11. An Optimal Design Problem for Two-dimensional Composite Materials. A Constructive Approach

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We consider an optimal design problem, when it is necessary to locate circular inclusions of fixed size having a given conductivity in a material of fixed shape and a given conductivity in such a way that with prescribed concentration of the inclusions and prescribed external field the whole composite material has extremal conductivity in a given direction. The problem is reformulated in terms of a boundary value problem for analytic functions with unknown geometrical parameters and constraints. The boundary value problem in special cases is solved in closed form and the optimization problem is reduced to the classical problem of the extrema of a continuous function on a compact set.

KEY WORDS: optimal design problem, effective conductivity, boundary value problems, complex potential

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1. INTRODUCTION

We refer to optimal design problems when it is necessary to determine an optimal composite in a given class of admissible composites. The investigation of optimal design problems for composite conductive materials is of mathematical and practical interest. The problems for optimal composites are usually formulated as variational problems of minimization of stored energy in the composite material (see e.g. [1, 2, 5, 6] and papers cited therein). If unknown variables (e.g. the form of inclusions, their locations, size, etc.) and/or constrains have geometric nature we deal with shape optimization problems (or optimal design problems). Typical physical constrains are the given conductivities of the components, condition of ideal contact on the boundary matrix-inclusions, and the external field outside the composite. The latter two conditions can be represented in the form of the boundary value problems for certain potentials.

For periodic composites the optimal design problems coincide with the problem of optimization of effective conductivity tensor [6] in the so called representative cell. We have to mention also a number of approaches which are devoted to the study of laminate composites (see e.g. [13]), fibre composites (see e.g. [8]), composites with the reach microstructure (see e.g. [16]), nano-composites (see e.g. [14, 15]), etc.

This paper contains the description of a new approach in the study of optimal effective properties of the plane composite materials. In contrast to highly developed theory based on the weak and variational statement of the corresponding problems, this approach is oriented on the construction of the analytic solutions and even (when it is possible) the closed form solution. To show the perspectivity of this approach we use very simple examples, although several situations of more general type were considered recently as well.

We propose another statement of optimal design problems. The principal difference is that we fix shape and size of the inclusions. In particular, we consider a bounded domain occupied by a host material with N inclusions (N can be equal to unity). Hence, each inclusion has a positive concentration in the bulk material. It is known that if the characteristic size ε of the inclusion tends to zero (simultaneously N tends to infinity with fixed concentration of inclusions), the homogenization theory [7] can be applied. Such an approach does not work for our model.

We also discuss optimal design problems for unbounded domains. In this case for simple external field the considered problem is equivalent to the problem of optimization of the effective conductivity of dilute composite materials, when concentration of inclusions is sufficiently small. Anyway the problem that we arrive at could not be handled via the homogenization method.

The main mathematical difficulty which we try to overcome is that at the moment an analytic solution of the R-linear boundary value problem (Markushevich's problem in another sources) is not known. Moreover, the physically relevant statement of the optimal design problem in potential case leads to mixed boundary conditions (different kind of boundary value problems on each component of the boundary). By using our approach we could overcome these difficulties at least in the case of very important model situations. Among the achievements presented in the paper we ought to mention the discovered phenomena of "packing" of inclusions in the optimal composites.

Our study is useful in technical applications, because this problem corresponds to the following engineering task. A designer has at his disposal a material of given shape on the boundary of which a prescribed external field is applied. Let the designer also have inclusions of a given shape and size. It is necessary to locate these inclusions with a fixed concentration in such a way that the conductivity in the fixed direction will be maximal (minimal). So using our formulas a designer can project complex fibre composite materials to reach optimal properties.

In Section 2 we formulate the problem, *i.e.*, describe an objective functional and the constraints in terms of the complex potentials satisfying boundary value problems. A simpler form of the problem follows after certain calculations. Section 3 is devoted to the solution of the problem in the case of weakly inhomogeneous materials. Special attention is paid to the case of external field determined by the gradient $(1,0)$. Our results are illustrated in Section 4 by a number of special cases. We have solved a problem for a bounded domain with one inclusion and a problem for an unbounded domain with two inclusions. Final remarks and discussion are presented in Section 5.

2. FORMULATION OF THE OPTIMAL DESIGN PROBLEM

2.1. Boundary value problem. Let us consider a bounded simply connected domain Q in the extended complex plane $\mathbb{C} \cup \infty$. Let $\mathbb{D}_k$ $(k = 1, 2, \ldots, N)$ be mutually disjoint disks $\mathbb{D}_k := \{z \in \mathbb{C} : |z - a_k| < r\}$ of equal radii belonging to Q. Hence, Q is divided onto the multiply connected domain $D = Q - \bigcup_{k=1}^{N} \mathbb{D}_k$ (matrix) and the non-overlapping disks $\mathbb{D}_k$ (inclusions).

We study the conductivity of the composite material, when the domains D and $\mathbb{D}_k$ are occupied by materials of conductivities λ_m and λ_i, respectively. Suppose that the conductive field is potential, *i.e.*, there exists a potential $u(z)$ satisfying the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad \bigcup_{k=1}^{N} \mathbb{D}_k \cup D. \tag{2.1}$$

Let the conjugation conditions (conditions of ideal contact or transmission conditions)

$$u^+ = u^-, \quad \lambda_m \frac{\partial u^+}{\partial n} = \lambda_i \frac{\partial u^-}{\partial n} \quad \text{on} \quad \mathbb{T}_k, \quad k = 1, 2, \ldots, N, \tag{2.2}$$

be valid. Here $\frac{\partial}{\partial n}$ is the outward normal derivative, the circle $\mathbb{T}_k := \{z \in \mathbf{C} : |z - a_k| = r\}$ is the boundary of $\mathbb{D}_k$ and

$$u^+(t) := \lim_{\substack{z \to t \\ z \in D}} u(z), \qquad u^-(t) := \lim_{\substack{z \to t \\ z \in \mathbb{D}_k}} u(z),$$

for $t \in \mathbb{T}_k$, $k = 1, 2, \ldots, N$.

We also assume that $u(z)$ satisfies a boundary condition on ∂Q. Let $h(t)$ be a given Hölder continuous function on the curve ∂Q. The following boundary conditions are considered: the Dirichlet condition

$$u(t) = h(t), \quad t \in \partial Q, \tag{2.3}$$

the Neumann condition

$$\frac{\partial u}{\partial n} = h_1(t), \quad t \in \partial Q, \tag{2.4}$$

and the mixed condition when the Dirichlet and Neumann conditions are given on different parts of ∂Q.

It is possible to introduce two types of the complex potentials corresponding to the problem (2.1)–(2.4) [12]. The first type has the form

$$\varphi(z) = u(z) + iv(z), \quad z \in D; \qquad \varphi_k(z) = \frac{\lambda_i + \lambda_m}{2\lambda_m}(u_k(z) + iv_k(z)), \quad z \in \mathbb{D}_k,$$

where the functions $\varphi(z)$, $\varphi_k(z)$ are analytic in D, $\mathbb{D}_k$ respectively and continuously differentiable in the closures of the considered domains. Observe that the function $\varphi(z)$ is single valued in D due to its definition and condition (2.2). Then two real conditions (2.2) on each $\mathbb{T}_k$ can be written in terms of these potentials

$$\varphi(t) = \varphi_k(t) - \rho\overline{\varphi_k(t)}, \quad |t - a_k| = r, \quad k = 1, 2, \dots, N, \tag{2.5}$$

where $\rho = (\lambda_i - \lambda_m)/(\lambda_i + \lambda_m)$ is a contrast parameter (see e.g. [3]), known also as Bergmann parameter. The relation (2.5) is a special case of so called $\mathbb{R}$–linear condition. The Dirichlet condition (2.3) becomes

$$\operatorname{Re}\varphi(t) = h(t), \quad t \in \partial Q. \tag{2.6}$$

The function $h(t)$ can be considered as the real part of a function $f(t)$ analytically continued into the domain Q. Then (2.6) can be written in the form of the homogeneous condition

$$\operatorname{Re}(\varphi(t) - f(t)) = 0, \quad t \in \partial Q. \tag{2.7}$$

Remark. The function $f(z)$ can be considered as a solution of the problem (2.5)–(2.6) with $\rho = 0$ (or more exactly, of the Schwarz problem (2.6) for the domain Q). In this case the potential $f(z)$ does not depend on $\lambda_i = \lambda_m$.

The second type of the potentials is defined by the derivatives

$$\psi(z) = \varphi'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, \quad \psi_k(z) = \varphi_k'(z). \tag{2.8}$$

Then (2.2) implies the following $\mathbb{R}$–linear condition

$$\psi(t) = \psi_k(t) + \rho\left(\frac{r}{t - a_k}\right)^2\overline{\psi_k(t)}, \quad |t - a_k| = r, \quad k = 1, 2, \dots, N. \tag{2.9}$$

Here we use the relation [12]

$$[\overline{\psi_k(t)}]' = -\rho\left(\frac{r}{t - a_k}\right)^2\overline{\psi_k(t)}, \quad |t - a_k| = r. \tag{2.10}$$

Let $n(t) = n_1 + in_2$ be the outward normal vector to ∂Q expressed by complex values. Then the Neumann condition (2.4) takes the form

$$\operatorname{Re}\{n(t)\psi(t)\} = f_1(t), \quad t \in \partial Q, \tag{2.11}$$

since

$$\frac{\partial u}{\partial n} = n_1\frac{\partial u}{\partial x} + n_2\frac{\partial u}{\partial y} = \operatorname{Re}(n_1 + in_2)\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right),$$

where $f_1(z)$ is a solution to the Schwarz problem

$$\operatorname{Re} f_1(t) = h_1(t), \quad t \in \partial Q, \tag{2.12}$$

for the bounded simply connected domain Q. Thus the problem (2.12) can be reformulated as the special case of the homogeneous Riemann-Hilbert problem

$$\mathrm{Re}\{n(t)\psi(t) - f_1(t)\} = 0, \quad t \in \partial Q. \tag{2.13}$$

2.2. Optimal design problem. We now consider the problem (2.1)–(2.2), i.e., (2.5)–(2.7) in terms of complex potentials. Local relation between the flux $\boldsymbol{q}$ and the gradient ∇u is given by the state law (Fourier's law in thermal conductivity or Ohm's law in electric problems)

$$\boldsymbol{q} = \begin{cases} \lambda_m \nabla u & \text{in} \quad D, \\ \lambda_i \nabla u & \text{in} \quad \bigcup\limits_{k=1}^{N} \mathbb{D}_k. \end{cases} \tag{2.14}$$

We introduce the value λ_e by the relation

$$\lambda_e F = \lambda_m \iint_D \frac{\partial u}{\partial x}\, dx\, dy + \lambda_i \sum_{k=1}^{N} \iint_{\mathbb{D}_k} \frac{\partial u_k}{\partial x}\, dx\, dy, \tag{2.15}$$

where

$$F = \iint_Q \mathrm{Re}\, f'(z)\, dx\, dy. \tag{2.16}$$

In the right-hand part of (2.15) we have the x–coordinate of the total flux passing through the domain Q. In the left-hand part we have the x–coordinate of the total flux of the potential $f(z)$ multiplied by a new value λ_e which will be called the conductivity of the domain Q in the x–direction. The direction x is fixed only for definiteness, and the conductivity of the domain Q in any other direction can be considered in the same way. One can consider the function $f(z)$ as a solution to the homogeneous Schwarz problem (2.7) for the domain Q, i.e., for a homogenized material with a constant conductivity λ_e. Then (2.15) can be considered as equality of the fluxes in the x–direction of the composite and homogenized materials.

Further, we normalize the given function $f(z)$ assuming that

$$F = \iint_Q \mathrm{Re}\, f'(z) dx dy = \int_{\partial Q} \mathrm{Re}\, f(t)\, dy = \int_{\partial Q} h(t)\, dy = |Q|. \tag{2.17}$$

In particular if the external field is determined by the potential $f(z) = z$ then (2.17) is fulfilled. Therefore if $u(z)$ and $u_k(z)$ satisfy the problem (2.1)–(2.2), (2.12) for the square Q, the relation (2.15) coincides with the definition of the effective conductivity of a composite material in the x–direction (for details see [11] and [4]).

Using Green's formula and the mean value theorem of the harmonic function theory one can rewrite (2.15) in the following form

$$\lambda_e F = \lambda_m \int_{\partial Q} u\, dy + \pi r^2 (\lambda_i - \lambda_m) \sum_{k=1}^{N} \frac{\partial u_k}{\partial x}(a_k), \tag{2.18}$$

We can replace the potential $u(z)$ by $\operatorname{Re} f(z)$ on ∂Q. Using (2.10) we write (2.18) in the form

$$\frac{\lambda_e}{\lambda_m} = 1 + \pi r^2 \frac{\lambda_i - \lambda_m}{\lambda_m} \sum_{k=1}^{N} \frac{\partial u_k}{\partial x}(a_k). \tag{2.19}$$

For our purposes it is convenient to rewrite (2.19) in terms of the complex potentials

$$\frac{\lambda_e}{\lambda_m} = 1 + 2\rho\nu|Q| \frac{1}{N} \sum_{k=1}^{N} \operatorname{Re} \varphi_k'(a_k), \tag{2.20}$$

where $\nu = \dfrac{N\pi r^2}{|Q|}$ is the concentration of the inclusions in Q.

In the present paper we are interested in the following optimal design problems:

Problem A. Let Q be a given bounded domain in $\mathbb{C}$, $f(z)$ be a given function analytic in Q and the fixed constant ρ satisfy the inequality $|\rho| < 1$. Let N disks of the fixed radius r be embedded in Q. Let $\varphi(z)$ and $\varphi_k(z)$ satisfy the problem (2.5), (2.7).

The question is to locate these disks in such a way that the functional λ_e from (2.20) possesses the maximum (minimum) value.

In this statement the disks have to be mutually disjoint but possibly touching each other and the boundary of Q.

If the boundary of Q is far away from all inclusions we can model this case putting ∂Q to infinity. Then instead of the boundary condition on ∂Q we assume that $f(z)$ has a singularity at infinity. In this case we also use the formula (2.20) assuming that $\nu|Q|$ is a given positive number.

Problem B. Let $\nu|Q|$ be a given positive number, $f(z)$ be a given function analytic in $\mathbb{C}$ and the fixed constant ρ satisfy the inequality $|\rho| < 1$. Let N disks of the fixed equal radius r be embedded in $\mathbb{C}$. Let $\psi(z)$ and $\psi_k(z)$ satisfy the problem

$$\psi(t) = \psi_k(t) + \rho \left(\frac{r}{t - a_k} \right)^2 \overline{\psi_k(t)} - f'(t), \quad |t - a_k| = r_k, \quad k = 1, 2, \ldots, N, \tag{2.21}$$

where $\psi_k(t)$ is analytic in $\mathbb{D}_k$, $\psi(z)$ is analytic in the complement of all $\mathbb{D}_k$ in the extended complex plane $\mathbb{C} \cup \{\infty\}$ and $\psi(\infty) = 0$.

The question is to locate these disks in such a way that the functional λ_Q from (2.20) possesses the maximum (minimum) value.

We have to mention that for fixed positions of the inclusions problem (2.5), (2.7) as well as problem (2.21) has a unique solution (cf. [12, Cor. 2.3]).

In order to study Problems A and B it is sufficient to investigate the value

$$\sigma = \sum_{k=1}^{N} \operatorname{Re} \varphi_k'(a_k) \quad \text{or} \quad \sigma = \sum_{k=1}^{N} \operatorname{Re} \psi_k(a_k) \tag{2.22}$$

instead of λ_e because only this part of

$$\frac{\lambda_e}{\lambda_m} = 1 + 2\rho\nu \frac{|Q|}{N} \sigma$$

depends on the locations of the inclusions, i.e., depends on the parameters a_k. Therefore, Problem A can be formulated also as follows:

Problem $\mathbf{A}_0$. Let Q be a given domain in $\mathbb{C}$, $f(z)$ be a given function analytic in Q and the fixed constant ρ satisfy the inequality $|\rho| < 1$. Let N disks of the fixed radius r be embedded in Q. Let $\varphi(z)$ and $\varphi_k(z)$ satisfy the problem (2.5), (2.7).

The question is to locate these disks in such a way that the functional σ from (2.22) possesses the maximum (minimum) value.

Along similar lines Problem B_0 corresponding to Problem B can be formulated.

Remark. If inclusions have another shape different from the circular one, the objective functional σ becomes

$$\sigma = \iint_P \operatorname{Re} \psi_k(z)\, dx\, dy,$$

where P is the union of all inclusions, $\psi_k(z)$ is the potential in the k–th inclusion.

Theorem 1. *For any fixed value of contrast parameter ρ ($|\rho| < 1$) and a fixed radius of inclusions r there exists an optimal configuration which solves Problem A (Problem A_0).*

Proof. The Problem A_0 means optimization of the functional λ_e in (2.22) on the set of the solutions of (2.9). In [12] the solution to $\mathbb{R}$–linear problem (2.9) was solved in analytic form. The components $\psi_k(z) = \psi_k(z; \rho, r^2, a_1, \ldots, a_N))$ of this solution are analytic in ρ, r^2 and continuous in $a_1, \ldots, a_N$. Therefore the functional λ_e is continuous with respect to the variables $a_1, \ldots, a_N$ and hence bounded on the set

$$\Omega := \{(a_1, \ldots, a_N) \in \mathbb{C}^N : \ a_j \in Q, \ j = 1, \ldots, N; \ |a_k - a_j| \geq 2r, \ k \neq j\}. \quad (2.23)$$

To find optimal configuration solving the Problem A it is sufficient now to find an extremum of the real-valued function λ_e of N complex variables $a_1, \ldots, a_N$ in the bounded closed domain Ω. The existence of the later problem follows from the classical Weierstrass Theorem for a compact domain in $\mathbb{C}^N$. □

3. WEAKLY INHOMOGENEOUS MATERIALS

In this Section we provide a deeper study of Problem B for weakly inhomogeneous materials, *i.e.*, for materials with small contrast parameter ρ.

At the beginning we assume that the location of the disks $\mathbb{D}_k$ is known. Using [12] one can represent the functions $\psi(z)$, $\psi_k(z)$ in the form

$$\begin{aligned} \psi(z) &= \psi^{(0)}(z) + \rho\psi^{(1)}(z) + O(\rho^2), \\ \psi_k(z) &= \psi_k^{(0)}(z) + \rho\psi_k^{(1)}(z) + O(\rho^2), \quad \rho \to 0. \end{aligned} \quad (3.1)$$

Let us substitute (3.1) in (2.9) and compare the coefficients at ρ^0 and ρ^1. The first relation is then the jump problem for a multiply connected domain

$$\psi^{(0)}(t) = \psi_k^{(0)}(t) - f'(t), \quad |t - a_k| = r, \quad k = 1, 2, \ldots, N, \quad (3.2)$$

with respect to $\psi^{(0)}(z)$ and $\psi_k^{(0)}(z)$ analytic in D and $\mathbb{D}_k$ respectively. Since $f(z)$ is by assumption analytic in Q, then the unique solution for (3.2) has the form

$$\psi^{(0)}(z) = 0, \quad \psi_k^{(0)}(z) = f'(z), \quad k = 1, 2, \ldots, N. \quad (3.3)$$

The second relation gives another jump problem after substituting (3.3)

$$\psi^{(1)}(t) = \psi_k^{(1)}(t) + \left(\frac{r}{t-a_k}\right)^2 \overline{f'(t)}, \quad |t-a_k| = r, \quad k = 1, 2, \ldots, N. \tag{3.4}$$

Due to relation (2.8), between complex potentials φ and ψ one can consider instead of (3.4) the problem

$$\varphi^{(1)}(t) = \varphi_k^{(1)}(t) - \overline{f(t)}, \quad |t-a_k| = r, \quad k = 1, 2, \ldots, N, \tag{3.5}$$

with respect to the complex potentials $\varphi^{(1)}(z)$ and $\varphi_k^{(1)}(z)$ for which $(\varphi^{(1)}(z))' = \psi^{(1)}(z)$, $(\varphi_k^{(1)}(z))' = \psi_k^{(1)}(z)$ and $\varphi^{(1)}(\infty) = 0$ (see (2.8) and (2.10)). Let us note that

$$\overline{f(t)} = \overline{f\left(\frac{r^2}{\overline{t-a_k}} + a_k\right)}$$

is analytically continued to $|z-a_k| > r$, since $f(z)$ is analytic in $|z-a_k| < r$. Then the unique solution to (3.5) can be found in the form

$$\begin{aligned} \varphi^{(1)}(z) &= -\sum_{m=1}^{n} \overline{f\left(\frac{r^2}{\overline{z-a_m}} + a_m\right)}, \quad z \in Q, \\ \varphi_k^{(1)}(z) &= -\sum_{m \neq k} \overline{f\left(\frac{r^2}{\overline{z-a_m}} + a_m\right)}, \quad z \in \mathbb{D}_k, \quad k = 1, 2, \ldots, N, \end{aligned} \tag{3.6}$$

where in the sum $\sum\limits_{m \neq k}$ the index m runs over 1 to N except $m \neq k$. Therefore,

$$\psi_k^{(1)}(z) = \sum_{m \neq k} \left(\frac{r}{z-a_k}\right)^2 \overline{f'\left(\frac{r^2}{\overline{z-a_m}} + a_m\right)}, \quad z \in \mathbb{D}_k, \quad k = 1, 2, \ldots, N. \tag{3.7}$$

Substituting (3.3), (3.7) in (3.1) and further to (2.22) we obtain

$$\sigma = \mathrm{Re}\left(\sum_{k=1}^{N} f'(a_k) + \rho \sum_{k=1}^{N} \sum_{m \neq k} \left(\frac{r}{a_k - a_m}\right)^2 \overline{f'\left(\frac{r^2}{\overline{a_k - a_m}} + a_m\right)}\right) + O\left(\rho^2\right). \tag{3.8}$$

Further investigation of the extremal values of σ can be performed by the standard calculus techniques.

In the case $f(z) = z$ we obtain from (3.8)

$$\sigma = N + \rho\sigma_1 + O\left(\rho^2\right),$$

where

$$\sigma_1 = \sum_{k=1}^{N} \sum_{m \neq k} \mathrm{Re}\, \frac{1}{(a_k - a_m)^2}. \tag{3.9}$$

depends on the locations of the centers a_k, and the last term is comparatively small with respect to others.

Theorem 2. *Let ρ be sufficiently small and $f(z) = z$. Then there exists a configuration of inclusions for which the functional λ_e possesses the maximum (minimum) value, i.e. there exists a solution to the Problem B.*

Each inclusion corresponding to this solution touches at least one other inclusion.

Proof. For ρ being sufficiently small one can neglect the last term in (3.8). Then only changing part of the functional λ_e we have to consider the functional σ_1 defined in (3.9). It is the real-valued function of N complex variables $(a_1, \ldots, a_N) \in \Omega$, where Ω is defined in the previous section by relation (2.23). The domain Ω is a compact subset of $\mathbb{C}^N$. Hence σ_1 possesses its maximum (minimum) in Ω due to the Weierstrass Theorem. It follows from the definition of the domain Ω that σ_1 is a harmonic function in Ω. Therefore extremal points of the function σ_1 belong to the boundary of Ω by the Maximum Principle for harmonic functions of several variables in a compact domain in $\mathbb{C}^N$. Thus the second statement of the theorem follows. □

4. SOME EXAMPLES AND PARTICULAR CASES

4.1. Problem a with one inclusion. Following the previous section we consider first Problem A for weakly inhomogeneous materials, *i.e.*, in the case of small contrast parameter ρ.

Let the domain Q be the unit disc $Q = \mathbb{U}$ (it can be supposed without loss of generality due to conformal equivalence of the form of the considered boundary value problems). In the case $N = 1$ the domain Q is divided by the circle $\mathbb{T}(a, r) := \{z \in \mathbb{C} : |z - a| = r\}$ with the fixed radius r onto two parts: $D^- = \{z \in \mathbb{C} : |z - a| < r\}$ and the doubly connected domain $D^+ = \mathbb{U} \setminus D^-$. Let a real-valued function $h(t)$ and a complex-valued function $f(t)$ be given Hölder continuous functions on $\partial Q = \mathbb{T}$ and $\mathbb{T}(a, r) := \partial D^-$ respectively,

$$\int_{\mathbb{T}} h(t)\, dy = 1. \tag{4.1}$$

The question is to locate the disk D^- in $\mathbb{U}$ in such a way that

$$\sigma := \operatorname{Re} \varphi'(a) \to \max(\min), \tag{4.2}$$

i.e., the functional σ possesses the maximum (minimum) value on the set of the solutions of the following boundary value problems:

$$\operatorname{Re} \varphi(t) = h(t), \quad t \in \partial\mathbb{T}, \tag{4.3}$$

$$\varphi^+(t) - \varphi^-(t) = \overline{f(t)}, \quad t \in \mathbb{T}(a, r), \tag{4.4}$$

Here we use the same notation for the complex potentials in matrix and in inclusions; $\varphi(z)$ corresponds to $\varphi^{(1)}(z)$ and $\varphi_1^{(1)}(z)$ from (3.5).

Let us first assume that the location of a point a is known, solve mixed boundary value problem (4.3)–(4.4) and then consider the corresponding optimization problem.

The Cauchy type integral

$$\Phi(z) := \frac{1}{2\pi i}\int_{\mathbb{T}(a,r)} \frac{\overline{f(\tau)}}{\tau - z}\,d\tau, \quad z \in D^+ \cup D^- \tag{4.5}$$

solves the jump problem (4.4). Introduce the unknown function $\Psi(z) := \varphi(z) - \Phi(z)$, $z \in \mathbb{U}$. It is analytic in $\mathbb{U}$, because $\Psi^+(t) - \Psi^-(t) = 0$ on $\mathbb{T}(a,r)$. The real part of the boundary value of this function on the unit circle $\mathbb{T}$ is equal to

$$\operatorname{Re}\Psi(t) = \operatorname{Re}(\varphi(t) - \Phi(t)) = h(t) - \operatorname{Re}\Phi(t), \quad t \in \mathbb{T}. \tag{4.6}$$

The equality (4.6) is a Schwarz problem for the unit disc $\mathbb{U}$ with respect to the function $\Psi(z)$. Its solution has the form

$$\Psi(z) := \frac{1}{\pi i}\int_{\mathbb{T}} \frac{h(t)}{t-z}\,dt - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{h(t)}{t}\,dt - \frac{1}{\pi i}\int_{\mathbb{T}} \frac{\operatorname{Re}\Phi(t)}{t-z}\,dt + \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\operatorname{Re}\Phi(t)}{t}\,dt + ic_0, \tag{4.7}$$

where c_0 is an arbitrary real constant. Then the solution to the problem (4.3)–(4.4) is given by the formula

$$\varphi(z) = \Psi(z) + \Phi(z), \tag{4.8}$$

where $\Psi(z)$ and $\Phi(z)$ are defined in (4.7) and (4.4) respectively. Hence

$$\varphi'(a) := \frac{1}{\pi i}\int_{\mathbb{T}} \frac{f(t)}{(t-a)^2}\,dt - \frac{1}{\pi i}\int_{\mathbb{T}} \frac{\operatorname{Re}\Phi(t)}{(t-a)^2}\,dt + \frac{1}{2\pi i}\int_{\mathbb{T}(a,r)} \frac{\overline{f(t)}}{(t-a)^2}\,dt. \tag{4.9}$$

Therefore, the problem (4.2)–(4.4) is reduced to the determination of extremum of the function $F(\xi,\eta) := \operatorname{Re}\varphi'(\xi + i\eta)$, $a = \xi + i\eta$, in the closed disk $K := \{(\xi,\eta) : \xi^2 + \eta^2 \le (1-r)^2\}$.

In the case

$$f(z) = z \tag{4.10}$$

the integral (4.5) becomes

$$\Phi(z) := \frac{1}{2\pi i}\int_{\mathbb{T}(a,r)} \frac{\overline{\tau}}{\tau - z}\,d\tau = \begin{cases} \overline{a}, & z \in D^+, \\ -\dfrac{r^2}{z-a}, & z \in D^-. \end{cases} \tag{4.11}$$

In this case we have $\operatorname{Re}\Psi(t) = \operatorname{Re}H(t) + r^2\operatorname{Re}\dfrac{1}{t-a}$ on the unit circle, where $H(z)$ is a solution to the following Schwarz problem for the unit disc

$$\operatorname{Re}H(t) = h(t), \qquad t \in \mathbb{T}.$$

The symmetry relation on the unit circle $\overline{t} = 1/t$ yields

$$\operatorname{Re}\Psi(t) = \operatorname{Re}\left(H(t) + \frac{r^2}{\overline{t-a}}\right) = \operatorname{Re}\left(H(t) + \frac{r^2}{1/t - \overline{a}}\right) = h(t) + \operatorname{Re}\frac{r^2 t}{1-\overline{a}t}, \quad t \in \mathbb{T}. \tag{4.12}$$

Hence $\Psi(z) = H(z) + \frac{r^2 z}{1-\overline{a}z} + ic_0$ and $\varphi'(z) = H'(z) + \frac{r^2}{(1-\overline{a}z)^2}$. The optimal values of the functional

$$\sigma = \operatorname{Re}\varphi'(a) = \operatorname{Re}H'(a) + \frac{r^2}{(1-|a|^2)^2} = \operatorname{Re}\frac{1}{2\pi i}\int_{\mathbb{T}} h(\tau)\frac{2\tau}{(\tau-a)^2}\,d\tau + \frac{r^2}{(1-|a|^2)^2}$$

depend now on the given function h.

Example. Assume $h(t) \equiv \operatorname{Re} t$. Then $\sigma = \operatorname{Re}\varphi'(a) = 1 + \frac{r^2}{(1-|a|^2)^2}$ depends only on $|a|$, $0 \le |a| \le 1-r$. It is obvious that the maximal value of the functional σ is equal to $\max_a \sigma = 1 + \frac{1}{(2-r)^2}$ (reached at $|a| = 1-r$).

4.2. Problem B with two inclusions. Consider the Problem B with the arbitrary value of the contrast parameter ρ. We assume additionally that $N = 2$ and $f(z) = z$. Therefore we need to find an optimal values of the functional

$$\sigma = \operatorname{Re}\{\varphi_1'(a_1) + \varphi_2'(a_2)\} \tag{4.13}$$

on the set of all analytic functions with $\varphi(\infty) \in \mathbb{C}$, satisfying the boundary condition

$$\varphi^+(t) = \varphi^-(t) + \rho\overline{\varphi^-(t)} + \overline{t}, \quad t \in L, \tag{4.14}$$

where $L \equiv L_1 \cup L_2$, $L_k = \mathbb{T}(a_k, r) = \{z \in \mathbb{C} : |z - a_k| = r\}$, $k = 1, 2$, are two circles with the fixed radius r. Location of their centers a_1, a_2 ($|a_1 - a_2| \ge 2r$) has to be determined.

Let z be an arbitrary point on the complex plane $\mathbb{C}$. Denote by the $z^*_{(k)}$ the point symmetric to z with respect to the circle $\mathbb{T}(a_k, r)$, *i.e.* $z^*_{(k)} = \frac{r^2}{\overline{z - a_k}} + a_k$, $k = 1, 2$.

Introduce the function

$$\Omega(z) = \begin{cases} \varphi_1(z) + \rho\overline{\varphi_2(z^*_{(2)})} - \overline{z^*_{(1)}}, & z \in D_1 \cup L_1, \\ \varphi_2(z) + \rho\overline{\varphi_1(z^*_{(1)})} - \overline{z^*_{(2)}}, & z \in D_2 \cup L_2, \\ \varphi(z) + \rho\overline{\varphi_1(z^*_{(1)})} + \rho\overline{\varphi_2(z^*_{(2)})}, & z \in D. \end{cases} \tag{4.15}$$

By construction the function $\Omega(z)$ is analytic in the domains D_1, D_2 and D. From the boundary condition (4.14) it follows that the function $\Omega(z)$ has no jump across the circles $\mathbb{T}(a_k, r)$, $k = 1, 2$, *i.e.* $\Omega^+(t) - \Omega^-(t) = 0$ because

$$\Omega^+(t) - \Omega^-(t) = \varphi_1(t) - \rho\overline{\varphi_1(t)} - \overline{t} - \varphi(t) = 0.$$

From the Liouville theorem it follows that

$$\Omega(z) = -\frac{r^2}{\overline{z - a_1}} - \frac{r^2}{\overline{z - a_2}} + c = -\overline{z^*_{(1)}} - \overline{z^*_{(2)}} + c_0.$$

Then we obtain the system of functional equations

$$\begin{cases} \varphi_1(z) = -\rho\overline{\varphi_2(z^*_{(2)})} - \overline{z^*_{(2)}} + c_0, & z \in \overline{D_1}, \\ \varphi_2(z) = -\rho\overline{\varphi_1(z^*_{(1)})} - \overline{z^*_{(1)}} + c_0, & z \in \overline{D_2} \end{cases} \tag{4.16}$$

From the second equation of the system we can find the values $\overline{\varphi_2(z^*_{(2)})}$ making symmetry with respect to the second circle

$$\overline{\varphi_2(z^*_{(2)})} = -\rho\varphi_1((z^*_{(2)})^*_{(1)}) - (z^*_{(2)})^*_{(1)} + \overline{c_0} = \rho\varphi_1(\alpha(z)) - \alpha(z) + \overline{c_0}. \tag{4.17}$$

Then

$$\varphi_1(z) = \rho^2\varphi_1[\alpha(z)] + h_1(z), \quad z \in \overline{D_1}, \tag{4.18}$$

where $\alpha(z)$ is the combination of two symmetries:

$$\alpha(z) = (z^*_{(2)})^*_{(1)} = \frac{r^2(z - a_2)}{r^2 + (\overline{a_2 - a_1})(z - a_2)} + a_1, \tag{4.19}$$

and inhomogeneous term $h_1(z) = \rho^2\alpha(z) - \dfrac{r^2}{z - a_2} - \overline{a_2} - \rho\overline{c_0} + c_0$. Applying the method of successive approximation (also called the Banach–Cacciopoli Fixed Point Theorem) we obtain that the solution $\varphi_1(z)$ in $\overline{D_1}$ can be represented in the form of the following series, uniformly converging in $\overline{D_1}$:

$$\varphi_1(z) = \sum_{k=0}^{\infty} \rho^{2k} h_1[\alpha_k(z)] = h_1(z) + \rho^2 h_1[\alpha(z)] + \ldots, \quad z \in \overline{D_1}, \tag{4.20}$$

where $\alpha_k(z)$ denotes the composition of k copies of $\alpha(z)$.

The absolute and uniform convergence of (4.20) is verified e.g. in [12]. The main observation leads to the conclusion that $|\alpha'(z)| < 1$ in $\overline{D_1}$. The same argument yields the following representation of the solution $\varphi_2(z)$:

$$\varphi_2(z) = \sum_{k=0}^{\infty} \rho^{2k} h_2[\beta_k(z)] = h_2(z) + \rho^2 h_2[\beta(z)] + \ldots, \quad z \in \overline{D_2}, \tag{4.21}$$

where $\beta_k(z)$-k-th iteration of the mapping

$$\beta(z) = (z^*_{(1)})^*_{(2)} = \frac{r^2(z - a_1)}{r^2 + (\overline{a_1 - a_2})(z - a_1)} + a_2,$$

$$h_2(z) = \rho^2\beta(z) - \frac{r^2}{z - a_1} - \overline{a_1} - \rho\overline{c_0} + c_0.$$

To study the properties of the functional (4.13) we calculate now the derivatives of the functions φ_1, φ_2. By the symmetry of their definition it is sufficient to calculate $\varphi_1'(a_1)$:

$$\varphi_1'(a_1) = h_1'(a_1) + \sum_{k=1}^{\infty} \rho^{2k} h_1'[\alpha_k(a_1)] \cdot \alpha'[\alpha_{k-1}(a_1)] \cdot \ldots \cdot \alpha'(a_1)$$

$$= \left[\rho^2\alpha'(a_1) + \frac{r^2}{(a_1 - a_2)^2}\right] \left(1 + \sum_{k=1}^{\infty} \rho^{2k} \cdot \alpha'[\alpha_{k-1}(a_1)] \cdot \ldots \cdot \alpha'(a_1)\right). \tag{4.22}$$

where

$$\alpha'(z) = \frac{r^4}{[(\overline{a_1 - a_2})(z - a_2) - r^2]^2}, \quad \alpha'(a_1) = \frac{r^4}{[(\overline{a_1 - a_2})(a_1 - a_2) - r^2]^2}. \quad (4.23)$$

Then by the direct calculation one can prove.

Lemma. *For each $k = 0, 1, \ldots$, the function $\gamma_k := (\overline{a_1 - a_2})(\alpha_k(a_1) - a_2)$ is a real-valued function depending on the parameter $A = |a_1 - a_2|$. Moreover γ_k is positive for $A > 2r$.*

Proof. The statement is evident for $k = 0$. For $k = 1$ one has from (4.23)

$$\alpha(a_1) - a_2 = -\frac{r^2(a_1 - a_2)}{(\overline{a_1 - a_2})(a_1 - a_2) - r^2} + (a_1 - a_2) = c(A)(a_1 - a_2),$$

$$c(A) := \frac{A^2 - 2r^2}{A^2 - r^2}.$$

Hence lemma is proven for $k = 1$. The final conclusion follows from the induction argument. □

Theorem 3. *For all fixed $|\rho| < 1$ effective functional possesses its maximal value on the set of the solutions of the boundary value problem occurs when circular inclusions are touching each other and located along real axis. The effective functional possesses its minimal value on the set of the solutions of the boundary value problem when circular inclusions are touching each other and located along imaginary axis.*

Proof. First we note that the values $\varphi_1'(a_1)$ and $\varphi_2'(a_2)$ are equal. Hence the functional σ can be represented in the form

$$\sigma = 2\left[\rho^2\alpha'(a_1) + \operatorname{Re}\frac{r^2}{(a_1 - a_2)^2}\right] + \left[1 + \operatorname{Re}\frac{r^2}{(a_1 - a_2)^2}\right] \cdot \sum_{k=1}^{\infty} \rho^{2k}\alpha_k'(a_1)$$
$$=: \chi_0(a_1, a_2, \rho) + \chi_1(a_1, a_2, \rho) \cdot \psi(a_1, a_2, \rho). \quad (4.24)$$

For all a_1, a_2, $|a_1 - a_2| \geq 2r$ and all $z \in \overline{D_1}$ we have

$$|\alpha'(z)| \leq \frac{r^4}{(\min_{z \in \overline{D_1}} |(\overline{a_1 - a_2})(z - a_2) - r^2|)^2} \leq \frac{r^4}{(2r^2 - r^2)^2} \leq 1.$$

Since $\alpha(\overline{D_1}) \subset D_1$, then $|\alpha_k'(z)| \leq 1$, $\forall z \in \overline{D_1}$, $\forall k = 1, 2, \ldots$ Thus the series $\psi(a_1, a_2, \rho)$ in (4.24) converges for all $|\rho| < 1$ uniformly with respect to a_1, a_2, $|a_1 - a_2| \geq 2r$.

It follows from Lemma that all terms in the series $\psi(a_1, a_2, \rho)$ are positive. Therefore for each fixed $|\rho| < 1$ the function $\psi(a_1, a_2, \rho)$ possesses its maximal values on the boundary of the domain $|a_1 - a_2| \geq 2r$, *i.e.*

$$0 < \psi(a_1, a_2, \rho) \leq \max_{|a_1 - a_2| \geq 2r} \psi(a_1, a_2, \rho) = \psi(a_1, a_2, \rho)\big|_{|a_1 - a_2| = 2r}.$$

From the other side for each fixed ρ the expressions $\chi_0(a_1, a_2, \rho), \chi_1(a_1, a_2, \rho)$ in (4.24) possess their maximal (positive) values on the boundary of the domain $|a_1 - a_2| \geq 2r$ if $a_1 - a_2 = \pm 2r$ and minimal (negative) values on the boundary of the domain $|a_1 - a_2| \geq 2r$ if $a_1 - a_2 = \pm 2ri$. It proves our theorem. □

5. CONCLUSIONS

The method of complex potentials is applied to optimal design problems under different constraints in the case of circular inclusions.

Examples show that in some cases the optimal configurations for the problems in a bounded domain or in the whole plane can be obtained either in the case of symmetric location of the inclusions or in the case of percolation, *i.e.*, when the inclusions either touch each other or the boundary of the matrix.

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