 1, 2, 3$. Let $Z_2 = \{u_1, u_2, w_1, w_2\}$. We then see that $e(G[Z_2]) \leq 1$, a contradiction.

Hence, $d \geq 5$ and $e(G[S]) \leq 13 - 2d \leq 3$, meaning that $\delta \leq 1$. For the sake of a contradiction, assume that $e(G[S]) = 2$ or 3 and $E(G[S])$ is not a perfect matching of size 3. It follows that $G[S]$ is isomorphic to one of the following graphs: the union of two disjoint edges, a path of length 2 or 3, a 3-star, and the disjoint union of a path of length 2 and an isolated edge. In every one of these 5 graphs, observe that there always exists an independent set Z of size 4 in S. It follows that $d = \max_{Z:Z\subseteq S,|Z|=4}\{6 - e(G[Z])\} = 6$ and therefore $e(G[S]) \leq 13 - 2d = 1$, a contradiction. $\dashv$

Claim 2. For $i = 1, 2$, G has a minor L_i on $V(G_i)$ obtained by contracting edges that have at least one end in $G_{3-i} - G_i$ such that

(1) $e(L_i) - e(G_i) \geq 7$ if $e(G[S]) \leq 1$, and $e(L_i) - e(G_i) \geq 6$ if $e(G[S]) = 3$, and

(2) there exists a vertex in $L_i[S]$ adjacent to all other vertices in S in L_i.

Proof of Claim 2. We will prove (1) and (2) hold for $i = 1$, and the case of $i = 2$ will follow by symmetry. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. Without loss of generality, assume that $E(G[S]) = \{s_1s_2\}$ if $e(G[S]) = 1$ in all cases, and $E(G[S]) = \{s_1s_2, s_3s_4, s_5s_6\}$ if $e(G[S]) = 3$. Since $\delta(G) \geq 10$, $|G_2 - G_1| \geq 5$. Choose two distinct vertices $x, y \in V(G_2 - G_1)$. Since G is 6-connected, there exist 6 paths $P_1, ..., P_6$ in G_2 between $\{x\}$ and S that are disjoint except for x. Without loss of generality, assume that P_j links x and s_j for $j = 1, ..., 6$.

Assume that every P_j has length exactly 1 for a moment. Since $G - x$ is 5-connected, there exist five paths $Q_1, ..., Q_5$ in $G_2 - x$ between $\{y\}$ to S that are disjoint except for y. If $e(G[S]) \leq 1$, without loss of generality, assume s_4 is an end of Q_1. By contracting the edge xs_3, contracting all vertices on Q_1 to s_4, and contracting other Q_j paths properly, we could obtain a minor L_1 of G on $V(G_1)$, which is isomorphic to some graph obtained from G_1 by making s_3 adjacent to all other vertices in S and making s_4 adjacent to at least four other vertices in S. It follows that $e(L_1) - e(G_1) \geq 8$, and s_3 is adjacent to all other vertices in S in the graph L_1. If $e(G[S]) = 3$, without loss of generality, assume s_3 is an end of Q_1. By contracting the edge xs_1, contracting all vertices on Q_1 to s_3, and contracting other Q_j paths properly, we could obtain a minor L_1 of G on $V(G_1)$, which is isomorphic to some graph obtained from G_1 by making s_1 adjacent to all other vertices in S and s_3 adjacent to at least two vertices other than s_1 and s_4 in S. It follows that $e(L_1) - e(G_1) \geq 6$.

We can now assume P_j has length at least 2 for some $j \in \{1, ..., 6\}$. Since now, for every path P, we will use $I(P)$ to denote the set of internal vertices of P.

Consider $e(G[S]) \leq 1$ first. Without loss of generality, we can assume one of P_1, P_2, P_3 has length at least 2. Let $X_1 = I(P_1) \cup I(P_2) \cup I(P_3)$ and $Y_1 = V(P_4 \cup P_5 \cup P_6 - x)$, and note that $X_1 \neq \emptyset$ and $Y_1 \neq \emptyset$. Since G is 6-connected, $C_1 = \{x, s_1, s_2, s_3\}$ is not a cut of G. It follows that there exists a path Q_1 in $G_2 - C_1$ that links some $x_1 \in X_1$ and

$y_1 \in Y_1$ such that no internal vertex of Q_1 is in X_1 or Y_1. Without loss of generality, assume $y_1 \in V(P_4 - x)$. Let $X_2 = X_1 \cup V(P_4 \cup Q_1) - \{x, s_4\}$ and $Y_2 = V(P_5 \cup P_6) - \{x\}$. Note that $X_2 \neq \emptyset$ and $Y_2 \neq \emptyset$, and that $C_2 = \{x, s_1, s_2, s_3, s_4\}$ is not a cut of G. It follows that there exists a path Q_2 in $G_2 - C_2$ linking some $x_2 \in X_2$ and $y_2 \in Y_2$ such that no internal vertex of Q_2 is in X_2 or Y_2. Without loss of generality, assume $y_2 \in V(P_5 - x)$. Observe that Q_1 and Q_2 are internally disjoint, and that Q_j is internally disjoint from P_i for every $j \in \{1, 2\}$, $i \in \{1, 2, 3, 4, 5, 6\}$. For $j = 1, 2$, contract edges on Q_j such that Q_j eventually becomes a path of length 1 linking x_j and y_j. Then, contract edges on $P_1, ..., P_6$ such that vertices in $V(P_i) - \{x\}$ are identified as one vertex at s_i for $i = 1, ..., 5$, and all vertices of P_6 are identified as one vertex at s_6. It follows that in the resulting graph, there is an edge between s_6 and every other vertex in S, an edge between s_4 and $\{s_1, s_2, s_3\}$, and an edge between s_5 and $\{s_1, s_2, s_3, s_4\}$. Let L_1 be the resulting graph induced on $V(G_1)$. We see that $e(L_1) - e(G_i) \geq 5 + 2 = 7$, and s_6 is adjacent to every other vertex in S in L_1.

Now, consider $e(G[S]) = 3$ and $E(G[S]) = \{s_1s_2, s_3s_4, s_5s_6\}$. Without loss of generality, assume $I(P_1) \neq \emptyset$. Let $X_1 = I(P_1) \cup I(P_2)$ and $Y_1 = V(\bigcup_{i=3}^{6} P_i - x)$. Similarly to the previous case, since $C_1 = \{x, s_1, s_2\}$ is not a cut of G, there exists a path Q_1 in $G_2 - C_1$ linking some $x_1 \in X_1$ and $y_1 \in Y_1$ such that no internal vertex of Q_1 is in X_1 or Y_1. Without loss of generality, assume $y_1 \in V(P_3 - x)$. Let $X_2 = X_1 \cup V(P_3 \cup P_4 \cup Q_1) - \{x, s_3, s_4\}$, $Y_2 = V(P_5 \cup P_6) - \{x\}$, and $C_2 = \{x, s_1, s_2, s_3, s_4\}$. Since C_2 is not a cut of G, there exists a path Q_2 linking some $x_2 \in X_2$ and $y_2 \in Y_2$ such that no internal vertex of Q_2 is in X_2 or Y_2. Without loss of generality, assume $y_2 \in V(P_5 - x)$. By contracting edges on Q_j such that Q_j becomes a path of length 1 for $j = 1, 2$ and contracting edges on $P_1, ..., P_6$ such that every internal vertex of P_i gets identified to s_i for $i = 1, ..., 6$, and x gets identified to s_6, we can obtain a resulting graph in which there is an edge between s_6 and every other vertex in S, an edge between s_3 and $\{s_1, s_2\}$, and an edge between s_5 and $\{s_1, s_2, s_3, s_4\}$. Let L_1 be the induced subgraph of the resulting graph on $V(G_1)$. It follows that $e(L_1) - e(G_1) \geq 4 + 2 = 6$, and that s_6 is adjacent to every other vertex in S in L_1. $\dashv$

For $i = 1, 2$, define d_i to be the maximum number of edges that can be added to $G[S]$ by contracting edges that have at least one end in $G_{3-i} - G_i$, and let J_i be a graph on $V(G_i)$ obtained from contracting edges that have at least one end in $G_{3-i} - G_i$ such that $e(J_i) = e(G_i) + d_i$. For $i = 1, 2$, define r_i and d'_i as follows: $r_i = 1$ if $J_i \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$, and $r_i = 0$ otherwise; $d'_i = 7$ if $e(G[S]) \leq 1$, and $d'_i = 6$ if $e(G[S]) = 3$. By Claim 2, $d_i \geq d'_i$ for both $i = 1, 2$. By Lemma 3.2.1, it follows that

$$d_1 + d_2 + e(G[S]) \leq 8|S| - 35 + r_1 + r_2 = 13 + r_1 + r_2 \leq 15.$$

Claim 3. If $r_i = 1$ for some $i \in \{1, 2\}$, then $e(G[S]) = 3$, $d_1 = d_2 = 6$, $J_i \cong K_{2,3,3,3,3}$, and $J_i[S] \cong K_{3,3}$.

Proof of Claim 3. Without loss of generality assume $r_1 = 1$ and $J_1 \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$. Observe that every induced subgraph of $K_{2,2,2,2,2,3}$ on 6 vertices has at least 11 edges, and every induced subgraph of $K_{2,3,3,3,3}$ on 6 vertices has at least 9 edges. It follows that $e(G[S]) + d_1 = e(J_1[S]) \geq 9$ and therefore $d_2 \leq 6$. Recall that $e(G[S]) \leq 3$ and $d_i \geq d'_i \geq 6$ for $i = 1, 2$ by Claim 1 and Claim 2. It follows that $e(G[S]) = 3$ and $d_1 = d_2 = 6$, which then implies that $J_1 \cong K_{2,3,3,3,3}$ and $J_1[S] \cong K_{3,3}$. ⊣

Now, if $e(G[S]) \leq 1$, by Claim 3 we know that $r_1 = r_2 = 0$ and therefore $d_1 + d_2 + e(G[S]) \leq 13$, a contradiction to the fact that $d_i \geq d'_i = 7$ for $i = 1, 2$. We may then assume $e(G[S]) = 3$. By Claim 2, $d_i \geq d'_i = 6$ for $i = 1, 2$. Since $d_1 + d_2 + e(G[S]) \leq 13 + r_1 + r_2$, it follows that $d_i = d'_i = 6$ and $r_i = 1$ for both $i = 1, 2$. By Claim 2, for both $i = 1, 2$ there exists a minor L_i of G on $V(G_i)$ obtained by contracting edges of G that have at least one end in $G_{3-i} - G_i$ such that $e(L_i) \geq e(G_i) + 6$, and that there exists a vertex in S adjacent to all other vertices in S in L_i. Since $d_i = d'_i = 6$, we may choose J_i to be equal to L_i for both $i = 1, 2$. This is then a contradiction to Claim 3 that $J_1[S] \cong J_2[S] \cong K_{3,3}$, since no vertex in $K_{3,3}$ is adjacent to all other vertices in S. □

3.3 Bounding Minimum Degree

We proved earlier in Lemma 3.1.1 that $\delta(G) \geq 10$. In this section, we will show $\delta(G) \geq 11$.

Lemma 3.3.1. $\delta(G) \geq 11$.

Proof. For the sake of a contradiction, assume that there exists a vertex $x \in V(G)$ such that $d(x) \leq 10$. By Lemma 3.1.1, $d(x) \geq 10$ and $\delta(N(x)) \geq 8$. It follows that $d(x) = \delta(G) = 10$, and that $N(x)$ contains a subgraph isomorphic to $K_{2,2,2,2,2}$. Let $N(x) = \bigcup_{i=1}^{t}\{s_i, t_i\}$, and assume that $N[s_i] \cap N(x) \supseteq N(x) - \{t_i\}$ and $N[t_i] \cap N(x) \supseteq N(x) - \{s_i\}$ for all $i = 1, 2, 3, 4, 5$.

Note that if there exist at most two non-edges in $N(x)$, then $N[x]$ would have a subgraph isomorphic to $K_7 + K_{2,2}$, which has a K_{10} minor, a contradiction. It follows there exist at least three non-edges in $N(x)$, meaning that $e(N[x]) \leq \binom{11}{2} - 3 = 52 < 53 = 8 \cdot 11 - 35$. Since $e(G) = 8n - 35$, it follows that $|G - N[x]| > 0$. Let (A_1, A_2) be a non-trivial separation of $(G, N[x])$ of minimum order. Let $S = A_1 \cap A_2$. Since $(N[x], V(G) - \{x\})$ is a non-trivial 10-separation of $(G, N[x])$, we know that $|S| \leq 10$. By Lemma 3.2.9, it follows that $7 \leq |S| \leq 10$.

For $i = 1, 2$, let $G_i = G[A_i]$, and let J_i be a minor of G on $V(G_i)$ that can be obtained from G by contracting edges that have at least one end in $G_{3-i} - G_i$ such that $d_i = e(J_i) - e(G_i)$ is maximum. Let r be the number of graphs among J_1 and J_2 that are isomorphic to $K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3,3}$. By Lemma 3.2.1, we know that

$$d_1 + d_2 + e(G[S]) \leq 8|S| - 35 + r.$$

Since $\delta(G) = 10$, by the minimality of $|S|$ and Lemma 3.2.3, if $|S| \leq 8$ then for every $Z \subseteq S$ with $|Z| = 4$ we know $G[(A_2 - A_1) \cup Z]$ has a K_4 minor rooted at Z. By Lemma 3.2.4, $G[S]$ contains an independent set of size 3 or two disjoint non-edges. It follows that $d_1 \geq 2$ if $|S| \leq 8$.

Observe that by the minimality of $|S|$, $x \notin S$ and that there exist $|S|$ disjoint paths $P_1, ..., P_{|S|}$ in $G_1 - \{x\}$ linking $N(x)$ and S. Let $U \subseteq N(x)$ be the subset of vertices that are ends of $P_1, ..., P_{|S|}$ in $N(x)$. We now prove the following claim that $|S| \geq 8$, and that if $|S| = 8$, then $G[U] \cong K_{2,2,2,2}$.

Claim 1. $|S| \geq 8$; and if $|S| = 8$, then $G[U] \cong K_{2,2,2,2}$.

Proof of Claim 1. First assume that $|S| = 7$, for the sake of a contradiction. Recall that this implies $d_1 \geq 2$. With $|S| = 7$, we have $d_1 + d_2 + e(G[S]) \leq 21 + r$. Since $|U| = |S| = 7$, it follows that $G[U]$ has a subgraph isomorphic to $K_{1,2,2,2}$. Furthermore, note that if there exist $s_i, t_i \in U$ that are not adjacent for some $i \in \{1, 2, 3, 4, 5\}$, then s_i, t_i each are adjacent to all four vertices in $N[x] - U$. Therefore, by contracting edges between U and $N[x] - U$ properly and then contracting each path among $P_1, ..., P_7$ to a single vertex, we would obtain a clique on S. It follows that $J_2[S] \cong K_7$ and $d_2 + e(G[S]) = \binom{7}{2} = 21$. Since neither $K_{2,2,2,2,2,3}$ nor $K_{2,3,3,3,3}$ contains a clique of size 7, $J_2 \ncong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$, and therefore $r \leq 1$. It follows that $d_1 \leq 21 + r - (d_2 + e(G[S])) = 21 + r - 21 = r \leq 1$, a contradiction to the fact that $d_1 \geq 2$.

We may now assume that $|S| = 8$, and it suffices to prove $G[U] \cong K_{2,2,2,2}$. With $|S| = 8$, we have $d_1 + d_2 + e(G[S]) \leq 29 + r$. Since $|U| = |S| = 8$, $G[U]$ contains a subgraph isomorphic to $K_{2,2,2,2}$. For the sake of a contradiction, assume $G[U] \ncong K_{2,2,2,2}$.

We next prove that $J_1 \cong K_{2,2,2,2,2,3}$. Note that $G[U] \ncong K_{2,2,2,2}$ implies that $G[U]$ contains a subgraph isomorphic to $K_{1,1,2,2,2}$. Also note that every two non-adjacent vertices in U are adjacent to all vertices in $N[x] - U$, where $|N[x] - U| = 3$ as $|U| = |S| = 8$. Therefore, by contracting edges between U and $N[x] - U$ properly and then contracting each path among $P_1, ..., P_8$ to a single vertex, we would obtain a clique on S. It follows that $J_2[S] \cong K_8$, $d_2 + e(G[S]) = \binom{8}{2} = 28$, and $r \leq 1$. Hence, $d_1 \leq 29 + r - (d_2 + e(G[S])) = r + 1 \leq 2$. Since $d_1 \geq 2$, it follows that $d_1 = 2$, $r = 1$, and therefore $J_1 \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$. Note that $J_1 \supseteq G_1$ which has $N[x]$ as a subgraph. Since $N_G[x]$ contains a subgraph isomorphic to $K_{1,2,2,2,2,2}$ and $K_{2,3,3,3,3}$ does not contain a subgraph isomorphic to

$K_{1,2,2,2,2,2}$, it follows that $J_1 \cong K_{2,2,2,2,2,3}$.

Now, since $J_1 \cong K_{2,2,2,2,2,3}$ does not have a clique of size 7, we know that $N_{J_1}[x] = N_G[x] \cong K_{1,2,2,2,2,2}$. Let the two vertices in $V(G_1) - N[x]$ be u and w. Since u, w are not contained in $N(x)$ and $x \notin S$, x is adjacent to neither u nor w in J_1. It follows that $\{x, u, w\}$ is exactly the 3-island of $J_1 \cong K_{2,2,2,2,2,3}$, meaning that these three vertices are pairwise non-adjacent in both G_1 and J_1. Since $N_{J_1}[x] = N_{G_1}[x]$ and $e(J_1) = e(G_1)+d_1 = e(G_1) + 2$, we know that the two edges in $E(J_1 - G_1)$ each has exactly one end in $\{u, w\}$, and therefore $\{u, w\} \cap S \neq \emptyset$.

Let $S' = S - \{u, w\}$. Assume $\{u, w\} \subseteq S$ for a moment. Note that at most two vertices in S are not adjacent to both u, w, meaning that $|(S' - N_G(u)) \cup (S' - N_G(w))| \leq 2$. Choose distinct vertices $y_1, y_2 \in S'$ such that $(S' - N_G(u)) \cup (S' - N_G(w)) \subseteq \{y_1, y_2\}$. Note that $e(G[\{u, w, y_1, y_2\}]) \leq 3$. On the other hand, since $|S| = 8$ and $d_1 = 2$, by Lemma 3.2.3, $e(G[Z]) \geq 6 - 2 = 4$ for every $Z \subseteq S$ with $|Z| = 4$, a contradiction. We may then assume that $S \cap \{u, w\} = \{u\}$, without loss of generality. Note this then implies that w is adjacent to every vertex in $N(x)$. Since $|S'| = 7$, without loss of generality we assume that $\{s_1, t_1, s_2, t_2\} \subseteq S'$. By Lemma 3.2.3, the graph G_1' obtained from G_1 by adding the edges s_1t_1 and s_2t_2 is a minor of G. Then, in the graph $G_1' - \{u\}$, by contracting the edges ws_3 and s_4s_5 we could obtain a K_{10} minor. This means that $G > G_1' - \{u\} > K_{10}$, a contradiction. $\dashv$

By Claim 1, without loss of generality, we can assume that $\{s_1, t_1, ..., s_4, t_4\} \subseteq U$. For each $i = 1, 2, 3, 4, 5$, if s_i (or t_i) is in U, we let the vertex in S corresponding to it via the paths $P_1, ..., P_{|S|}$ be s_i' (or t_i', respectively). Let $I = \{1, 2, 3, 4\}$ if $|S| \leq 9$ and $I = \{1, 2, 3, 4, 5\}$ if $|S| = 10$.

Observe that if there exist three disjoint paths Q_1, Q_2, Q_3 internally contained in $A_2 - A_1$ such that, for some distinct indices $i, j, k \in I$, Q_1 links s_i', t_i', Q_2 links s_j', t_j', and Q_3 links s_k', t_k', then by contracting the paths $Q_1, Q_2, Q_3, P_1, ..., P_{|S|}$ properly, we can obtain a resulting graph on $N[x]$ that contains a subgraph isomorphic to $K_7 + K_{2,2} > K_{10}$, which is a contradiction. Therefore, it is enough to prove the existence of such three disjoint paths Q_1, Q_2, Q_3, meaning that it suffices to show that $(G[(A_2 - A_1) \cup X], X)$ is 3-linked for some $X = \{s'_i, t'_i, s'_j, t'_j, s'_k, t'_k\}$ where i, j, k are distinct indices in I.

Let $\mathcal{X} = \{X \subseteq S : X = \{s'_i, t'_i, s'_j, t'_j, s'_k, t'_k\}$ where $i, j, k \in I$ are distinct$\}$. For the sake of a contradiction, we may assume that for every $X \in \mathcal{X}$, $(G[(A_2 - A_1) \cup X], X)$ is not 3-linked. Let $S' = \bigcup_{i \in I}\{s'_i, t'_i\} \subseteq S$ and $t = |S - S'|$. Note that $t = 0$ if $|S| = 8$ or 10, and $t = 1$ if $|S| = 9$. We now prove a few inequalities in the next claim.

Claim 2. The following statements are true:

(1) $\binom{|I|-1}{2} e(S', A_2 - A_1) + \binom{|I|}{3} e(G[A_2 - A_1]) \leq \binom{|I|}{3} \cdot (5|A_2 - A_1| + 3)$,

(2) $(10 - t)|A_2 - A_1| \leq e(S', A_2 - A_1) + 2e(G[A_2 - A_1])$,

(3) $(8 - t)|A_2 - A_1| + 1 \leq e(S', A_2 - A_1) + e(G[A_2 - A_1])$.

Proof of Claim 2. Let $X \in \mathcal{X}$. By the minimality of $|S|$, $(G[(A_2 - A_1) \cup X], X)$ does not have separation of order at most 5. Since $(G[(A_2 - A_1) \cup X], X)$ is not 3-linked, by Theorem 2.3.2 we know that $\rho_{G[(A_2 - A_1) \cup X]}(A_2 - A_1) \leq 5|A_2 - A_1| + 3$. Since $X \in \mathcal{X}$ was arbitrary and $|\mathcal{X}| = \binom{|I|}{3}$, we have

$$\sum_{X \in \mathcal{X}} \rho_{G[(A_2 - A_1) \cup X]}(A_2 - A_1) \leq \binom{|I|}{3} \cdot (5|A_2 - A_1| + 3).$$

Since every pair $\{s_i, t_i\} \subseteq S'$ is contained in exactly $\binom{|I|-1}{2}$ sets in $\mathcal{X}$, it follows that

$$\begin{aligned}\binom{|I|-1}{2} e(S', A_2 - A_1) + \binom{|I|}{3} e(G[A_2 - A_1]) &= \sum_{X \in \mathcal{X}} \rho_{G[(A_2 - A_1) \cup X]}(A_2 - A_1) \\ &\leq \binom{|I|}{3} \cdot (5|A_2 - A_1| + 3),\end{aligned}$$

and this proves (1).

Note $t = |S - S'|$. Since $\delta(G) = 10$, every vertex in $A_2 - A_1$ has at least $10 - t$

neighbors in $G[(A_2 - A_1) \cup S']$. It follows that

$$(10-t)|A_2 - A_1| \leq \sum_{v \in A_2 - A_1} d_{G[(A_2-A_1)\cup S']}(v) = e(S', A_2 - A_1) + 2e(G[A_2 - A_1]),$$

and this proves (2).

To see (3), first observe that $e(S, A_2 - A_1) + e(G[A_2 - A_1]) = e(G) - e(G_1)$. Since $G > G_1$, we know $e(G_1) \leq 8|G_1| - 34$. Assume $e(G_1) \geq 8|G_1| - 35$, and note this means that G_1 is isomorphic to some exceptional graph. By Lemma 3.2.4, there exist two distinct pairs of non-adjacent vertices $\{a_1, b_1\}$ and $\{a_2, b_2\}$ in $G[S]$, i.e. $\{a_1, b_1\} \neq \{a_2, b_2\}$ and a_i, b_i are not adjacent in G for $i = 1, 2$. By the minimality of $|S|$, we know that $G > G_1 + \{a_ib_i\}$ for both $i = 1, 2$. On the other hand, by Lemma 2.5.7, there exists at most one pair of non-adjacent vertices in G_1 such that after adding an edge between them, the resulting graph does not have a K_{10} minor. By the minimality of S, it follows that $G > G_1 + \{a_ib_i\} > K_{10}$ for some $i \in \{1, 2\}$, a contradiction. Hence, we conclude that $e(G_1) \leq 8|G_1| - 36$, and therefore

$$e(S, A_2-A_1)+e(G[A_2-A_1]) = e(G)-e(G_1) \geq (8n-35)-(8|G_1|-36) = 8|A_2-A_1|+1.$$

Since $e(S', A_2-A_1)+e(G[A_2-A_1]) = e(S, A_2-A_1)+e(G[A_2-A_1])-e(S-S', A_2-A_1)$, and $e(S - S', A_2 - A_1) \leq |S - S'| \cdot |A_2 - A_1| = t|A_2 - A_1|$, it follows that

$$e(S', A_2 - A_1) + e(G[A_2 - A_1]) \geq (8-t)|A_2 - A_1| + 1,$$

which completes the proof of (3). $\dashv$

Claim 3. $|S| = 10$ and $|A_2 - A_1| \leq 4$.

Proof of Claim 3. We first assume $|S| \leq 9$, which means that $|I| = 4$. By (1) of Claim 2,

$$3e(S', A_2 - A_1) + 4e(G[A_2 - A_1]) \leq 4(5|A_2 - A_1| + 3) = 20|A_2 - A_1| + 12.$$

Observe that

$$\begin{aligned}3e(S', A_2 - A_1) + 4e(G[A_2 - A_1]) =& \big(e(S', A_2 - A_1) + 2e(G[A_2 - A_1])\big) \\ &+ 2\big((e(S', A_2 - A_1) + e(G[A_2 - A_1])\big).\end{aligned}$$

By (2) and (3) of Claim 2, we have

$$(10 - t)|A_2 - A_1| + 2\big((8 - t)|A_2 - A_1| + 1\big) \leq 20|A_2 - A_1| + 12,$$

which can be simplified to $(6 - 3t)|A_2 - A_1| \leq 10$. If $|S| = 8$, then $t = 0$ and thus $|A_2 - A_1| \leq 1$, a contradiction to the fact that $\delta(G) = 10$.

If $|S| = 9$, then $t = 1$ and thus $|A_2 - A_1| \leq 3$. Since $\delta(G) = 10$, $|A_2 - A_1| = 2$ or 3. If $|A_2 - A_1| = 2$, then every vertex in $A_2 - A_1$ is adjacent to all vertices in S and the other vertex in $A_2 - A_1$. Without loss of generality, assume that $s_5 \in U$ and $t_5 \notin U$. By contracting each path P_i to a single vertex for $i = 1, ..., 9$ and contracting the edges xs_1, s_2s_3, and s_4t_5, we would obtain a K_{10} minor, a contradiction. If $|A_2 - A_1| = 3$, let v_1, v_2, v_3 be the three vertices in $A_2 - A_1$. Since $\delta(G) = 10$, v_i is adjacent to at least 7 vertices in S' for $i = 1, 2, 3$. This means that for each $i = 1, 2, 3$, there exists three distinct indices $i^1, i^2, i^3 \in I$ such that v_i is adjacent to s'_{i^j} and t'_{i^j} for all $j = 1, 2, 3$. By relabeling vertices in S', we may assume that v_i is adjacent to both s'_i and t'_i for $i = 1, 2, 3$. By contracting each path P_i to a single vertex for $i = 1, ..., 9$ and contracting each edge $v_js'_j$ for $j = 1, 2, 3$, we would obtain a resulting graph on $N[x]$ that contains a subgraph isomorphic to $K_7 + K_{2,2} > K_{10}$, a contradiction.

We now assume $|S| = 10$, meaning $|I| = 5$, $t = 0$, and $S = S'$. By (1) of Claim 2,

$$6e(S, A_2 - A_1) + 10e(G[A_2 - A_1]) \leq 10(5|A_2 - A_1| + 3) = 50|A_2 - A_1| + 30.$$

Note that

$$6e(S, A_2 - A_1) + 10e(G[A_2 - A_1]) = 4\big(e(S, A_2 - A_1) + 2e(G[A_2 - A_1])\big) + 2\big(e(S, A_2 - A_1) + e(G[A_2 - A_1])\big).$$

By (2) and (3) of Claim 2, it follows that

$$4\big(10|A_2 - A_1|\big) + 2\big(8|A_2 - A_1| + 1\big) \leq 50|A_2 - A_1| + 30.$$

This means that $6|A_2 - A_1| \leq 28$, and therefore $|A_2 - A_1| \leq 4$. ⊣

Claim 3 shows that $|S| = 10$ and $|A_2 - A_1| \leq 4$. Note that we may just choose $A_1 = N[x]$, $A_2 = V(G) - \{x\}$, and $S = N(x)$. It follows that $|G - N[x]| \leq 4$. Let $l = |G - N[x]|$, and let $v_1, ..., v_l$ be the vertices in $G - N[x]$.

If $l = 1$, then the only vertex v_1 in $G - N[x]$ is adjacent to every vertex in $N(x)$ since $\delta(G) = 10$. Observe that $e(N(x)) = e(G) - d(x) - d(v_1) = 8 \cdot 12 - 35 - 10 - 10 = 41$, meaning $N(x) \cong K_{1,1,2,2,2,2}$. It follows that $G \cong K_{1,1,2,2,2,2,2}$, which is an exceptional graph, a contradiction.

If $l = 2$, since $\delta(G) = 10$, we know v_i is adjacent to at least 9 vertices in $N(x)$ for $i = 1, 2$. Without loss of generality, we can assume that v_1, v_2 each are adjacent to s_i and t_i for $i = 1, 2, 3$. Note that if $s_i t_i \in E(G)$ for some $i \in \{1, 2, 3, 4, 5\}$, then by contracting $v_1 s_j$ and $v_2 s_k$ for some distinct $j, k \in \{1, 2, 3\}$ we would obtain a new graph on $N[x]$ that has a subgraph isomorphic to $K_7 + K_{2,2} > K_{10}$, a contradiction. Hence, $s_i t_i \notin E(G)$ for all $i = 1, ..., 5$. It follows that $N[x] \cong K_{1,2,2,2,2,2}$, and therefore $e(N(x), G - N[x]) + e(G[\{v_1, v_2\}]) = e(G) - e(K_{1,2,2,2,2,2}) = 8 \cdot 13 - 35 - 50 = 19$. Since $\delta(G) = 10$, it follows that v_1, v_2 are adjacent and they each are adjacent to exactly 9 vertices in $N(x)$. Without loss of generality, assume s_5 is the unique vertex in $N(x)$ not adjacent to v_1, and that either s_4 or t_5 is the one that is not adjacent to v_2. If s_4, v_2 are not adjacent, then by contracting

xs_1, s_2s_3, and s_4s_5 we could obtain a K_{10} minor, a contradiction. If t_5, v_2 are not adjacent, the $G \cong K_{2,2,2,2,2} + C_5$, which is an exceptional graph, again a contradiction.

We may then assume $l = 3$ or 4. Say s_j and t_j form a *pair* for $j = 1, 2, 3, 4, 5$. Since $d(v_i) \geq 10$ for each i, each v_i has at least $10 - (l - 1) = 11 - l$ neighbors in $N(x)$. Note that $11 - l \geq 7$ since $l \leq 4$, and this means that each v_i is a common neighbor for at least two pairs in $N(x)$. Also note that if some v_i is a common neighbor for 3 pairs in $N(x)$, then, by relabeling the vertices in $N(x)$ and $G - N[x]$, we may assume that v_i is a common neighbor for s_i and t_i for $i = 1, 2, 3$. By contracting v_is_i for $i = 1, 2, 3$, we would then obtain a new graph having a subgraph isomorphic to $K_7 + K_{2,2} > K_{10}$, a contradiction. Therefore, we may assume that every v_i is a common neighbor for exactly two pairs in $N(x)$, meaning that $l = 4$ and every v_i has exactly 7 neighbors in $N(x)$. Since $d(v_i) \geq 10$ for every $i = 1, 2, 3, 4$, it follows that $G - N[x] \cong K_4$. Therefore, $e(N[x]) = e(G) - (e(N(x), G - N[x]) + e(G - N[x])) = (8 \cdot 15 - 35) - (4 \cdot 7 + \binom{4}{2}) = 85 - 32 = 53$, meaning that $N[x] \cong K_7 + K_{2,2} > K_{10}$, a contradiction. $\square$

CHAPTER 4

MAIN TECHNICAL LEMMA

The goal of the entire Chapter 4 is to prove Lemma 4.1.1, the main technical lemma in our proof for Theorem 1.1.5.

4.1 Statements and Proof Outline

We state the main technical lemma and give an outline of its proof in this section.

Lemma 4.1.1. *Let $x \in V(G)$ such that $11 \leq d(x) \leq 15$. Let M be the subset of vertices of $N(x)$ that are not adjacent to all other vertices of $N(x)$, i.e. $M = \{v \in N(x) : vu \notin E(G)$ for some $u \in N(x) - \{v\}\}$. Then, for every component K of $G - N[x]$, there exists some component K' of $G - N[x]$ such that $N(K') \cap M \not\subseteq N(K)$.*

To prove Lemma 4.1.1, for the sake of a contradiction, assume that there exists some $x \in V(G)$ with $11 \leq d(x) \leq 15$ and a component K of $G - N[x]$ such that for every component K' of $G - N[x]$, $N(K') \cap M \subseteq N(K)$ where $M = \{v \in N(x) : vu \notin E(G)$ for some $u \in N(x) - \{v\}\}$. We choose such a pair (x, K) such that $d(x)$ is minimum over all choices.

The rest of our proof for Lemma 4.1.1 can be outlined as follows.

In Section 4.2, we first prove $M \subseteq N(K)$ and $N(x) \not\succ K_8 \cup K_1$ in Lemma 4.2.1 and Lemma 4.2.2. Notice that $11 \leq d(x) \leq 15$ and $\delta(N(x)) \geq 8$ by Lemma 3.1.1. It follows that $N(x)$ is isomorphic to some graph H such that (i) $11 \leq |H| \leq 15$, (ii) $\delta(H) \geq 8$, and (iii) $H \not\succ K_8 \cup K_1$. Note that there are only finitely many graphs satisfying (i)-(iii). In Lemma 4.2.3, we present all edge-minimal graphs satisfying (i)-(iii), which are generated by a computer program. There are precisely 101 such graphs, up to isomorphism, and we call them *problem graphs*. It remains to show that if $N(x)$ has a subgraph isomorphic to some problem graph, then we can contract edges that have at least one end in $G - N[x]$ such that the resulting graph on $N[x]$ has a K_{10} minor. It turns out that if $N(x)$ has a subgraph isomorphic to the three problem graphs $K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$, we would need to spend some more effort to find the desired K_{10} minor; and if $N(x)$ does not have a subgraph isomorphic to those three graphs, a K_{10} minor is relatively easier to be found.

In Section 4.3, we consider the case that $N(x)$ has a subgraph $N' \cong K_{2,3,3,3}$, $K_{3,3}+C_5$, or $K_{4,4,4}$. We first prove that $N(x) = N'$, so $N(x)$ itself is isomorphic to one of the three problem graphs in Lemma 4.3.1. We then prove $|G - N[x]| \geq 3$ in Lemma 4.3.2 and a quite technical result on 2-separations of each component of $G - N[x]$ in Lemma 4.3.4. Then, we show that a K_{10} minor can be found if $G - N[x]$ is 2-connected in Lemma 4.3.5 and Lemma 4.3.6, and that a K_{10} minor can be found if $G - N[x]$ is NOT 2-connected in Lemma 4.3.7.

In Section 4.4, we consider the case that $N(x)$ does NOT have a subgraph isomorphic to $K_{2,3,3,3}$, $K_{3,3}+C_5$, or $K_{4,4,4}$. We use computer programs to verify that every problem graph that is NOT isomorphic to $K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$ satisfies one of the properties (A1) and (A2) in Lemma 4.4.1 and one of the properties (B1)-(B6) in Lemma 4.4.3. Finally, we use properties (A1) and (A2) to show that a K_{10} minor can be found if $G - N[x]$ is 2-connected and has at least two vertices in Lemma 4.4.2, and we use properties (B1)-(B6) to show that a K_{10} minor can be found otherwise in Lemma 4.4.4.

4.2 Problem Graphs

We will prove $M \subseteq N(K)$ in Lemma 4.2.1 and $N(x) \not\geq K_8 \cup K_1$ in Lemma 4.2.2.

Lemma 4.2.1. *$M \subseteq N(K)$.*

Proof. For the sake of a contradiction, assume that $M - N(K) \neq \emptyset$.

We first observe that for every $v \in M - N(K)$, v does not have any neighbor in $G - N[x]$, since otherwise there would exist some component K' of $G - N[x]$ such that $y \in N(K') \cap M$ but $y \notin N(K)$, which is a contradiction to the choice of x and K.

Choose $y \in M - N(K)$ such that $d(y)$ is minimum. Let $M_y = \{v \in N(y) : vu \notin E(G)$ for some $u \in N(y) - \{v\}\}$. By the previous observation, it follows that y has no neighbor in $G - N[x]$ and therefore $N[y] \subseteq N[x]$. Since $y \in M$, y is not adjacent to every vertex in $N(x)$, and it follows that $d(y) < d(x)$. Let J be the component of $G - N[x]$ that contains K. We will complete the proof by considering the following two cases: $N(x) - N[y] \subseteq V(J)$, or $N(x) - N[y] \not\subseteq V(J)$.

Case 1: $N(x) - N[y] \subseteq V(J)$.

Since $d(y) < d(x)$, by the choice of x and K we know that J is not the only component of $G - N[y]$. Let J' be any component of $G - N[y]$ such that $J' \neq J$. Since J' is chosen arbitrarily, it suffices to show a contradiction by proving that $N(J') \cap M \subseteq N(J)$.

Observe that since $N[y] \subseteq N[x]$, $G - N[x]$ is an induced subgraph of $G - N[y]$. Since $N(x) - N[y] \subseteq V(J)$, every vertex in $G - N[y]$ but not in $G - N[x]$ is contained in the component J of $G - N[y]$. It follows that J' itself is also a component of $G - N[x]$. By the choice of x and K, it follows that $N(J') \cap M \subseteq N(K)$. Since $N[y] \subseteq N[x]$, we know that $x \notin M_y$ and therefore $M_y \subseteq M$. It follows that

$$N(J') \cap M_y \subseteq N(J') \cap M \subseteq N(K).$$

Now, observe that since $V(K) \subseteq V(J)$, every neighbor of K is either in J or in $N(J)$, and therefore

$$N(J') \cap M_y \subseteq N(K) = \big(N(K) \cap V(J)\big) \cup \big(N(K) \cap N(J)\big).$$

Notice that $N(J') \cap M_y \subseteq N(y)$ and $N(K) \cap V(J) \subseteq V(J)$ which is disjoint from $N(y)$. It follows that

$$N(J') \cap M_y \subseteq N(K) \cap N(J) \subseteq N(J).$$

Case 2: $N(x) - N[y] \not\subseteq V(J)$.

Let H be a component of $N(x) - N[y] \cup V(J)$. Note that $H \subseteq G - N[y] \cup V(J)$, meaning that H is contained in some component of $G - N[y]$ disjoint from J.

We will first show that H itself is a component of $G - N[y]$. For every $z \in V(H)$, note that $z \in N(x) - N[y]$, meaning that $z \in M$ and therefore $z \in M - N(K)$. By the observation at the beginning of this proof, it follows that z has no neighbor in $G - N[x]$. It follows that $N(H) \subseteq N[x]$. Let J' be the component of $G - N[y]$ that contains H. Note that if $V(J' - H) \neq \emptyset$, since $N[y] \subseteq N[x]$ then every vertex in $J' - H$ must be contained in some component K' of $G - N[x]$ such that $K' \neq K$, a contradiction to the fact that $N(H) \subseteq N[x]$. Hence, we conclude that H itself is a component of $G - N[y]$.

Now, since $V(K) \subseteq V(J)$ and J, H are disjoint components of $G - N[y]$, it follows that for every $z \in V(H)$ we have $z \notin N(K)$ and therefore $z \in M - N(K)$. By the choice of y, $d(z) \geq d(y)$ for every $z \in V(H)$. Let $t = |H|$. Assume $t = 1$ for a moment, and let z^* be the unique vertex in H. It follows that $N(H) = N(z^*) = N(y)$. Since $d(y) < d(x)$, this means that (y, H) contradicts the choice of (x, K). Therefore, $t \geq 2$. On the other hand, since $H \subseteq N(x) - N[y] = N(x) - (N[y] - \{x\})$, we know that $t \leq d(x) - d(y) \leq 15 - 11 = 4$. Hence, we have $2 \leq t \leq 4$.

Now, let $L = G[N[y] \cup V(H)]$. Note $|L| = d(y) + t + 1$, and that

$$e(L) = d(y) + e(N(y)) + e(N(y), V(H)) + e(H).$$

Note that $x \in N(y)$ is adjacent to all other vertices in $N(y)$. Since $\delta(N(y)) \geq 8$, we know $\delta(N(y) - \{x\}) \geq 7$. It follows that

$$e(N(y)) = e(\{x\}, N(y) - \{x\}) + e(N(y) - \{x\}) \geq d(y) - 1 + \frac{1}{2} \cdot 7(d(y) - 1) = \frac{9}{2}d(y) - \frac{9}{2}.$$

For every $z \in V(H)$, since $N(z) \subseteq V(H) \cup N(y)$ and $d(z) \geq d(y)$, we have

$$e(N(y), V(H)) + 2e(H) = \sum_{z \in V(H)} d(z) \geq \sum_{z \in V(H)} d(y) = td(y),$$

and therefore $e(N(y), V(H)) + e(H) \geq td(y) - e(H) \geq td(y) - \binom{t}{2}$. Hence, we have

$$e(L) \geq d(y) + \frac{9}{2}d(y) - \frac{9}{2} + td(y) - \binom{t}{2}$$

$$= (\frac{11}{2} + t)d(y) - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{9}{2}$$

$$= 8(d(y) + t + 1) + (t - \frac{5}{2})d(y) - \frac{1}{2}t^2 - \frac{15}{2}t - \frac{25}{2}$$

$$= 8|L| + (t - \frac{5}{2})d(y) - \frac{1}{2}t^2 - \frac{15}{2}t - \frac{25}{2}.$$

Let $G_1 = G[V(J) \cup N(J)]$, $S = N(J)$, and $G_2 = G - V(J)$. Observe that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = G[S]$, and $L \subseteq G_2$. Let d_2 be the maximum number of edges that can be added to $G[S]$ by contracting edges that have at least one edge in $V(G_1 - G_2)$. Let L' be a graph with $V(L') = V(L)$ that can be obtained from G by deleting vertices in $V(G_2 - L)$ and contracting edges that have at least one end in J such that $e(L') = e(L) + d_2$. We then have

$$e(L') = e(L) + d_2 \geq 8|L'| + (t - \frac{5}{2})d(y) - \frac{1}{2}t^2 - \frac{15}{2}t - \frac{25}{2} + d_2.$$

If $t = 3$ or 4, then $t - \frac{5}{2} > 0$. Since $d(y) \geq 11$,

$$e(L') \geq 8|L'| + 11(t - \frac{5}{2}) - \frac{1}{2}t^2 - \frac{15}{2}t - \frac{25}{2} + d_2 = 8|L'| - \frac{1}{2}(t - \frac{7}{2})^2 - \frac{271}{8} + d_2 = 8|L'| - 34 + d_2.$$

If $t = 2$, note $N(y) \subseteq N[x] - \{y\} - H$, meaning that $d(y) \leq d(x) - t \leq 15 - 2 = 13$. Then,

$$e(L') \geq 8|L'| - \frac{1}{2}13 - \frac{1}{2}2^2 - \frac{15}{2}2 - \frac{25}{2} + d_2 = 8|L'| - 36 + d_2.$$

Note that $|L'| = d(y) + t + 1 \geq 11 + 2 + 1 = 14 > 8$. If $e(L') \geq 8|L'| - 35$, then by induction we know that L' is isomorphic to some exceptional graph. If $e(L') \geq 8|L'| - 35$, then by induction we know that L' is isomorphic to some exceptional graph. This is not possible, because x is adjacent to all other vertices in L', and there is no exceptional graph on at least 14 vertices in which there is a vertex adjacent to all other vertices. It follows that $e(L') \leq 8|L'| - 36$, which could happen when $t = 2$ and $d_2 = 0$. However, by (3) of Lemma 3.2.4, the connectivity of $G_1 - G_2 = J$ implies that by contracting all of J to a vertex in S that has the minimum degree inside $G[S]$ we would obtain at least one extra edge on $G[S]$. It follows that $d_2 \geq 1$, a contradiction. □

Lemma 4.2.2. $N(x) \not> K_8 \cup K_1$.

Proof. For the sake of a contradiction, assume $N(x) > K_8 \cup K_1$. Choose $y \in N(x)$ such that $N(x) - \{y\} > K_8$. We may assume that y is not adjacent to some vertex in $N(x)$, since otherwise $N(x) > K_9$ which then implies $G > K_{10}$, a contradiction. It follows that $y \in M \subseteq N(K)$. By contracting all vertices in K to y, in the resulting graph on $N(x)$, y would be adjacent to all other vertices in M, meaning that y would be adjacent to all other vertices in $N(x)$. It follows that the new graph on $N(x)$ has a K_9 minor and therefore $G > K_{10}$, a contradiction. □

Lemma 4.2.3 (computer-assisted)**.** $N(x)$ *has a subgraph that is isomorphic to one of the 101 graphs listed in Appendix.*

We call each one of these 101 graphs a *problem graph*.

4.3 $K_{2,3,3,3}$, $K_{3,3} + C_5$, and $K_{4,4,4}$

Now, assume that $N(x)$ contains a subgraph N' isomorphic to $K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$ such that $|N(x)| = |N'|$.

We first define three types of minors of G. For a minor H of $G - x$ rooted at $N(x)$, say H is a *minor of G of type I* if there exist distinct vertices $s_1, s_2, s_3, t_1, t_2, t_3 \in N(x)$ such that for some distinct islands C_1, C_2, C_3 of N', $s_i, t_i \in C_i$ and $s_i t_i \in E(H) - E(N')$ for $i = 1, 2, 3$; say H is a *minor of G of type II* if there are distinct vertices x_1, x_2, x_3, y_1, y_2 such that $x_1, x_2, x_3 \in C_1$ and $y_1, y_2 \in C_2$ for some distinct islands C_1, C_2 of N' and $x_1x_2, x_1x_3, x_2x_3, y_1y_2 \in E(H)-E(N')$; say H is a *minor of G of type III* if $N' \cong K_{3,3}+C_5$ and there are distinct vertices $x_1, x_2, y_1, y_2, y_3, y_4 \in N(x)$ such that x_1, x_2 are contained in a 3-island of N', y_i for $i = 1, 2, 3, 4$ are in the 5-island of N', and that $x_1x_2, y_1y_2, y_2y_3, y_3y_4 \in E(H) - E(N')$.

Observe that since $N' \cong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$, if G has a minor H of one of the three types defined above, then $G > K_{10}$, a contradiction. Hence, G does not have a minor of type I, type II, or type III.

Lemma 4.3.1. $N(x) \cong N'$.

Proof. For the sake of a contradiction, assume that $N(x) \neq N'$.

We first prove that $G - N[x] \neq \emptyset$. Let $E' = E(N(x)) - E(N')$. Note that the end vertices of edges in E' are in at most two islands of N', since otherwise G would have a minor of type I, a contradiction. If $G - N[x] = \emptyset$, then $|E'| = e(G) - (d(x) + e(N')) = 8 \cdot |G| - 35 - (d(x) + e(N'))$. This means that if $N' \cong K_{2,3,3,3}$, then $|E'| = (8 \cdot 12 - 35) - (11 + 45) = 5$; if $N' \cong K_{3,3} + C_5$, then $|E'| = (8 \cdot 12 - 35) - (11 + 44) = 6$; and if $N' \cong K_{4,4,4}$, then $|E'| = (8 \cdot 13 - 35) - (12 + 48) = 9$. One can then check that in all cases, G would have a minor of type II or type III, a contradiction. Hence, $G - N[x] \neq \emptyset$.

Now, let $s_1, t_1 \in N(x)$ be such that $s_1t_1 \in E(N') - E(G)$, and let C_1 be the island of N' containing s_1, t_1. Recall that K is a component of $G - N[x]$ such that $M \subseteq N(K)$ where $M = \{v \in N(x) : vu \notin E(G) \text{ for some } u \in N(x) - \{v\}\}$.

We next show that $N' - C_1 = N(x) - C_1$. For the sake of a contradiction, assume that there exist $s_2, t_2 \in N(x)$ and an island C_2 of $N' - C_1$ such that $s_2, t_2 \in C_2$ and $s_2t_2 \in E(N(x)) - E(N')$. Then, we see that $N' - C_1 \cup C_2 \cong N(x) - C_1 \cup C_2$, since otherwise G would have a minor of type I, a contradiction. Since N' has at least three islands in all cases, there exist two vertices $s_3, t_3 \in N(x) - C_1 \cup C_2$ that are not adjacent to each other in both N' and $N(x)$. It follows that $s_3, t_3 \in M \subseteq N(K)$. By contracting all of K to one of s_3 and t_3, we would then obtain a resulting graph on $N(x)$ that has edges s_1t_1, s_2t_2, s_3t_3, meaning that G has a minor of type I, a contradiction. Therefore, $N' - C_1 = N(x) - C_1$. Observe that in all cases, every vertex in N' has some non-neighbor in it. It follows that $N(x) - C_1 \subseteq M \subseteq N(K)$.

In the rest of the proof, we consider the case $|K| \geq 2$ and the case that $|K| = 1$.

Case 1: $|K| \geq 2$.

Observe that since $N' \cong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$, in all cases there exists a subset of vertices $X = \{x_1, x_2, x_3\} \subseteq N(x) - C_1$ such that $N'[X] = G[X] \cong \overline{K_3}$. Let C_2 be the island of N' containing X, and note that $X \subseteq C_2 \subseteq N(K)$. Note that $G[V(K) \cup X]$ is connected, and it does not have a K_3 minor rooted at X, since otherwise G would have a minor of type II due to the edge s_1t_1, a contradiction. By Lemma 2.2.1, $G[V(K) \cup X]$ has a cut vertex w, and that there are components J_1, J_2, J_3 of $G[V(K) \cup X] - \{w\}$ such that $x_i \in V(J_i)$ for $i = 1, 2, 3$. Notice that $w \in V(K)$, as K itself is a connected subgraph of G.

Since $|K| \geq 2$, $K - \{w\} \neq \emptyset$. Without loss of generality, assume that either $J_1 - \{x_1x\} \neq \emptyset$ or $K' = G[V(K) \cup X] - V(J_1 \cup J_2 \cup J_3) \cup \{w\} \neq \emptyset$. Let L be a non-trivial component of $J_1 - \{x_1\}$ if $J_1 - \{x_1x\} \neq \emptyset$, and let L be a component of K' if $K' \neq \emptyset$. In both cases, notice that L is a non-trivial subgraph of $K - V(J_2 \cup J_3) \cup \{w, x_1\}$ such that $N(L) \cap (V(G) - N[x]) \subseteq \{w\}$, meaning that L has at most one neighbor in $G - N[x]$. Since G is 7-connected, L has at least six neighbors in $N(x)$. Observe that L does not have non-adjacent neighbors in some island C_3 of $N(x) - C_1 \cup C_2$, since otherwise we could just contract all of $V(J_2 \cup J_3 - \{x_3\}) \cup \{w\}$ to x_2 and contract all of L to one of its non-adjacent neighbors in C_3 to obtain a minor of G of type I, a contradiction. It follows that

$$|N(L) \cap C_1| \geq 6 - |N(L) \cap C_2| - \omega(N' - C_1 \cup C_2).$$

If $N' \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, then $C_2 = X$ and $N(L) \cap C_2 = N(L) \cap X \subseteq \{x_1\}$, meaning that $|N(L) \cap C_2| \leq 1$. If C_1 is an independent set, then one can observe that in all cases we have $\omega(N' - C_1 \cup C_2) = 2$ and thus $|N(L) \cap C_1| \geq 6-1-2 = 3$. It follows that C_1 is an independent set of size 3 and $C_1 \subseteq N(L)$. Then, by contracting all of L to the unique vertex in $C_1 - \{s_1, t_1\}$ and contracting all of $V(J_2 \cup J_3 - \{x_3\}) \cup \{w\}$ to x_2, we then obtain a minor of G of type II, a contradiction. It follows that C_1 is not an independent set, and this means that $N' = K_{3,3} + C_5$ and $N'[C_1]$ is the 5-cycle. Note now $N' - C_1 \cup C_2$ is simply an independent set of size 3, so $\omega(N' - C_1 \cup C_2) = 1$. Therefore, $|N(L) \cap C_1| \geq 4$. Without loss of generality, we can assume that $s_1, s_3, t_3 \in C_1$ are all neighbors of L where s_3, t_3 are in the positions such that $s_1s_3, s_3t_3 \notin E(N')$. Then, by contracting all of L to s_3 we can then obtain a resulting graph that includes edges s_1t_1, s_1s_3, s_3t_3. By further contracting all of $V(J_2 \cup J_3 - \{x_3\}) \cup \{w\}$ to x_2, we would then obtain a minor of G of type III, a contradiction.

We may then assume that $N' \cong K_{4,4,4}$. In this case, every island of N' is an independent set of size 4, and thus $|N(L) \cap C_2| \leq |\{x_1\} \cup (C_2 - X)| = 2$ and $\omega(N' - C_1 \cup C_2) = 1$. It follows that $|N(L) \cap C_1| \geq 6 - 2 - 1 = 3$. Note that if L has all four vertices in C_1 as its neighbors, then by contracting all of L to one vertex in $C_1 - \{s_1, t_1\}$, we would then obtain a clique of size 3 in C_1, and this implies that G has a minor of type II due to J_2, J_3, and w. It follows that L has exactly three neighbors in C_1, two neighbors in C_2, one neighbor in $N(x) - C_1 \cup C_2$, and one neighbor in $G - N[x]$, and this means that $|N(L)| = 7$. By Lemma 3.2.3, it follows that $G[V(L) \cup (N(L) \cap C_1)]$ has a K_3 minor rooted at $N(L) \cap C_1$. Therefore, G has a minor of type II due to J_2, J_3, and w, a contradiction.

Case 2: $|K| = 1$.

Case 2.1: $G - N[x]$ is disconnected.

Let K' be a component of $G - N[x]$ such that $K \neq K'$. Assume for a moment that there exists an island C_2 of $N(x) - C_1$ such that K' has some non-adjacent neighbors $s_2, t_2 \in C_2$. Since N' has at least three islands in all cases, there exists an island C_3

of $N' - C_1 \cup C_2$. Let $s_3, t_3 \in C_3$ be two non-adjacent vertices, and note this means that $\{s_3, t_3\} \subseteq M \subseteq N(K)$. Then, by contracting all of K' to s_2 and contracting all of K to s_3, we would then obtain a minor of G of type I, a contradiction. Therefore, $N(K') \cap (N(x) - C_1)$ is a clique. One can then check that unless $N' \cong K_{3,3} + C_5$ and C_1 is the 5-cycle in it, $|N(K')| \leq |C_1| + \omega(N(x) - C_1) \leq 6$, which is a contradiction to the 7-connectivity of G. It follows that $N' \cong K_{3,3} + C_5$ and $N'[C_1]$ is a 5-cycle. Due to the 7-connectivity of G, K' has exactly one neighbor in each 3-island of N' and has all five vertices in C_1 as its neighbors. Let $s_2, t_2 \in C_1 - \{s_1, t_1\}$ be such that $s_1s_2, s_2t_2 \notin E(N')$. Then, by contracting all of K' to s_2 and contracting all of K to any vertex in $N(x) - C_1$, we would then obtain a minor of type III, a contradiction.

Case 2.2: $G - N[x]$ is connected.

Since $|K| = 1$, let y be the unique vertex in K, and let $t = e(G[C_1]) - e(N'[C_1]))$. We then have $8n - 35 = e(G) = d(x) + d(y) + e(N') + t$, and therefore

$$t = (8n - 35) - (d(x) + d(y) + e(N')).$$

If $N' \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, then $n = 13$ and $d(x) = d(y) = 11$ since $\delta(G) \geq 11$. It follows that $t = 8 \cdot 13 - 35 - (11 + e(N')) - 11 = 47 - e(N')$. If $N' \cong K_{2,3,3,3}$, then $e(N') = 45$ and therefore $t = 2$. This implies that C_1 is an 3-island of N', and that $G[C_1]$ is a path of length 2. It follows that $G \cong K_{1,2,2,2,3,3}$, which is an exceptional graph, a contradiction. If $N' \cong K_{3,3} + C_5$, then $e(N') = 44$ and therefore $t = 3$. If C_1 is a 3-island, then $G \cong K_3 + K_{2,3} + C_5 > K_{10}$, a contradiction. If $N'[C_1] \cong C_5$, note $G[C_1]$ is either a path of length 3 or a disjoint union of an edge and a path of length 2. It follows that $G \cong K_{1,1,2,3,3} + \overline{P_3}$ or $K_{1,2,2,2,3,3}$. This is a contradiction, since $K_{1,1,2,3,3} + \overline{P_3} > K_{10}$ and $K_{1,2,2,2,3,3}$ is an exceptional graph.

If $N' \cong K_{4,4,4}$, then $n = 14$, $d(x) = 12$, $d(y) \leq 12$, and $e(N') = 48$. It follows that $t \geq (8 \cdot 14 - 35) - 12 - 48 - 12 = 5$. Since every island of N' is a 4-island, $G[C_1]$

has at least five edges on 4 vertices, meaning that there is a clique of size 3 on C_1 in G. Note that $V(K) = \{y\}$ has all vertices in $N(x) - C_1$ as its neighbors. By contracting an edge between y and any vertex in $N(x) - C_1$, we would then obtain a minor of type II, a contradiction. $\square$

Lemma 4.3.2. $|G - N[x]| \geq 3$.

Proof. For the sake of a contradiction, assume $|G - N[x]| \leq 2$. By Lemma 4.3.1, we know that $N(x) = N' \cong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$. Notice that if $N(x) \cong K_{4,4,4}$, then every two adjacent vertices in $N(x)$ have exactly five common neighbors in $G - N[x]$, so they must have at least three common neighbors in $G - N[x]$, a contradiction to the assumption that $|G - N[x]| \leq 2$. Hence, either $N(x) \cong K_{2,3,3,3}$ or $N(x) \cong K_{3,3} + C_5$.

If $N(x) \cong K_{2,3,3,3}$, then observe that every two adjacent vertices each contained in some 3-island of $N(x)$ have exactly 6 common neighbors in $G - N[x]$ and therefore have at least two common neighbors in $G - N[x]$. It follows that $|G - N[x]| = 2$, and that every vertex in a 3-island of $N(x)$ is adjacent to both vertices in $G - N[x]$. Let $V(G) - N[x] = \{a, b\}$. Note now $|G| = 14$, and

$$d(a) + d(b) - e(G[\{a, b\}]) = (8 \cdot 14 - 35) - e(K_{1,2,3,3,3}) = 21.$$

Since $\delta(G) \geq 11$, $ab \in E(G)$ and a, b each have exactly 10 neighbors in $N(x)$. Recall that every vertex in a 3-island of $N(x)$ is adjacent to both a and b. Since $\delta(G) \geq 11$ and every vertex in a 3-island of $N(x)$ has exactly 10 neighbors in $N[x]$, it follows that the set of edges between $\{a, b\}$ and the 2-island of $N(x)$ is precisely a perfect matching. It follows that $G \cong K_{3,3,3} + C_5$, which is an exceptional graph, a contradiction.

If $N(x) \cong K_{3,3} + C_5$, then observe that every two vertices from distinct islands of $N(x)$ have exactly six common neighbors in $N[x]$ and thus have at least two common neighbors in $G - N[x]$. It follows that $|G - N[x]| = 2$, $|G| = 14$, and every vertex in $G - N[x]$ is adjacent to all vertices in $N(x)$. Since $e(G) = 8 \cdot 14 - 35$, one can then check that there is no edge between the two vertices in $G - N[x]$ and thus $G \cong K_{3,3,3} + C_5$, again a contradiction since $K_{3,3,3} + C_5$ is an exceptional graph. □

If $N(x) \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, let T_1, T_2 be two distinct 3-islands of $N(x)$; if $N(x) \cong K_{4,4,4}$, let T_1, T_2 be two disjoint independent sets of size 3 each in a 4-island of $N(x)$. In all cases, let $T_1 = \{x_1, x_2, x_3\}$ and $T_2 = \{y_1, y_2, y_3\}$.

Lemma 4.3.3. *In all cases, the following statements are true:*

(1) There is no clique of size 5 in $N(x)$.

(2) For all $i, j \in \{1, 2, 3\}$, x_i, y_j have at least two common neighbors in $G - N[x]$.

Proof. (1) is simply true, as there is no clique of size 5 in $K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$. To see (2) is true, note that for any $i, j \in \{1, 2, 3\}$, $x_i y_j \in E(G)$ and they have at least eight common neighbors in G by Lemma 3.1.1. One can observe that x_i, y_j have at most five common neighbors in $N(x)$ in all cases, due to the construction of T_1, T_2. It follows that x_i, y_j have at most six common neighbors in $N(x)$ and therefore at least two common neighbors in $G - N[x]$. □

Let $H \subseteq G$ be a subgraph, and let $S \subseteq V(H)$ be a subset of vertices. Say a vertex $v \in V(G)$ is *associated with S with respect to H* if there is a path P linking v and some vertex $u \in S$ such that P is otherwise disjoint from H.

4.3.1 Proof of Lemma 4.3.4

Lemma 4.3.4. *Let (A, B) be a 2-separation of a component of $G - N[x]$ such that $A \cap B = \{a, b\}$, $B - A \neq \emptyset$, and there is a path linking a, b in $G[B]$. If no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$, then there exist two disjoint paths in $G[B]$ such that one links a, b and the other one links a neighbor of y_1 and a neighbor of y_2 for some non-adjacent vertices $y_1, y_2 \in N(x) \cap N(B - A)$.*

Proof. Say a 2-separation (A', B') of $G[A \cup B]$ satisfies property $\mathcal{P}$ if there exist two disjoint paths in $G[B']$ such that one links a' and b' and the other one links a neighbor of y_1' and a neighbor of y_2' for some $y_1', y_2' \in N(x) \cap N(B' - A')$ such that $y_1'y_2' \notin E(G)$. For the sake of a contradiction, assume that (A, B) fails property $\mathcal{P}$ such that $|B|$ is minimum among all 2-separations of $G[A, B]$ that fail property $\mathcal{P}$. With these assumptions, we make the following claim.

Claim 1. The following statements are true:

(1) $N(x)$ does not contain a clique of size more than 4.

(2) For every $v \in B - A$, $|N(v) \cap N(x)| \leq 4$ and $d_{G[B]}(v) \geq 7$.

(3) For any $a - b$ path P in $G[B]$, there is no ≤ 2-separation (B_1, B_2) of $G[B]$ such that $V(P) \subseteq B_1$ and $B_2 - B_1 \neq \emptyset$.

(4) There is no cut vertex of $G[B]$ that separates a and b.

Proof of Claim 1. (1) is simply true because $N(x) \cong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$ by Lemma 4.3.1, and none of these three graphs has a clique of size more than 4. Then, (2) and (3) follow immediately from (1), due to the facts that $\delta(G) \geq 11$, G is 7-connected, and that no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$.

To see (4) is true, for the sake of a contradiction, assume that there is a cut vertex $w \in B - A$ of $G[B]$ that separates a, b. Let B_1, B_2 be the components of $G[B] - \{w\}$ such that $a \in V(B_1)$ and $b \in V(B_2)$. Since there is a path linking a, b in $G[B]$, there exists a path P_a lining a, w in $G[B_1 \cup \{w\}]$ and a path P_b linking b, w in $G[B_2 \cup \{w\}]$. By the minimality of $|B|$ when choosing (A, B), we know that $B = V(B_1 \cup B_2) \cup \{w\}$. By (2), w has at least 7 neighbors in B and thus one of $B_1 - \{a\}$ and $B_2 - \{b\}$ is non-empty. Without loss of generality, say $B_1 - \{a\} \neq \emptyset$. Let $A' = A \cup B - (B_1 - \{a\})$ and $B' = B_1 \cup \{w\}$. Notice that (A', B') is a 2-separation of $G[A \cup B]$ such that $B' - A' \neq \emptyset$ and there is a path linking the two vertices a, w in $A' \cup B'$. Then by the minimality of $|B|$ when choosing (A, B), it follows that (A', B') satisfies property $\mathcal{P}$, meaning that there exist two disjoint paths in $G[B']$ such that one links a, w and the other one links a neighbor of y_1' and a neighbor of y_2'

for some non-adjacent vertices $y_1', y_2' \in N(x) \cap N(B' - A')$. Then, by extending the $a - w$ path along P_b to make it an $a - b$ path, we would then have a path linking a, b and a path linking y_1', y_2' that are disjoint from each other. This means that (A, B) satisfies property $\mathcal{P}$, a contradiction. It follows that $B_1 = \{a\}$ and therefor $B_2 = \{b\}$ by symmetry, meaning that $B - A = \{w\}$. Since $\delta(G) \geq 11$, w has at least 9 neighbors in $N(x)$, a contradiction to (2). ⊣

Since G is 7-connected, $B - A$ has at least 5 neighbors in N'. By (1) of Claim 1, there exist two non-adjacent vertices $v_1, v_2 \in N(x) \cap N(B - A)$. Let u_1, u_2 be neighbors of v_1, v_2 in $B - A$, respectively. Note that $u_1 \neq u_2$, as no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$.

Claim 2. There exist two internally disjoint $a-b$ paths L_1, L_2 in $G[B]$ such that $u_1 \in V(L_1)$ and $u_2 \in V(L_2)$.

Proof of Claim 2. By (4) of Claim 1, there exist two disjoint internally disjoint $a-b$ paths in $G[B]$. Let them be Q_1 and Q_2. Observe that if u_1, u_2 are both included in Q_1, then $u_1Q_1u_2$ would be a $u_1 - u_2$ path that is disjoint from the $a - b$ path Q_2, which implies property $\mathcal{P}$, a contradiction. By symmetry, it follows that $|V(Q_i \cap \{u_1, u_2\}| \leq 1$ for $i = 1, 2$, so we may assume that one of u_1, u_2 is not included in $V(Q_1 \cup Q_2)$.

Next, we show that we may assume that one of Q_1, Q_2 goes through u_1. To see it, assume that $u_1 \notin V(Q_1 \cup Q_2)$. By (3) of Claim 1, there exist three paths S_1, S_2, S_3 linking u_1 and $V(Q_1 \cup Q_2)$ in $G[B]$ that are pairwise disjoint except for u_1. Let s_i be the end of S_i in $V(Q_1 \cup Q_2)$ for $i = 1, 2, 3$. Notice that, without loss of generality, we can assume that s_1, s_2 are both on Q_1, and that Q_1 goes through a, s_1, s_2, b in order. (It is possible that $s_i = a$ or b for $i = 1, 2$, but it does not matter.) Replace Q_1 with the path $aQ_1s_1 \cup S_1 \cup S_2 \cup s_2Q_1b$, and it follows that Q_1 now goes through u_1.

Now, since $u_1 \in V(Q_1)$ and one of u_1, u_2 is not included in $V(Q_1 \cup Q_2)$, it follows that $u_2 \notin V(Q_1 \cup Q_2)$. Again by (4) of Claim 1, there exist three paths R_1, R_2, R_3 linking u_2

and $V(Q_1 \cup Q_2)$ in $G[B]$ that are pairwise disjoint except for u_2. Let r_i be the end of R_i in $V(Q_1 \cup Q_2)$ for $i = 1, 2, 3$. If $r_i \in V(Q_1) - \{a, b\}$ for some $i \in \{1, 2, 3\}$, then we would have the $u_1 - u_2$ path $R_i \cup r_i Q_1 u_1$ being disjoint from the $a - b$ path Q_2, implying property $\mathcal{P}$, a contradiction. It follows that $r_i \in V(Q_2)$ for $i = 1, 2, 3$. Without loss of generality, say Q_2 goes through a, r_1, r_2, b in order (possible that $r_1 = a$ or $r_2 = b$). Then, the $a - b$ paths $L_1 = Q_1$ and $L_2 = aQ_2r_1 \cup R_1 \cup R_2 \cup r_2Q_2b$ are as desired. $\dashv$

Claim 3. There exists a non-trivial 3-separation (D, E) of $G[B]$ such that

(1) $a, b \in D$ and $N(v_i) \cap B \subseteq D$ for $i = 1, 2$,

(2) $G[E]$ and $G[E - D]$ are both connected, and

(3) there is no non-trivial $\leq$ 3-separation of $(G[E], D \cap E)$.

Proof of Claim 3. We first prove that $G[B]$ is not a planar graph. By (2) of Claim 1, $d_{G[B]}(v) \geq 11 - 4 = 7$ for all $v \in B - A$, and it follows that

$$e(G[B]) \geq \frac{7}{2}|B - A| + \frac{1}{2}\big(d_{G[B]}(a) + d_{G[B]}(b)\big).$$

If $G[B]$ is planar, then

$$\frac{7}{2}|B - A| + \frac{1}{2}\big(d_{G[B]}(a) + d_{G[B]}(b)\big) \leq e(G[B]) \leq 3|B| - 6 = 3|B - A|,$$

which means that $\frac{1}{2}|B - A| + \frac{1}{2}\big(d_{G[B]}(a) + d_{G[B]}(b)\big) \leq 0$, a contradiction.

Let H be the multigraph obtained from $G[B \cup \{v_1, v_2\}]$ by adding the four edges an_1, n_1b, bn_2, n_2a and eliminating the edge v_1v_2 if $v_1v_2 \in E(G)$. Since $G[B]$ is a subgraph of H, H is not planar either. Observe that any $v_1 - v_2$ path P in H such that $a, b \notin V(P)$ has a subgraph P' that links a neighbor of v_1 and a neighbor of v_2. Since (A, B) does not satisfy property $\mathcal{P}$, it follows that there do not exist two disjoint paths in H such that one links v_1 and v_2 and the other one links a and b. Let C be the cycle in H that goes through a, v_1, b, v_2 in order. By Theorem 2.3.1, there exists a non-trivial C-reduction of H that can be drawn on the plane such that C bounds the infinite region. This means that there exists a non-trivial ≤ 3-separation (D', E') of $(H, V(C))$. Choose (D', E') such that $|E'|$ is minimum over all such ≤ 3-separations.

Let $D = D' \cap B$ and $E = E'$. It follows that (D, E) is a ≤ 3-separation of $G[B]$ such that $a, b \in D$ and $E - D \neq \emptyset$. (2) and (3) can then be simply implied by the minimality of $|E'|$ when choosing (D', E'). Observe that if $N(v_i) \cap B \subseteq D$ for $i = 1, 2$, then we know $D - E \neq \emptyset$ since a, b, u_1, u_2 are four distinct vertices, and therefore (D, E) is a non-trivial separation of $G[B]$. Hence, in the remaining of the proof, it suffices to show that $|D \cap E| = 3$ and $N(v_i) \cap B \subseteq D$ for $i = 1, 2$.

To see $|D \cap E| = 3$, assume $|D \cap E| \leq 2$ for the sake of a contradiction. Since $a, b \in D$, it follows that at least one of the $a - b$ paths L_1 and L_2 is included in D completely, a contradiction to (3) of Claim 1. Since $D \cap E = (D' \cap E') \backslash \{v_1, v_2\}$ and $|D' \cap E'| \leq 3$, it follows that $|D' \cap E'| = 3$ and $v_1, v_2 \in D' - E'$. Therefore, no neighbor of v_1 or v_2 is included in $E' - D'$. Since $E - D = E' - D'$, it follows that $N(v_i) \cap B \subseteq D$ for $i = 1, 2$. $\dashv$

The next goal is to prove that $V(L_i) \cap (E - D) \neq \emptyset$ for $i = 1, 2$ in Claim 5. To prove it, we need to introduce a few definitions first and make some observations first in Claim 4.

Let $Y = \{y \in N(x) : v_1 y, v_2 y \in E(G)\} \subseteq N(x)$. We will need to consider explicit positions of v_1, v_2 in N', and observe that it suffices for us to consider the following five cases, up to isomorphism:

Case 1: $N(x) \cong K_{2,3,3,3}$ and v_1, v_2 are both contained in a 3-island.

Case 2: $N(x) \cong K_{2,3,3,3}$ and v_1, v_2 are both contained in the 2-island.

Case 3: $N(x) \cong K_{3,3} + C_5$ and v_1, v_2 are both contained in a 3-island.

Case 4: $N(x) \cong K_{3,3} + C_5$ and v_1, v_2 are both contained in the 5-island.

Case 5: $N(x) \cong K_{4,4,4}$ and v_1, v_2 are both contained in a 4-island.

Claim 4. The following statements are true.

(1) $G[Y]$ does not contain a clique of size 4.

(2) Let $v_3, v_4 \in Y$ be non-adjacent and $Y' = \{y \in Y : v_3y, v_4y \in E(G)\} \subseteq Y$. Then, $G[Y']$ does not have a clique of size 3.

(3) In Case 2 and Case 5, $N(x) - \{v_1, v_2\}$ does not have a clique of size 4.

(4) In Cases 1, 3, and 4, if $Z \subseteq N(x) - \{v_1, v_2\}$ such that $G[Z] \cong K_4$, then there exist vertices $v_1', v_2' \in Z$ such that $v_iv_i' \notin E(G)$ for $i = 1, 2$, where v_1' and v_2' are not necessarily distinct.

Proof of Claim 4. In Cases 1-5, $G[Y]$ is isomorphic to $K_{2,3,3}$, $K_{3,3,3}$, $\overline{K_3} + C_5$, $K_{1,3,3}$, and $K_{4,4}$ respectively. Since none of these five graphs have a clique of size 4, it follows that (1) is true. Observe that $N(x) - \{v_1, v_2\} \cong K_{3,3,3}$ in Case 2, and that $N(x) - \{v_1, v_2\} \cong K_{2,4,4}$ in Case 5. Since neither $K_{3,3,3}$ nor $K_{2,4,4}$ has a clique of size 4, it follows that (3) is true.

To see (2), observe that v_3, v_4 are in the same island of $G[Y]$, as they are non-adjacent. In Case 3, $G[Y] \cong \overline{K_3} + C_5$ and therefore $G[Y'] \cong C_5$ or $K_{1,3}$. Neither of these two graphs has a clique of size 3, meaning that (2) is true in Case 3. In other cases, observe that $G[Y]$ has at most three islands and each island is an independent set. This means that no vertex in Z is in the same island with v_3 and v_4 in $G[Y]$. By (1), $G[Y']$ does not contain a clique of size 3.

To see (4), let C_0 be the island of $N(x)$ that contains v_1 and v_2. Observe that C_0 is an independent set of size 3 in Case 1 and Case 3. Also observe that $N(x) - C_0 \cong K_{2,3,3}$ in Case 1 and $N(x) - C_0 \cong \overline{K_3} + C_5$ in Case 3, and both of these two graphs have their maximum clique of size 3. It follows that if $Z \subseteq N(x) - \{v_1, v_2\}$ such that $G[Z] \cong K_4$, then Z must contain the unique vertex in $C_0 - \{v_1, v_2\}$ which is adjacent to neither of v_1, v_2. In Case 4, if $Z \subseteq N(x) - \{v_1, v_2\}$ such that $G[Z] \cong K_4$, then Z must contain exactly one vertex in each one of the two 3-islands and two vertices in the 5-island C_0. Since the two vertices in $Z \cap C_0$ are adjacent to each other, it follows that one of them is not adjacent to v_1 and the other one is not adjacent to v_2. ⊣

Claim 5. $V(L_i) \cap (E - D) \neq \emptyset$ for $i = 1, 2$.

Proof of Claim 5. For the sake of a contradiction, assume that $V(L_1) \subseteq D$. By Claim 3, we know that $v_1, v_2 \notin N(E-D) \cap N(x)$. Since $G[E-D]$ is connected by Claim 3, $N(E-D) \cap N(x)]$ must be a clique, because otherwise there would be a path in $G[E-D]$ linking neighbors of two non-adjacent vertices in $N(x)$ which is disjoint from the $a-b$ path L_1, meaning that property $\mathcal{P}$ holds, a contradiction. By (1) of Claim 1 and the 7-connectivity of G, it follows that $N(E-D) \cap N(x) \cong K_4$. By (3) of Claim 4, Case 2 and Case 5 are not possible, so it remains to consider Cases 1, 3, and 4. By (4) of Claim 4, we can choose $v_i' \in N(E-D) \cap N(x)$ for $i = 1, 2$ such that $v_i v_i' \notin E(G)$, where v_1', v_2' are not necessarily distinct.

Assume $V(L_2) \cap (E-D) \neq \emptyset$ for a moment. Since $G[E-D]$ is connected, there exists a path $R \subseteq G[E-D]$ linking some $u_2' \in N(v_2') \cap (E-D)$ and a vertex on L_2 such that R is otherwise disjoint from L_2. We can then extend R to obtain a $v_2 - v_2'$ path which is disjoint from the $a-b$ path L_1. This implies property $\mathcal{P}$ is true, a contradiction.

We may then assume that $V(L_1 \cup L_2) \subseteq D$. Let $r \in E-D$. By (3) of Claim 1, there is no 2-cut of $G[B]$ separating r from $V(L_1 \cup L_2)$. It follows that there exist three paths linking r and $V(L_1 \cup L_2)$ that are pairwise disjoint except for r and are disjoint from $V(L_1 \cup L_2)$ otherwise. Note that at least one of these three paths has an end in $V(L_1 \cup L_2) - \{a, b\}$. Without loss of generality, assume that one of these three paths has an end in $V(L_1) - \{a, b\}$. It follows that there is a path R_1 linking r and u_1 that is disjoint from L_2. Since $G[E-D]$ is connected and contains some vertex $u_1' \in N(v_1')$, it follows that there exists a path R_2 linking u_1 and u_1' that is disjoint from the $a-b$ path, meaning that property $\mathcal{P}$ holds true, a contradiction. $\dashv$

By Claim 5, since the end vertices a, b of L_i are both contained in D, we know $|V(L_i) \cap (D \cap E)| \geq 2$. Since $|D \cap E| = 3$, without loss of generality, we can assume that $a \in D \cap E$, $b \in D - E$, and $|V(L_i) \cap (D \cap E) - \{a\}| = 1$ for $i = 1, 2$. Let $D \cap E = \{w_1, w_2, w_3\}$ where $a = w_3$ and w_i is the unique vertex in $V(L_i) \cap (D \cap E) - \{a\}$ for $i = 1, 2$.

Claim 6. There exists a $(D \cap E)$-tripod in $G[E]$ such that every leg of the tripod is trivial.

Proof of Claim 6. We first prove that $G[E]$ is a non-planar graph. By (2) of Claim 1, every vertex in $E-D$ has at least 7 neighbors in $G[E]$. It follows that

$$e(G[E]) = \frac{1}{2}\sum_{v\in E} d_{G[E]}(v) \geq \frac{1}{2}\sum_{v\in D\cap E} d_{G[E]}(v)+\frac{7}{2}|E-D| = \frac{1}{2}\sum_{v\in D\cap E} d_{G[E]}(v)+\frac{7}{2}|E|-\frac{21}{2}.$$

If $G[E]$ is planar, then

$$\frac{1}{2}\sum_{v\in D\cap E} d_{G[E]}(v) + \frac{7}{2}|E| - \frac{21}{2} \leq e(G[E]) \leq 3|E| - 6,$$

meaning that $\sum_{v\in D\cap E} d_{G[E]}(v)+|E| \leq 9$. Since $E-D \neq \emptyset$ and every vertex in $E-D$ has at least 7 neighbors in $G[E]$, we know that $|E| \geq 8$. It follows that $\sum_{v\in D\cap E} d_{G[E]}(v) \leq 1$, meaning that some vertex in $D\cap E$ has no neighbor in $E-D$, a contradiction to Claim 5.

Now, recall that there is no non-trivial $\leq$ 3-separation of $(G[E], D\cap E)$ by (3) of Claim 3. By Lemma 2.4.3 and (4) of Lemma 2.4.4, there exists some $(D\cap E)$-tripod $T \subseteq G[E]$ that is split by some 3-separation (E_1, E_2) of $G[E]$. This means that $D\cap E \subseteq \mathcal{L}(T) \subseteq E_1$ and $E_2 - E_1 \neq \emptyset$. Since there is no non-trivial $\leq$ 3-separation of $(G[E], D\cap E)$, it follows that $D\cap E = \mathcal{L}(T) = E_1$ and therefore every leg of T is trivial. Hence, the $(D\cap E)$-tripod T is as desired. $\dashv$

Claim 7. $N(E-D)\cap N(x) \subseteq Y$ where $Y = \{y \in N(x) : v_1y, v_2y \in E(G)\}$ as defined before Claim 4.

Proof of Claim 7. For the sake of a contradiction, assume that $v_1' \in N(E-D)\cap N(x)$ is not adjacent to v_1.

By Claim 6, let $T \subseteq G[E]$ be a $(D\cap E)$-tripod such that every leg of T is trivial. Let $p, q \in V(T) - D\cap E$ and paths P_i, Q_i for $i = 1, 2, 3$ be such that T is the union of the internally disjoint paths $P_1, P_2, P_3, Q_1, Q_2, Q_3$ such that P_i links p, w_i and Q_i links q, w_i for $i = 1, 2, 3$.

Note that $v_1' \in N(E-D)\cap N(x)$ has some neighbor $r_1 \in E-D$. By (2) of Claim 1,

r_1 has at least 7 neighbors in the connected subgraph $G[E]$. Therefore, without loss of generality, we can assume that there is a path R linking r_1 and some $r_1' \in V(P_1 \cup P_2 \cup P_3) - D \cap E$ that is disjoint from T otherwise. Recall that u_1 is a neighbor of v_1 in $B - A$ included in the path L_1. By Claim 3, $u_1 \in V(L_1) \cap D - \{b\}$. It follows that $R \cup P_1 \cup P_2 \cup P_3 - \{a, w_2\}$ has a subpath R' linking r_1 and w_1 that is disjoint from $Q_2 \cup Q_3$. Then, $R' \cup u_1 L_1 w_1$ would be a path linking r_1, u_1 which is disjoint from the $a - b$ path $Q_3 \cup Q_2 \cup w_2 L_2 b$, meaning that property $\mathcal{P}$ holds, a contradiction. $\dashv$

Claim 8. Let $T \subseteq G[E]$ be a $(D \cap E)$-tripod such that every leg of T is trivial. Let $p, q \in V(T) - D \cap E$ and paths P_i, Q_i for $i = 1, 2, 3$ be such that T is the union of the internally disjoint paths $P_1, P_2, P_3, Q_1, Q_2, Q_3$ such that P_i links p, w_i and Q_i links q, w_i for $i = 1, 2, 3$. If there exist non-adjacent vertices $y_1, y_2 \in N(E - D) \cap N(x)$, then every neighbor of y_1 or y_2 in $E - D$ is not associated with $V(P_1 \cup P_2 \cup Q_1 \cup Q_2) - \{p, q\}$ with respect to T.

Proof of Claim 8. Let $t_i \in E - D$ be a neighbor of y_i for $i = 1, 2$. For the sake of a contradiction, assume that t_1 is associated with $V(P_1 \cup P_2 \cup Q_1 \cup Q_2) - \{p, q\}$. Without loss of generality, say t_1 is associated with $P_1 - \{p\}$.

For convenience, let $P' = P_1 \cup P_2 \cup P_3 - D \cap E$ and $Q' = Q_1 \cup Q_2 \cup Q_3 - D \cap E$. By (2) of Claim 1, t_1 has at least 7 neighbors in the connected subgraph $G[E]$, so t_2 is associated with $V(P')$ or $V(Q')$ with respect to T. If t_2 is associated with $V(P')$ with respect to T, then there exists a path linking t_1 and t_2 that is disjoint from $Q_1 \cup Q_2 \cup Q_3$. If t_2 is associated with $V(Q')$ with respect to T, then there is a path linking t_1, t_2 that goes through w_1 (possibly q as well) and is disjoint from $P_2 \cup P_3$. In both cases, we can find a $t_1 - t_2$ path that is disjoint from some $a - b$ path, which implies property $\mathcal{P}$, a contradiction. $\dashv$

Now, let $T \subseteq G[E]$ be a fixed $(D \cap E)$-tripod. Let vertices p, q and paths P_i, Q_i be labeled as in Claim 8. Since G is 7-connected and $G[E - D]$ is connected, we know that $|N(E - D) \cap N(x)| \geq 4$. By Claim 7 and (1) of Claim 4, $N(E - D) \cap N(x)$ is not a clique. Hence, we choose non-adjacent vertices $v_3, v_4 \in N(E-D) \cap N(x)$ and vertices $u_3, u_4 \in E-D$ such that $u_i v_i \in E(G)$ for $i = 3, 4$.

Claim 9. There exists a ≤ 2-separation (E_1, E_2) of $G[E]$ such that $V(P_1 \cup P_2 \cup Q_1 \cup Q_2) \subseteq E_1$ and $a \in E_2 - E_1$.

Proof of Claim 9. For the sake of a contradiction, assume that there exist three paths R_1, R_2, R_3 in $G[E]$ linking a and $V(P_1 \cup P_2 \cup Q_1 \cup Q_2)$ that are pairwise disjoint except for a. Let r_i be the end of R_i on $P_1 \cup P_2 \cup Q_1 \cup Q_2$ for $i = 1, 2, 3$. Due to Corollary 2.3.4 and the existence of the paths P_3, Q_3, we may choose R_1, R_2, R_3 such that $p = r_2$ and $q = r_3$. Then, without loss of generality, assume that $r_1 \in V(P_1) - \{p\}$.

Observe that $T' = P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R_2 \cup R_3 \subseteq G[E]$ is also a $(D \cap E)$-tripod such that every leg of T' is trivial. By Claim 8 and the construction of u_3, u_4, it follows that both u_3, u_4 are associated with $V(R_2 \cup R_3) - \{a\}$ with respect to T'. It follows that, for $i = 3, 4$, there exists some path $S_i \subseteq G[E]$ linking u_i and some vertex $V(R_2 \cup R_3) - \{a\}$ that is otherwise disjoint from T'. Observe that $V(S_i) \cap V(R_1) = \emptyset$ for $i = 3, 4$, since otherwise u_i would be associated with $V(P_1) - \{p\}$ with respect to T', a contradiction to Claim 8. Therefore, we can find a subpath of $S_3 \cup S_4 \cup P_2 \cup R_2 \cup Q_2 \cup R_3 - \{a\}$ linking u_3 and u_4 that is disjoint from the $a-b$ path $R_1 \cup r_1 P_1 w_1 \cup w_1 L_1 b$. This implies property $\mathcal{P}$, a contradiction. $\dashv$

By Claim 9, we choose (E_1, E_2) to be a ≤ 2-separation of $G[E]$ such that $V(P_1 \cup P_2 \cup Q_1 \cup Q_2) \subseteq E_1$ and $a \in E_2 - E_1$ such that $|E_2|$ is maximum over all choices. Observe that $|E_1 \cap E_2| = 2$ due to the paths P_3 and Q_3. Furthermore, we can write $E_1 \cap E_2 = \{p_0, q_0\}$ such that $p_0 \in V(P_3) - \{a\}$, $q_0 \in V(Q_3) - \{a\}$, $V(pP_3p_0 \cup qQ_3q_0) \subseteq E_1$, and $V(p_0P_3a \cup q_0Q_3a) \subseteq E_2$.

Claim 10. Every vertex in $N(E_1 - \{p_0, q_0, w_1, w_2\}) \cap N(x)$ is adjacent to all other vertices in $N(E-D) \cap N(x)$.

Proof of Claim 10. Assume that there exist two vertices $y_1, y_2 \in N(E-D) \cap N(x)$ such that $y_1y_2 \notin E(G)$. It suffices to prove that neither y_1 nor y_2 has a neighbor in $E_1 - \{p_0, q_0, w_1, w_2\}$. Observe that if $p = p_0$ and $q = q_0$, then $V(P_3 \cup Q_3) \subseteq E_2$, and it follows that y_1, y_2 have no neighbor in $E_1 - E_2$ by Claim 8. Therefore, without loss of generality, we may assume that $p \neq p_0$.

Let $W = V(P_1 \cup P_2 \cup Q_1 \cup Q_2)$, just for convenience. By the maximality of $|E_2|$ when choosing (E_1, E_2), there is no non-trivial $\leq$ 2-separation (F_1, F_2) of $G[E_1]$ such that $W \subseteq F_1$ and $\{p_0, q_0\} \subseteq F_2$. By Corollary 2.3.5, it follows that there exist internally disjoint paths S_1, S_2, S_3 in $G[E_1]$ satisfying the following properties: (1) S_i for $i = 1, 2$ each link p_0 and some vertex in W, (2) S_3 links q_0 and some vertex in W, and (3) the end vertices of S_1, S_2, S_3 in W are distinct and include both p, q. Observe that in all cases, there exist distinct vertices $p', q' \in W - \{w_1, w_2, p_0\}$ and seven internally disjoint paths P'_1, P'_2, Q'_1, Q'_2, P''_3, Q''_3, R in $G[E_1]$ such that (i) P'_i for $i = 1, 2$ links p', w_i, (ii) Q'_i for $i = 1, 2$ links q', w_i, (iii) P''_3 links p', p_0, (iv) Q''_3 links q', q_0, and (v) R links p_0 and some $r \in V(P'_1 \cup P'_2 \cup Q'_1 \cup Q'_2) - \{p', q'\}$. Let $P'_3 = P''_3 \cup p_0P_3p$ and $Q'_3 = Q''_3 \cup q_0Q_3q$. It follows that $T' = \bigcup_{i=1,2,3}(P'_i \cup Q'_i)$ is a $(D \cap E)$-tripod in $G[E]$ such that every leg of T' is trivial, and that R is a path linking p_0, r that is disjoint from T' otherwise.

Let $t_1, t_2 \in E - D$ be neighbors of y_1, y_2, respectively. Note that it suffices to prove $t_1, t_2 \notin E_1 - E_2$. For the sake of a contradiction, assume that $t_1 \in E_1 - E_2$. Note that for both $i = 1, 2$, $t_i \in E - D$, meaning that t_i has at least $11 - 4 - 3 = 4$ neighbors in the connected subgraph $G[E - D]$ by Claim 1, and therefore there exists some non-empty $S_i \subseteq V(T') - E \cap D$ such that t_i is associated with S_i with respect to T'. By Claim 8, t_i for $i = 1, 2$ is not associated with $V(P'_1 \cup P'_2 \cup Q'_1 \cup Q'_2) - \{p', q'\}$ with respect to T'. Since $t_1 \in E_1 - E_2$, without loss of generality, assume that t_1 is associated with $V(p'P'_3p_0) - \{p_0\}$ with respect to T'. Note that t_2 is not associated with $V(P'_3) - \{a\}$ with respect to T', since otherwise there would exist a path linking t_1, t_2 that is disjoint from an $a - b$ path obtained by extending $Q'_2 \cup Q'_3$, which implies property $\mathcal{P}$, a contradiction. It follows that t_2 is associated with $V(Q'_3) - \{a\}$ with respect to T'.

Recall that R is a path linking p_0 and $r \in V(P_1' \cup P_2' \cup Q_1' \cup Q_2') - \{p', q'\}$ that is otherwise disjoint from T'. Without loss of generality, assume that either $r \in V(P_1) - \{p'\}$ or $r \in V(Q_1') - \{q'\}$. Note that R is disjoint from T_1 and T_2, since otherwise t_1 or t_2 would be associated with $V(P_1' \cup P_2' \cup Q_1' \cup Q_2') - \{p', q'\}$ with respect to T', a contradiction to Claim 8.

Let b_1, b_2 be the end vertices of paths T_1, T_2, respectively, on $p'P_3'p_0 - \{p_0\}$ and $Q_3' - \{a\}$, respectively. Let $O_1 = T_1 \cup b_1P_3'p' \cup P_2' \cup Q_2' \cup q'Q_3'b_2 \cup T_2$. Then, observe that in both the case $r \in V(P_1) - \{p'\}$ and the $r \in V(Q_1') - \{q'\}$, O_1 is a $t_1 - t_2$ path disjoint from the subgraph $O_2' = (P_1' \cup Q_1' - \{p', q'\}) \cup R \cup aP_3'p_0$. If $r \in V(P_1') - \{p'\}$, let $O_2 = aP_3'p_0 \cup R \cup rP_1'w_1 \cup w_1L_1b$; and if $r \in V(Q_1') - \{q'\}$, let $O_2 = aP_3'p_0 \cup R \cup rQ_1'w_1 \cup w_1L_1b$. In both cases, we see that O_2 is a subpath of O_2' linking a and b, and therefore O_1, O_2 are disjoint paths. This implies property $\mathcal{P}$ again, a contradiction. $\dashv$

To finish the proof, we first show that $E_1 = \{w_1, w_2, p, q\}$. Note that it suffices to prove that $E_1 = \{w_1, w_2, p_0, q_0\}$. For the sake of a contradiction, assume that $E_1 - \{w_1, w_2, p_0, q_0\} \neq \emptyset$. Let K_1 be a component of $G[E_1] - \{w_1, w_2, p_0, q_0\}$. Since G is 7-connected, $|N(K_1) \cap N(x)| \geq 3$. Recall that $Y = \{y \in N(x) : v_1y, v_2y \in E(G)\}$. Since $V(K_1) \subseteq E - D$, we have $N(K_1) \cap N(x) \subseteq N(E - D) \cap N(x) \subseteq Y$ by Claim 7. By Claim 10, $N(K_1) \cap N(x)$ is a clique such that every vertex in it is adjacent all other vertices in $N(E - D) \cap N(x)$. Recall that $v_3, v_4 \in N(E - D) \cap N(x)$ are non-adjacent by construction. It follows that $N(K_1) \cap N(x)$ is a clique contained in the subset $Y' = \{y \in Y : v_3y, v_4y \in E(G)\}$. By (2) of Claim 4, it follows that $|N(K_1) \cap N(x)| \leq 2$, a contradiction.

Now, $p \in E - D$ has at least 3 neighbors in E_1, and that p has at least 4 neighbors in $N(x)$. Since $\delta(G) \geq 11$, p has at least 4 neighbors in $E_2 - E_1$, meaning that $E_2 - E_1 \cup \{a\} \neq \emptyset$. Let $F_1 = E_1 \cup \{a\}$ and $F_2 = E_2$. It follows that (F_1, F_2) is a non-trivial 3-separation of $(G[E], D \cap E)$, a contradiction to (3) of Claim 3. $\square$

4.3.2 Proof of Lemma 4.3.5

Lemma 4.3.5. *If $G - N[x]$ is 2-connected and no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$, then $G - x$ has a minor J rooted at $N(x)$ such that $J > K_9$.*

Proof. For the sake of a contradiction, assume that such a minor J does not exist. Recall the definitions of $T_1 = \{x_1, x_2, x_3\}$ and $T_2 = \{y_1, y_2, y_3\}$ right before Section 4.3.1. By Lemma 4.3.3, for all $i, j \in \{1, 2, 3\}$, x_i, y_j have at least two common neighbors in $G - N[x]$. Since no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$, there exist a subset of three vertices $X = \{v_1, v_2, v_3\} \subseteq V(G) - N[x]$ and a subset of nine vertices $\mathcal{A} \subseteq V(G) - N[x] \cup X$ such that v_i for $i = 1, 2, 3$ is a common neighbor for x_i, y_i, and that every vertex in $\mathcal{A}$ is a common neighbor for x_i, y_j for some unique ordered pair (i, j) where $i, j \in \{1, 2, 3\}$.

Our proof for Lemma 4.3.5 is a bit lengthy. An outline of the proof is as follows. In Claim 1 and Claim 2, we will make a series of observations on the structure of $G - N[x]$, given that it is 2-connected and no vertex in it is adjacent to two non-adjacent vertices in $N(x)$. In Claim 3, we find an X-tripod T satisfying a few desired extremal properties. Then in Claim 4 and Claim 5, we prove that T has at least one non-trivial leg. Without loss of generality, assume that the leg L_1 of T on v_1 is non-trivial. In Claim 6-Claim 10, we prove that there is a non-trivial T-bridge B_1 that attaches to v_1 and exactly one of v_2, v_3 as its only attachment on T outside L_1. Without loss of generality, say B_1 attaches to v_2. Let $\mathcal{A}_3 \subseteq \mathcal{A}$ be the subset of vertices in $\mathcal{A}$ that are adjacent to x_3 or y_3. We will then use the T-bridge B_1 to show that every vertex in $\mathcal{A}_3$ is not associated with $T - \{v_2\} \cup V(L_3)$ in Claim 11-Claim 14. Finally, we use these vertices in $\mathcal{A}_3$ and the T-bridge B_1 to show a minor of type II exists, a contradiction.

We start the proof with observing a few properties of $G - N[x]$.

Claim 1. The following statements are true:

(1) For every two vertices $a_1, a_2 \in \mathcal{A}$, there exist distinct vertices $v_1, v_2 \in T_j$ for some $j \in \{1, 2\}$ such that $a_i v_i \in E(G)$ for $i = 1, 2$.

(2) There do not exist two disjoint connected subgraphs G_1, G_2 of $G - N[x]$ and a subset $X' \subseteq V(G_1)$ of three vertices such that for some $j \in \{1, 2\}$, each vertex in X' has a unique neighbor in T_j, G_1 has a K_3-minor rooted at X', and G_2 has two non-adjacent neighbors in $N(x) - T_j$.

(3) There do not exist two disjoint connected subgraphs G_1, G_2 of $G - N[x]$ such that $X \subseteq V(G_1)$, G_1 has a K_3-minor rooted at X, and G_2 has two non-adjacent neighbors in $N(x)$.

(4) If $G_1 \subseteq G - N[x]$ such that $X \subseteq V(G_1)$ and G_1 has a K_3-minor rooted at X, then $|V(K) \cap \mathcal{A}| \leq 1$ for every component K of $G - N[x] \cup V(G_1)$.

Proof of Claim 1. One can observe that (1) is simply true by the definition of $\mathcal{A}$, and that (3) and (4) immediately follow (1) and (2). So it suffices to prove (2). For the sake of a contradiction, assume that such subgraphs G_1, G_2 and $X' \subseteq V(G_1)$ exist. Without loss of generality, say each vertex in X' has a unique neighbor in T_1, and $w_1, w_2 \in N(x) - T_1$ are non-adjacent and both neighbors of G_2. It follows that $G[V(G_1) \cup T_1]$ has a K_3 minor rooted at T_1. By contracting edges in $G[V(G_1) \cup T_1]$ properly and contracting all vertices in G_2 to one of w_1, w_2, we could then obtain a minor of G of type II, a contradiction. Therefore, (2) is true. $\dashv$

In the next claim, we make observations on some properties of X-tripods in $G - N[x]$.

Claim 2. If $G - N[x]$ has an X-tripod T as a subgraph, then the following statements are true.

(1) T has a K_3-minor rooted at X.

(2) For every component K of $G - (N[x] \cup V(T))$, $N(K) \cap N(x)$ is a clique.

(3) Every non-trivial T-bridge in $G - N[x]$ has at least three attachments on T.

(4) Let vertices p, q and z_i for $i = 1, 2, 3$ and paths L_i, P_i, Q_i for $i = 1, 2, 3$ be labeled for T as in Definition 2.4.2. Then for each $i \in \{1, 2, 3\}$, there exists at most one vertex in

$\mathcal{A}$ associated with $V(P_i) - \{z_i, p\}$ with respect to T and at most one vertex in $\mathcal{A}$ associated with $V(Q_i) - \{z_i, q\}$ with respect to T.

Proof of Claim 2. One can easily check that (1) is true, and that (2) can be implied by (1) and Claim 1.

To prove (3), recall that $N(x)$ has no clique og size greater than 4 by Lemma 4.3.3. By (2), it follows that every non-trivial T-bridge B haw at most four neighbors in $N(x)$. Since G is 7-connected, it follows that B has at least three attachments on T.

To see (4) is true, let $G_1 = T - (V(P_i)\backslash\{z_i, p\})$. Observe that G_1 has a K_3 minor rooted at X. Let K be the component of $G - (N[x] \cup V(G_1))$ such that $P_i - \{z_i, p\} \subseteq K$. It follows that $|V(K) \cap \mathcal{A}| \leq 1$, meaning that at most one vertex in $\mathcal{A}$ is associated with $V(P_i) - \{z_i, p\}$ with respect to T. By symmetry, it follows that at most one vertex in $\mathcal{A}$ is associated with $V(P_i) - \{z_i, p\}$ with respect to T. $\dashv$

Claim 3. There exists an X-tripod T in $G - N[x]$ split by some 3-separation of $G - N[x]$ such that every T-bridge in $G - N[x]$ is stable, and that there is no X-tripod in $G - N[x]$ that can be obtained from T by a tripod-transformation.

Proof of Claim 3. Let (A_1, A_2) be a 2-separation of $(G - N[x], X)$ such that $|A_1|$ is minimum. Note that we know there exists some 2-separation, as we know $|G - N[x]| \geq 3$ by Lemma 4.3.2 and we do not require the 2-separation to be non-trivial. By the choice of (A_1, A_2), there is no non-trivial ≤ 2-separation of $(G[A_1], X)$.

We next prove that $G[A_1]$ is non-planar. By Lemma 4.3.3, there is no clique of size 5 in $N(x)$. Since no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$, it follows that every vertex in $V(G) - N[x]$ has at most 4 neighbors in $N(x)$. Since $\delta(G) \geq 11$ by Lemma 3.1.1, we have $d_{G-N[x]}(v) \geq 7$ for every $v \in V(G) - N[x]$, meaning that all but at most 2 vertices in $G[A_1]$ has degree at least 7 inside $G[A_1]$. Therefore, $e(G[A_1]) \geq \frac{7}{2}(|A_1| - 2) = \frac{7}{2}|A_1| - 7$. If $G[A_1]$ is planar, then

$$\frac{7}{2}|A_1| - 7 \leq e(G[A_1]) \leq 3|A_1| - 6,$$

meaning that $|A_1| \leq 2$. This is a contradiction, since $X \subseteq A_1$ and therefore has cardinality at least 3.

Now, since $G[A_1]$ is non-planar and $(G[A_1], X)$ has no non-trivial $\leq$ 2-separation, by Lemma 2.4.3 and Lemma 2.4.5, there exists an X-tripod T in $G[A_1]$ such that some 3-separation of $G[A_1]$ splits T, every T-bridge in $G[A_1]$ is stable, and that there is no X-tripod in $G[A_1]$ that can be obtained from T by a tripod-transformation. It now suffices to prove that $A_2 - A_1 = \emptyset$.

For the sake of a contradiction, assume that $A_2 - A_1 \neq \emptyset$. Let a, b be the two vertices in $A_1 \cap A_2$. Then, since $G - N[x]$ is 2-connected, there exists some a-b path in $G[A_2]$. By Lemma 4.3.4, there exists two disjoint paths P, Q in $G[A_2]$ such that P links a, b and Q links a neighbor of w_1 and a neighbor of w_2 for some non-adjacent vertices $w_1, w_2 \in N(x)$. Then, the path Q excludes a and b and is therefore disjoint from the X-tripod T in $G-N[x]$, a contradiction to (2) of Claim 2. $\dashv$

Now, fix $T \subseteq G - N[x]$ to be an X-tripod and fix (A, B) to be a 3-separation of $G - N[x]$ such that (A, B) splits T, every T-bridge in $G - N[x]$ is stable, and that there is no X-tripod in $G - N[x]$ that can be obtained from T by a tripod-transformation. Let vertices $z_1, z_2, z_3, p, q \in V(T)$ and paths L_1, L_2, L_3, P_1, P_2, P_3, Q_1, Q_2, Q_3 are labeled as in Definition 2.4.2 for T.

Claim 4. If every leg of T is trivial, then there is a trisection (A_1, A_2, A_3) of order 2 of $G - N[x]$ such that $\{p, q\} = A_1 \cap A_2 \cap A_3$ and $V(P_i \cup Q_i) \subseteq A_i$ for $i = 1, 2, 3$.

Proof of Claim 4. For the sake of a contradiction, assume that there exists a path R linking some $r_1 \in V(P_1 \cup Q_1) - \{p, q\}$ and $r_2 \in V(P_2 \cup Q_2) - \{p, q\}$ that is otherwise disjoint from T. Note that up to symmetry, we can assume that $r_1 \in V(P_1) - \{p\}$ and either $r_2 \in V(P_2) - \{p\}$ or $V(Q_2) - \{q\}$. One can observe that both $T \cup R - \{p\}$ and $T \cup R - \{q\}$ have a K_3 minor rooted at X in all cases. It follows that at most one vertex in $\mathcal{A}$ is associated with s for $s \in \{p, q\}$ with respect to T.

Note that $|\mathcal{A}| = 9$, and it follows that at least 7 vertices in $\mathcal{A}$ are associated with

$V(T) - \{p, q\}$, with respect to T. By (4) of Claim 2, there exists some vertex $a_0 \in \mathcal{A}$ that is not associated with $V(T) - X$ with respect to T. Since $X \cap \mathcal{A} = \emptyset$, it follows that a_0 is contained in some non-trivial T-bridge $B_0 \subseteq G - N[x]$ whose set of attachments is a subset of X. By Claim 2, the set of attachments of B_0 on T is precisely X. Note that $P_1 \cup P_2 \cup P_3 \cup B_1$ now has a K_3-minor rooted at X, and therefore there is at most one vertex in $\mathcal{A}$ associated with $V(Q_1 \cup Q_2 \cup Q_3) - X$ with respect to T. By symmetry, there is at most one vertex in $\mathcal{A}$ associated with $V(P_1 \cup P_2 \cup P_3) - X$ with respect to T, too.

By Claim 2, it follows that there exist 7 vertices $a_1, ..., a_7 \in \mathcal{A}$ such that for $i = 1, ..., 7$, each a_i is contained in a unique non-trivial T-bridge $B_i \subseteq G - N[x]$ whose attachments on T are precisely v_1, v_2, v_3. Without loss of generality, assume that a_i is a common neighbor for x_3 and y_i for $i = 1, 2, 3$. It follows that $P_1 \cup P_2 \cup B_1 - \{v_3\}$ has a K_3 minor rooted on v_1, v_2, a_1 and $B_2 \cup B_3 - \{v_1, v_2\}$ contains a path linking a_2, a_3. By contracting edges that have at least one end in $P_1 \cup P_2 \cup B_1 - \{v_3\}$ properly and contracting all of $B_2 \cup B_3 - \{v_1, v_2\}$ to y_2, we can then obtain a clique on T_1 and the edge $y_2 y_3$ in T_2, meaning G has a minor of type II, a contradiction. ⊣

Claim 5. Some leg of T is non-trivial.

Proof of Claim 5. For the sake of a contradiction, assume every leg of T is trivial for some X-tripod $T \subseteq G - N[x]$. Let vertices $p, q \in V(T)$ and paths P_1, P_2, P_3, Q_1, Q_2, Q_3 be labeled as in Definition 2.4.2 for T. By Claim 4, there is a trisection (A_1, A_2, A_3) of order 2 of $G - N[x]$ such that $\{p, q\} = A_1 \cap A_2 \cap A_3$ and $V(P_i \cup Q_i) \subseteq A_i$ for $i = 1, 2, 3$. Since $|\mathcal{A}| = 9$, $|\mathcal{A} \cap (A_i - \{p, q\})| \geq 3$ for some $i \in \{1, 2, 3\}$. Without loss of generality, say there are distinct vertices $a_1, a_2, a_3 \in \mathcal{A}$ that are all contained $A_1 - \{p, q\}$.

Assume for a moment that there exist pairwise disjoint subsets $S_1, S_2, S_3 \subseteq A_1 - \{v_1, p, q\}$ such that for $i = 1, 2, 3$, $a_i \in S_i$, $G[S_i]$ is connected, and $\{v_1, p, q\} \subseteq N(S_i)$. If a_1 is adjacent to x_1, then note that the subgraph $G_1 = G[S_1 \cup V(P_2 \cup Q_2 \cup Q_3)]$ has a K_3 minor rooted at $\{a_1, v_2, v_3\}$. By contracting edges inside this subgraph properly and contracting edges between T_1 and $\{a_1, v_2, v_3\}$, we can then obtain a clique on T_1. Note that the subgraph $G[S_2 \cup S_3 \cup \{v_1\}]$ is disjoint from G_2. It follows that $N(a_i) \cap T_2 = \{y_1\}$ for both $i = 2, 3$, since otherwise we would be able to obtain a minor of G of type II, a contradiction. Without loss of generality, we then assume that a_2 is adjacent to x_2 and y_1. Now since a_2 is adjacent to y_1, by the same argument as above, a_1, a_3 are both adjacent to x_1. It follows that a_3 is adjacent to both x_1 and y_1. Using the same argument again, we then have a_i is adjacent to both x_1, y_1 for all $i = 1, 2, 3$, a contradiction to the construction of $\mathcal{A}$. Hence, a_1 is not adjacent to x_1, and it follows that a_i is not adjacent to x_1 or y_1 for all $i = 1, 2, 3$. Without loss of generality, we can then assume that $a_1x_2, a_1y_2, a_2x_2, a_2y_3 \in E(G)$. It follows that the subgraph of G induced on $S_1 \cup S_2 \cup \{v_1, p\}$ has a K_3 minor rooted at $\{v_1, a_1, a_2\}$. By contracting edges in $G[T_2 \cup S_1 \cup S_2 \cup \{v_1, p\}]$ properly we can then obtain a clique on T_2, and by contracting edges in $G[\{x_2, x_3\} \cup V(Q_2 \cup Q_3)]$ properly we can obtain the edge x_2x_3. This again shows that G has a minor of type II, a contradiction.

Therefore, it now suffices to prove that there exist pairwise disjoint subsets $S_1, S_2, S_3 \subseteq A_1 - \{v_1, p, q\}$ such that for $i = 1, 2, 3$, $a_i \in S_i$, $G[S_i]$ is connected, and $\{v_1, p, q\} \subseteq N(S_i)$. Let $\mathcal{A}' \subseteq \{a_1, a_2, a_3\}$ be the subset of vertices that are not associated with $V(T) - \{v_1, p, q\}$. Note that $|\mathcal{A}'| \leq 3$. Also note that $a_i \in A_1 - \{p, q, v_1\}$ for all $i = 1, 2, 3$, which means that every vertex in $\mathcal{A}'$ is contained in a unique non-trivial T-bridge inside $G[A_1]$ whose set of attachments on T is a subset of $\{v_1, p, q\}$. By (3) of Claim 2, the attachments of this bridge are precisely v_1, p, q. It follows that if $|\mathcal{A}'| = 3$, then we have found the desired three pairwise disjoint subsets already. So we may assume that $|A'| \leq 2$. Notice that every vertex in $\mathcal{A}'$ is associated with $V(P_1) - \{v_1, p\}$ or $V(Q_1) - \{v_1, q\}$ with respect to T. By (4) of Claim 2, $|\mathcal{A}'| \geq 1$ and thus $|\mathcal{A}'| = 1$ or 2. We will discuss the case $|\mathcal{A}'| = 1$ and the case $|\mathcal{A}'| = 2$ separately in the rest of this proof.

Case 1: $|\mathcal{A}'| = 2$.

Without loss of generality, assume that $\mathcal{A}' = \{a_1, a_2\}$, and that a_3 is associated with $V(P_1 \cup Q_1) - \{v_1, p, q\}$. Let $B_i \subseteq G[A_1]$ for $i = 1, 2$ be the non-trivial T-bridge such that $a_i \in V(B_i)$. Note that the attachments of B_i for $i = 1, 2$ on T are precisely v_1, p, q.

Also note that $a_3 \notin V(B_1 \cup B_2)$, since otherwise one of a_1, a_2 would be associated with $V(P_1 \cup Q_1) - \{v_1, p, q\}$ with respect to T as well, a contradiction to the fact that $\mathcal{A}' = \{a_1, a_2\}$.

Let $H = (T - V(P_1 \cup Q_1)) \cup B_1 \cup B_2$, and let R_1 be a path linking v_1, p and R_2 be a path linking v_1, q such that R_1, R_2 are internally disjoint and $V(R_i) \subseteq V(B_i)$ for $i = 1, 2$. Let T' be the graph obtained from T by substituting P_1, Q_1 with R_1, R_2, respectively. Observe that $T' \subseteq H$ is an X-tripod such that a_1, a_2 are associated with $R_1 - \{v_1, p\}$ and $R_2 - \{v_1, q\}$, respectively, with respect to T'. Note that $a_3 \notin V(H)$ and therefore $a_3 \notin V(T')$. By Claim 2, a_3 is contained in a non-trivial H-bridge $B_3 \subseteq G[A_1]$ whose attachments on H are all included in $V(B_1 \cup B_2)$. Notice that B_3 has no attachment in $V(B_i) - \{v_1, p, q\}$, since otherwise a_3 would be associated with $V(R_1) - \{v_1, p\}$ or $V(R_2) - \{v_1, q\}$ with respect to T', a contradiction to (4) of Claim 2. By (3) of Claim 2, it follows that the attachments of B_3 on H are precisely v_1, p, q. Let $S_i = V(B_i) - \{v_1, p, q\}$ for $i = 1, 2, 3$, and it follows that S_1, S_2, S_3 are as desired.

Case 2: $|\mathcal{A}'| = 1$.

Without loss of generality, assume that $\mathcal{A}' = \{a_1\}$, a_2 is associated with $V(P_1) - \{v_1, p\}$, and a_3 is associated with $V(Q_1) - \{v_1, q\}$. Let $B_1 \subseteq G[A_1]$ be the non-trivial T-bridge whose attachments are precisely v_1, p, q. Let T' be the X-tripod obtained from T by replacing P_1 with a $v_1 - p$ path P_1' that in internally contained in B_1. Let $H = (T - V(P_1)) \cup B_1$. Note that $T' \subseteq H$ and $a_2 \notin V(H)$ is not associated with $V(P_1') - \{v_1, p\}$ or $V(Q_1) - \{v_1, q\}$ with respect to T'. It follows that a_2 is contained in non-trivial H-bridge B_2 such that the attachments of B_2 on H are precisely v_1, p, q. Let $H' = (T' - V(Q_1)) \cup B_2$, and let T'' be the X-tripod obtained from T' by replacing Q_1 with a $v_1 - q$ path Q_1' that is internally contained in B_2. It follows that $T'' \subseteq H'$ and $a_3 \notin V(H')$ is contained in a non-trivial H'-bridge B_3 such that the attachments of B_3 on H are precisely v_1, p, q. It follows that $S_i = V(B_i) - \{v_1, p, q\}$ for $i = 1, 2, 3$ are as desired. $\dashv$

Now, let $R_i = V(L_i \cup P_i \cup Q_i) - \{p, q, v_i\}$ for $i = 1, 2, 3$.

Claim 6. At least two vertices in $\mathcal{A}$ are associated with R_i with respect to T for some $i \in \{1,2,3\}$ such that $z_i \neq v_i$.

Proof of Claim 6. We first prove that there exist at most two vertices in $\mathcal{A}$ that are not associated with $V(T) - X$ with respect to T. For the sake of a contradiction, assume that there are distinct $a_1, a_2, a_3 \in \mathcal{A}$ that are not associated with $V(T) - X$ with respect to T. Note that $X \cap \mathcal{A} = \emptyset$. By Claim 2, a_i for each $i = 1,2,3$ is contained in a unique non-trivial T-bridge B_i such that set of attachments of B_i on T is equal to X. Without loss of generality, assume that a_1 is adjacent to x_3. Note that $L_1 \cup L_2 \cup P_1 \cup P_2 \cup B_1 - \{v_3\}$ has a K_3 minor rooted at $\{v_1, v_2, x_3\}$, and therefore by contracting this subgraph properly and contracting edges between T_1 and $\{v_1, v_2, x_3\}$ we can obtain a clique on T_1. Since v_3 is adjacent to y_3, it follows that a_2, a_3 are both adjacent to y_3. Since a_2, a_3 are both adjacent to y_3, by the same argument that we applied to a_1, it follows that a_1, a_2, a_3 are all adjacent to x_3. This means that a_2, a_3 are both common neighbors for x_3, y_3, a contradiction to the construction of $\mathcal{A}$.

Since $|\mathcal{A}| = 9$, it follows that at least 7 vertices in $\mathcal{A}$ are associated with $V(T) - X$. Note $V(T) - X = R_1 \cup R_2 \cup R_3 \cup \{p, q\}$, and therefore at least 5 vertices in $\mathcal{A}$ are associated with $R_1 \cup R_2 \cup R_3$. Note that if $z_i \neq v_i$ for some $i \in \{1,2,3\}$, we may assume that at most one vertex in $\mathcal{A}$ is associated with R_i. If $z_i = v_i$ for some $i \in \{1,2,3\}$, then by (4) of Claim 2, at most one vertex in $\mathcal{A}$ is associated with $V(P_i) - \{v_i, p\}$ and at most one vertex in $\mathcal{A}$ is associated with $V(Q_i) - \{v_i, q\}$, and therefore at most two vertices in $\mathcal{A}$ are associated with R_i. By Claim 5, without loss of generality, we can assume that $v_1 \neq z_1$, $v_i = z_i$ for $i = 2,3$, and for any subset of vertices $V' \in \{R_1, V(P_2) - \{v_2, p\}, V(Q_2) - \{v_2, q\}, V(P_3) - \{v_3, p\}, V(Q_3) - \{v_3, q\}\}$, there exists exactly one vertex in $\mathcal{A}$ associated with V'. In particular, note that there exist two vertices $a_1, a_2 \in \mathcal{A}$ that are both associated with $V(Q_2 \cup Q_3) - \{v_2, v_3\}$.

Note that $\{z_1, v_2, v_3\}$ now separates $V(L_1)$ from $V(P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3)$ in $G - N[x]$. Since $G - N[x]$ is 2-connected, there exist two paths S_1, S_2 linking v_1 and

$\{z_1, v_2, v_3\}$ such that S_1, S_2 are disjoint except for v_1 and they are both internally disjoint from $T - V(L_1)$. It follows that $G_1 = P_1 \cup P_2 \cup P_3 \cup S_1 \cup S_2$ has a K_3 minor rooted at X. Note that $a_, a_1 \in \mathcal{A}$ are both associated with $V(Q_2 \cup Q_3) - \{v_2, v_3\}$ with respect to T, and that G_1 is disjoint from $V(Q_2 \cup Q_3) - \{v_2, v_3\}$. This is then a contradiction to (4) of Claim 1. ⊣

By Claim 5 and Claim 6, without loss of generality, assume that $v_1 \neq z_1$ and there exist two vertices $b_1, b_2 \in \mathcal{A}$ such that b_1, b_2 are both associated with R_1. For $i = 1, 2$, let W_i be a path in $G - N[x]$ linking b_i and some vertex in R_1 that is otherwise disjoint from T. It follows that $G[R_1] \cup W_1 \cup W_2$ contains a path linking b_1, b_2 in $G - N[x]$.

From Claim 7 to Claim 10, we will show that there exists some non-trivial T-bridge attaching to v_1, and that every such T-bridge attaches to exactly one of v_2 and v_3 as its only attachment on T outside L_1.

Claim 7. The following statements are true.

(1) There does not exist a non-trivial T-bridge with attachments $u_i \in V(L_i)$ for $i = 1, 2, 3$ such that $u_i \neq z_i$ for some $i \in \{1, 2, 3\}$.

(2) There do not exist two disjoint paths S_1, S_2 in $G - N[x]$ such that for some distinct indices $j, k \in \{1, 2, 3\}$, S_1 links v_j and some vertex on $L_k - \{v_k\}$, S_2 links v_k and some vertex on $L_j - \{v_j\}$, and that S_1, S_2 are both internally disjoint from T.

(3) If a non-trivial T-bridge D attaches to v_1 and at least two vertices on $L_2 \cup L_3$, then $b_i \in V(D) - V(T)$ for some $i \in \{1, 2\}$.

(4) If a non-trivial T-bridge D attaches to v_1 and a vertex on $L_j - \{v_j\}$ for some $j \in \{2, 3\}$ and there is a path S linking v_j and some vertex on L_{5-j} such that S is otherwise disjoint from T, then $b_i \in V(D) - V(T)$ for some $i \in \{1, 2\}$.

Proof of Claim 7. (1) and (2) are simply true since there is no X-tripod that can be obtained from T by any tripod-transformation.

To see (3), assume some non-trivial T-bridge D attaches to v_1 and at least two vertices on $L_2 \cup L_3$. Then notice that $G_1 = (D \backslash (L_1 - \{v_1\})) \cup L_2 \cup L_3 \cup P_2 \cup P_3$ has a K_3 minor rooted at X and is disjoint from R_1. If b_1, b_2 neither are in $D - V(T)$, then $G_2 = G[R_1] \cup W_1 \cup W_2$ would be disjoint from G_1 and contain a path linking b_1 and b_2, a contradiction to (4) of Claim 1. It follows that $b_i \in V(D) - V(T)$ for some $i \in \{1, 2\}$.

To see (4), without loss of generality, assume that a non-trivial T-bridge D attaches to v_1 and a vertex on $L_2 - \{v_2\}$, and that a path S links v_2 and some vertex on L_3 such that S is otherwise disjoint from T. By (1), D has no attachment on L_3 and therefore S is internally disjoint from $T \cup D$. Then, observe that $G_1' = (D \backslash (L_1 - \{v_1\})) \cup L_2 \cup L_3 \cup S \cup P_2 \cup P_3$ has a K_3 minor rooted at X and is disjoint from R_1. By a similar argument as above, it follows that $b_i \in V(D) - V(T)$ for some $i \in \{1, 2\}$. ⊣

Claim 8. The following statements are true.

(1) If $v_i \neq z_i$ for $i \in \{1, 2, 3\}$, then v_i has at least 6 neighbors in $A - V(L_i)$.

(2) There exists a non-trivial T-bridge attaching to v_1.

Proof of Claim 8. We first prove (1). Since (A, B) splits T and $v_i \neq z_i$, we know that $v_i \in A - B$. Since no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$, by Lemma 4.3.3, v_i has at most four neighbors in $N(x)$. Since every T-bridge is stable, v_i has exactly one neighbor on L_i. As $\delta(G) \geq 11$ by Lemma 3.1.1 and $v_i \in A - B$, it follows that v_i has at least 6 neighbors in $A - V(L_i)$.

We now prove (2). Since $v_1 \neq z_1$, by (1) it follows that v_1 has at least 6 neighbors in $A - V(L_1)$. If there is no non-trivial T-bridge attaching to v_1, then v_1 must have at least 6 neighbors on $L_2 \cup L_3$. It follows that the subgraph $G[\{v_1\} \cup V(L_2 \cup L_3 \cup P_2 \cup P_3)] \subseteq G - N[x]$ has K_3 minor rooted at X and is disjoint from $G[R_1] \cup W_1 \cup W_2$ which contains a path linking b_1, b_2, a contradiction to (3) of Claim 1. Hence, there exists some non-trivial T-bridge attaching to v_1. ⊣

Claim 9. If a non-trivial T-bridge B_1 attaches to v_1 and some $u_2 \in V(L_j) - \{v_j\}$ for some $j \in \{2, 3\}$, then the following statements are true.

(1) There exists a path S linking v_j and v_1 or some vertex on L_{5-j}. If v_1 is an end of S,

then B_1 attaches to v_j and S is internally contained in $B_1 - V(T)$.

(2) $b_i \in V(B_1) - V(T)$ for some $i \in \{1, 2\}$.

(3) Neither one of b_1, b_2 is a common neighbor for x_1 and y_1.

(4) b_1, b_2 are the only vertices in $\mathcal{A}$ that are associated with $(L_1 - \{v_1\}) \cup (\mathcal{R}(T) - \{z_2, z_3\})$.

Proof of Claim 9. Without loss of generality, assume that $u_2 \in V(L_2) - \{v_2\}$.

We first prove (1) and (2). Note that there is a v_1-u_2 path internally contained in $B_1 - V(T)$. By (2) of Claim 7, there is no path linking v_2 and some vertex on $L_1 - \{v_1\}$ that is otherwise disjoint from $T \cup B_1$. Note that the fact that B_1 attaches to $u_2 \in V(L_2) - \{v_2\}$ implies that $v_2 \neq z_2$. By (1) of Claim 8, v_2 has at least 6 neighbors in $A - (V(L_1 \cup L_2) \backslash \{v_1\})$. It follows that there exists a path S linking v_2 and either v_1 or some vertex on L_3 such that S is otherwise disjoint from $T \cup B_1$. Moreover, if v_1 is an end of S, then B_1 attaches to v_2 and S is internally contained in $B_1 - V(T)$. Then, due to the existence of the path S, by (3) -(4) in Claim 7, it follows that $b_i \in V(B_1) - V(T)$ for some $i \in \{1, 2\}$. We are now done with proving (1) and (2).

Now assume that $b_1 \in V(B_1) - V(T)$ without loss of generality. Since b_1 is associated with $R_1 = V(L_1 \cup P_1 \cup Q_1) - \{v_1, p, q\}$ and (A, B) splits T, we know that B_1 attaches to some $u_1 \in V(L_1) - \{v_1\}$. Recall that W_i for $i = 1, 2$ is a path linking b_i and some vertex in R_1 in $G - N[x]$ that is otherwise disjoint from T.

To prove (3), note that $(T - \{v_1\}) \cup W_2$ has a K_3 minor rooted at $\{b_2, v_2, v_3\}$ and is disjoint from some path linking v_1 and b_1 inside B_1. If b_2 is a common neighbor for x_1 and y_1, then b_1 is not adjacent to both x_1 and y_1 by the definition of $\mathcal{A}$ and thus v_1 and b_1 have distinct neighbors in at least one of T_1 and T_2. This is then a contradiction to (2) of Claim 1. Therefore, b_2 is not a common neighbor for x_1, y_1. Note that either S is contained in the T-bridge B_1, or that S links v_2 and some vertex on L_3 and is otherwise disjoint from $T \cup B_1$. In both cases, $G_1 = (B_1 - V(L_1)) \cup S \cup L_2 \cup L_3 \cup P_2 \cup P_3$ has a K_3 minor rooted at $\{b_1, v_2, v_3\}$ and is disjoint from $G_2 = G[R_1 \cup \{v_1\}] \cup W_2$ which contains a path linking

v_1 and b_2, again a contradiction to (2) of Claim 1. To conclude, b_1 is neither a common neighbor for x_1 and y_1.

To prove (4), let $G_1' = (B_1 - R_1) \cup S \cup L_2 \cup L_3 \cup P_2 \cup P_3$. Observe that G_1' has a K_3 minor rooted at X and is disjoint from b_2 and $G[R_1] \cup Q_2 \cup Q_3 - \{z_2, z_3\}$, which includes all vertices in R_1. Since $b_2 \notin V(G_1')$ and b_2 is associated with R_1, by (4) of Claim 1, b_2 is the only vertex in $\mathcal{A} - \{b_1\}$ that is associated with $R_1 \cup V(Q_2 \cup Q_3) - \{z_2, z_3\}$. Let $G_1'' = (B_1 - R_1) \cup S \cup L_2 \cup L_3 \cup Q_2 \cup Q_3$. Then notice that G_1' has a K_3 minor rooted at X too and is disjoint from b_2 and $G[R_1] \cup P_2 \cup P_3 - \{z_2, z_3\}$. Therefore, b_2 is also the only vertex in $\mathcal{A} - \{b_1\}$ that is associated with $R_1 \cup V(P_2 \cup P_3) - \{z_2, z_3\}$. Combining the two observations, we conclude that b_1, b_2 are the only vertices in $\mathcal{A}$ associated with $R_1 \cup \mathcal{R}(T) - \{z_2, z_3\}$, the union of $(R_1 \cup V(P_2 \cup P_3) - \{z_2, z_3\})$ and $(R_1 \cup V(Q_2 \cup Q_3) - \{z_2, z_3\})$. $\dashv$

Claim 10. Every non-trivial T-bridge attaching to v_1 attaches to exactly one of v_2, v_3 as its only attachment on T outside L_1.

Proof of Claim 10. Let B_1 be any T-bridge attaching to v_1. Since every T-bridge is stable, B_1 has some attachment on $L_2 \cup L_3$. By (1) of Claim 7, attachments of B_1 outside L_1 are all included in one of L_2 and L_3. Without loss of generality, say attachments of B_1 outside L_1 are all on L_2. It then suffices to prove that B_1 does not attach to any vertex on $L_2 - \{v_2\}$. For the sake of a contradiction, assume that B_1 attaches to some $u_2 \in V(L_2) - \{v_2\}$, and it follows that (1)-(4) in Claim 9 are true. Let S be a path linking v_2 and v_1 or a vertex on L_3 as described in (1) of Claim 9.

Let $a_1 \in \mathcal{A}$ be the vertex adjacent to both x_1 and y_1. By Claim 9, $a_1 \neq b_1$ or b_2, and a_1 is not associated with $(L_1 - \{v_1\}) \cup (\mathcal{R}(T) - \{z_2, z_3\})$. Assume for a moment that there exist distinct $m_1, m_2 \in V(L_2 \cup L_3)$ and two paths M_1, M_2 in $G - N[x]$ such that M_i for $i = 1, 2$ links a_1 and m_i and is otherwise disjoint from T. It follows that $L_2 \cup L_3 \cup M_1 \cup M_2 \cup P_2 \cup P_3$ has a K_3 rooted minor at $\{a_1, v_2, v_3\}$ and is disjoint from the subgraph $B_1 - V(L_2)$ which contains a path linking v_1 and b_1. Since a_1 is adjacent to both x_1, y_1, we know v_1 and b_1 have distinct neighbors in T_1 or T_2. This is then a contradiction to (2) of Claim 1.

Hence, a_1 is associated with at most one vertex on $L_2 \cup L_3$. Note that $a_1 \notin X$ and every non-trivial T-bridge has at least three attachments on T. Since a_1 is not associated with $(L_1 - \{v_1\}) \cup (\mathcal{R}(T) - \{z_2, z_3\})$, it follows that a_1 is simply a vertex on $L_i - \{v_i\}$ for some $i \in \{2, 3\}$. In the remaining proof, we will discuss the case $a_1 \in V(L_2) - \{v_2\}$ and the case $a_1 \in V(L_3) - \{v_3\}$ separately.

Case 1: $a_1 \in V(L_2) - \{v_2\}$**.**

In this case, first observe that if the path S links v_2 and some vertex on L_3, then $L_2 \cup L_3 \cup S \cup P_2 \cup P_3$ would have a K_3 minor rooted at $\{a_1, v_2, v_3\}$ and be disjoint from some v_1-b_1 path contained in $B_1 - V(L_2)$, a contradiction to (2) of Claim 1. By (1) of Claim 9, S links v_1, v_2 and B_1 attaches to v_2. Now, note that $(B_1 - V(L_1)) \cup L_2$ has a K_3 minor rooted at $\{a_1, v_2, b_1\}$ and is disjoint from the v_1-v_3 path $L_1 \cup L_3 \cup P_1 \cup P_3$. It follows that b_1 is not adjacent to x_3 or y_3. Note that b_1 is not a common neighbor for x_1 and y_1 by (3) of Claim 9. It follows that b_1 is adjacent to one of x_2 and y_2. Then, one can observe that subgraph $(B_1 - V(L_2)) \cup L_3 \cup P_1 \cup P_3$ has a K_3 minor rooted at $\{v_1, b_1, v_3\}$ and is disjoint from the path $v_2 L_2 a_1$ linking a_1 and v_2, a contradiction to (2) of Claim 1.

Case 2: $a_1 \in V(L_3) - \{v_3\}$**.**

In this case, first observe that $B_1 \cup L_1 \cup L_2 \cup P_1 \cup Q_1 \cup P_2$ has a K_3 minor rooted at $\{v_1, v_2, b_1\}$ and is disjoint from the path $v_3 L_3 a_1$. By (2) of Claim 1, b_1 is not adjacent to x_3 or y_3. By (3) of Claim 9, b_1 is not a common neighbor for x_1, y_1. So without loss of generality, we can assume that b_1 is adjacent to x_2.

We next prove that there is no path linking v_3 and some vertex in $L_1 \cup L_2 - \{v_1\}$ that is otherwise disjoint from T. If a path S' links v_3 and a vertex on $L_1 - \{v_1\}$ and is otherwise disjoint from T, then the subgraph $S' \cup (L_1 - \{v_1\}) \cup L_2 \cup L_3 \cup \mathcal{R}(T)$ would have a K_3 minor rooted at $\{a_1, v_2, v_3\}$ and is disjoint from some v_1-b_1 path contained in B_1, a contradiction to (2) of Claim 1. If a path S'' links v_3 and some vertex on L_2 and is otherwise disjoint from T, then $L_2 \cup L_3 \cup S' \cup P_2 \cup P_3$ would have a K_3 minor rooted at $\{a_1, v_2, v_3\}$ and is disjoint from some v_1-b_1 path in B_1, again a contradiction to (2) of Claim 1.

Now, since $a_1 \in V(L_3) - \{v_3\}$, we know $v_3 \neq z_3$. By (1) of Claim 8, v_3 has at least 6 neighbors in $A - V(L_3)$. As there is no path linking v_3 and some vertex in $L_1 \cup L_2 - \{v_1\}$ that is otherwise disjoint from T, there is a non-trivial T-bridge $B_2 \neq B_1$ attaching to v_1 and v_3. Since (A, B) splits T and every T-bridge has at least three attachments on T, B_2 also attaches to some vertex on $L_3 - \{v_3\}$. It follows that $B_2 \cup L_2 \cup L_3 \cup P_2 \cup P_3$ has a K_3 minor rooted at X and is disjoint from $G[R_1] \cup W_1 \cup W_2$ which contains a path linking b_1 and b_2, a contradiction to (4) of Claim 1. $\dashv$

Recall that every non-trivial T-bridge has at least three attachments on T. By Claim 10, since the X-tripod T is split, every T-bridge attaches to some vertex on $L_1 - \{v_1\}$. By (2) Claim 8, we now fix a vertex $v_1' \in V(L_1) - \{v_1\}$ and a non-trivial T-bridge B_1 attaching to v_1 such that there is no non-trivial T-bridge B_1' attaching to a vertex on $v_1' L_1 z_1 - \{v_1'\}$. By Claim 10, assume that v_2 is the only attachment of B_1 on T outside L_1. Let $L_1' = v_1 L_1 v_1'$ for notation. Let $\mathcal{A}_3 \subseteq \mathcal{A}$ be the subset of five vertices in $\mathcal{A}$ that are adjacent to x_3 or y_3. In Claim 11-Claim 14, we will show that every vertex in $\mathcal{A}_3$ is not associated with anywhere on T outside $V(L_3) \cup \{v_2\}$ due to the existence of the T-bridge B_1.

Claim 11. $(V(B_1) - V(T)) \cap \mathcal{A}_3 = \emptyset$.

Proof of Claim 11. For the sake of a contradiction, assume there exists some $c \in (V(B_1) - V(T)) \cap \mathcal{A}_3$. Without loss of generality, say c is adjacent to x_3. For $i = 1, 2$, let $a_i \in \mathcal{A}$ be the vertex adjacent to both x_2 and y_i. By Claim 2, $a_i \notin V(B_1) - V(T)$ for $i = 1, 2$.

If a_i for some $i \in \{1, 2\}$ is associated with $(V(L_1) - \{v_1\}) \cup (V(P_1 \cup P_2 \cup P_3) - \{z_2, z_3\})$, then $G_1 = L_1 \cup (B_1 - \{v_2\}) \cup P_1 \cup P_2 \cup P_3 - \{z_2, z_3\}$ would have a K_3 minor rooted at $\{v_1, a_i, c\}$. Note that $G_2 = L_2 \cup L_3 \cup Q_2 \cup Q_3$ is a path linking v_2, v_3 that is disjoint from G_1. By contracting edges in G_1, G_2 and edges between T_i and G_i for $i = 1, 2$ properly, we can then obtain a clique on T_1 and the edge $y_2 y_3$ in T_2, a contradiction. Therefore, a_1, a_2 are not associated with $(V(L_1) - \{v_1\}) \cup (V(P_1 \cup P_2 \cup P_3) - \{z_2, z_3\})$. By symmetry, they are not associated wtih $(V(L_1) - \{v_1\}) \cup (V(Q_1 \cup Q_2 \cup Q_3) - \{z_2, z_3\})$ either. It follows that a_1, a_2 are not associated with $(V(L_1) - \{v_1\}) \cup (\mathcal{R}(T) - \{z_2, z_3\})$.

Now note that $L_1 \cup L_2 \cup B_1 \cup P_1 \cup P_2$ has a K_3 minor rooted at $\{v_1, v_2, c\}$. Since v_3 is adjacent to y_3, it follow that neither a_1 nor a_2 is associated with $V(L_3)$. Therefore, a_1, a_2 are not associated with $V(T) - V(L_2) \cup \{v_1\}$. Notice that $\{a_1, a_2\} \cap \{v_1, v_2\} = \emptyset$. By Claim 2, a_1, a_2 are both associated with $V(L_2) - \{v_2\}$. It follows that for $i = 1, 2$, there exists some path S_i linking a_i and some vertex on $L_2 - \{v_2\}$ such that S_i is otherwise disjoint from $T \cup B_1$. Let $G_1' = L_1 B_1 \cup \cup P_1 \cup P_3 \cup L_3$ and $G_2' = (L_2 - \{v_2\}) \cup S_1 \cup S_2$. It follows that G_1', G_2' are disjoint connected subgraphs of $G - N[x]$ such that G_1' has a K_3 minor rooted at X, and that G_2' contains a path linking a_1, a_2, a contradiction to (4) of Claim 1. ⊣

Claim 12. There is no vertex in $\mathcal{A}_3$ associated with $V(L_1') - \{v_1\}$.

Proof of Claim 12. For the sake of a contradiction, assume that there exists some $c \in \mathcal{A}_3$ associated with $V(L_1') - \{v_1\}$. Without loss of generality, say c is adjacent to x_3. By Claim 11, $c \notin V(B_1) - V(T)$. Let S_c be a path linkling c and some vertex in $V(L_1') - \{v_1\}$ such that S_c is otherwise disjoint from $T \cup B_1$. Let $G_1 = B_1 \cup L_1' \cup S_c$, and note that G_1 has a K_3 minor rooted at $\{v_1, v_2, c\}$ and is disjoint from the connected subgraph $T - (V(L_1') \cup \{v_2\})$ of $G - N[x]$. By contracting edges in G_1 and edges between T_1 and G_1, we can obtain a clique on T_1. Since $v_3 \in V(T) - (V(L_1') \cup \{v_2\})$, by (2) of Claim 1, no vertex adjacent to y_1 or y_2 is associated with $V(T) - (V(L_1') \cup \{v_2\})$.

Let $\mathcal{A}' \subseteq \mathcal{A}$ be the subset of vertices that are adjacent to y_1 or y_2 and are not contained in $B_1 - V(T)$. Notice that $|\mathcal{A}'| \geq 5$, and that every vertex in $\mathcal{A}'$ is either a vertex on L_1' or contained in a non-trivial T-bridge whose attachments are contained in $V(L_1') \cup \{v_2\}$. Notice that $B_1 \cup T - (V(L_1') \backslash \{v_1, v_1'\})$ has a K_3 minor rooted at X and is disjoint from the interior of L_1'. By (4) of Claim 1, at most one vertex in $\mathcal{A}$ is associated with the interior of L_1'. Also, at most one vertex in $\mathcal{A}'$ is equal to v_1'. Since every non-trivial T-bridge is stable and has at least three attachments on T, it follows that there exist three distinct vertices $a_i \in \mathcal{A}'$ for $i = 1, 2, 3$ each contained in a unique non-trivial T-bridge D_i whose attachments on T are precisely v_1, v_2, and v_1'.

Assume for a moment that a_1 is adjacent to x_3. Recall that $A' \cap (V(B_1) - V(T)) = \emptyset$ by definition. Let $G_1 = B_1 \cup D_1 - (V(L_1') \setminus \{v_1\})$. Then observe that G_1 has a K_3 minor rooted at $\{v_1, v_2, a_1\}$ and therefore a clique on T_1 can be obtained by contracting edges in G_1 and edges between G_1 and T_1 properly. Let $G_2 = (D_2 - \{v_1, v_2\}) \cup v_1' L_1 z_1 \cup L_3 \cup P_1 \cup P_3$, and note that G_2 contains a path linking a_2, v_3 and is disjoint from G_1. Since a_2 is adjacent to y_1 or y_2, by the definition of $\mathcal{A}'$, it follows that by contracting edges in G_2 and edges between T_2 and G_2 properly, we can then obtain the edge $y_2 y_3$, which then implies a minor of type II, a contradiction. By symmetry, it follows that none of a_1, a_2, a_3 is adjacent to x_3.

Now, each a_i for $i = 1, 2, 3$ is a common neighbor for one of x_1, x_2 and one of y_1, y_2. Without loss of generality, assume that a_1 is a common neighbor for x_j and y_j for some $j \in \{1, 2\}$. If $j = 1$, let $G_1 = (D_1 - \{v_1\}) \cup v_1' L_1 z_1 \cup L_2 \cup L_3 \cup P_1 \cup P_2 \cup P_3$, $G_2 = D_2 - \{v_1', v_2\}$, and $X' = \{a_1, v_2, v_3\}$. Then, G_1, G_2 are disjoint subgraphs of $G - N[x]$ such that G_1 has a K_3 minor rooted at X' and G_2 contains a path linking v_1 and a_2. Since a_1 is adjacent to both x_1 and y_1, a_2 is adjacent to x_2 or y_2. This is then a contradiction to (2) of Claim 1. If $j = 2$, let $G_1 = (D_1 - \{v_2\}) \cup L_1 \cup L_3 \cup P_1 \cup P_3$, $G_2 = D_2 - \{v_1, v_1'\}$, and $X' = \{v_1, a_1, v_3\}$. Then, G_1 has a K_3 minor rooted at X' and G_2 contains a path linking v_2 and a_2. Since a_1 is adjacent to x_2 and y_2, a_2 is adjacent to x_1 or y_1. Again, a contradiction to (2) of Claim 1. $\dashv$

Claim 13. There is no vertex in $\mathcal{A}_3$ associated with $V(v_1' L_1 z_1) \cup \mathcal{R}(T) - \{z_2, z_3\}$.

Proof of Claim 13. For the sake of a contradiction, assume that $c \in \mathcal{A}_3$ is associated with $V(v_1' L_1 z_1) \cup \mathcal{R}(T) - \{z_2, z_3\}$. Without loss of generality, assume that c is adjacent to x_3 and is associated with $V(v_1' L_1 z_1 \cup P_1 \cup P_2 \cup P_3) - \{z_2, z_3\}$. Let $s \in V(v_1' L_1 z_1 \cup P_1 \cup P_2 \cup P_3) - \{z_2, z_3\}$ such that there exists a path S_1 linking c and s that is otherwise disjoint from T, and let S_2 be the subpath of $v_1' L_1 z_1 \cup P_1 \cup P_2 \cup P_3 - \{z_2, z_3\}$ linking v_1' and s. Note that we may choose c, s and the paths S_1, S_2 such that no vertex in $\mathcal{A}_3$ is associated with $V(S_2) - \{s\}$, and subject to this, $|S_1|$ is minimum.

Now, let $G_1 = B_1 \cup L_1' \cup S_1 \cup S_2$. Observe that G_1 has a K_3 minor rooted at $\{v_1, v_2, c\}$ and $v_3 \notin V(G_1)$. By (2) of Claim 1, no vertex adjacent to y_1 or y_2 in $G - N[x]$ is associated with $V(T) - V(G_1)$. By the definition of $\mathcal{A}_3$, there exists some $c' \in \mathcal{A}_3 - \{c\}$ that is adjacent to y_1 or y_2. Observe that $V(G_1) \cap V(T) = \{v_2\} \cup V(L_1') \cup V(S_2)$, so either $c' \in \{v_2\} \cup V(L_1') \cup V(S_2)$ or c' is contained in a non-trivial T-bridge whose attachments on T are all included in $\{v_2\} \cup V(L_1') \cup V(S_2)$. By Claim 13 and the choice of c, s and S_1, S_2, no vertex in $\mathcal{A}_3$ is associated with $V(L_1') \cup V(S_2) - \{v_1, s\}$ and $c' \neq s$. Since $\{v_1, v_2\} \cap \mathcal{A} = \emptyset$ and every non-trivial T-bridge has at least three attachments on T, it follows that c' is contained in a non-trivial T-bridge, say B', whose attachments on T are precisely v_1, v_2, and s.

Since (A, B) splits T and $v_1 \in A - B$, it follows that B' is contained in A and therefore $s \in V(v_1' L_1 z_1)$. By Claim 12, c is not associated with v_1', so $s \in V(v_1' L_1 z_1) - \{v_1'\}$. It follows that B' is a non-trivial T-bridge whose attachments on T are precisely v_1, v_2, and s where $s \in V(v_1' L_1 z_1) - \{v_1'\}$, a contradiction to the choice of B_1 and v_1'. $\dashv$

Claim 14. There is no vertex in $\mathcal{A}_3$ associated with v_1 or $V(L_2) - \{v_2\}$.

Proof of Claim 14. Note that if some vertex in $\mathcal{A}_3$ is associated with v_1, then it is contained in $V(D) - V(T)$ for some non-trivial T-bridge D, since $v_1 \notin \mathcal{A}$. By Claim 10, since (A, B) splits T and every non-trivial T-bridge has at least three attachments on T, D attaches to some vertex on $L_1 - \{v_1\}$ and therefore some vertex in $\mathcal{A}_3$ is associated with $V(L_1) - \{v_1\}$, a contradiction to Claim 12. Therefore, no vertex in $\mathcal{A}_3$ associated with v_1.

It remains to prove that no vertex is associated with $V(L_2) - \{v_2\}$. Notice that $B_1 \cup L_1 \cup L_3 \cup P_1 \cup P_3$ has a K_3 minor rooted at X and is disjoint from $L_2 - \{v_2\}$. If two vertices $c_1, c_2 \in \mathcal{A}_3$ are both associated with $V(L_2) - \{v_2\}$, note that neither of them is contained in B_1, as B_1 has no attachment on $L_2 - \{v_2\}$. It follows that we can extend some subpath of $L_2 - \{v_2\}$ to a $c_1 - c_2$ path that is disjoint from $B_1 \cup L_1 \cup L_3 \cup P_1 \cup P_3$, a contradiction to (4) of Claim 1. Therefore, at most one vertex in $\mathcal{A}_3$ is associated with $V(L_2) - \{v_2\}$.

For the sake of a contradiction, assume $c \in \mathcal{A}_3$ is associated with $V(L_2) - \{v_2\}$, and let S be a path linking c and some vertex on $L_2 - \{v_2\}$ such that S is otherwise disjoint from T. Without loss of generality, say c is adjacent to x_3. By the definition of $\mathcal{A}_3$, there exists some $c' \in \mathcal{A}_3$ such that $c' \neq c$ and c' is adjacent to x_3 and one of y_1, y_2. Notice that c' is not associated with $V(T) - (V(L_3) \cup \{v_2\})$ by Claim 11-Claim 13 and the fact that c is the only vertex in $\mathcal{A}_3$ associated with $V(L_2) - \{v_2\}$. Since $v_2 \notin \mathcal{A}_3$ and every T-bridge is stable, it follows that c' is associated with $V(L_3)$, meaning that there exists a path S' linking c' and some vertex on L_3 such that S' is otherwise disjoint from T. Let $G_1 = B_1 \cup L_1 \cup L_2 \cup S \cup P_1 \cup P_2$ and $G_2 = L_3 \cup S'$. Then, G_1 has a K_3 minor rooted at X and G_2 contains a path linking c' and v_3, a contradiction to (3) of Claim 1. $\dashv$

By Claim 11-Claim 14, every vertex in $\mathcal{A}_3$ is either a vertex on $L_3 - \{v_3\}$ or contained in $D - V(T)$ for some non-trivial T-bridge D attaching to v_2 and at least two vertices on L_3, as every non-trivial T-bridge is stable and has at least three attachments on T. It follows that every vertex in $\mathcal{A}_3$ is associated with $V(L_3) - \{v_3\}$. Choose $c \in \mathcal{A}_3$ and $u_3 \in V(L_3) - \{v_3\}$ such that some c-u_3 path S is disjoint from T except for u_3 and no vertex in $\mathcal{A}_3$ is associated with $V(u_3 L_3 z_3) - \{u_3\}$, and subject to these, $|S|$ is minimum. Without loss of generality, say c is adjacent to x_3. Notice that $G_1 = B_1 \cup L_1 \cup S \cup u_3 L_3 z_3 \cup P_1 \cup P_3$ has a K_3 minor rooted at $\{v_1, v_2, c\}$ and is disjoint from $v_1 L_3 u_3 - \{u_3\}$, so a clique on T_1 can be obtained by contracting edges in G_1 and edges between T_1 and G_1 properly. By the definition of $\mathcal{A}_3$, choose $c' \in A_3$ such that $c' \neq c$ and c' is adjacent to x_3 and one of y_1, y_2. It follows that c' is not associated with $V(v_1 L_3 u_3) - \{u_3\}$, since otherwise there would exist a path linking v_3 and c' disjoint from G_1, a contradiction to (2) of Claim 1. Since c' is associated with some vertex on $L_3 - \{v_3\}$ and no vertex in $\mathcal{A}_3$ is associated with $V(u_3 L_3 z_3) - \{u_3\}$ by the choice of u_3, it follows that c' associated with exactly u_3 on L_3. Moreover, $c' = u_3$, as every non-trivial T-bridge is stable and thus has at least two attachments on L_3. By the minimality of $|S|$ when choosing c' and u_3, this means that we should have chosen c' and u_3 instead of c and u_3, a contradiction. $\square$

4.3.3 Proof of Lemma 4.3.6

Lemma 4.3.6. *Suppose $G - N[x]$ is 2-connected. If there exists a vertex $u \in V(G) - N[x]$ that is common neighbor for two non-adjacent vertices in $N(x)$, then $G - x$ has a minor J rooted at $N(x)$ such that $J > K_9$.*

Proof. For $i = 1, 2$, let C_i be the island of $N(x)$ such that $T_i \subseteq C_i$.

Claim 1. There do not exist vertices distinct $a, b \in V(G) - N[x]$ such that a is adjacent to at least two vertices in each one of T_1, T_2, and that b has at least five neighbors in $T_1 \cup T_2$.

Proof of Claim 1. For the sake of a contradiction, assume that such $a, b \in V(G) - N[x]$ exist, where b is adjacent to all vertices in T_1 and at least two vertices in T_2 without loss of generality. Since $|G - N[x]| \geq 3$ by Lemma 4.3.2, there exists some component L of $G - N[x] \cup \{a, b\}$. Since $G - N[x]$ is 2-connected and G is 7-connected, a, b are both neighbors of L and $|N(L) \cap N(x)| \geq 5$. By Lemma 4.3.3, there is no clique of size 5 in $N(x)$. Therefore, there exist $z_1, z_2 \in N(L) \cap N(x)$ that are not adjacent to each other.

Note that if $\{z_1, z_2\} \cap (C_1 \cup C_2) = \emptyset$, then by contracting an edge between T_1 and a, an edge between T_2 and b, and contracting all of L to z_1, we would then obtain a minor of G of type I, a contradiction. It follows that $\{z_1, z_2\} \subseteq C_i$ for some $i = 1, 2$.

Assume for a moment that z_1, z_2 are both contained in C_1. Note that $|C_1 - T_1| \leq 1$, so at least one of z_1, z_2 is contained in T_1. Since $|T_1| = 3$, without loss of generality, assume that $z_1 = x_1$ and $z_2 \neq x_2$. This means that z_1, z_2, x_2 are three distinct vertices in C_1. Note that b is adjacent to x_1, x_2, x_3 and $b \in N(L)$, so by contracting all of L to z_2 and contracting the edge bx_2, we would then obtain a clique on $\{z_1, z_2, x_2\}$. Then, by contracting an edge between a and a neighbor of it in T_2, we then obtain a minor of G of type II, a contradiction.

We may then assume $\{z_1, z_2\} \subseteq C_2$. Note that a has at least two neighbors in T_1, so without loss of generality, we say a is adjacent to x_1 and x_2. Then, by contracting ax_1 and bx_3 and contracting all of L to z_1, we would then obtain a clique on T_1 and the edge z_1z_2 in C_2. This means that we obtained a minor of G of type II again, a contradiction. $\dashv$

Claim 2. There exist distinct vertices $v_1, v_2, v_3 \in V(G) - N[x] \cup \{u\}$ such that v_i is a common neighbor of x_i and $y_{\sigma(i)}$ for $i = 1, 2, 3$, for some permutation $\sigma \in S_3$.

Proof of Claim 2. Note that by Lemma 4.3.3, every vertex in T_1 and every vertex in T_2 have at least two common neighbors in $G - N[x]$.

By Claim 1, there do not exist two vertices $G - N[x]$ that are both adjacent to all six vertices in $T_1 \cup T_2$. This means that $|(V(G) - N[x]) \cap \left(\bigcup_{i=1,2,3} N(x_i) \cap N(y_{\sigma_i})\right)| \geq 3$ for every permutation $\sigma \in S_3$. Therefore, there are three distinct vertices $v_1, v_2, v_3 \in V(G) - N[x]$ such that v_i is a common neighbor for x_i and y_i for $i = 1, 2, 3$. Similarly, there are distinct vertices $u_1, u_2, u_3 \in V(G) - N[x]$ such that $u_1 \in N(x_1) \cap N(y_2)$, $u_2 \in N(x_2) \cap N(y_3)$, and $u_3 \in N(x_3) \cap N(y_1)$. Hence, we may assume that $u \in \{v_1, v_2, v_3\}$ and $u \in \{u_1, u_2, u_3\}$.

Without loss of generality, assume that $u = v_1$. Note that if there exists some $v_4 \in V(G) - \{v_1, v_2, v_3\}$ such that v_4 is adjacent to both x_1 and y_1, then v_2, v_3, v_4 are as desired. Therefore, we may assume that all common neighbors of x_1, y_1 in $G - N[x]$ are included in $\{v_1, v_2, v_3\}$. Since x_1, y_1 have at least two common neighbors in $G - N[x]$, without loss of generality, we assume that v_2 is adjacent to both x_1 and y_1. By the same argument as above, it follows that all common neighbors of x_2, y_2 are contained in $\{v_1, v_2, v_3\}$ and that one of v_1 and v_3 is a common neighbor for x_2, y_2.

Assume for a moment that v_3 is a common neighbor for x_2 and y_2. Then, by the previous argument, one of v_1 and v_2 is a common neighbor for x_3 and y_3. Observe that v_2 is not adjacent to x_3 or y_3, since otherwise this would be a contradiction to Claim 1 due to the fact that v_3 has at least two neighbors in T_i for both $i = 1, 2$. It follows that v_1 is adjacent to both x_3 and y_3. Since $G - N[x]$ is 2-connected, there exists a path P linking v_2 and v_3 in $G - N[x]$ that does not include v_1. Note that v_1 is adjacent to both x_1, x_3 in T_1, v_2 is adjacent to $y_1, y_2 \in T_2$, and that v_3 is adjacent to $y_2, y_3 \in T_2$. By contracting the edges v_1x_1, v_2y_1, v_3y_3 and contracting the path P to a single edge, we would then obtain a minor of type II of G, a contradiction.

Hence, v_1 is a common neighbor for x_2 and y_2. Notice now both v_1 and v_2 are adjacent to all four vertices x_1, y_1, x_2, y_2, meaning that they each have at least two neighbors in T_i for $i = 1, 2$. By Claim 1, we can then make the following observation.

Observation. *The following statements are true.*

(1) v_3 has at most four neighbors in $T_1 \cup T_2$.

(2) v_i is not adjacent to x_3 or y_3 for $i = 1, 2$.

Recall that $u_1, u_2, u_3 \in V(G) - N[x]$ are distinct vertices such that $u_1 \in N(x_1) \cap N(y_2)$, $u_2 \in N(x_2) \cap N(y_3)$, and $u_3 \in N(x_3) \cap N(y_1)$. Also recall that $u \in \{u_1, u_2, u_3\}$. By (2) of Observation, $u = v_1$ is not adjacent to x_3 or y_3. This means that $u \neq u_2$ or u_3, and therefore $u_1 = u = v_1$. If there exists some $u_4 \in V(G) - \{u_1, u_2, u_3\}$ such that u_4 is a common neighbor for x_1 and y_2, then we would have u_2, u_3, u_4 as desired. Since x_1, y_2 have at least two common neighbors in $G - N[x]$, we may assume that one of their common neighbors in $G - N[x]$ is u_2 or u_3.

If u_2 is a common neighbor for x_1, y_2, then u_2 is adjacent to both x_2 and y_2. Since all common neighbors of x_2, y_2 in $G - N[x]$ are contained $\{v_1, v_2, v_3\}$, we know that $u_2 \in \{v_1, v_2, v_3\}$. Note that $u_2 \neq v_1$ since $v_1 = u = u_1$ and $u_1 \neq u_2$. Since v_2 is not adjacent to y_3 by (2) of Observation but u_2, y_3 are adjacent, it follows that $u_2 \neq v_2$ and therefore $u_2 = v_3$. This means that $u_2 = v_3$ is adjacent to all of x_1, x_2, y_2, x_3, y_3, a contradiction to (1) of Observation due to the fact that u_1 is adjacent to all four of x_1, y_1, y_2. If u_3 is a common neighbor for x_1, y_2, then u_3 is adjacent to both x_1 and y_1. Since all common neighbors of x_1, y_1 in $G - N[x]$ are contained $\{v_1, v_2, v_3\}$ and $v_1 = u = u_1 \neq u_3$, it follows that $u_3 = v_2$ or v_3. Then, we can again find contradictions to (2) and (1) of Observation in cases $u_3 = v_2$ and $u_3 = v_1$, respectively. $\dashv$

By Claim 2, without loss of generality (by relabeling the vertices in T_1, T_2), we can assume that v_i is a common neighbor for x_i and y_i for $i = 1, 2, 3$. Let $w_1, w_2 \in N(x)$ be the two non-adjacent vertices that have u as a common neighbor. Without loss of generality, assume that $w_1, w_2 \notin C_1$.

Note that this means $G[(V(G) - N[x] \cup \{u\}) \cup T_1]$ does not have a K_3 minor rooted at T_1, since otherwise we would obtain a minor of G of type II due to the common neighbor u for $w_1, w_2 \notin T_1$. Since $G - N[x]$ is 2-connected, we know $G - N[x] \cup \{u\}$ is connected. By Lemma 2.2.1, since T_1 is an independent set, it follows that $G[(V(G) - N[x] \cup \{u\}) \cup T_1]$ has a cut vertex $w \in V(G) - N[x] \cup \{u\}$, and that there are distinct components J_1, J_2, J_3 of $G[(V(G) - N[x] \cup \{u\}) \cup T_1] - \{w\}$ such that $x_i \in V(J_i)$ for $i = 1, 2, 3$. Without loss of generality, assume that $w \neq v_1$ or v_2, and it is possible that $w = v_3$. Since v_i is adjacent to x_i for $i = 1, 2, 3$, it follows that $v_1 \in V(J_1)$, $v_2 \in V(J_2)$, and $v_3 \in V(J_3) \cup \{w\}$. For $i = 1, 2$, let L_i be the component of $J_i - \{x_i\}$ such that $v_i \in L_i$. If $w = v_3$, then let $L_3 = \{v_3\} = \{w\}$; otherwise, let L_3 be the component of $J_3 - \{x_3\}$ such that $v_3 \in L_3$. Note that since $G - N[x]$ is 2-connected,$\{u, w\} \subseteq N(L_i)$ for $i = 1, 2$, and that $\{u, w\} \subseteq N(L_3)$ if $w \neq v_3$.

Claim 3. The following statements are true.

(1) $x_2, x_3 \notin N(L_1)$, $x_1, x_3 \notin N(L_2)$, and if $w \neq v_3$ then $x_1, x_2 \notin V(L_3)$.

(2) $V(G) - N[x] \cup \{u\} = V(L_1 \cup L_2 \cup L_3) \cup \{w\}$.

Proof of Claim 3. Since $w \in V(G) - N[x] \cup \{u\}$ is a cut vertex of $G[(V(G) - N[x] \cup \{u\}) \cup T_1]$ and J_1, J_2, J_3 are distinct components of $G[(V(G) - N[x] \cup \{u\}) \cup T_1] - \{w\}$ such that $x_i \in V(J_i)$ for $i = 1, 2, 3$, we know that $N(J_i - \{x_i\}) \cap T_1 = \{x_i\}$ for $i = 1, 2$, and that if $w \neq v_3$ then $N(J_3 - \{x_3\}) \cap T_1 = \{x_3\}$. By the definition of L_i for $i = 1, 2, 3$, it follows that (1) is true.

To prove (2), for the sake of a contradiction, assume that there is some component L' of $G - N[x] \cup \{u, w\}$ such that $V(L') \cap \{v_1, v_2, v_3\} = \emptyset$. Note that u, w are the only neighbors of L' in $G - N[x]$. Since G is 7-connected and $N(x)$ does not have a clique of size 5, it follows that L' has at least 5 neighbors in $N(x)$ and therefore it has two non-adjacent neighbors $r_1, r_2 \in N(x)$. Note that we can fix $j \in \{1, 2\}$ such that $r_1, r_2 \notin C_j$. Then, observe that we can contract edges in $G[V(L_1 \cup L_2 \cup L_3) \cup \{u, w\}]$ property to become a cycle that goes through v_1, v_2, v_3, and this means that $G[[V(L_1 \cup L_2 \cup L_3) \cup \{u, w\} \cup T_j]$

has a K_3 minor rooted at T_j. Since L' has non-adjacent neighbors $r_1, r_2 \in N(x) - T_j$, it follows that G has a minor of type II, a contradiction. $\dashv$

In the rest of the proof, we consider the case $w = v_3$ and the case $w \neq v_3$ separately.

Case 1: $w = v_3$**.**

By Claim 3, there is no neighbor of x_3 in L_1 or L_2 and therefore $N(x_3) \cap (V(G) - N[x]) \subseteq \{u, w\}$. Note that x_3 and y_i for $i = 1, 2, 3$ have at least two common neighbors in $G - N - [x]$. It follows that u and w are precisely the common neighbors for x_3 and y_i in $G - N[x]$ for $i = 1, 2, 3$. Hence, both u and w are adjacent to x_3 and all three vertices in T_2.

Assume for a moment that $y_3 \in N(L_1)$. Then, by contracting all of L_1 to y_1 and contracting the edge y_2u, we would then obtain a clique on T_2. By contracting all of $L_2 \cup \{w\}$ to x_2, we can obtain the edge x_2x_3. It follows that G has a minor of type II, a contradiction. This means that $y_3 \notin N(L_1)$, and furthermore by symmetry we know $y_3 \notin N(L_2)$ either.

By (2) of Claim 3, we have $N(y_3) \cap (V(G) - N[x]) \subseteq \{u, w\}$. For $i = 1, 2, 3$, since there are at least two common neighbors for x_i and y_3 in $G - N[x]$, it follows that they are precisely u and w. This means that u and w each are adjacent to all six vertices in $T_1 \cup T_2$, a contradiction to Claim 1.

Case 2: $w \neq v_3$**.**

By (1) of Claim 3, we know that $x_2, x_3 \notin N(L_1)$, $x_2, x_3 \notin N(L_2)$, and $x_1, x_2 \notin V(L_3)$. Assume for a moment that $y_2, y_3 \notin N(L_1)$, $y_1, y_3 \notin N(L_2)$, and $y_1, y_2 \notin N(L_3)$. This would then imply that u and w are precisely the common neighbors for x_i and y_j in $G - N[x]$ for $i, j \in \{1, 2, 3\}$ such that $i \neq j$. This means that u and w each are adjacent to all six vertices in $T_1 \cup T_2$, a contradiction to Claim 1. It follows that $y_i \in N(L_j)$ for some $i, j \in \{1, 2, 3\}$ such that $i \neq j$. Without loss of generality, we assume that $y_2 \in N(L_1)$.

Recall that $w_1, w_2 \in N(x)$ are two non-adjacent neighbors of u, and we previously assumed that they are not in C_1 without loss of generality. Now observe that if w_1, w_2

are contained in an island of $N(x)$ outside $C_1 \cup C_2$, then by contracting all of L_1 to y_1, contracting all of $L_2 \cup L_3 \cup \{w\}$ to x_2, and contracting the edge uw_2, we would then obtain a minor of G of type I, a contradiction. It follows that $w_1, w_2 \in C_2$. In the rest of the proof, we will consider the cases $|\{w_1, w_2\} \cap \{y_1, y_2\}| = 0, 1$, or 2, separately.

Assume $|\{w_1, w_2\} \cap \{y_1, y_2\}| = 0$, and note that this is only possible when $N(x) \cong K_{4,4,4}$ and $\{w_1, w_2\} = C_2 - \{y_1, y_2\}$.Without loss of generality, say $w_1 = y_3$ and y_4 is the unique vertex in $C_2 - T_2$. By contracting all of L_i to y_i for $i = 1, 2$ and contracting all of $L_3 \cup \{w\}$ to y_3, we first obtain a clique on $T_1 = \{y_1, y_2, y_3\}$. Then, by contracting the edge uy_4, we see that y_4 would then be adjacent to all three vertices y_1, y_2, y_3 due to the fact that $u \in N(L_i)$ for $i = 1, 2, 3$. This means that the resulting graph now on $N[x]$ is isomorphic to $K_5 + K_{4,4}$, which has a K_{10} minor. It follows that $G > K_{10}$, a contradiction.

Assume $|\{w_1, w_2\} \cap \{y_1, y_2\}| = 1$. Then, without loss of generality, assume that $y_1 = w_1$ and $y_2 \neq w_2$. Notice that $y_1 = w_1, y_2$, and w_2 are now three distinct vertices in C_2. By contracting contracting the edge uw_2, contracting all of L_1 to y_2, and contracting all of $L_2 \cup L_3 \cup \{w\}$, we would then obtain a clique on $\{y_1, y_2, w_2\}$ in C_2 and the edge x_2x_3 in C_1. This means that G has a minor of type II, a contradiction.

Finally, assume $|\{w_1, w_2\} \cap \{y_1, y_2\}| = 2$. Then, without loss of generality, assume that $w_1 = y_1$ and $w_2 = y_2$. Note that if $y_3 \in N(L_1)$, then by contracting all of L_1 to y_3, contracting the edge uy_2, and contracting all of $L_2 \cup L_3 \cup \{w\}$ to x_2, we would then obtain a clique on T_2 and the edge x_2x_3 in T_1. This means that G has a minor of type II, a contradiction. Therefore, $y_3 \notin N(L_1)$. Since $y_3 \notin N(L_1 \cup L_2)$ by Claim 3, $N(y_3) \cap (V(G) - N[x]) \subseteq \{u, w\}$. Since x_1 and y_3 have at least two common neighbors in $G - N[x]$, it follows that their common neighbors in $G - N[x]$ are exactly u and w, and this means that u is in fact adjacent to all three vertices in T_2. Then, by contracting all of L_1 to y_1, contracting the edge uy_3, and contracting all of $L_2 \cup L_3 \cup \{w\}$ to x_2, we can obtain a clique on T_2 and the edge x_2x_3 in T_1. This means that G has a minor of type II, a contradiction. □

4.3.4 Proof of Lemma 4.3.7

Lemma 4.3.7. *If $G - N[x]$ is not 2-connected, then $G - x$ has a minor J rooted at $N(x)$ such that $J > K_9$.*

Proof. Recall that by Lemma 4.2.1, $M \subseteq N(K)$ where K is a component of $G - N[x]$ and $M = \{v \in N(x) : vu \notin E(G)$ for some $u \in N(x) - \{v\}\}$. Since $N(x) \cong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$ by Lemma 4.3.1, we see that $N(x) = M$ and therefore $N(K) = N(x)$.

If $G - N[x]$ is disconnected, let L_1, L_2 be two distinct components of $G - N[x]$ such that $N(L_1) = N(x)$ and, subject to that, $|L_2|$ is maximum. If $G - N[x]$ has a cut vertex w, let L_1, L_2 be two distinct components of $G - N[x] \cup \{w\}$ such that $|L_1| + |L_2|$ are maximum among all choices. In both cases, let $A_i = V(L_i) \cap N(x)$ and $H_i = G[A_i \cup V(L_i)]$ for $i = 1, 2$. We first make some simple observations in Claim 1 and Claim 2.

Claim 1. The following statements are true:

(1) L_1, L_2 are disjoint connected induced subgraphs of G, and $V(H_1 \cap H_2) = A_1 \cap A_2 \subseteq N(x)$.

(2) $|A_i| \geq 6$ for $i = 1, 2$.

(3) $A_i \subseteq N_G(L_i)$ and $|N_G(L_i) - A_i| \leq 1$ for $i = 1, 2$.

(4) $|L_i| \geq 2$ for some $i \in \{1, 2\}$.

Proof of Claim 1. (1) is simply true by the construction of L_i, A_i, and H_i for $i = 1, 2$. Observe that in all cases, L_i has at most one neighbor in $G - N[x]$ for $i = 1, 2$, and it follows that (2) and (3) are true since G is 7-connected.

To prove (4), for the sake of a contradiction, assume that $|L_i| = 1$ for $i = 1, 2$. By the choice of L_1 and L_2, it follows that either $G - N[x]$ is a star or $V(G) - N[x]$ is just an independent set in G. Recall that $|G - N[x]| \geq 3$ by Lemma 4.3.2. Therefore, in both cases, there exist three distinct vertices $v_1, v_2, v_3 \in V(G) - N[x]$ such that v_1, v_2 each have at most one neighbor in $G - N[x]$, and v_3 has at most two neighbors in $G - N[x]$. Since $\delta(G) \geq 11$, v_1, v_2 each have at least 10 neighbors in $N(x)$ and v_3 has at least 9 neighbors in $N(x)$. Since $N(x) \cong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$, we observe that v_1, v_2 each have non-adjacent neighbors in three distinct islands of $N(x)$, and that v_3 has non-adjacent neighbors in at least two distinct islands of $N(x)$. Hence, there exist distinct vertices $s_1, t_1, s_2, t_2, s_3, t_3 \in N(x)$ such that s_i, t_i for $i = 1, 2, 3$ are non-adjacent, contained in a distinct island of $N(x)$, and both adjacent v_i. By contracting edges $v_i s_i$ for $i = 1, 2, 3$, we would then obtain a minor of G of type I, a contradiction. $\dashv$

Now, let $G_1 = L_1$. Let $G_2 = L_2$ if $G - N[x]$ is disconnected, and let $G_2 = G - N[x] \cup V(L_1)$ if $G - N[x]$ has a cut vertex.

Claim 2. The following statements are true:

(1) G_1, G_2 are disjoint connected induced subgraphs of $G - N[x]$.

(2) $A_1 = N(G_1) \cap N(x)$, $A_2 \subseteq N(G_2) \cap N(x)$.

(3) $N(x) = (N(G_1) \cap N(x)) \cup (N(G_2) \cap N(x)) = A_1 \cup (N(G_2) \cap N(x))$.

Proof of Claim 2. (1) and (2) are simply due to the construction of L_1, L_2 and G_1, G_2. If $G - N[x]$ is disconnected, recall that we chose L_1 such that $N(L_1) = N(x)$, and therefore (3) is true. If $G - N[x]$ has a cut vertex, then $G_1 \cup G_2 = G - N[x]$. Since $V(G - N[x]) = N(x)$, it follows that (3) is true. $\dashv$

Claim 3. If $G[A_i]$ for some $i \in \{1, 2\}$ does not have an independent set of size 3, then the following statements are true.

(1) For $S \subseteq A_i$ such that $|S| \geq 6$, there exists a subset $Z = \{z_1, z_2, z_3, z_4\} \subseteq S$ of size 4 such that $z_1 z_2, z_3 z_4 \notin E(G)$.

(2) $|A_i| \leq 9$, where the equality holds only if $N(x) \cong K_{3,3}+C_5$ and $G[A_i] \cong K_{2,2}+C_5$.

Proof of Claim 3. Let $S \subseteq A_i$ be such that $|S| \geq 6$ be arbitrary. Since there is no dependent set of size 3 in $G[A_i]$, A_i includes at most two vertices in each island of $N(x)$ that is an independent set, and so does S.

Since $|S| \geq 6$, we see that if $N(x) \cong K_{2,3,3,3}$, then $G[A_i]$ and $G[S]$ each are isomorphic to one of $K_{2,2,2}$, $K_{1,2,2,2}$, or $K_{2,2,2,2}$; and if $N(x) \cong K_{4,4,4}$, then $G[A_i] = G[S] \cong K_{2,2,2}$.

This shows that $N(x) \cong K_{2,3,3,3}$ or $K_{3,3}+C_5$, then the desired subset Z exists and $|A_i| \leq 8$.

If $N(x) \cong K_{3,3} + C_5$, let C_1, C_2 be the two 3-islands and let C_3 be the 5-island. Since A_i does not contain an independent set of size 3, $|A_i \cap C_j| \leq 2$ for $j = 1, 2$. Furthermore, we may assume $|A_i \cap C_j| \leq 1$ for some $j\ in \{1, 2\}$, since otherwise we can find a subset $Z \subseteq A_i \cap (C_1 \cup C_2)$ of size 4 that is as desired. Without loss of generality, say $|A_i \cap C_1| \leq 2$ and $|A_i \cap C_2| \leq 1$. Since $|A_i| \geq 6$, it follows that $|A_i \cap C_3| \geq 3$ and thus there are non-adjacent vertices $s_3, t_3 \in A_i \cap C_3$. Now, if $|A_i \cap C_1| = 2$, then $Z = (A_i \cap C_1) \cup \{s_3, t_3\}$ is as desired. If $|A_i \cap C_1| \leq 1$, then $|A_i \cap C_3| \geq 4$. It follows that any subset $Z \subseteq A_i \cap C_3$ such that $|Z| = 4$ satisfies that $G[Z]$ is a path of length 3 and therefore is as desired. Finally, if $|A_i| \geq 9$, one can simply observe that this is only possible if $|A_i| = 9$ and $G[A_i] \cong K_{2,2} + C_5$. ⊣

Claim 4. The following statements are true about A_i for both $i = 1, 2$.

(1) If $|A_i| \leq \delta(G) - 3$, there exists a subset $S \subseteq A_i$ such that $|S| \geq 6$ and for every $Z \subseteq S$ such that $|Z| = 4$, $G[V(L_i) \cup Z]$ has a K_4 minor rooted at Z.

(2) If $G[A_i]$ does not contain an independent set of size 3, then there exists $Z = \{z_1, z_2, z_3, z_4\} \subseteq A_i$ such that $z_1z_2, z_3z_4 \notin E(G)$ and $G[V(L_i) \cup Z]$ has a K_4 minor rooted at Z.

Proof of Claim 4. Recall that we defined $H_i = G[A_i \cup V(L_i)]$ for $i = 1, 2$. Let (A, B) be a separation of $(H_i), A_i)$ such that (i) $B - A \neq \emptyset$, (ii) $|A \cap B|$ is minimum subject to (i), and (iii) $|B|$ is minimum subject to (i) and (ii). Since $(A', B') = (A_i, V(L_i))$ is a separation of (H_i, A_i) where $B' - A' \neq \emptyset$, it follows that $|A \cap B| \leq |A_i|$. Note that $B - A \subseteq V(L_i)$. By Claim 1, we know $|N_G(B - A)| \leq |A \cap B| + 1$. Since G is 7-connected, $|N_G(B - A)| \geq 7$ and therefore $|A \cap B| \geq 6$.

By the minimality of $|A \cap B|$, there exist disjoint paths $P_1, ..., P_{|A \cap B|}$ linking A_i and $A \cap B$ in $G[A]$. Let $S \subseteq A_i$ be the collection of end vertices of the disjoint paths $P_1, ..., P_{|A \cap B|}$ in A_i. It follows that $|S| = |A \cap B| \geq 6$. By the minimality of $|A \cap B|$ and $|B|$ when choosing (A, B), there is no non-trivial separation of $(G[B], A \cap B)$ of order at most $|A \cap B|$.

Since $A \cap B \subseteq N_G(B-A)$, there is no non-trivial separation of $(G[(B-A) \cup N_G(B-A)], N_G(B-A))$ of order at most $|A \cap B|$. Since $|N_G(B-A)| \leq |A \cap B|+1$, it follows that there is no non-trivial separation of $(G[(B-A) \cup N_G(B-A)], N_G(B-A))$ of order at most $|N_G(B-A)|-1$.

To prove (1), assume that $\delta(G)-3$, and we will show that S is as desired. Let $Z \subseteq S$ be any subset such that $|Z|=4$. Without loss of generality, say P_1, P_2, P_3, P_4 are the disjoint paths whose end vertices in S are Z. Let $Z_1' \subseteq A \cap B$ be the collection of end vertices of P_1, P_2, P_3, P_4 in $A \cap B$. Note that by Claim 1, $|A_i| \leq \delta(G)-3$ implies that $|N_G(B-A)| \leq \delta(G)-2$. Since there is no non-trivial separation of $(G[(B-A) \cup N_G(B-A)], N_G(B-A))$ of order at most $|N_G(B-A)|-1$, by Lemma 3.2.3 it follows that $G[(B-A) \cup Z_1]$ has a K_4 minor rooted at Z_1. Due to the disjoint paths P_1, P_2, P_3, P_4 linking Z and Z_1 in $G[A]$, $G[V(L_i) \cup Z]$ has a K_4 minor rooted at Z. This completes the proof of (1).

To prove (2), assume that $G[A_i]$ does not contain an independent set of size 3. By Claim 3, $|A_i| \leq 9$ and there exists $Z = \{z_1, z_2, z_3, z_4\} \subseteq S$ such that $z_1z_2, z_3z_4 \notin E(G)$. Again without loss of generality, assume that for $j=1,2,3,4$, P_i links $z_j \in Z \subseteq A_i$ and $z_j' \in A \cap B$. Let $Z_1 = \{z_1', z_2', z_3', z_4'\} \subseteq A \cap B$. Observe that if $|N_G(B-A)| \leq \delta(G)-2$, then by the same argument above we can show that $G[(B-A) \cup Z_1]$ has a K_4 minor rooted at Z_1, and therefore $G[V(L_i) \cup Z]$ has a K_4 minor rooted at Z due to the disjoint paths P_1, P_2, P_3, P_4. Thus, we may assume that $|N_G(B-A)| \geq \delta(G)-1 \geq 10$ as $\delta(G) \geq 11$. Note that by Claim 1, $|N_G(B-A)| \leq |A_i|+1 \leq 10$. If follows that $|N_G(B-A)| = 10$ and $|A_i| = 9$. By Claim 3, this is only possible if $N(x) \cong K_{3,3}+C_5$ and $G[A_i] \cong K_{2,2}+C_5$.

Now, let $Y_1' \subseteq A_i$ be the union of the two 2-islands of $G[A_i]$, and let Y_2' be four vertices in the 5-islands of $G[A_i]$. Without loss of generality, assume that $Z = Y_1'$ and the disjoint paths P_1, P_2, P_3, P_4 each have an end vertex in Y_1'. Let $Y_1 \subseteq A \cap B$ be the collection of end vertices of these paths in $A \cap B$. Also assume that P_5, P_6, P_7, P_8 each have an end vertex in Y_2', and let $Y_2 \subseteq A \cap B$ be the collection of end vertices of these paths in $A \cap B$. Note that $G[Y_1']$ and $G[Y_2']$ each contain two disjoint pairs of non-adjacent vertices, so it suffices to prove that one of Y_1' and Y_2' is as desired.

By Lemma 3.2.3, one of (2a) and (2b) is true with respect to $G[(B-A)\cup N_G(B-A)]$ and Y_1. First assume that (2a) is true with respect to $G[(B-A)\cup N_G(B-A)]$ and Y_1. Note that $G[Y_1'] \cong K_{2,2}$, so there exist $y_1, y_2 \in Y_1'$ such that $y_1y_2 \in E(G)$. Without loss of generality, say P_j for $j=1,2$ links $y_j \in Y_1'$ and $y_j' \in Y_j$. By (2a), $G[(B-A)\cup Y_1]$ has an H-minor rooted at Y_1' such that $H \cup \{y_1'y_2'\} \cong K_4$. Due to the edge $y_1y_2 \in E(G)$ and the disjoint paths P_1, P_2, P_3, P_4 linking Y_1' and Y_1, it follows that $G[V(L_i)\cup Y_1']$ has a K_4 minor rooted at Y_1'. This completes the proof of (2) as $G[Y_1'] \cong K_{2,2}$ contains two disjoint pairs of non-adjacent vertices. We may then assume (2b) is true with respect to $G[(B-A)\cup N_G(B-A)]$ and Y_1. Since $Y_1 \cap Y_2 = \emptyset$, by (2b) it follows that $G[(B-A)\cup Y_2]$ has a K_4 minor rooted at Y_2, implying that $G[V(L_i)\cup Y_2']$ has a K_4 minor rooted at Y_2' due to the disjoint paths P_5, P_6, P_7, P_8. This again completes the proof of (2) as $G[Y_2'] \cong \overline{K_4^-}$ contains two disjoint pairs of non-adjacent vertices. $\dashv$

Claim 5. Let C_1, C_2 be two islands of $N(x)$ that are not necessarily distinct. Suppose that for some $i \in \{1,2\}$, there is a subset $Z = \{s_1, t_1, s_2, t_2\} \subseteq A_i$ of size 4 such that $s_j, t_j \in C_j$ for $j=1,2$ and $G[V(L_i)\cup Z]$ has a K_4 minor rooted at Z. Then, the following statements are true.

(1) $A_{3-i} - C_1 \cup C_2$ is a clique.

(2) If $C_1 \neq C_2$ and C_j is an independent set for some $j \in \{1,2\}$, then $|A_{3-i} \cap C_j| \leq \max\{2, |C_j|-1\}$.

(3) If $C_1 \neq C_2$ and C_1, C_2 are both independent sets, then there exists a subset $Z' = \{s_1', t_1', s_2', t_2'\} \subseteq A_{3-i}$ of size 4 such that $s_j', t_j' \in C_j$ for $j=1,2$ and $G[(V(L_{3-i})\cup Z']$ has a K_4 minor rooted at Z'.

Proof of Claim 5. To see (1) is true, assume for the sake of a contradiction that there exist $s_3, t_3 \in A_{3-i} - C_1 \cup C_2$ such that $s_3t_3 \notin E(G)$. Note that if $C_1 \neq C_2$, then by contracting edges in $G[V(L_i)\cup Z]$ properly to obtain a clique on $Z = \{s_1, t_1, s_2, t_2\}$ and contracting all of L_{3-i} to s_3, we would then obtain a resulting graph on $N[x]$ that contains edges s_it_i

for $i = 1, 2, 3$. This means that G has a minor of type I, a contradiction. It follows that $C_1 = C_2$, and this means that either $N(x) \cong K_{3,3} + C_5$ and $C_1 = C_2$ is the 5-island in it, or that $N(x) \cong K_{4,4,4}$ and $C_1 = C_2$ is a 4-island in it. In the former case, by contracting edges in $G[V(L_i) \cup Z]$ properly to obtain a clique on $Z = \{s_1, t_1, s_2, t_2\}$ and contracting all of L_{3-i} to s_3, we would then obtain a minor of G of type III, a contradiction. In the latter case, we see that the graph obtained from $N(x) \cong K_{4,4,4}$ by making one of its islands a clique is isomorphic to $K_4 + K_{4,4}$, which has a K_9 minor. It follows that by contracting edges in $G[V(L_i) \cup Z]$ properly to obtain a clique on Z, the resulting graph on $N[x]$ would have a K_{10} minor, a contradiction.

To see (2) is true, without loss of generality, assume that C_1 is an independent set. Note that if $|A_{3-i} \cap C_1| > \max\{2, |C_1| - 1\}$, then $|C_1| \geq |A_{3-i} \cap C_1| \geq 3$. Let $r_1 \in A_{3-i} \cap C_1$ such that $r_1 \neq s_1$ or t_1. Then, by contracting edges in $G[V(L_i) \cup Z]$ properly to obtain a clique on $Z = \{s_1, t_1, s_2, t_2\}$ and contracting all of L_{3-i} to r_1, we would then obtain a clique on $\{s_1, t_1, r_1\}$ in C_1 and the edge s_2t_2, meaning that G has a minor of type II, a contradiction.

It remains to prove (3). If $N(x) \cong K_{2,3,3,3}$, let C_3, C_4 be the two islands of $N(x) - C_1 \cup C_2$. We would then have $|A_{3-i} \cap C_j| \leq 1$ for $j = 3, 4$ by (1) and $|A_{3-i} \cap C_j| \leq 2$ for $j = 1, 2$ by (1). Since $|A_{3-i}| \geq 6$, it follows that $|A_{3-i} \cap C_j| = 2$ for $j = 1, 2$, $|A_{3-i} \cap C_j| = 1$ for $j = 3, 4$, and $|A_{3-i}| = 6$. By (1) of Claim 4, $Z' = A_{3-i} \cap (C_1 \cup C_2)$ is as desired. If $N(x) \cong K_{2,3,3,3}$, then C_1, C_2 are precisely the two 3-islands of $N(x)$. Let C_3 be the 5-island of $N(x)$. Note that the maximum independent set in a 5-cycle has size 2. By (1) and (2), it follows that $|A_{3-i} \cap C_j| \leq 2$ for $j = 1, 2, 3$. Since $|A_{3-i}| \geq 6$, it follows that $|A_{3-i}| = 6$ and $|A_{3-i} \cap C_j| = 2$ for $j = 1, 2, 3$. By (1) of Claim 4, $Z' = A_{3-i} \cap (C_1 \cup C_2)$ is as desired. Finally, if $N(x) \cong K_{4,4,4}$, let C_3 be the 4-island of $N(x) - C_1 \cup C_2$. By (1) and (2), it follows that $|A_{3-i} \cap C_j| \leq 3$ for $j = 1, 2$ and $|A_{3-i} \cap C_3| \leq 1$. Note that this shows that $|A_{3-i}| \leq 7$ and $|S \cap C_j| \geq 2$ for every $S \subseteq A_{3-i}$. By Claim 1, $|A_{3-i}| \leq 7$ implies that $|N_G(L_{3-i})| \leq |A_{3-i}| + 1 \leq 8 \leq \delta(G) - 3$ as $\delta(G) \geq 11$. Therefore, by (1) of Claim 4, there exists some $Z' = \{s'_1, t'_1, s'_2, t'_2\} \subseteq A_{3-i}$ such that $s'_j, t'_j \in C_j$ for $j = 1, 2$. $\dashv$

Claim 6. For $i = 1, 2$, for every subset $Z \subseteq A_i$ of size 4 that is the union of two disjoint pairs of non-adjacent vertices, $G[V(L_i) \cup Z]$ does not have a K_4 minor rooted at Z.

Proof of Claim 6. For the sake of a contradiction, assume that for some $i \in \{1, 2\}$, $G[V(L_i) \cup Z]$ has a K_4 minor rooted at Z where $Z \subseteq A_i$ such that Z has size 4 and is the union of two disjoint pairs of non-adjacent vertices. Let $Z = \{s_1, t_1, s_2, t_2\}$ where $s_j t_j \notin E(G)$ for $j = 1, 2$. Let C_j be the island of $N(x)$ containing s_j and t_j for $j = 1, 2$. Note it is possible that $C_1 = C_2$, and we will consider the case $C_1 = C_1$ and the $C_1 \neq C_2$ separately in the rest of the proof.

Case 1: $C_1 = C_2$**.**

Note that $C_1 = C_2$ means that either $N(x) \cong K_{3,3} + C_5$ and $C_1 = C_2$ is the the 5-island of $N(x)$, or that $N(x) \cong K_{4,4,4}$ and $C_1 = C_2 = Z$ is the 4-island of $N(x)$. Observe that in the latter case, we can just contract edges in $G[V(L_i) \cup Z]$ to obtain a K_4 minor rooted at Z, and the resulting graph on $N[x]$ would be isomorphic to $K_5 + K_{4,4}$ which has a K_{10} minor, a contradiction.

We may then assume that $N(x) \cong K_{3,3} + C_5$ and $C_1 = C_2$ is the the 5-island of $N(x)$. By (1) Claim 5, $A_{3-i} - C_1 \cup C_2$ is a clique, meaning that A_{3-i} has at most one vertex in each of the 3-island of $N(x)$. Since $|A_{3-i}| \geq 6$, we have $|A_{3-i} \cap C_1| \geq 4$. Furthermore, this means that for every $S \subseteq A_{3-i}$ such that $|S| \geq 6$, S contains two disjoint pairs of non-adjacent vertices. By Claim 4, there exists a subset $Z' \subseteq A_{3-i} \cap C_1$ of size 4 such that $G[(V(L_{3-i}) \cup Z']$ has a K_4 minor rooted at Z'. We can then apply the same argument back to A_i to show that A_i has at most one vertex in each 3-island of $N(x)$. It follows that A_1 and A_2 each have at most one vertex in each 3-island of $N(x)$. By Claim 2 , $N(x) \subseteq A_1 \cup N(G_2)$, and it follows that $N(G_2)$ contains at least two vertices in each 3-island of $N(x)$. Therefore, by contracting edges that have at least one end in L_1 properly to obtain a clique of size 4 in the 5-island of $N(x)$ and contracting all of G_2 to a neighbor of it in a 3-island of $N(x)$, we would eventually obtain a minor of G of type III, a contradiction.

Case 2: $C_1 \neq C_2$**.**

We will consider the case that one of C_1, C_2 is not an independent set and the case that both C_1, C_2 are independent sets separately.

Case 2.1: One of C_1, C_2 is not an independent set

Observe that one of C_1, C_2 is not an independent set only if $N(x) \cong K_{3,3} + C_5$ and one of C_1, C_2 is the 5-island. Without loss of generality, assume that C_1 is a 3-island and C_2 is a 5-island. Let C_3 be the 3-island in $N(x) - C_1 \cup C_2$. By (1) and (2) of Claim 5, we know that $|A_{3-i} \cap C_3| \leq 1$ and $|A_{3-i} \cap C_1| \leq 2$. Since $|A_{3-i}| \geq 6$, it follows that $3 \leq |A_{3-i} \cap C_2| \leq 5$. Furthermore, observe that for every $S \subseteq A_{3-i}$ such that $|S| \geq 6$, S has some subset of size 4 that is the union of two disjoint pairs of non-adjacent vertices. By (1) of Claim 4, there exists a subset $Z' \subseteq A_{3-i}$ such that Z' is the union of two disjoint pairs of non-adjacent vertices and that $G[(V(L_{3-i}) \cup Z']$ has a K_4 minor rooted at Z'. Since $|A_{3-i} \cap C_3| \leq 1$, $|Z' \cap C_3| \leq 1$. By applying (1) in Claim 5 back to A_i, we see that $|A_i \cap C_3| \leq 3$. Now by Claim 2 , $N(x) \subseteq A_1 \cup N(G_2)$, and therefore $N(G_2)$ contains at least two vertices in C_3. Then, by contracting edges that have at least one end in L_1 to obtain a K_4 minor rooted at Z or Z' and contracting all of G_2 to one of its vertex in C_3, we would then obtain a minor of G of type I or type III, a contradiction.

Case 2.2: C_1, C_2 are both independent sets.

By (3) of Claim 5, there exists a subset $Z' = \{s'_1, t'_1, s'_2, t'_2\} \subseteq A_{3-i}$ of size 4 such that $s'_j, t'_j \in C_j$ for $j = 1, 2$ and $G[(V(L_{3-i}) \cup Z']$ has a K_4 minor rooted at Z'. Observe that regardless of which graph $N(x)$ is isomorphic to, there exists an island C_3 of $N(x) - C_1 \cup C_2$ such that $|C_3| \geq 3$. By (1) of Claim 5, it follows that for both $j = 1, 2$, $|A_j \cap C_3| \leq 1$ if C_3 is an independent set, and that $|A_j \cap C_3| \leq 2$ if $G[C_3] \cong C_5$. By Claim 2 , we know that $C_3 \subseteq A_1 \cup N(G_2)$, and therefore $|N(G_2) \cap C_3| \geq 2$ if C_3 is an independent set, and $|N(G_2) \cap C_3| \geq 3$ if $G[C_3] \cong C_5$. In both cases, observe that G_2 has two non-adjacent neighbors $s_3, t_3 \in C_3$. By contracting edges that have at least one end in L_1 properly to obtain a K_4 minor rooted at one of Z and Z' and contracting all of G_2 to s_3, we would then obtain a type I, a contradiction. $\dashv$

By Claim 6 and (2) of Claim 4, we can now conclude that A_i contains an independent set of size 3 for both $i = 1, 2$.

Claim 7. If $|L_i| \geq 2$ for some $i \in \{1, 2\}$ and $X \subseteq A_i$ such that $G[X] \cong \overline{K_3}$, then $G[V(L_i) \cup X]$ has a K_3 minor rooted at X.

Proof of Claim 7. Let x_1, x_2, x_3 be the three vertices in X. For the sake of a contradiction, assume that $G[V(L_i) \cup X]$ does not have a K_3 minor rooted at X. By Lemma 2.2.1, there exists a cut vertex $u \in V(L_i)$ of $G[V(L_i) \cup X]$, and there are distinct components J_1, J_2, J_3 of $G[V(L_i) \cup X] - \{u\}$ such that $x_j \in V(J_j)$ for $j = 1, 2, 3$.

Since $|L_i| \geq 2$, there exists some component R of $L_i - \{u\}$. Observe that due to the components J_1, J_2, J_3 of $G[V(L_i) \cup X] - \{u\}$ where $x_j \in V(J_j)$ for $j = 1, 2, 3$, we can assume that $R \cap J_2 = R \cap J_3 = \emptyset$ without loss of generality. Therefore, $|N(R) \cap X| \leq 1$. We also have $|N(R) \cap (V(G) - N[x])| \leq 2$ since $|N(L_i) \cap (V(G) - N[x])| \leq 1$. Since G is 7-connected, it follows that $|N(R) \cap (N(x) - X)| \geq 4$. Let C_X be the island of $N(x)$ that contains X, and note that $X = C_X$ if $N(x) \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, and that C_X is a 4-island if $N(x) \cong K_{4,4,4}$. It follows that $|N(R) \cap (N(x) - C_X)| \geq 4$ if $N(x) \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, and $|N(R) \cap (N(x) - C_X)| \geq 3$ if $N(x) \cong K_{4,4,4}$. Observe that $N(x) - C_X \cong K_{2,3,3}$ or $\overline{K_3} + C_5$ if $N(x) \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, and $N(x) \cong K_{4,4}$ if $N(x) \cong K_{4,4,4}$. Hence, in all cases, R has two non-adjacent neighbors $r_1, r_2 \in N(x) - C_X$. By contracting all of $G[V(J_2 \cup J_3) \cup \{w\} - \{x_3\}]$ to x_2 and contracting all of R to r_1, we would then obtain a clique of size four on $\{x_2, x_3, r_1, r_2\}$ where $x_2x_3, r_1r_2 \notin E(G)$, a contradiction to Claim 6. $\dashv$

Claim 8. Suppose $|L_i| \geq 2$ for some $i \in \{1, 2\}$. Then, $|A_{3-i}| = 6$. Furthermore, for any $X \subseteq A_i$ such that $G[X] \cong \overline{K_3}$, let C_X is the island of $N(x)$ that contains X, then $C_X \subseteq A_{3-i}$ and $A_{3-i} - C_X$ is a maximum clique in $N(x) - C_X$.

Proof of Claim 8. Note that we showed that there exists some independent set of size 3 in A_i. Let $X \subseteq A_i$ that $G[X] \cong \overline{K_3}$ be arbitrary. We will prove that $|A_{3-i}| = 6$, $C_X \subseteq A_{3-i}$, and $A_{3-i} - C_X$ is a maximum clique in $N(x) - C_X$. By Claim 7, $G[V(L_i) \cup X]$ has a K_3 minor rooted at X. This implies that $A_{3-i} - C_X$ is a clique, since otherwise we would have a minor of G of type II, which is a contradiction. It follows that $|A_{3-i} - C_X| \leq 3$ if $N(x) \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, and $|A_{3-i} - C_X| \leq 2$ if $N(x) \cong K_{4,4,4}$. Since $|A_{3-i}| \geq 6$ by Claim 1, it follows that A_{3-i} is precisely the union of C_X and a maximum clique of $A_{3-i} - C_X$. Therefore, $|A_{3-i}| = 6$, $C_X \subseteq A_{3-i}$, and $A_{3-i} - C_X$ is a maximum clique in $N(x) - C_X$. $\dashv$

To finish the proof, we first show that $|L_i| \geq 2$ for $i = 1, 2$. For the sake of a contradiction, assume that $|L_i| = 1$ for some $i \in \{1, 2\}$. Since $\delta(G) \geq 11$ and L_i has at most one neighbor in $G - N[x]$, we have $|A_i| = |N(L_i) \cap N(x)| \geq 10$. By Claim 1, $|L_i| = 1$ implies that $|L_{3-i}| \geq 2$. It follows that $|A_i| = 6$ by Claim 8, a contradiction.

Now, let $X \subseteq A_1$ such that $G[X] \cong \overline{K_3}$, and let C_1 be the island of $N(x)$ that contains X. By Claim 8, since $|L_1| \geq 2$, it follows that $X \subseteq C_1 \subseteq A_2$. Since $|L_2| \geq 2$ and $X \subseteq A_2$, by Claim 8 again, $A_1 - C_1$ is a maximum clique in $N(x) - C_1$. Note that in all cases, there exists some island C_2 of $N(x) - C_1$ that is an independent set of size at least 3. Since $A_1 - C_1$ is a maximum clique in $N(x) - C_1$, we have $|A_1 \cap C_2| = 1$. By Claim 1, $N(x) = A_1 \cup (N(G_2) \cap N(x))$. It follows that there exist non-adjacent vertices $y_1, y_2 \in N(G_2) \cap N(x)$ that are both contained in C_2. By Claim 7, we can contract edges that have at least one end in L_1 to obtain a clique on X in C_1. Then, by contracting all of G_2 to one of y_1, y_2, we would then obtain the edge $y_1 y_2$ in C_2. This shows that G has a minor of type II, a contradiction. $\square$

4.4 Other Problem Graphs

Lemma 4.4.1 (computer-assisted)**.** *Let H be a problem graph such that $H \ncong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$. Then, there exists a subset $Z = \{a_1, a_2, b_1, b_2\} \subseteq V(H)$ of size 4 such* *that $a_1b_1, a_2b_2 \in E(H)$, a_i, b_i for $i = 1, 2$ share at most $4+i$ common neighbors in H, and one of the following statements is true:*

(A1) a_2, b_2 share at most 5 common neighbors in H, and that there exists some $z \in Z$ and $v \in V(H)-Z$ such that v has at most 9 neighbors in H and $H \cup \{a_1a_2, a_1b_2, b_1a_2, b_1b_2, zv\} > K_9$.

(A2) b_2 has at most 8 neighbors in H, there exists some $v \in V(H)-Z$ such that a_2 and v are adjacent and share at most 6 common neighbors in H, and that $H \cup \{a_1a_2, a_1b_2, a_1v, b_1b_2\} > K_9$.

Lemma 4.4.2. *If $G - N[x]$ is 2-connected or has at most two vertices, then $N(x) \ncong H$ where H is a problem graph and $H \ncong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$.*

Proof. For the sake of a contradiction, assume that $N(x) \cong H$ for some problem graph H such that $H \ncong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$. By Lemma 4.4.1, there exists a subset $Z = \{a_1, a_2, b_1, b_2\} \subseteq V(H)$ of size 4 such that $a_1b_1, a_2b_2 \in E(G)$, a_i, b_i for $i = 1, 2$ share at most $4 + i$ common neighbors in $N(x)$, and one of the properties (A1) and (A2) in Lemma 4.4.1 is true.

Since a_i, b_i share at most $4 + i$ common neighbors in $N(x)$ for $i = 1, 2$, they share at most $5 + i$ common neighbors in $N[x]$ and thus share at least $3 - i$ common neighbors in $G - N[x]$. This means that there exist distinct vertices $u_1, u_2 \in V(G) - N[x]$ that are both common neighbors for a_1 and b_1. In the rest of the proof, we will consider the case (A1) is true and the case (A2) is true separately.

Case 1: (A1) in Lemma 4.4.1 is true.

In this case, a_2, b_2 share at most 5 common neighbors in $N(x)$, and that there exists some $z \in Z$ and $v \in N(x) - Z$ such that v has at most 9 neighbors in $N(x)$ and $N(x) \cup \{a_1a_2, a_2b_2, b_1a_2, b_1b_2, zv\} > K_9$. To show a contradiction, it suffices to prove that we can contract some edges that have at least one end in $G - N[x]$ in a way to obtain the edges $a_1a_2, a_2b_2, b_1a_2, b_1b_2, zv$ in $N(x)$, since this would then imply $G - \{x\} > K_9$ and thus $G > K_{10}$.

Observe that the fact that a_2, b_2 share at most 5 common neighbors in $N(x)$ means that they share at least two common neighbors in $G - N[x]$, so there exist distinct vertices $w_1, w_2 \in N(G)$ that are both common neighbors for a_2 and b_2. Note that a_i, b_i now share at least two common neighbors in $G - N[x]$ for both $i = 1, 2$, so without loss of generality, we can assume z to be equal to any vertex in Z, say $z = a_2$. Since $|G - N[x]| \leq 2$ or $G - N[x]$ is 2-connected, there exist two disjoint paths Q_1, Q_2 between $\{u_1, u_2\}$ and $\{w_1, w_2\}$ in $G - N[x]$. Without loss of generality, assume that Q_i joins u_i, w_i for $i = 1, 2$. Since v has at most 9 neighbors in $N(x)$, it has some neighbor in $G - N[x]$. So without loss of generality, we can assume that there exists a path R linking v and some vertex on Q_1 such that R is contained in $G - N[x]$ except for v and R is disjoint from Q_2. Then, by contracting all of $V(Q_1 \cup R) - \{v\}$ to $a_2 = z$, we would obtain the edges a_1a_2, b_1a_2, and $a_2v = zv$, and by contracting all of $V(Q_2)$ to b_2, we would then obtain a_1b_2 and b_1b_2.

Case 2: (A2) in Lemma 4.4.1 is true.

In this case, b_2 has at most 8 neighbors in $N(x)$, there exists some $v \in N(x) - Z$ such that a_2 and v are adjacent and share at most 6 common neighbors in $N(x)$, and that $N(x) \cup \{a_1a_2, a_1b_2, a_1v, b_1b_2\} > K_9$. To show a contradiction, it suffices to prove that we can contract some edges that have at least one end in $G - N[x]$ in a way to obtain the edges $a_1a_2, a_1b_2, a_1v, b_1b_2$.

Since a_2 and v are adjacent and share at most 6 common neighbors in $N(x)$, they share at most 7 common neighbors in $N[x]$ and thus at least one common neighbor in $G - N[x]$. Let $w_1 \in V(G) - N[x]$ be a common neighbor for a_2 and v. Since b_2 has at most 8 neighbors in $N(x)$, it has at most 9 neighbors in $N[x]$ and therefore at least two neighbors in $G - N[x]$ due to the fact that $\delta(G) \geq 11$. Therefore, there exists some $w_2 \in V(G) - N[x]$ such that $w_1 \neq w_2$ and $b_2w_2 \in E(G)$. Recall that a_i, b_i share at least $3 - i$ common neighbors in $G - N[x]$ for $i = 1, 2$, meaning that there exists some common neighbor of a_2, b_2 in $G - N[x]$. Therefore, if w_1 is not a common neighbor for a_2, b_2, we choose w_2 to be a common neighbor for them. This means that one of w_1, w_2 is a common neighbor for

a_2, b_2.

Again since $|G - N[x]| \leq 2$ or $G - N[x]$ is 2-connected, there exist two disjoint paths Q_1, Q_2 between $\{u_1, u_2\}$ and $\{w_1, w_2\}$ in $G - N[x]$, and without loss of generality we assume that Q_i joins u_i, w_i for $i = 1, 2$. If w_1 is a common neighbor for a_2, b_2, then $w_1 \in V(Q_1)$ is adjacent to all three of a_2, b_2, v. This means that by contracting all of Q_1 to a_1, we can obtain edges a_1a_2, a_1b_2, and a_1v. Then, by contracting all of Q_2 to b_1, we can obtain the edge b_1b_2. We may now assume that w_2 is a common neighbor for a_2, b_2. Then, by contracting all of Q_1 to a_1 we can obtain edges a_1a_2 and a_1v, and by contracting all of Q_2 to b_2 we can obtain edges a_1b_2 and a_2b_2. □

Lemma 4.4.3 (computer-assisted)**.** *Let H be a problem graph such that $H \ncong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$. Let M be the subset of vertices in H that are not adjacent to all other vertices in H, i.e. $M = \{v \in V(H) : \exists u \in V(H) - \{v\}$ such that $vu \notin E(H)\}$. Then, for every $B_1, B_2 \subseteq V(H)$ such that $|B_i| \geq 6$ for $i = 1, 2$, $M \subseteq B_1 \cup B_2$, and neither $H[B_1]$ nor $H[B_2]$ is a clique, one of the following statements is true:*

(B1) There exist $b_1 \in B_1 - B_2$ and $b_2 \in B_2 - B_1$ such that b_1, b_2 are adjacent and share at most 6 common neighbors in H.

(B2) There exist $b_1 \in B_1$ and $b_2 \in B_2$ such that $H' > K_9$, where H' is the graph obtained from H by making b_1 adjacent to all other vertices in B_1 and making b_2 adjacent to all other vertices in B_2.

(B3) For some $i \in \{1, 2\}$, $|B_i| \leq 8$ and for every $B_i' \subseteq B_i$ such that $|B_i'| \geq 6$, there exist some $Z \subseteq B_i'$ with $|Z| = 4$ and $b \in B_{3-i}$ such that $H' > K_9$, where H' is the graph obtained from H by making Z a clique and making b adjacent to all other vertices in B_{3-i}.

(B4) One of B_1, B_2 is contained in the other such that $|B_1 \cap B_2| = 6$, and there exists some $Z \subseteq B_1 \cap B_2$ such that $|Z| = 4$ and $e(H[B_1 \cap B_2]) - e(H[Z]) - \delta(H[B_1 \cap B_2]) \geq 6$.

(B5) One of B_1, B_2 is contained in the other such that $|B_1 \cap B_2| = 6$ and $G[B_1 \cap B_2] \cong K_6^-$.

(B6) (B_1, B_2) is a non-trivial separation of H of order $k \leq 7$ such that $e(H[B_1 \cap B_2]) = 4k - 20 + \binom{k-5}{2}$, and that edges with at least one end in $B_i - B_{3-i}$ for $i = 1, 2$ can be

contracted in a way such that the new graph on $B_1 \cap B_2$ has at most 3 non-edges.

Lemma 4.4.4. *If $G - N[x]$ is not 2-connected and $|G - N[x]| \geq 3$, then $N(x) \ncong H$ where H is a problem graph and $H \ncong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$.*

Proof of Lemma 4.4.4. For the sake of a contradiction, assume that $N(x) \cong H$ for some problem graph H that is not isomorphic to $K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$. Let $M = \{v \in N(x) : \exists u \in N(x) - \{v\}$ such that $vu \notin E(G)\} \subseteq N(x)$. Recall that by the choice of x and Lemma 4.2.1, there exists a component K of $G - N[x]$ such that $M \subseteq N(K)$ and $N(K') \cap M \subseteq N(K)$ for every component K' of $G - N[x]$.

If $G - N[x]$ is disconnected, choose G_1 to be one component of $G - N[x]$ and let $G_2 = G - N[x] \cup V(G_1)$. If $G - N[x]$ is connected and has a cut vertex w, choose G_1 to be one component of $G - N[x] \cup \{w\}$ and let $G_2 = G - N[x] \cup V(G_1)$. Let $B_i = N(G_i) \cap N(x)$ for $i = 1, 2$ in both cases. We now make the following observation.

Observation. *The following statements are true.*

(1) $V(G_1) \cap V(G_2) = \emptyset$ and $V(G_1) \cup V(G_2) = V(G) - N[x]$.

(2) $|N(G_i) \cap N(x)| = |B_i| \geq 6$ for $i = 1, 2$.

(3) If $|B_i| = 6$ for some $i \in \{1, 2\}$, then $G - N[x]$ is connected and has a cut vertex.

(4) $M \subseteq B_1 \cup B_2$.

(5) $M \subseteq B_i$ for some $i \in \{1, 2\}$ if $G - N[x]$ is disconnected.

(6) For $i = 1, 2$, there exists a connected induced subgraph L_i of G_i such that $|N_G(L_i) \cap (V(G) - N[x])| \leq 1$ and $|N_G(L_i) \cap N(x)| \geq 6$.

(7) Neither $G[B_1]$ nor $G[B_2]$ is a clique.

Proof of Observation. (1)-(3) are simply true due to the construction of G_1, G_2 and the 7-connectivity of G. (4) and (5) true because there exists some component K of $G - N[x]$ such that $M \subseteq N(K)$ by Lemma 4.2.1.

To see (6) is true, let $L_1 = G_1$ in both cases. If $G - N[x]$ is disconnected, let L_2 be one single component of G_2; and if $G - N[x]$ has a cut vertex w, let L_2 be a component of

$G_2 - \{w\}$. Since G is 7-connected, L_1, L_2 are as desired and therefore (6) is true in both cases.

To see (7) is true, let $L_i \subseteq G_i$ for $i = 1, 2$ be as in (6), and let $A_i = N_G(L_i) \cap N(x) \subseteq B_i$ for $i = 1, 2$. Then, notice that A_i for both $i = 1, 2$ is a separator of G if G is disconnected, and that $A_i \cup \{w\}$ is a separator of G if w is a cut vertex of $G - N[x]$ that defines G_1 and G_2. By Lemma 3.2.4, $G[A_i]$ is a clique for neither $i = 1, 2$. Since $A_i \subseteq B_i$ for $i = 1, 2$, it follows that neither $G[B_1]$ nor $G[B_2]$ is a clique. □

Hence, by (2), (4), and (7) in Observation and Lemma 4.4.3, one of the properties in Lemma 4.4.3 is true about B_1 and B_2. We will consider each one of them separately in the rest of this proof.

Case 1: (B1) in Lemma 4.4.3 is true

In this case, there exist $b_1 \in B_1 - B_2$ and $b_2 \in B_2 - B_1$ such that b_1, b_2 are adjacent and share at most 6 common neighbors in $N(x)$, meaning that they share at most 7 common neighbors in $N[x]$ and thus at least one common neighbor $u \in V(G) - N[x]$. Since $b_1 \in B_1 - B_2$ and $b_2 \in B_2 - B_1$, we know $b_1 \notin N(G_2)$ and $b_2 \notin N(G_1)$. It follows that $u \notin V(G_1) \cup V(G_2)$, a contradiction to (1) in Observation.

Case 2: (B2) in Lemma 4.4.3 is true

In this case, there exist $b_1 \in B$ and $b_2 \in B_2$ such that $J > K_9$, where J is the graph obtained from $N(x)$ by making b_1 adjacent to all other vertices in B_1 and making b_2 adjacent to all other vertices in B_2. This means that by contracting all of G_1 to $b_1 \in B_1$ and contracting all of G_2 to $b_2 \in B_2$, we can then obtain a resulting graph on $N(x)$ that has a K_9 minor, and therefore $G > K_{10}$, a contradiction.

Case 3: (B3) in Lemma 4.4.3 is true

In this case, $|B_i| \leq 8$ for some fixed $i \in \{1, 2\}$, and for every $B_i' \subseteq B_i$ such that $|B_i'| \geq 6$, there exist some $Z \subseteq B_i'$ with $|Z| = 4$ and $b \in B_{3-i}$ such that $J > K_9$, where J is the graph obtained from $N(x)$ by making Z a clique and making b adjacent to all other vertices in B_{3-i}.

By (6) in Observation, choose a connected induced subgraph L_i of G_i such that $|N_G(L_i) \cap (V(G) - N[x])| \leq 1$ and $|N_G(L_i) \cap N(x)| \geq 6$. Then, in the graph $G[V(L_i) \cup N_G(L_i)]$, choose a separation (X, Y) of $(G[V(L_i) \cup N_G(L_i)], N_G(L_i))$ such that $Y - X \neq \emptyset$ and $|X \cap Y|$ is minimum over all choices of (X, Y). Notice that the minimality of $|X \cap Y|$ implies that there exist disjoint paths $P_1, ..., P_{|X \cap Y|}$ linking $N_G(L_i)$ and $X \cap Y$ in $G[X]$. Note that $Y - X \neq \emptyset$ and $N_G(Y - X) = X \cap Y$. Since G is 7-connected, it follows that $|X \cap Y| = |N_G(Y - X)| \geq 7$.

Since $|N_G(L_i) \cap (V(G) - N[x])| \leq 1$, without loss of generality, we can assume that the end vertices of $P_1, ..., P_{|X \cap Y|-1}$ in $N_G(L_i)$ are all contained in $N_G(L_i) \cap N(x)$. Let $U \subseteq N_G(L_i) \cap N(x)$ and $U' \subseteq X \cap Y$ be the sets of end vertices of $P_1, ..., P_{|X \cap Y|-1}$ in $N_G(L_i) \cap N(x)$ and $X \cap Y$, respectively. Note that $|U| = |U'| = |X \cap Y| - 1 \geq 6$. Since $U \subseteq N_G(L_i) \cap N(x) \subseteq N_G(G_i) \cap N(x) = B_i$, it follows that U is a subset of B_i of size at least 6. By (B3) in Lemma 4.4.3, there exist $Z \subseteq U$ with $|Z| = 4$ and $b \in B_{3-i}$ such that $J > K_9$, where J is the graph obtained from $N(x)$ by making Z a clique and making b adjacent to all other vertices in B_{3-i}. Without loss of generality, say vertices in Z are precisely the end vertices of P_1, P_2, P_3, P_4 in U, and say Z' is the set of end vertices of P_1, P_2, P_3, P_4 in $U' \subseteq X \cap Y$.

Note now we have $|N_G(L_i) \cap (V(G) - N[x])| \leq 1$, $N_G(L_i) \cap N(x) \subseteq B_i$, and $|B_i| \geq 8$. It then follows that

$$|N_G(L_i)| \leq |N_G(L_i) \cap N(x)| + 1 \leq |B_i| + 1 \leq 9 \leq \delta(G) - 2.$$

By the choice of (X, Y), since the trivial separation $(X', Y') = (N_G(L_i), V(L_i) \cup N_G(L_i))$ satisfies that $Y' - X' \neq \emptyset$, we know that $|X \cap Y| \leq |X' \cap Y'| = |N_G(L_i)|$ due to the minimality of $|X \cap Y|$. It follows that $|X \cap Y| \leq \delta(G) - 2$. Observe that the choice of (X, Y) also implies that there is no non-trivial separation of $(G[Y], X \cap Y)$ of order at most $|X \cap Y| - 1$. Hence, by Lemma 3.2.3, $G[(Y - X) \cup Z']$ has a K_4 minor rooted at Z'. Now, by contracting all of G_{3-i} to $b \in B_{3-i}$ and by contracting edges in $G[Y]$ properly and contracting each path P_i for $i = 1, 2, 3, 4$ to a single vertex, we can then eventually obtain a resulting graph on $N(x)$ that contains J as a subgraph. Since $J > K_9$, it follows that $G - \{x\} > K_9$ and therefore $G > K_{10}$, a contradiction.

Case 4: (B4) in Lemma 4.4.3 is true

In this case, $B_i \subseteq B_{3-i}$ for some fixed $i \in \{1, 2\}$ such that $|B_i| = 6$ and there exists some $Z \subseteq B_1 \cap B_2$ such that $|Z| = 4$ and $e(G[B_1 \cap B_2]) - e(G[Z]) - \delta(G[B_1 \cap B_2]) \geq 6$. By (3) and (6) of Observation and by the 7-connectivity of G, $G - N[x]$ is connected and has a cut vertex w, and that $N(G_i) = B_i \cup \{w\}$ is a minimum separator of G.

Let $H_1 = G[V(G_i) \cup B_i \cup \{w\}]$ and $H_2 = G - V(G_i)$. Notice that H_1, H_2 defines a non-trivial 7-separation of G, where $V(H_1 \cap H_2) = B_i \cup \{w\}$ is a separator of order 7 of G. By Lemma 3.2.3, $G[V(G_i) \cup Z]$ has a rooted-K_4 minor at Z and therefore we can contract edges that have at least one end in G_i to Z properly to obtain $6 - e(G[Z])$ extra edges. Let this new graph on $V(H_2)$ be H_2'. By Lemma 3.2.4, $H_2' \ngeq K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$. On the other hand, note that $x \in V(H_2 - H_1)$ and $G_{3-i} - \{w\} \subseteq H_2 - H_1$. Since $B_i \cup \{w\}$ is a minimum separator of G, by contracting all vertices in one component of $G_{3-i} - \{w\}$ to w, we can have w adjacent to all six vertices in B_i in the new graph. Furthermore, let $u \in B_i$ such that $d_{G[B_i]}(u) = \delta(G[B_i])$. Since $B_i \subseteq N(x)$, by contracting the edge xu we can then have u adjacent to all other vertices in B_i. Let $\delta = \delta(G[B_i])$. Then, by contracting all vertices in one component of $G_{3-i} - \{w\}$ to w and contracting the edge xu, we are able to obtain $6 + e(G[B_i]) + (5 - \delta) = 11 + e(G[B_i]) - \delta$ edges on $B_i \cup \{w\}$ in the new graph.

By Lemma 3.2.1, it follows that

$$8 \cdot 7 \geq 35 + (6 - e(G[Z])) + (11 + e(G[B_i]) - \delta) - 1 = 51 + e(G[B_i]) - e(G[Z]) - \delta,$$

meaning that $e(G[B_i]) - e(G[Z]) - \delta \leq 5$, a contradiction to the inequality $e(G[B_i]) -$

$e(G[Z]) - \delta \geq 6$ in property (B4).

Case 5: (B5) in Lemma 4.4.3 is true

In this case, $B_i \subseteq B_{3-i}$ for some $i \in \{1, 2\}$ such that $|B_i| = 6$ and $e(G[B_i]) \cong K_6^-$. By (3) and (6) in Observation and by the 7-connectivity of G, it follows that $G - N[x]$ is connected and has a cut vertex w such that $N(G_i) = B_i \cup \{w\}$. Notice that $N(G_i) = B_i \cup \{w\}$ is a minimum separator of G. Since $G[B_i] \cong K_6^-$, there is a unique missing edge in $G[B_i]$. Let $t = d_{N(G_i)}(w)$. It follows that $e(N(G_i)) = 14 + t$. Since $N(G_i) = B_i \cup \{w\}$ is a minimum separator of G, we know that by contracting all vertices in any component of $G - N(G_i)$ to w, we can obtain $6 - t$ extra edges on $N(G_i)$. By Lemma 3.2.8, it follows that

$$8 \cdot 7 \geq 35 + 2(6 - t) + 14 + t = 61 - t.$$

This means that $t \geq 5$ and therefore w has at most one non-neighbor in B_i. By (1) in Lemma 3.2.4, w has exactly one non-neighbor in B_i. Since $G[B_i] \cong K_6^-$, there exists some $Z \subseteq B_i \cup \{w\}$ such that $|Z| = 4$ and that the graph obtained from $N(G_i)$ by making Z a clique is isomorphic to K_7, a contradiction to (2) in Lemma 3.2.4.

Case 6: (B6) in Lemma 4.4.3 is true

In this case, (B_1, B_2) is a non-trivial separation of $N(x)$ of order $k \leq 7$ such that $e(G[B_1 \cap B_2]) = 4k - 20 + \binom{k-5}{2}$, and that edges with at least one end in $B_i - B_{3-i}$ for $i = 1, 2$ can be contracted in a way such that the new graph on $B_1 \cap B_2$ has at most 3 non-edges.

Notice that if $G - N[x]$ is disconnected, then $(B_1 \cap B_2) \cup \{x\}$ separates $V(G_1) \cup (B_1 - B_2)$ from $V(G_2) \cup (B_2 - B_1)$; and if $G - N[x]$ has a cut vertex w, then $(B_1 \cap B_2) \cup \{x, w\}$ separates $V(G_1) \cup (B_1 - B_2)$ from $(V(G_2) - \{w\}) \cup (B_2 - B_1)$. Let $H_1 = G[V(G_1) \cup B_1 \cup \{x\}]$ in the former case, and let $H_1 = G[V(G_1) \cup B_1 \cup \{x, w\}]$ in the latter case. In both cases, let $H_2 = G - V(G_1) \cup (B_1 - B_2)$. It follows that $(V(H_1), V(H_2))$ is a separation of G in both cases. Let $S = V(H_1 \cap H_2)$. Notice that if $G - N[x]$ is disconnected, then $S = (B_1 \cap B_2) \cup \{x\}$ and $|S| = k+1$; and if $G - N[x]$ has a cut vertex w, then $S = (B_1 \cap B_2) \cup \{x, w\}$ and $|S| = k+2$. We will again apply the inequality in Lemma 3.2.1 to this separation to show contractions.

We first prove an upper bound to $e(G[S])$ in both cases. Since $B_1 \cap B_2 \subseteq N(x)$, we know that x is adjacent to all vertices in S. Therefore, if $G - N[x]$ is disconnected, then $e(G[S]) = e(G[B_1 \cap B_2]) + k$; and if $G - N[x]$ has a cut vertex w, then $e(G[S]) = e(G[B_1 \cap B_2]) + k + d_{G[S]}(w) \leq e(G[B_1 \cap B_2]) + 2k$, where the last inequality is due to the facts that $w \notin N(x)$ and $d_{G[S]}(w) \leq |B_1 \cap B_2| = k$. Since $e(G[B_1 \cap B_2]) = 4k - 20 + \binom{k-5}{2}$, it follows that

$$e(G[S]) \leq (4 + |S| - k)k - 20 + \binom{k-5}{2}$$

in both cases.

We now prove a lower bound to the number of edges in H_i' for $i = 1, 2$, a supgraph on $V(H_i)$ that can be obtained from G by contracting edges that have at least one end in $H_{3-i} - H_i$. In the case $G - N[x]$ has a cut vertex w, note that $w \in S = V(H_1 \cap H_2)$ and for both $i = 1, 2$, $G_i - \{w\} \subseteq H_i - H_{3-i}$ and w is a neighbor for every component of $G_i - \{w\}$. This means that for $i = 1, 2$, by contracting all of $G_i - \{w\}$ to w, we would have w to be adjacent to all vertices in $B_1 \cap B_2 \subseteq B_i$ in the new graph on S. Recall that for $i = 1, 2$, edges with at least one end in $B_i - B_{3-i}$ can be contracted in a way such that the new graph on $B_1 \cap B_2$ has at most 3 non-edges. Therefore, for $i = 1, 2$, there is a supgraph H_i' on $V(H_i)$ obtained by contracting edges that have at least one end in $H_{3-i} - H_i$ such that if $G - N[x]$ is disconnected, then

$$e(H_i'[S]) = e(H_i'[B_1 \cap B_2]) + d_{H_i'[S]}(x) \geq \binom{k}{2} - 3 + k;$$

and that if $G - N[x]$ has a cut vertex, then

$$e(H_i'[S]) = e(H_i'[B_1 \cap B_2]) + d_{H_i'[S]}(x) + d_{H_i'[S]}(w) \geq \binom{k}{2} - 3 + 2k.$$

To summarize, in both cases we have

$$e(H_i'[S]) \geq \binom{k}{2} - 3 + (|S| - k)k.$$

By Lemma 3.2.1, we know that $8|S| \geq 33 + e(H_1'[S]) + e(H_2'[S]) - e(G[S])$. Due to the upper bound for $e(G[S])$ and the lower bounds for $e(H_1'[S])$ and $e(H_2'[S])$ above, it follows that

$$8|S| \geq 33 + 2(\binom{k}{2} - 3 + (|S| - k)k) - ((4 + |S| - k)k - 20 + \binom{k-5}{2}) = 47 + (|S| - 5)k - \binom{k-5}{2}.$$

If $G - N[x]$ is disconnected, then $|S| = k + 1$ and thus

$$8(k+1) \geq 47 + (k-4)k - \binom{k-5}{2} = 47 + k^2 - 4k - \binom{k-5}{2},$$

meaning that $k^2 - 12k - \binom{k-5}{2} + 39 \leq 0$, a contradiction to the fact that $k \leq 7$.

If $G - N[x]$ has a cut vertex, then $|S| = k + 2$ and thus

$$8(k+2) \geq 47 + (k-3)k - \binom{k-5}{2} = 47 + k^2 - 3k - \binom{k-5}{2},$$

again a contradiction to the fact that $k \leq 7$. □

CHAPTER 5
CONCLUSION

Finally, in this chapter we apply Lemma 4.1.1, the main technical lemma, to complete the proof for Theorem 1.1.5.

Proof of Theorem 1.1.5. By Lemma 3.1.1 and Lemma 3.3.1, there exists a vertex $x \in V(G)$ with $11 \leq d(x) \leq 15$. We choose such a vertex x with a component K of $G - N[x]$ such that $|K|$ is minimum over all choices of x and K. In the next Claim, we prove that every vertex in K has degree at least 16 in G.

Claim 1. For every $y \in V(K)$, $d_G(y) \geq 16$.

Proof of Claim 1. Choose $y \in V(K)$ such that $d_G(y)$ is minimum among all vertices in K. For the sake of a contradiction, assume that $d_G(y) \leq 15$. Note that $x \in V(G) - N[y]$, as x, y are not adjacent to each other. Let L be the component of $G - N[y]$ that contains x, and let $M_y \subseteq N(y)$ be the subset of vertices that are not adjacent to all other vertices in $N(y)$.

Since $d_G(y) \leq 15$, by Lemma 4.1.1 there exists some vertex $z \in M_y - N(L)$. Note this implies that $z \notin N(x)$ and therefore $z \in V(K) - \{y\}$. By the choice of y, we know $d_G(z) \geq d_G(y)$. Since $z \in M_y$ is not adjacent to some vertex in $N(y)$, z must be adjacent to some $z' \in V(G) - N[y] \cup V(L)$. Let L' be the component of $G - N[y]$ that contains z', and note that $L \neq L'$ since $z \notin N(L)$. It follows that $z' \notin N(x)$. Since z' is a neighbor of z and $z \in V(K)$, it follows that $z' \in V(K)$ as well. Furthermore, since $x \in V(L)$ and L, L' are two distinct components of $G - N[y]$, it follows that $V(L') \cap N(x) = \emptyset$ and thus some component of $G - N[x]$ includes all vertices in L'. Since $z' \in V(L')$ is contained in K, it follows that $V(L') \subseteq V(K)$. Notice that $y \in V(K)$ and $y \notin V(L')$, and it follows that $|L'| < |K|$. Since $d_G(y) \leq 15$, the fact that $|L'| < |K|$ is a contradiction to the choice of x and K. $\dashv$

Now, let $G_1 = G - V(K)$ and $G_2 = G[N(K) \cup V(K)]$. Let d_2 be the maximum number of edges that can be added to G_2 by contracting edges that have at least one end in $G_1 - G_2$, and let J_2 be a minor of G on $V(G_2)$ such that $e(J_2) = e(G_2) + d_2$. By Claim 1, $d_G(y) \geq 16$ for every $y \in V(K)$. Since $|K| > 0$, we know $\Delta(J_2) \geq \Delta(G_2) \geq 16$, and therefore $J_2 \ncong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$. Since J_2 is a proper minor of G, it follows that $e(J_2) \leq 8|G_2| - 35$. Let $\delta = \delta(N(K))$ and choose $z \in N(K)$ such that $d_{N(K)}(z) = \delta$. Note that $N(K) \subseteq N(x)$, and therefore by contracting the edge xz we could have the new vertex adjacent to all other vertices in $N(K)$. This shows that $d_2 \geq |N(K)| - 1 - \delta$, and therefore

$$e(G_2) = e(J_2) - d_2 \leq (8|G_2| - 35) - (|N(K)| - 1 - \delta) = 8|K| + 7|N(K)| + \delta - 34.$$

By Claim 1, $d_G(y) \geq 16$ for every $y \in V(K)$, and this implies that $16|K| \leq \sum_{y \in V(K)} d_G(y) = 2e(K) + e(K, N(K))$. Note that $e(N(K)) \geq \frac{1}{2}\delta|N(K)|$. For simplicity, let $k = |K|$ and $N = |N(K)|$. It follows that

$$e(K) = \big(2e(K) + e(K, N(K))\big) + e(N(K)) - e(G_2)$$

$$\geq 16k + \frac{1}{2}\delta N - \big(8k + 7N + \delta - 34\big)$$
$$= 8k + \frac{1}{2}\delta(N - 2) - 7N + 34.$$

Since $\delta(N(x)) \geq 8$, $\delta = \delta(N(K)) \geq \delta(N(x)) - (d(x) - |N(K)|) \geq 8 - d(x) + N$. Let $d = d(x)$. It follows that

$$e(K) \geq 8k + \frac{1}{2}(8 - d + N)(N - 2) - 7N + 34.$$

Observe that

$$2\cdot\Big(\frac{1}{2}(8-d+N)(N-2)-7N+34\Big) = N^2-(8+d)N+2d(x)+52 = \big(N-\frac{1}{2}(8+d)\big)^2-\frac{1}{4}\big(d+4\big)^2+40.$$

Therefore,

$$2e(K) \geq 16k + \big(N - \frac{1}{2}(8+d)\big)^2 - \frac{1}{4}\big(d+4\big)^2 + 40.$$

Assume $|K| \geq 8$ for a moment. Since $G > K$, we know that $e(K) \leq 8k - 34$, and therefore

$$16k + \big(N - \frac{1}{2}(8+d)\big)^2 - \frac{1}{4}\big(d+4\big)^2 + 40 \leq 2e(K) \leq 16k - 68,$$

meaning that $\big(N-\frac{1}{2}(8+d)\big)^2 - \frac{1}{4}\big(d+4\big)^2 + 108 \leq 0$, and therefore $-\frac{1}{4}\big(d+4\big)^2 + 108 \leq 0$. Note that $d = d(x) \leq 15$. It follows that $-\frac{1}{4}\big(d+4\big)^2 + 108 \geq -\frac{19^2}{4} + 108 > 0$, a contradiction.

We may then assume $|K| \leq 7$. Note that

$$16k + \big(N - \frac{1}{2}(8+d)\big)^2 - \frac{1}{4}\big(d+4\big)^2 + 40 \leq 2e(K) \leq 2\binom{k}{2} = k^2 - k.$$

It follows that

$$k^2 - 17k \geq \big(N - \frac{1}{2}(8+d)\big)^2 - \frac{1}{4}\big(d+4\big)^2 + 40 \geq 0 - \frac{(15+4)^2}{4} + 40 = -\frac{201}{4},$$

where the second inequality is due to $d = d(x) \leq 15$. Since $k^2 - 17k \geq -\frac{201}{4}$ and $k \leq 7$, it follows that $k \leq 3$. By Lemma 4.1.1, we know $|N(K)| \leq d(x) - 1 \leq 14$. Since $d_G(y) \geq 16$ for every $y \in K$, it follows that $|K| = k = 3$, $|N(K)| = 14$, $d(x) = 15$, and that every vertex in K is adjacent to all other vertices in $G_2 = G[V(K) \cup N(K)]$.

Recall that $e(G_2) + d_2 = e(J_2) \leq 8|G_2| - 35$. Since $|G_2| = 3 + 14 = 17$, it follows that $e(G_2) + d_2 \leq 8 \cdot 17 - 35 = 101$. Therefore,

$$e(N(K)) + d_2 = \big(e(G_2) + d_2\big) - \big(e(K) + e(K, N(K))\big) \leq 101 - (3 + 3 \cdot 14) = 56.$$

Note that $\delta \geq \delta(N(x)) - 1 \geq 7$. This means that $e(N(K)) \geq \frac{1}{2}\delta|N(K)| \geq \frac{1}{2} \cdot 7 \cdot 14 = 49$, and thus $d_2 \leq 56 - 49 = 7$. If $\delta \geq 8$, then $e(N(K)) \geq \frac{1}{2} \cdot 8 \cdot 14 = 56$. It follows that $d_2 = 0$, $e(N(K)) = 56$, and $d_{N(K)}(v) = 8$ for every $v \in N(K)$. This then implies that, by contracting an edge between x and any vertex in $N(K)$, we would obtain exactly 5 extra edges on $N(K)$, meaning $d_2 \geq 5$, a contradiction. We then conclude that $\delta = 7$. Note now $d_2 \geq |N(K)| - 1 - \delta = 14 - 1 - 7 = 6$. Since $d_2 \leq 7$, it follows that $d_2 = 6$ or 7 and $e(N(K)) = 49$ or 50.

Now, note that $N(K)$ is either a 7-regular graph or obtained from a 7-regular graph by adding one more edge. Let $U = \{u \in N(K) : d_{N(K)}(u) = 7\}$, and notices that $|U| = 12$ or 14. Choose distinct vertices $u_1, u_2 \in U$ such that $u_1u_2 \notin E(G)$. Let w be the unique vertex in $N(x) - N(K)$. Since $\delta(N(x)) \geq 8$, every vertex $v \in N(K)$ such that $d_{N(K)}(v) = 7$ must be adjacent to w and therefore $U \subseteq N(w)$. Observe that $|N(K) - N[u_1]| = 14 - 1 - 7 = 6$, and $|N(w) \cap N(K) - N[u_2]| \geq |U - N[u_2]| \geq |U| - |N[u_2]| \geq 12 - 8 = 4$. Therefore, by contracting xu_1 and wu_2, we can obtain $|N(K) - N[u_1]| + |N(w) \cap N(K) - N[u_2]| - 1 \geq 6 + 4 - 1 = 9$ extra edges on $N(K)$. It follows that $d_2 \geq 9$, a contradiction to the fact that $d_2 \leq 7$. □

Appendices

APPENDIX A
PROBLEM GRAPHS

We present the 101 problem graphs explicitly here, as mentioned in Lemma 4.2.3. There are 13 problems graphs on 11 vertices, 35 problem graphs on 12 vertices, 33 problem graphs on 13 vertices, 11 problem graphs on 14 vertices, and 9 problem graphs on 15 vertices.

Here is how to read the problem graphs in this appendix: For $k = 11, 12, 13, 14, 15$, for each problem graph on k vertices, the vertices are 0-indexed, and we give the full list of neighbors of each vertex from vertex-0 to vertex-(k-1). For example, the third line of the matrix for graph 1 on 11 vertices, as shown below, says "2 : 3 4 5 6 7 8 9 10", and this means that the neighborhood of vertex-2 in this graph is precisely the set of vertices indexed 3, 4, 5, 6, 7, 8, 9, 10 in this graph.

A.1 Problem Graphs on 11 vertices

There are 13 problem graphs on 11 vertices, up to isomorphism.

Graph 1, on 11 vertices

0 : 3 4 5 6 7 8 9 10

1 : 3 4 5 6 7 8 9 10

2 : 3 4 5 6 7 8 9 10

3 : 0 1 2 6 7 8 9 10

4 : 0 1 2 6 7 8 9 10

5 : 0 1 2 6 7 8 9 10

6 : 0 1 2 3 4 5 9 10

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 3 4 5 9 10

9 : 0 1 2 3 4 5 6 7 8

10 : 0 1 2 3 4 5 6 7 8

Graph 2, on 11 vertices

0 : 3 4 5 6 7 8 9 10

1 : 3 4 5 6 7 8 9 10

2 : 3 4 5 6 7 8 9 10

3 : 0 1 2 6 7 8 9 10

4 : 0 1 2 6 7 8 9 10

5 : 0 1 2 6 7 8 9 10

6 : 0 1 2 3 4 5 8 10

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 3 4 5 6 10

9 : 0 1 2 3 4 5 7 10

10 : 0 1 2 3 4 5 6 7 8 9

Graph 3, on 11 vertices

0 : 3 4 5 6 7 8 9 10

1 : 3 4 5 6 7 8 9 10

2 : 3 4 5 6 7 8 9 10

3 : 0 1 2 6 7 8 9 10

4 : 0 1 2 6 7 8 9 10

5 : 0 1 2 6 7 8 9 10

6 : 0 1 2 3 4 5 8 9

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 3 4 5 6 10

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 7 8

Graph 4, on 11 vertices

0 : 3 4 5 6 7 8 9 10

1 : 3 4 5 6 7 8 9 10

2 : 3 4 5 6 7 8 9 10

3 : 0 1 2 5 6 7 9 10

4 : 0 1 2 6 7 8 9 10

5 : 0 1 2 3 7 8 9 10

6 : 0 1 2 3 4 8 9 10

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 4 5 6 9 10

9 : 0 1 2 3 4 5 6 7 8

10 : 0 1 2 3 4 5 6 7 8

Graph 5, on 11 vertices

0 : 2 4 5 6 7 8 9 10

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 6 7 8 9 10

3 : 1 4 5 6 7 8 9 10

4 : 0 1 2 3 6 8 9 10

5 : 0 1 2 3 7 8 9 10

6 : 0 1 2 3 4 8 9 10

7 : 0 1 2 3 5 8 9 10

8 : 0 1 2 3 4 5 6 7

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7

Graph 6, on 11 vertices

0 : 2 3 5 6 7 8 9 10

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 6 7 8 9 10

3 : 0 1 5 6 7 8 9 10

4 : 1 2 5 6 7 8 9 10

5 : 0 1 2 3 4 7 9 10

6 : 0 1 2 3 4 8 9 10

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 3 4 6 9 10

9 : 0 1 2 3 4 5 6 7 8

10 : 0 1 2 3 4 5 6 7 8

Graph 7, on 11 vertices

0 : 2 3 5 6 7 8 9 10

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 6 7 8 9 10

3 : 0 1 5 6 7 8 9 10

4 : 1 2 5 6 7 8 9 10

5 : 0 1 2 3 4 7 8 10

6 : 0 1 2 3 4 8 9 10

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 3 4 5 6 10

9 : 0 1 2 3 4 6 7 10

10 : 0 1 2 3 4 5 6 7 8 9

Graph 8, on 11 vertices

0 : 2 3 4 6 7 8 9 10

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 6 7 8 9 10

3 : 0 1 5 6 7 8 9 10

4 : 0 1 2 6 7 8 9 10

5 : 1 2 3 6 7 8 9 10

6 : 0 1 2 3 4 5 8 9

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 3 4 5 6 10

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 7 8

Graph 9, on 11 vertices

0 : 2 3 4 5 7 8 9 10

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 6 7 8 9 10

3 : 0 1 5 6 7 8 9 10

4 : 0 1 2 6 7 8 9 10

5 : 0 1 2 3 7 8 9 10

6 : 1 2 3 4 7 8 9 10

7 : 0 1 2 3 4 5 6 10

8 : 0 1 2 3 4 5 6 10

9 : 0 1 2 3 4 5 6 10

10 : 0 1 2 3 4 5 6 7 8 9

Graph 10, on 11 vertices

0 : 2 3 4 5 7 8 9 10

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 6 7 8 9 10

3 : 0 1 5 6 7 8 9 10

4 : 0 1 2 6 7 8 9 10

5 : 0 1 2 3 7 8 9 10

6 : 1 2 3 4 7 8 9 10

7 : 0 1 2 3 4 5 6 9

8 : 0 1 2 3 4 5 6 10

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 8

Graph 11, on 11 vertices

0 : 2 3 4 5 6 8 9 10

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 6 7 8 9 10

3 : 0 1 5 6 7 8 9 10

4 : 0 1 2 6 7 8 9 10

5 : 0 1 2 3 7 8 9 10

6 : 0 1 2 3 4 8 9 10

7 : 1 2 3 4 5 8 9 10

8 : 0 1 2 3 4 5 6 7

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7

Graph 12, on 11 vertices

0 : 2 3 4 5 6 7 9 10

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 6 7 8 9 10

3 : 0 1 5 6 7 8 9 10

4 : 0 1 2 6 7 8 9 10

5 : 0 1 2 3 7 8 9 10

6 : 0 1 2 3 4 8 9 10

7 : 0 1 2 3 4 5 9 10

8 : 1 2 3 4 5 6 9 10

9 : 0 1 2 3 4 5 6 7 8

10 : 0 1 2 3 4 5 6 7 8

Graph 13, on 11 vertices

0 : 2 3 4 5 6 7 8 9

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 6 7 8 9 10

3 : 0 1 5 6 7 8 9 10

4 : 0 1 2 6 7 8 9 10

5 : 0 1 2 3 7 8 9 10

6 : 0 1 2 3 4 8 9 10

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 3 4 5 6 10

9 : 0 1 2 3 4 5 6 7

10 : 1 2 3 4 5 6 7 8

A.2 Problem Graphs on 12 vertices

There are 35 problem graphs on 12 vertices, up to isomorphism.

Graph 1, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 8 9 10 11

5 : 0 1 2 3 8 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 8 9 10 11

8 : 0 1 2 3 4 5 6 7

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7

11 : 0 1 2 3 4 5 6 7

Graph 2, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 7 9 10 11

5 : 0 1 2 3 8 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 9 10 11

8 : 0 1 2 3 5 6 10 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

Graph 3, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 7 8 10 11

5 : 0 1 2 3 8 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 9 10 11

8 : 0 1 2 3 4 5 6 11

9 : 0 1 2 3 5 6 7 11

10 : 0 1 2 3 4 5 6 7

11 : 0 1 2 3 4 5 6 7 8 9

Graph 4, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 7 8 10 11

5 : 0 1 2 3 8 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 9 10 11

8 : 0 1 2 3 4 5 6 10

9 : 0 1 2 3 5 6 7 11

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 9

Graph 5, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 7 8 9 10

5 : 0 1 2 3 8 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 9 10 11

8 : 0 1 2 3 4 5 6 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7

11 : 0 1 2 3 5 6 7 8

Graph 6, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 7 8 10 11

5 : 0 1 2 3 7 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 5 10 11

8 : 0 1 2 3 4 6 10 11

9 : 0 1 2 3 5 6 10 11

10 : 0 1 2 3 4 5 6 7 8 9

11 : 0 1 2 3 4 5 6 7 8 9

Graph 7, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 7 8 9 11

5 : 0 1 2 3 7 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 5 10 11

8 : 0 1 2 3 4 6 10 11

9 : 0 1 2 3 4 5 6 11

10 : 0 1 2 3 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8 9

Graph 8, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 7 8 9 10

5 : 0 1 2 3 7 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 5 10 11

8 : 0 1 2 3 4 6 10 11

9 : 0 1 2 3 4 5 6 11

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 5 6 7 8 9

Graph 9, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 7 8 10 11

5 : 0 1 2 3 7 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 3 4 6 10 11

9 : 0 1 2 3 5 6 7 11

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 8 9

Graph 10, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 7 8 10 11

5 : 0 1 2 3 7 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 3 4 6 9 11

9 : 0 1 2 3 5 6 7 8

10 : 0 1 2 3 4 5 6 7

11 : 0 1 2 3 4 5 6 8

Graph 11, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 6 8 9 10

5 : 0 1 2 3 7 9 10 11

6 : 0 1 2 3 4 8 9 11

7 : 0 1 2 3 5 9 10 11

8 : 0 1 2 3 4 6 10 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 7 8

11 : 0 1 2 3 5 6 7 8

Graph 12, on 12 vertices

0 : 4 5 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 4 5 6 7 8 9 10 11

4 : 0 1 2 3 6 8 9 11

5 : 0 1 2 3 7 8 9 10

6 : 0 1 2 3 4 8 10 11

7 : 0 1 2 3 5 9 10 11

8 : 0 1 2 3 4 5 6 10

9 : 0 1 2 3 4 5 7 11

10 : 0 1 2 3 5 6 7 8

11 : 0 1 2 3 4 6 7 9

Graph 13, on 12 vertices

0 : 3 4 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 5 6 7 8 9 10 11

4 : 0 1 2 6 8 9 10 11

5 : 1 2 3 7 8 9 10 11

6 : 0 1 2 3 4 9 10 11

7 : 0 1 2 3 5 9 10 11

8 : 0 1 2 3 4 5 10 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

Graph 14, on 12 vertices

0 : 3 4 6 7 8 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 5 6 7 8 9 10 11

4 : 0 1 2 6 7 8 10 11

5 : 1 2 3 7 8 9 10 11

6 : 0 1 2 3 4 9 10 11

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 3 4 5 9 11

9 : 0 1 2 3 5 6 7 8

10 : 0 1 2 3 4 5 6 7

11 : 0 1 2 3 4 5 6 8

Graph 15, on 12 vertices

0 : 3 4 5 6 7 9 10 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 5 6 7 8 9 10 11

4 : 0 1 2 7 8 9 10 11

5 : 0 1 2 3 8 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 9 10 11

8 : 1 2 3 4 5 6 10 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

Graph 16, on 12 vertices

0 : 3 4 5 6 7 8 9 10

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 5 6 7 8 9 10 11

4 : 0 1 2 7 8 9 10 11

5 : 0 1 2 3 8 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 9 10 11

8 : 0 1 2 3 4 5 6 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7

11 : 1 2 3 4 5 6 7 8

Graph 17, on 12 vertices

0 : 3 4 5 6 7 8 9 10

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 5 6 7 8 9 10 11

4 : 0 1 2 6 7 9 10 11

5 : 0 1 2 3 8 9 10 11

6 : 0 1 2 3 4 8 9 10

7 : 0 1 2 3 4 8 10 11

8 : 0 1 2 3 5 6 7 11

9 : 0 1 2 3 4 5 6 11

10 : 0 1 2 3 4 5 6 7

11 : 1 2 3 4 5 7 8 9

Graph 18, on 12 vertices

0 : 3 4 5 6 7 8 9 11

1 : 4 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 4 5 6 7 8 10 11

4 : 0 1 2 3 7 8 9 10

5 : 0 1 2 3 8 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 2 3 4 9 10 11

8 : 0 1 2 3 4 5 6 10

9 : 0 1 2 4 5 6 7 11

10 : 1 2 3 4 5 6 7 8

11 : 0 1 2 3 5 6 7 9

Graph 19, on 12 vertices

0 : 3 4 6 7 8 9 10 11

1 : 3 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 1 6 7 8 9 10 11

4 : 0 2 6 7 8 9 10 11

5 : 1 2 6 7 8 9 10 11

6 : 0 1 2 3 4 5 9 10

7 : 0 1 2 3 4 5 10 11

8 : 0 1 2 3 4 5 10 11

9 : 0 1 2 3 4 5 6 11

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 7 8 9

Graph 20, on 12 vertices

0 : 3 4 6 7 8 9 10 11

1 : 3 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 1 6 7 8 9 10 11

4 : 0 2 6 7 8 9 10 11

5 : 1 2 6 7 8 9 10 11

6 : 0 1 2 3 4 5 9 10

7 : 0 1 2 3 4 5 9 11

8 : 0 1 2 3 4 5 10 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 8

11 : 0 1 2 3 4 5 7 8

Graph 21, on 12 vertices

0 : 3 4 5 6 8 9 10 11

1 : 3 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 1 6 7 8 9 10 11

4 : 0 2 6 7 8 9 10 11

5 : 0 1 2 7 8 9 10 11

6 : 0 1 2 3 4 9 10 11

7 : 1 2 3 4 5 9 10 11

8 : 0 1 2 3 4 5 10 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

Graph 22, on 12 vertices

0 : 3 4 5 6 7 8 10 11

1 : 3 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 1 6 7 8 9 10 11

4 : 0 2 6 7 8 9 10 11

5 : 0 1 2 7 8 9 10 11

6 : 0 1 2 3 4 9 10 11

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 3 4 5 9 11

9 : 1 2 3 4 5 6 7 8

10 : 0 1 2 3 4 5 6 7

11 : 0 1 2 3 4 5 6 8

Graph 23, on 12 vertices

0 : 3 4 5 6 7 8 9 10

1 : 3 4 5 6 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 1 6 7 8 9 10 11

4 : 0 1 2 6 7 8 9 11

5 : 0 1 2 7 8 9 10 11

6 : 0 1 2 3 4 9 10 11

7 : 0 2 3 4 5 9 10 11

8 : 0 1 2 3 4 5 10 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 5 6 7 8

11 : 1 2 3 4 5 6 7 8

Graph 24, on 12 vertices

0 : 3 4 5 6 8 9 10 11

1 : 3 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 1 4 6 7 9 10 11

4 : 0 2 3 7 8 9 10 11

5 : 0 1 2 7 8 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 1 2 3 4 5 9 10 11

8 : 0 1 2 4 5 6 10 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

Graph 25, on 12 vertices

0 : 3 4 5 7 8 9 10 11

1 : 3 5 6 7 8 9 10 11

2 : 4 5 6 7 8 9 10 11

3 : 0 1 4 6 7 8 9 10

4 : 0 2 3 6 8 9 10 11

5 : 0 1 2 6 7 8 10 11

6 : 1 2 3 4 5 8 9 10

7 : 0 1 2 3 5 9 10 11

8 : 0 1 2 3 4 5 6 11

9 : 0 1 2 3 4 6 7 11

10 : 0 1 2 3 4 5 6 7

11 : 0 1 2 4 5 7 8 9

Graph 26, on 12 vertices

0 : 2 4 5 6 7 9 10 11

1 : 3 4 6 7 8 9 10 11

2 : 0 4 5 6 8 9 10 11

3 : 1 5 6 7 8 9 10 11

4 : 0 1 2 7 8 9 10 11

5 : 0 2 3 7 8 9 10 11

6 : 0 1 2 3 8 9 10 11

7 : 0 1 3 4 5 9 10 11

8 : 1 2 3 4 5 6 10 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

Graph 27, on 12 vertices

0 : 2 4 5 6 7 9 10 11

1 : 3 4 6 7 8 9 10 11

2 : 0 4 5 7 8 9 10 11

3 : 1 5 6 7 8 9 10 11

4 : 0 1 2 6 8 9 10 11

5 : 0 2 3 7 8 9 10 11

6 : 0 1 3 4 8 9 10 11

7 : 0 1 2 3 5 9 10 11

8 : 1 2 3 4 5 6 10 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

Graph 28, on 12 vertices

0 : 2 4 5 6 8 9 10 11

1 : 3 4 6 7 8 9 10 11

2 : 0 4 5 7 8 9 10 11

3 : 1 5 6 7 8 9 10 11

4 : 0 1 2 6 7 9 10 11

5 : 0 2 3 7 8 9 10 11

6 : 0 1 3 4 8 9 10 11

7 : 1 2 3 4 5 9 10 11

8 : 0 1 2 3 5 6 10 11

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

Graph 29, on 12 vertices

0 : 2 4 5 6 7 8 9 10

1 : 3 4 6 7 8 9 10 11

2 : 0 4 5 6 7 8 10 11

3 : 1 5 6 7 8 9 10 11

4 : 0 1 2 5 8 9 10 11

5 : 0 2 3 4 8 9 10 11

6 : 0 1 2 3 7 8 9 11

7 : 0 1 2 3 6 9 10 11

8 : 0 1 2 3 4 5 6 10

9 : 0 1 3 4 5 6 7 11

10 : 0 1 2 3 4 5 7 8

11 : 1 2 3 4 5 6 7 9

Graph 30, on 12 vertices

0 : 2 3 5 6 8 9 10 11

1 : 3 4 5 6 7 8 10 11

2 : 0 4 5 7 8 9 10 11

3 : 0 1 5 6 7 9 10 11

4 : 1 2 6 7 8 9 10 11

5 : 0 1 2 3 7 8 10 11

6 : 0 1 3 4 8 9 10 11

7 : 1 2 3 4 5 9 10 11

8 : 0 1 2 4 5 6 10 11

9 : 0 2 3 4 6 7 10 11

10 : 0 1 2 3 4 5 6 7 8 9

11 : 0 1 2 3 4 5 6 7 8 9

Graph 31, on 12 vertices

0 : 2 3 5 6 8 9 10 11

1 : 3 4 5 6 7 8 9 11

2 : 0 4 5 7 8 9 10 11

3 : 0 1 5 6 7 9 10 11

4 : 1 2 6 7 8 9 10 11

5 : 0 1 2 3 7 8 10 11

6 : 0 1 3 4 8 9 10 11

7 : 1 2 3 4 5 9 10 11

8 : 0 1 2 4 5 6 10 11

9 : 0 1 2 3 4 6 7 11

10 : 0 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8 9

Graph 32, on 12 vertices

0 : 2 3 5 6 8 9 10 11

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 7 8 9 10 11

3 : 0 1 5 6 7 9 10 11

4 : 1 2 6 7 8 9 10 11

5 : 0 1 2 3 7 8 10 11

6 : 0 1 3 4 8 9 10 11

7 : 1 2 3 4 5 9 10 11

8 : 0 1 2 4 5 6 10 11

9 : 0 1 2 3 4 6 7 11

10 : 0 1 2 3 4 5 6 7 8

11 : 0 2 3 4 5 6 7 8 9

Graph 33, on 12 vertices

0 : 2 3 5 6 8 9 10 11

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 7 8 9 10 11

3 : 0 1 5 6 7 8 10 11

4 : 1 2 6 7 8 9 10 11

5 : 0 1 2 3 7 8 9 10

6 : 0 1 3 4 8 9 10 11

7 : 1 2 3 4 5 9 10 11

8 : 0 1 2 3 4 5 6 11

9 : 0 1 2 4 5 6 7 11

10 : 0 1 2 3 4 5 6 7

11 : 0 2 3 4 6 7 8 9

Graph 34, on 12 vertices

0 : 2 3 5 6 8 9 10 11

1 : 3 4 5 6 7 8 9 10

2 : 0 4 5 7 8 9 10 11

3 : 0 1 5 6 7 8 10 11

4 : 1 2 6 7 8 9 10 11

5 : 0 1 2 3 7 8 9 11

6 : 0 1 3 4 8 9 10 11

7 : 1 2 3 4 5 9 10 11

8 : 0 1 2 3 4 5 6 10

9 : 0 1 2 4 5 6 7 11

10 : 0 1 2 3 4 6 7 8

11 : 0 2 3 4 5 6 7 9

Graph 35, on 12 vertices

0 : 2 3 5 6 7 8 9 10

1 : 3 4 5 6 7 8 10 11

2 : 0 4 5 7 8 9 10 11

3 : 0 1 5 6 7 9 10 11

4 : 1 2 6 7 8 9 10 11

5 : 0 1 2 3 7 8 9 11

6 : 0 1 3 4 8 9 10 11

7 : 0 1 2 3 4 5 9 10

8 : 0 1 2 4 5 6 10 11

9 : 0 2 3 4 5 6 7 11

10 : 0 1 2 3 4 6 7 8

11 : 1 2 3 4 5 6 8 9

A.3 Problem Graphs on 13 vertices

There are 33 problem graphs on 13 vertices, up to isomorphism.

Graph 1, on 13 vertices

0 : 5 6 7 8 9 10 11 12

1 : 5 6 7 8 9 10 11 12

2 : 5 6 7 8 9 10 11 12

3 : 5 6 7 8 9 10 11 12

4 : 5 6 7 8 9 10 11 12

5 : 0 1 2 3 4 10 11 12

6 : 0 1 2 3 4 10 11 12

7 : 0 1 2 3 4 10 11 12

8 : 0 1 2 3 4 10 11 12

9 : 0 1 2 3 4 10 11 12

10 : 0 1 2 3 4 5 6 7 8 9

11 : 0 1 2 3 4 5 6 7 8 9

12 : 0 1 2 3 4 5 6 7 8 9

Graph 2, on 13 vertices

0 : 5 6 7 8 9 10 11 12

1 : 5 6 7 8 9 10 11 12

2 : 5 6 7 8 9 10 11 12

3 : 5 6 7 8 9 10 11 12

4 : 5 6 7 8 9 10 11 12

5 : 0 1 2 3 4 9 11 12

6 : 0 1 2 3 4 9 11 12

7 : 0 1 2 3 4 10 11 12

8 : 0 1 2 3 4 10 11 12

9 : 0 1 2 3 4 5 6 10

10 : 0 1 2 3 4 7 8 9

11 : 0 1 2 3 4 5 6 7 8

12 : 0 1 2 3 4 5 6 7 8

Graph 3, on 13 vertices

0 : 5 6 7 8 9 10 11 12

1 : 5 6 7 8 9 10 11 12

2 : 5 6 7 8 9 10 11 12

3 : 5 6 7 8 9 10 11 12

4 : 5 6 7 8 9 10 11 12

5 : 0 1 2 3 4 9 10 11

6 : 0 1 2 3 4 9 10 12

7 : 0 1 2 3 4 9 11 12

8 : 0 1 2 3 4 10 11 12

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 6 8

11 : 0 1 2 3 4 5 7 8

12 : 0 1 2 3 4 6 7 8

Graph 4, on 13 vertices

0 : 5 6 7 8 9 10 11 12

1 : 5 6 7 8 9 10 11 12

2 : 5 6 7 8 9 10 11 12

3 : 5 6 7 8 9 10 11 12

4 : 5 6 7 8 9 10 11 12

5 : 0 1 2 3 4 8 9 10

6 : 0 1 2 3 4 9 10 11

7 : 0 1 2 3 4 10 11 12

8 : 0 1 2 3 4 5 11 12

9 : 0 1 2 3 4 5 6 12

10 : 0 1 2 3 4 5 6 7

11 : 0 1 2 3 4 6 7 8

12 : 0 1 2 3 4 7 8 9

Graph 5, on 13 vertices

0 : 4 5 6 7 9 10 11 12

1 : 4 5 6 7 9 10 11 12

2 : 4 6 7 8 9 10 11 12

3 : 5 6 7 8 9 10 11 12

4 : 0 1 2 8 9 10 11 12

5 : 0 1 3 8 9 10 11 12

6 : 0 1 2 3 9 10 11 12

7 : 0 1 2 3 9 10 11 12

8 : 2 3 4 5 9 10 11 12

9 : 0 1 2 3 4 5 6 7 8

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

12 : 0 1 2 3 4 5 6 7 8

Graph 6, on 13 vertices

0 : 3 4 5 6 9 10 11 12

1 : 4 5 6 7 9 10 11 12

2 : 5 6 7 8 9 10 11 12

3 : 0 6 7 8 9 10 11 12

4 : 0 1 7 8 9 10 11 12

5 : 0 1 2 8 9 10 11 12

6 : 0 1 2 3 9 10 11 12

7 : 1 2 3 4 9 10 11 12

8 : 2 3 4 5 9 10 11 12

9 : 0 1 2 3 4 5 6 7 8

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

12 : 0 1 2 3 4 5 6 7 8

Graph 7, on 13 vertices

0 : 3 4 6 8 9 10 11 12

1 : 4 5 6 7 9 10 11 12

2 : 5 6 7 8 9 10 11 12

3 : 0 5 6 7 9 10 11 12

4 : 0 1 7 8 9 10 11 12

5 : 1 2 3 8 9 10 11 12

6 : 0 1 2 3 9 10 11 12

7 : 1 2 3 4 9 10 11 12

8 : 0 2 4 5 9 10 11 12

9 : 0 1 2 3 4 5 6 7 8

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

12 : 0 1 2 3 4 5 6 7 8

Graph 8, on 13 vertices

0 : 3 4 5 6 8 9 11 12

1 : 5 6 7 8 9 10 11 12

2 : 5 6 7 8 9 10 11 12

3 : 0 4 5 7 8 9 10 12

4 : 0 3 6 7 8 10 11 12

5 : 0 1 2 3 7 9 11 12

6 : 0 1 2 4 9 10 11 12

7 : 1 2 3 4 5 10 11 12

8 : 0 1 2 3 4 9 10 11

9 : 0 1 2 3 5 6 8 10

10 : 1 2 3 4 6 7 8 9

11 : 0 1 2 4 5 6 7 8

12 : 0 1 2 3 4 5 6 7

Graph 9, on 13 vertices

0 : 3 4 6 7 8 9 11 12

1 : 4 5 6 8 9 10 11 12

2 : 4 5 6 8 9 10 11 12

3 : 0 5 6 7 9 10 11 12

4 : 0 1 2 7 8 9 10 11

5 : 1 2 3 7 8 10 11 12

6 : 0 1 2 3 7 9 10 12

7 : 0 3 4 5 6 8 10 12

8 : 0 1 2 4 5 7 11 12

9 : 0 1 2 3 4 6 10 11

10 : 1 2 3 4 5 6 7 9

11 : 0 1 2 3 4 5 8 9

12 : 0 1 2 3 5 6 7 8

Graph 10, on 13 vertices

0 : 3 4 5 6 8 9 10 11

1 : 4 5 6 7 8 10 11 12

2 : 4 5 6 7 8 10 11 12

3 : 0 5 6 7 8 9 10 12

4 : 0 1 2 7 8 9 10 12

5 : 0 1 2 3 9 10 11 12

6 : 0 1 2 3 9 10 11 12

7 : 1 2 3 4 8 9 11 12

8 : 0 1 2 3 4 7 10 11

9 : 0 3 4 5 6 7 11 12

10 : 0 1 2 3 4 5 6 8

11 : 0 1 2 5 6 7 8 9

12 : 1 2 3 4 5 6 7 9

Graph 11, on 13 vertices

0 : 3 4 5 7 8 10 11 12

1 : 4 5 6 7 8 9 10 11

2 : 5 6 7 8 9 10 11 12

3 : 0 4 6 7 9 10 11 12

4 : 0 1 3 7 8 9 11 12

5 : 0 1 2 6 8 9 10 12

6 : 1 2 3 5 8 9 10 11

7 : 0 1 2 3 4 10 11 12

8 : 0 1 2 4 5 6 9 11

9 : 1 2 3 4 5 6 8 12

10 : 0 1 2 3 5 6 7 12

11 : 0 1 2 3 4 6 7 8

12 : 0 2 3 4 5 7 9 10

Graph 12, on 13 vertices

0 : 3 4 6 7 9 10 11 12

1 : 3 5 6 8 9 10 11 12

2 : 4 5 7 8 9 10 11 12

3 : 0 1 6 7 9 10 11 12

4 : 0 2 6 8 9 10 11 12

5 : 1 2 7 8 9 10 11 12

6 : 0 1 3 4 9 10 11 12

7 : 0 2 3 5 9 10 11 12

8 : 1 2 4 5 9 10 11 12

9 : 0 1 2 3 4 5 6 7 8

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

12 : 0 1 2 3 4 5 6 7 8

Graph 13, on 13 vertices

0 : 3 4 6 7 9 10 11 12

1 : 3 5 6 8 9 10 11 12

2 : 4 5 7 8 9 10 11 12

3 : 0 1 6 7 9 10 11 12

4 : 0 2 6 8 9 10 11 12

5 : 1 2 7 8 9 10 11 12

6 : 0 1 3 4 8 9 11 12

7 : 0 2 3 5 9 10 11 12

8 : 1 2 4 5 6 10 11 12

9 : 0 1 2 3 4 5 6 7

10 : 0 1 2 3 4 5 7 8

11 : 0 1 2 3 4 5 6 7 8

12 : 0 1 2 3 4 5 6 7 8

Graph 14, on 13 vertices

0 : 3 4 6 8 9 10 11 12

1 : 3 5 6 7 9 10 11 12

2 : 4 5 7 8 9 10 11 12

3 : 0 1 6 7 9 10 11 12

4 : 0 2 6 8 9 10 11 12

5 : 1 2 7 8 9 10 11 12

6 : 0 1 3 4 9 10 11 12

7 : 1 2 3 5 9 10 11 12

8 : 0 2 4 5 9 10 11 12

9 : 0 1 2 3 4 5 6 7 8

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

12 : 0 1 2 3 4 5 6 7 8

Graph 15, on 13 vertices

0 : 3 4 7 8 9 10 11 12

1 : 3 5 6 8 9 10 11 12

2 : 4 5 6 7 9 10 11 12

3 : 0 1 6 7 9 10 11 12

4 : 0 2 6 8 9 10 11 12

5 : 1 2 7 8 9 10 11 12

6 : 1 2 3 4 9 10 11 12

7 : 0 2 3 5 9 10 11 12

8 : 0 1 4 5 9 10 11 12

9 : 0 1 2 3 4 5 6 7 8

10 : 0 1 2 3 4 5 6 7 8

11 : 0 1 2 3 4 5 6 7 8

12 : 0 1 2 3 4 5 6 7 8

Graph 16, on 13 vertices

0 : 3 4 5 6 9 10 11 12

1 : 3 5 6 7 8 9 11 12

2 : 4 5 7 8 9 10 11 12

3 : 0 1 6 7 8 10 11 12

4 : 0 2 6 8 9 10 11 12

5 : 0 1 2 7 9 10 11 12

6 : 0 1 3 4 8 9 11 12

7 : 1 2 3 5 8 10 11 12

8 : 1 2 3 4 6 7 11 12

9 : 0 1 2 4 5 6 11 12

10 : 0 2 3 4 5 7 11 12

11 : 0 1 2 3 4 5 6 7 8 9 10

12 : 0 1 2 3 4 5 6 7 8 9 10

Graph 17, on 13 vertices

0 : 3 4 5 6 8 9 11 12

1 : 3 5 6 7 9 10 11 12

2 : 4 5 7 8 9 10 11 12

3 : 0 1 6 7 9 10 11 12

4 : 0 2 7 8 9 10 11 12

5 : 0 1 2 6 8 9 11 12

6 : 0 1 3 5 8 10 11 12

7 : 1 2 3 4 9 10 11 12

8 : 0 2 4 5 6 10 11 12

9 : 0 1 2 3 4 5 7 10

10 : 1 2 3 4 6 7 8 9

11 : 0 1 2 3 4 5 6 7 8

12 : 0 1 2 3 4 5 6 7 8

Graph 18, on 13 vertices

0 : 3 4 5 7 8 9 11 12

1 : 3 6 7 8 9 10 11 12

2 : 4 5 6 7 8 10 11 12

3 : 0 1 5 6 9 10 11 12

4 : 0 2 6 7 8 9 11 12

5 : 0 2 3 7 8 10 11 12

6 : 1 2 3 4 9 10 11 12

7 : 0 1 2 4 5 9 10 11

8 : 0 1 2 4 5 9 10 12

9 : 0 1 3 4 6 7 8 10

10 : 1 2 3 5 6 7 8 9

11 : 0 1 2 3 4 5 6 7

12 : 0 1 2 3 4 5 6 8

Graph 19, on 13 vertices

0 : 3 4 6 7 8 10 11 12

1 : 3 5 6 8 9 10 11 12

2 : 4 5 7 8 9 10 11 12

3 : 0 1 5 7 8 10 11 12

4 : 0 2 6 7 9 10 11 12

5 : 1 2 3 7 9 10 11 12

6 : 0 1 4 8 9 10 11 12

7 : 0 2 3 4 5 10 11 12

8 : 0 1 2 3 6 10 11 12

9 : 1 2 4 5 6 10 11 12

10 : 0 1 2 3 4 5 6 7 8 9

11 : 0 1 2 3 4 5 6 7 8 9

12 : 0 1 2 3 4 5 6 7 8 9

Graph 20, on 13 vertices

0 : 3 4 6 7 8 10 11 12

1 : 3 5 6 7 9 10 11 12

2 : 4 5 6 7 8 10 11 12

3 : 0 1 5 8 9 10 11 12

4 : 0 2 6 7 8 9 10 11

5 : 1 2 3 8 9 10 11 12

6 : 0 1 2 4 7 9 10 12

7 : 0 1 2 4 6 9 11 12

8 : 0 2 3 4 5 9 10 11

9 : 1 3 4 5 6 7 8 12

10 : 0 1 2 3 4 5 6 8

11 : 0 1 2 3 4 5 7 8

12 : 0 1 2 3 5 6 7 9

Graph 21, on 13 vertices

0 : 3 4 6 7 9 10 11 12

1 : 3 5 6 8 9 10 11 12

2 : 4 5 6 8 9 10 11 12

3 : 0 1 5 7 8 10 11 12

4 : 0 2 6 7 8 9 11 12

5 : 1 2 3 7 8 9 10 12

6 : 0 1 2 4 7 9 10 11

7 : 0 3 4 5 6 8 10 11

8 : 1 2 3 4 5 7 11 12

9 : 0 1 2 4 5 6 10 12

10 : 0 1 2 3 5 6 7 9

11 : 0 1 2 3 4 6 7 8

12 : 0 1 2 3 4 5 8 9

Graph 22, on 13 vertices

0 : 3 4 6 7 8 10 11 12

1 : 3 5 6 8 9 10 11 12

2 : 4 5 7 8 9 10 11 12

3 : 0 1 5 6 7 10 11 12

4 : 0 2 6 7 8 9 11 12

5 : 1 2 3 7 9 10 11 12

6 : 0 1 3 4 8 9 11 12

7 : 0 2 3 4 5 10 11 12

8 : 0 1 2 4 6 9 10 12

9 : 1 2 4 5 6 8 11 12

10 : 0 1 2 3 5 7 8 12

11 : 0 1 2 3 4 5 6 7 9

12 : 0 1 2 3 4 5 6 7 8 9 10

Graph 23, on 13 vertices

0 : 3 4 6 7 8 10 11 12

1 : 3 5 6 8 9 10 11 12

2 : 4 5 7 8 9 10 11 12

3 : 0 1 5 6 7 10 11 12

4 : 0 2 6 7 8 9 11 12

5 : 1 2 3 7 9 10 11 12

6 : 0 1 3 4 8 9 11 12

7 : 0 2 3 4 5 10 11 12

8 : 0 1 2 4 6 9 10 11

9 : 1 2 4 5 6 8 11 12

10 : 0 1 2 3 5 7 8 12

11 : 0 1 2 3 4 5 6 7 8 9

12 : 0 1 2 3 4 5 6 7 9 10

Graph 24, on 13 vertices

0 : 3 4 6 7 8 9 10 12

1 : 3 5 6 8 9 10 11 12

2 : 4 5 7 8 9 10 11 12

3 : 0 1 5 6 7 9 11 12

4 : 0 2 6 7 8 10 11 12

5 : 1 2 3 7 9 10 11 12

6 : 0 1 3 4 8 10 11 12

7 : 0 2 3 4 5 9 11 12

8 : 0 1 2 4 6 9 10 11

9 : 0 1 2 3 5 7 8 10

10 : 0 1 2 4 5 6 8 9

11 : 1 2 3 4 5 6 7 8

12 : 0 1 2 3 4 5 6 7

Graph 25, on 13 vertices

0 : 2 4 5 6 8 9 10 11

1 : 3 4 6 7 8 9 11 12

2 : 0 4 5 7 8 9 10 12

3 : 1 5 6 7 8 10 11 12

4 : 0 1 2 6 9 10 11 12

5 : 0 2 3 7 8 9 10 11

6 : 0 1 3 4 8 10 11 12

7 : 1 2 3 5 9 10 11 12

8 : 0 1 2 3 5 6 9 12

9 : 0 1 2 4 5 7 8 11

10 : 0 2 3 4 5 6 7 12

11 : 0 1 3 4 5 6 7 9

12 : 1 2 3 4 6 7 8 10

Graph 26, on 13 vertices

0 : 2 4 5 6 8 9 11 12

1 : 3 4 6 7 8 9 10 12

2 : 0 4 5 7 9 10 11 12

3 : 1 5 6 7 8 10 11 12

4 : 0 1 2 6 8 9 10 12

5 : 0 2 3 7 8 10 11 12

6 : 0 1 3 4 8 9 11 12

7 : 1 2 3 5 9 10 11 12

8 : 0 1 3 4 5 6 10 12

9 : 0 1 2 4 6 7 11 12

10 : 1 2 3 4 5 7 8 12

11 : 0 2 3 5 6 7 9 12

12 : 0 1 2 3 4 5 6 7 8 9 10 11

Graph 27, on 13 vertices

0 : 2 4 5 6 8 9 11 12

1 : 3 4 6 7 8 9 10 11

2 : 0 4 5 7 9 10 11 12

3 : 1 5 6 7 8 10 11 12

4 : 0 1 2 6 8 9 10 12

5 : 0 2 3 7 8 10 11 12

6 : 0 1 3 4 8 9 11 12

7 : 1 2 3 5 9 10 11 12

8 : 0 1 3 4 5 6 10 12

9 : 0 1 2 4 6 7 11 12

10 : 1 2 3 4 5 7 8 12

11 : 0 1 2 3 5 6 7 9

12 : 0 2 3 4 5 6 7 8 9 10

Graph 28, on 13 vertices

0 : 2 4 5 6 8 9 11 12

1 : 3 4 6 7 8 9 10 12

2 : 0 4 5 7 9 10 11 12

3 : 1 5 6 7 8 10 11 12

4 : 0 1 2 6 8 9 10 11

5 : 0 2 3 7 8 10 11 12

6 : 0 1 3 4 8 9 11 12

7 : 1 2 3 5 9 10 11 12

8 : 0 1 3 4 5 6 10 12

9 : 0 1 2 4 6 7 11 12

10 : 1 2 3 4 5 7 8 12

11 : 0 2 3 4 5 6 7 9

12 : 0 1 2 3 5 6 7 8 9 10

Graph 29, on 13 vertices

0 : 2 4 5 6 8 9 11 12

1 : 3 4 6 7 8 9 10 11

2 : 0 4 5 7 9 10 11 12

3 : 1 5 6 7 8 10 11 12

4 : 0 1 2 6 8 9 10 11

5 : 0 2 3 7 8 10 11 12

6 : 0 1 3 4 8 9 11 12

7 : 1 2 3 5 9 10 11 12

8 : 0 1 3 4 5 6 10 12

9 : 0 1 2 4 6 7 11 12

10 : 1 2 3 4 5 7 8 12

11 : 0 1 2 3 4 5 6 7 9

12 : 0 2 3 5 6 7 8 9 10

Graph 30, on 13 vertices

0 : 2 4 5 6 8 9 11 12

1 : 3 4 6 7 8 9 10 12

2 : 0 4 5 7 9 10 11 12

3 : 1 5 6 7 8 10 11 12

4 : 0 1 2 6 8 9 10 11

5 : 0 2 3 7 8 10 11 12

6 : 0 1 3 4 8 9 11 12

7 : 1 2 3 5 9 10 11 12

8 : 0 1 3 4 5 6 10 12

9 : 0 1 2 4 6 7 11 12

10 : 1 2 3 4 5 7 8 11

11 : 0 2 3 4 5 6 7 9 10

12 : 0 1 2 3 5 6 7 8 9

Graph 31, on 13 vertices

0 : 2 4 5 6 7 9 10 12

1 : 3 4 6 7 8 10 11 12

2 : 0 4 5 7 8 9 10 11

3 : 1 5 6 7 8 9 11 12

4 : 0 1 2 6 8 9 10 11

5 : 0 2 3 7 8 9 11 12

6 : 0 1 3 4 9 10 11 12

7 : 0 1 2 3 5 10 11 12

8 : 1 2 3 4 5 9 10 12

9 : 0 2 3 4 5 6 8 11

10 : 0 1 2 4 6 7 8 12

11 : 1 2 3 4 5 6 7 9

12 : 0 1 3 5 6 7 8 10

Graph 32, on 13 vertices

0 : 2 4 5 6 7 9 10 11

1 : 3 4 6 7 8 9 11 12

2 : 0 4 5 7 8 9 10 12

3 : 1 5 6 7 8 10 11 12

4 : 0 1 2 6 9 10 11 12

5 : 0 2 3 7 8 9 10 11

6 : 0 1 3 4 8 10 11 12

7 : 0 1 2 3 5 9 10 12

8 : 1 2 3 5 6 9 11 12

9 : 0 1 2 4 5 7 8 11

10 : 0 2 3 4 5 6 7 12

11 : 0 1 3 4 5 6 8 9

12 : 1 2 3 4 6 7 8 10

Graph 33, on 13 vertices

0 : 2 4 5 6 8 9 11 12

1 : 3 4 6 7 8 9 10 12

2 : 0 4 5 7 9 10 11 12

3 : 1 5 6 7 8 9 10 11

4 : 0 1 2 6 7 9 11 12

5 : 0 2 3 7 8 10 11 12

6 : 0 1 3 4 8 9 10 11

7 : 1 2 3 4 5 10 11 12

8 : 0 1 3 5 6 9 10 12

9 : 0 1 2 3 4 6 8 11

10 : 1 2 3 5 6 7 8 12

11 : 0 2 3 4 5 6 7 9

12 : 0 1 2 4 5 7 8 10

A.4 Problem Graphs on 14 vertices

There are 11 problem graphs on 14 vertices, up to isomorphism.

Graph 1, on 14 vertices

0 : 3 4 6 7 10 11 12 13

1 : 4 5 7 8 10 11 12 13

2 : 5 6 8 9 10 11 12 13

3 : 0 6 7 9 10 11 12 13

4 : 0 1 7 8 10 11 12 13

5 : 1 2 8 9 10 11 12 13

6 : 0 2 3 9 10 11 12 13

7 : 0 1 3 4 10 11 12 13

8 : 1 2 4 5 10 11 12 13

9 : 2 3 5 6 10 11 12 13

10 : 0 1 2 3 4 5 6 7 8 9

11 : 0 1 2 3 4 5 6 7 8 9

12 : 0 1 2 3 4 5 6 7 8 9

13 : 0 1 2 3 4 5 6 7 8 9

Graph 2, on 14 vertices

0 : 3 5 6 7 9 10 11 12

1 : 4 5 7 8 9 10 12 13

2 : 4 6 7 8 9 11 12 13

3 : 0 5 6 8 9 10 11 13

4 : 1 2 7 8 10 11 12 13

5 : 0 1 3 6 9 10 12 13

6 : 0 2 3 5 9 11 12 13

7 : 0 1 2 4 8 10 11 12

8 : 1 2 3 4 7 10 11 13

9 : 0 1 2 3 5 6 12 13

10 : 0 1 3 4 5 7 8 11

11 : 0 2 3 4 6 7 8 10

12 : 0 1 2 4 5 6 7 9

13 : 1 2 3 4 5 6 8 9

Graph 3, on 14 vertices

0 : 3 5 6 7 8 10 12 13

1 : 4 7 8 9 10 11 12 13

2 : 4 7 8 9 10 11 12 13

3 : 0 5 6 7 9 10 11 13

4 : 1 2 5 6 8 9 11 12

5 : 0 3 4 6 8 10 11 12

6 : 0 3 4 5 9 10 12 13

7 : 0 1 2 3 8 9 11 13

8 : 0 1 2 4 5 7 11 12

9 : 1 2 3 4 6 7 11 13

10 : 0 1 2 3 5 6 12 13

11 : 1 2 3 4 5 7 8 9

12 : 0 1 2 4 5 6 8 10

13 : 0 1 2 3 6 7 9 10

Graph 4, on 14 vertices

0 : 3 4 6 8 9 11 12 13

1 : 3 5 6 7 10 11 12 13

2 : 4 5 7 8 9 10 12 13

3 : 0 1 6 7 9 11 12 13

4 : 0 2 6 8 9 10 12 13

5 : 1 2 7 8 10 11 12 13

6 : 0 1 3 4 9 10 12 13

7 : 1 2 3 5 9 11 12 13

8 : 0 2 4 5 10 11 12 13

9 : 0 2 3 4 6 7 12 13

10 : 1 2 4 5 6 8 12 13

11 : 0 1 3 5 7 8 12 13

12 : 0 1 2 3 4 5 6 7 8 9 10 11

13 : 0 1 2 3 4 5 6 7 8 9 10 11

Graph 5, on 14 vertices

0 : 3 4 6 8 9 11 12 13

1 : 3 5 6 7 10 11 12 13

2 : 4 5 7 8 9 10 12 13

3 : 0 1 6 7 9 11 12 13

4 : 0 2 6 8 9 10 12 13

5 : 1 2 7 8 10 11 12 13

6 : 0 1 3 4 9 10 12 13

7 : 1 2 3 5 9 11 12 13

8 : 0 2 4 5 10 11 12 13

9 : 0 2 3 4 6 7 10 11

10 : 1 2 4 5 6 8 9 11

11 : 0 1 3 5 7 8 9 10

12 : 0 1 2 3 4 5 6 7 8

13 : 0 1 2 3 4 5 6 7 8

Graph 6, on 14 vertices

0 : 3 4 7 8 9 11 12 13

1 : 3 5 6 8 10 11 12 13

2 : 4 5 6 7 9 10 12 13

3 : 0 1 6 7 9 11 12 13

4 : 0 2 6 8 9 10 12 13

5 : 1 2 7 8 10 11 12 13

6 : 1 2 3 4 9 10 12 13

7 : 0 2 3 5 9 11 12 13

8 : 0 1 4 5 10 11 12 13

9 : 0 2 3 4 6 7 10 11

10 : 1 2 4 5 6 8 9 11

11 : 0 1 3 5 7 8 9 10

12 : 0 1 2 3 4 5 6 7 8

13 : 0 1 2 3 4 5 6 7 8

Graph 7, on 14 vertices

0 : 3 4 7 8 9 11 12 13

1 : 3 5 6 8 10 11 12 13

2 : 4 5 6 7 9 10 11 12

3 : 0 1 6 7 8 11 12 13

4 : 0 2 6 8 9 10 11 12

5 : 1 2 7 9 10 11 12 13

6 : 1 2 3 4 8 10 11 12

7 : 0 2 3 5 9 11 12 13

8 : 0 1 3 4 6 9 10 13

9 : 0 2 4 5 7 8 10 13

10 : 1 2 4 5 6 8 9 13

11 : 0 1 2 3 4 5 6 7

12 : 0 1 2 3 4 5 6 7

13 : 0 1 3 5 7 8 9 10

Graph 8, on 14 vertices

0 : 2 4 6 7 8 10 11 13

1 : 3 5 6 7 9 10 11 12

2 : 0 4 6 8 9 11 12 13

3 : 1 5 7 8 9 10 12 13

4 : 0 2 6 8 10 11 12 13

5 : 1 3 7 9 10 11 12 13

6 : 0 1 2 4 8 10 11 12

7 : 0 1 3 5 9 10 11 13

8 : 0 2 3 4 6 10 12 13

9 : 1 2 3 5 7 11 12 13

10 : 0 1 3 4 5 6 7 8

11 : 0 1 2 4 5 6 7 9

12 : 1 2 3 4 5 6 8 9

13 : 0 2 3 4 5 7 8 9

Graph 9, on 14 vertices

0 : 2 4 6 7 8 9 11 12

1 : 3 5 6 7 9 10 11 13

2 : 0 4 6 8 9 10 11 12

3 : 1 5 7 8 9 10 12 13

4 : 0 2 6 8 10 11 12 13

5 : 1 3 7 9 10 11 12 13

6 : 0 1 2 4 8 10 11 13

7 : 0 1 3 5 9 11 12 13

8 : 0 2 3 4 6 10 12 13

9 : 0 1 2 3 5 7 11 12

10 : 1 2 3 4 5 6 8 13

11 : 0 1 2 4 5 6 7 9

12 : 0 2 3 4 5 7 8 9

13 : 1 3 4 5 6 7 8 10

Graph 10, on 14 vertices

0 : 2 4 6 7 8 9 10 12

1 : 3 5 6 7 8 10 11 13

2 : 0 4 6 8 9 10 11 12

3 : 1 5 7 8 9 10 11 13

4 : 0 2 6 8 9 11 12 13

5 : 1 3 7 9 10 11 12 13

6 : 0 1 2 4 8 11 12 13

7 : 0 1 3 5 9 10 12 13

8 : 0 1 2 3 4 6 11 13

9 : 0 2 3 4 5 7 10 12

10 : 0 1 2 3 5 7 9 12

11 : 1 2 3 4 5 6 8 13

12 : 0 2 4 5 6 7 9 10

13 : 1 3 4 5 6 7 8 11

Graph 11, on 14 vertices

0 : 2 4 6 7 8 10 11 13

1 : 3 5 6 7 8 9 11 12

2 : 0 4 6 8 9 10 12 13

3 : 1 5 7 8 9 10 11 12

4 : 0 2 6 8 10 11 12 13

5 : 1 3 7 9 10 11 12 13

6 : 0 1 2 4 8 9 12 13

7 : 0 1 3 5 9 10 11 13

8 : 0 1 2 3 4 6 11 12

9 : 1 2 3 5 6 7 12 13

10 : 0 2 3 4 5 7 11 13

11 : 0 1 3 4 5 7 8 10

12 : 1 2 3 4 5 6 8 9

13 : 0 2 4 5 6 7 9 10

A.5 Problem Graphs on 15 vertices

There are 9 problem graphs on 15 vertices, up to isomorphism.

Graph 1, on 15 vertices

0 : 3 4 6 8 10 12 13 14

1 : 4 5 7 9 10 12 13 14

2 : 5 6 8 9 11 12 13 14

3 : 0 7 8 10 11 12 13 14

4 : 0 1 6 9 10 12 13 14

5 : 1 2 7 9 11 12 13 14

6 : 0 2 4 8 9 12 13 14

7 : 1 3 5 10 11 12 13 14

8 : 0 2 3 6 11 12 13 14

9 : 1 2 4 5 6 12 13 14

10 : 0 1 3 4 7 12 13 14

11 : 2 3 5 7 8 12 13 14

12 : 0 1 2 3 4 5 6 7 8 9 10 11

13 : 0 1 2 3 4 5 6 7 8 9 10 11

14 : 0 1 2 3 4 5 6 7 8 9 10 11

Graph 2, on 15 vertices

0 : 3 4 7 8 9 11 12 14

1 : 3 5 6 8 9 10 12 13

2 : 4 5 6 7 9 10 12 13

3 : 0 1 6 7 9 11 13 14

4 : 0 2 6 8 9 11 13 14

5 : 1 2 7 8 10 11 12 13

6 : 1 2 3 4 9 10 13 14

7 : 0 2 3 5 10 11 12 14

8 : 0 1 4 5 10 11 12 14

9 : 0 1 2 3 4 6 12 13

10 : 1 2 5 6 7 8 12 14

11 : 0 3 4 5 7 8 13 14

12 : 0 1 2 5 7 8 9 10

13 : 1 2 3 4 5 6 9 11

14 : 0 3 4 6 7 8 10 11

Graph 3, on 15 vertices

0 : 3 4 5 8 10 11 12 13

1 : 4 6 7 8 9 10 13 14

2 : 5 6 7 8 9 11 12 14

3 : 0 6 7 9 10 11 12 13

4 : 0 1 5 8 10 11 13 14

5 : 0 2 4 8 10 11 12 14

6 : 1 2 3 7 9 10 12 14

7 : 1 2 3 6 9 11 13 14

8 : 0 1 2 4 5 9 12 13

9 : 1 2 3 6 7 8 12 13

10 : 0 1 3 4 5 6 12 14

11 : 0 2 3 4 5 7 13 14

12 : 0 2 3 5 6 8 9 10

13 : 0 1 3 4 7 8 9 11

14 : 1 2 4 5 6 7 10 11

Graph 4, on 15 vertices

0 : 3 4 6 7 9 10 12 13

1 : 3 5 6 8 9 11 12 14

2 : 4 5 7 8 10 11 13 14

3 : 0 1 6 7 9 10 12 13

4 : 0 2 6 8 9 11 12 14

5 : 1 2 7 8 10 11 13 14

6 : 0 1 3 4 9 10 12 13

7 : 0 2 3 5 9 11 12 14

8 : 1 2 4 5 10 11 13 14

9 : 0 1 3 4 6 7 12 13

10 : 0 2 3 5 6 8 12 14

11 : 1 2 4 5 7 8 13 14

12 : 0 1 3 4 6 7 9 10

13 : 0 2 3 5 6 8 9 11

14 : 1 2 4 5 7 8 10 11

Graph 5, on 15 vertices

0 : 3 4 6 7 9 11 12 13

1 : 3 5 6 8 10 11 13 14

2 : 4 5 7 8 9 10 12 14

3 : 0 1 6 7 9 11 13 14

4 : 0 2 6 8 9 10 12 13

5 : 1 2 7 8 10 11 12 14

6 : 0 1 3 4 9 10 12 13

7 : 0 2 3 5 9 11 12 14

8 : 1 2 4 5 10 11 13 14

9 : 0 2 3 4 6 7 12 14

10 : 1 2 4 5 6 8 12 13

11 : 0 1 3 5 7 8 13 14

12 : 0 2 4 5 6 7 9 10

13 : 0 1 3 4 6 8 10 11

14 : 1 2 3 5 7 8 9 11

Graph 6, on 15 vertices

0 : 3 4 6 7 9 11 12 14

1 : 3 5 6 8 10 11 12 13

2 : 4 5 7 8 9 10 13 14

3 : 0 1 6 7 9 11 12 13

4 : 0 2 6 8 9 10 12 14

5 : 1 2 7 8 10 11 13 14

6 : 0 1 3 4 9 10 12 13

7 : 0 2 3 5 9 11 13 14

8 : 1 2 4 5 10 11 12 14

9 : 0 2 3 4 6 7 12 13

10 : 1 2 4 5 6 8 13 14

11 : 0 1 3 5 7 8 12 14

12 : 0 1 3 4 6 8 9 11

13 : 1 2 3 5 6 7 9 10

14 : 0 2 4 5 7 8 10 11

Graph 7, on 15 vertices

0 : 3 4 6 7 9 11 12 14

1 : 3 5 6 8 10 11 13 14

2 : 4 5 7 8 9 10 12 13

3 : 0 1 6 7 9 11 13 14

4 : 0 2 6 8 9 10 12 14

5 : 1 2 7 8 10 11 12 13

6 : 0 1 3 4 9 10 13 14

7 : 0 2 3 5 9 11 12 13

8 : 1 2 4 5 10 11 12 14

9 : 0 2 3 4 6 7 12 13

10 : 1 2 4 5 6 8 13 14

11 : 0 1 3 5 7 8 12 14

12 : 0 2 4 5 7 8 9 11

13 : 1 2 3 5 6 7 9 10

14 : 0 1 3 4 6 8 10 11

Graph 8, on 15 vertices

0 : 3 4 6 7 10 11 12 14

1 : 3 5 6 8 9 10 13 14

2 : 4 5 7 8 9 11 12 13

3 : 0 1 6 7 9 11 13 14

4 : 0 2 6 8 10 11 12 13

5 : 1 2 7 8 9 10 12 14

6 : 0 1 3 4 9 10 12 13

7 : 0 2 3 5 9 11 12 14

8 : 1 2 4 5 10 11 13 14

9 : 1 2 3 5 6 7 12 13

10 : 0 1 4 5 6 8 12 14

11 : 0 2 3 4 7 8 13 14

12 : 0 2 4 5 6 7 9 10

13 : 1 2 3 4 6 8 9 11

14 : 0 1 3 5 7 8 10 11

Graph 9, on 15 vertices

0 : 3 4 6 8 9 11 12 13

1 : 3 5 6 7 10 11 13 14

2 : 4 5 7 8 9 10 12 14

3 : 0 1 6 7 9 11 12 13

4 : 0 2 6 8 9 10 12 14

5 : 1 2 7 8 10 11 13 14

6 : 0 1 3 4 9 10 12 13

7 : 1 2 3 5 9 11 13 14

8 : 0 2 4 5 10 11 12 14

9 : 0 2 3 4 6 7 12 13

10 : 1 2 4 5 6 8 12 14

11 : 0 1 3 5 7 8 13 14

12 : 0 2 3 4 6 8 9 10

13 : 0 1 3 5 6 7 9 11

14 : 1 2 4 5 7 8 10 11

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